

# Gyrokinetic simulations of Trapped Ion Modes for ITG instabilities

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# Ion Temperature Gradient Instability (ITG) and trapped ion modes (TIM) (1)

→ Low frequency ion-temperature-gradient-driven instabilities responsible for ion turbulence in tokamaks.

→ Trapped particles have unfavorable magnetic drift leading to the trapped particle instability i.e. a low-frequency interchange-type instability

## Ion Temperature Gradient Instability (ITG) and trapped ion modes (TIM) (2)

→ Instabilities characterized by frequencies of the order of the trapped particle precession frequency and radial scales of the order of several « banana » widths.

→ Because of their large scale and in spite of their low-frequencies, these instabilities could lead to a huge transport corresponding to a Bohm-like confinement.

## Ion Temperature Gradient Instability (ITG) and trapped ion modes (TIM) (3)

→ It is well known that turbulent transport can be greatly reduced in the presence of a negative magnetic shear

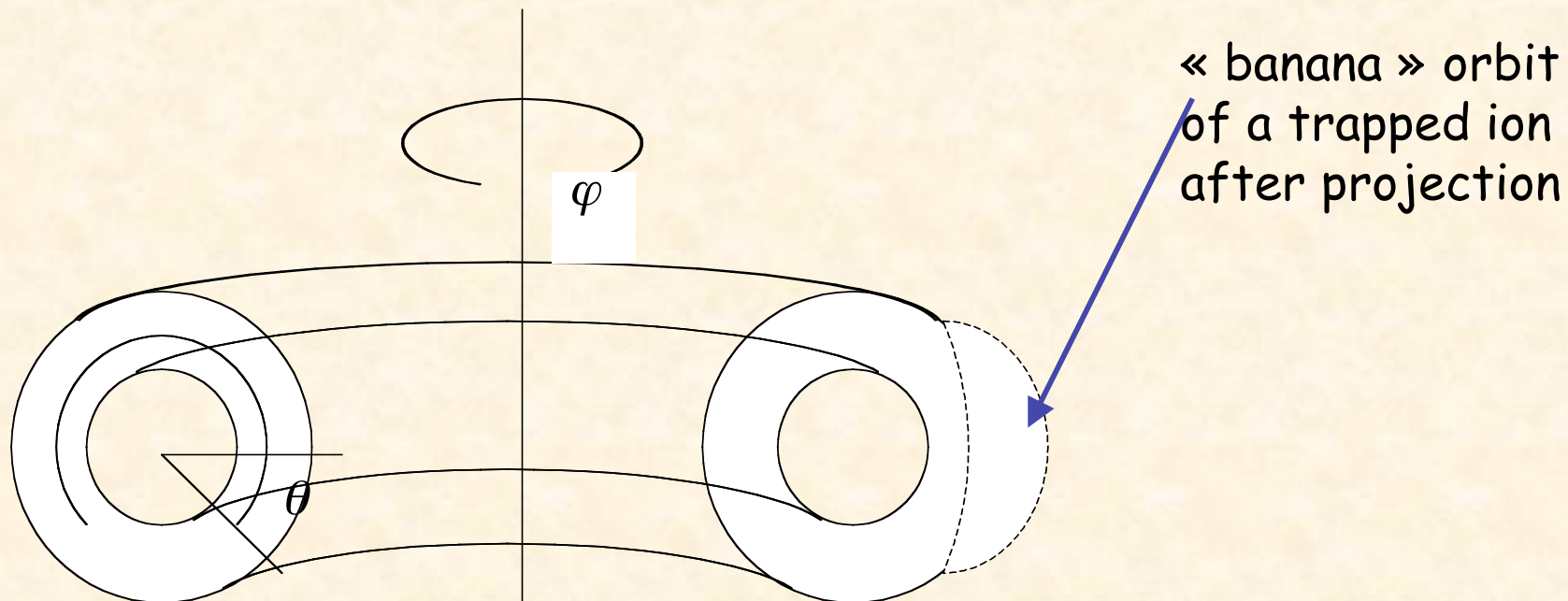
→ Magnetic shear describes the dependence of field line direction on the coordinate across closed flux surfaces

Where  $q$  is the safety factor

$$s = \frac{r_0}{2\pi} \left( \frac{dq}{dr} \right)_{r_0}$$



# Hamiltonian formalism in angle-action



- Classic phase space  $(\mathbf{x}, \mathbf{p})$  may be described by a system of three action variables  $(J_1, J_2, J_3)$  and their three corresponding angular variables  $(\phi_1, \phi_2, \phi_3)$

# TIM model



• The motion of trapped ion is thus characterized by three invariants:

- For cyclotron motion at frequency  $\omega_1 = \omega_c$   $\mu = \frac{mv_{\perp}^2}{2B(\mathbf{x}_G)}$
- For bounce motion at frequency  $\omega_2 = \omega_b$   $E = \frac{1}{2}mv_{G\parallel}^2 + \mu B(\mathbf{x}_G)$
- For precession motion at frequency  $\omega_3 = \omega_d$  (toroidal kinetic momentum)

$$M = e\psi(x_G) + mR(x_G)v_{G\varphi}$$

## TIM model

- Trajectory of trapped particle may be described by (Garbet et al JCP 87, 249 (1990) )

$$\begin{aligned}\theta_G &= \hat{\theta}(\mathbf{J}, \phi_2) \\ \varphi_G &= \phi_3 + q(r_0)\hat{\theta}(\mathbf{J}, \phi_2) + \hat{\varphi}(\mathbf{J}, \phi_2) \\ r_G &= r_0 + \hat{r}(\mathbf{J}, \phi_2)\end{aligned}$$

- Functions  $\hat{\theta}(\mathbf{J}, \phi_2)$  and  $\hat{\varphi}(\mathbf{J}, \phi_2)$  are periodic and represent respectively the bounce motion and the deviation from the regular precession motion.

# TIM model

- The low frequency response of ion plasma is usually obtained by gyro-averaging the Vlasov equation over the cyclotron phase  $\phi_1$ :

$$\mu = \frac{mv_{\perp}^2}{2B(\mathbf{x}_G)} \quad \text{becomes then an adiabatic invariant}$$

- For trapped particles, it is also possible to make a second average over now the bounce motion. Finally we have transformed a Vlasov problem of six independent variables into a 2D  $(\phi_3, \psi)$  phase space where the precession phase is  $\phi_3 = \varphi - q\theta$  and  $\psi$  is the poloidal flux (divided by  $2\pi$ ).

## TIM model

- Each adiabatic invariant reduces the dimensionality by a factor two and a trapped particle is fully described by the position of the « banana center »

$$\omega_k = \frac{d\phi_k}{dt} = \frac{\partial H}{\partial J_k}$$
$$\frac{dJ_k}{dt} = 0 \quad \text{for } k = 1, 2$$

- The system is then described by an ensemble of distribution functions of type  $f = f_{k,E}(\psi, \phi_3, t)$ , which numerically allows a parallelization along the direction of adiabatic invariants (indeed simple parameters)

## Bounce motion

- For a trapped ion, as  $\frac{d\phi_2}{dt} = \omega_b$ , the revolution of the banana motion is completed within a time

$$T_b = \frac{2\pi}{\omega_b} = \int_{-\theta_0}^{+\theta_0} \frac{q_0 R_0 d\theta}{\sqrt{2E/m} \sqrt{1 - \lambda b(\theta)}}$$

where  $1 - \lambda b(\theta) = \frac{mv_{\parallel}^2}{2E}$  with

$$\lambda = \frac{\mu B_0}{E}$$

The bounce frequency is then

$$\omega_b = \sqrt{\frac{2E}{m}} \frac{1}{q_0 R_0} \bar{\omega}_b(\kappa)$$

Where the dimensionless quantity is

With the complete elliptic function integral  $\bar{\omega}_b(\kappa) \approx \frac{\pi\sqrt{\epsilon}}{2\sqrt{2}} \frac{1}{K(\kappa)}$

$$K(\kappa)$$



## Toroidal precession motion

- The precession frequency  $\frac{d\phi_3}{dt} = \omega_d$  is determined by the average over the bounce motion, we have:

$$\omega_d(\kappa; s) = \frac{q_0 E}{e B_0 r_0 R_0} \bar{\omega}_d(\kappa; s)$$

with

$$\bar{\omega}_d(\kappa; s) = \frac{2E(\kappa)}{K(\kappa)} - 1 + 4s \left( \frac{E(\kappa)}{K(\kappa)} + \kappa^2 - 1 \right)$$

- is the « trapping » parameter

$$\kappa^2 = \sin^2 \left( \frac{\theta_0}{2} \right) = \frac{1 - \lambda}{2\varepsilon\lambda}$$

# Bounce-averaged Vlasov model (1)

- In the gyrokinetic approach, the evolution of an ensemble of particles is described by the distribution function  $f(\phi, \mathbf{J}, t)$  which obeys the Vlasov equation:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + [f, H] = 0 \quad \text{where} \quad [f, H] = \frac{\partial f}{\partial \phi} \frac{\partial H}{\partial \mathbf{J}} - \frac{\partial f}{\partial \mathbf{J}} \frac{\partial H}{\partial \phi}$$

- In a first approach, let us assume that the Hamiltonian is the sum of the equilibrium Hamiltonian plus a perturbation term:

$$H(\phi, \mathbf{J}, t) = H_0(\mathbf{J}) + \sum_{\mathbf{n}, \omega} \delta H_{\mathbf{n}, \omega}(\mathbf{J}) e^{i(\mathbf{n} \cdot \phi - \omega t)}$$

and

$$f(\phi, \mathbf{J}, t) = F_0(\mathbf{J}) + \sum_{\mathbf{n}, \omega} \delta f_{\mathbf{n}, \omega}(\mathbf{J}) e^{i(\mathbf{n} \cdot \phi - \omega t)}$$

- And using a local equilibrium  $F_0(\mathbf{J}) = \frac{n_0(J_3, t)}{(2\pi m T_0)^{3/2}} e^{-\left(\frac{H_0(\mathbf{J})}{T_0}\right)}$

## Bounce-averaged Vlasov model (2)

- It is then possible to write the perturbation term in the form

$$\delta f_{\mathbf{n},\omega}(\mathbf{J}) = -\frac{F_0(\mathbf{J})}{T_0} \delta H_{\mathbf{n},\omega}(\mathbf{J}) + \frac{F_0(\mathbf{J})}{T_0} \frac{\omega - \mathbf{n} \cdot \boldsymbol{\omega}^*}{\omega - \mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J}) + i0^+} \delta H_{\mathbf{n},\omega}$$

Adiabatic part

resonant wave-particle contribution

where  $\boldsymbol{\omega}^* = T_0 \left[ \frac{1}{n_0} \frac{dn_0}{dJ_3} + \frac{1}{T_0} \frac{dT_0}{dJ_3} \left( \frac{H_0}{T_0} - \frac{3}{2} \right) \right]$  is the diamagnetic frequency

• We have use

$$\frac{d\mathbf{J}}{dt} = -\frac{\partial H_0}{\partial \boldsymbol{\phi}} = 0$$

$$\frac{d\boldsymbol{\phi}}{dt} = \frac{\partial H_0}{\partial \mathbf{J}} = \boldsymbol{\Omega}(\mathbf{J}) = (\omega_c, \omega_b, \omega_d)$$

## Bounce-averaged Vlasov model (3)

- The term  $\mathbf{n} \cdot \boldsymbol{\Omega}(\mathbf{J})$  is reduced to  $n_3 \omega_d(\kappa; s) = n \omega_d(\kappa; s)$
- In tokamaks, we have

$$\omega \approx n_3 \omega_d \approx \omega^* \approx \frac{\rho_c}{a} n_2 \omega_b \approx \left( \frac{\rho_c}{a} \right)^2 n_1 \omega_c$$

This procedure of reduction (by suppressing the high-frequency terms) may be generalized by introducing the distribution function at the requester order

$$f = F_0(\mathbf{J}) \left( -\frac{\delta H}{T_0(J_3)} + \delta g \right)$$

- The gyro-average leads to « smooth » quantities

$$\overline{\delta H}(-\omega_{d0} t, \phi_3, \mathbf{J}) = \oint \frac{d\phi_1}{2\pi} \oint \frac{d\phi_2}{2\pi} \delta H(\phi, \mathbf{J})$$

## Bounce-averaged Vlasov model (4)

- The expression of the « particle density » may be obtained via

$$n_i(\mathbf{x}, t) = -\frac{n_0}{T_0} \delta H(\mathbf{x}, t) + \int d^3 p F_0(\mathbf{x}, \mathbf{p}) \{J_0 \overline{\delta g}\}$$

$\overline{\delta g} = J_0 \delta g$  introduces an operator which transform  $\delta g$  into its global gyro-average (bounce motion plus cyclotron motion).

The gyrokinetic equation can be simplified in the case where the density gradient vanishes and the temperature diamagnetic frequency


$$\omega^* = n_3 T_0 \frac{dn_0}{n_0 dJ_3} \quad \text{is much smaller than the precession frequency}$$



## Bounce-averaged Vlasov model (4)

It is then to write the ion distribution function in the form

$$f = \frac{1}{T_0^{3/2}} e^{-\frac{E}{T_0}} \{1 + \overline{\delta g}\} \quad \text{with} \quad \overline{\delta g} = J_0 \delta g$$

 Gyro-average operator

Normalizing the energy to the equilibrium temperature, the gyro-averaged Vlasov equation becomes:

$$\frac{\partial f_{\kappa,E}}{\partial t} + \omega_d(\kappa; s) E \frac{\partial f_{\kappa,E}}{\partial \phi_3} + \frac{\partial J_0 U}{\partial \psi} \frac{\partial f_{\kappa,E}}{\partial \phi_3} - \frac{\partial J_0 U}{\partial \phi_3} \frac{\partial f_{\kappa,E}}{\partial \psi} = 0$$

Which is coupled with

$$C \{U - \langle U \rangle_\psi\} - C \alpha \nabla^2 U = \bar{n}_i(\phi_3, \psi, t) - 1$$



## Bounce-averaged Vlasov model (4)

- Gyro-average operator  $J_0 = j_0(k_\perp \rho_c) j_0(k_\psi \delta_b)$

The present implementation of this operator is based on a Padé approximation of the Bessel function

$$J_0 = \left( 1 + \frac{E \delta_b^2}{4} \partial_\psi^2 \right) \left( 1 + \frac{E \rho_c^2}{4} \partial_{\phi_3}^2 \right)$$

The « banana center » particle density is then given by

$$\bar{n}_i(\phi_3, \psi, t) = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} dE \sqrt{E} \int_0^1 d\kappa \kappa K(\kappa) J_0 f_{\kappa, E}$$

Consequently we have reduced the 6D Vlasov equation into a set of 2D Vlasov equations (indeed  $N_E \times N_\kappa$  equations), acting on a 2D phase space:  $\phi_3 = \varphi - q\theta$ , the precession angle and poloidal flux

## Linear analysis (1)

- Dispersion relation for TIM is obtained by linearizing the reduced Vlasov equation with :

$$f_{\kappa,E}(\phi_3, \psi, t) = F_0(\psi) + \sum_n \hat{f}_n(\psi) e^{i(n\phi_3 - \omega t)}$$

and

$$U(\phi_3, \psi, t) = \sum_n \hat{U}_n(\psi) e^{i(n\phi_3 - \omega t)}$$

where the equilibrium distribution function and potential are taken as

$$F_0(\psi) = e^{-E} \left[ 1 + \Delta\tau \omega_d(\kappa; s) \left( E - \frac{3}{2} \right) \psi \right] \quad \text{and} \quad U_0(\psi) = 0$$

$$\Delta\tau = \frac{\Delta\psi}{T_0} \frac{dT_0}{d\psi} \quad \text{is the normalized temperature gradient}$$

## Linear analysis (2)

- Following standard Landau analysis of Vlasov equation

yields 
$$\hat{f}_n = \frac{\Delta\tau n\omega_d(\kappa; s) \left(E - \frac{3}{2}\right) e^{-E}}{n\omega_d(\kappa; s) E - \omega} J_0 \hat{U}_n$$

The quasi-neutrality condition (assuming the polarization contribution Negligible gives (with the usual Landau prescription on the imaginary part of frequency):

$$C\hat{U}_n = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} dE \sqrt{E} \int_0^1 d\kappa \kappa K(\kappa) \frac{\Delta\tau n \left(E - \frac{3}{2}\right) e^{-E}}{E - \frac{\omega}{n\omega_d(\kappa; s)}} J_0^2 \hat{U}_n$$

## Linear analysis (3)

- Resolution in the case of a marginal solution (corresponding to real frequency) leads to the determination of the threshold of ITG instability for TIM:

$$\Delta\tau_s = \frac{C}{\xi}$$

with

$$\xi = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} dE \sqrt{E} \int_0^1 d\kappa \kappa K(\kappa) e^{-E} j_0^2(\pi\delta_b E) j_0^2(\pi\rho_c E)$$

We have thus for frequencies:

$$\omega = \frac{3}{2} n \langle \omega_d(\kappa; s) \rangle_{\kappa}$$

The potential term obeys

$$\widehat{U}_n - \frac{\Delta\tau}{C} \frac{2}{\sqrt{\pi}} \int_0^{+\infty} dE \sqrt{E} \int_0^1 d\kappa \kappa K(\kappa) e^{-E} J_0^2 \widehat{U}_n = 0$$

## Linear analysis (4)

- The marginal solution for perturbation term for potential is

$$\widehat{U}_n(\psi) = U_0 \sin(l\pi\psi) \quad \text{with } l = 1, 2, 3, \dots$$

The case  $l=1$  corresponds to the lowest gradient and the dispersion relation can be expressed as follows (with a frequency shift given by

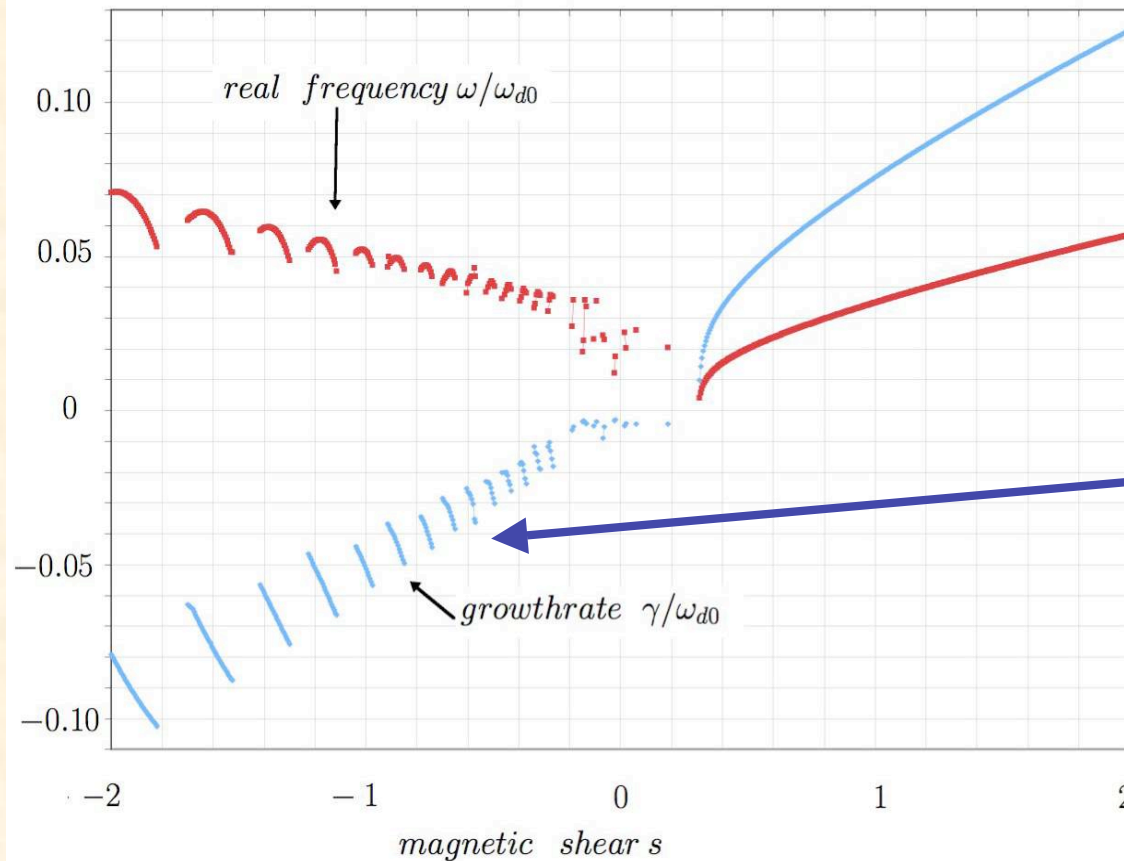
$$\Delta\omega = \frac{\omega}{n} - \frac{3}{2}\omega_d(\kappa; s) \quad \text{and assuming } J_0^2 = 1$$

$$\frac{1}{2} \left( \frac{\Delta\tau_s}{\Delta\tau} - 1 \right) = \int_0^1 d\kappa \frac{\kappa K(\kappa)}{\omega_d(\kappa; s)} \left[ 1 + \sqrt{\frac{3}{2} + \frac{\Delta\omega}{\omega_d(\kappa; s)}} Z \left( \sqrt{\frac{3}{2} + \frac{\Delta\omega}{\omega_d(\kappa; s)}} \right) \right]$$

Z being the function of Fried and Conte



# Numerical resolution of the dispersion relation of TIM



$$C = 0.10$$

$$\frac{\delta_b}{\Delta\psi} = 0.01$$

- Stabilizing element of ITG:
- Reversed magnetic shear



## Vlasov simulations (1)

- The code solves, in a self-consistent way, the gyrokinetic Vlasov equation, together with the quasi-neutrality condition:

$$\frac{\partial f_{\kappa,E}}{\partial t} + \omega_d(\kappa; s) E \frac{\partial f_{\kappa,E}}{\partial \phi_3} + \frac{\partial J_0 U}{\partial \psi} \frac{\partial f_{\kappa,E}}{\partial \phi_3} - \frac{\partial J_0 U}{\partial \phi_3} \frac{\partial f_{\kappa,E}}{\partial \psi} = 0$$

$$C \{ U - \langle U \rangle_\psi \} = \frac{2}{\sqrt{\pi}} \int_0^{+\infty} dE \sqrt{E} \int_0^1 d\kappa \kappa K(\kappa) J_0 f_{\kappa,E} - 1$$

The numerical method is based on a standard semi-lagrangian scheme  
Allowing a very fine resolution in phase space and the start-up of the  
ITG instability from the roundoff errors .

## Vlasov simulations (2)

- We used normalized quantities: the time is normalized to the inverse drift frequency  $\omega_{d0}^{-1}$ , the poloidal flux is given in  $\Delta\psi = r_0 R_0 \frac{B_0}{q_0}$  units, and the potential is expressed in  $T_0 / e$  units.

The initial condition is described by:

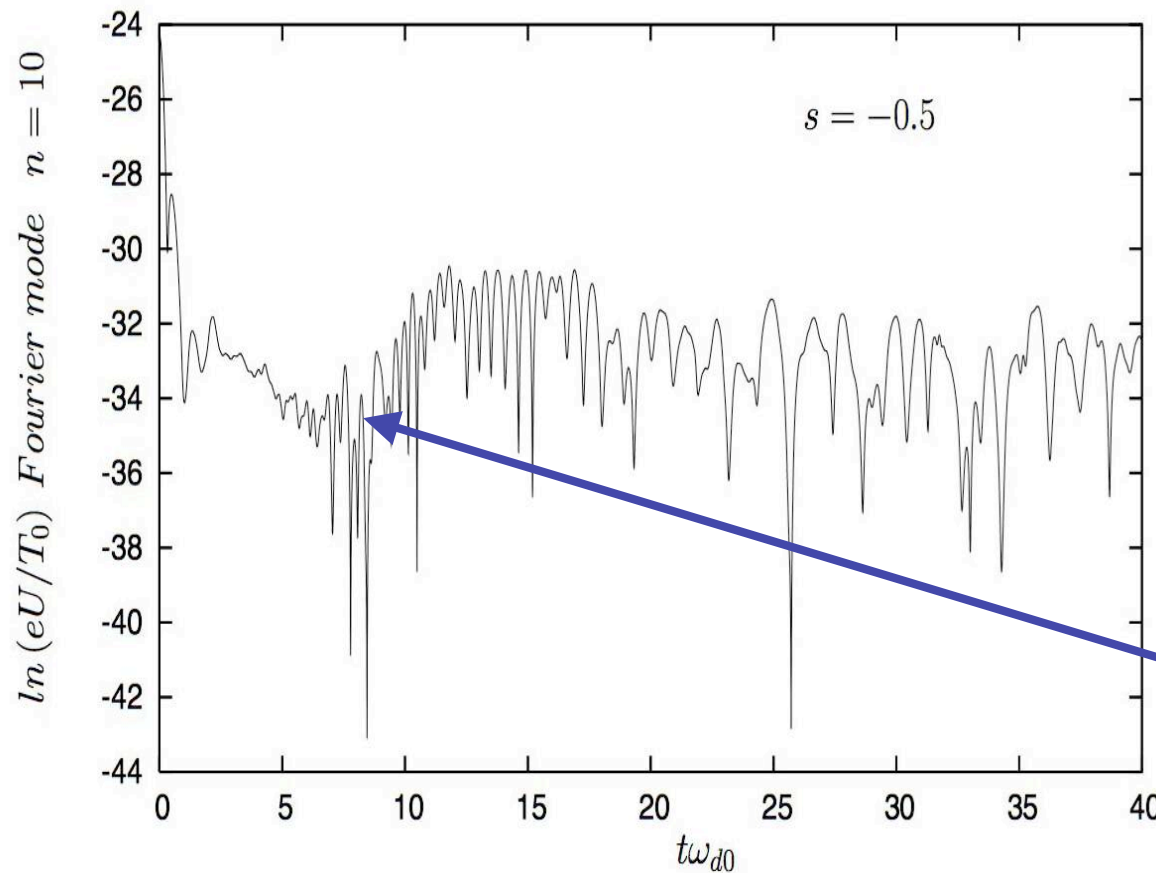
$$f_{\kappa,E}(\phi_3, \psi, t=0) = e^{-E} \left[ 1 + \Delta\tau\omega_d(\kappa; s) \left( E - \frac{3}{2} \right) \psi \right] + e^{-E} \Delta\tau J_0 U_0 \sin \pi\psi \cos 10\phi_3$$

We introduce a perturbation on the toroidal number  $n=10$

The simulation was carried out with a temperature gradient chosen just above the threshold of ITG:  $\Delta\tau - \Delta\tau_s = 0.0075$

With a ITG threshold of  $\Delta\tau_s = 0.100075$

# Numerical results (1)



$$C = 0.10 \quad \frac{\delta_b}{\Delta\psi} = 0.01$$

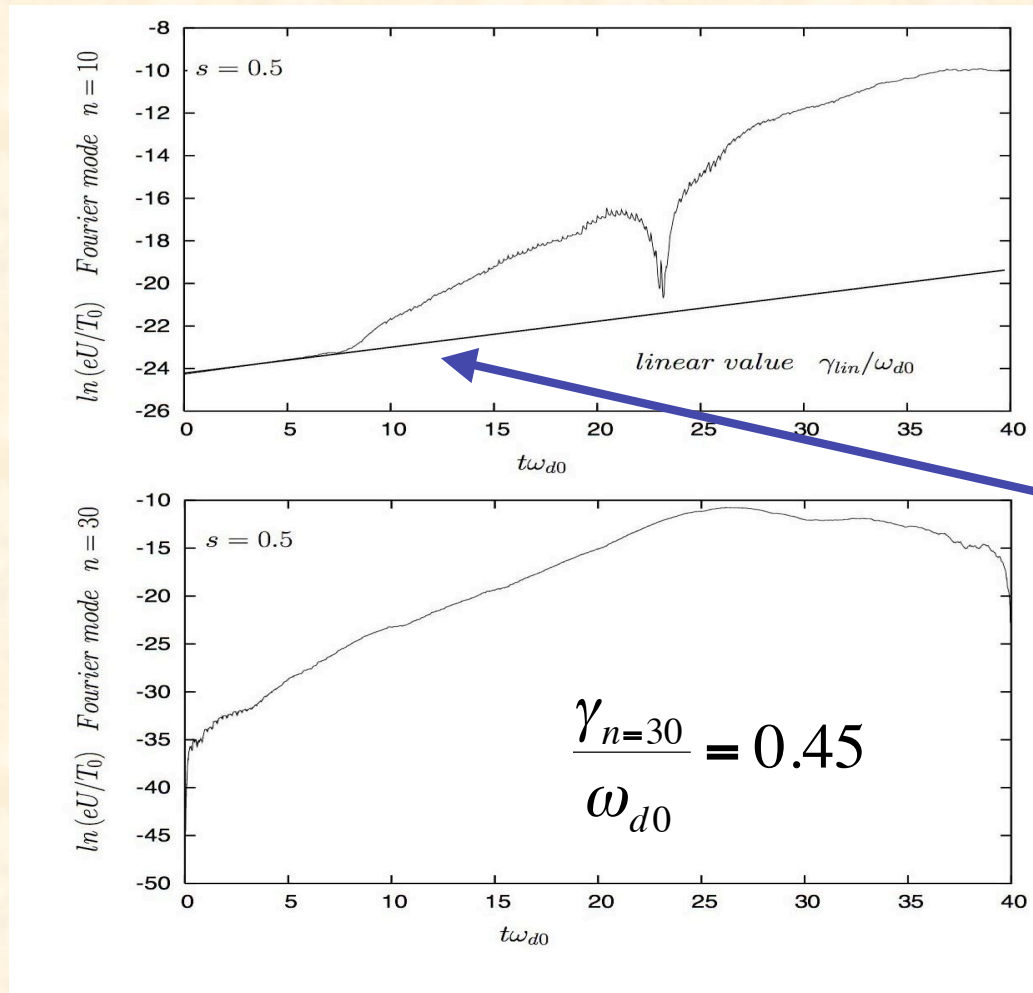
$$N_E \times N_\kappa = 128 \times 16$$

$$N_{\phi_3} \times N_\psi = 512 \times 128$$

- For reversal magnetic shear of  $S = -0.50$  we observe the stabilization (damping of the initially perturbed mode)

## Numerical results (2)

- we choose to increase the magnetic shear to value  $s=+0.50$



$$N_E \times N_\kappa = 128 \times 16$$

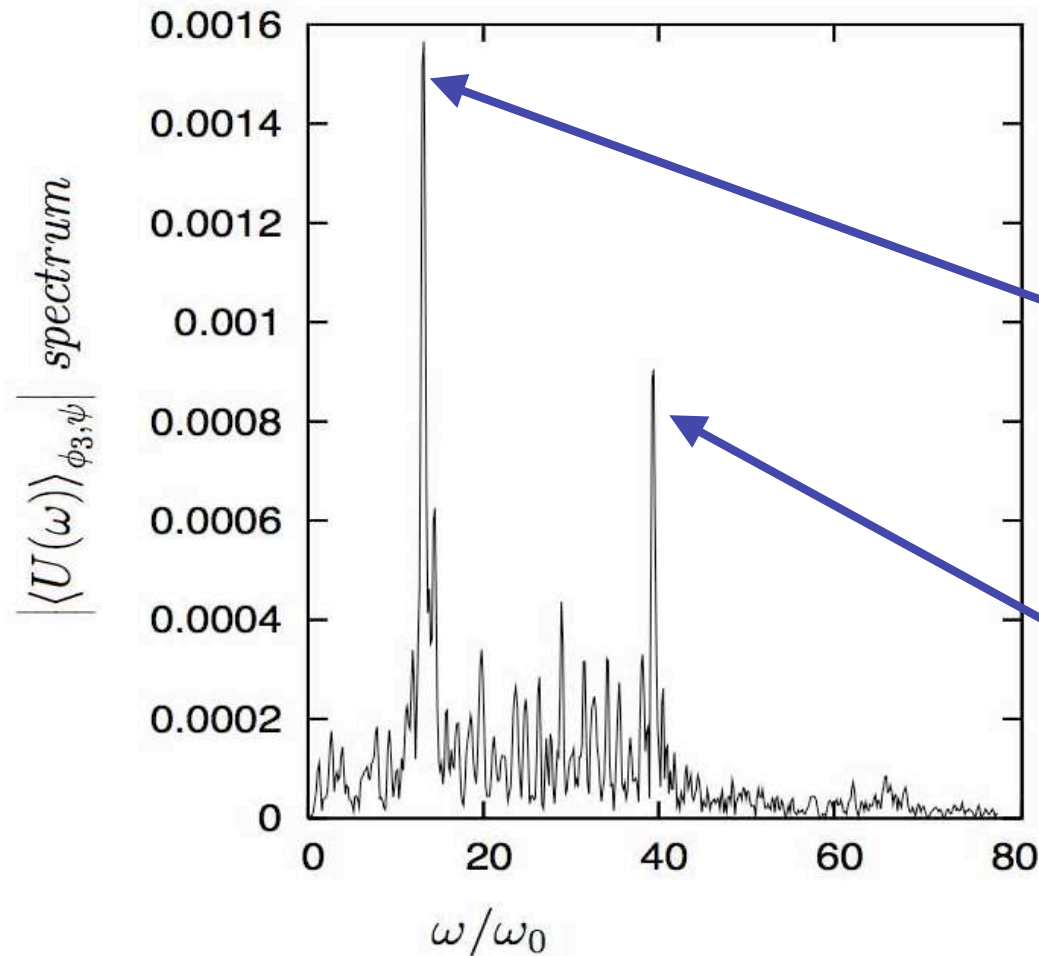
$$N_{\phi_3} \times N_\psi = 512 \times 128$$

we observe the linear phase plus a nonlinear mode coupling with the Growth of mode  $n=30$

$$\frac{\gamma_{n=10}}{\omega_{d0}} = 0.045$$

$$\frac{\gamma_{num}}{\omega_{d0}} \approx 0.05$$

## Numerical results (3)



$$s = 0.50$$

$$N_E \times N_\kappa = 128 \times 16$$

$$N_{\phi_3} \times N_\psi = 512 \times 128$$

Initial perturbed mode  
n=10

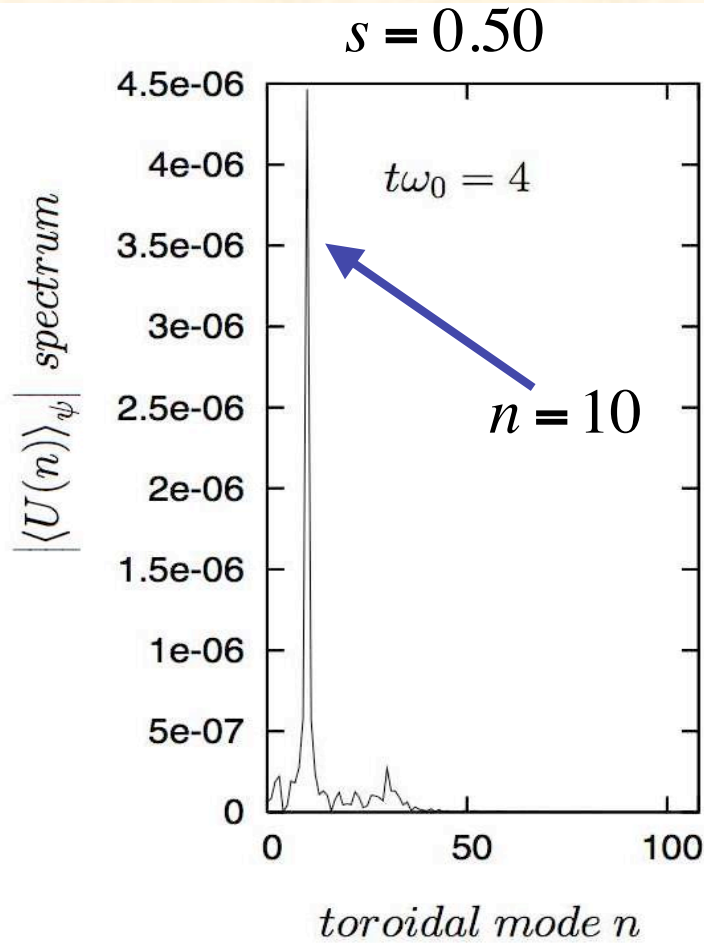
$$\omega_{n=10} = \frac{3}{2} n \langle \omega_d \rangle \approx 13 \omega_0$$

Mode n=30 with

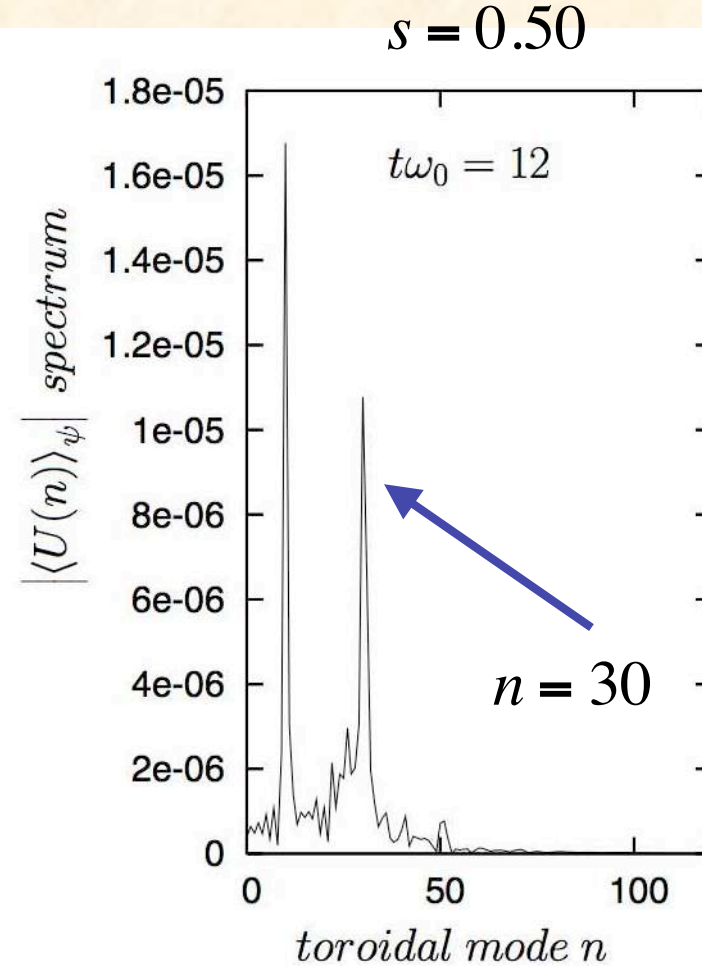
$$\omega_{n=30} = \frac{3}{2} n \langle \omega_d \rangle \approx 40 \omega_0$$



# Numerical results (4)



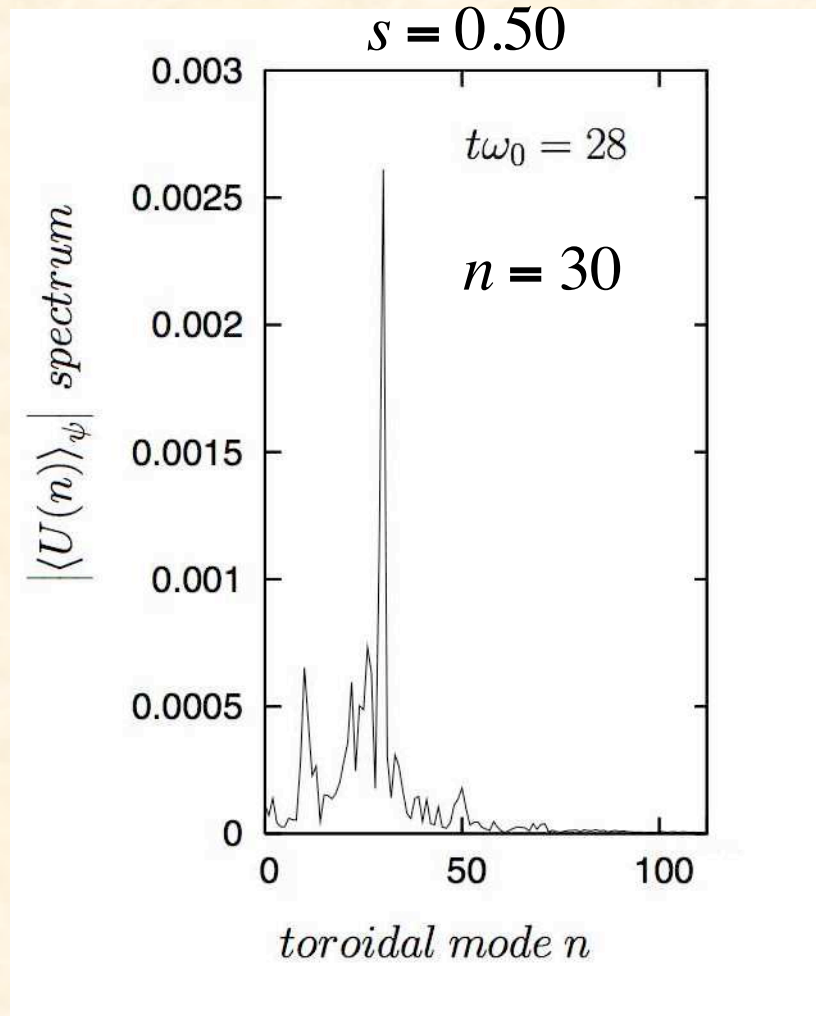
Initial perturbed mode  $n=10$



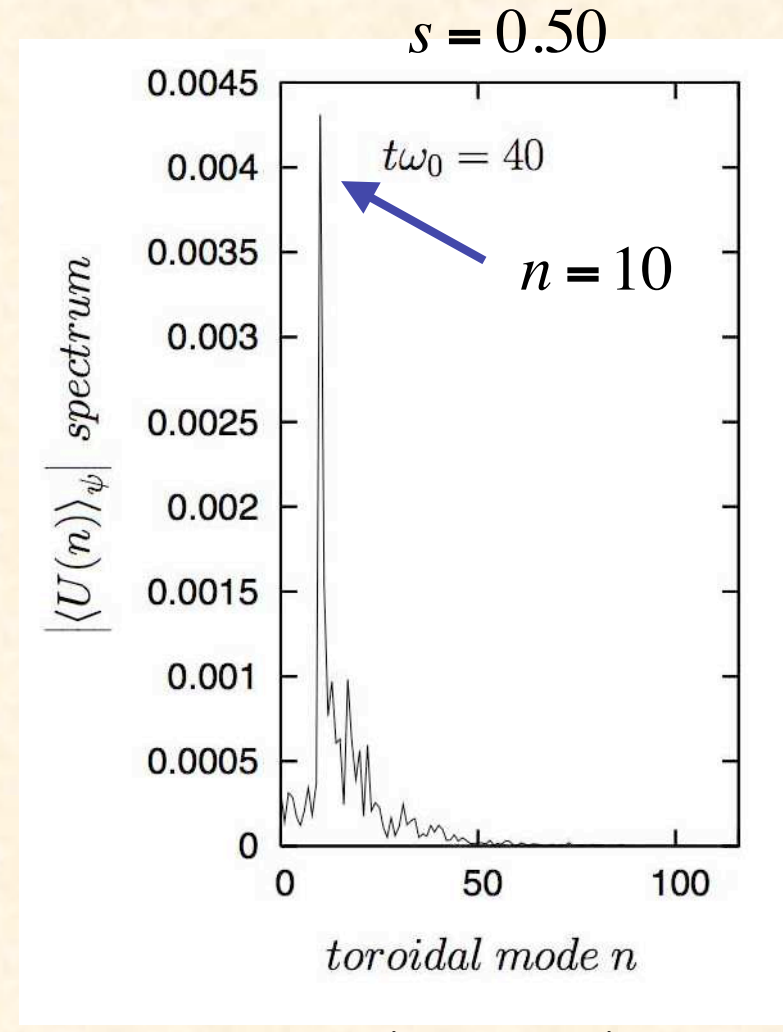
Two dominant modes



# Numerical results (4)

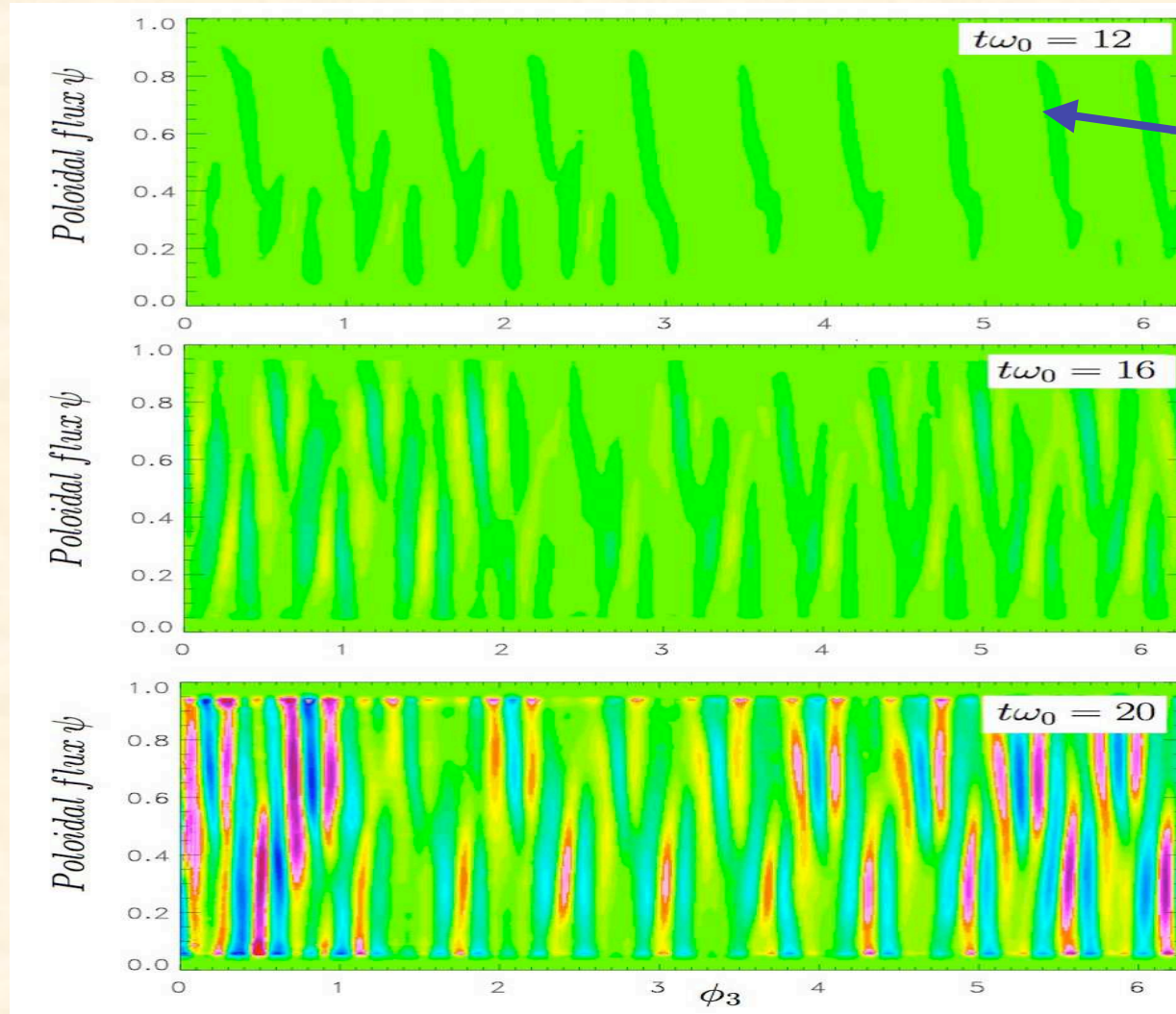


Second mode becomes dominant



Inverse cascade to mode  $n=10$

# Phase space representation of U (1)

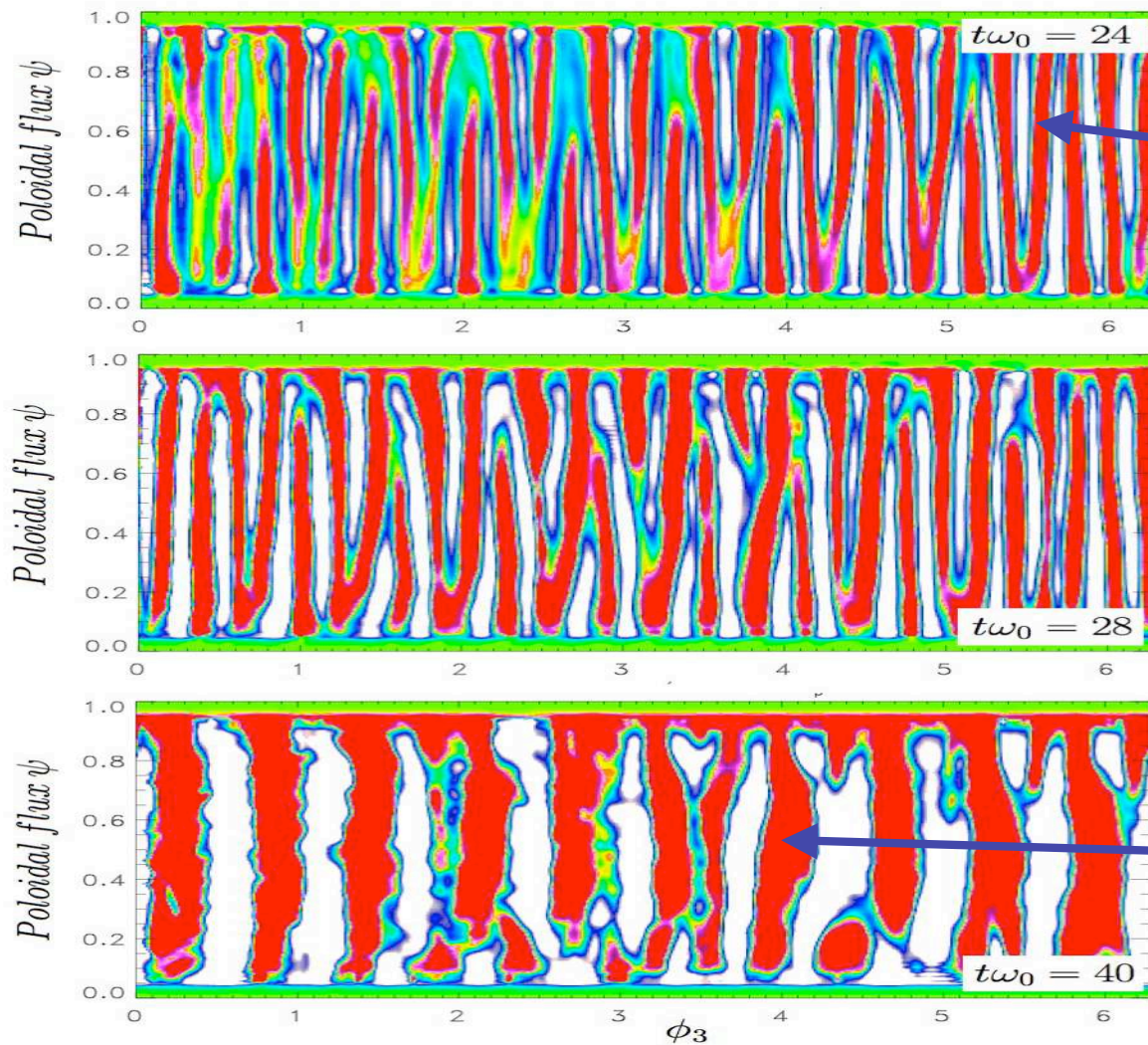


$$s = 0.50$$

Excitation of mode  $n=10$

Mode coupling

# Phase space representation of U (2)



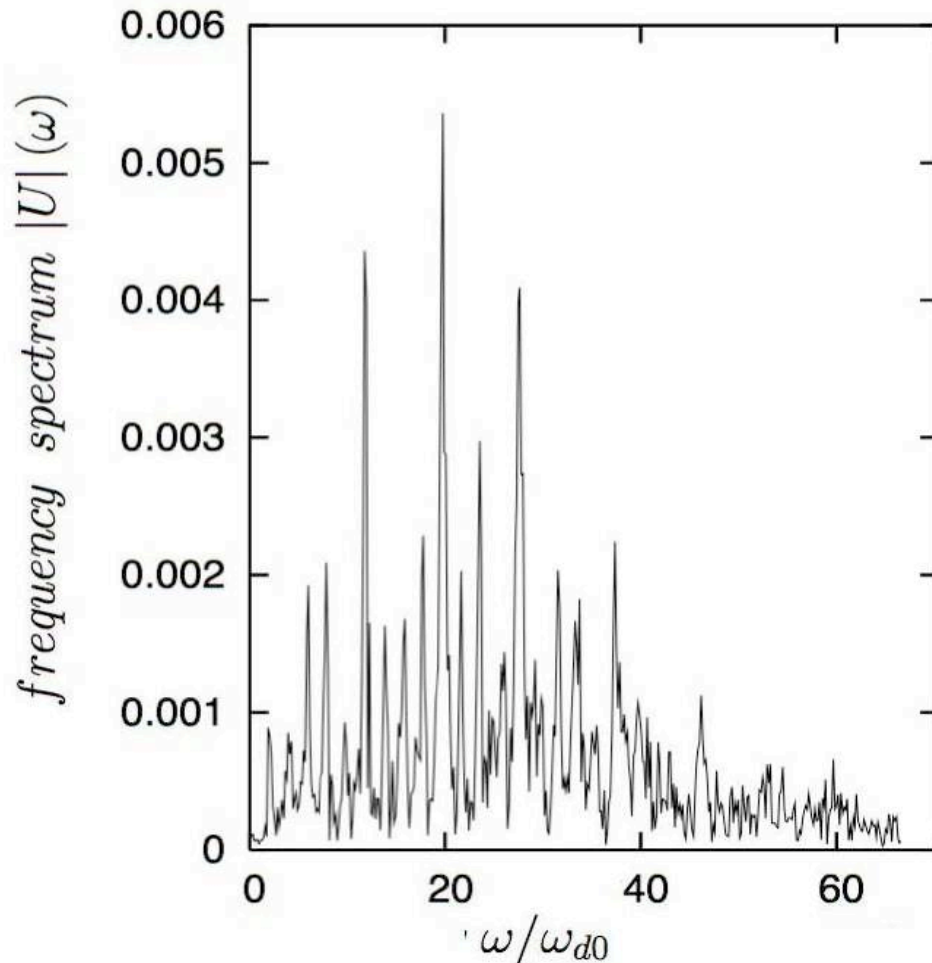
$$s = 0.50$$

Strong growth  
of mode  $n=30$  in  
red

Mode  $n=10$   
Becomes  
dominant



## Last example with a magnetic shear of $s=1$



- several resonant three-wave interactions seem implicated

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2$$

$$\omega(\mathbf{k}) = \omega_1(\mathbf{k}_1) + \omega_2(\mathbf{k}_2)$$

- Full-developped turbulence by increasing the magnetic shear parameter

## Conclusion

- ITG driven Trapped Ion Modes (TIM) are studied by solving a Vlasov equation averaged over the cyclotron and bounce motions in presence of a magnetic shear
- Kinetic effects as resonant ave-particle interactions are accurately described
- The magnetic shear appears to play a major role in the stabilization process of the ITG instability as predicted by Kadomtsev and Pogutse (Nuclear Fusion 1971)

The introduction of the magnetic shear allow us to study the different Regimes of turbulent transport from state described by a small number of toroidal modes till large band for full-developped turbulence