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# Mesh adaptation by modifying the node positions

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**Abstract.** An adaptive meshing technique is proposed. For a given number of nodes, the method allows a better precision, with a better repartition of the position of the nodes. It is based on the analysis of the influence of the node position and edge connection on the functional of the problem

**PACS.** 41.20.Gz Magnetostatics – 41.20.-q Electric, magnetic, and electromagnetic fields

## 1 Introduction

Once the modelisation of the problem is done, the set of equations written and its transformation into the weak form is realized, the practical precision of the chosen discretization is in question.

A relatively low number of nodes allows to ensure a fair precision [1], whatever is the discretization. But to gain even more precision from this previous step, it is necessary to include in the mesh an extremely high number of nodes.

In order to get more accuracy, many different methods, mostly based on estimator errors have been implemented [2]. Most of them are based on the conservative laws [3], or on dual formulation (in magnetic vector potential and  $T - \Omega$ ) with a behaviour law between the induction and the magnetic field [4].

Instead of computing the error, an idea is to study the node movement where the error is big. This so-called moving finite element technique has been invented by Miller [5,6], and has been used in time-dependent partial differential equations in fluid mechanics [7–9]. It is here adapted to a non time-dependent electromagnetic problem.

The aim of the paper is to study the adaptation of the mesh to the solution, for a constant number of nodes. This leads to discuss what is an adapted support for discretization directly related to the studied functional : what is the impact of the position of the nodes and of the edge connection on the precision of the result.

## 2 Position of the Problem

The study is carried out on a simple example which is the bifilar line : it is composed of two parallel conductors of circular section, one supplied by a positive current  $+I$ , the other by the same current in return  $-I$ . The aim is to calculate its inductance  $L_0$  by unity of length.

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In the general case an analytical solution is not known. If the aim is to get the error, a dual formulation can be computed at the end to enclose the real value of the inductance.

Any modification (inflation of the node number, displacement of the nodes) leads to increase the coenergy functional, thus to decrease the error. If increasing the node number leads to a constant value of the inductance, then the solution is assumed to be exact. In the same way, for the moving finite elements with a constant number of nodes, if the nodes do not move anymore, the error is assumed to be as low as possible for this number of nodes.

For this example, the analytical solution of the problem is known, so that it allows to give the accuracy of any other method. Even if there is no singularity in this problem, the error committed with a regular mesh is big, unless the density of the mesh is high.

The basis of the study is the bidimensional linear magnetostatic model. The coenergy functional is :

$$\overline{W}(I, a) = \int_D \left[ ja - \frac{1}{2} \frac{(\nabla a)^2}{\mu_0} \right] \quad (1)$$

where  $a$  is the magnetic vector potential (MVP),  $j$  the current density in the conductors and  $\mu_0$  the permeability of the air.

The coenergy function is defined by the computation of the MVP  $a_\infty$  that maximizes the coenergy functional :

$$\overline{W}(I) = \max_{a \in \mathcal{H}^1(D)} \overline{W}(I, a) = \overline{W}(I, a_\infty) \quad (2)$$

where  $\mathcal{H}^1(D)$  is the (Sobolev) space of the functions which the square of their gradients can be integrated.

To discretize the functional  $\overline{W}$ , a necessary step is to choose the geometrical parameters of the mesh, among which the position of the nodes ( $\mathbf{x} = [(x_1, y_1) \dots (x_N, y_N)]$ ), where  $N$  is the constant total number of nodes.

It is also necessary to choose the connections between these nodes  $c = (c_1 \dots c_{ar})$ .  $ar$  is the total number of edges

which is, by using the Euler relations  $3N - 3N_0 - 3$ , for a triangular mesh of connex polygons of  $N$  nodes, where  $N_0$  are on the border of the mesh.

To be clear, an application  $\mathcal{A}$  can be introduced as following :

$$\mathcal{A} : \mathcal{H}^1(D) \times (\mathbb{R}^2)^N \times ((1\dots N)^2)^{ar} \rightarrow \mathcal{H}^1(D) \quad (3)$$

$$(a, \mathbf{x}, c) \mapsto \mathcal{A}(a, \mathbf{x}, c) = A$$

If  $\mathcal{V}$  is the space image of  $\mathcal{H}^1(D)$  through the application  $\mathcal{A}$  :

$$\mathcal{V}(\mathbf{x}, c) = \mathcal{A}(a, \mathbf{x}, c) \quad (4)$$

then the discrete version of the problem (1) consists in substituting  $\mathcal{V}$  to  $\mathcal{H}^1(D)$ .

A field  $A$  is an element of the space  $\mathcal{V}(\mathbf{x}, c)$  if its values at the node  $n$  :  $A_n = A(x_n, y_n)$  are given, and the value  $A(x, y)$  is deduced elsewhere by linear interpolation on the triangles formed by the edges.

The functional  $\overline{W}$  is then written as :

$$\overline{W}(I, A) = L(A_1, \dots, A_N, \mathbf{x}, c) - Q(A_1, \dots, A_N, \mathbf{x}, c) \quad (5)$$

where  $L$  is a linear form in  $A$  corresponding to the source terms, and  $Q$  is a quadratic form. The functional  $\overline{W}(I, A)$  depends on the values of the potential at the nodes ( $A_1 \dots A_N$ ), the positions at the nodes ( $\mathbf{x}_1 \dots \mathbf{x}_n$ ), and also the connections between the nodes  $c$ .

The finite element method consists in maximizing  $\overline{W}$  with respect to ( $A_1 \dots A_N$ ), with the positions at the nodes  $\mathbf{x}_1 \dots \mathbf{x}_n$  and the connections between the nodes  $c$  given.

### 3 Analysis of the solution for a given mesh

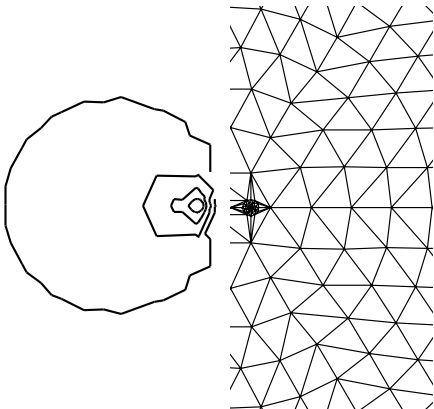


Fig. 1. Initial mesh and field lines - 400 nodes

In order to compare the performances of the methods that modify the support of discretization, the same initial support has to be used. A possible criterion to make this initial support is to display the nodes in order to have elements of equal surfaces and of angles near a mean value, thus avoiding elongated elements [10].

As the mesh generation allows a large range of possibilities, the inner nodes can be displayed in regular geometrical shapes. Rectangles, squares or concentric circles can be used. Other geometries like concentric ellipses, which can be a fair approximation for the field lines near the conductors, have also been tested.

The aim is not to find an initial support of discretization which is optimal for the studied functional. In this particular case, the analytical solution is known, and it would be possible to display the nodes with a density decreasing with the radius to adapt roughly the support to the functional. The aim is to find a technique which can be generalized to other problems, that do not have a known analytical solution.

So, any initial mesh can be used. As the outer border is a circle, the initial mesh with nodes displayed in concentric circles (Fig. 1) has been chosen. Moreover, the nodes in the conductors are displayed in the same way for all the further computations. To avoid the truncation of the field to the outer border, a dipolar equivalent condition is set on this border.

The inductance by meter of length for two parallel wires, at a distance  $d$  from each other, and of which the radius of the wires is  $r$ , is :

$$L_0 = \frac{\mu_0}{2\pi} \left( \frac{1}{2} + 2 \ln \left( \frac{d}{r} \right) \right) \quad (H/m)$$

With a ratio  $d/r$  equal to 5, the inductance per meter is  $7.43 \cdot 10^{-7} (H/m)$ .

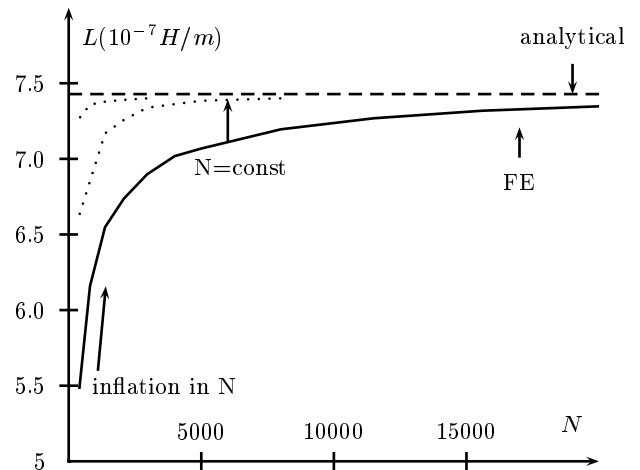


Fig. 2. Computation of the inductance with the FE method

Table 1. Precision of the FE computation

Nb of nodes	400	1370	2960	5000	11500	20000
Precision (%)	26.	12.	7.2	4.9	2.3	1.2

For the finite element method, with 5000 nodes, the precision is under 5 %, which begins to be satisfying (Tab. 1). But if the required precision is 1 per 1000, then 62000 nodes have to be inserted.

The main idea is to increase the precision when the number of nodes is given, in order to get the FE computation curve of Fig. 2 closer to the analytical value, for any number of nodes, and not to get a certain precision by going along the FE computation curve with an inflation of the number of nodes.

A first complementary approach is to consider that the connections between the nodes are free : the arguments in order to maximize the functional  $\overline{W}$  are the connections and the MVP values at the nodes  $(A_1 \dots A_N, c)$ .

#### 4 Modification of the connections

The initial mesh is given, on which  $\max_{(A_1 \dots A_N)} \overline{W}(I, A)$  is computed. The edges which do not belong to the border of a domain are considered. Each edge is part of two adjacent elements  $e_1$  and  $e_2$  forming a polygon  $P$  (Fig. 3). The value of the potential vector is fixed at the four nodes. The incidence of the changes in the connections on the functional  $\overline{W}$  is discussed.

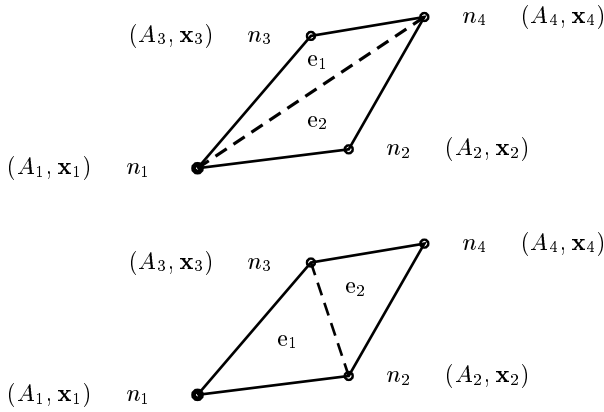


Fig. 3. Modification of the connection

There are two possibilities for the connection of the inner edge in the polygon : either  $n1 - n4$  or  $n2 - n3$ . To maximize  $\overline{W}$ , it is necessary to minimize the contribution of the polygon  $P$  to the energy  $W$  :

$$W(A_n, c) = \frac{1}{2} \int_{\Delta_{e_1}} \frac{(\nabla A(x, y))^2}{\mu_0} + \frac{1}{2} \int_{\Delta_{e_2}} \frac{(\nabla A(x, y))^2}{\mu_0} \quad (6)$$

where the edge between  $e_1$  and  $e_2$  is defined by the connection  $c$ .

An analysis of the problem shows that the edge of shortest length between  $n1 - n4$  and  $n2 - n3$  will give this minimum, whatever are the values of the potential at the nodes.

Then the problem of maximizing the functional with the arguments  $(A_1 \dots A_N, c)$  can be decoupled : it can be maximized with the connections, and then with  $(A_1 \dots A_N)$ .

If the nodes of the initial mesh are displayed in squares or rectangles, the connection argument is not operating :

the energy of the polygon is exactly the same for both possible connections.

The making of the initial mesh is indeed most of the time based on a quality criterion : the position of the nodes are put so that the surfaces of the elements have almost the same value, which is also the case of the angles of the elements. In this case, the connection modification is useless : regular meshes with inner points displayed in concentric circles or ellipses also showed no improvement.

In fact, such an approach could be possible if the inner nodes were inserted at random to make the initial mesh.

As the maximization of the functional  $\overline{W}$  with respect to  $(A_1 \dots A_N, c)$  has been tested, the maximization with respect to  $(A_1 \dots A_N, \mathbf{x})$  is detailed, i.e. the effect of the modification of the position of the nodes on the functional.

#### 5 Modification of the position of the nodes

Because the dependency of the functional  $\overline{W}$  in  $\mathbf{x}$  is not quadratic, a relaxation method component by component is chosen. A starting solution with the first approach should not be too far from the real solution.

The initial mesh is given, on which  $\max_{(A_1 \dots A_N)} \overline{W}(I, A)$  is computed. The position and the value of the potential vector of the border nodes and of the nodes in the conductor domains are fixed.

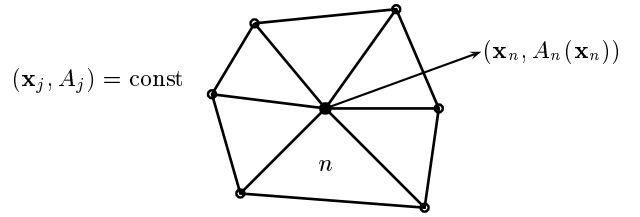


Fig. 4. Variation of the position of the node  $n$  on  $P$

Each node  $n$  is then considered, for which the polygon  $P$  is defined.  $P$  is formed by the elements to which  $n$  belongs (Fig. 4).  $\overline{W}(I, A)$  is maximized with respect to  $(A_n, x_n, y_n) = (A_n, \mathbf{x}_n)$ , the potential vectors and positions of the other nodes of the polygon being fixed. The contribution of these elements to  $\overline{W}$  is  $-W(\mathbf{x}_n, A_n)$  :

$$W(\mathbf{x}_n, A_n) = \sum_{el \in P} \frac{1}{2} \int_{\Delta_{el}} \frac{(\nabla A(x, y))^2}{\mu_0} \quad (7)$$

Then, this is the same as minimizing the energy  $W$  with respect to  $(A_n, \mathbf{x}_n)$ . For the value  $A_n^\infty$  which minimizes  $W$ , we have for the zero order and first order respectively (where  $\nabla_{\mathbf{x}} A_n^\infty$  are the derivatives of  $A_n^\infty$  with respect to the position) :

$$\frac{\partial W}{\partial a}(\mathbf{x}_n, A_n^\infty) = 0 \quad (8)$$

$$\nabla_{\mathbf{x}} \left( \frac{\partial W}{\partial a}(\mathbf{x}_n, A_n^\infty) \right) + \left( \frac{\partial^2 W}{\partial a^2}(\mathbf{x}_n, A_n^\infty) \right) \nabla_{\mathbf{x}} A_n^\infty = 0 \quad (9)$$

The Taylor expansion of the energy at the second order can be expressed with a minimum of calculus as :

$$W(\mathbf{x}_n + \delta\mathbf{x}, A_n(\mathbf{x}_n + \delta\mathbf{x})) = W(\mathbf{x}_n, A_n(\mathbf{x}_n)) + \nabla_{\mathbf{x}} W(\mathbf{x}_n, A_n(\mathbf{x}_n)) \cdot \delta\mathbf{x} + \frac{1}{2} \delta\mathbf{x}^t \left[ \nabla_{\mathbf{x}}^2 W - \left( \frac{\partial^2 W}{\partial a^2} \right) (\nabla_{\mathbf{x}} A_n^\infty) (\nabla_{\mathbf{x}} A_n^\infty)^t \right] \delta\mathbf{x} \quad (10)$$

As the positions and MVPs of the neighbors of  $n$  are fixed, the local FE problem is solved on  $P$ , and gives the MVP  $A_n$ . Thus, the MVP can be obtained for any position  $\mathbf{x}_n$ . The starting position is the former position of the node  $n$ . The energy minimization is done by using its Taylor expansion at the second order (10), with Newton like methods : it leads to new node coordinates  $\mathbf{x}_n$ . The new value of the MVP  $A_n$  can be computed again. If the new coordinates are out of  $P$ , which happens very rarely, then the former position is kept.

## 5.1 Results

The modification of the node positions is an iterative method : an iteration corresponds to moving the position of all the nodes (except these in the conductor domains) one by one. A better precision is obtained with 800 nodes and 100 iterations than with 5000 nodes for the initial mesh. With 10 iterations and 2000 nodes, the precision is the same (Fig. 5) The precision increases rapidly with ten iterations (Tab. 2). The results for 50 iterations are close to these for 100 iterations. For 50 iterations and 400 nodes, the computation time is about 30% higher than for the FE method with the same precision (this moving node procedure has not been optimized).

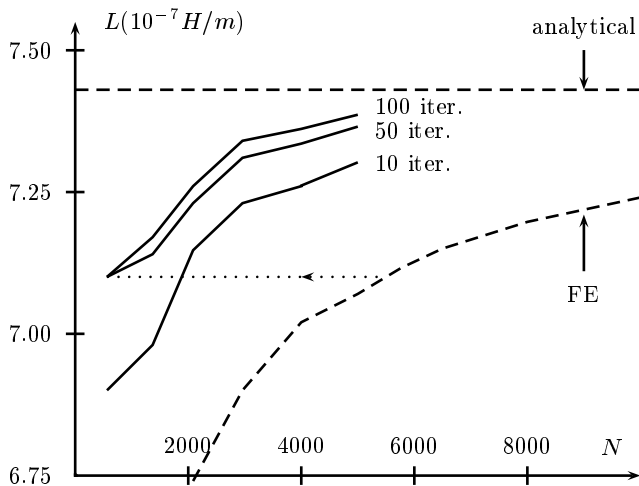


Fig. 5. Modification of the node positions

The mesh modification shows that the nodes are roughly in concentric circles around the conductors (Fig. 6 left). If many iterations are required to get a better precision, then the quality of the elements will be low as far as the

Table 2. Accuracy of the FE method with modification of the node positions

Nb of nodes	400	800	1370	2086	2960	5000
0 iter. (%)	26.	17.	12.	9.4	7.2	4.9
10 iter (%)	14.	8.7	6.2	4.0	2.8	1.8
50 iter (%)	11.	6.3	4.0	2.8	1.7	0.98
100 iter (%)	11.	3.7	3.6	2.4	1.3	0.70

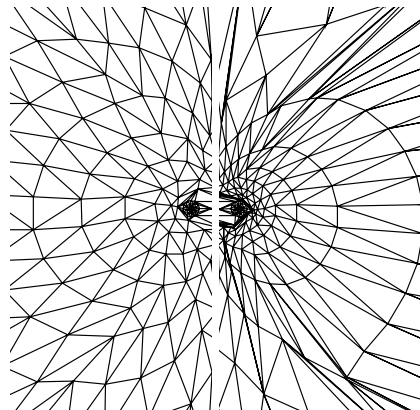


Fig. 6. Final mesh (400 nodes) : 10 iterations (left) to 100 iterations (right) with modification of the node positions (missing edges are due to the clipping)

angles and the surfaces of the elements are concerned (Fig. 6 right).

The node movement is all the more efficient as the number of nodes is low : the geometrical modifications of the mesh can be quicker and more important. To be able to make even quicker modifications, and give more possibilities to the geometry of the mesh, the connections of the nodes have to be analyzed in addition to the node movement.

## 6 Modification of the node positions and connections

The connection modification is necessary when the modification of the node position method leads to nodes which cannot be moved anymore because of the geometry obtained (Fig. 6). The analysis of the node position and connection modification can then be decoupled.

The connection modification as described in the previous section does not work : if an element is elongated, then its connection will not be changed, since the edge is already the one of shortest length.

In order to ease the movement of the nodes, an angle criterion is tested (Fig. 7). If the angle  $\theta'$  of the adjacent element is greater than  $\theta$ , then the connection of the edge  $n_1 - n_2$  is modified into  $n_3 - n_4$  (Fig. 7). This connection

modification is regularly tested after several iterations of the node position modification method.

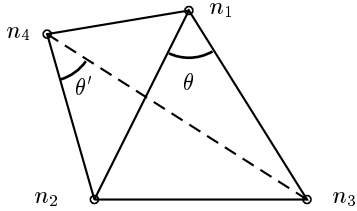


Fig. 7. Edge connection : angle criterion

## 6.1 Results

If the number of nodes is high, the results are the same as those of the previous method for the first iterations : only a few edges need to be reconnected (Fig. 8 left). But when the number of iterations increases, the node movement has a better precision for a given number of iterations (Fig. 9). With 5000 nodes, the same precision can be obtained as the initial mesh with 40000 nodes (Tab. 3).

If the number of nodes is low, this method is much more efficient, even for a low number of iterations. With 400 nodes, the same precision can be obtained as the initial mesh with 11500 nodes. The quality of the mesh is much higher, even for a high number of iterations (Fig. 8 right). Even if this moving node procedure with reconnection has not been fully optimized, the computation time is about 60% lower than for a FE method with the same precision.

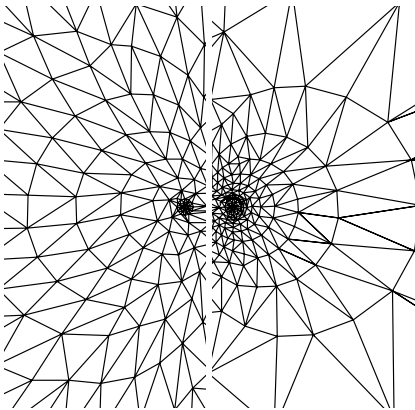


Fig. 8. Final mesh (400 nodes) : 10 iterations (left) to 100 iterations (right) with modification of the node positions and connections

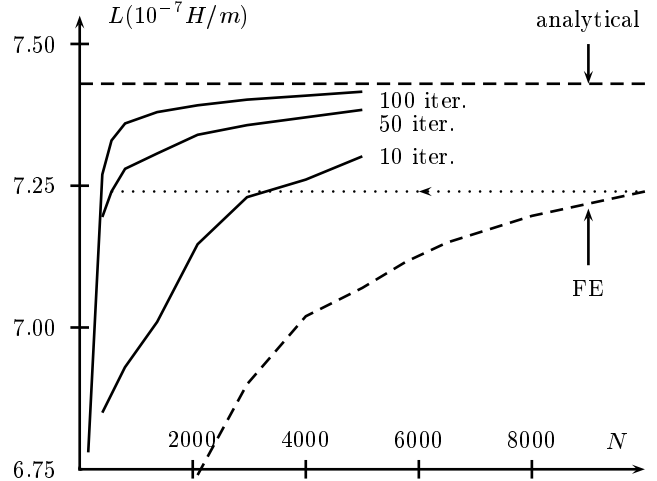


Fig. 9. Modification of the node positions and connections

Table 3. Accuracy of the FE method with modification of the node positions and connections

Nb of nodes	400	800	1370	2086	2960	5000
0 iter. (%)	26.	17.	12.	9.4	7.2	4.9
10 iter (%)	7.9	6.8	5.8	3.9	2.8	1.8
50 iter (%)	3.3	2.1	1.8	1.31	1.1	0.72
100 iter (%)	2.2	1.1	0.76	0.62	0.48	0.29

## 7 Modification of the node positions and connections and nodes removing

Up to now, the number of nodes was set constant. It is possible to reduce the number of nodes by edge fusion [11]. The choice was made to cancel the nodes themselves, with a local approximation. The question is to wonder, if a few nodes can be removed or not, in order to accelerate the iterations, without losing too much accuracy.

The polygon formed by the elements which the node  $n$  belongs to is considered (Fig. 10). The positions of the nodes and values of the MVP are fixed at all the nodes. The contribution to the energy of the polygon without the node  $n$  (the inner edges are displayed in dashed lines) is compared to the previous one with the node  $n$  (the inner edges are in filled lines). If they are close enough, the node  $n$  can be removed.

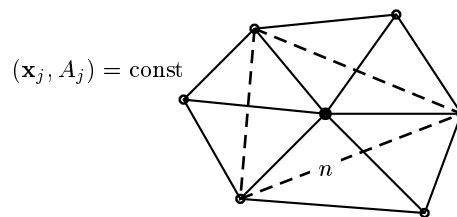


Fig. 10. Cancelling the node  $n$

**7.1 Results**

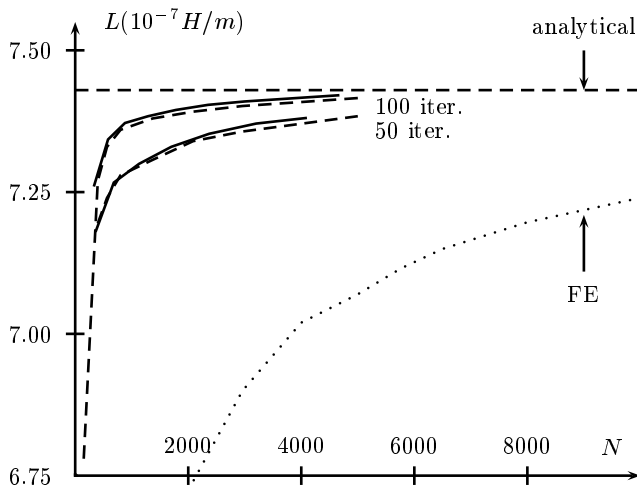
If the number of nodes is reduced, the same accuracy cannot be obtained as the one with the constant initial number of nodes (Tab. 4). But the accuracy with node cancelling is better than with the constant final number of nodes. The computation time is slightly more important than the method without node cancelling.

**Table 4.** Accuracy of the FE method with modification of the node positions and connections, and node cancelling

Nb of nodes	400	800	1370	2086	2960	5000
0 iter. (%)	26.	17.	12.	9.4	7.2	4.9
Nb of nodes	360	680	1130	1700	2370	4100
50 iter (%)	3.5	2.3	1.9	1.5	1.2	0.76
Nb of nodes	330	580	880	1290	1780	3000
100 iter (%)	2.4	1.3	0.88	0.72	0.57	0.37

Fig. 11 shows the difference between the results of the addition of the node cancelling and the method without node cancelling (in dashed lines).

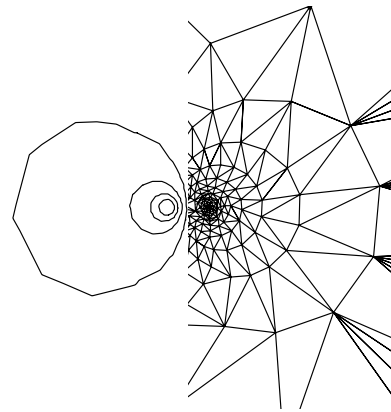
The mesh is less regular than the one without node removing (Fig. 12). Nodes near the outer border have been eliminated.



**Fig. 11.** Modification of the node positions and connections and node cancelling

**8 Conclusion**

The technique which is used, is to try to modify the support of discretization, for a constant number of nodes. By maximizing the functional of the coenergy, expressed



**Fig. 12.** Final mesh (330 nodes) : 100 iterations with modification of the node positions and connections, and node cancelling (70)

as a function of the node positions in the mesh, it is possible to obtain more precision.

In order to make more important changes in this support, the edge connection is used in addition to the modification of the node positions. This method is much more efficient for a low number of nodes : when compared with the initial mesh, the number of nodes is reduced by 25 for the same precision, and the computation time has been largely reduced.

For a greater number of nodes, a node removing operation can be used to gain computation time without losing much precision.

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