A multi-stream model for Weibel-type instabilities in the relativistic regime

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Topics

1. Vlasov plasmas
2. The Multi-Stream model
3. Current Filamentation Instability
4. The Weibel instability
5. Conclusions

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The problem

- As a fundamental issue, the Weibel instability or the current filamentation instability (CFI) have been the subject of research interest for half a century.
- Both processes are associated with the generation of quasi-static intense magnetic field

- Weibel (PRL 2, 83 (1959)) introduces a mechanism based on thermal anisotropy
- Fried (Phys. Fluids 2, 337 (1959)) proposes a counter-streaming based instability that propagates along the perpendicular direction with respect to the beam

Vlasov model

- Halfway between the N body model and the usual hydrodynamical one the Vlasov equation (supplemented by the Poisson-Maxwell equations) describes different media and problems:

- Magnetic reconnection in astrophysics and tokamak plasmas
- Gyrokinetic transport
- Laser-plasma interaction (parametric instabilities and Weibel-type instabilities, Self-induced transparency for overdense plasmas, …)
- Relativistic plasmas (particle accelerators, …)

Vlasov plasmas

- From a physical point of view: what situations and which physical systems are described by Vlasov model with particular attention to Weibel-type instabilities?
- From a mathematical point of view: what problems can be solved analytically?

- Another point deals with the numerical aspect of the problem and is connected with the huge field of computer simulation: we will concentrate on what, in our opinion, is a central problem putting aside technical and often non trivial aspects.

Phase space properties (1)

We consider the motion of N interacting particles in phase space (q,p)

At time t_0 we draw a closed contour in phase space defining a volume $dS_0 = \oint dqdp$

At later time t the particles on the initial contour have moved and define a new contour the volume of which is

$$\oint dqdp = dS$$

If the motion is Hamiltonian (i.e. $\frac{dq}{dt} = \frac{\partial H}{\partial p} \quad and \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$)

Then: $\boxed{dS = dS_0}$

Phase space properties (2)

Conservation of the number of particle inside the contour

$$f(q,p,t)dS = f(q_0,p_0,t)dS_0 \quad and \quad dS_0 = dS$$

We deduce $\quad f(q,p,t) = f(q_0,p_0,t) \quad or \quad \dfrac{Df}{Dt} = 0$

Where $\quad \dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + \dfrac{dq}{dt}\dfrac{\partial}{\partial q} + \dfrac{dp}{dt}\dfrac{\partial}{\partial p}$

Taking into account $\quad \dfrac{dq}{dt} = v \quad and \quad \dfrac{dp}{dt} = F$

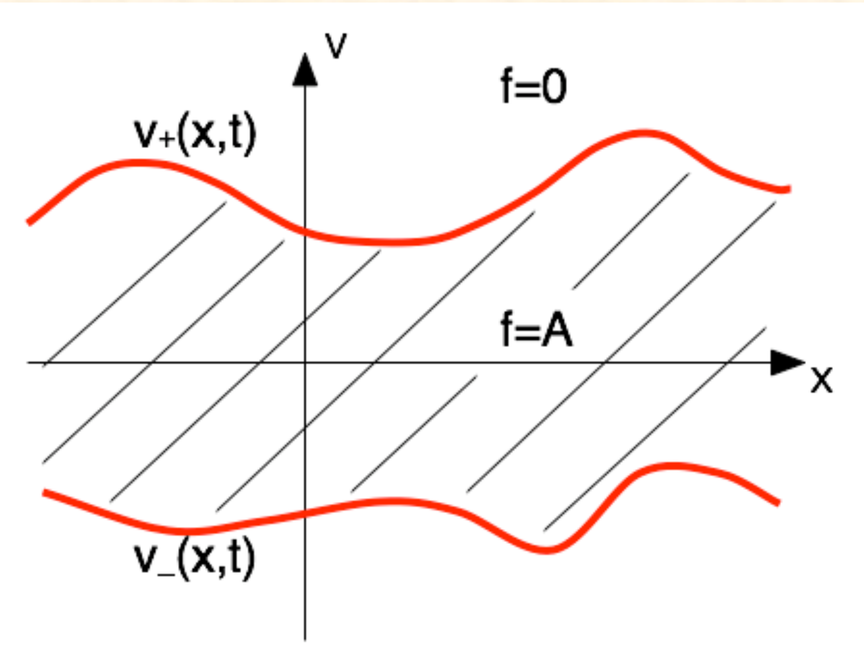
We obtain $\quad \boxed{\dfrac{\partial f}{\partial t} + v\dfrac{\partial f}{\partial q} + F\dfrac{\partial f}{\partial p} = 0}$: Liouville or Vlasov EQUATION

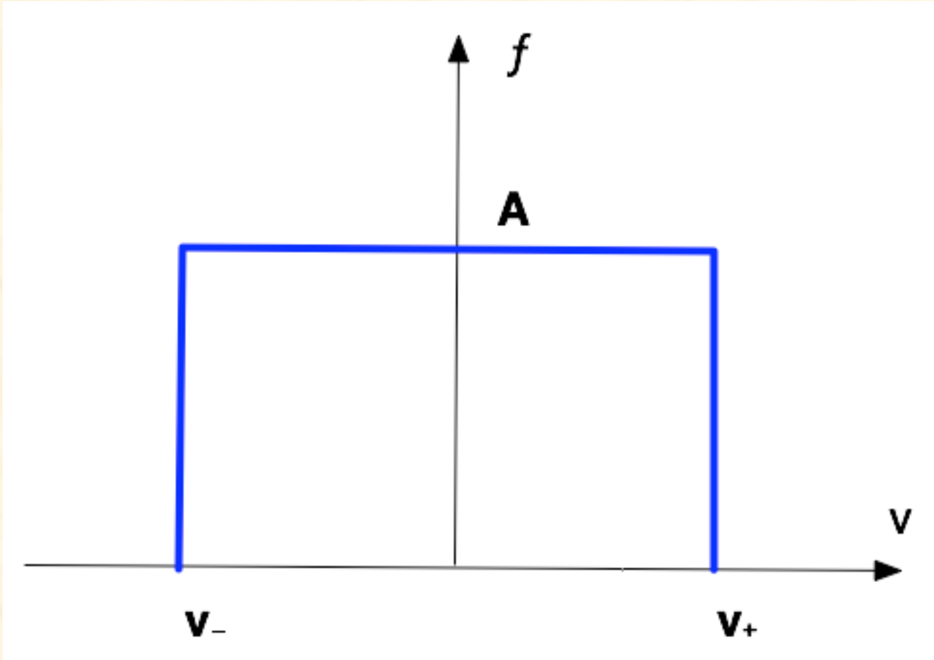
A Curiosity: the « Water-Bag » model

Consider a 1D plasma (2D phase space)

Initial condition
$$f(x,v,0) = A \quad v_- < v < v_+$$
$$= 0 \quad \text{elsewhere}$$

As a result of the Liouville's equation *f(x,v,t)* remains equal to *A*

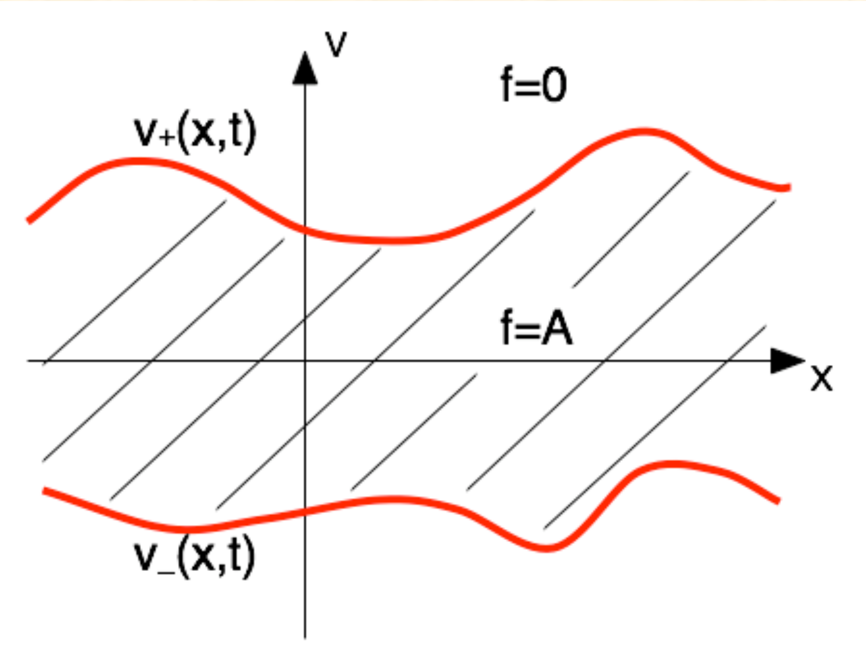




The Water-Bag model (2)

The problem in entirely described by the two contours $v_{\pm}(x,t)$

(single valued functions)



$$\frac{Dv_+}{Dt} = \frac{\partial v_+}{\partial t} + v_+ \frac{\partial v_+}{\partial x} = \frac{eE(x,t)}{m}$$

$$\frac{Dv_-}{Dt} = \frac{\partial v_-}{\partial t} + v_- \frac{\partial v_-}{\partial x} = \frac{eE(x,t)}{m}$$

+ Poisson equation

$$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0} A(v_+ - v_-) - \frac{en_0}{\varepsilon_0}$$

Connection with the hydrodynamical model

Density

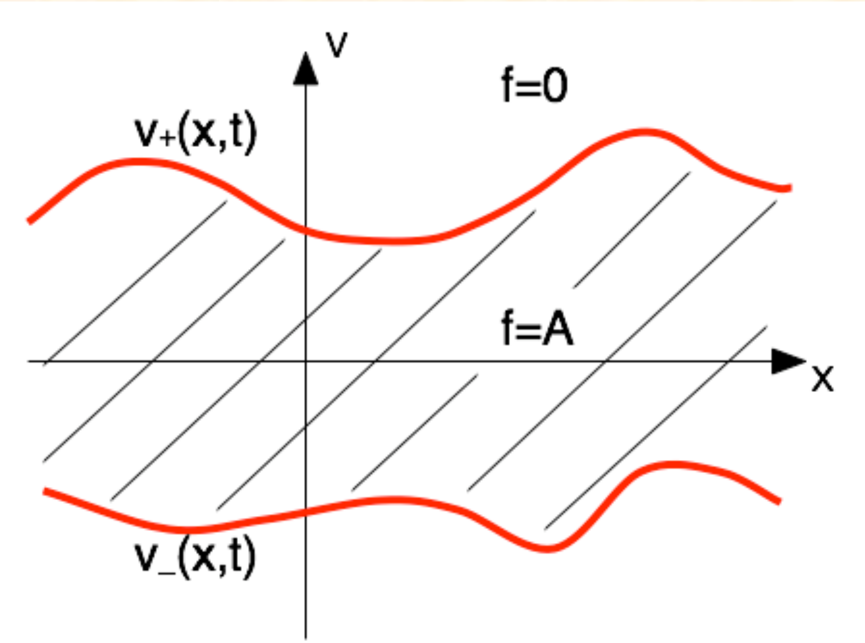
$$n(x,t) = A(v_+ - v_-)$$

Mean velocity

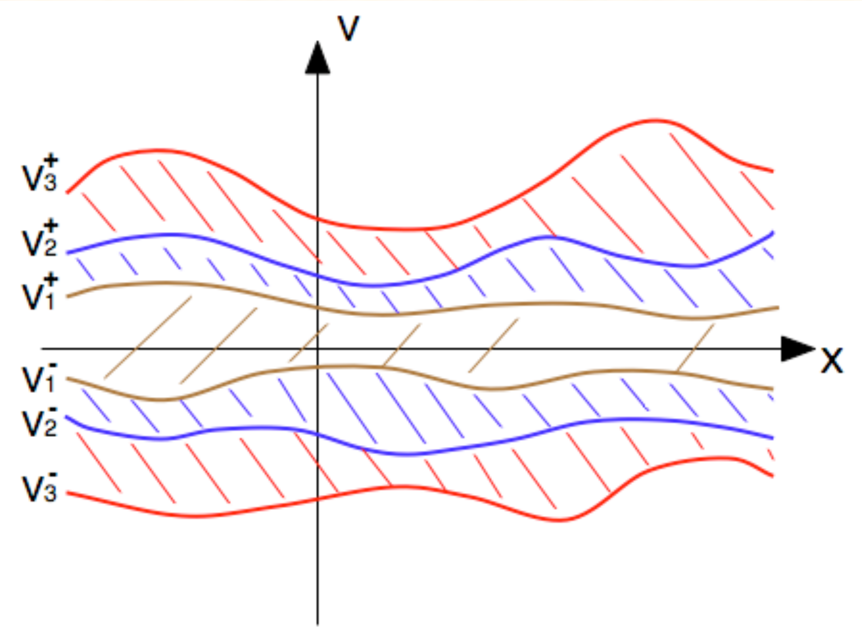
$$u(x,t) = \frac{1}{2}(v_+ + v_-)$$

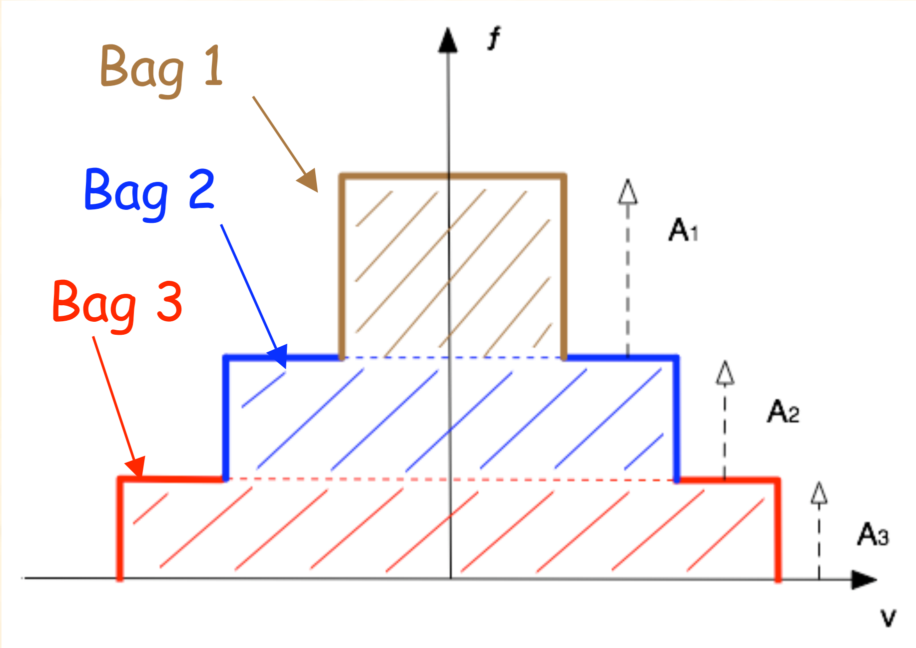
$$\frac{\partial v_+}{\partial t} + v_+ \frac{\partial v_+}{\partial x} = \frac{eE(x,t)}{m}$$
$$\frac{\partial v_-}{\partial t} + v_- \frac{\partial v_-}{\partial x} = \frac{eE(x,t)}{m}$$

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -\frac{1}{nm}\frac{\partial P}{\partial x} + \frac{eE}{m}$$
$$Pn^{-3} = \frac{m}{12A^2}$$



Possible extension to N « bags »





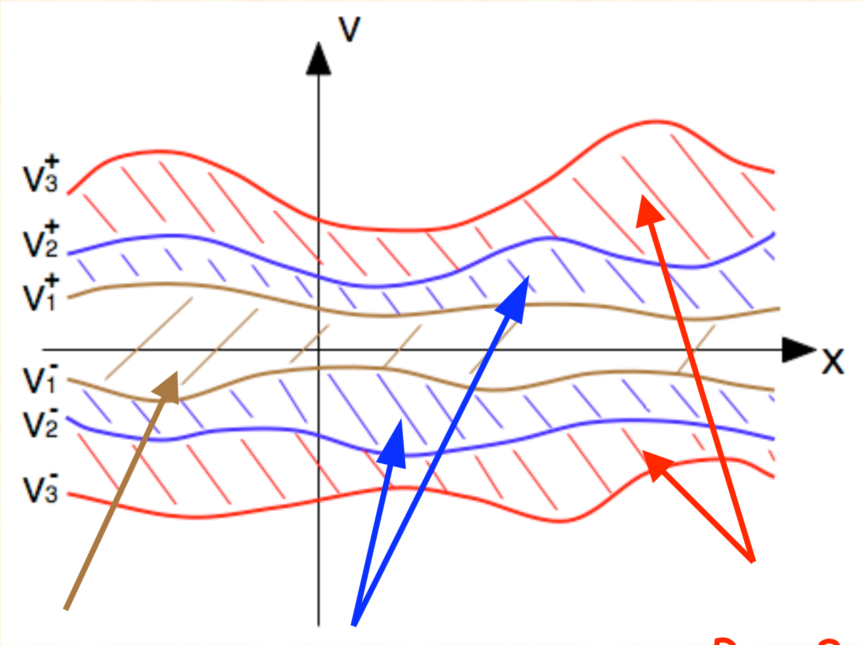
$$f(x,v,t) = \sum_{j=1}^{N} A_j \left(\Upsilon(v - v_j^-(x,t) - \Upsilon(v - v_j^+(x,t)) \right)$$

$$\Upsilon(x) = 0 \text{ si } x < 0\,;\ \Upsilon = 1 \text{ si } x > 0 \text{ (Heaviside)}$$

Multi-fluids and MWB

density: $$n_j(x,t) = A_j\left(v_j^+ - v_j^-\right)$$

Mean velocity: $$u_j(x,t) = \frac{1}{2}\left(v_j^+ + v_j^-\right)$$



Bag 1 Bag 2 Bag 3

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}\left(n_j u_j\right) = 0$$

$$\frac{\partial u_j}{\partial t} + u_j \frac{\partial u_j}{\partial x} = -\frac{1}{n_j m}\frac{\partial P_j}{\partial x} + \frac{eE}{m}$$

$$P_j n_j^{-3} = \frac{m}{12 A_j^2}$$

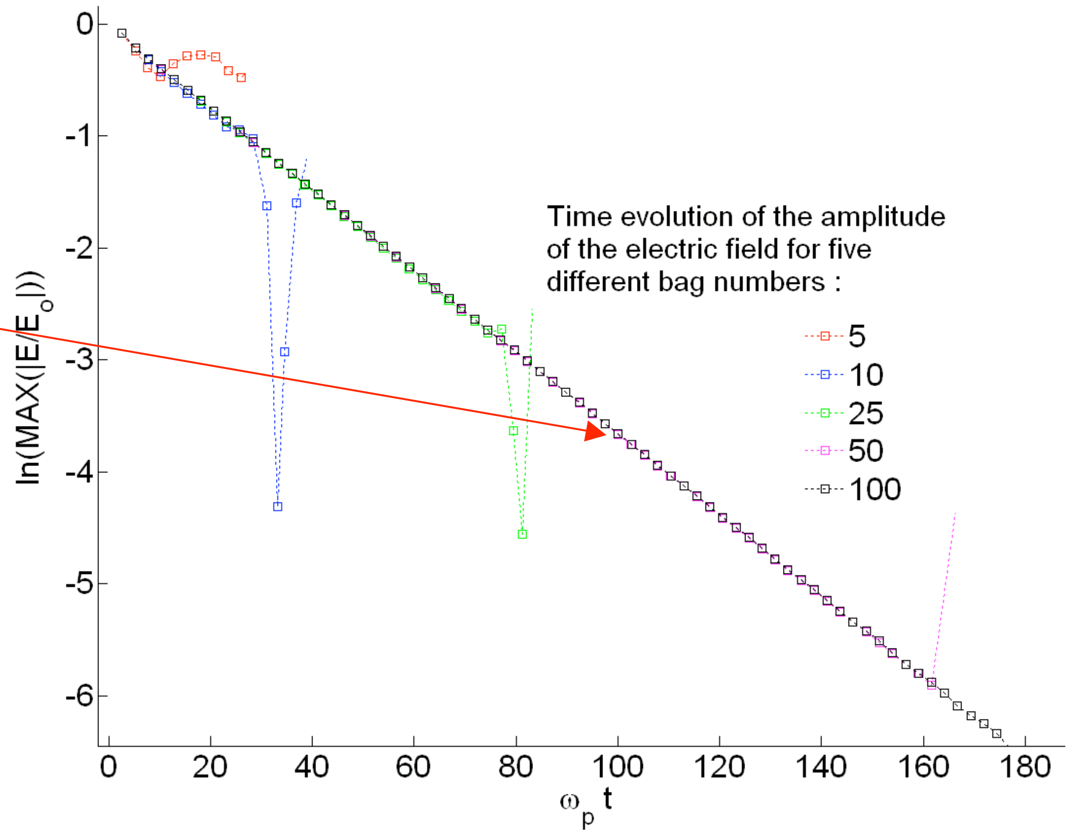
The coupling is made through Poisson: $$\frac{\partial E}{\partial x} = \frac{e}{\varepsilon_0}\left(\sum_{j=1}^{N} n_j - n_0\right)$$

MWB: Linear Landau damping

$$\varepsilon = 10^{-2} \quad k\lambda_D = 0,7$$

$$\frac{\gamma_{Landau}}{\omega_p} = 0.37$$

Slope 0,367



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Modeling Vlasov-Maxwell system for Weibel-type instabilities

Vlasov equation for the electronic distribution function F

$$\frac{\partial F}{\partial t} + \frac{p_x}{m\gamma}\frac{\partial F}{\partial x} + e\left(\mathbf{E} + \frac{\mathbf{p}\times\mathbf{B}}{m\gamma}\right)\cdot\frac{\partial F}{\partial \mathbf{p}} = 0$$

$$\gamma = \left(\frac{p_x^2 + \mathbf{p}_\perp^2}{m^2c^2}\right)^{1/2}$$

$F = F(x, p_x, \mathbf{p}_\perp, t) \Rightarrow \quad D_x = 1$ but $D_p = 3$

Does not point to Vlasov Codes!

Par chance we have an invariant!

Particle motion (1)

Let us come back to particle motion and consider the Hamiltonian of an electron in an electromagnetic field

$$\mathrm{H} = mc^2\left[1+\frac{(\mathbf{P}_c - e\mathbf{A}_\perp)^2}{m^2c^2}\right]^{1/2} + e\phi(x,t)$$

Using Coulomb gauge $\nabla\cdot\mathbf{A}=0 \quad \Rightarrow \quad \mathbf{A}=\mathbf{A}_\perp$

$\mathbf{P}_c = \mathbf{p} + e\mathbf{A}_\perp$ is the canonical momentum

Hamilton equations $\quad \dfrac{dP_{cx}}{dt} = -\dfrac{\partial H}{\partial x} \qquad \dfrac{d\mathbf{P}_{c\perp}}{dt} = -\nabla_\perp \mathrm{H} = 0$

Particle motion (2)

Now let us divide the electron population into a finite number N of groups j

Each group has the same perpendicular constant canonical momentum $\mathbf{C}_j$

The hamiltonian of the *particle for bunch j* is:

$$\mathrm{H}_j = mc^2\left[1+\frac{p_x^2}{m^2c^2}+\frac{\left(C_j - eA_\perp(x,t)\right)^2}{m^2c^2}\right]^{1/2} + e\phi(x,t)$$

Vlasov equation for *j*-particles (1)

Now we have to solve *N* Vlasov equations in the 2D phase space (x, p_x)

$$\boxed{\frac{Df_j}{Dt} = \frac{\partial f_j}{\partial t} + \left[f_j, H_j\right] = 0}$$

where

$$\boxed{\left[f_j, H_j\right] = \frac{\partial f_j}{\partial x}\frac{\partial H_j}{\partial p_x} - \frac{\partial f_j}{\partial p_x}\frac{\partial H_j}{\partial x}}$$

Finally we have

$$\boxed{\frac{\partial f_j}{\partial t} + \frac{p_x}{m\gamma_j}\frac{\partial f_j}{\partial x} + \left(eE_x - \frac{1}{2m\gamma_j}\frac{\partial}{\partial x}\left(\mathbf{C}_j - e\mathbf{A}_j\right)^2\right)\frac{\partial f_j}{\partial p_x} = 0}$$

The f_j's are coupled by Maxwell's equations for scalar and vector potential

Vlasov equation for *j*-particles (2)

The coupling of the multi-stream model with Maxwell's equations is straighforward:

$$\frac{\partial E_x}{\partial x} = \frac{e}{\varepsilon_0}\left\{n(x,t) - n_0\right\} \quad \text{and} \quad \frac{\partial^2 \mathbf{A}_\perp}{\partial t^2} - c^2 \frac{\partial^2 \mathbf{A}_\perp}{\partial x^2} = \frac{\mathbf{J}_\perp}{\varepsilon_0}$$

Through the data of the source terms:

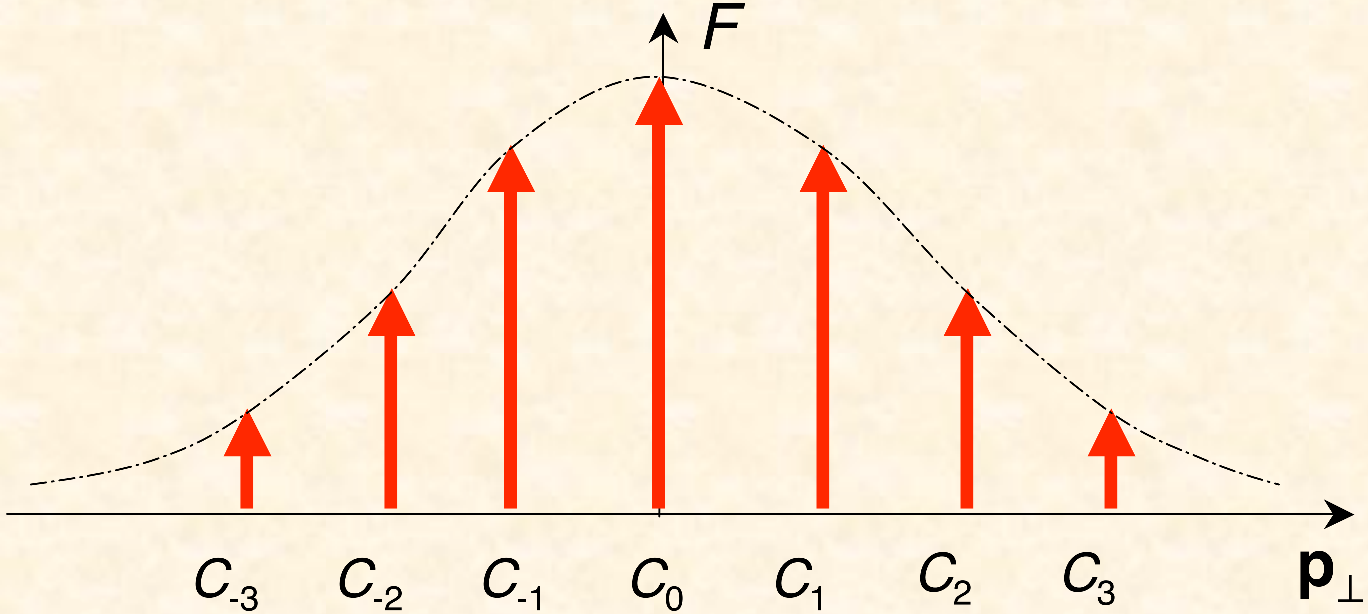
Density
$$\boxed{n(x,t) = \sum_{j=1}^{N} \int_{-\infty}^{+\infty} dp_x f_j(x,p_x,t)}$$

Current
$$\boxed{\mathbf{J}_\perp = \sum_{j=1}^{N} \mathbf{J}_{\perp j}(x,t) = \frac{e}{m}\left(\mathbf{C}_j - e\mathbf{A}_\perp\right)\int_{-\infty}^{+\infty}\left(f_j/\gamma_j\right)dp_x}$$

Vlasov equation for *j*-particles (3)

The full distribution function can be written as a sum of Dirac delta masses

$$F(x,p_x,\mathbf{p}_\perp,t)=\sum_{j=1}^{N} f_j(x,p_x,t)\,\delta\Big(\mathbf{p}_\perp-\big(\mathbf{C}_j-e\mathbf{A}_\perp(x,t)\big)\Big)$$



Connection with the multi-fluid model (1)

We have N Vlasov equations for each stream j

$$\boxed{\frac{\partial f_j}{\partial t} + \frac{p_x}{m\gamma_j}\frac{\partial f_j}{\partial x} + \left(eE_x - \frac{1}{2m\gamma_j}\frac{\partial}{\partial x}\left(\mathbf{C}_j - e\mathbf{A}_j\right)^2 \right)\frac{\partial f_j}{\partial p_x} = 0}$$

By considering the moments of these Vlasov equations,
It is possible to build a (closed) multi-fluid model:

Without loss of generality, we assume now :

$$\boxed{\mathbf{A}_\perp = A_y(x,t)\mathbf{e}_y \quad and \quad \mathbf{P}_{c\perp} = C_j\mathbf{e}_y = const}$$

The plasma is « cold » along p_x

Connection with the multi-fluid model (2)

For each stream j:

Continuity equation

$$\boxed{\frac{\partial n_j}{\partial t}+\frac{\partial}{\partial x}\left(\frac{n_j u_j}{m\bar{\gamma}_j}\right)=0}$$

Euler equation

$$\boxed{\frac{\partial u_j}{\partial t}+\frac{u_j}{m\bar{\gamma}_j}\frac{\partial u_j}{\partial x}=eE_x-\frac{1}{2m\bar{\gamma}_j}\frac{\partial}{\partial x}\left(C_j-eA_y\right)^2}$$

With a mean Lorentz factor

$$\bar{\gamma}_j=\sqrt{1+\frac{u_j^2}{m^2c^2}+\frac{\left(C_j-eA_y(x,t)\right)^2}{m^2c^2}}$$

Connection with the multi-fluid model (3)

For each stream j:

self-consistently with

Poisson equation

$$\frac{\partial E_x}{\partial x} = \frac{e}{\varepsilon_0}\left(\sum_{j=1}^{N} n_j(x,t) - n_0\right)$$

vector potential

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} + \sum_{j=1}^{N}\frac{e^2}{m\varepsilon_0}\rho_j\right)A_y = \frac{e}{m\varepsilon_0}\sum_{j=1}^{N} C_j\rho_j$$

using a « density » of

$$\rho_j = \int_{-\infty}^{+\infty}\frac{f_j dp_x}{\gamma_j(x,p_x,t)}$$

Note that the system is closed.

Connection with the multi-fluid model (4)

Let us consider an expansion around an equilibrium characterised by a mean density n_{0j} (for the stream j):

Fluctuations for density, Momentum and fields: $\delta n_j, \delta u_j, \delta A_y, \delta E_x$

By linearizing and performing Fourier transform in time and space:

$$\partial / \partial t = -i\omega \quad and \quad \partial / \partial x = ik$$

Connection with the multi-fluid model (5)

Continuity equation $$-i\omega\delta n_j + \frac{ikn_{0j}}{m\Gamma_{0j}}\delta u_j = 0$$

Euler's equation $$-i\omega\delta u_j = e\delta E_x + \frac{iek}{m\Gamma_{0j}}C_j\delta A_y$$

Poisson's equation $$ik\delta E_x = \frac{e}{\varepsilon_0}\sum_{j=1}^{N}\delta n_j$$

and $$\left(-\omega^2 + k^2c^2 + \sum_{j=1}^{N}\frac{\omega_{pj}^2}{\Gamma_{0j}}\right)\delta A_y - \sum_{j=1}^{N}\frac{\omega_{pj}^2 C_j^2}{m^2c^2\Gamma_{0j}^3}\delta A_y = \frac{e}{m\varepsilon_0}\sum_{j=1}^{N}\frac{C_j}{\Gamma_{0j}}\delta n_j$$

We have assumed $\boxed{\sum_{j=1}^{N} n_{0j} = n_0}$ and $\boxed{\sum_{j=1}^{N}\frac{C_j n_{0j}}{\Gamma_{0j}} = 0}$

Dispersion relation for Multi-fluids

We have indeed approximated

$$\bar{\gamma}_j(x,t) \approx \Gamma_{0j}^{-1} + \delta\bar{\gamma}_j \approx \Gamma_{0j}^{-1} + \frac{eC_j\delta A_y}{m^2c^2\Gamma_{0j}^3}$$

where

$$\Gamma_{0j} = \sqrt{1 + \frac{C_j^2}{m^2c^2}}$$

$$\omega_{pj}^2 = \frac{n_{0j}e^2}{m\varepsilon_0}$$

The relativistic dispersion relation is then:

$$\boxed{\left(1 - \sum_{j=1}^{N} \frac{\omega_{pj}^2}{\omega^2\Gamma_{0j}}\right)\left\{-\omega^2 + k^2c^2 + \sum_{j=1}^{N}\frac{\omega_{pj}^2}{\Gamma_{0j}^3} + \frac{k^2c^2}{\omega^2}\sum_{j=1}^{N}\frac{\omega_{pj}^2}{\Gamma_{0j}^3}\frac{C_j^2}{m^2c^2}\right\} = -\frac{k^2c^2}{\omega^2}\left(\sum_{j=1}^{N}\frac{\omega_{pj}^2}{\Gamma_{0j}^2}\frac{C_j}{mc}\right)^2}$$

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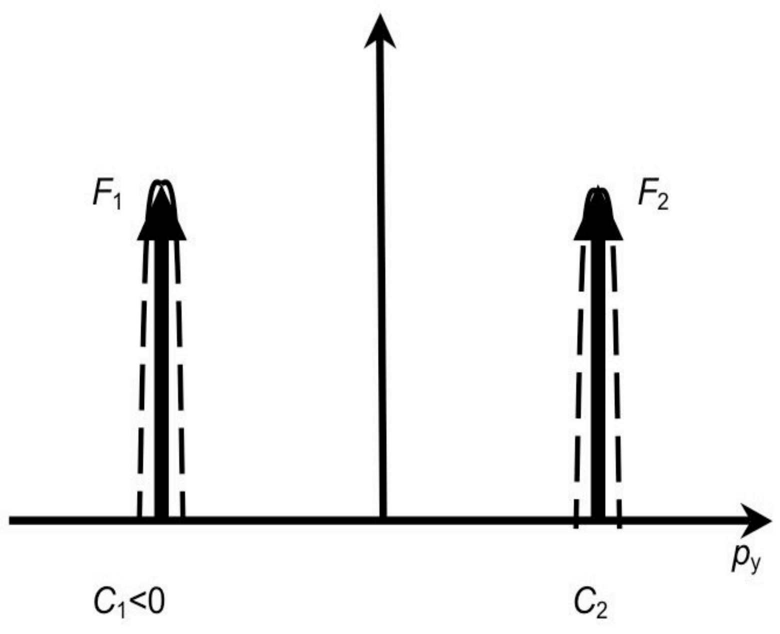
The CFI case

The dispersion relation for N=2 corresponds to the solution obtained by Pegoraro et al

Pegoraro et al
Phys. Scripta T63,262 (1996)



$$\left(1-\sum_{j=1}^{N}\frac{\omega_{pj}^2}{\omega^2\Gamma_{0j}}\right)\left\{-\omega^2+k^2c^2+\sum_{j=1}^{N}\frac{\omega_{pj}^2}{\Gamma_{0j}^3}+\frac{k^2c^2}{\omega^2}\sum_{j=1}^{N}\frac{\omega_{pj}^2}{\Gamma_{0j}^3}\frac{C_j^2}{m^2c^2}\right\}=$$

$$-\frac{k^2c^2}{\omega^2}\left(\sum_{j=1}^{N}\frac{\omega_{pj}^2}{\Gamma_{0j}^2}\frac{C_j}{mc}\right)^2$$

Semi-lagrangian Vlasov simulations:

The reduction Hamiltonian technique is tested by full 1D2V Vlasov-Maxwell simulations based on

$$\frac{\partial F}{\partial t}+\frac{p_x}{m\gamma}\frac{\partial F}{\partial x}+e\left(E_x+\frac{p_y B_z}{m\gamma}\right)\frac{\partial F}{\partial p_x}+e\left(E_y-\frac{p_x B_z}{m\gamma}\right)\frac{\partial F}{\partial p_y}=0$$

with $\frac{\partial B_z}{\partial t}=-\frac{\partial E_y}{\partial x}$ and $\frac{\partial E_y}{\partial t}=-c^2\frac{\partial B_z}{\partial x}-\frac{J_y}{\varepsilon_0}$

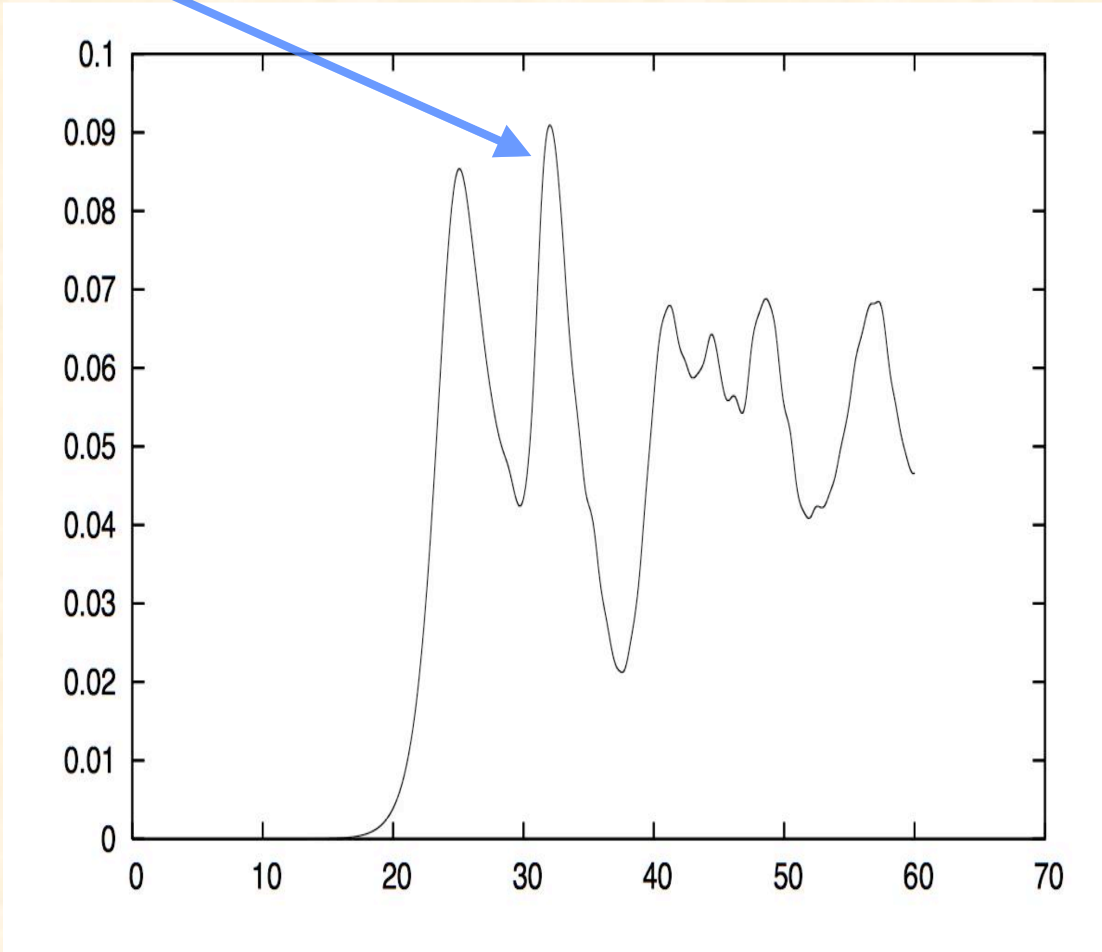
The source terms are:

$$n(x,t)=\iint F\left(x,p_x,p_y\right)dp_x dp_y$$

$$J_y(x,t)=-e\iint\frac{p_y}{m\gamma}F\left(x,p_x,p_y\right)dp_x dp_y$$

Current Filamentation Instability

- High intense magnetic field



Physical Parameters

Te=3keV

B_0=0.0001

Momentum of beams

P_{01}=0.90 mc

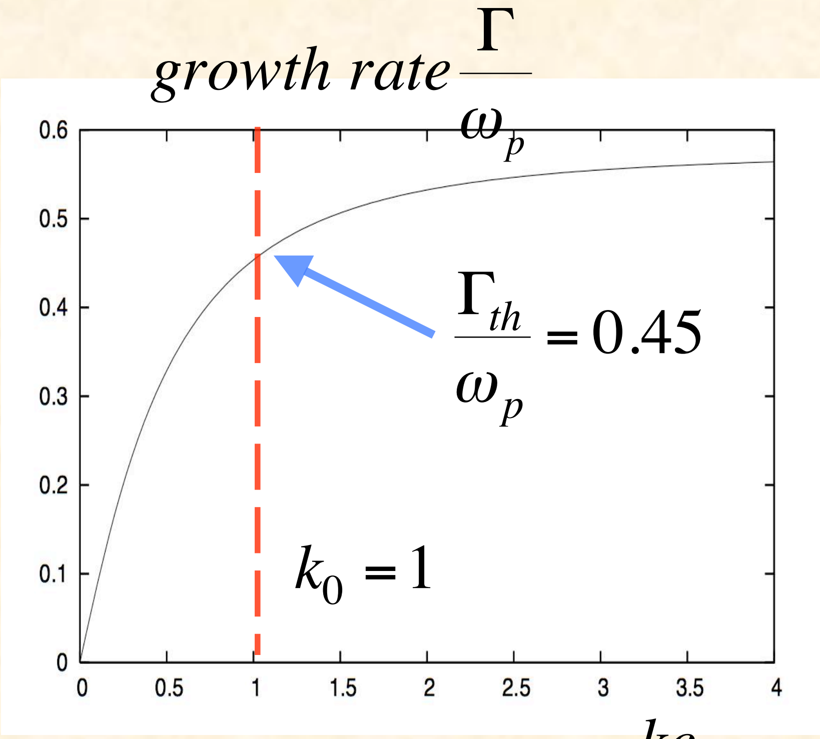
P_{02}=-0.90 mc

Density of beams

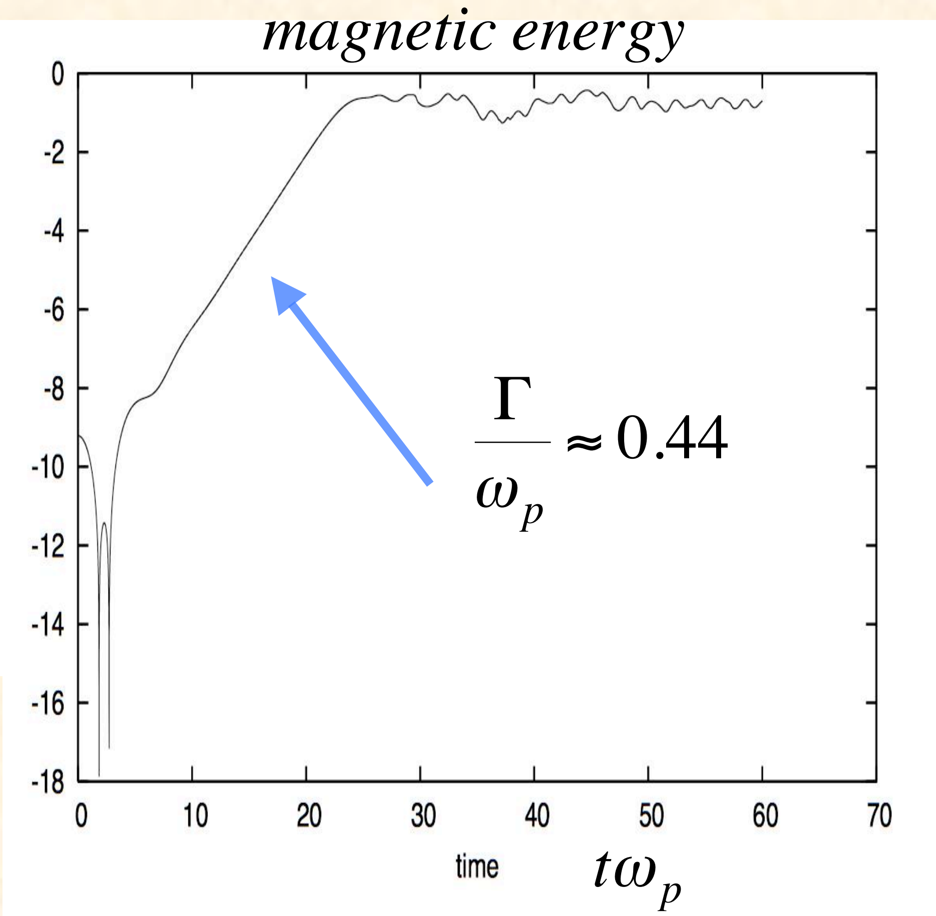
n_{01}=0.5 n_0

n_{02}=0.5 n_0

CFI (2)



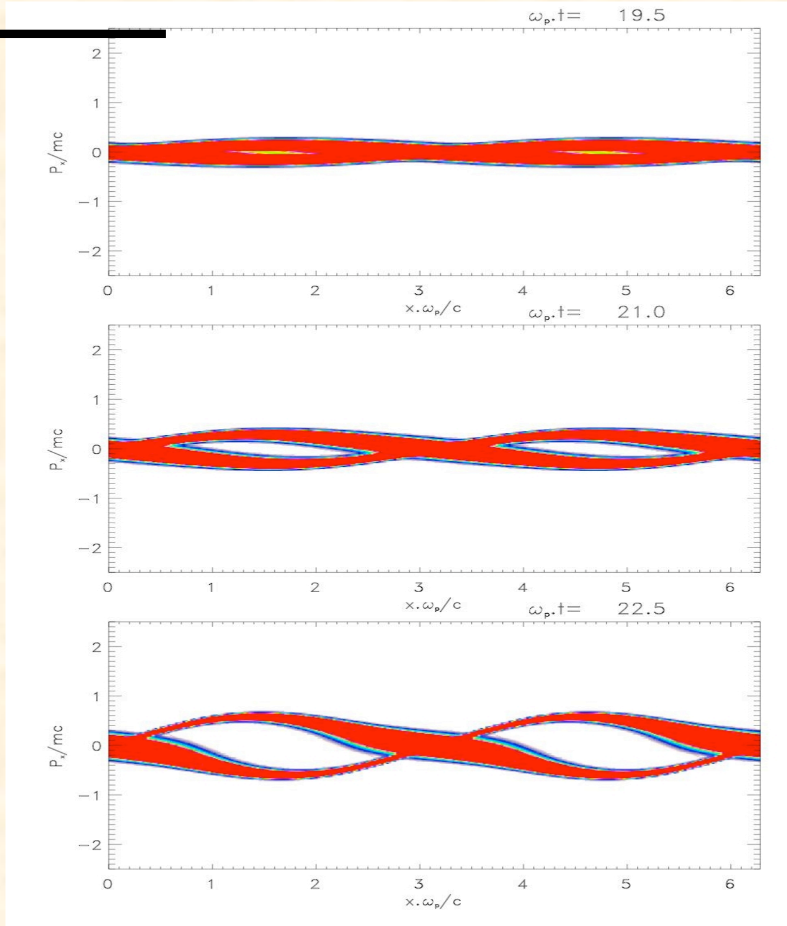
Perturbation introduced on k_0=1



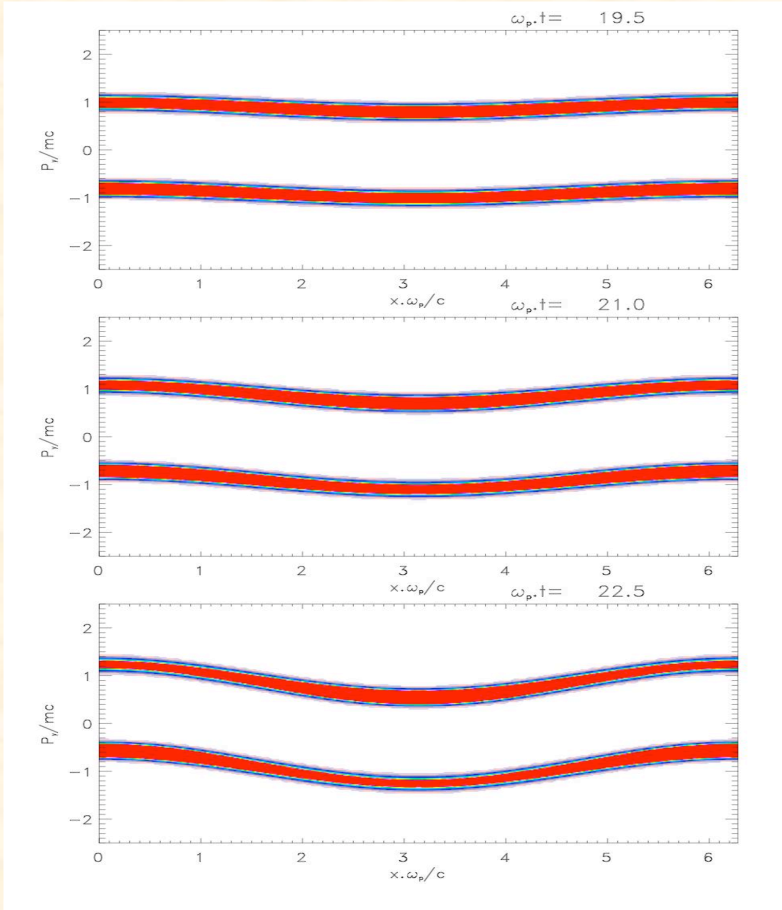
$1D2V\ Vlasov - Maxwell$

CFI- 1D2V Vlasov

X-P_x phase space



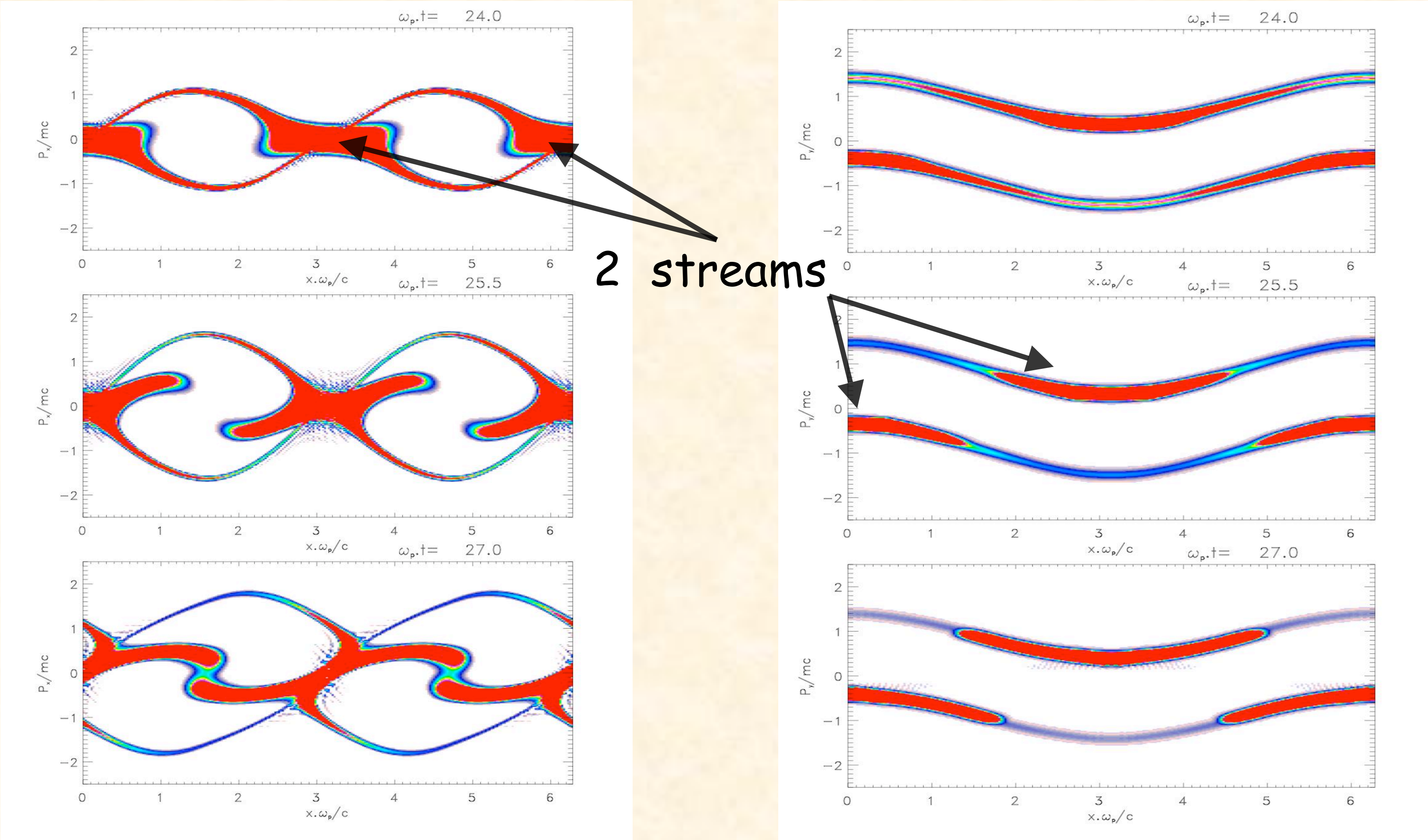
X-P_y phase space



CFI- 1D2V simulation (2)

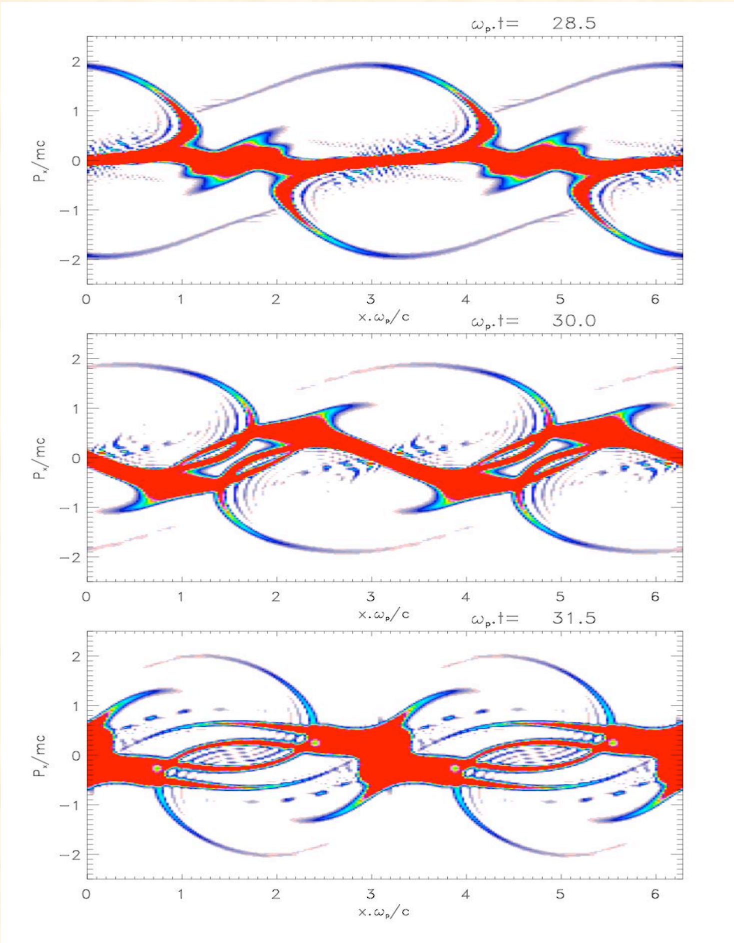
X-P_x phase space

X-P_y phase space

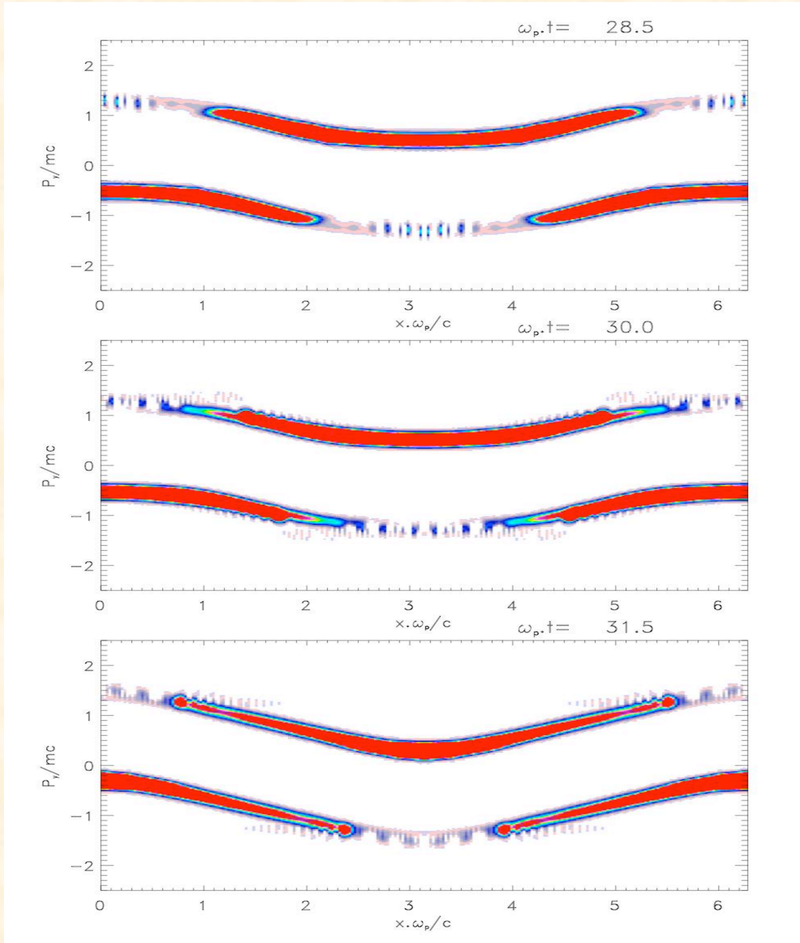


CFI- 1D2V simulation (3)

X-P_x phase space

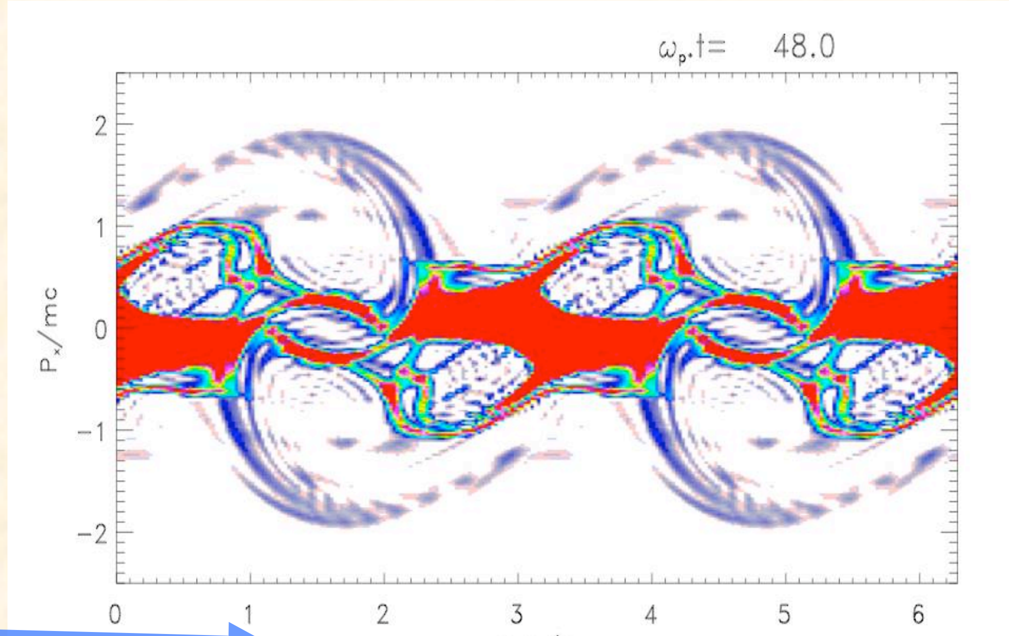


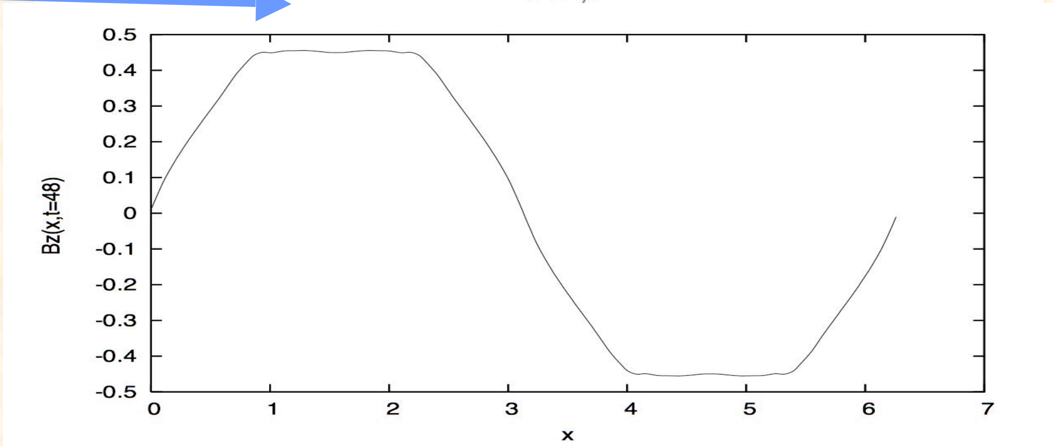
X-P_y phase space



CFI - 1D2V simulation (4)

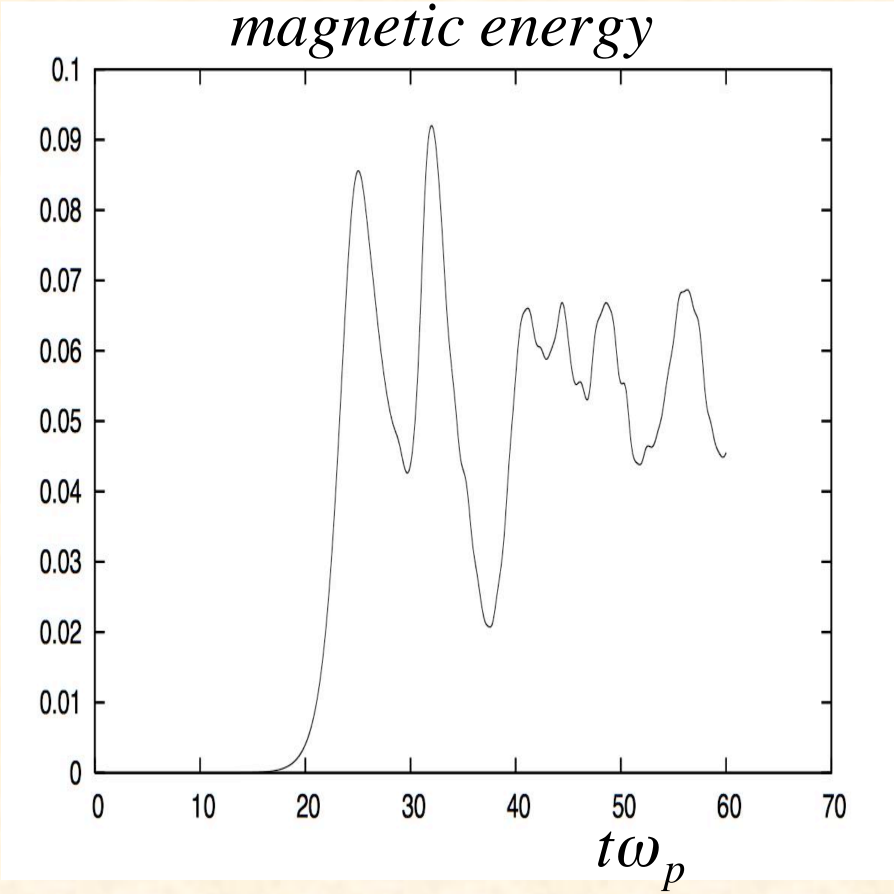
Region of strong magnetic field is very complex



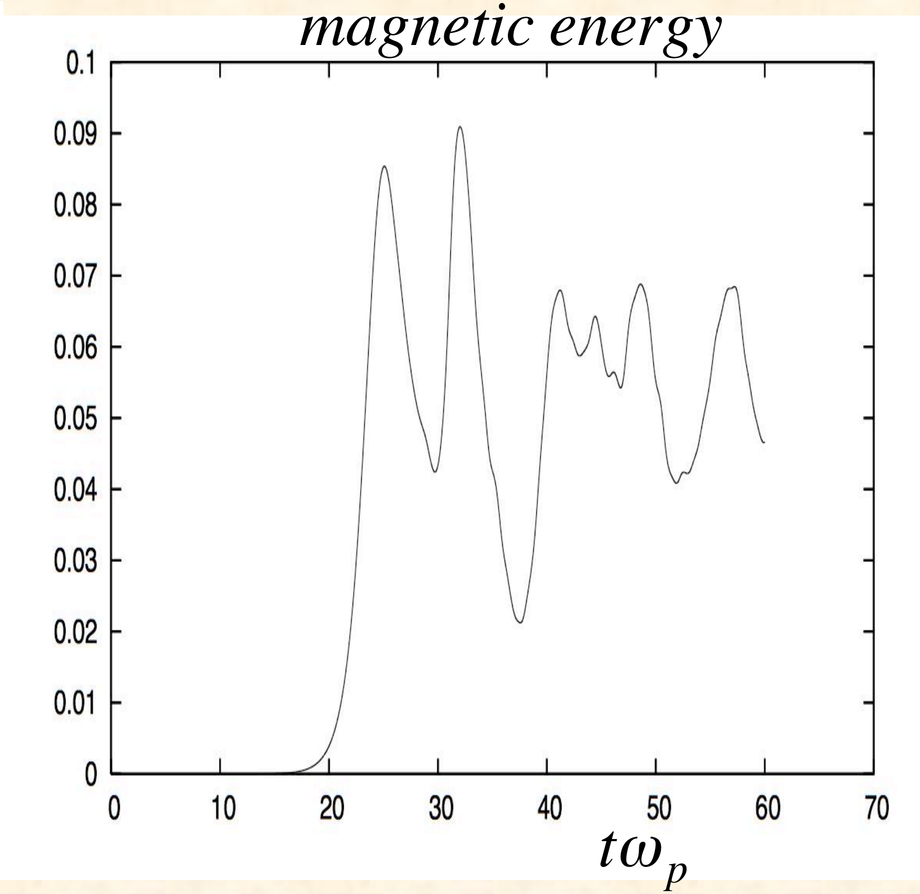


Comparison with the multi-stream model (1)

Multi- stream Vlasov model (N=2)



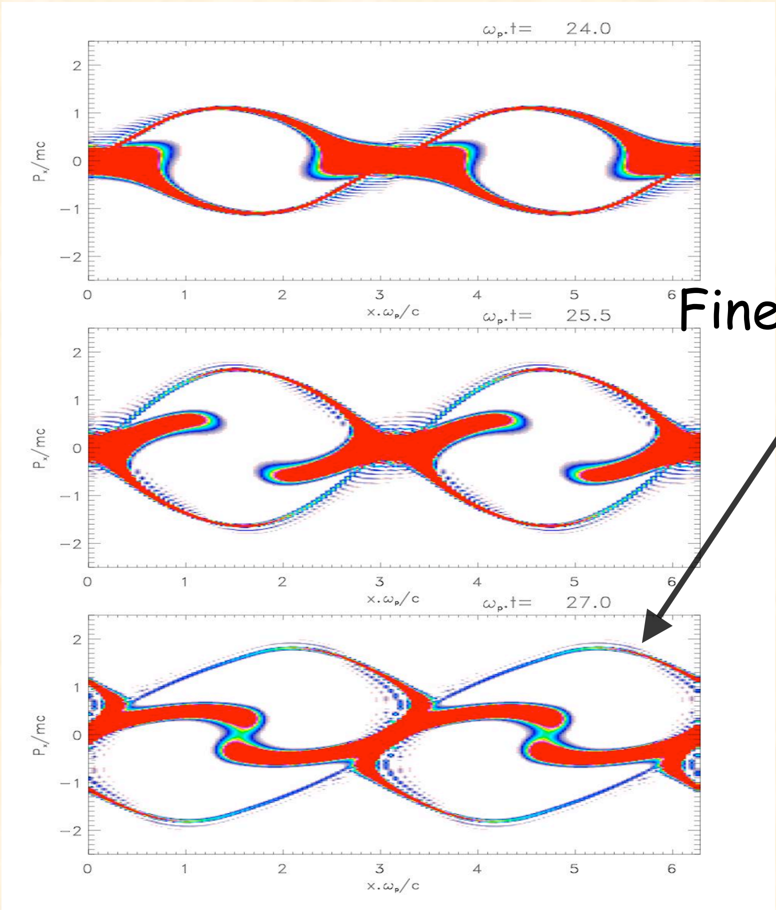
Full 1D2V Vlasov model



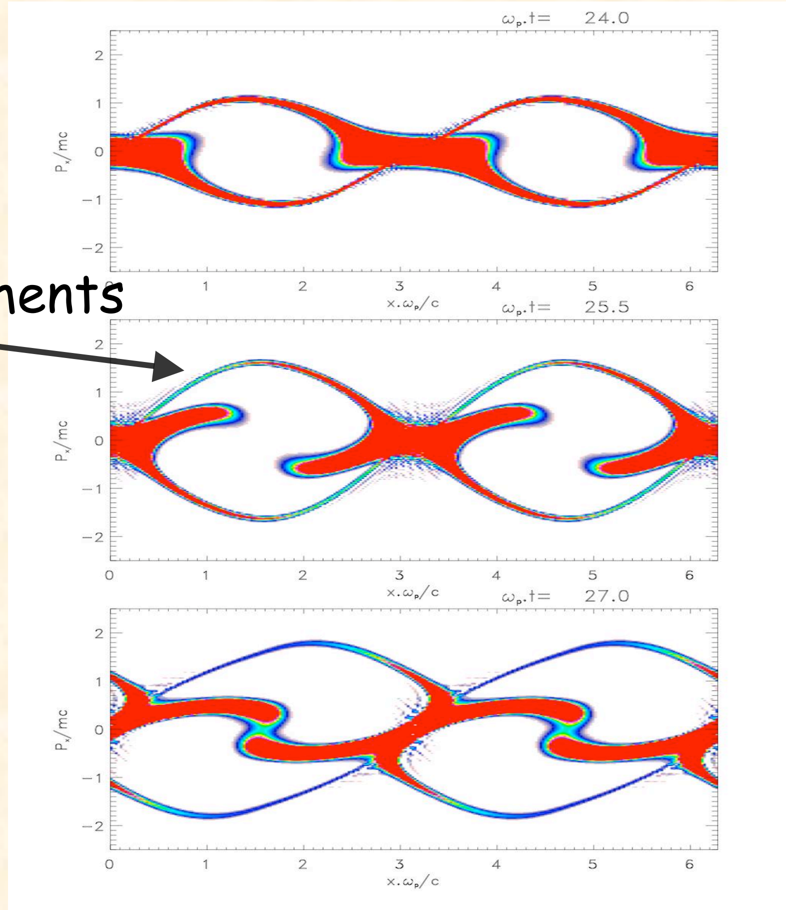
Comparison with the multi-stream model (2)

Multi- stream Vlasov model (N=2)

Full 1D2V Vlasov model

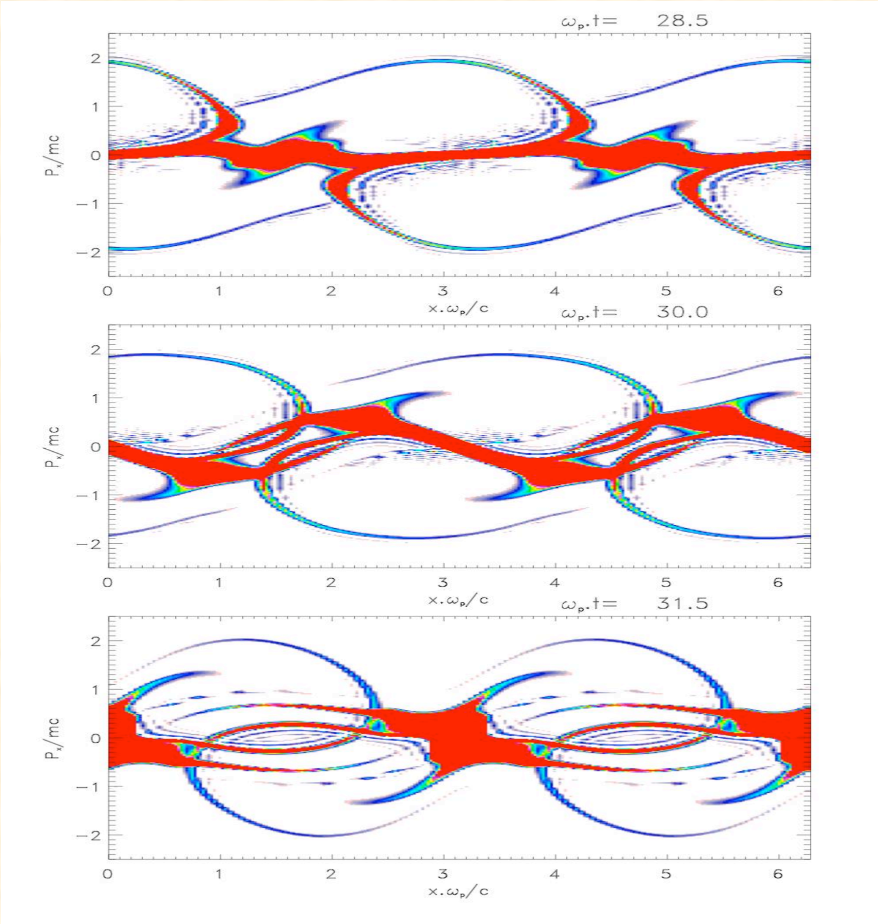


Fine filaments

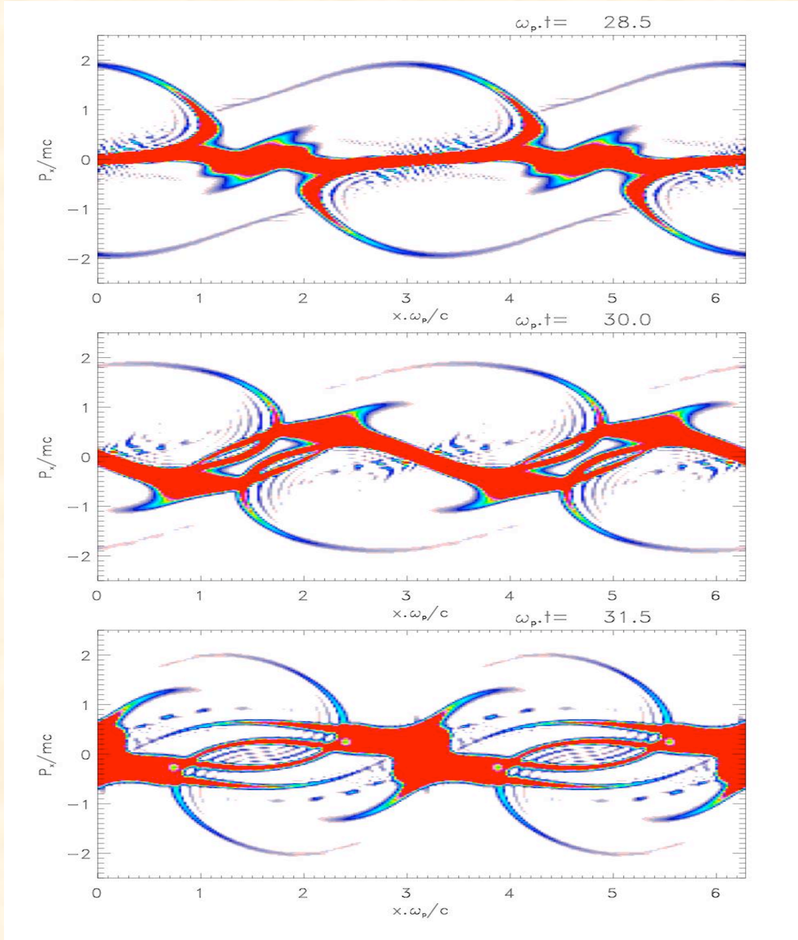


Comparison 1D1V and 1D2V (4)

Multi- stream Vlasov model (N=2)

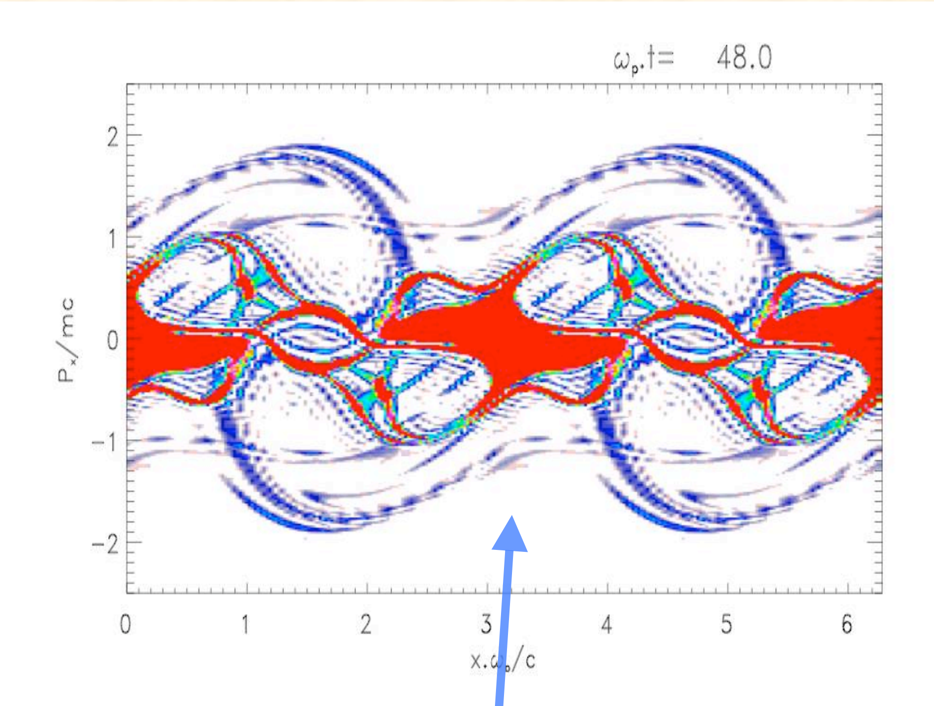


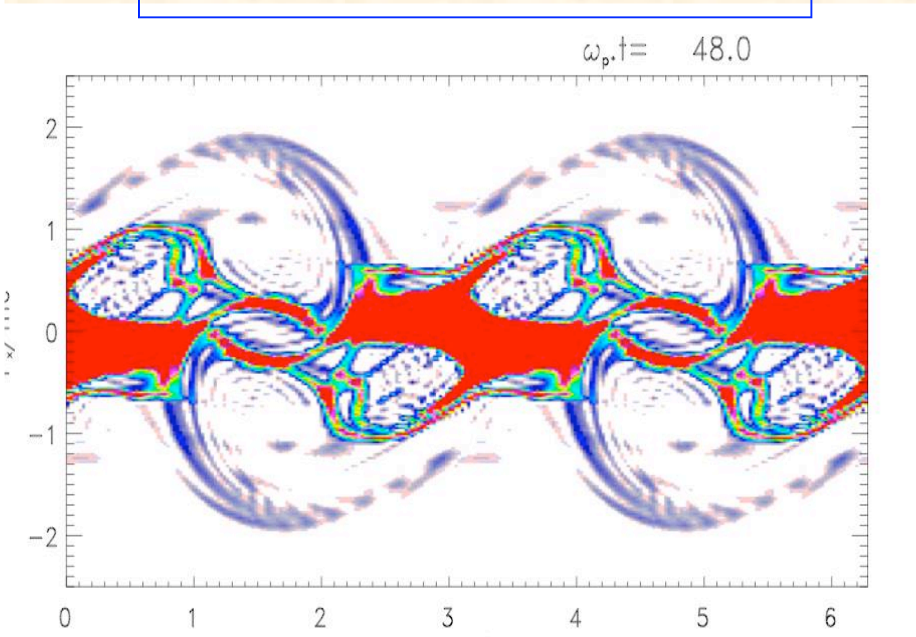
Full 1D2V Vlasov model



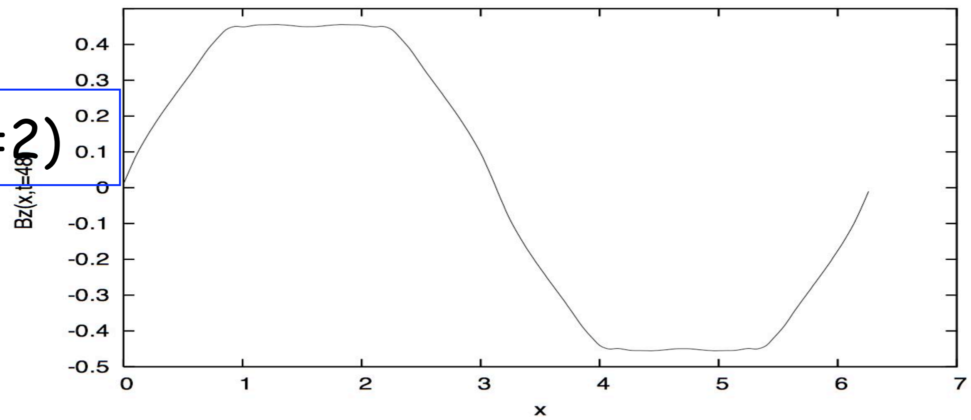
Comparison 1D1V and 1D2V (4)

Full 1D2V Vlasov model





Multi- stream Vlasov model (N=2)



Topics

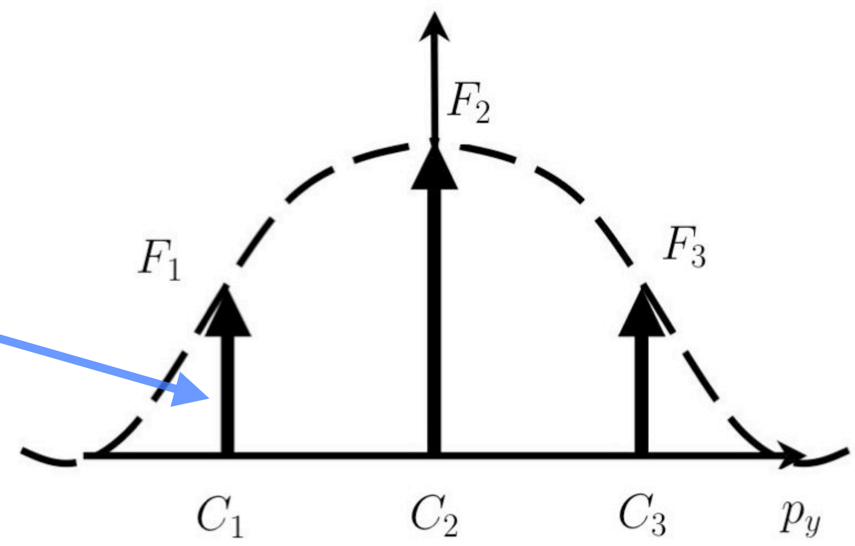
1. Vlasov plasmas
2. The Multi-Stream model
3. Current Filamentation Instability
4. The Weibel instability
5. Conclusions

Extension to the Weibel case

We can recover the Weibel dispersion relation induced by a temperature anisotropy by taking only N=3 streams

$$\boxed{F\left(x,p_x,p_y,t=0\right)=n_0\delta(p_x)\sum_{j=1}^{N}F_j\delta\left(p_y-C_j\right)}$$

We have to determine Fj and Cj



Standard Weibel case (2)

In the non relativistic regime

$$\omega_p^2 + k^2c^2 - \omega^2 + \frac{k^2c^2}{\omega^2}\sum_{j=1}^{N}\omega_{pj}^2\frac{C_j^2}{m^2c^2} = 0$$

Let us now consider a Maxwellian of density n_0 and Thermal velocity v_{th}:

We can write

$$\iint F(p_x, p_y)dp_x dp_y = n_0$$

$$\iint p_y^2 F(p_x, p_y)dp_x dp_y = n_0 m^2 v_{th}^2$$

$$\iint p_y^4 F(p_x, p_y)dp_x dp_y = 3 n_0 m^4 v_{th}^4$$

Standard Weibel case (3)

In the non relativistic regime

$$\boxed{\omega_p^2 + k^2c^2 - \omega^2 + \frac{k^2c^2}{\omega^2}\sum_{j=1}^{N}\omega_{pj}^2\frac{C_j^2}{m^2c^2} = 0}$$

For the different streams :

$$\boxed{\sum_{j=1}^{3} F_j = 1 \quad and \quad \sum_{j=1}^{3} C_j F_j = 0}$$

$$\sum_{j=1}^{3} F_j C_j^2 = F_1 C_1^2 + F_2 C_2^2 + F_3 C_3^2 = 2F_1 C_1^2 = m^2 v_{th}^2$$

$$\sum_{j=1}^{3} F_j C_j^4 = F_1 C_1^4 + F_2 C_2^4 + F_3 C_3^4 = 2F_1 C_1^4 = 3m^4 v_{th}^4$$

Standard Weibel case (4)

In the non relativistic regime

$$\boxed{\omega_p^2 + k^2c^2 - \omega^2 + \frac{k^2c^2}{\omega^2}\sum_{j=1}^{N}\omega_{pj}^2\frac{C_j^2}{m^2c^2} = 0}$$

The solution is straightforward

$$\boxed{\begin{array}{ll} F_1 = F_3 = \dfrac{1}{6} & \text{and} \quad F_2 = \dfrac{2}{3} \\ C_3 = -C_1 = \sqrt{3}mv_{th} & \text{and} \quad C_2 = 0 \end{array}}$$

And finally

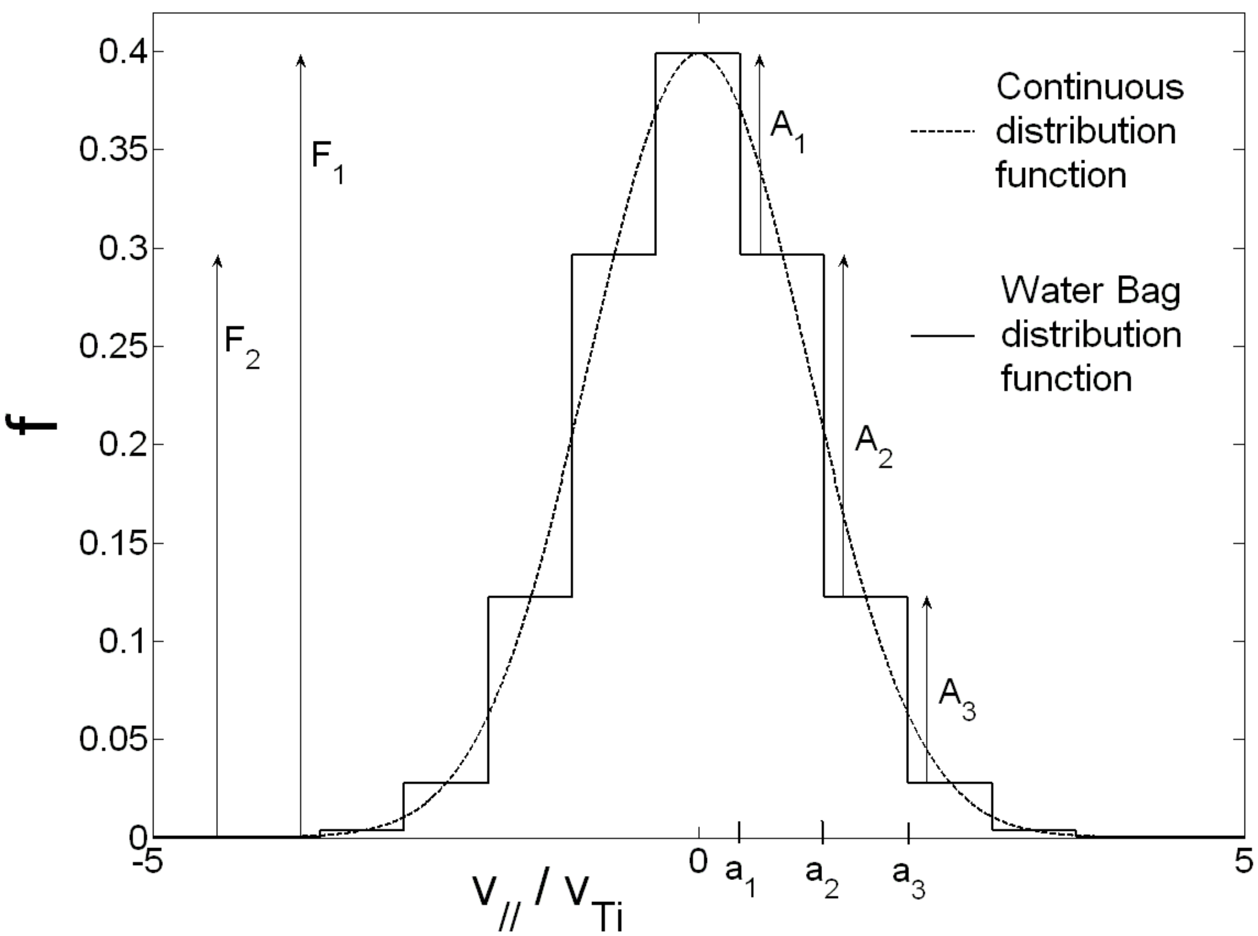
$$\omega^2 - \omega_p^2 - k^2c^2 - \omega_p^2\frac{k^2v_{th}^2}{\omega^2} = 0$$

Weibel -type instabilities ?

Strong relativistic effects are importants: difficulties To take into account these effects: different distributions in the literature (semi-relativistic bi-Maxwellians, Maxwell -Jüttner fonction, Water-bag, …)

How bluid the « streams » for a general distribution function ?
A first response is maybe given in the standard fluid model in the sense of « moments » of f

Example: constructing a five bag model



Topics

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Conclusions

- Mathematical analogy between Weibel instability and CFI allows us to build the Multi-stream model
- « Multi-streams » as an ensemble of Dirac-masses: a kinetic Vlasov-type model using the technique of reduction of the Hamiltonian formalism
- An exact method allowing to describe complex physics with only a small number of « streams » (equivalence in the sense of moments of f)

Open questions

- An interesting analogy with the Water-Bag model: offers the possibility to couple the multi-stream model with WB:

For instance Multi- stream for perpendicular temperature plus WB for logitudinal temperature

- The lack of the exact invariance of the canonical momentum in 2D spatial system precludes the extension of the model in 2D
- BUT we have the possibility then to replace the exact invariant by a approximate invariant (adiabatic invariant as in gyrokinetic code ?)