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Université de Lorraine  
Institut Élie Cartan

Traineeship report of  
Master 2 Mathématiques Fondamentales et Appliquées  
Spécialité: Probabilités

Tutor: Ivan Nourdin, full professor, Université de Lorraine

## Invariance and Universality Principles for Chaotic Polynomial of Free and Standard Random Variables

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### **Abstract**

We present in this report some results about the convergence of the multi-linear homogeneous sums of random variables (also called Chaotic Polynomial), following the invariance principle recently established by Mossel, O'Donnell and Oleszkiewicz [2], the universality principle developed by Nourdin, Pecatti and Reinert in [3], and the results developed by Déya and Nourdin in the free probability framework [1]. Moreover, we provide an extension of these results by defining an interesting family of these Chaotic Polynomials which have the same but no gaussian or semicircular asymptotically distribution.

## Résumé

On présente ici des résultats récents sur la convergence des sommes multi-linéaires homogènes (qu'on appelle aussi sommes chaotiques) en se basant sur le principe d'invariance établie par Mossel, O'Donnell et Oleszkiewicz [2], ainsi que sur le principe d'universalité développé par Nourdin, Peccati et Reinert dans [3]. Par la suite, on établie une extension de ces résultats en définissant une famille de polynômes chaotiques présentant une caractéristique intéressante d'équivalence en loi limite.

## Introduction

Le Principe d'invariance, développé par Mossel, O'Donnell et Oleszkiewicz dans [2], établie que, sous certaines conditions, les distributions de sommes multi-linéaires homogènes sont de plus en plus proches que les influences de ces sommes diminuent. En d'autres termes, on peut, à partir de ce résultat, construire des polynômes chaotiques (sommés multi-linéaires homogènes) ayant des distributions presque équivalentes pour toutes les familles de variables aléatoires indépendantes et identiquement distribuées, ce qui constitue une propriété tout à fait remarquable.

Par la suite, Nourdin, Peccati et Reinert [3], en combinant le principe d'invariance de Mossel et al [2] et le célèbre théorème des moments d'ordre quatre (Fourth Moment Theorem), établie par Nualart et Peccati [4], ont découvert un phénomène surprenant: le Phénomène d'universalité des Chaos de Wiener: d'après ce résultat, l'approximation gaussienne de tout polynôme chaotique est entièrement caractérisée par l'approximation gaussienne du chaos de Wiener correspondant à ce polynôme.

Dans [1], Déya et Nourdin ont développé une extension des principes d'invariance et d'universalité au cadre des espaces de probabilités Libres.

Nous proposons dans ce manuscrit une présentation des principes d'invariance et d'universalité, en définissant le contexte général de ces principes ainsi que quelques idées prolongeant la réflexion sur ces résultats.

Dans le premier chapitre, nous proposons une présentation des principes d'invariance et d'universalité dans le cadre des espaces de probabilité libres en se basant sur les travaux de Déya et Nourdin [1]. On proposera aussi une introduction de la théorie des probabilités libres. Pour finir, on étudiera une famille

de polynômes chaotiques ayant des propriétés intéressantes dont on parlera de la notion d'universalité totale.

Dans le second chapitre, on prolonge les résultats établies au premier chapitre au cas des espaces de probabilités standards. on définira aussi une famille équivalente de polynômes chaotiques ayant la propriété d'universalité totale. On finira par développer quelques réflexions et perspectives.

## Introduction

Mossel, O'Donnell and Oleszkiewicz [2], have given a very interesting principle showing that under mild conditions the distribution of multi-linear polynomials with low influences and bounded degree is essentially close for all product spaces. Thereafter, Nourdin, Peccati and Reinert [3], by combining the results given in [2] with those developed by Nualart and Peccati[4], essentially, the Fourth Moment Theorem, have discovered a so called universality phenomenon for chaotic sums of random variables. The authors have proved that the normal approximations of any multi-linear homogeneous sums are completely characterized and assessed by those of its Wiener chaos counterpart.

Déya and Nourdin [1] have given an extension of results established by Mossel et al [2] and those developed by Nourdin, Peccati and Reinert in [3], in the context of free probability space.

In the first chapter, we present the invariance and universality principles in the free probability framework. We will pay particular attention to some Chaotic Polynomial of free random variables, for which we will talk about 'Total Universality' phenomenon as an extension of the results developed in [1], and we will show some interesting properties of these quantities.

Thereafter, we extend these results in the context of standard probability space, and then we define some particular multi-linear homogeneous sums that enjoy the same Total Universality phenomenon. To do so, we keep the same kind of reasoning and, we use some results and definitions used in [3].

Since our results and definitions will be essentially related to the multi linear homogeneous sums, we begin by defining these quantities

**Definition 1.** (*Multi-linear homogeneous sums*) Fix some integers  $N, d \geq 2$  and

write  $[N] = 1, \dots, N$ . Let  $X = \{x_i : i \geq 1\}$  be a collection of centred independent random variables, and let  $f : [N]^d \rightarrow \mathbb{R}$  be a symmetric function vanishing on diagonals (i.e.,  $f(i_1, \dots, i_d) = 0$  whenever there exist  $k \neq j$  such that  $i_k = i_j$ ). The multi-linear homogeneous sum, of order  $d$ , based on  $f$  and on the first  $N$  elements of  $X$  is defined as:

$$\begin{aligned} Q_d(N, f, X) &= Q_d(X) \\ &= \sum_{1 \leq i_1, \dots, i_d \leq N} f(i_1, \dots, i_d) x_{i_1} \dots x_{i_d} \\ &= d! \sum_{i_1, \dots, i_d \subset [N]^d} f(i_1, \dots, i_d) x_{i_1} \dots x_{i_d} \\ &= d! \sum_{1 \leq i_1 < \dots < i_d \leq N} f(i_1, \dots, i_d) x_{i_1} \dots x_{i_d} \end{aligned}$$

The Chaotic Polynomials play a central role in several branches of probability theory and stochastic analysis. For instance:

- In the case  $d = 2$ , homogeneous sums are indeed standard quadratic forms with no diagonal terms. Limit theorems for quadratic forms have been the object of an intense study since 30 years. An exhaustive discussion about the best available results on the approximation of the laws of quadratic forms can be found in Gotze and Tikhomirov [10].

- By setting  $X_i = H_{q_i}(G_i)$ , where  $G_i$  is an independent and identically distributed centred standard Gaussian sequence and  $H_q$  the Hermite polynomial of degree  $q$ , one obtains the basic building blocks of the so-called Gaussian Wiener chaos, whose properties are at the core of Malliavin calculus (See [11]).

- By setting  $X_i = J_{q_i}(P_i)$ , where  $J_q$  indicates the Charlier polynomial of degree  $q$  and  $P_i$  the values of a centred Poisson measure over disjoint sets, one obtains a class of random elements generating the Poisson Wiener chaos (see [12], [13], [14], [15]).

- By setting  $X$  to be a Rademacher sequence (that is, the  $X_i$ 's are i.i.d and such that  $P(X_1 = 1) = P(X_1 = -1) = \frac{1}{2}$ ) then  $Q_d(X)$  are the constitutive elements of the so-called Walsh chaos, playing a central role in harmonic analysis (See [16]), probability on Banach spaces (See [17]), quantum probability (See [18]), social choice theory and theoretical computer science (See [2]).

Also, we will use the notion of influence for an element of a given function. Let us give the definition of this notion,

**Definition 2.** The influence of the  $i$ th coordinate on  $f$ , denoted  $\text{Inf}_i(f)$  is defined as follows:

$$\text{Inf}_i(f) = P_x[f(x_1, \dots, x_n) \neq f(x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_n)]$$

Another definition of influence of the coordinate  $i$  is

$$\text{Inf}_i(f) = \sum_{\{i_2, \dots, i_d\} \in [N]^{d-1}} f^2(i, i_2, \dots, i_d) = \frac{1}{(d-1)!} \sum_{1 \leq i_2, \dots, i_d \leq N} f^2(i, i_2, \dots, i_d)$$

In other words, what is the influence of changing  $i$  in the output of the function  $f$ , i.e. What is the probability that if we change the  $i$ th coordinate, the output of  $f$  will be changed?

# Chapter 1

## Total Universality for some Chaotic Polynomial of free random variables

Since our discussion in this chapter will be essentially in the free probability space, let us first present some basic definitions and main results about this theory rely primarily on the paper of Dan Voiculescu [5], which gives a simple but a very clear introduction. For the more precise and general definitions and properties, see [6].

### 1.1 Free Probability Background

The free probability theory is a new concept developed by Dan Voiculescu around 1985. Its goal is to understand special classes of Von Neumann Algebras. This theory brings together many fields of mathematics: operator algebra, random matrices, combinatorics, representation theory of symmetric groups. The basis of free probability theory is the study of non commutative random variables.

Free probability was developed as a kind of probability theory for random variables which do not commute, like matrices. At start, Voiculescu was motivated by the resolution of operator algebra classification problems. He has invented his free probability theory to understand the structure of many families of random matrices. Indeed, using this theory we could, with a low chance of being wrong, predict the shape of the spectrum of the sum of two large Hermitian matrices, knowing only the spectrum of each. Free probability also has applications outside of mathematics, in particular in electrical engineering, digital communications, nuclear physics, mathematical finance. This is because many situations in these fields can often be modelled by random matrices. When the size of the matrices become large, free probability theory is an invaluable

tool for describing the asymptotic behaviour of many systems.

A further event that has transformed the free probability theory was Voiculescu's discovery in 1991 that many families of random matrices satisfy asymptotically the freeness relation (which is an analogue of the notion of independence from classical probability theory). This result represents a very important result about the structure of operator algebras and also it brought new concepts and tools into random matrix theory.

Free probability theory can be described as non commutative probability theory endowed with the definition of free independence. Let us explain these terms:

### 1.1.1 Non commutative probability theory

In non commutative probability theory, we replace the ring of numerical random variables in the classical probability theory, by a possibly non commutative algebra  $\mathcal{A}$  over  $\mathbb{C}$  with unit  $1 \in \mathcal{A}$ , which is endowed with a linear expectation functional  $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ , such that  $\varphi(1) = 1$ .  $(\mathcal{A}, \varphi)$  is a non commutative probability space and elements  $a \in \mathcal{A}$  are non commutative random variables.

The distribution of a random variable  $a \in \mathcal{A}$  is the linear map  $\mu_a : \mathbb{C}[X] \rightarrow \mathbb{C}$  that  $\mu_a(P) = \varphi(P(a))$ . The information encoded in  $\mu_a$  is the same as giving the collection of moments  $\varphi(a^n)_{n>0}$ .

### 1.1.2 Free tracial probability space

The usual and more restricted context for free probability theory is the free tracial probability space  $(M, \tau)$ , where  $M$  is a von Neumann algebra and we impose on the linear expectation functional  $\tau$  the property that it is a trace. (i.e. it has the property that  $\tau(ST) = \tau(TS)$  for all  $S, T \in M$ ). This means that  $M$  is a self-adjoint algebra of bounded operators on a Hilbert space  $\mathcal{H}$  (i.e.  $T \in M \Rightarrow T^* \in M$ ) which is weakly closed (i.e., if for some  $(T_i)_{i \in I}$  in  $M$ , we have  $\langle T_i h, k \rangle \rightarrow \langle Th, k \rangle$  for all  $h, k \in \mathcal{H}$ , then  $T \in M$ ).

### 1.1.3 Free independence

A family of sub-algebras  $(\mathcal{A}_i)_{i \in I}$ , with  $1 \in \mathcal{A}_i$ , in  $(\mathcal{A}, \varphi)$  is freely independent if

$$\varphi(a_1 \dots a_n) = 0$$

whenever  $\varphi(a_j) = 0$ ,  $1 \leq j \leq n$  and  $a_j \in \mathcal{A}_{i(j)}$  with  $i(j) \neq i(j+1)$ ,  $1 \leq j \leq n-1$ . A family of subsets  $(\omega_i)_{i \in I}$  in  $(\mathcal{A}, \varphi)$  is freely independent if the algebras  $\mathcal{A}_i$  generated by  $\{1\} \cup \omega_i$  are freely independent.

The above definition means that products of centred variables, such that consecutive ones are in different algebras, have expectation zero. Note that this does not preclude that  $i(j) = i(k)$  as long as  $|j - k| \geq 2$ , but only the immediate consecutive.

#### 1.1.4 Some basic laws in Free Probability theory

There exist a parallel between the classical probability theory and the free probability theory, and so, a parallel between the fundamental distributions in both theories:

a) The role of the Gaussian distribution in free probability theory is held by the semi-circle distribution, which is a distribution with compact support.

The semicircular distribution  $S(m, \sigma^2)$  with mean  $m \in \mathbb{R}$  and variance  $\sigma^2 > 0$  is the probability distribution

$$S(m, \sigma^2) = \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - (x - m)^2} \mathbf{1}_{\{|x - m| \leq 2\sigma\}} dx$$

If  $m = 0$ , the semicircular distribution is symmetric around 0, and therefore its odd moments all vanish. A simple calculation shows that the even centred moments are given by Catalan numbers: for non negative integers  $k$ ,

$$\int_{m-2\sigma}^{m+2\sigma} (x - m)^{2k} S(m, \sigma^2) dx = C_k \sigma^{2k}$$

where  $C_k = \frac{(2k)!}{(k+1)!k!}$

b) For the Poisson distribution, the free correspondent is a distribution related to the semi-circle law. It is also a compactly supported distribution which has at most one atom.

c) The free Cauchy distribution is the Cauchy distribution itself, i.e. the free correspondent is the same as the classical law.

The semicircular distribution occurs in random matrix theory, where Wigner discovered that it is the limit distribution of eigenvalues of large hermitian gaussian matrices. Similarly, the free Poisson distribution also occurs in random matrix theory, and it is the limit distribution of eigenvalues for matrices of the form  $X * X$ , where  $X$  is a rectangular gaussian matrix, it is the Pastur-Marchenko distribution.

With this basic introduction to a free probability theory, we could state the two main results developed by Déya and Nourdin in [1], which will be the basis

to show our results.

### 1.1.5 Free Brownian Motion

A free Brownian motion  $S = (S_t)_{t \geq 1}$  is a non commutative stochastic process, it's a one-parameter family of self-adjoint operators  $S_t$  in a  $W^*$ -probability space  $(\mathcal{A}, \varphi)$ , with the following characteristics:

- $\{\mathcal{A}_t : t \geq 1\}$  is a filtration of von Neumann sub-algebras of  $\mathcal{A}$ .
- (i)  $S_0 = 0$ ;
- (ii)  $S_{t_2} - S_{t_1}$  follows the semi circular distribution of variance  $t_2 - t_1$  for all  $t_1 \leq t_2$ ;
- (iii) For all  $0 \leq t_1 \leq t_2 \leq \dots \leq \infty$ , the increments  $S_{t_1}, S_{t_2} - S_{t_1}, \dots, S_{t_n} - S_{t_{n-1}}$  are freely independent.

Then, the free Brownian motion is the analogue, in the free probability framework, of Standard Brownian motion.

### 1.1.6 The Wigner Chaos

We define the Wigner Chaos as follow:

for every  $q \geq 1$ , the  $q$ th Wigner Chaos associated with a free Brownian motion  $S$  is the random variable  $I_q(f)$ ,  $f \in L^2(\mathbb{R}_+^q)$ , such that:

$$I_q(f) = (S_{b_1} - S_{a_1}) \dots (S_{b_q} - S_{a_q})$$

for every function  $f$  having the form

$$f(t_1, \dots, t_q) = 1_{(a_1, b_1)}(t_1) \times \dots \times 1_{(a_q, b_q)}(t_q)$$

where the intervals  $(a_i, b_i), i = 1, \dots, q$  are pairwise disjoint.

## 1.2 Invariance Principle and Universality in the free probability space

We present here the results obtained and proved by Déya and Nourdin in [1], namely, the Invariance principles and Universality phenomenon for homogeneous sums of free random variables, and discuss some examples and counterexamples.

Before proceeding, we give both results without proof, see [1] to get the proofs and more details.

**Theorem 1.** (See [1]) Let  $(\mathcal{A}, \varphi)$  be a free tracial probability space. Let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be two sequences of centred free random variables with unit variance (that is, such that  $\varphi(X_i^2) = \varphi(Y_i^2) = 1$  for all  $i$ ), satisfying moreover

$$\sup_{i \geq 1} \varphi(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} \varphi(|Y_i|^r) < \infty$$

for all  $r \geq 1$ , where  $|X| = \sqrt{X * X}$ .

Fix  $d \geq 1$ , and consider a sequence of functions  $f_N : \{1, \dots, N\}^d \rightarrow \mathbb{R}$  satisfying the following three assumptions for each  $N$  and each  $i_1, \dots, i_d = 1, \dots, N$ :

- (i) (mirror-symmetry)  $f_N(i_1, \dots, i_d) = f_N(i_d, \dots, i_1)$ ;
- (ii) (vanishing on diagonals)  $f_N(i_1, \dots, i_d) = 0$  if  $i_k = i_l$  for some  $k \neq l$ .
- (iii) (normalization)  $\sum_{j_1, \dots, j_d=1}^N f_N(j_1, \dots, j_d)^2 = 1$ .

Also, set

$$Q_N(x_1, \dots, x_N) = \sum_{i_1, \dots, i_d=1}^N f_N(i_1, \dots, i_d) x_{i_1} \dots x_{i_d} \quad (1.1)$$

and

$$\text{Inf}_i(f_N) = \sum_{l=1}^d \sum_{j_1, \dots, j_{d-1}=1}^N f_N(j_1, \dots, j_{l-1}, i, j_l, \dots, j_{d-1})^2, i = 1, \dots, N. \quad (1.2)$$

Then, for any integer  $m \geq 1$ ,

$$\varphi(Q_N(X_1, \dots, X_N)^m) - \varphi(Q_N(Y_1, \dots, Y_N)^m) = O(\tau_N^{\frac{1}{2}}) \quad (1.3)$$

where  $\tau_N = \max_{1 \leq i \leq N} \text{Inf}_i(f_N)$ .

Theorem 1 is an extension, in the free probability framework, of the invariance principles developed by Mossel, O'Donnell and Oleszkiewicz [2]. It essentially shows that the distributions of multi-linear homogeneous sums of free random variables close as the influence of all  $i$ th coordinate on  $f_N, 1 \leq i \leq N$ , denoted  $\text{Inf}_i(f_N)$ , decreases.

By combining Theorem 1 with the main finding of [3], the authors in [1] have proved the following Theorem, called the 'free universality phenomenon':

**Theorem 2.** (See [1]) Let  $(\mathcal{A}, \varphi)$  be a free tracial probability space. Let  $S_1, S_2, \dots$  be a sequence of centred free semicircular random variables with unit variance (i.e,  $S(0, 1)$ )

Fix  $d \geq 2$ , and consider a sequence of functions  $f_N : \{1, \dots, N\}^d \rightarrow \mathbb{R}$  satisfying the following three assumptions for each  $N$  and each  $i_1, \dots, i_d = 1, \dots, N$ :

- (i) (full symmetry)  $f_N(i_1, \dots, i_d) = f_N(i_{\sigma(1)}, \dots, i_{\sigma(d)})$  for all  $\sigma \in \mathfrak{S}_d$ ;  
(ii) (vanishing on diagonals)  $f_N(i_1, \dots, i_d) = 0$  if  $i_k = i_l$  for some  $k \neq l$ .  
(iii) (normalization)  $\sum_{j_1, \dots, j_d=1}^N f_N(j_1, \dots, j_d)^2 = 1$ .

Also, let  $Q_N(x_1, \dots, x_N)$  be the polynomial in non-commuting variables given by (1). Then, the following two conclusions are equivalent as  $N \rightarrow \infty$ :

- (A)  $Q_N(S_1, \dots, S_N) \xrightarrow{law} S(0, 1)$ ;  
(B)  $Q_N(X_1, \dots, X_N) \xrightarrow{law} S(0, 1)$  for any sequence  $X_1, X_2, \dots$  of identically distributed and centred random variables with unit variance.

The main result of Theorem 2 is the implication (A)  $\Rightarrow$  (B), it means that if we have a condition (A), i.e. if  $Q_N(S_1, \dots, S_N)$  converges (in distribution) to a semicircular law  $S(0, 1)$ , then for any sequence of free and identically distributed random variables  $X_1, X_2, \dots$ , we have that  $Q_N(X_1, \dots, X_N)$  converges to  $S(0, 1)$ , i.e. the semicircular behaviour of  $Q_N(X_1, \dots, X_N)$  is asymptotically insensitive to the distribution of its summands.

The question asked by the authors, (and another could be immediately asked by every reader) are:

1. Why a semicircular distribution? i.e. Why the convergence of  $Q_N(S_1, \dots, S_N)$  towards a semicircular  $S(0, 1)$  it is enough to guarantee the same convergence no matter the distribution of the sequence of free random variables  $X_1, X_2, \dots$ ?
2. Which (does exists?) are the distributions that enjoy such a universality phenomenon? i.e. does it exist another distribution that can play the same role than semicircular one in Theorem 2?

The answer to the first question is in the proof of Theorem 2. Let us review this proof (see [1] for details):

We begin by defining a useful result:

**Proposition (Multiplication formula)**(See [19]).

For every function  $f \in L^2(\mathbb{R}_+^p)$  and  $g \in L^2(\mathbb{R}_+^q)$ , where  $p, q \geq 1$ , we have

$$I_p(f)I_q(g) = \sum_{r=0}^{p \wedge q} I_{p+q-2r}(f \overset{r}{\frown} g)$$

where  $f \overset{r}{\frown} g$  is the  $r$ th contraction of  $f$  and  $g$ , defined as the element of  $L^2(\mathbb{R}_+^{p+q-2r})$ , by

$$f \overset{r}{\frown} g(t_1, \dots, t_{p+q-2r})$$

$$= \int_{\mathbb{R}_+^{p+q-2r}} f(t_1, \dots, t_{p-r}, x_1, \dots, x_r) g(x_r, \dots, x_1, t_{p-r+1}, \dots, t_{p+q-2r}) dx_1 \dots dx_r$$

Let  $S_1, S_2, \dots \sim \mathcal{S}(0, 1)$  be freely independent. Fix  $d \geq 2$ , and consider a sequence of functions  $f_N : \{1, \dots, N\} \rightarrow \mathbb{R}$  satisfying the assumptions (ii) and (iii) of Theorem 2.

Let also  $Q_N(x_1, \dots, x_N)$  be the chaotic polynomial in non-commuting variables. Set  $e_i = 1_{[i-1, i]} \in L^2(\mathbb{R}_+)$ ,  $i \geq 1$ . For each  $N$ , one has

$$Q_N(S_1, \dots, S_N) \stackrel{law}{=} Q_N(I_1(e_1), \dots, I_1(e_N)).$$

By applying the multiplication formula and taking into account assumption (ii) of Theorem 2, we find,

$$Q_N(I_1(e_1), \dots, I_1(e_N)) = I_d(g_N),$$

where

$$g_N = \sum_{i_1, \dots, i_d=1}^N f_N(i_1, \dots, i_d) e_{i_1} \otimes \dots \otimes e_{i_d}.$$

Due to the assumptions (ii) and (iii),  $g_N$  is mirror-symmetric and has an  $L^2(\mathbb{R}_+^d)$ -norm equal to 1. Using Theorem 1.3 and Theorem 1.6 of [7], we deduce the following equivalence, as  $N \rightarrow \infty$ :

$$Q_N(S_1, \dots, S_N) \stackrel{law}{\rightarrow} \mathcal{S}(0, 1) \Leftrightarrow \|g_N\|_{L^2(\mathbb{R}_+^{2d-2r})}^r \rightarrow 0$$

for all  $r \in \{1, \dots, d-1\}$

Then, it results from combination of Theorem 1.3 [7] and Theorem 1.6 [7] a direct equivalence between the convergence of Wigner stochastic integrals to  $\mathcal{S}(0, 1)$  and the vanishing of all non trivial contractions.

On the other hand, in [1], Déya and Nourdin, give a direct relationship between the vanishing of all non trivial contractions and the fact that the quantity  $\tau_N$  given in Theorem 1 vanishes as  $N \rightarrow \infty$ , and therefore the equivalence (i) and (ii) in Theorem 2. In other words, the simple answer to the first question is essentially based in the fourth moment Theorem ([4] and [7]). (See the paper [1] and [7])

For the second question, unfortunately, the answer is more complicated, and is still yet an open problem. Indeed, to illustrate the problem, we review the example given by Déya and Nourdin in [1]. Fix  $d = 2$  and consider

$$Q_N(x_1, \dots, x_N) = \frac{1}{\sqrt{2N-2}} \sum_{i=2}^N (x_1 x_i + x_i x_1), N \geq 2$$

Let  $S_1, S_2, \dots$  be a sequence of free  $S(0, 1)$  random variables and let  $X_1, X_2, \dots$  be a sequence of free Rademacher random variables (that is, the law of  $X_1$  is given by  $\frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ ). Then, using the free central limit theorem, it is clear (since  $\frac{1}{\sqrt{N-1}} \sum_{i=2}^N X_i \rightarrow S(0, 1)$ ) that on the one hand:

$$\begin{aligned} Q_N(X_1, \dots, X_N) &= \frac{1}{\sqrt{2}} X_1 \left( \frac{1}{\sqrt{N-1}} \sum_{i=2}^N X_i \right) + \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{N-1}} \sum_{i=2}^N X_i \right) X_1 \\ &\xrightarrow{\text{law}} \frac{1}{\sqrt{2}} (X_1 S_1 + S_1 X_1), \text{ as } N \rightarrow \infty, \end{aligned}$$

with  $X_1$  and  $S_1$  freely independent. By Proposition 1.10 and identity (1.10) of Nica and Speicher [9], it turns out that  $\frac{1}{\sqrt{2}}(X_1 S_1 + S_1 X_1) \sim S(0, 1)$ . But, on the other hand,

$$\begin{aligned} Q_N(S_1, \dots, S_N) &= \frac{1}{\sqrt{2}} S_1 \left( \frac{1}{\sqrt{N-1}} \sum_{i=2}^N S_i \right) + \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{N-1}} \sum_{i=2}^N S_i \right) S_1 \\ &\stackrel{\text{law}}{=} \frac{1}{\sqrt{2}} (S_1 S_2 + S_2 S_1). \end{aligned}$$

The random variable  $\frac{1}{\sqrt{2}}(S_1 S_2 + S_2 S_1)$  being not  $S(0, 1)$  distributed. Indeed, its law is the so called tetilla law (see [8]), we deduce that one cannot replace the role played by the semicircular distribution in Theorem 1.4 by the Rademacher distribution.

Let us discuss this example in more detail, and let us show why it is not surprising that this multi-linear sum cannot enjoy the universality phenomenon in the sense of Theorem 2 (i.e in the sense of asymptotic semicircular behaviour). Now we take a look about the structure of this multi linear sum rather than the distribution of its summands. Indeed, we can write the previous multi linear sum in the same way as for (1).i.e. Fix  $d = 2$ , and let, for  $N \geq 2$ :

$$Q_N(x_1, \dots, x_N) = \sum_{i_1, i_2=1}^N f_N(i_1, i_2) x_{i_1} x_{i_2}$$

with

$$f_N(i_1, i_2) = \begin{cases} \frac{1}{\sqrt{2N-2}} & \text{for } i_1 = 1, i_1 \neq i_2 \\ \frac{1}{\sqrt{2N-2}} & \text{for } i_2 = 1, i_1 \neq i_2 \\ 0 & \text{otherwise} \end{cases}$$

Clearly, the function  $f_N(i_1, i_2)$  satisfies all conditions (i), (ii), (iii) of Theorem 2. We know, by combining properties (15) and (16) in [1], that if

$$\max_{\{i=1, \dots, N\}} \sum_{k_2, \dots, k_d=1}^N f_N(i, k_2, \dots, k_d)^2 \rightarrow 0, \text{ as } N \rightarrow \infty$$

then,  $Q(S_1, \dots, S_N) \rightarrow S(0, 1)$  and there is no universality phenomenon (in the sense of semicircular behaviour). From this, for the last example, we have:

$$\max_{\{i=1, \dots, N\}} \sum_{k_2, \dots, k_d=1}^N f_N(i, k_2, \dots, k_d)^2 = (N-1) \frac{1}{2N-2} = \frac{1}{2} \rightarrow 0$$

So we can deduce, without study of the convergence of  $Q(S_1, \dots, S_N)$ , that this multi-linear sum cannot enjoy an Universality Phenomenon in the sense of semicircular behaviour (Theorem [2]).

We can generalize this by:

**Corollary 1.** Fix  $d \geq 1$ , and consider a sequence of functions  $f_N : \{1, \dots, N\}^d \rightarrow \mathbb{R}$  satisfying the following three assumptions for each  $N$  and each  $i_1, \dots, i_d = 1, \dots, N$ :

- (i) (mirror-symmetry)  $f_N(i_1, \dots, i_d) = f_N(i_d, \dots, i_1)$ ;
- (ii) (vanishing on diagonals)  $f_N(i_1, \dots, i_d) = 0$  if  $i_k = i_l$  for some  $k \neq l$ .
- (iii) (normalization)  $\sum_{j_1, \dots, j_d=1}^N f_N(j_1, \dots, j_d)^2 = 1$ .

Let  $Q(x_1, \dots, x_N)$  be defined as in (1). If, for some  $i = 1, \dots, N$ , we have that:

$$\sum_{k_2, \dots, k_d=1}^N f_N(i, k_2, \dots, k_d)^2 \rightarrow 0,$$

as  $N \rightarrow \infty$ . Then:

1.  $Q_N(S_1, \dots, S_N) \rightarrow S(0, 1)$ . i.e. The multi linear homogeneous sum  $Q_N(S_1, \dots, S_N)$  does not converge to the standard semicircular distribution.

2. There is no universality phenomenon in the sense of semicircular behaviour. (In the sense of Theorem 2).

*Proof.* We know that (see (16) in [1]),

$$\|g_N \overset{1}{\frown} g_N\|_{\mathbb{L}_+^2} \geq \max_{i=1, \dots, N} \sum_{k_2, \dots, k_d=1}^N f_N(i, k_2, \dots, k_d)^2$$

with  $\|g_N \overset{1}{\frown} g_N\|_{\mathbb{L}_+^2}$  defined as in (1.12) (see below). Since we suppose that,  $\sum_{k_2, \dots, k_d=1}^N f_N(i, k_2, \dots, k_d)^2 \rightarrow 0$ , as  $N \rightarrow \infty$ , then by (1.10) (See below) we conclude that  $Q_N(S_1, \dots, S_N)$  is not standard semicircular distributed.  $\square$

So, when we have a polynomial in non-commuting variables given by (1.1), we can immediately know, if it's susceptible to welcome an universality phenomenon in the sense of Theorem 2. Of course, this is a necessary but absolutely not a sufficient condition.

Now, another question naturally arises: if a given polynomial in non-commuting variables given by (1.1) does not welcome an universality phenomenon in the sense of Theorem 2, can it welcome an universality phenomenon with other distribution, i.e. can we have the following universality?:

'Let  $X_1, X_2, \dots$  be a sequence of free random variables distributed following a given free distribution  $Z$  ( $Z$  being not the semicircular law), let  $Q_N(x_1, \dots, x_N)$  be defined as in (1.1) and let  $R$  be another given distribution, so if:  
 $Q_N(X_1, \dots, X_N) \xrightarrow{law} R$ , then  
 $Q_N(Y_1, \dots, Y_N) \xrightarrow{law} R$ , for any sequence  $Y_1, Y_2, \dots$  of free identically distributed random variables.'

In other words, does another universality phenomenon exist when there is no universality phenomenon in the sense of Theorem 2 (i.e. in the sense of semicircular law ?).

In the next section, we present our main results. As an element of response to the last question, we begin by the bi-dimensional case, and therefore generalize for all dimensions.

## 1.3 Total universality for some Chaotic Polynomial

We present in this section a particular Chaotic Polynomial, that enjoys interesting properties, essentially the fact that this polynomial has always the same asymptotically behaviour with any sequence of random variables of different distributions. We begin by defining this polynomial and its properties in the bi-dimensional case. Thereafter, we generalize the results to the multi-dimensional case.

### 1.3.1 The bi-dimensional case

We propose to begin by presenting our results, and discuss therefore around this result.

**Proposition 1.** *Let  $(\mathcal{A}, \varphi)$  be a free tracial probability space. Let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be two sequences of centred free random variables with unit variance*

(that is, such that  $\varphi(X_i^2) = \varphi(Y_i^2) = 1$  for all  $i$ ), satisfying moreover

$$\sup_{i \geq 1} \varphi(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} \varphi(|Y_i|^r) < \infty$$

for all  $r \geq 1$ , where  $|X| = \sqrt{X * X}$

Assume  $N > 2$ . Let  $Z$  be a given distribution in the free probability space  $(\mathcal{A}, \varphi)$ , then the following two conclusions are equivalent as  $N \rightarrow \infty$ :

$$(A) \quad \sum_{i_1, i_2=1; i_1 \neq i_2}^N \frac{1}{\sqrt{N(N-1)}} X_{i_1} X_{i_2} \xrightarrow{\text{law}} Z$$

$$(B) \quad \sum_{i_1, i_2=1; i_1 \neq i_2}^N \frac{1}{\sqrt{N(N-1)}} Y_{i_1} Y_{i_2} \xrightarrow{\text{law}} Z$$

In other words:

$$\sum_{i_1, i_2=1; i_1 \neq i_2}^N \frac{1}{\sqrt{N(N-1)}} X_{i_1} X_{i_2} \xrightarrow{\text{law}} \sum_{i_1, i_2=1; i_1 \neq i_2}^N \frac{1}{\sqrt{N(N-1)}} Y_{i_1} Y_{i_2}; \text{ as } N \rightarrow \infty \quad (1.4)$$

*Proof.* In the spirit of the definition of  $Q(x_1, \dots, x_N)$  in (1.1), let us define for  $d = 2$ ,  $P(x_1, \dots, x_N)$  as

$$P(x_1, \dots, x_N) = \sum_{i_1, i_2=1}^N k_N(i_1, i_2) x_{i_1} x_{i_2} \quad (1.5)$$

with,

$$k_N(i_1, i_2) = \begin{cases} \frac{1}{\sqrt{N(N-1)}} & \text{if } i_1 \neq i_2 \\ 0 & \text{otherwise} \end{cases} \quad (1.6)$$

We have, by definition, that  $k_N(i_1, i_2)$  satisfies the condition (i), (ii) of Theorem 1. In other hand, it's easy to verify the condition (iii) of the same theorem. Indeed:

$$\sum_{i_1, i_2=1}^N k_N(i_1, i_2)^2 = N(N-1) \left( \frac{1}{\sqrt{N(N-1)}} \right)^2 = 1$$

Let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be two sequences of centred free random variables with unit variance (that is, such that  $\varphi(X_i^2) = \varphi(Y_i^2) = 1$  for all  $i$ ), satisfying moreover

$$\sup_{i \geq 1} \varphi(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} \varphi(|Y_i|^r) < \infty$$

for all  $r \geq 1$ , where  $|X| = \sqrt{X * \bar{X}}$ . Then, by applying Theorem 1, and since conditions (i), (ii), (iii) are satisfied, we get:

$$\varphi(P_N(X_1, \dots, X_N)^m) - \varphi(P_N(Y_1, \dots, Y_N)^m) = O(\tau_N^{\frac{1}{2}}) \quad (1.7)$$

where,  $\tau_N = \max_{1 \leq i \leq N} \text{Inf}_i(k_N)$

Let us now calculate the quantity  $\text{Inf}_i(k_N)$  defined by (1.2). We have, for each  $1 \leq i \leq N$ :

$$\text{Inf}_i(k_N) = \sum_{l=1}^2 \sum_{j_1}^N k_N(j_1, \dots, j_{l-1}, i, j_l, \dots, j_1)^2$$

Therefore, for each  $1 \leq i \leq N$

$$\begin{aligned} \text{Inf}_i(k_N) &= \sum_{j_1}^N k_N(i, j_1)^2 + \sum_{j_1}^N k_N(j_1, i)^2 \\ &= N - 1 \frac{1}{N(N-1)} + N - 1 \frac{1}{N(N-1)} = \frac{2}{N} \end{aligned}$$

And, we have,

$$\tau_N = \max_{1 \leq i \leq N} \text{Inf}_i(k_N) = \frac{2}{N}$$

But, as  $N \rightarrow \infty$ , we have that

$$\tau_N^{\frac{1}{2}} \rightarrow 0 \quad (1.8)$$

Finally, combining (1.7) and (1.8), we conclude the proof.  $\square$

**Corollary 2.** *Assume  $N > d$ . Let  $(\mathcal{A}, \varphi)$  be a free tracial probability space. Let  $X_1, X_2, \dots$  be a sequence of centred free random variables with unit variance (that is, such that  $\varphi(X_i^2) = 1$  for all  $i$ ), satisfying moreover*

$$\sup_{i \geq 1} \varphi(|X_i|^r) < \infty$$

and for all  $r \geq 1$ , where  $|X| = \sqrt{X * \bar{X}}$

Then,

$$\sum_{i_1, i_2=1; i_1 \neq i_2}^N \frac{1}{\sqrt{N(N-1)}} X_{i_1} X_{i_2} \xrightarrow{\text{law}} S(0, 1) \quad (1.9)$$

In other words,

$$\sum_{i_1, i_2=1; i_1 \neq i_2}^N \frac{1}{\sqrt{N(N-1)}} X_{i_1} X_{i_2}$$

never converges to the semicircular distribution  $S(0, 1)$ .

*Proof.* We have from (15) in [1] that,

$$P_N(S_1, \dots, S_N) \xrightarrow{\text{law}} S(0, 1) \Leftrightarrow \|g_N \overset{r}{\prec} g_N\|_{\mathbb{L}^{2d-2r}(\mathbb{R}_+^2)} \rightarrow 0 \quad (1.10)$$

for all  $r \in \{1, \dots, d-1\}$ , with  $Q(x_1, \dots, x_N)$  given by (1), and, for  $e_i = 1_{[i-1, i]} \in \mathbb{L}^2(\mathbb{R}_+)$ ,  $i \geq 1$ ,

$$g_N = \sum_{i_1, \dots, i_d}^N f_N(i_1, \dots, i_d) e_{i_1} \otimes \dots \otimes e_{i_d} \quad (1.11)$$

Also, from [1], we have that, for  $r = d-1$ ,

$$\|g_N \stackrel{d-1}{\frown} g_N\|_{\mathbb{L}_+^2} = \sqrt{\sum_{i, j=1}^N \left( \sum_{k_2, \dots, k_d=1}^N f_N(i, k_2, \dots, k_d) f_N(k_d, \dots, k_2, j) \right)^2} \quad (1.12)$$

Replacing in (1.12)  $f_N$  by  $k_N$  defined in (1.6), and by taking  $d = 2$ , we obtain,

$$\begin{aligned} \|g_N \stackrel{1}{\frown} g_N\|_{\mathbb{L}_+^2} &= \sqrt{N(N-1) \left( \frac{N-2}{N(N-1)} \right)^2 + N \left( \frac{N-1}{N(N-1)} \right)^2} \\ &= \sqrt{\frac{(N-2)^2}{N(N-1)} + \frac{1}{N}} \end{aligned} \quad (1.13)$$

observe that,

$$\|g_N \stackrel{1}{\frown} g_N\|_{\mathbb{L}_+^2} \rightarrow 0,$$

as  $N \rightarrow \infty$ ,

then, by (1.10), we conclude that  $P(S_1, \dots, S_N) \not\rightarrow S(0, 1)$ , i.e. it being not standard semicircular distributed.

Moreover, by Proposition 1, we know that  $P(S_1, \dots, S_N) \stackrel{law}{=} P(X_1, \dots, X_N)$  for any sequence of free random variables  $X_1, X_2, \dots$ . Then we conclude that  $P(X_1, \dots, X_N)$  never converges to the semicircular distribution  $S(0, 1)$ .  $\square$

This result gives us an interesting indication. Indeed, in the last section we have presented the universality phenomenon in the sens of Déya and Nourdin [1]. The authors showed the fact that if a given chaotic sum following (1.1) with semi circular entries ( $Q(S_1, \dots, S_N)$ , with  $S_1, S_2, \dots \sim S(0, 1)$ ) asymptotically converges to a standard semicircular distribution, then this sum still converges to  $S(0, 1)$  for any other entries with other distributions (i.e.,  $Q(X_1, \dots, X_N)$ , with  $X_1, X_2, \dots$  freely independent and identically distributed). The question was, does there exist another distribution that could play the same role than  $S(0, 1)$ ? The result presented here gives us a partial answer to this question:

Indeed, for the chaotic sum presented above in the bi-dimensional case and below for the general case, we have always the same asymptotic distribution no matter what the distribution of entries is and, on the other hand, no matter what the asymptotically distribution of this sum is (following the different dimensions, if it changes of course!). So through this sum, all free probability distributions are asymptotically equivalent (in law). Moreover, this sum never converges to a standard semicircular distribution. Which gives us a polynomial with unique convergence far from the semicircular one. Also, it's interesting to try to find

the asymptotic convergence for such quantities. We use the notion of 'Total Universality' to keep the same spirit of notations in [1] and [3]. Moreover, the notion of 'Chaotic' in the case of sum presented in Proposition 1 takes all its literally meaning. Indeed, the sum 'smooths' all free probability distributions to get an unique one !

Moreover, to answer the question: does there exist a distribution playing the same role of  $S(0, 1)$  in Theorem 2, we could say that: there exist, at least, a chaotic sum enjoying a close phenomenon without a restriction of the nature of asymptotically law.

In the next section, we develop exactly the same principles in the general case.

### 1.3.2 The multi-linear case

**Proposition 2.** *Let  $(\mathcal{A}, \varphi)$  be a free tracial probability space. Let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be two sequences of centred free random variables with unit variance (that is, such that  $\varphi(X_i^2) = \varphi(Y_i^2) = 1$  for all  $i$ ), satisfying moreover*

$$\sup_{i \geq 1} \varphi(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} \varphi(|Y_i|^r) < \infty$$

for all  $r \geq 1$ , where  $|X| = \sqrt{X * X}$

Assume  $N > d$ . Let  $Z$  be a given distribution in the free probability space  $(\mathcal{A}, \varphi)$ , then the following two conclusions are equivalent as  $N \rightarrow \infty$ :

$$(C) \quad \sum_{\substack{i_1, \dots, i_d=1 \\ i_1 \neq \dots \neq i_d}}^N \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} X_{i_1} X_{i_2} \dots X_{i_d} \xrightarrow{law} Z$$

$$(D) \quad \sum_{\substack{i_1, \dots, i_d=1 \\ i_1 \neq \dots \neq i_d}}^N \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} Y_{i_1} Y_{i_2} \dots Y_{i_d} \xrightarrow{law} Z$$

In other words, setting  $K(N) = \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}}$ , we have that, as  $N \rightarrow \infty$ ,

$$\sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d=1}^N K(N) X_{i_1} \dots X_{i_d} \stackrel{law}{=} \sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d}^N K(N) Y_{i_1} \dots Y_{i_d}; \quad (1.14)$$

*Proof.* The proof in the multi-linear case follows the same way as in the bi-dimensional case. Indeed, we define a multi-linear homogeneous sum  $R(x_1, \dots, x_N)$  as

$$R(x_1, \dots, x_N) = \sum_{i_1, \dots, i_d=1}^N h_N(i_1, \dots, i_d) x_{i_1} \dots x_{i_d} \quad (1.15)$$

with,

$$h_N(i, j) = \begin{cases} \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} & \text{if } i_1 \neq i_2 \dots \neq i_d \\ 0 & \text{otherwise} \end{cases} \quad (1.16)$$

As in the case  $d = 2$ , the conditions (i), (ii) of Theorem 1 are trivial. Also, we have,

$$\sum_{i_1, \dots, i_d=1}^N h_N(i_1, \dots, i_d)^2 = N(N-1)\dots(N-d+1) \left( \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} \right)^2 = 1$$

Let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be two sequences of centred free random variables with unit variance (that is, such that  $\varphi(X_i^2) = \varphi(Y_i^2) = 1$  for all  $i$ ), satisfying moreover

$$\sup_{i \geq 1} \varphi(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} \varphi(|Y_i|^r) < \infty$$

for all  $r \geq 1$ , where  $|X| = \sqrt{X * X}$ . Then, by applying Theorem 1, and since conditions (i), (ii), (iii) are satisfied, we find that:

$$\varphi(R_N(X_1, \dots, X_N)^m) - \varphi(R_N(Y_1, \dots, Y_N)^m) = O(\tau_N^{\frac{1}{2}}) \quad (1.17)$$

where,  $\tau_N = \max_{1 \leq i \leq N} \text{Inf}_i(h_N)$

On the other hand, we have that, for each  $1 \leq i \leq N$ ,

$$\begin{aligned} \text{Inf}_i(h_N) &= \sum_{l=1}^d \sum_{j_1, \dots, j_{d-1}=1}^N h_N(j_1, \dots, j_{l-1}, i, j_l, \dots, j_{d-1})^2 \\ &= d(N-1)\dots(N-d+1) \left( \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} \right)^2 \\ &= \frac{d}{N} \end{aligned}$$

Then,

$$\text{Inf}_i(h_N) \rightarrow 0,$$

as  $N \rightarrow \infty$ , and so,

$$\tau_N \rightarrow 0, \text{ as } N \rightarrow \infty \quad (1.18)$$

Combining (1.17) and (1.18), we find the desired result.  $\square$

**Corollary 3.** Assume  $N > d$ . Let  $(\mathcal{A}, \varphi)$  be a free tracial probability space. Let  $X_1, X_2, \dots$  be a sequence of centred free random variables with unit variance (that is, such that  $\varphi(X_i^2) = 1$  for all  $i$ ), satisfying moreover

$$\sup_{i \geq 1} \varphi(|X_i|^r) < \infty$$

and for all  $r \geq 1$ , where  $|X| = \sqrt{X * X}$

Then,

$$\sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d}^N \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} X_{i_1} X_{i_2} \dots X_{i_d} \xrightarrow{law} S(0, 1) \quad (1.19)$$

In other words,

$$\sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d}^N \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} X_{i_1} X_{i_2} \dots X_{i_d}$$

never converges to the semicircular distribution  $S(0, 1)$ .

*Proof.* The proof in the multi-dimensional case follows exactly the same schemes as in the bi-dimensional case.

Replacing in (1.12)  $f_N$  by  $h_N$  defined in (1.16), we obtain,

$$\begin{aligned} \|g_N \overset{d-1}{\frown} g_N\|_{\mathbb{L}_+^2} &= \sqrt{N(N-1) \left( \frac{(N-2)(N-3)\dots(N-d+1)(N-d)}{N(N-1)\dots(N-d+1)} \right)^2 + N \left( \frac{(N-1)(N-2)\dots(N-d+1)}{N(N-1)\dots(N-d+1)} \right)^2} \\ &= \sqrt{\frac{(N-d)^2}{N(N-1)} + \frac{1}{N}} \end{aligned} \quad (1.20)$$

observe that,

$$\|g_N \overset{d-1}{\frown} g_N\|_{\mathbb{L}_+^2} \rightarrow 0, \text{ as } N \rightarrow \infty$$

then, by (1.10), we conclude that  $R(S_1, \dots, S_N) \not\xrightarrow{law} S(0, 1)$ , i.e. it's not standard semicircular distributed.

Moreover, by Proposition 1, we know that  $R(S_1, \dots, S_N) \xrightarrow{law} R(X_1, \dots, X_N)$  for any sequence of free and identically random variables  $X_1, X_2, \dots$ . Then we conclude that  $R(X_1, \dots, X_N)$  never converges to the semicircular distribution  $S(0, 1)$ .  $\square$

## Chapter 2

# Total Universality for some Chaotic Polynomial of the standard random variables

We extend, in the standard probability space framework, the notion of "Total Universality" we have established in the preceding chapter. And we prove that there also exist a chaotic sum in the standard probability space, which enjoys this phenomenon. Indeed, through these sums, all probability distributions converge asymptotically to the same non gaussian law.

The chaotic sums of free random variables presented in the precedent chapter, always converge to the same law. Moreover, this law is not a semi-circular one, which is quit interesting because the most results of convergence in the free probability framework turn around the semi-circular distribution. So, this results may let us consider that other laws could play the same role as semi circular law.

Since the semi-circular distribution plays the same role in free probability framework as a gaussian one in standard probability space, we discuss here the same result in the standard probability framework by replacing the semicircular law by gaussian law. We present now our main results which are an extension of those of Proposition 2 and Corollary 3.

### 2.1 Total universality for some chaotic polynomial of standard random variables

We begin by presenting our results. After we propose to discuss and try to give an interpretation. To finish, we present the proofs using some results established in [2] and [3].

**Proposition 3.** *Let  $(\Omega, \mathcal{F}, P)$  be a classical probability space. Let  $X_1, X_2, \dots$*

and  $Y_1, Y_2, \dots$  be two sequences of independent centred random variables with unit variance (that is, such that  $E(X_i^2) = E(Y_i^2) = 1$  for all  $i$ ), satisfying moreover

$$\sup_{i \geq 1} E(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} E(|Y_i|^r) < \infty$$

for all  $r \geq 1$

Assume  $N > d$ , set

$$\Theta_N(x_1, \dots, x_N) = \sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d}^N \frac{1}{\sqrt{d!N(N-1)\dots(N-d+1)}} x_{i_1} x_{i_2} \dots x_{i_d} \quad (2.1)$$

Let  $Z$  be a given distribution in the probability space  $(\Omega, \mathcal{F}, P)$ , then the following two conclusions are equivalent as  $N \rightarrow \infty$ :

$$(i) \Theta_N(X_1, \dots, X_N) = \sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d}^N \frac{1}{\sqrt{d!N(N-1)\dots(N-d+1)}} X_{i_1} X_{i_2} \dots X_{i_d} \xrightarrow{\text{law}} Z$$

$$(ii) \Theta_N(Y_1, \dots, Y_N) = \sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d}^N \frac{1}{\sqrt{d!N(N-1)\dots(N-d+1)}} Y_{i_1} Y_{i_2} \dots Y_{i_d} \xrightarrow{\text{law}} Z$$

In other words, we have that, as  $N \rightarrow \infty$ ,

$$\Theta_N(X_1, \dots, X_N) \stackrel{\text{law}}{=} \Theta_N(Y_1, \dots, Y_N) \quad (2.2)$$

So from this proposition we deduced that the chaotic sums presented above in (2.1) always converge to the same distribution, no matter what is the distribution of entries. Moreover, thanks to Mossel, O'Donnell and Oleszkiewicz[2], another interesting point is the fact that the entries, i.e. the sequence of independent random variables  $X_1, X_2, \dots$  do not necessarily have the same distribution!

Now, in the corollary below, we show that the chaotic polynomial (2.1) does not converge to a standard gaussian distribution.

**Corollary 4.** *Assume  $N > d$ . Let  $(\Omega, \mathcal{F}, P)$  be a standard probability space. Let  $X_1, X_2, \dots$  be a sequence of independent centred random variables with unit variance (that is, such that  $E(X_i^2) = 1$  for all  $i$ ), satisfying moreover*

$$\sup_{i \geq 1} E(|X_i|^r) < \infty$$

for all  $r \geq 1$ ,

Then,

$$\Theta_N(X_1, \dots, X_N) \xrightarrow{law} \mathcal{N}(0, 1) \quad (2.3)$$

with  $\Theta_N(x_1, \dots, x_N)$  defined as in (2.1) and  $\mathcal{N}(0, 1)$  is the standard gaussian law.

In other words,

$$\sum_{i_1, \dots, i_d=1; i_1 \neq \dots \neq i_d}^N \frac{1}{\sqrt{N(N-1)\dots(N-d+1)}} X_{i_1} X_{i_2} \dots X_{i_d}$$

never converges to the standard gaussian distribution  $\mathcal{N}(0, 1)$ .

Then, we conclude that the chaotic polynomial defined in (2.1) always converges to the same non standard gaussian distribution for every sequence of independent and centred random variables.

The polynomial defined in (2.1) does not, in itself, represent any particular quantity (for the time being). But what is interesting in this quantity is, on the one hand, the fact that it always converges toward the same distribution for any sequence of random variables satisfying the assumptions of Corollary 3. On the other hand, the most interesting thing, is the fact that this convergence is not toward the gaussian law. Since most results about the convergence of such polynomials are around the gaussian distribution (in particular the multi linear central limit theorem), this polynomial represents a case where we have the same, but not gaussian convergence for a chaotic polynomial.

Before to prove the above results, we propose to give some helpful results and definitions borrowed from the works of Nourdin et al [3] and those from Mossel et al [2].

## 2.2 Definitions and background

We begin by the invariance principle in the version of E.Mossel, R.O'Donnell and K.Oleszkiewicz established in [2].

**Theorem 3.** (See [2]) *Let  $(\Omega, \mathcal{F}, P)$  be a classical probability space. Let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be two sequences of centred random variables with unit variance (that is, such that  $E(X_i^2) = E(Y_i^2) = 1$  for all  $i$ ), satisfying moreover*

$$\sup_{i \geq 1} E(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} E(|Y_i|^r) < \infty$$

for all  $r \geq 1$

Fix  $d \geq 1$ , and consider a sequence of functions  $f_N : \{1, \dots, N\}^d \rightarrow \mathbb{R}$  satisfying the following two assumptions for each  $N$  and each  $i_1, \dots, i_d = 1, \dots, N$ :

- (i) (full symmetry)  $f_N(i_1, \dots, i_d) = f_N(i_{\sigma(1)}, \dots, i_{\sigma(d)})$ ;
- (ii) (normalization)  $d! \sum_{j_1, \dots, j_d=1}^N f_N(j_1, \dots, j_d)^2 = 1$ .

Also, set

$$Q_N(x_1, \dots, x_N) = \sum_{i_1, \dots, i_d=1}^N f_N(i_1, \dots, i_d) x_{i_1} \dots x_{i_d} \quad (2.4)$$

and

$$\text{Inf}_i(f_N) = \sum_{j_2, \dots, j_d=1}^N f_N(i, j_2, \dots, j_d)^2, i = 1, \dots, N. \quad (2.5)$$

Then, for any integer  $m \geq 1$ ,

$$E(Q_N(X_1, \dots, X_N)^m) - E(Q_N(Y_1, \dots, Y_N)^m) = O(\tau_N^{\frac{1}{2}}) \quad (2.6)$$

where  $\tau_N = \max_{1 \leq i \leq N} \text{Inf}_i(f_N)$ .

This result, named invariance principle, shows that under mild conditions the distribution of multi linear polynomials with low influences and bounded degree is essentially invariant for all product spaces. This principle constitute the first base of our proof.

Let us now define the notion of 'contraction' defined on discrete sets of the type  $[N]^d, N \geq 1$ , which play an important role when studying homogeneous sums. These objects enjoy a number of useful combinatorial properties.

**Definition 3.** Fix  $d, N \geq 2$ . Let  $f : [N]^d \rightarrow \mathbb{R}$  be a symmetric function vanishing on diagonals. For every  $r = 0, \dots, d$ , the contraction  $f \star_r f$  is the function on  $[N]^{2d-2r}$  given by:

$$f \star_r f(j_1, \dots, j_{2d-2r}) = \sum_{1 \leq a_1, \dots, a_r \leq N} f(a_1, \dots, a_r, j_1, \dots, j_{d-r}) f(a_1, \dots, a_r, j_{d-r+1}, \dots, j_{2d-2r}). \quad (2.7)$$

The symmetrization of  $f \star_r f$  is written  $f \tilde{\star}_r f$ .

To finish, and before proceeding to the prove of Proposition 2 and Corollary 2, let us present the following weak and one dimensional version of Central Limit Theorem for chaotic sums established in [3].

**Proposition 4.** (See [3]) Let  $G_1, G_2, \dots$  be a sequence of standard gaussian random variables independent and identically distributed.

Fix  $d \geq 2$ , and consider a sequence of functions  $f_N : \{1, \dots, N\}^d \rightarrow \mathbb{R}$  satisfying the following three assumptions for each  $N$  and each  $i_1, \dots, i_d = 1, \dots, N$ :

- (i) (full symmetry)  $f_N(i_1, \dots, i_d) = f_N(i_{\sigma(1)}, \dots, i_{\sigma(d)})$  for all  $\sigma \in \mathfrak{S}_d$ ;  
(ii) (vanishing on diagonals)  $f_N(i_1, \dots, i_d) = 0$  if  $i_k = i_l$  for some  $k \neq l$ .

Also, set

$$Q_N(x_1, \dots, x_N) = \sum_{i_1, \dots, i_d=1}^N f_N(i_1, \dots, i_d) x_{i_1} \dots x_{i_d} \quad (2.8)$$

Suppose moreover that,  $E[Q_N(G_1, \dots, G_N)^2] \rightarrow 1$ . Then the following two assertions are equivalent, as  $N \rightarrow \infty$

- (1)  $Q_N(G_1, \dots, G_N)$  converges in law to a centred gaussian random variable with variance  $v$ .  
(2) For every  $r = 1, \dots, d-1$ ,  $\|f_N \star_r f_N\|_{2d-2r} \rightarrow 0$ .

We can now proceed to proof our above stated results.

## 2.3 Proofs

We begin by the proof of Proposition 3,

Set

$$g_N(x_{i_1}, \dots, x_{i_d}) = \begin{cases} \frac{1}{\sqrt{d!N(N-1)\dots(N-d+1)}} & \text{for } i_1 \neq i_2 \neq \dots \neq i_d \\ 0 & \text{Otherwise} \end{cases}$$

So, we can write,

$$\Theta_N(x_1, \dots, x_N) = \sum_{i_1, \dots, i_d=1}^N g_N(i_1, \dots, i_d) x_{i_1} \dots x_{i_d}$$

It is easy to verify that  $g_N(i_1, \dots, i_d)$  verifies the condition (i) and (ii) of Theorem 3. Let  $(\Omega, \mathcal{F}, P)$  be a classical probability space. Let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be two sequences of independent centred random variables with unit variance (that is, such that  $E(X_i^2) = E(Y_i^2) = 1$  for all  $i$ ), satisfying moreover

$$\sup_{i \geq 1} E(|X_i|^r) < \infty$$

and

$$\sup_{i \geq 1} E(|Y_i|^r) < \infty$$

for all  $r \geq 1$

Then, from (2.6), we have,

$$E(\Theta_N(X_1, \dots, X_N)^m) - E(\Theta_N(Y_1, \dots, Y_N)^m) = O(\tau_N^{\frac{1}{2}}) \quad (2.9)$$

with,

$$\tau_N = \max_{1 \leq i \leq N} \text{Inf}_i(g_N) \quad (2.10)$$

and,

$$\text{Inf}_i(g_N) = \sum_{j_2, \dots, j_d=1}^N g_N(i, j_2, \dots, j_d)^2, i = 1, \dots, N. \quad (2.11)$$

On the other hand, let compute  $\text{Inf}_i(g_N)$ . From (2.11), we have, for  $i = 1, \dots, N$ ,

$$\text{Inf}_i(g_N) = (N-1)(N-2)\dots(N-d+1) \left( \frac{1}{\sqrt{d!N(N-1)(N-2)\dots(N-d+1)}} \right)^2 = \frac{1}{d!N}$$

So,

$$\text{Inf}_i(g_N) \rightarrow 0, \quad (2.12)$$

as  $N \rightarrow \infty$  Then  $\tau_N \rightarrow 0$  as  $N \rightarrow \infty$ . From (2.9), we conclude that,

$$E(\Theta_N(X_1, \dots, X_N)^m) - E(\Theta_N(Y_1, \dots, Y_N)^m) = 0$$

which completes the proof.

Now, we proceed to the proof of Corollary 4. We know that,

$$\| f_N \star_{d-1} f_N \|_2^2 = \sum_{1 \leq i, k \leq N} \left[ \sum_{1 \leq i_2, \dots, i_d \leq N} f_N(i, i_2, \dots, i_d) f_N(k, i_2, \dots, i_d) \right]^2$$

then, let compute  $\| g_N \star_r g_N \|_{2d-2r}$  for  $r = d - 1$ , with

$$g_N(x_{i_1}, \dots, x_{i_d}) = \begin{cases} \frac{1}{\sqrt{d!N(N-1)\dots(N-d+1)}} & \text{for } i_1 \neq i_2 \neq \dots \neq i_d \\ 0 & \text{Otherwise} \end{cases}$$

we have:

$$\begin{aligned} \| g_N \star_{d-1} g_N \|_2^2 &= \sum_{1 \leq i, k \leq N} \left[ \sum_{1 \leq i_2, \dots, i_d \leq N} g_N(i, i_2, \dots, i_d) g_N(k, i_2, \dots, i_d) \right]^2 \\ &= N^2 \left[ (N-2)(N-3)\dots(N-d+1)(N-d) \frac{1}{\sqrt{d!N(N-1)\dots(N-d+1)}} \frac{1}{\sqrt{d!N(N-1)\dots(N-d+1)}} \right]^2 \\ &= N^2 \frac{(N-d)^2}{d!N^2(N-1)^2} \\ &\rightarrow \frac{1}{d!} \rightarrow 0 \end{aligned} \tag{2.13}$$

as  $N \rightarrow \infty$ .

Then, by using Proposition 4 of [3], we conclude that  $\Theta_N(G_1, \dots, G_N)$  does not converge to the standard gaussian law, with  $G_1, G_2, \dots$  a sequence of standard gaussian random variables independent and identically distributed.

On the other hand, using Proposition 3, we conclude that  $\Theta_N(X_1, \dots, X_N)$  never converges to the standard gaussian law, with  $X_1, X_2, \dots$  a sequence of independent random variables with unit variance, satisfying moreover

$$\sup_{i \geq 1} E(|X_i|^r) < \infty$$

for all  $r \geq 1$ , which completes the proof of Corollary 3.

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