HAL
open science

Rational extensions of the trigonometric Darboux-Poschl-Teller potential based on para-Jacobi polynomials

B. Bagchi, Y. Grandati, C. Quesne

To cite this version:

B. Bagchi, Y. Grandati, C. Quesne. Rational extensions of the trigonometric Darboux-Poschl-Teller potential based on para-Jacobi polynomials. Journal of Mathematical Physics, 2015, 56 (6), pp.062103. 10.1063/1.4922017 . hal-01513176

HAL Id: hal-01513176

https://hal.univ-lorraine.fr/hal-01513176

Submitted on 9 May 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Rational extensions of the trigonometric Darboux-Pöschl-Teller potential based on para-Jacobi polynomials

B. Bagchi, Y. Grandati, and C. Quesne

Citation: Journal of Mathematical Physics **56**, 062103 (2015); doi: 10.1063/1.4922017
View online: http://dx.doi.org/10.1063/1.4922017
View Table of Contents: http://aip.scitation.org/toc/jmp/56/6
Published by the American Institute of Physics

JOURNAL OF MATHEMATICAL PHYSICS **56**, 062103 (2015)

Rational extensions of the trigonometric Darboux-Pöschl-Teller potential based on para-Jacobi polynomials

B. Bagchi,[1,a)] Y. Grandati,[2,b)] and C. Quesne[3,c)]
[1]*Department of Applied Mathematics, University of Calcutta, 92 Acharya Prafulla Chandra Road, Kolkata, India*
[2]*Equipe BioPhysStat, LCP A2MC, Université de Lorraine-Site de Metz, 1 Blvd. D. F. Arago, F-57070 Metz, France*
[3]*Physique Nucléaire Théorique et Physique Mathématique, Université Libre de Bruxelles, Campus de la Plaine CP229, Boulevard du Triomphe, B-1050 Brussels, Belgium*

(Received 24 November 2014; accepted 22 May 2015; published online 8 June 2015)

The possibility for the Jacobi equation to admit, in some cases, general solutions that are polynomials has been recently highlighted by Calogero and Yi, who termed them para-Jacobi polynomials. Such polynomials are used here to build seed functions of a Darboux-Bäcklund transformation for the trigonometric Darboux-Pöschl-Teller potential. As a result, one-step regular rational extensions of the latter depending both on an integer index n and on a continuously varying parameter λ are constructed. For each n value, the eigenstates of these extended potentials are associated with a novel family of λ-dependent polynomials, which are orthogonal on $]-1,1[$. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4922017]

I. INTRODUCTION

The construction of new solvable systems is an interesting problem by itself in mathematical physics. In the quantum framework, the techniques of Supersymmetric Quantum Mechanics (SUSY QM)[1–3] have proved to be one of the simplest and efficient tools to achieve this goal. From the analytical point of view, these basically correspond to a single or multi-step Darboux transformations.[4,5] Applying such transformations to a class of known solvable systems that are guided by primary translationally shape invariant potentials (TSIPs),[1–3,6,7] we can generate chains of new potentials, called extensions, which are (quasi)isospectral to the original potential and whose eigenfunctions are entirely determined in an explicit way.

Of particular interest are the rational extended potentials, i.e., extensions which are rational in an appropriate variable. It appears that those obtained from the three confining TSIPs (namely, the harmonic oscillator, isotonic oscillator, and trigonometric Darboux-Pöschl-Teller (TDPT) potentials) have eigenfunctions which coincide (up to a gauge factor) with the exceptional orthogonal polynomials recently discovered by Gómez-Ullate, Kamran, and Milson.[8–12] During the last five years, this topic has been the subject of an intense research activity (see Ref. 13 and references therein).

In general, for a given primary TSIP, the possible rational extensions are characterized by one or two multi-indices, which define in a unique way the corresponding extended potentials.[14–18] Then, until now, the rationality condition appears quite restrictive, by excluding the possibility to generate families of potentials depending on one or several continuous parameters, which is a very interesting feature of SUSY QM.[19,20] The one (n-)parameter isospectral families are in particular closely connected to one (multi-)soliton solutions of some nonlinear evolution equations.

a)bbagchi123@gmail.com
b)grandati@univ-lorraine.fr
c)cquesne@ulb.ac.be

 © 2015 AIP Publishing LLC

In this paper, we show that for the TDPT potential with some specific underlying parameters, it is nevertheless possible to generate one-parameter families of rational extensions via one-step state-adding Darboux transformations. To achieve this goal, we use seed functions based on para-Jacobi polynomials.[21] Although noticed by Szegö in his classical treatise,[22] the possibility for the Jacobi equation to admit in some cases a general solution that is a polynomial has not been considered in the other standard treatises on orthogonal polynomials. It has been highlighted only recently by Calogero and Yi,[21] who introduced the so-called para-Jacobi polynomials. This result has been now extended by Calogero, who has considered in particular a more general second-order linear differential equation, featuring an arbitrary number of free parameters, for which all the solutions are polynomials,[23] and has also defined the concept of generalized hypergeometric polynomials.[24]

The paper is organized as follows. We start by recalling the essential features of Darboux transformations and of the TDPT in Secs. II and III, respectively. In Sec. IV, we introduce the para-Jacobi polynomials and specify the domain of values of the free parameter for which they are nodeless on the considered interval. In Sec. V, we then use the corresponding eigenfunctions as seed functions for state-adding Darboux-Bäcklund transformations to build a set of extensions of the TDPT potentials, which are regular, rational, and dependent on one continuous parameter. In addition, we determine explicitly the eigenstates of these extended potentials and give plots for several values of the parameter in the first non-trivial case of the extension. Finally, Sec. VI contains the conclusion.

II. DARBOUX TRANSFORMATIONS AND EXTENDED SOLVABLE POTENTIALS

A formal eigenfunction $\psi_E(x)$ for the eigenvalue E of the one-dimensional Hamiltonian $\widehat{H} = -d^2/dx^2 + V(x)$, $x \in I \subset \mathbb{R}$, is a solution of the Schrödinger equation

$$\psi_E''(x) + (E - V(x))\,\psi_E(x) = 0. \tag{2.1}$$

In the following, we simply call eigenfunctions such formal eigenfunctions, employing eigenstates for the eigenfunctions satisfying given Dirichlet boundary conditions at the limits of the definition interval I. With such Dirichlet boundary conditions on I, $\widehat{H}$ is supposed to admit a discrete spectrum of energies and eigenstates $(E_n, \psi_n)_{n\in\{0,\ldots,n_{\max}\}\subseteq\mathbb{N}}$ where, without loss of generality, we can always take the ground level of $\widehat{H}$ at zero ($E_0 = 0$). We suppose also that in the considered domain of eigenvalues, the eigenfunctions can be indexed by a real spectral parameter μ ($E \to E_\mu, \psi_E \to \psi_\mu(x)$) which takes integer values $\mu = n \in \mathbb{N}$ when E_n is a bound-state energy.

Starting from an eigenfunction $\psi_\nu(x)$ of $\widehat{H}$, we define the first-order operator $\widehat{A}(w_\nu)$ by

$$\widehat{A}(w_\nu) = d/dx + w_\nu(x), \tag{2.2}$$

where $w_\nu(x) = -\psi_\nu'(x)/\psi_\nu(x)$. Then, if $W(y_1, \ldots, y_m \mid x)$ denotes the Wronskian of the functions $y_1, \ldots, y_m$,[25]

$$W(y_1, \ldots, y_m \mid x) = \begin{vmatrix} y_1(x) & \ldots & y_m(x) \\ \ldots & & \ldots \\ y_1^{(m-1)}(x) & \ldots & y_m^{(m-1)}(x) \end{vmatrix}, \tag{2.3}$$

for $\mu \neq \nu$, the function defined via the so-called Darboux-Crum formula

$$\psi_\mu^{(\nu)} = \widehat{A}(w_\nu)\,\psi_\mu(x) = \frac{W(\psi_\nu, \psi_\mu \mid x)}{\psi_\nu(x)} \tag{2.4}$$

is a solution of the Schrödinger equation

$$\psi_\mu^{(\nu)\prime\prime}(x) + \left(E_\mu - V^{(\nu)}(x)\right)\psi_\mu^{(\nu)}(x) = 0, \tag{2.5}$$

with the same energy E_μ as in Eq. (2.1), but with a modified potential

$$V^{(\nu)}(x) = V(x) + 2w_\nu'(x). \tag{2.6}$$

We call $V^{(\nu)}(x)$ an extension of $V(x)$ and the correspondence

$$\begin{pmatrix} V(x) \\ \psi_\mu(x) \end{pmatrix} \overset{\widehat{A}(w_\nu)}{\rightarrow} \begin{pmatrix} V^{(\nu)}(x) \\ \psi_\mu^{(\nu)}(x) \end{pmatrix} \tag{2.7}$$

is called a Darboux-Bäcklund Transformation (DBT). The eigenfunction ψ_ν is the seed function of the DBT $\widehat{A}(w_\nu)$.

Note that $\widehat{A}(w_\nu)$ annihilates ψ_ν and, consequently, Eq. (2.4) is not valid for $\mu = \nu$. Nevertheless, we can readily verify that $1/\psi_\nu(x)$ is an eigenfunction of $V^{(\nu)}(x)$ for the eigenvalue E_ν. Consequently, by extension, we define the "image" by $\widehat{A}(w_\nu)$ of the seed eigenfunction ψ_ν itself as

$$\psi_\nu^{(\nu)}(x) \sim 1/\psi_\nu(x). \tag{2.8}$$

In general, the transformed potential $V^{(\nu)}(x)$ is singular at the nodes of $\psi_\nu(x)$ and for integer values n of ν, $V^{(n)}$ is regular only when $n = 0$, that is, when the seed function is the ground state of $\widehat{H}$, which corresponds exactly to the usual SUSY partnership in quantum mechanics.[1–3]

We can nevertheless envisage to use as seed function an eigenfunction associated to an eigenvalue in the disconjugacy sector of Eq. (2.1), here a negative eigenvalue.[17] Indeed, in this sector, every solution of this equation has at most one (simple) zero on I[26,27] and, moreover, the existence of nodeless solutions of this equation is ensured.

To prove that a given solution $\psi_\nu(x)$ belongs to this category, it is sufficient to determine its signs at the boundaries of I. If they are identical, then ψ_ν is nodeless and if they are opposite, then ψ_ν presents a unique zero on I. In the first case, $V^{(\nu)}(x)$ constitutes then a regular (quasi)isospectral extension of $V(x)$.

III. TDPT POTENTIAL

The TDPT potential (with zero ground-state energy) is defined on $x \in]0, \pi/2[$ by

$$V(x;\alpha,\beta) = \frac{(\alpha+1/2)(\alpha-1/2)}{\sin^2 x} + \frac{(\beta+1/2)(\beta-1/2)}{\cos^2 x} - (\alpha+\beta+1)^2 \tag{3.1}$$

and is a confining potential for $|\alpha|, |\beta| > 1/2$, i.e., in the region bounded by the singularities of $V(x;\alpha,\beta)$ at $x = 0$ and $\pi/2$. Only in the case $|\alpha| = |\beta|$, the potential hole is symmetrical.

Consider a solution $\psi_E(x;\alpha,\beta)$ of the Schrödinger equation associated to this potential

$$\left(\widehat{H}(x;\alpha,\beta) - E\right)\psi_E(x;\alpha,\beta) = 0, \tag{3.2}$$

where

$$\widehat{H}(x;\alpha,\beta) = -\frac{d^2}{dx^2} + V(x;\alpha,\beta). \tag{3.3}$$

Introducing the gauge transformation

$$\psi_E(x;\alpha,\beta) = \psi_0(x;\alpha,\beta)\, y_E(x;\alpha,\beta), \tag{3.4}$$

with

$$\psi_0(x;\alpha,\beta) = (\sin x)^{\alpha+1/2}(\cos x)^{\beta+1/2}, \tag{3.5}$$

which is nonzero on $]0, \pi/2[$ and satisfies the Schrödinger equation at "zero energy"

$$\widehat{H}(x;\alpha,\beta)\psi_0(x;\alpha,\beta) = 0, \tag{3.6}$$

we obtain for $y_E(x;\alpha,\beta)$ the following second-order linear ordinary differential equation (ODE):

$$y_E''(x;\alpha,\beta) + 2(\log\psi_0(x;\alpha,\beta))' y_E'(x;\alpha,\beta) + E y_E(x;\alpha,\beta) = 0, \tag{3.7}$$

where the prime denotes a derivative with respect to x.

Using the change of variable $z = \cos 2x$, $z \in]-1,1[$, the above equation becomes (with a dot denoting a derivative with respect to z)

$$\left(1-z^2\right)\ddot{y}_E(z;\alpha,\beta) - [(\alpha+\beta+2)z+(\alpha-\beta)]\,\dot{y}_E(z;\alpha,\beta) + \frac{E}{4}y_E(z;\alpha,\beta) = 0, \tag{3.8}$$

in which we recognize a hypergeometric equation.[28]

Writing $E = E_\nu = 4\nu(\nu+\alpha+\beta+1)$, we arrive at

$$\begin{aligned}&\left(1-z^2\right)\ddot{y}_\nu(z;\alpha,\beta) - [(\alpha+\beta+2)z+(\alpha-\beta)]\,\dot{y}_\nu(z;\alpha,\beta)\\ &\quad+\nu(\nu+\alpha+\beta+1)\,y_\nu(z;\alpha,\beta) = 0,\end{aligned} \tag{3.9}$$

which, for integer values of the spectral parameter $\nu = n$ and values of the α,β parameters in the interval $]-1,+\infty[$, admits the Jacobi polynomial[22,28]

$$\begin{aligned}P_n^{(\alpha,\beta)}(z) &= \frac{1}{2^n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n+\alpha}{k}\binom{n+\beta}{n-k}(1-z)^{n-k}(1+z)^k\\ &= \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+\beta+1)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(n+\alpha+\beta+1+k)}{2^k\Gamma(\alpha+1+k)}(1-z)^k\\ &= \frac{(-1)^n\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(n+\alpha+\beta+1+k)}{2^k\Gamma(\beta+1+k)}(1+z)^k\end{aligned} \tag{3.10}$$

as a solution analytic at the origin $z = 0$.

Note that the standard Gauss form for the hypergeometric equation[28] is obtained from Eq. (3.8) via the second change of variable $w = (1-z)/2$, $w \in]0,1[$,

$$\begin{aligned}&\left\{w(1-w)\frac{d^2}{dw^2} + [(\alpha+1)-(\alpha+\beta+2)w]\frac{d}{dw} + \nu(\nu+\alpha+\beta+1)\right\}\\ &\quad\times y_\nu(w;\alpha,\beta) = 0.\end{aligned} \tag{3.11}$$

From the preceding results, we deduce that, for $|\alpha|,|\beta| > 1/2$, the physical spectrum of the TDPT potential associated to the asymptotic Dirichlet boundary conditions

$$\psi\left(0^+;\alpha,\beta\right) = 0 = \psi\left(\left(\frac{\pi}{2}\right)^-;\alpha,\beta\right) \tag{3.12}$$

is given in terms of Jacobi polynomials $P_n^{(\alpha,\beta)}$ by

$$\begin{cases}E_n(\alpha,\beta) = (\alpha_n+\beta_n+1)^2-(\alpha+\beta+1)^2 = 4n(\alpha+\beta+1+n)\\ \psi_n(x;\alpha,\beta) = \psi_0(x;\alpha,\beta)\,P_n^{(\alpha,\beta)}(\cos 2x)\end{cases},\ n\in\mathbb{N}, \tag{3.13}$$

with $(\alpha_n,\beta_n) = (\alpha+n,\beta+n)$.

The dispersion relation, E_ν as a function of ν, is a convex parabola with zeros at $\nu = 0$ and $\nu = -(\alpha+\beta+1)$ and the disconjugacy sector corresponds to the values of ν in between these two boundaries.

From the point of view of pure parameters' transformations, we have three possible discrete symmetries for $V(x;\alpha,\beta)$, which are given by

(a)

$$(\alpha,\beta)\xrightarrow{\Gamma_+}(-\alpha,\beta),\quad \begin{cases}V(x;\alpha,\beta)\xrightarrow{\Gamma_+}V(x;\alpha,\beta)+4\alpha(\beta+1),\\ \psi_n(x;\alpha,\beta)\xrightarrow{\Gamma_+}\phi_{n,+}(x;\alpha,\beta)=\psi_n(x;-\alpha,\beta),\end{cases} \tag{3.14}$$

(b)

$$(\alpha,\beta)\xrightarrow{\Gamma_-}(\alpha,-\beta),\quad \begin{cases}V(x;\alpha,\beta)\xrightarrow{\Gamma_-}V(x;\alpha,\beta)+4\beta(\alpha+1),\\ \psi_n(x;\alpha,\beta)\xrightarrow{\Gamma_-}\phi_{n,-}(x;\alpha,\beta)=\psi_n(x;\alpha,-\beta),\end{cases} \tag{3.15}$$

(c)

$$(\alpha,\beta)\overset{\Gamma_3=\Gamma_+\circ\Gamma_-}{\longrightarrow}(-\alpha,-\beta),\quad \begin{cases} V(x;\alpha,\beta)\overset{\Gamma_3}{\longrightarrow}V(x;\alpha,\beta)+4(\alpha+\beta),\\ \psi_n(x;\alpha,\beta)\overset{\Gamma_3}{\longrightarrow}\phi_{n,3}(x;\alpha,\beta)=\psi_n(x;-\alpha,-\beta).\end{cases} \tag{3.16}$$

In the (α,β) plane, Γ_+ and Γ_- correspond, respectively, to the reflections with respect to the axes $\alpha=0$ and $\beta=0$. Combining the transformation of coordinates to the parameters' transformations, we have a supplementary symmetry given by

$$(x;\alpha,\beta)\overset{\Omega=S\otimes P}{\longrightarrow}(\xi;\beta,\alpha),\quad \begin{cases} V(x;\alpha,\beta)\overset{\Omega}{\longrightarrow}V(x;\alpha,\beta),\\ \phi_{n,+}(x;\alpha,\beta)\overset{\Omega}{\longrightarrow}\phi_{n,+}(\xi;\beta,\alpha)=(-1)^n\phi_{n,-}(x;\alpha,\beta),\end{cases} \tag{3.17}$$

where

$$x\in[0,\pi/2]\overset{S}{\longrightarrow}\xi=\frac{\pi}{2}-x\in[0,\pi/2] \tag{3.18}$$

is the reflection on the coordinate axis with respect to the point $\frac{\pi}{4}$ ($\frac{\pi}{4}-x\overset{S}{\longrightarrow}x-\frac{\pi}{4}$), and

$$(\alpha,\beta)\overset{P}{\longrightarrow}(\beta,\alpha) \tag{3.19}$$

is the reflection with respect to the principal diagonal in the (α,β) parameters' plane.

The functions $\phi_{n,i}$, $i=+,-,3$, satisfy the respective equations

$$\widehat{H}(\alpha,\beta)\,\phi_{n,i}(x;\alpha,\beta)=\mathcal{E}_{n,i}(\alpha,\beta)\phi_{n,i}(x;\alpha,\beta)\,, \tag{3.20}$$

with

$$\begin{cases} \mathcal{E}_{n,+}(\alpha,\beta)=E_{n-\alpha}(\alpha,\beta)=4(n-\alpha)(n+\beta+1),\\ \mathcal{E}_{n,-}(\alpha,\beta)=E_{n-\beta}(\alpha,\beta)=\mathcal{E}_{n,+}(\beta,\alpha),\\ \mathcal{E}_{n,3}(\alpha,\beta)=E_{-(n+1)}(\alpha,\beta)=-4(n+1)(\alpha+\beta-n).\end{cases} \tag{3.21}$$

The eigenvalue $\mathcal{E}_{n,3}$ is negative for $n<\alpha+\beta$, while for $i=+$ and $i=-$, the disconjugacy condition $\mathcal{E}_{n,i}(\alpha,\beta)\le 0$ imposes that the constraints $\alpha>n$ and $\beta>n$ be satisfied, respectively.

IV. DISCONJUGATED SEED FUNCTIONS ASSOCIATED TO PARA-JACOBI POLYNOMIALS

Suppose that α and β are two positive integers,

$$\alpha=N\in\mathbb{N}^*,\quad \beta=M\in\mathbb{N}^*, \tag{4.1}$$

and apply the Γ_3 symmetry. The function $\phi_{n,3}(x;N,M)=\psi_n(x;-N,-M)$ is a solution of the Schrödinger equation

$$\left(-\frac{d^2}{dx^2}+V(x;N,M)-\mathcal{E}_{n,3}(N,M)\right)\phi_{n,3}(x;N,M)=0, \tag{4.2}$$

where

$$\mathcal{E}_{n,3}(N,M)=E_{-(n+1)}(N,M)<0, \tag{4.3}$$

if $n<N+M$.

But $\phi_{n,3}(x;N,M)=\psi_n(x;-N,-M)$ can also be written as

$$\phi_{n,3}(x;N,M)=\psi_0(z;-N,-M)\,y_n(z;-N,-M)\,, \tag{4.4}$$

where, up to a constant factor,

$$\psi_0(z;-N,-M)=(1-z)^{(-N+1/2)/2}(1+z)^{(-M+1/2)/2} \tag{4.5}$$

and

$$\begin{aligned}&\left(1-z^2\right)\ddot{y}_n\left(z;-N,-M\right)-\left[\left(-N-M+2\right)z+\left(-N+M\right)\right]\dot{y}_n\left(z;-N,-M\right)\\&\quad+n\left(n-N-M+1\right)y_n\left(z;-N,-M\right)=0.\end{aligned}\tag{4.6}$$

In this case, as noticed by Szegö[22] and emphasized by Calogero and Yi,[21] for values of n such that

$$\frac{N+M}{2}\leq n<N+M,\tag{4.7}$$

the general solution of the equation for $y_n\left(z;-N,-M\right)$ is a polynomial, called **para-Jacobi polynomial**, which has the (monic) form[21]

$$\begin{aligned}p_n^{(-N,-M)}(z;\lambda)=&\frac{(-2)^n(n-M)!n!}{(2n-M-N)!}\sum_{k=0}^{n-M}\frac{(-1)^{n-k}(2n-M-N-k)!}{k!(n-M-k)!(n-k)!}\left(\frac{1+z}{2}\right)^{n-k}\\&+\lambda\frac{(-2)^n(2n-M-N+1)!(M+N-n-1)!}{(n-N)!}\\&\times\sum_{k=2n-M-N+1}^{n}\frac{(-1)^{n-k}(k-n+M-1)!}{k!(k+N+M-2n-1)!(n-k)!}\left(\frac{1+z}{2}\right)^{n-k},\end{aligned}\tag{4.8}$$

λ being an arbitrary real parameter.

In other words, we have a basis of two polynomial eigenfunctions

$$\begin{cases}\Theta_{n,1}^{(-N,-M)}(z)=\sum_{k=M}^{n}\dfrac{(-1)^k(n-M-N+k)!}{2^k k!(k-M)!(n-k)!}(1+z)^k,\\ \Theta_{n,2}^{(-N,-M)}(z)=\sum_{k=0}^{N+M-n-1}\dfrac{(-1)^k(M-1-k)!}{2^k k!(N+M-n-1-k)!(n-k)!}(1+z)^k,\end{cases}\tag{4.9}$$

with $N,M>0$ and

$$\begin{cases}\Theta_{n,1}^{(-N,-M)}(-1)=0,\\ \Theta_{n,2}^{(-N,-M)}(-1)=\dfrac{(M-1)!}{(N+M-n-1)!n!}>0.\end{cases}\tag{4.10}$$

It results that the general solution of Eq. (4.2) can be written as ($z=\cos 2x$)

$$\psi_n\left(x;-N,-M;\lambda\right)=\psi_0\left(x;-N,-M\right)p_n^{(-N,-M)}\left(z;\lambda\right),\tag{4.11}$$

with

$$\begin{aligned}p_n^{(-N,-M)}(z;\lambda)=&\frac{(-2)^n(n-M)!n!}{(2n-M-N)!}\Theta_{n,1}^{(-N,-M)}(z)\\&+\lambda\frac{(-2)^n(2n-M-N+1)!(M+N-n-1)!}{(n-N)!}\Theta_{n,2}^{(-N,-M)}(z).\end{aligned}\tag{4.12}$$

Moreover, from Eq. (4.9), we have

$$\begin{aligned}\dot{\Theta}_{n,1}^{(-N,-M)}(z)&=\sum_{j=M-1}^{n-1}\frac{(-1)^{j+1}(n-M-N+j+1)!}{2^{j+1}j!(j+1-M)!(n-j-1)!}(1+z)^j\\&=-\frac{1}{2}\Theta_{n-1,1}^{(-(N-1),-(M-1))}(z).\end{aligned}\tag{4.13}$$

More generally, for $i=1,2$,

$$\dot{\Theta}_{n,i}^{(-N,-M)}(z)=-\frac{1}{2}\Theta_{n-1,i}^{(-N+1,-M+1)}(z),\tag{4.14}$$

which gives

$$\dot{p}_n^{(-N,-M)}(z;\lambda)=np_{n-1}^{(-N+1,-M+1)}(z;\lambda'),\tag{4.15}$$

where

$$\lambda' = \frac{M+N-n-1}{n}\lambda. \tag{4.16}$$

For $\frac{N+M}{2} \leq n < N+M$, $\psi_n(x;-N,-M;\lambda)$ is disconjugated on $z \in]-1,1[$ and consequently admits at most one zero on this interval. To ensure the absence of node, we have to verify that the sign of $\psi_n(x;-N,-M;\lambda)$ at the boundaries of the interval is the same. This is the case if

$$\operatorname{sign}\left(p_n^{(-N,-M)}(1;\lambda)\right) = \operatorname{sign}\left(p_n^{(-N,-M)}(-1;\lambda)\right). \tag{4.17}$$

But due to the symmetry given in Eq. (3.19), we have[21]

$$p_n^{(-N,-M)}(-z;\lambda) = (-1)^n p_n^{(-M,-N)}\left(z;\widetilde{\lambda}\right), \tag{4.18}$$

where

$$\begin{aligned}\widetilde{\lambda} &= (-1)^{2n-N-M+1}\lambda \\ &\quad + (-1)^{n-M}\frac{n!\,(n-M)!\,(n-N)!}{(2n-N-M)!\,(2n-N-M+1)!(N+M-n-1)!}.\end{aligned} \tag{4.19}$$

This implies

$$p_n^{(-N,-M)}(-1;\lambda) = \lambda\frac{(-2)^n\,(2n-N-M+1)!\,(M-1)!}{(n-N)!\,n!} \tag{4.20}$$

and

$$\begin{aligned}p_n^{(-N,-M)}(1;\lambda) &= (-1)^n p_n^{(-M,-N)}\left(-1;\widetilde{\lambda}\right) \\ &= (-1)^n\widetilde{\lambda}\frac{(-2)^n\,(2n-N-M+1)!\,(N-1)!}{(n-M)!\,n!}.\end{aligned} \tag{4.21}$$

The absence of node is then obtained if λ and $(-1)^n\widetilde{\lambda}$ have the same sign. In the following, we note that

$$\lambda_n^{(-N,-M)} \equiv \frac{n!\,(n-M)!\,(n-N)!}{(2n-N-M)!\,(2n-N-M+1)!(N+M-n-1)!} > 0. \tag{4.22}$$

Case (i)

Suppose, first, that we take $\lambda > 0$. We must then have

$$(-1)^n\widetilde{\lambda} = (-1)^{n-N-M+1}\lambda + (-1)^M\lambda_n^{(-N,-M)} > 0, \tag{4.23}$$

that is, if $n-N-M$ is odd,

$$\lambda > (-1)^{M+1}\lambda_n^{(-N,-M)}. \tag{4.24}$$

This is always achieved when M is even ($n-N$ is odd) and it necessitates to take

$$\lambda > \lambda_n^{(-N,-M)} \tag{4.25}$$

when M is odd ($n-N$ is even). In the case where $n-N-M$ is even, we deduce in the same way that the condition of equality of signs can never be reached if M is odd ($n-N$ is odd) and imposes to take

$$0 < \lambda < \lambda_n^{(-N,-M)} \tag{4.26}$$

when M is even ($n-N$ is even).

Case (ii)

If we take $\lambda < 0$, we must have

$$(-1)^n\widetilde{\lambda} = (-1)^{n-N-M+1}\lambda + (-1)^M\lambda_n^{(-N,-M)} < 0, \tag{4.27}$$

that is, if $n-N-M$ is odd,

$$\lambda < (-1)^{M+1}\lambda_n^{(-N,-M)}. \tag{4.28}$$

This is always achieved when M is odd ($n-N$ is even) and it necessitates to take

$$|\lambda| > \lambda_n^{(-N,-M)} \tag{4.29}$$

when M is even ($n-N$ is odd). In the case where $n-N-M$ is even, we deduce in the same way that the condition of equality of signs can never be reached if M is even ($n-N$ is even) and imposes to take

$$|\lambda| < \lambda_n^{(-N,-M)} \tag{4.30}$$

when M is odd ($n-N$ is odd).

To summarize, $\psi_n(x;-N,-M;\lambda)$ has no node in the four following cases:

(i) M is even, $n-N$ is odd, $\lambda < -\lambda_n^{(-N,-M)}$ or $\lambda > 0$;

(ii) M is even, $n-N$ is even, $0 < \lambda < \lambda_n^{(-N,-M)}$;

(iii) M is odd, $n-N$ is even, $\lambda < 0$ or $\lambda > \lambda_n^{(-N,-M)}$;

(iv) M is odd, $n-N$ is odd, $-\lambda_n^{(-N,-M)} < \lambda < 0$.

V. REGULAR RATIONAL EXTENSIONS

Suppose λ satisfies the conditions mentioned above, then $\psi_n(x;-N,-M;\lambda)$ can be used as seed function to build a state-adding DBT, which generates a regular rational extension of $V(x;N-1,M-1)$. It is given by (see Eqs. (2.6) and (4.11))

$$\begin{aligned}\widetilde{V}^{(n)}(x;N,M;\lambda) &= V(x;N,M) - 2(\log(\phi_{n,3}(x;N,M)))'' \\ &= V(x;N,M) - 2(\log(\psi_n(x;-N,-M;\lambda)))'' \\ &= V(x;N-1,M-1) - 4(N+M) - 2\Big(\log\Big(p_n^{(-N,-M)}(z;\lambda)\Big)\Big)'',\end{aligned} \tag{5.1}$$

since

$$\begin{aligned}\widetilde{V}^{(0)}(x;N,M) &= V(x;N,M) - 2(\log(\psi_0(x;-N,-M)))'' \\ &= V(x;N-1,M-1) + E_{-1}(N,M).\end{aligned} \tag{5.2}$$

On taking into account of relations

$$\frac{d}{dx} = -2\sqrt{1-z^2}\frac{d}{dz}, \qquad \frac{d^2}{dx^2} = 4\left(1-z^2\right)\frac{d^2}{dz^2} - 4z\frac{d}{dz}, \tag{5.3}$$

and Eqs. (4.15) and (4.16), we arrive at

$$\begin{aligned}&\widetilde{V}^{(n)}(x;N,M;\lambda) \\ &\quad = V(x;N-1,M-1) - 4(N+M) - 8n(n-1)\left(1-z^2\right)\frac{p_{n-2}^{(-N+2,-M+2)}(z;\lambda'')}{p_n^{(-N,-M)}(z;\lambda)} \\ &\qquad + 8n^2\left(1-z^2\right)\left(\frac{p_{n-1}^{(-N+1,-M+1)}(z;\lambda')}{p_n^{(-N,-M)}(z;\lambda)}\right)^2 + 8nz\frac{p_{n-1}^{(-N+1,-M+1)}(z;\lambda')}{p_n^{(-N,-M)}(z;\lambda)},\end{aligned} \tag{5.4}$$

where

$$\lambda'' = \frac{(M+N-n-1)(M+N-n-2)}{n(n-1)}\lambda. \tag{5.5}$$

For $\widetilde{V}^{(n)}(x;N,M;\lambda)$ to be a confining potential on $]0,\pi/2[$, we need to impose that $V(x;N-1,M-1)$ has this property, which is achieved for $N,M > 3/2$, thence $N,M \geq 2$. For such N,M values, we note that $\widetilde{V}^{(n)}(x;N,M;\lambda)$ is actually strongly repulsive in both 0 and $\pi/2$, since the singularities are there of the type g/x^2 ($g \geq 3/4$) and $g/(\pi/2-x)^2$ ($g \geq 3/4$), respectively. This means that at each extremity, only one basis solution is quadratically integrable. This choice limits

(see Eq. (4.7)) the possible values of n to $n \geq 2$, but in this case, the corresponding Hamiltonian $\widetilde{\widetilde{H}}^{(n)}(x; N, M; \lambda)$ is essentially self-adjoint.[29,30] The simplest example corresponds to $n = N = M = 2$, in which case

$$p_2^{(-2,-2)}(z; \lambda) = z^2 + 2(1-\lambda)z + 1, \tag{5.6}$$

with $0 < \lambda < \lambda_2^{(-2,-2)} = 2$, which is example (6b) in Ref. 21. The initial potential is

$$V(x; 2, 2) = \frac{15}{1-z^2} - 25 \tag{5.7}$$

and the partner potential reads

$$\begin{aligned} \widetilde{V}^{(2)}(x; 2, 2; \lambda) = {} & \frac{3}{1-z^2} - 16\frac{(\lambda-1)z + 2\lambda^2 - 4\lambda - 1}{z^2 + 2(1-\lambda)z + 1} \\ & + 64\lambda(\lambda-2)\frac{(1-\lambda)z + 1}{\left[z^2 + 2(1-\lambda)z + 1\right]^2} - 25. \end{aligned} \tag{5.8}$$

The ground-state energy of the latter is equal to $E_{-3}(2, 2) = -24$. In Fig. 1, the corresponding potentials are plotted for different values of the λ parameter.

Using the standard properties of Wronskians[25]

$$\begin{cases} W(uy_1, \ldots, uy_m \mid x) = u(x)^m W(y_1, \ldots, y_m \mid x), \\ W(y_1, \ldots, y_m \mid x) = \left(\dfrac{dz}{dx}\right)^{m(m-1)/2} W(y_1, \ldots, y_m \mid z), \end{cases} \tag{5.9}$$

the eigenstates of $\widetilde{V}^{(n)}(x; N, M; \lambda)$ are given by ($k \geq 0$)

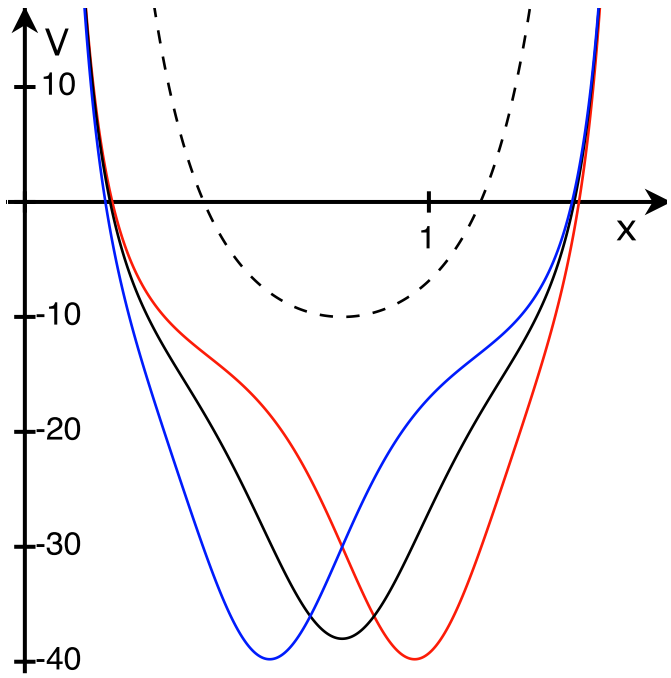

FIG. 1. Plot of $\widetilde{V}^{(2)}(x; 2, 2; \lambda)$ as a function of x, for $\lambda = 1/2$ (red line), $\lambda = 1$ (black solid line), and $\lambda = 3/2$ (blue line). The partner $V(x; 2, 2)$ is also shown (black dashed line).

$$\begin{aligned}\widetilde{\psi}_k^{(n)}(x;N,M;\lambda) &= \frac{W(\psi_n(x;-N,-M;\lambda),\psi_k(x;N,M)\mid x)}{\psi_n(x;-N,-M;\lambda)}\\ &\propto \psi_0(x;-N,-M)(1-z)^{1/2}(1+z)^{1/2}\\ &\quad\times\frac{W\big(p_n^{(-N,-M)}(z;\lambda),(1-z)^N(1+z)^M P_k^{(N,M)}(z)\mid z\big)}{p_n^{(-N,-M)}(z;\lambda)},\end{aligned} \tag{5.10}$$

that is, on using derivation properties of Jacobi polynomials[22,28] and Eq. (4.15)

$$\widetilde{\psi}_k^{(n)}(x;N,M;\lambda) \propto \frac{\psi_0(x;N-1,M-1)}{p_n^{(-N,-M)}(z;\lambda)} Q_k^{(n)}(z;N,M;\lambda), \tag{5.11}$$

where

$$\begin{aligned}Q_k^{(n)}(z;N,M;\lambda) &= \left(1-z^2\right)\left(\frac{k+M+N+1}{2}P_{k-1}^{(N+1,M+1)}(z)\,p_n^{(-N,-M)}(z;\lambda)\right.\\ &\quad\left.-\,np_{n-1}^{(-N+1,-M+1)}(z;\lambda')\,P_k^{(N,M)}(z)\right)\\ &\quad-\left((N+M)z+N-M\right)P_k^{(N,M)}(z)\,p_n^{(-N,-M)}(z;\lambda).\end{aligned} \tag{5.12}$$

Moreover, since

$$\widetilde{\psi}_{-(n+1)}^{(n)}(x;N,M;\lambda) = \frac{1}{\psi_n(x;-N,-M;\lambda)} = \frac{\psi_0(x;N-1,M-1)}{p_n^{(-N,-M)}(z;\lambda)} \tag{5.13}$$

satisfies the Dirichlet boundary conditions and the square integrability one, it is an eigenstate of $\widetilde{V}^{(n)}(x;N,M;\lambda)$ with an energy $E_{-(n+1)}(N,M)$ and the DBT $\widehat{A}(w_{n,3})$ is state-adding. To summarize, the spectrum of the extended potential is ($k\in\{-(n+1),0,1,\ldots\}$)

$$\begin{cases} E_k(N,M),\\ \widetilde{\psi}_k^{(n)}(x;N,M;\lambda) \propto \dfrac{\psi_0(x;N-1,M-1)}{p_n^{(-N,-M)}(z;\lambda)} Q_k^{(n)}(z;N,M;\lambda),\end{cases} \tag{5.14}$$

where $Q_{-n-1}^{(n)}(z;N,M;\lambda)=1$ and where the $Q_{k\geq 0}^{(n)}$ are given in Eq. (5.12).

Due to the orthogonality properties of the $\widetilde{\psi}_k^{(n)}$, we deduce that the $Q_k^{(n)}(z;N,M;\lambda)$ are orthogonal polynomials on $]-1,1[$ with respect to the measure

$$\mu_n^{(-N,-M)}(z;\lambda) = \frac{(1-z)^{N-1}(1+z)^{M-1}}{\left(p_n^{(-N,-M)}(z;\lambda)\right)^2}. \tag{5.15}$$

VI. CONCLUSION

In this article, we have shown that it is possible to build one-step regular rational extensions of the TDPT potential depending both on an integer index n and on a continuously varying parameter λ. This is achieved by using, for the underlying Darboux transformations, seed functions which are associated to the para-Jacobi polynomials of Calogero and Yi . To each value of n corresponds then a novel family of λ-dependent polynomials which are orthogonal on $]-1,1[$.

Note that some continuously parametrized rationally extended potentials can also be obtained by using an extension scheme based on confluent DBT.[31]

The multistep version of the results presented here is in progress and will be the subject of a forthcoming paper.

ACKNOWLEDGMENTS

For one of us (B.B.), it is a great pleasure to thank Yves Grandati as well as the other members of BioPhysStat group at Université de Lorraine-Metz for their warm hospitality. The remaining two of us (Y.G. and C.Q.) would like to thank F. Calogero for some very interesting discussions.

[1] F. Cooper, A. Khare, and U. Sukhatme, "Supersymmetry and quantum mechanics," Phys. Rep. **251**, 267–385 (1995); e-print arXiv:hep-th/9405029.
[2] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetry and Quantum Mechanics* (World Scientific, Singapore, 2001).
[3] B. Bagchi, *Supersymmetry in Quantum and Classical Mechanics* (Chapman & Hall/CRC, Boca Raton, FL, 2001).
[4] G. Darboux, "Sur une proposition relative aux équations linéaires," Compt. Rend. Acad. Sci. **94**, 1456–1459 (1882).
[5] M. M. Crum, "Associated Sturm-Liouville systems," Q. J. Math. Oxford Ser. 2 **6**, 121–127 (1955).
[6] L. Gendenshtein, "Derivation of exact spectra of the Schrödinger equation by means of supersymmetry," JETP Lett. **38**, 356–359 (1983).
[7] Y. Grandati and A. Bérard, "Rational solutions for the Riccati-Schrödinger equations associated to translationally shape invariant potentials," Ann. Phys. (N. Y.) **325**, 1235–1259 (2010).
[8] D. Gómez-Ullate, N. Kamran, and R. Milson, "An extended class of orthogonal polynomials defined by a Sturm-Liouville problem," J. Math. Anal. Appl. **359**, 352–367 (2009); e-print arXiv:0807.3939.
[9] D. Gómez-Ullate, N. Kamran, and R. Milson, "An extension of Bochner's problem: Exceptional invariant subspaces," J. Approx. Theory **162**, 987–1006 (2010); e-print arXiv:0805.3376.
[10] C. Quesne, "Exceptional orthogonal polynomials, exactly solvable potentials and supersymmetry," J. Phys. A **41**, 392001 (2008); e-print arXiv:0807.4087.
[11] C. Quesne, "Solvable rational potentials and exceptional orthogonal polynomials in supersymmetric quantum mechanics," SIGMA **5**, 084 (2009); e-print arXiv:0906.2331.
[12] S. Odake and R. Sasaki, "Infinitely many shape invariant potentials and new orthogonal polynomials," Phys. Lett. B **679**, 414–417 (2009); e-print arXiv:0906.0142.
[13] D. Gómez-Ullate, Y Grandati, and R. Milson, "Extended Krein-Adler theorem for the translationally shape invariant potentials," J. Math. Phys. **55**, 043510 (2014); e-print arXiv:1309.3756.
[14] D. Gómez-Ullate, N. Kamran, and R. Milson, "Two-step Darboux transformations and exceptional Laguerre polynomials," J. Math. Anal. Appl. **387**, 410–418 (2012); e-print arXiv:1103.5724.
[15] S. Odake and R. Sasaki, "Exactly solvable quantum mechanics and infinite families of multi-indexed orthogonal polynomials," Phys. Lett. B **702**, 164–170 (2011); e-print arXiv:1105.0508.
[16] C. Quesne, "Rationally-extended radial oscillators and Laguerre exceptional orthogonal polynomials in kth-order SUSYQM," Int. J. Mod. Phys. A **26**, 5337–5347 (2011); e-print arXiv:1110.3958.
[17] Y. Grandati, "Multistep DBT and regular rational extensions of the isotonic oscillator," Ann. Phys. (N. Y.) **327**, 2411 (2012); e-print arXiv:1108.4503.
[18] Y. Grandati and C. Quesne, "Disconjugacy, regularity of multi-indexed rationally-extended potentials, and Laguerre exceptional polynomials," J. Math. Phys. **54**, 073512 (2013); e-print arXiv:1211.5308.
[19] W.-Y. Keung, U. P. Sukhatme, Q. Wang, and T. D. Imbo, "Families of strictly isospectral potentials," J. Phys. A **22**, L987–L992 (1989).
[20] Q. Wang, U. P. Sukhatme, W.-Y. Keung, and T. D. Imbo, "Solitons from supersymmetry," Mod. Phys. Lett. A **5**, 525–530 (1990).
[21] F. Calogero and G. Yi, "Can the *general* solution of the second-order ODE characterizing Jacobi polynomials be *polynomial*?," J. Phys. A **45**, 095206 (2012).
[22] G. Szegö, *Orthogonal Polynomials* (American Mathematical Society, Providence, RI, 1939).
[23] F. Calogero, "A linear second-order ODE with only polynomial solutions," J. Differ. Equations **255**, 2130–2135 (2013).
[24] F. Calogero, "Can the generalized hypergeometric equation feature several independent polynomial solutions?," J. Phys. A **47**, 045205 (2014).
[25] T. Muir, *A Treatise on the Theory of Determinants* (Dover, New York, 1960), revised and enlarged by W. H. Metzler.
[26] P. Hartman, *Ordinary Differential Equations* (John Wiley, New York, 1964).
[27] W. A. Coppel, *Disconjugacy* (Springer, Berlin, 1971).
[28] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953).
[29] W. M. Frank, D. J. Land, and R. M. Spector, "Singular potentials," Rev. Mod. Phys. **43**, 36–98 (1971).
[30] L. Lathouwers, "The Hamiltonian $H = (-1/2)\, d^2/dx^2 + x^2/2 + \lambda/x^2$ reobserved," J. Math. Phys. **16**, 1393–1395 (1975).
[31] Y. Grandati and C. Quesne, "Confluent chains of DBT: Enlarged shape invariance and new orthogonal polynomials," SIGMA (submitted); e-print arXiv:1503.07747.