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# The Japanese pioneering contribution to investigation of modular structures

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## Abstract

Monoarchetypal modular crystal structures are built by regularly juxtaposing modules obtained from the same archetype. At the interface the coordination environment may be altered, leading to a chemical modulation that accompanies the structural variability. When this does not occur the result is a series of polytypes, whose building modules are not limited to layers, although the latter represent the most common examples. The symmetry theory of polytypes was developed by the Order–Disorder (OD) school, which however considered only layers as building modules, and only those polytypes in which pairs of layers are geometrically equivalent. Well before the OD school, Ito in Japan recognized the need of a wider algebraic category, that he called “twinned space groups”, in which he considered only digonal operations. His approach was later expanded by Sadanaga, who recognized that Ito’s “twinned space groups” are space groupoids. In this article the contribution of the Japanese school to the field is traced back, and it is shown that the fundamental concepts later developed in a more systematic way, were already present, in a nutshell, in the works of Ito and Sadanaga; finally, a slightly revised classification of monoarchetypal modular structures is proposed which better reflects the original contribution by the Japanese school.

## 1. Introduction

A crystal structure can be decomposed into elementary building blocks of variable size. In case of non-molecular structures, the smallest and universally known building block is the coordination polyhedron. Bigger and more complex building blocks occur in modular structures, which are built by juxtaposing one or more types of modules. The modules are three-dimensional but less than triperiodic components that can ideally be described as cuts from a structurally and chemically homogeneous parent structure: the *archetype*, which does not necessarily correspond to a real

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structure, or is not necessarily defined in a unique way, although the module is. For example, in the case of structures based on stacking of layers of equal spheres, the archetype is a structure with periodicity of one layer and whose stacking mode is AA; clearly, this is a structure with less than maximal compactness, which is however realized in a number of derivative structures in which the trigonal prismatic cavities between two layers are partially occupied. If the requirement of maximal compactness has instead to be fulfilled, one can take either the  $2H$  (*hcp*) structure or the  $3C$  (*ccp*) structure as archetype: the module is then obtained as a cut at one-half or one-third across the unit cell.

The global symmetry of the modular structure is described by an ordinary space group, whereas the building modules possess only a subperiodic group  $G_m^n$ , where  $n$  is the number of dimensions of the space and  $m$  the number of directions along which the module is periodic ( $m < n$ ). We are interested in three-dimensional structures; the subperiodic group of the modules can therefore be  $G_0^3$  (point groups, for 0-periodic modules: bricks or blocks),  $G_1^3$  (rod groups, for 1-periodic modules: rods or chains),  $G_2^3$  (layer groups, for 2-periodic modules: layers or sheets).

The modules can be obtained from chemically different archetypes: the resulting structures are known as polyarchetypal structures (diarchetypal being the most frequent case) and are evidently also heterochemical (different composition) with respect to each of the archetypes. Polyarchetypal modular structures correspond to polysomatic series, as defined by Thompson [1]. When the modules are obtained from the same archetype, they are known as monoarchetypal structures. The resulting structure may or may not have the same chemical composition as the archetype; in particular, modifications of the coordination at the interface between two module can alter the chemical composition of the modular structure. In this article we specifically address monoarchetypal structures and present a revised classification scheme with respect to that in [2].

Monoarchetypal modular structures are known as *cell-twins* [3], a term issued from the similarity with the operations mapping individuals (domains) in twins. The generation of a series of structures from the juxtaposition of modules is obtained by varying the relative orientation and/or position of adjacent modules. The monoarchetypal nature of the modular structure leaves open a number of structural possibilities:

- the juxtaposition may substantially alter the coordination polyhedra at the interface between neighbour modules, resulting in the occurrence of what has been called a “chemical stress” [4];
- cell-twins built by modules of the same size and without chemical stress are *polytypes*;
- cell-twins with chemical stress accompanied or not by a chemical change have been called *heterochemical* and *isochemical chemical twins* respectively [5].

Cell-twinning has been identified as structure-building mechanism in which modules taken from one and the same archetype are juxtaposed in such a way that at the interface atomic sites for both modules coalesce or new atomic sites are created, and this allows for a chemical change which becomes more important the thinner the modules are. This mechanism has been called *tropochemical cell-twinning* [3] and explains the occurrence of morphotropism [6] in modular structures. A special case corresponds to modules obtained as cuts of different width from the same archetype, as in the case of the lillianite homologous series. In this series, the archetype is the galena (PbS) structure, and the modules are obtained from cuts corresponding to planes of the (311) family. Juxtaposing modules with the same width and filling by Bi the coordination sites at the interface between modules, one gets different structures like lillianite, heyrovskite and galeno-bismuthite. Instead, when the modules have different width, one obtained the components of the so-called “phase V”, which spans a region from 66.7 to 73.5 mol% Bi<sub>2</sub>S<sub>3</sub>. These cell-twins have been called contracted twins [3]; they can be considered as an intermediate between monoarchetypal and

polyarchetypal modular structures because the building modules are different although obtained from the same archetype.

For our analysis we need to briefly remind the reader of the matrix representation of isometries (distance-preserving transformations). Two points in space with coordinates  $|x,y,z\rangle$  and  $|x',y',z'\rangle$  are related by an operation that includes a rotation or rotoinversion component (the linear part) and a translational component. This operation can be represented as a matrix-column pair  $(\mathbf{W},\mathbf{w})$ :

$$\mathbf{W}|x,y,z\rangle + \mathbf{w} = |x',y',z'\rangle \quad (1)$$

where  $\mathbf{W}$  is a  $3\times 3$  unimodular matrix representing the linear part of the operation and  $\mathbf{w}$  is a  $3\times 1$  matrix (column matrix) representing the translational part. In the following, we deal with special cases of isometries, namely symmetry operations compatible with the existence of a periodic lattice, *i.e.* crystallographic symmetry operations.

A point-group operation consists of the linear part  $\mathbf{W}$  only; a space-group operation includes also a translational part  $\mathbf{w}$ , possibly zero. The linear part  $\mathbf{W}$  corresponds to an operation of order  $n = 1, 2, 3, 4$  or  $6$ , which means that  $\mathbf{W}^n = \mathbf{I}$ , where  $\mathbf{I}$  is the identity matrix. Accordingly,  $(\mathbf{W},\mathbf{w})^n = (\mathbf{I},\mathbf{t})$ , where  $\mathbf{t}$  is a full lattice translations. The matrix-column pair can be synthetically represented by the Seitz notation  $\{\mathbf{R}|\mathbf{v}\}$ , where  $\mathbf{R}$  is the Hermann-Mauguin symbol of the linear part and  $\mathbf{v}$  is the translation part [7]. We will use this notation in the analysis of some examples.

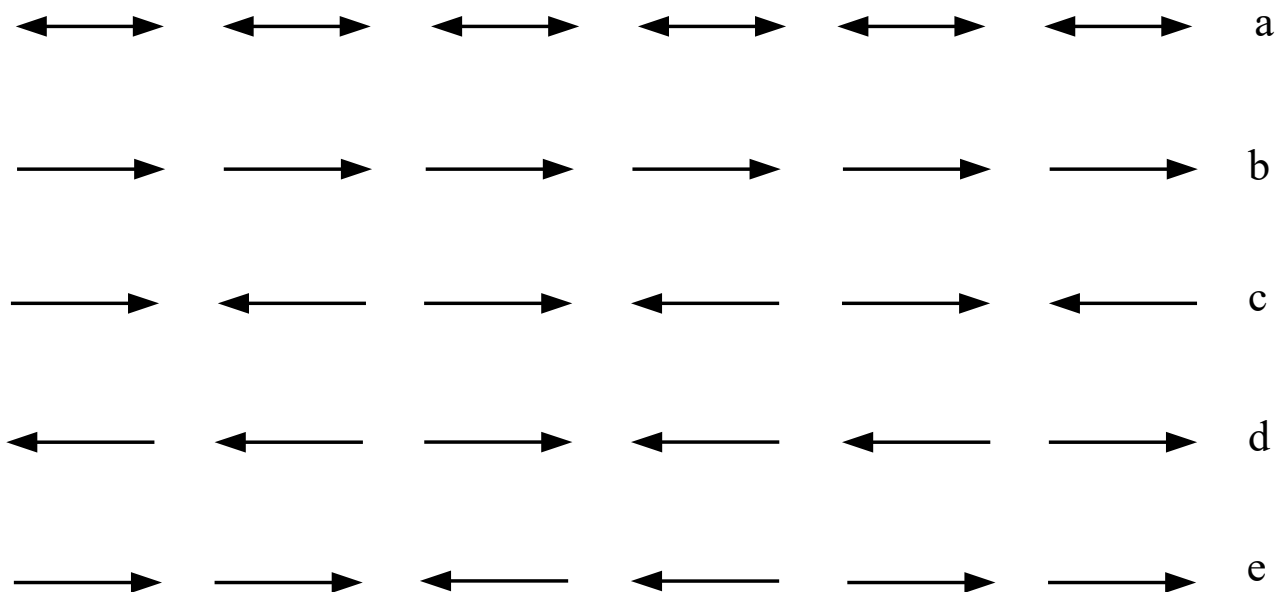
In a twin, the individuals are related by an isometry that does not contain any translational component: it is a crystallographic point-group operation that maps the *orientation* of the individuals (see, e.g., [8]). In a cell-twin, the modules are related by isometries that do contain translational components: they map the *orientation* and the *position* of the modules. If the linear part  $\mathbf{W}$  corresponds to an operation of order  $n$ , then:

$$(\mathbf{W},\mathbf{w})^p = (\mathbf{I},\boldsymbol{\tau}), \boldsymbol{\tau} = \mathbf{t}/q, p,q \text{ integers} \quad (2)$$

where  $\mathbf{t}$  is a lattice vector of the cell-twin. As we are going to show, the classification of modular structures introduced by Sadanaga [9] actually depends on whether  $q = 1$  or  $q > 1$ .

As stated above, isochemical monoarchetypal modular structures without chemical stress are 101  
polytypes. According to the official definition [10], polytypes are obtained by “stacking layers of 102  
(nearly) identical structure and composition” so that the resulting triperiodic structures “differ only 103  
in their stacking sequence”. The building modules are clearly requested to be layers, although “a 104  
more general definition of polytypism that includes 'rod' and 'block' polytypes may become 105  
necessary in the future”. Based on this explicit statement, in the following we identify polytypes 106  
with isochemical monoarchetypal modular structures built by only one kind of modules, 107  
independently on whether these modules are 2-, 1- or 0-periodic. 108

The symmetry theory of polytypes built by diperiodic modules (layers or sheets) has been presented 109  
in various publications; in particular, an extensive treatment has been introduced for layer structures 110  
by the so-called OD (Order-Disorder) theory [11]. An OD structure is a structure in which layers are 111  
stacked one above the other in a regular way so that pairs of layers are geometrically equivalent; the 112  
stacking can give rise to periodic or non-periodic structures. Therefore, not all polytypes are OD 113  
structures: polytypes in which layer pairs are not all equivalent do not satisfy the definition of an 114  
OD structure ([12]; see also a discussion in [13]). (**Figure 1**). Although the official definition [10] 115  
does not explicitly mention non-periodic structures, the term “disordered polytypes” does appear in 116  
the reports, so that one can consider that all OD structures are polytypes. 117



**Figure 1** Schematic representation of monoarchetypal modular structures built by the stacking of diperiodic modules (layers). Double and simple arrows represent apolar and polar layers respectively. Polytypes a, b and c are OD structures (classified as type I, II and III respectively), because the layer pairs are geometrically equivalent (the interlayer region has an arrow head or an arrow tail on each side). Polytypes d and e are not, because the layer pairs are not all equivalent due to the the presence of interlayer regions with an arrow head on one side and an arrow tail on the opposite side (modified after [12]).

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The mapping of each layer onto itself and onto different layers is realized by isometries that, in general, do not act on the whole crystal space, but only on a subspace of it. These are called, respectively, local and partial operations [9]. In the OD theory, they are both called “partial” operations, because algebraically they are both partial functions, but divided into  $\lambda$ -operation and  $\sigma$ -operations, respectively [11], where the Greek letter  $\lambda$  is reminiscent of “layer”; further, operations are divided into  $\tau$  and  $\rho$ , depending on whether they keep the orientation of the layer with respect to the stacking direction or invert it. Local and partial operations give rise to an algebraic structure more general than a group, which is known as a groupoid in the sense of Brandt [14] (see Appendix)<sup>1</sup>. Knowing the structure of the layer (and thus its local, or  $\lambda$ , operations) and the possible stacking operations (i.e. the possible partial, or  $\sigma$ , operations) one can get a fictitious structure, known as the *family structure*, obtained by applying to the layer all the possible partial operations so

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<sup>1</sup> Brandt’s original paper is often incorrectly quoted as 1926, whereas it was published one year later.

that they are “prolonged” to global operation of the family structure, as explained below. These 131  
operations are the generators – although not, in general, a minimal set of generators, of the 132  
symmetry group of the family structure, which is atomically overpopulated and contains unrealistic 133  
bond distances, but can be seen as the common prototype from which all the possible structures are 134  
obtained by selecting some of the possible partial operations. 135

Well before the development of the OD theory, Ito [15] addressed the problem of isochemical 136  
monoarchetypal modular structures, starting from the case of pyroxenes, where both monoclinic and 137  
orthorhombic polymorphs can be described as obtained by the stacking of (somewhat idealized) 138  
layers (a modern analysis has been presented in [16]). Unfortunately, the bulk of the theory was 139  
published in an article in Japanese [17] and remains still today essentially unknown outside Japan. 140

Later, Ito [18] extended this approach to a large number of isochemical monoarchetypal modular 141  
structures occurring in minerals; the theoretical description was however limited to a few pages in 142  
the introduction, used terms that did not follow in the literature, and remains somehow imprecise. 143

Sadanaga, who was a student of Ito, influenced by the approach of the OD school, tried to extend 144  
Ito’s approach in a more general and formal way. He called the archetype *fundamental structure*, the 145  
module *substructure* and the structures obtained by stacking the modules *complex structures*; 146  
unfortunately, he introduced some contradictory definitions which resulted in a somewhat confused 147  
scheme which, instead of bringing forward Ito’s innovative ideas eventually resulted in 148  
misunderstandings for the few readers who have had access to the rare journals where the series of 149  
articles (partly in Japanese) were published. 150

The purpose of this article is to amend the (largely unknown) analysis proposed by the Japanese 151  
school, going back to Ito’s original idea, but including the precise and rigorous algebraic definitions 152  
that were missing in the previous literature, and provide a revised classification of monoarchetypal 153  
modular structures. Accordingly, and for ease of comparison, we adopt here the term “complex 154  
structure” used by Sadanaga [9] to indicate a modular structure. 155



## 2. Module-mapping operations 156

A monoarchetypal modular structure (“complex structure”) is a three-dimensional triperiodic 157  
structure consisting of subperiodic modules (“substructures”) which are either strictly identical or 158  
nearly identical to each other in configuration. The complex structure is therefore determined by the 159  
structure of the module and the way in which the modules are arranged one with respect to the 160  
others. 161

We consider the operations mapping the various modules. These operations are of two types [9]: 162

1. *local ( $\lambda$ ) operations*: isometries mapping a module to itself; they are symmetry operations of 163  
the module, defined only for module on which they act, *i.e.* in the subspace spanned by the 164  
module itself; 165
2. *partial ( $\sigma$ ) operations*: isometries mapping different modules; a given partial operation is 166  
defined only for the pair of modules to which it applies and does not act elsewhere in the 167  
crystal space. 168

Exactly like the infinite number of atoms in a crystal structure is represented by the finite number in 169  
the unit cell, the infinitely many modules that build up a modular structure can be represented by a 170  
finite number. If  $\mathcal{G}_m^3$  is the symmetry group of an  $m$ -periodic three-dimensional object ( $m < 3$ ), a 171  
module can be fully contained in a unit cell only for  $m = 0$ ; when  $m = 1$  or  $m = 2$  the modules are 172  
rods (chains) or layers (sheets) respectively and are therefore infinite in one or two directions 173  
respectively so that a unit cell cannot actually *contain* any module. We can however identify a finite 174  
number of modules to a unit cell by counting the number of modules that cross the unit cell in the 3- 175  
 $m$  directions of missing periodicity, *i.e.* which are not related by a full lattice translation in that or 176  
those direction(s). The other modules are then obtained by full lattice translations. In the following, 177  
we will denote the  $i$ -th module as  $X_i$  where  $i = 0, 1, \dots, k$ , so that  $k+1$  modules are identified in the 178  
unit cell of the triperiodic structure. The subperiodic group of the  $i$ -th module,  $\mathcal{G}_{m,i}^3$ , contains the 179  
local ( $\lambda$ ) operations of  $X_i$  and constitutes its stabilizer (group of mappings that leave the module 180

invariant); it will be indicated as  $N_i$  for *nucleus* [19]. Further, let us denote a partial operation 181  
mapping  $X_i$  to  $X_0$  as  $\Phi(i \rightarrow 0)$ . The target module of  $\Phi(i \rightarrow 0)$  is the reference module  $X_0$ , whereas 182  
the source module varies with  $i$ , *i.e.* the operations are meant to be applied from the left: 183

$$\Phi(i \rightarrow 0)X_i = X_0. \quad (3) \quad 184$$

The operation  $\Phi(0 \rightarrow 0)$  maps  $X_0$  to  $X_0$ , *i.e.* it is a local operation of  $X_0$ :  $\Phi(0 \rightarrow 0) \in N_0$ . The 185  
product of a local operation  $\Phi(0 \rightarrow 0)$  and a partial operation  $\Phi(i \rightarrow 0)$  is again a partial operation 186  
that maps the same pair of modules for a different but equivalent configuration of the target module 187  
 $X_0$ . Hence, the product  $N_0\Phi(i \rightarrow 0)$  is defined. The complete set of all the mappings to  $X_0$  obtained 188  
by making  $i$  a running index gives what Loewy [20] has called the *Mischgruppe* of  $X_0$ , denoted  $M_0$  189  
below, a term which has received several translations. Hereafter we use the term *mixed group* 190  
suggested by Dixon [19], which is close to the original German<sup>2</sup>. For the sake of brevity, the 191  
notation  $\Phi(i \rightarrow 0)$  is simplified to  $\Phi_i$  whenever this does not lead to ambiguity. The mixed group of 192  
 $X_0$  is therefore defined as: 193

$$M_0 = \cup_i N_0\Phi_i \quad (4) \quad 194$$

The index  $i$  runs over all the infinite modules, but in the following we explicitly show only the 195  
interval  $[0,k]$  of the modules identified in a single unit cell, the other modules being obtained by full 196  
lattice translations. When  $i = 0$ , the product  $N_0\Phi_i$  is simply  $N_0$ , the subperiodic group containing the 197  
local operations of  $X_0$ , because  $\Phi_0 = \Phi(0 \rightarrow 0) \in N_0$ . When  $i \neq 0$ ,  $N_0\Phi_i$  is the set of all the partial 198  
operations mapping any  $X_i \neq X_0$  ( $i=1,2,\dots,k$ ) to  $X_0$ . This set is called the *shell* of the mixed group 199  
 $M_0$ , denoted  $H_0$ . Incidentally, it does not form a group. The mixed group can therefore be 200  
decomposed into the nucleus and the shell: 201

$$M_0 = N_0 \cup_{i=1,k} N_0\Phi_i = N_0 \cup H_0 \quad (5) \quad 202$$

The inverse of a partial operation  $\Phi_i = \Phi(i \rightarrow 0)$  is the partial operation  $\Phi_i^{-1} = \Phi(0 \rightarrow i)$  that maps 203  
 $X_0$  to  $X_i$ . The product of a partial operation  $\Phi(0 \rightarrow i)$  and a local operation  $\Phi(0 \rightarrow 0)$  is again a 204

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2 To be noted that despite its name, a “mixed group” is not a group but a set of operations.

partial operation that maps the same pair of modules for a different but equivalent configuration of the source module  $X_0$ . Hence, the product  $\Phi(0 \rightarrow i)N_0 = \Phi^{-1}(i \rightarrow 0)N_0$  is defined. It follows that  $\Phi_j^{-1}N_0\Phi_i$  is defined as the set of all the partial operations mapping  $X_i$  to  $X_j$  via  $X_0$ . If we take  $j = i \neq 0$ , this set becomes  $\Phi_i^{-1}N_0\Phi_i$  and contains all the operations mapping a module  $X_i \neq X_0$  to itself via  $X_0$ . Hence  $\Phi_i^{-1}N_0\Phi_i$  is the set of the local operations of  $X_i$ , *i.e.*  $N_i$ . Indeed,  $\Phi_i^{-1}N_0\Phi_i$  is simply the conjugation of  $N_0$  by  $\Phi_i$  and conjugation is a similarity transformation. All  $N_i$  are isomorphic to each other, differing for the orientation and/or position in space of their symmetry elements.

By applying the partial operation  $\Phi_j^{-1}$  to  $M_0$ , the mixed group of  $X_0$ , we obtain:

$$\Phi_j^{-1}M_0 = \Phi_j^{-1}N_0 \cup \Phi_j^{-1}H_0 = \Phi_j^{-1}N_0 \cup_i \Phi_j^{-1}N_0\Phi_i = \Phi_j^{-1}N_0\Phi_j \cup_{i \neq j} \Phi_j^{-1}N_0\Phi_i = N_j \cup_{i \neq j} \Phi_j^{-1}N\Phi_i = M_j \quad (6)$$

which is the set of all the local operations of  $X_j$  (*i.e.* the subperiodic group, or nucleus  $N_j$ ) and of the partial operations mapping any module but  $X_j$  to  $X_j$  (*i.e.* the shell  $H_j$ ). In other words,  $\Phi_j^{-1}M_0$  is simply  $M_j$ , the mixed group of  $X_j$ . Then,  $N_j$  and  $H_j$  are the nucleus and the shell of  $M_j$ . Taking the union of all the mixed groups, we obtain the groupoid of the complex structure:

$$\mathcal{D} = \cup_i M_i = \cup_{ij} \Phi_j^{-1}N_0\Phi_i \quad (7)$$

Since the modules are all of the same type, the groupoid is connected and corresponds to the definition of Brandt [14].  $\mathcal{D}$  contains the complete set of all the operations mapping any pair of modules,  $X_i$  and  $X_j$ , obtained by making not only  $i$  but also  $j$  a running index. The groupoid can be shown in a tabular form, where the index  $i$  runs on all the infinite modules. Exactly as in a space group the infinite number of symmetry operations can be represented by a finite set from which all the others are obtained by adding full lattice translations to the translations component of the representation  $(\mathbf{W}, \mathbf{w})$ , a groupoid can be represented by a finite table in which the index is limited to the  $k+1$  modules identified in a single unit cell, *i.e.* not related by full lattice translations:

$$\begin{aligned} \mathcal{D} = & \quad N_0 \cup N_0\Phi_1 \cup N_0\Phi_2 \cup \dots \cup N_0\Phi_i \cup \dots \cup N_0\Phi_k & 228 \\ & \cup \Phi_1^{-1}N_0 \cup N_1 \cup \Phi_1^{-1}N_0\Phi_2 \cup \dots \cup \Phi_1^{-1}N_0\Phi_i \cup \dots \cup \Phi_1^{-1}N_0\Phi_k & 229 \\ & \cup \Phi_2^{-1}N_0 \cup \Phi_2^{-1}N_0\Phi_1 \cup N_2 \cup \dots \cup \Phi_2^{-1}N_0\Phi_i \cup \dots \cup \Phi_2^{-1}N_0\Phi_k & 230 \end{aligned}$$

$\cup \overset{\dots}{\Phi_j^{-1}N_0} \cup \overset{\dots}{\Phi_j^{-1}N_0\Phi_1} \cup \overset{\dots}{\Phi_j^{-1}N_0\Phi_2} \cup \dots \cup \overset{\dots}{\Phi_j^{-1}N_0\Phi_i} \cup \dots \cup \overset{\dots}{\Phi_j^{-1}N_0\Phi_k}$	231 232 233
$\cup \overset{\dots}{\Phi_k^{-1}N_0} \cup \overset{\dots}{\Phi_k^{-1}N_0\Phi_1} \cup \overset{\dots}{\Phi_k^{-1}N_0\Phi_2} \cup \dots \cup \overset{\dots}{\Phi_k^{-1}N_0\Phi_i} \cup \dots \cup N_k$	234 235

The finite table representing the groupoid is composed by  $k+1$  lines, each representing a mixed group; each mixed group is composed by one nucleus and  $k$  cosets building one shell; the table is then composed by  $(k+1)^2$  blocks.

The operations in each mixed group map the various modules to the same target module. A partial or local operation  $\Phi$  is defined by its matrix representation  $(\mathbf{W}, \mathbf{w})$  and the substructure on which it acts and the structure on which the latter is mapped. Another operation  $\Phi'$  which has the same matrix representation  $(\mathbf{W}, \mathbf{w})$  as  $\Phi$  but acts on different substructure(s) is of the same *type* but differ from  $\Phi$  in the domain of action. If all the pairs of substructures are related by operations of the same type, *i.e.* which have the same matrix representation  $(\mathbf{W}, \mathbf{w})$ , then each pair of substructures is related by the same type of operation; altogether they define a global operation which has the same matrix representation  $(\mathbf{W}, \mathbf{w})$  and shows up in the space group of the complex structure. This is true at least for the identity operation of each module, which becomes the only identity operation of the complex structure, and the translations that correspond to the full lattice vectors of the complex structure.

### 3. The Japanese pioneering contribution to the classification of complex structures

Let  $A$  be the archetype (Sadanaga's "fundamental structure") from which the modules  $X$  (Sadanaga's "substructures") are obtained, and  $G_A$  its space group; let  $C$  be a monoarchetypal modular structure (Sadanaga's "complex structure") obtained by stacking modules  $X$  obtained from a cut in  $A$ , and  $G_C$  its space group. As we have seen, the full symmetry of  $C$  is described by a space groupoid  $\mathcal{D}_C$ , consisting of subperiodic groups  $N(X_i)$  – in the following shortened to  $N_i$  because of the monoarchetypal nature of the complex structure - which represent the symmetry of  $X_i$ ; the partial operations mapping different modules constitute the shell of each mixed group. The space

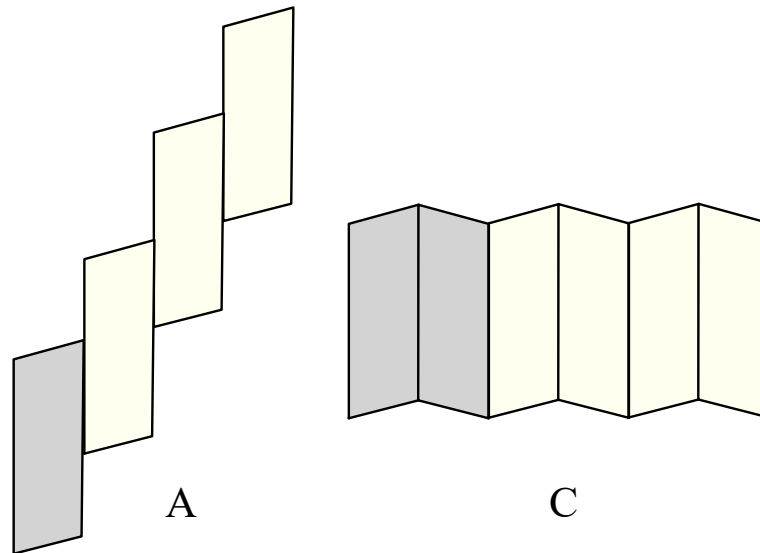
groupoid  $\mathcal{D}_C$  may actually degenerate into the space group of the complex structure  $G_C$  or not, 258  
depending on whether all or only some of the local and partial operations have continuation in the 259  
whole crystal space and define the corresponding global operations. 260  
Ito [17,18] was the first to recognize the presence of partial operations in crystal structures. Without 261  
the mathematical background of Brandt [14] and Loewy [20], he nevertheless understood that a 262  
wider algebraic structure, going beyond space groups, was needed to completely describe complex 263  
structures. His approach was however essentially inductive and did not allow him to obtain a 264  
complete picture. The complex structures studied by Ito were called “polysynthetic structures” by 265  
analogy with the polysynthetic twinning occurring in minerals like albite, and the term “twinning” 266  
was used to describe the geometric relation between the modules, an unfortunate choice that later 267  
led to repeated confusion in the literature; this was eventually corrected to “cell-twinning” [3]. That 268  
the choice of wording was inherited from a parallelism with the case of twinning is evident from the 269  
terminology introduced by Ito himself. The space groupoids (term that was unknown to Ito) which 270  
describe the complex structures he studied were called *twinned space groups*; they were obtained in 271  
the following way ([18], page 2). 272

*A twinned space group may be obtained by adding superimposed a group of operations, 273  
called a twinning group, onto one of the space groups of Schoenflies. The domain of 274  
operation of a twinning group is always a multiple (or submultiple) of that of the 275  
original group. A new cell is formed, governed by the operation of the twinning group. 276  
The unit cells (or multiple or submultiple thereof) of the original space group are so 277  
juxtaposed that only every other, instead of every, cell is in parallel and continuous 278  
position. At the boundary planes of the original cells twinned the operations of the 279  
twinning group prevail and those of original group are overridden by them and cease to 280  
be effective. [...] the symmetry elements that constitute the twinning group must be 281*

*either digonal (rotation or screw) axes or reflexion [sic] (with or without a glide) 282*  
*planes. Simple gliding must also be taken into account. 283*

Although somewhat imprecise when judged from a contemporary perspective, Ito's definition can 284  
be easily made more formal and rigorous. The "original group" is  $G_A$ . The "twinning group",  $G_T$  in 285  
the following, whose "symmetry elements" can only be "digonal", is a group having as generator(s) 286  
operation(s) obtained by prolonging the partial operation(s) mapping the modules X to global 287  
operations. The "domain of operation" of  $G_T$  is the subspace of the crystal space on which the 288  
partial operations act. Depending on whether the module X spans the whole unit cell of the 289  
fundamental structure A (**Figure 2**) or only party of it (**Figure 3**), the "domain of operation" of  $G_T$  is 290  
"a multiple" or "a submultiple" of that of  $G_A$ . In fact, a complex structure C being by definition 291  
composed by multiple copies of the module X, two situations may occur: 292

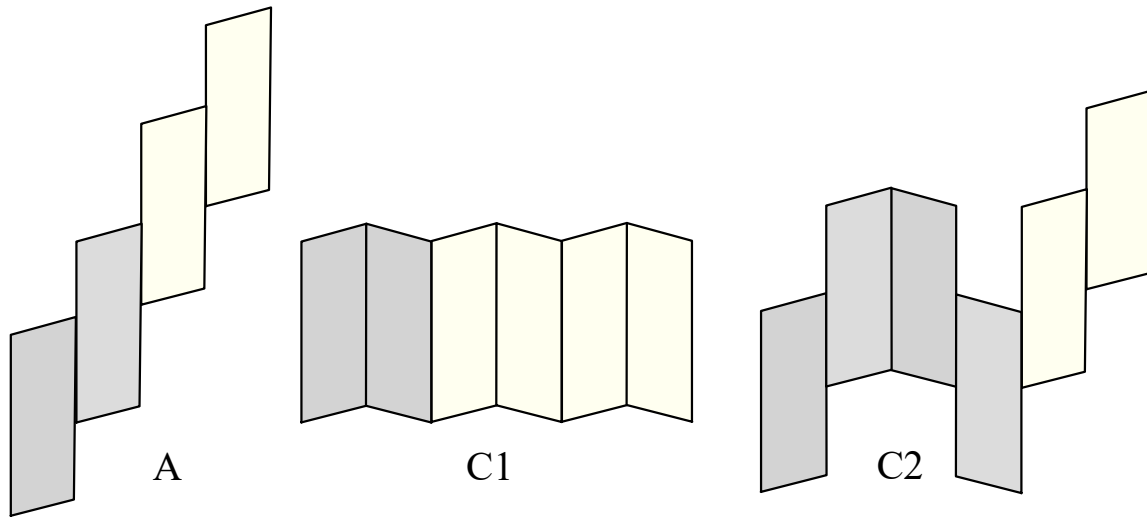
1. if only one copy of X can be identified in the unit cell of A, then the unit cell of C is larger 293  
than the unit cell of A. The operations mapping the various  $X_i$  (operations of  $G_T$ ) act at the 294  
boundaries of the unit cell of A and beyond, so that their "domain of operation" is a 295  
"multiple" of the unit cell of A (Figure 2); 296
2. if instead X is obtained as a cut from A, there are necessarily partial operations mapping 297  
modules  $X_i$  that act on a region inside the unit cell of A; this region is therefore a 298  
"submultiple" of the unit cell of A (Figure 3). 299



**Figure 2:** Example of fundamental structure A (left) built by juxtaposing modules X in a monotonous way. The unit formula of A corresponds to a single instance of X, so that the repetition period of A (grey region) is one-module thick (X). The complex structure C (right) is obtained by acting with a mirror reflection on X on the boundary of the unit cell of A. As a result, the repetition period of C corresponds to two modules ( $X_2$ ). Accordingly, the unit cell of C is twice larger than that of A. The grey regions identify the modules in a single unit cell; the unit cell itself is not drawn).

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**Figure 3:** Example of fundamental structure A (left) built by juxtaposing modules X in a monotonous way. The unit formula of A corresponds to two instances of X, so that the repetition period of A (grey region) is two-module thick ( $X_2$ ). Examples of modular structures whose shortest repetition period consists of two modules can be found in all layer structures with the topology of close packing of spheres. The complex structure C1 (middle) is obtained by acting with a mirror reflection on X in the middle of the A. As a result, the repetition period of C corresponds to two modules ( $X_2$ ) and the unit cell of C has the same width as that of A. The complex structure C2 (right) is obtained by acting with a translation and then with a mirror reflection on X. As a result, the repetition period of C corresponds to four modules ( $X_4$ ) and the unit cell of C is twice larger than that of A.

An example of case 2 above is found in pyroxenes, which were also the starting point of Ito's 303  
 research [15]. Clinoenstatite (the “fundamental structure” A), protoenstatite and orthoenstatite (two 304  
 “complex structures” C) can be synthetically represented as  $X_2^+$ ,  $X^+X^-$ ,  $X_2^+X_2^-$ , respectively, where 305  
 “+” or “-” indicate the displacement of the  $\text{SiO}_3$  chains either  $+c/3$  or  $-c/3$  along the  $a$  axis of the 306  
 fundamental structure A. Figure 3 is a schematic representation of this case, where the slope of the 307  
 module can be read as “+” or “-”. The module X is a layer which spans half of the unit cell of the 308  
 fundamental structure [16]. Examples of case 1 above are discussed in the following sub-sections. 309  
 The statement that the operations of  $\mathcal{G}_T$  “prevail” over those of  $\mathcal{G}_A$  “[a]t the boundary planes of the 310  
 original cells twinned” is easily understood considering that the groupoids  $\mathcal{D}_A$  and  $\mathcal{D}_C$  contain 311  
 different partial operations  $\Phi$ . Some of the operations of  $\mathcal{D}_A$  do not act on the modules X in C and 312  
 “are overridden [...] and cease to be effective”, so that the corresponding global operations in  $\mathcal{G}_A$  do 313



not appear in  $\mathcal{G}_C$ . They are replaced by operations of  $\mathcal{D}_C$ , which, prolonged to global operations, 314  
appear in  $\mathcal{G}_T$ . The nature of the groupoid was clearly present, although in rather vague terms, in Ito's 315  
original considerations, as shown also by the following passage ([18], page 5): 316

*A twinned space group may be identical with one of the 230 space groups of 317  
Schoenflies. Theoretically, however, this is not always the case and happens only when 318  
certain geometrical requisites are fulfilled by the original lattice. Generally we find that 319  
a twinned space group is very near, but distinctly different from, one of the space groups 320  
of Schoenflies, containing certain symmetry elements in excess. This is due to the 321  
difference of domain of operation of the twinning group and the original space group, 322  
upon which it acts to produce a twinned space group. 323*

Translated in modern language and correcting some confusion in the language (“symmetry 324  
elements” when the “symmetry operations” are actually meant), the above statement says that the 325  
groupoid  $\mathcal{D}$  (“twinned space group”) can degenerate to a space group, and therefore coincide with  $\mathcal{G}$  326  
 (“may be identical with one of the 230 space groups”); this happens when all the partial operations 327  
have a continuation in the whole crystal space. Otherwise,  $\mathcal{D}$  contains partial operations (“certain 328  
symmetry elements in excess”) that do not have a continuation (“due to the difference of domain of 329  
operation”). In symbols, Ito's definition corresponds to Equation (7) provided that  $\mathcal{D}$  is renamed 330  
 “twinned space group” and the operations  $\Phi_i$  are the isometries that, if prolonged to global 331  
operations, generate the “twinning group”  $\mathcal{G}_T$ . The OD idea of family structure was, in a nutshell, 332  
already present in Ito's mind: his “twinning group” would appear in the group of the family 333  
structure is all the possible partial operations, rather than those occurring in a specific structure, 334  
were used to build it. 335

Depending on the nature of the partial operation expressed in the basis of the modular structure, two 336  
scenarios become possible, that were later recognized by Sadanaga [9]: 337

1. partial operations correspond to  $q = 1$  ( $\tau = \mathbf{t}$ ) (Eq. 2); when they are prolonged to global operations, the group they generate not only corresponds to one of the 230 crystallographic types, but its unit cell coincides with that  $G_C$ :  $\mathcal{T}(G_T) = \mathcal{T}(G_C)$ .
2. partial operations correspond to  $q > 1$  ( $\tau = \mathbf{t}/p$ ) (Eq. 2); when they are prolonged to global operations, the group they generate has a unit cell that is a submultiple of that of  $G_C$ , so that the translation subgroup of  $G_T$  is a supergroup of the translation subgroup of  $G_C$ :  $\mathcal{T}(G_T) \supset \mathcal{T}(G_C)$ .

Ito's "twinning group" was restricted to digonal operations and by definition fell in case No. 1 above. A further restriction imposed by Ito [17] is that the "twinning group"  $G_T$  and the space group of the complex structure  $G_C$  do not contain operations with the same linear part  $\mathbf{W}$  whose symmetry elements are parallel. This unnecessary restriction comes from the analogy with the case of twins, in which the orientation of the individuals (domains) are related by operations that cannot belong to the point group of either individual.

Sadanaga attempted to provide a more precise definition of "twinned space group", and in doing so he actually ended up extending and generalizing Ito's results. Sadanaga [21, 22] considered a structure consisting of substructures which are either strictly equal or nearly equal to each other in configuration (which means only one type of module, possibly ignoring small differences due, e.g. to atomic substitutions) and called it a *superstructure*, a collective name to indicate both the fundamental and the complex structures, all made up by the same type of substructures. Removing the restriction imposed on the "twinning group"  $G_T$  by Ito [17], Sadanaga [9] identified the substructure (module) in the complex structure as a cut from the fundamental structure (archetype). He realised that the full symmetry of the complex structure can be described by a space groupoid, a special case of which is Ito's "twinned space group". Unfortunately, his definitions in different papers, but also within the same paper, are contradictory and easily bring confusion in the reader. In

particular, the shell (for which Sadanaga more often used the term “hull”: see [23] for a brief terminological survey) was not uniquely defined. In his papers we find two definitions.

D1. Sadanaga [9], caption of Figure 3 (where substructures are schematically represented by triangles), says that “the set of all operations that bring every triangle into superposition upon another constitutes the hull”; the same definition is found also in [21, 22];

D2. the same paper, right below Eq. 2, states that “ $h_i$ 's are the elements of  $H$ ” where  $h_i$  ( $\Phi_i$  in Eq. 3) are the partial operations which, applied to the nuclei  $N_i$ , build up the “hull” (shell)  $H$ , suggesting therefore that, contrary to the definition above, the latter is composed not by “all [partial] operations” but only by the representatives.

From the analysis of some concrete structures, he classified “complex structures” in two categories, depending on whether “every member is derived by a twinning group operating on the unit cells (or multiples or submultiples thereof) of the fundamental structure [...]” or “the set of complex structures contains a structure that has no twinning group to derive it from the fundamental structure” [9]. The twinning group was defined as the set-theoretical union  $e \cup H$ , where  $e$  is the identity operation (which is required to for a group but is obviously not found among the partial, i.e.  $\sigma$ , operations) and  $H$  is “hull” (shell). Clearly, if  $H$  contains operations corresponding to  $q > 1$  in Eq. 2, then  $H$  cannot form a group, even supplemented by the identity. A group can instead be obtained if the partial operations in definition D2 above are used as *generators*: in this case, there is no need to add the identity, because the set of generators includes, for each operation, also its inverse, and therefore the identity as well. Therefore, Sadanaga’s classification of monoarchetypal modular structures can be rationalized as follows:

1. polysynthetic structures, later renamed “Ito twins” [3], if the operation in  $H$  generate a group with the same translation sublattice as the resulting modular structure; this group generated in this way corresponds to Ito’s “twinning group”;

2. polytypes, if  $H$  contains operations corresponding to  $q > 1$  in Eq. 2, so the group generated 386  
by them has a translation sublattice which is a superlattice of that of the resulting modular 387  
structure. 388

The choice of the term “polytypes” to cover isochemical monoarchetypal modular structures 389  
*exclusive* of “polysynthetic structures” was unfortunate, because its meaning is more restricted with 390  
respect to the first official definition presented one year before [24], which covers both categories. 391  
A more serious problem occurs however when we compare the definition of “hull” and that of 392  
“twinning group”. In this respect, definition D1 is clearly incorrect, because it corresponds to 393  
extracting from the groupoid *all* the partial ( $\sigma$ ) operations. However, with the exception of the 394  
trivial case  $N_i = 1$ , the set-theoretical union of partial operations does not form a group, even when 395  
it is supplemented by the identity. In fact, a group must be closed under any combination of the 396  
elements of the set. The groupoid contains all the partial operations  $\Phi_s(i \rightarrow 0)$  and all the partial 397  
operations  $\Phi_t(0 \rightarrow j) = \Phi_t^{-1}(j \rightarrow 0)$ . For  $i = j$  the combination  $\Phi_s(0 \rightarrow i)\Phi_t(i \rightarrow 0)$  is a local operation 398  
of  $N_i$  and the combination  $\Phi_s(i \rightarrow 0)\Phi_t(0 \rightarrow i)$  is a local operation of  $N_0$ . However, with the 399  
exception of the identity, the “twinning group” does not contain any local operation, as it turns out 400  
to be the case for definition D1, unless  $N = 1$  or  $s = t$ . 401

Definition D2 does instead provide a group if, for each pair of substructures (modules), one 402  
representative  $\Phi_s(i \rightarrow 0)$  is selected as generator. In this case, in fact, the only local operations that 403  
are obtained by combining partial operations  $\Phi_s(0 \rightarrow i)\Phi_s(i \rightarrow 0)$  and  $\Phi_s(i \rightarrow 0)\Phi_s(0 \rightarrow i)$  are the 404  
identities of the respective nuclei, which always have a continuation in the whole crystal space: the 405  
only identity operation of the “twinning group”  $G_T$ . Ito’s idea of “twinning group” as formalized by 406  
Sadanaga corresponds therefore to definition D2. 407

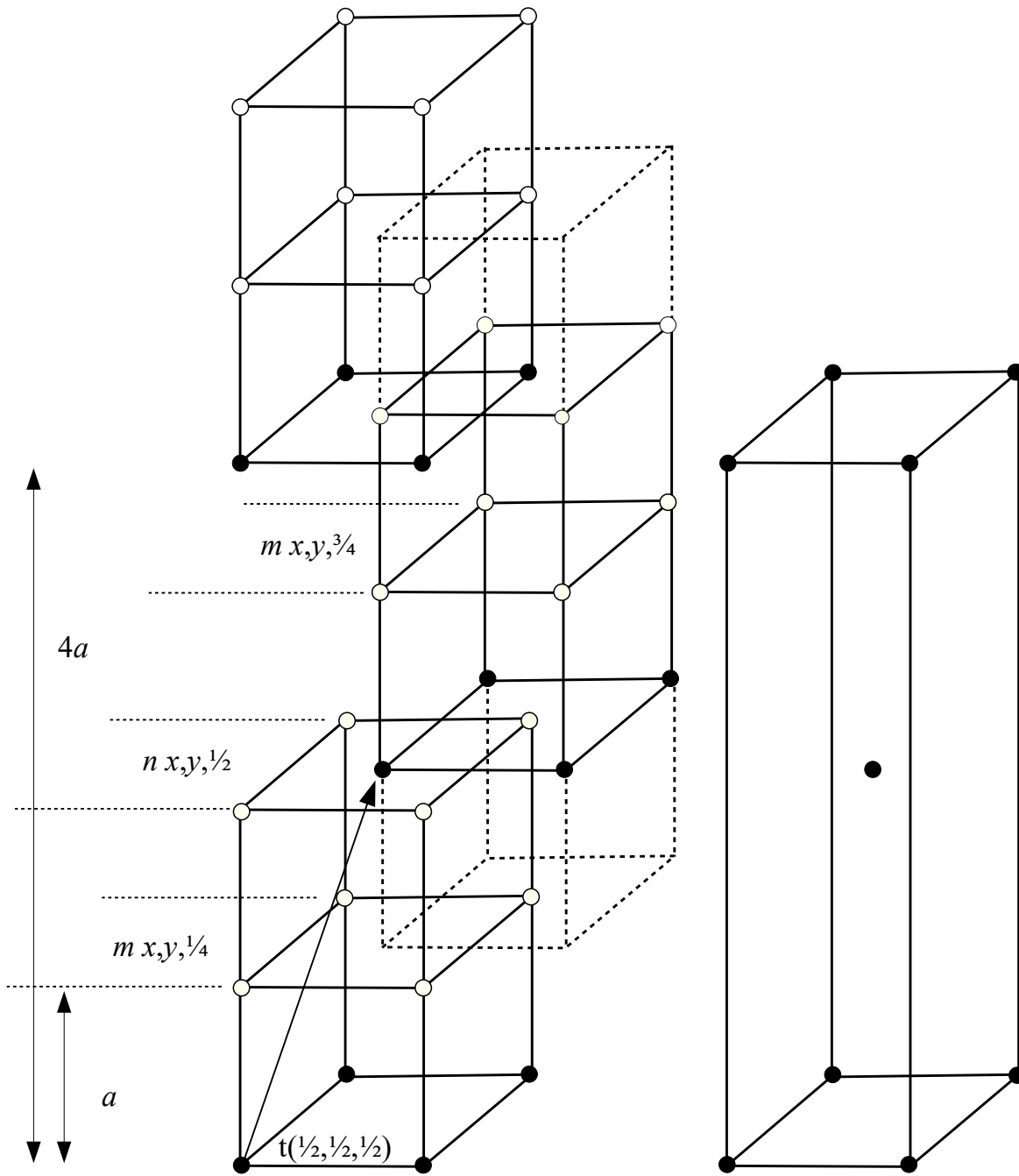
For the case of complex structures that were called “polytypes”, Sadanaga used the term 408  
“polytypism groupoid” for  $e \cup H$  instead of “twinning group” to emphasize that these operations do 409  
not form a group. However, if we adopt definition D2, for consistency, a group is always obtained 410

by using partial operations as generators but, because of the unusual translation component of the 411  
partial operations, the basis vectors of this group differ from those of the space group  $G_C$ . In fact,  $q$  412  
> 1 in Eq. (2) means that  $\tau < t$ , i.e. the periodicity defined by the partial operations, once prolonged 413  
to global operations, is shorter than the periodicity of the space group  $G_C$ . Sadanaga's classification 414  
does therefore make sense, but needs to be reformulated. For that, we re-analyse the same examples 415  
used by Sadanaga [9] as well as the simpler but enlightening example of the *hcp* polytype of sphere 416  
packing. 417

### 3.1 The modular structure of boleite and pseudoboleite 418

Pseudoboleite,  $Pb_{31}Cu_{24}Cl_{62}(OH)_{48}$ , is a rare secondary mineral commonly occurring as intergrowths 419  
with and as parallel overgrowths on boleite,  $KPb_{26}Ag_9Cu_{24}Cl_{62}(OH)_{48}$ . The former is tetragonal, 420  
space-group type  $I4/mmm$ , cell parameters  $a = 15.24 \text{ \AA}$ ,  $c = 30.74 \text{ \AA}$  [25]. The symmetry of boleite 421  
has long been a subject of discussion. C. Friedel [26] defined it as cubic, whereas G. Friedel [27] 422  
strongly denied even a cubic pseudo-symmetry and attributed to boleite a tetragonal symmetry with 423  
 $c/a$  ratio of 4. Hadding [28] classified boleite as cubic, and Gossner and Arm [29] assigned it to a 424  
space group of type  $Pm\bar{3}m$ . G. Friedel [30] replied insisting on his idea of tetragonal symmetry, 425  
with mimetic twinning simulating a cubic symmetry. Hocart [31], based on diffraction from rotating 426  
crystal, confirmed Friedel's interpretation [30], but Gossner [32] contested his conclusions on the 427  
basis of a larger number of collected diffractions. The matter was eventually settled much later, 428  
when Rouse [33] refined the crystal structure showing it is indeed cubic, space-group type  $Pm\bar{3}m$ , 429  
with cell parameter  $a = 15.29 \text{ \AA}$ , later refined to  $a = 15.288(2) \text{ \AA}$  [34]. Boleite corresponds therefore 430  
to the fundamental structure A in which only one module is identified in the unit cell, whereas 431  
pseudoboleite corresponds to a complex structure C with two layers identified in the unit cell (cf. 432  
Figure 2). At Ito's time, however, the correct symmetry of boleite was still unconfirmed, with a 433  
majority of opinions in favour of a tetragonal cell with  $c/a$  ratio of 4. Ito [18] started from this 434  
hypothesis and studied boleite with a Weissenberg camera, obtaining cell parameters  $a = 15.24 \text{ \AA}$ ,  $c$  435

= 60.82 Å. The structure he reported corresponds to a cell-twin of four cubic unit cells alternately 436  
 related by mirror reflections and  $z = 1/4$  and  $z = 3/4$  and an  $n$ -glide reflection at  $z = 1/2$  (**Figure 4**). 437  
 Sadanaga [9], probably unaware of the work by Rouse [33], adopted Ito's description, but in his 438  
 groupoid analysis he used  $Pm\bar{3}m$  as nucleus  $N_0$ , which is incorrect because a substructure is by 439  
 definition subperiodic and therefore the nucleus must correspond to a subperiodic group, not to a 440  
 space group. Sadanaga chose  $m_{x,y,1/4}$  (incorrectly indicated as mirror reflection at  $c/8$  instead of 441  
 $c/4$ ) and  $t(1/2,1/2,1/2)$  as partial operations: the latter results from the product of  $m_{x,y,1/4}$  and  $n_{x,y,1/2}$  442  
 (operations meant to be applied from the left). The tabular expression of the groupoid, Eq. 3 in 443  
 Sadanaga [9], is a  $3 \times 3$  matrix instead of a  $4 \times 4$  as it should be considering the interpretation of 444  
 boleite: this is because only one mirror reflection, at  $m_{x,y,1/4}$  was explicitly considered: the one at 445  
 $x,y,3/4$  was not included, as it should be if the tabular expression of the groupoid has to include 446  
 operations mapping modules identified in the same unit cell. Despite these errors, his analysis is 447  
 nevertheless worth considering because it gives a clue to understand the classification he proposed. 448  
 Sadanaga placed the groupoid obtained in this way in the category of "twinned space groups", 449  
 stating that " $e \cup H \cong Im$ ". As already discussed above, a group of type  $Im$  can be obtained by 450  
 considering one partial operation for each pair of substructures  $(i,0)$ , as well as its inverse; for 451  
 example, in the case under consideration the operations  $1$  (identity),  $t(\overline{1/2,1/2,1/2})$ ,  $m_{x,y,1/4}$ ,  $m_{x,y,3/4}$ , all 452  
 digonal, as well the operations obtained by adding full lattice translations (which are not shown 453  
 explicitly in the tabular for of the groupoid). This result corresponds to definition D2. 454  
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Figure 4: The structure of boleite as imagined by Ito [18] and Sadanaga [9], on the left, built by stacking a cubic unit cell, with partial operations  $m x, y, \frac{1}{4}$ ,  $n x, y, \frac{1}{2}$ ,  $m x, y, \frac{3}{4}$ . The translation  $t(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$  is obtained as combination of the first two operations and is prolonged to global operation, resulting in the  $tl$  unit cell on the right. White lattice nodes represent an intermediate step in the construction of the  $tl$  unit cell and do not appear in the latter, because they do not correspond to any global operation. The cubic unit cell was assigned to perclite, later discredited, whereas it actually corresponds to boleite itself. The  $tl$  unit cell corresponds to an old model of boleite (see the history of boleite structure determination in the text).

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The structure of boleite is however cubic, not tetragonal, and we have to show that boleite and pseudoboleite are built by the same layer, stacked differently. When extracting the layer, one of the three cubic axes of boleite becomes the direction of missing periodicity for the layer. In the following, we take the cubic  $c$  axis as this direction. Boleite and pseudoboleite are then obtained by translating this layer along the cubic  $[001]$  and  $[111]$  directions respectively. First of all, we need to show that these two structures do have a common substructure. Boleite is the fundamental structure A with a single module identified in the unit cell; pseudoboleite is the complex structure C with two modules identified in the unit cell. To compare the two structures, we express them in the common maximal subgroup, which is of type  $P4/mmm$  and is obtained in the following way. The structure of pseudoboleite is expressed in the klassengleiche subgroup of index 2 obtained by removing the  $I$ -centring translation. A model of pseudoboleite can be obtained from the module of boleite in the following way.

1. The structure of boleite is expressed in the  $P4/mmm$  general subgroup (index 6) from  $Pm\bar{3}m$  obtained by basis change  $\mathbf{a}, \mathbf{b}, 2\mathbf{c}$  with origin shift  $0, 0, \frac{1}{2}$  (the origin shift is required to match the  $z$  coordinates of the corresponding atoms in the two minerals).
2. The upper part ( $z > \frac{1}{2}$ ) of the structure, which corresponds to a single module of boleite and is related to the lower part by a translation of  $c/2$ , is shifted by a  $C$ -centring vector, which is tantamount to take an  $I$ -translated copy of the lower part.

In this setting, the model of pseudoboleite obtained from boleite can be compared with the structure of pseudoboleite, with a 1:1 correspondence of atomic sites (Table 1: hydrogen atoms are not included because of the uncertainty on their position). For such a comparison, one needs to:

1. ignore the difference in the chemical nature for some cations: K and Ag in pseudoboleite are all replaced by Pb in the model obtained from boleite;
2. one OH site in pseudoboleite is replaced by one Cl site in the model obtained from boleite and vice-versa.



The similarity of the two modules can be checked with the routine STRUCTURE RELATIONS at 485  
the Bilbao Crystallographic Server [35]. The maximal difference between corresponding atoms is 486  
0.6604 Å, despite the chemical difference (K or Ag vs. Pb) and the average difference on whole 487  
structure is only 0.0815 Å. There is therefore no doubt that the two models fit pretty well one 488  
another. Because of the chemical heterogeneity, albeit limited to a few atomic sites, in the layers 489  
building the two structures, boleite and pseudoboleite cannot be considered polytypes, but rather 490  
pseudopolytypes [36]. 491

Having shown that the two structures are indeed built by the same type of module, once some 492  
chemical difference is ignored, we can analyse the groupoid of both structures. The symmetry of the 493  
module is diperiodic and corresponds to layer group  $P(4/m)mm$  (notation after [37]: the parentheses 494  
identify the direction of missing periodicity), or  $p4/mmm$  (No. 61 in volume E of the International 495  
Tables for Crystallography: [38]). The partial operation mapping the two substructures in the unit 496  
cell of pseudoboleite is the body-centring translation. The groupoid corresponding to the 497  
pseudoboleite structure is obtained by the set-theoretical union  $\cup_{i,j=[0,1]} \Phi_j^{-1} N_0 \Phi_i$  where  $\Phi_0$  is the 498  
identity,  $\Phi_1 = t(\overline{1/2} \overline{1/2} \overline{1/2})$ , and  $N_0 = P(4/m)mm$  (Table 2). 499

For the diagonal terms, conjugation of  $P(4/m)mm$  with the partial operation  $\{1|\overline{1/2} \overline{1/2} \overline{1/2}\}$  gives back 500  
 $P(4/m)mm$  modulo a full lattice translation. This means that all the local operations in two modules 501  
have a continuation in the whole crystal space and give rise to a global operation of the same kind. 502  
The same holds also for the extra-diagonal terms, showing that the partial operations too have a 503  
continuation in the next layer. As a result, the groupoid of pseudoboleite degenerates to a space 504  
group, of type  $I4/mmm$ , *i.e.*  $G_C$ . The “twinning group” would therefore simply be  $G_T = I1$ . 505

### 3.2 The modular structure of zoisite and clinozoisite 506

Clinozoisite  $\text{Ca}_2\text{Al}_3(\text{SiO}_4)(\text{Si}_2\text{O}_7)\text{O}(\text{OH})$ , is a calcium aluminium hydroxy sorosilicate belonging to 507  
the epidote group of minerals. It crystallizes in a space group of type  $P12_1/m1$ , with cell parameters 508  
 $a = 8.87 \text{ \AA}$ ,  $b = 5.592 \text{ \AA}$ ,  $c = 10.114 \text{ \AA}$  and  $\beta = 115.4^\circ$  [39]. It has an orthorhombic dimorph, zoisite, 509

which is no longer considered a member of the epidote group [40]: it crystallises in a space group of type  $Pnma$ , with cell parameters  $a = 16.212 \text{ \AA}$ ,  $b = 5.555 \text{ \AA}$  and  $c = 10.034 \text{ \AA}$  [39]. Clinozoisite corresponds to the fundamental structure A in which only one module is identified in the unit cell. Zoisite corresponds to a complex structure C with two layers identified in the unit cell (cfr. Figure 2).

Ito [18] established a metric relation between the two minerals noting that  $2a_{\text{clino}}\cos(\beta_{\text{clino}}-\pi/2) = a_{\text{zoisite}}$ <sup>3</sup>. This metric relation does not represent, however, a valid change of basis: the corresponding direction, perpendicular to the  $c$  axis, would be irrational in the unit cell of clinozoisite, whereas the closest transformation,  $\mathbf{a}_{\text{zoisite}} = 2\mathbf{a}_{\text{clino}} + \mathbf{c}_{\text{clino}}$ , leads to a  $\beta_{\text{zoisite}}$  angle of about  $81^\circ$  instead of  $90^\circ$ . Nevertheless, if we ignore this significant angular deviation, we can follow Ito's original reasoning and Sadanaga's development. Similarly to what has been done for the case boleite-pseudoboleite, we build a structural model of zoisite from that of clinozoisite in the following way.

1. The structure of clinozoisite is expressed in the isomorphic  $P12_1/m$  subgroup (index 2) obtained by basis change  $2\mathbf{a}, \mathbf{b}, \mathbf{c}$  with origin shift  $\frac{1}{2}, 0, 0$  (the origin shift is required to match the  $x$  coordinates of the corresponding atoms in the two minerals).
2. The module common to the two dimorphs is located in the region  $\frac{1}{4} \leq x \leq \frac{3}{4}$ ; the model of the structure of zoisite is obtained by applying an  $n$ -glide reflection to this module at  $\frac{1}{4}, y, z$ .

In this case, the matching of atomic positions is somewhat more approximate, because of the difference in the  $\beta$  angle mentioned above. Nevertheless, the atomic pairing in Table 3, which corresponds to a maximal and average distance between corresponding atoms of  $0.7423 \text{ \AA}$  and  $0.3870 \text{ \AA}$  respectively, shows again that the two structures can be obtained by stacking of the same module, although it undergoes some degree of desymmetrization [41] in the real structure.

The diperiodic symmetry of the layer (nucleus  $N_0$ ) is  $P(1)2_1/m1$  (No. 15 in Volume E of the *International Tables for Crystallography* [38]). The partial operations are a translation along the  $a$

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<sup>3</sup> Ito actually used the cell parameters of epidote, isostructural with clinozoisite; moreover, he incorrectly wrote  $\cos\beta_{\text{epidote}}$  instead of  $\cos(\beta_{\text{epidote}}-\pi/2)$ .

axis in the case of clinozoisite and an  $n$ -glide reflection at  $\frac{1}{4}yz$  in the case of zoisite. The groupoid 534  
corresponding to the structure of zoisite is obtained by the set-theoretical union  $\cup_{i,j=\{0,1\}} \Phi_j^{-1}N_0\Phi_i$  535  
where  $\Phi_0$  is the identity,  $\Phi_1 = \{m_{100}|\frac{1}{2}\frac{1}{2}\frac{1}{2}\}$ , and  $N_0 = P(1)2_1/m1$  (Table 4). 536

Similarly to the case of pseudoboleite, all the operations have a continuation in the whole crystal 537  
space and are prolonged to global operations. The result is a space group of type  $Pnma$ , which is 538  
precisely that of zoisite. As the case of pseudoboleite, the groupoid of zoisite  $\mathcal{D}_C$  degenerates to 539  
space group  $\mathcal{G}_C$ . 540

Sadanaga [9] placed this groupoid again in the category of “twinned space groups”, stating that that 541  
“ $e \cup H \cong Pn$ ”. This is again correct only if  $H$  corresponds to definition D2. 542

#### 4 Polytypes beyond Ito twins 543

Space groupoids contain of all the isometries mapping substructures in the superstructure. All these 544  
isometries need not correspond to  $q = 1$  (Eq. 2). When it is the case, they correspond to Ito’s idea of 545  
“twinned space group” and his “twinning group” is then defined as the group having as generators 546  
the partial operations mapping the various modules. On the contrary, the presence of isometries 547  
which correspond to  $q > 1$  is realized when the intrinsic translational part of the operation (screw or 548  
glide component) is not compatible with the period of the structure. This was realized by Sadanaga 549  
[9] in the case of of the  $6H$  polytype of SiC. 550

Polytypes based on the close packing of equal spheres are obtained by three-dimensional stacking 551  
of diperiodic close packed sphere sheets according to the well known A-B-C mode. Spheres of 552  
equal size can be packed on a plane to form a densest sheet, the hexagonal sheet, in which each 553  
sphere is surrounded by six other spheres; the layer group is  $P(6/m)mm$  (**Figure 5**). A three- 554  
dimensional structure is obtained by stacking sheets along the out-of-plane direction. The second 555  
sheet can be put above the first sheet producing the stacking A-B (equivalent to A-C). A triperiodic 556  
structure is then obtained by repeating the stacking of layers in a zig-zag way along the direction of 557

missing periodicity of the layer. Filling the cavities formed by stacking of layers leads to a large number of derivative structures [42].

The partial operations mapping two successive layers are  $t(\frac{2}{3}\frac{1}{3}\mathbf{c}_0)$  for the stacking  $A \rightarrow B$  (or  $B \rightarrow C$  or  $C \rightarrow A$ ) and  $t(\frac{1}{3}\frac{2}{3}\mathbf{c}_0)$  for the stacking  $A \rightarrow C$  (or  $C \rightarrow B$  or  $B \rightarrow A$ ), where  $\mathbf{c}_0$  is the perpendicular distance between the plane of the layers. Sadanaga [9] considered the  $3C$  polytype of SiC (sequence ABC) as fundamental structure, and the  $6H$  polytype (ABCACB) as complex structure. The latter can be obtained from the former through a reflection  $m_{x,y,\frac{1}{2}}$  which transforms ABC to CBA followed by a translation  $\frac{1}{3}\frac{2}{3}0$  which transforms CBA to ACB. The partial operation would therefore be  $\{m_{001}|\frac{1}{3}\frac{2}{3}1\}$ , or  $g(\frac{1}{3}\frac{2}{3}0)_{x,y,\frac{1}{2}}$  in Hermann-Mauguin notation, for which  $q = 3$  (Eq. 2). Sadanaga [9] concluded that “the reflection followed by the shift will not produce an ordinary glide reflection, because the length of the shift is neither a half nor a quarter of the period in that direction”. However, if the partial operation  $\{m_{001}|\frac{1}{3}\frac{2}{3}1\}$  is prolonged to a global operation, it does generate a group, which is  $H11m$ , whose translations sublattice, because of the  $H$  centring, is a supergroup of index 3 of the translation sublattice of the  $6H$  polytype.

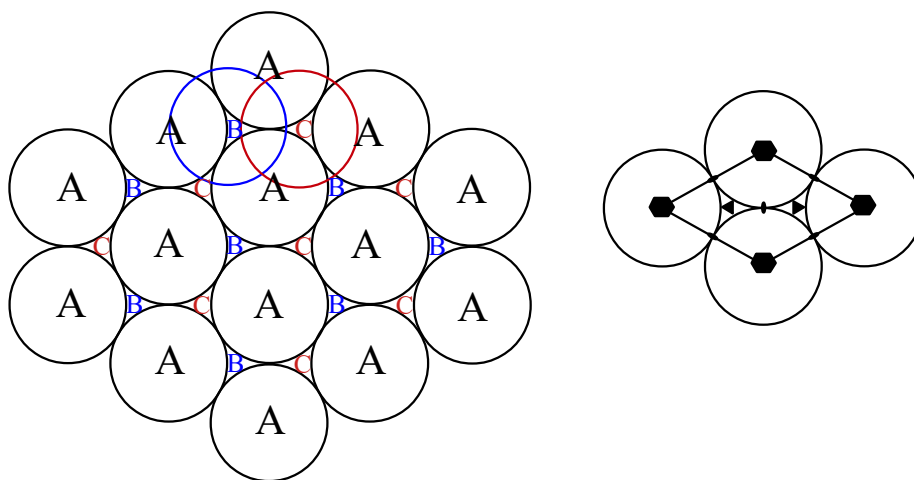


Figure 5: (Left) Geometry and notation of close packing of equal spheres. If the first layer is labelled A, two types of triangular cavities, B and C, with opposite orientation, exist. Positions indicated by the same letter are translationally equivalent, whereas positions indicated by different letters are not. (Right) A diperiodic close packed equal sphere sheet and its rotational symmetry elements acting on the plane of the sheet.

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An even simpler example is represented by the AB polytype of sphere packings, which can be considered as a complex structure obtained from a fundamental structure having a period of only one layer, A. In the AB polytype, the period corresponds to two layers and the partial operation is  $t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}})$ , which maps B onto A. We can derive the groupoid of AB polytype by set-theoretical union  $\cup_{i,j=\{0,1\}} \Phi_j^{-1}N_0\Phi_i$  where  $\Phi_0$  is the identity,  $\Phi_1 = t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}})$ , and  $N_0 = P(6/m)mm$ .

$$\mathcal{D}_C = \begin{array}{ccc} P(6/m)mm & P(6/m)mm t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}}) & \\ t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}})P(6/m)mm & t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}})P(6/m)mm t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}}) & \end{array} \quad (8)$$

The diagonal terms correspond to the two nuclei,  $N_A = P(6/m)mm$  and  $N_B = t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}})P(6/m)mm$ , *i.e.* the conjugation of  $N_A$  by the partial operation. The 24 operations of  $N_B$  (modulo the full lattice translations) are explicitly listed in Table 5, where operations in bold differ from those in  $N_A$  by full lattice translations. This means that the corresponding operations in  $N_A$  have a continuation in  $N_B$ , so that the diperiodic group of the layer pair therefore  $P(\overline{6})m2$ . Because the layers are stacked in an alternate way (A always followed by B, B always followed by A), every layer pair has the same diperiodic group  $P(\overline{6})m2$  so that all together the operations of this layer group are prolonged to global operations for the three-periodic polytype.

The next step consists in finding the operations in the extra-diagonal terms of the groupoid which also have continuation and therefore are prolonged to global operations. Because  $P(\overline{6})m2$  is a normal subgroup of  $P(6/m)mm$  (the basis vectors being in common, the index of the subgroup in the group is 2, which means that the subgroup is necessarily normal), all the operations in one of diagonal terms are present also in the other (up to translation): extending  $P(\overline{6})m2$  by any operation gives back a holohedral group. We therefore know that the space group is holohedral. There are however four types of holohedral hexagonal space group:  $P6/mmm$ ,  $P6/mcc$ ,  $P6_3/mcm$  and  $P6_3/mmc$  (No. 191 to 194). To find which one corresponds to the AB polytype one has to check the translation parts of the operations in the extra-diagonal terms of the groupoid which are prolonged to global operations. Table 6 shows the twelve operations obtained as  $P(\overline{6})m2t(\overline{\frac{2}{3}\frac{1}{3}\frac{1}{2}})$  in the first

column, and those obtained as  $P(\bar{6})m2t(\frac{2}{3}\frac{1}{3}\frac{1}{2})$  in the second column, i.e. the set of  $\Phi(B \rightarrow A)$  and  $\Phi(A \rightarrow B)$  operation the differ by a full lattice translation and, therefore, have a continuation. The whole set of global operations obtained in the two steps above correspond to  $P6_3/mmc$ , in an uncommon setting obtained by a shift of the origin of  $\frac{1}{3}\frac{2}{3}\frac{1}{4}$  as it can be verified by conjugating the matrices representing the symmetry operations of this group with  $t(\frac{1}{3}\frac{2}{3}\frac{1}{4})$ . This shift of the origin transforms the coordinates of the A and B layers from (000) and  $(\frac{2}{3}\frac{1}{3}\frac{1}{2})$  respectively to  $(\frac{1}{3}\frac{1}{3}\frac{1}{4})$  and  $(\frac{1}{3}\frac{2}{3}\frac{1}{4})$ , i.e. positions 2d and 2c in  $P6_3/mmc$ , which are precisely the atomic positions of the hcp structure.

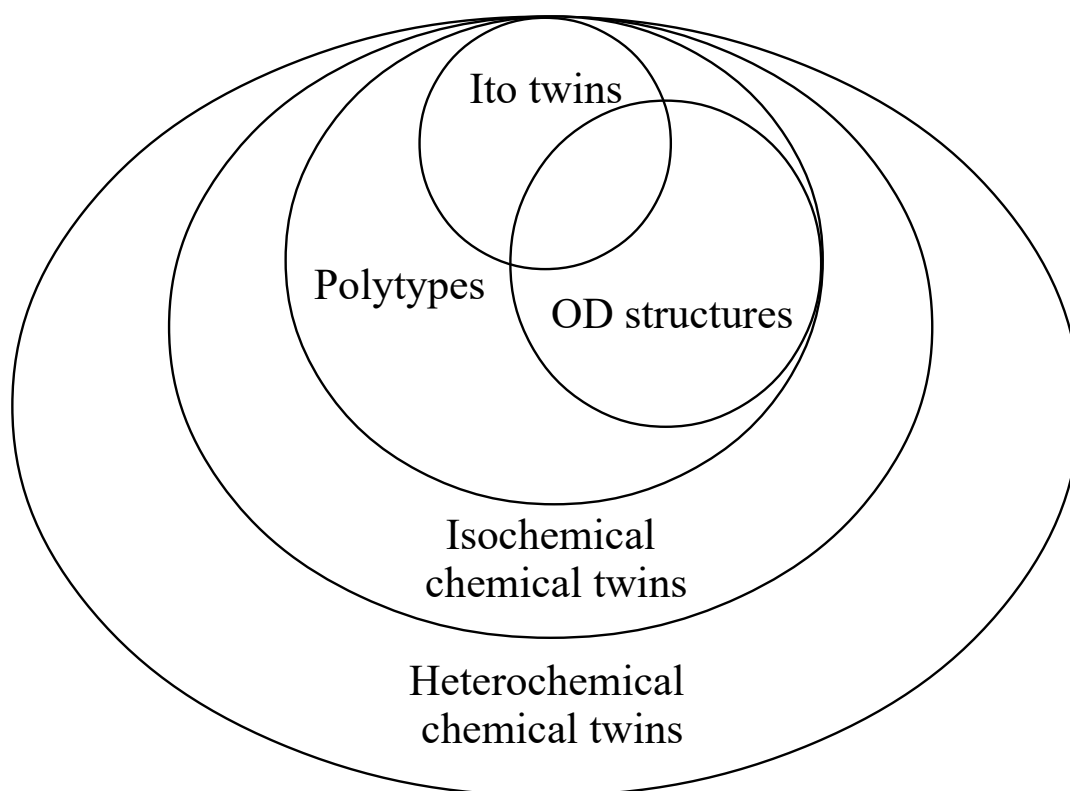
Let us now concentrate on the partial operations in the groupoid that do not have a continuation and are not prolonged to global operations (Table 7). In the groupoid of the AB polytype, the partial operations correspond to  $q = 3$  in Eq. (2) and require six application to obtain an operation of type  $(\mathbf{I}, \mathbf{t})$  i.e. a period three times larger that that of the AB polytype (six layers). These operations do not correspond to a standard Hermann-Mauguin symbol. For instance, the four three-fold screw rotations parallel to the hexagonal [001] direction ( $\{3^+_{001}|\frac{1}{3}\frac{1}{3}\frac{1}{2}\}$ ,  $\{3^-_{001}|\frac{1}{3}\frac{2}{3}\frac{1}{2}\}$ ,  $\{3^+_{001}|\frac{2}{3}\frac{1}{3}\frac{1}{2}\}$  and  $\{3^-_{001}|\frac{2}{3}\frac{2}{3}\frac{1}{2}\}$ ) have a screw component of  $\pm\frac{1}{2}$ . They can be noted as  $3_{\pm 3/2}$  meaning a  $360^\circ/N$  rotation with  $m/N$  translation part noted as  $N_m$ . The three twofold partial rotations in the (001) plane,  $\{2_{1\bar{1}0}|\frac{2}{3}\frac{1}{3}\frac{1}{2}\}$ ,  $\{2_{210}|\frac{2}{3}\frac{1}{3}\frac{1}{2}\}$  and  $\{2_{120}|\frac{2}{3}\frac{1}{3}\frac{1}{2}\}$  have a screw component of  $\frac{1}{6}$  (the first two) and  $\frac{1}{3}$  (the latter) and can be noted as  $2_{1/3}$  or  $2_{2/3}$ . None of these operations fits Ito's original idea of "twinning group": it is to deal with this type of situation that Sadanaga introduced the term "polytypism groupoid". However, one can generate a group from one representative of the partial operations, and the lattice translations  $G_C$ ; the resulting group depends on the choice of the partial operation but its translation sublattice is always a superlattice of the translation sublattice of the polytype:  $\mathcal{T}(G_T) \supset \mathcal{T}(G_C)$ . For example, by taking  $\{2_{1\bar{1}0}|\frac{2}{3}\frac{1}{3}\frac{1}{2}\}$  as generating operation one gets a space group of type  $H121$  (basis vectors in common with  $G_C$ ), which is an unconventional setting of  $C121$  to which it is transformed by a change of basis vectors  $\mathbf{a}+\mathbf{b}$ ,  $(-\mathbf{a}+\mathbf{b})/3$ ,  $\mathbf{c}$  and an origin shift  $0,0,\frac{1}{4}$ . The same result is obtained

by using one of the other two-fold rotations in the (001), as it should be expected from the fact that these operations belong to the same conjugacy class in the hexagonal system. If instead the threefold rotations are taken as generators, the result is a group of type  $P3$  with half the period along  $c$  with respect to the unit cell of the  $2H$  polytype.

Summarizing, the space groupoid of a modular structure can be classified in two types:

1. the partial operations generate a space group whose translation subgroup coincide with that of the modular structure (Ito's "twinning group");
2. the partial operations generate a space group whose translation subgroup is a supergroup of that of that of the modular structure (Sadanaga's "polytypism groupoid").

The above dichotomy exhausts the possibilities when the modules (substructures) are layers. However, in case of rods or bricks with non-trivial nuclei, operations corresponding to  $q = 1$  and operations corresponding to  $q > 1$  may actually coexist in a mixed group. One concrete example is the case of sepiolite [43], that can be interpreted as composed to rods 8-octahedra thick ( $Z_8$  modules), in which all the partial operations correspond to  $q = 1$ . However, these rods can be further subdivided into thinner rods 3-octahedra thick ( $Z_3$  modules). In this case the operations mapping  $Z_3$  modules in different regions of a same or different  $Z_8$  modules correspond to  $q > 1$ , whereas operations mapping  $Z_3$  modules in the same region of different  $Z_8$  modules correspond to  $q = 1$ . The presence of operations corresponding to  $q > 1$  was excluded by Ito, whose "twinned space groups" are space groupoids in which partial operations correspond to case 1 above. Modular structures in which the modules are related by operations defining Ito's "twinning group" and corresponding to Sadanaga's "polysynthetic structures" have been called "Ito twins" [3]: they represent a special subset of polytypes. **Figure 6** provides a revised classification with respect to the scheme proposed by [2].



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Figure 6: Euler diagram showing a revised classification of monoarchetypal modular structures, or cell-twins. The difference between polytypes and chemical twins consists in the absence or presence, respectively, of “chemical stress” (altered coordination) as the interface between modules. The latter are then classified in heterochemical and isochemical depending on whether the chemical stress does or does not result in a modification of the chemical composition. Concretely, isochemical chemical twins (after [5]) are monoarchetypal modular structures in which at the interface between modules new coordination sites are created but not occupied. Polytypes in which the partial operations mapping the modules are ordinary space-group operations, i.e. for which the group generated by partial operations corresponds to  $q = 1$  in Eq. (2), are called “Ito twins” (after [3]) and correspond to Ito’s and Sadanaga’s “polysynthetic structures”. OD structures are polytypes in which layer pairs are geometrically equivalent.

## 5. Discussion

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A complete analysis of modular crystal structures requires the identification of the space groupoid 648  
 which contains the full set of local and partial operations mapping each substructure (module) to 649  
 itself and to the other substructures. Ito was the first to realize the need of an algebraic structure 650  
 more general than a group to describe the crystal structure of a number minerals that share a 651  
 common module. Proceeding in an inductive way from the comparison with the well-know 652  
 phenomenon of twinning, he recognized that the same type of phenomenon accounts for the relation 653



between substructures in modular structures. His observations suggested that only digonal 654  
operations may actually occur; he however included also translation by a half period as a possible 655  
structure-building operation. He named “polysynthetic structures” the modular structures he 656  
studied, in analogy with the polysynthetic twinning shown by various minerals. Twinning is a 657  
macroscopic phenomenon relating the orientation of individuals or domains [8] and may or may not 658  
result in multiple individuals; cell-twinning instead acts at the unit-cell level and by definition 659  
produces a large number of modules in alternating orientation/position. The choice of the name 660  
“polysynthetic structures” seems therefore quite natural. All the modular structures studied by Ito 661  
were actually polytypes, or at least “configurational polytypes” [36], because of some deviation in 662  
the chemical composition. However, the restriction of partial operation to digonal operations covers 663  
only a subset of polytypes, and this subset has been later called “Ito twins” [3], in his name. As we 664  
have shown, the translation subgroup defined by the partial operations taken as generators coincide 665  
with translation subgroup of the modular structure, which is not the case when the partial operations 666  
are not digonal. Assigning this category a special name does therefore make sense and we consider 667  
that the term “Ito twins”, already present in the literature, should be preserved. On the other hand, 668  
the term “twinning space group” is today of only historical interest, being nothing else than a space 669  
groupoid, term which is well established in the literature. In the same way, obsolete terms like 670  
“twinning group” and “polytypism groupoid” are today a source of confusion and should not be 671  
preserved other than for their historical role. Indeed, Ito’s “twinning group”, introduced in 1938, in 672  
which the partial operations are prolonged to global operations, can be considered the ancestor of 673  
the idea of prolonging all the partial *and local* operations to global operations, introduced much 674  
later by the OD school [11], which produces the *family structure*, a fictitious structure containing all 675  
the possible positions and orientation of the substructures (modules). Sadanaga went further in this 676  
direction, by removing the restriction to digonal operations, and recognized that this extension 677  
results in a wider category. His unfortunate choice of the term “polytype” as exclusive of Ito’s 678

“polysynthetic structures”, as well as the fact that most of his research was published in rare journals hardly accessible outside Japan, did not help drawing the attention his results actually deserved. With this analysis, we hope to contribute to remediate the oblivion the Japanese research in the field has suffered.

## Appendix

The word “groupoid” has been used in the scientific literature with different meanings. Hausmann & Ore [44] defined a groupoid as set on which a binary operation acts, but neither the identity nor the inversion are included. Today, the term “magma” is preferred (see, e.g., [45]). A magma in which the binary operation is partial is a partial magma, which some authors (see, e.g., [46,47]) call “pargoid” or “partial groupoid” or “halfgroupoid”.

The term “groupoid” today is normally used to indicate a small category in which all the morphisms (mappings) are invertible. Groupoids according to Ehresmann [48] have no limitation on the kind of nuclei and are, in general, disconnected. A special case is that of Brandt groupoids, which are connected (or transitive) groupoids; this implies that in a Brandt groupoid the nuclei are all of the same type

Brandt groupoids are suitable to describe monoarchetypal modular structures; Ehresmann groupoids are instead necessary for polyarchetypal modular structures. In a similar way, twins and allotwins are described by Brandt and Ehresmann (point) groupoids respectively [23].

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**Table 1.** Comparison of the atomic coordinates of pseudoboleite (left) and of the model of pseudoboleite obtained from the structure of boleite (right), expressed in the maximal common  $P4/mmm$  subgroup (hydrogen atoms are not included because of the uncertainty on their position). Atomic pairing is determined by the closest position in the unit cell, ignoring the difference in the chemical nature of the atoms. K in pseudoboleite (Wyckoff position 2g) as well as all the Ag are replaced by Pb in the model obtained from boleite. one pair O-Cl is inverted.  $\Delta x$ ,  $\Delta y$ ,  $\Delta z$  are the differences of fractional atomic coordinates, d is the distance in Ångstroms between the corresponding atoms.

WP	Atom Pairings				Atomic Displacements							
	Atomic coordinates of pseudoboleite			Atomic coordinates of the pseudoboleite model obtained from boleite			$\Delta x$	$\Delta y$	$\Delta z$	d(Å)		
4n	Pb	0.7264	1/2	0	Pb	0.7260	1/2	0	-0.0004	0.0000	0.0000	0.0067
4i	Pb	1/2	0	0.8632	Pb	1/2	0	0.8649	0.0000	0.0000	0.0017	0.0514
8r	Pb	0.6970	0.3030	0.8485	Pb	0.6958	0.3042	0.8490	-0.0012	0.0012	0.0005	0.0301
2h	Pb	1/2	1/2	0.8868	Pb	1/2	1/2	0.8927	0.0000	0.0000	0.0059	0.1795
4m	Pb	0.2264	0	1/2	Pb	0.2260	0	1/2	-0.0004	0.0000	0.0000	0.0067
8t	Pb	1/2	0.7736	0.7500	Pb	1/2	0.7750	0.7421	0.0000	0.0014	-0.0079	0.2426
8s	Pb	0.7263	0	0.7500	Pb	0.7250	0	0.7579	-0.0013	0.0000	0.0079	0.2424
8r	Pb	0.1970	0.8030	0.3485	Pb	0.1958	0.8042	0.3490	-0.0012	0.0012	0.0005	0.0301
2g	Pb	0	0	0.3868	Pb	0	0	0.3927	0.0000	0.0000	0.0059	0.1795
4i	Pb	0	1/2	0.3632	Pb	0	1/2	0.3649	0.0000	0.0000	0.0017	0.0514
8p	Cl	0.6698	0.2104	0	O	0.6707	0.2099	0	0.0009	-0.0005	0.0000	0.0157
16u	O	0.3119	0.8120	0.9060	O	0.3118	0.8088	0.9057	-0.0001	-0.0032	-0.0003	0.0498
8s	O	0	0.3302	0.8948	O	0	0.3306	0.8949	0.0000	0.0004	0.0001	0.0068
8r	O	0.8120	0.1880	0.8441	O	0.8108	0.1892	0.8460	-0.0012	0.0012	0.0019	0.0650
8s	O	0.2104	0	0.8349	O	0.2108	0	0.8368	0.0004	0.0000	0.0019	0.0584
8t	O	1/2	0.8302	0.3948	O	1/2	0.8306	0.3949	0.0000	0.0004	0.0001	0.0068
8r	O	0.3120	0.6880	0.3441	O	0.3108	0.6892	0.3460	-0.0012	0.0012	0.0019	0.0650
8t	O	0.7104	1/2	0.3349	O	0.7108	1/2	0.3368	0.0004	0.0000	0.0019	0.0584
16u	O	0.8119	0.3120	0.4060	O	0.8118	0.3088	0.4057	-0.0001	-0.0032	-0.0003	0.0498
8q	O	0.1698	0.7104	1/2	O	0.1707	0.7099	1/2	0.0009	-0.0005	0.0000	0.0157
2g	K	0	0	0.7500	Pb	0	0	0.7284	0.0000	0.0000	-0.0216	0.6604
16u	Cu	0.0946	0.7443	0.8722	Cu	0.0953	0.7434	0.8733	0.0007	-0.0009	0.0011	0.0389
8r	Cu	0.7443	0.2557	0.9527	Cu	0.7435	0.2565	0.9529	-0.0008	0.0008	0.0002	0.0189
16u	Cu	0.5946	0.2443	0.3722	Cu	0.5953	0.2434	0.3733	0.0007	-0.0009	0.0011	0.0389
8r	Cu	0.2443	0.7557	0.4527	Cu	0.2435	0.7565	0.4529	-0.0008	0.0008	0.0002	0.0189

4l	O	0	0.3296	0	Cl	0	0.3265	0.0000	0.0000	-0.0031	0.0000	0.0474
8t	Cl	0.8684	1/2	0.9342	Cl	0.8704	1/2	0.9341	0.0020	0.0000	-0.0001	0.0307
8r	Cl	0.8778	0.1222	0.9389	Cl	0.8780	0.1220	0.9383	0.0002	-0.0002	-0.0006	0.0188
8r	Cl	0.6152	0.3848	0.9395	Cl	0.6163	0.3837	0.9408	0.0011	-0.0011	0.0013	0.0463
2g	Cl	0	0	0.8352	Cl	0	0	0.8261	0.0000	0.0000	-0.0091	0.2782
16u	Cl	0.1210	0.6152	0.8076	Cl	0.1219	0.6161	0.8076	0.0009	0.0009	0.0000	0.0195
8t	Cl	0.6694	1/2	0.8347	Cl	0.6736	1/2	0.8402	0.0042	0.0000	0.0055	0.1800
16u	Cl	0.6210	0.1152	0.3076	Cl	0.6219	0.1161	0.3076	0.0009	0.0009	0.0000	0.0195
8s	Cl	0.3684	0	0.4342	Cl	0.3704	0	0.4341	0.0020	0.0000	-0.0001	0.0307
2h	Cl	1/2	1/2	0.3352	Cl	1/2	1/2	0.3261	0.0000	0.0000	-0.0091	0.2782
8s	Cl	0.1694	0	0.3347	Cl	0.1736	0	0.3402	0.0042	0.0000	0.0055	0.1800
8r	Cl	0.8306	0.8306	0.7500	Cl	0.8579	0.8579	0.7512	0.0273	0.0273	0.0012	0.5914
8r	Cl	0.6316	0.6316	0.7500	Cl	0.6421	0.6421	0.7488	0.0105	0.0105	-0.0012	0.2300
4o	Cl	1/2	0.8296	1/2	Cl	1/2	0.8265	1/2	0.0000	-0.0031	0.0000	0.0474
8r	Cl	0.1152	0.8848	0.4395	Cl	0.1163	0.8837	0.4408	0.0011	-0.0011	0.0013	0.0463
8r	Cl	0.3778	0.6222	0.4389	Cl	0.3780	0.6220	0.4383	0.0002	-0.0002	-0.0006	0.0188
4l	Ag	0	0.1573	0	Pb	0	0.1547	0	0.0000	-0.0026	0.0000	0.0397
2f	Ag	0	1/2	0	Pb	0	1/2	0	0.0000	0.0000	0.0000	0.0000
2g	Ag	0	0	0.9214	Pb	0	0	0.9163	0.0000	0.0000	-0.0050	0.1544
2e	Ag	1/2	0	1/2	Pb	1/2	0	1/2	0.0000	0.0000	0.0000	0.0000
4o	Ag	1/2	0.6573	1/2	Pb	1/2	0.6547	1/2	0.0000	-0.0026	0.0000	0.0397
2h	Ag	1/2	1/2	0.4214	Pb	1/2	1/2	0.4163	0.0000	0.0000	-0.0050	0.1544
2h	Ag	1/2	1/2	0.7500	Pb	1/2	1/2	0.7716	0.0000	0.0000	0.0216	0.6604

**Table 2.** The space groupoid of pseudoboleite obtained by applying the partial operation  $t(\overline{1/2\ 1/2\ 1/2})$  to the nucleus  $P(4/m)mm$  expressing the diperiodic symmetry of layer common to boleite and pseudoboleite.

$P(4/m)mm$	$P(4/m)mm\ t(\overline{1/2\ 1/2\ 1/2})$
$\{1 000\}$	$\{t \overline{1/2\ 1/2\ 1/2}\}$
$\{2_{001} 000\}$	$\{2_{001} \overline{1/2\ 1/2\ 1/2}\}$
$\{4^+_{001} 000\}$	$\{4^+_{001} \overline{1/2\ 1/2\ 1/2}\}$
$\{4^-_{001} 000\}$	$\{4^-_{001} \overline{1/2\ 1/2\ 1/2}\}$

$\{2_{010} 000\}$	$\{2_{010} \overline{1/2 1/2 1/2}\}$
$\{2_{100} 000\}$	$\{2_{100} \overline{1/2 1/2 1/2}\}$
$\{2_{110} 000\}$	$\{2_{110} \overline{1/2 1/2 1/2}\}$
$\{2_{\bar{1}\bar{1}0} 000\}$	$\{2_{\bar{1}\bar{1}0} \overline{1/2 1/2 1/2}\}$
$\{\bar{1} 000\}$	$\{\bar{1} \overline{1/2 1/2 1/2}\}$
$\{m_{001} 000\}$	$\{m_{001} \overline{1/2 1/2 1/2}\}$
$\{4^+_{001} 000\}$	$\{4^+_{001} \overline{1/2 1/2 1/2}\}$
$\{4^-_{001} 000\}$	$\{4^-_{001} \overline{1/2 1/2 1/2}\}$
$\{m_{010} 000\}$	$\{m_{010} \overline{1/2 1/2 1/2}\}$
$\{m_{100} 000\}$	$\{m_{100} \overline{1/2 1/2 1/2}\}$
$\{m_{110} 000\}$	$\{m_{110} \overline{1/2 1/2 1/2}\}$
$\{m_{\bar{1}\bar{1}0} 000\}$	$\{m_{\bar{1}\bar{1}0} \overline{1/2 1/2 1/2}\}$
<b><math>t(\overline{1/2 1/2 1/2}) P(4/m)mm</math></b>	<b><math>t(\overline{1/2 1/2 1/2}) P(4/m)mm t(\overline{1/2 1/2 1/2})</math></b>
$\{t \overline{1/2 1/2 1/2}\}$	$\{1 0\}$
$\{2_{001} \overline{1/2 1/2 1/2}\}$	$\{2_{001} \bar{1}10\}$
$\{4^+_{001} \overline{1/2 1/2 1/2}\}$	$\{4^+_{001} \bar{1}00\}$
$\{4^-_{001} \overline{1/2 1/2 1/2}\}$	$\{4^-_{001} 0\bar{1}0\}$
$\{2_{010} \overline{1/2 1/2 1/2}\}$	$\{2_{010} \bar{1}0\bar{1}\}$
$\{2_{100} \overline{1/2 1/2 1/2}\}$	$\{2_{100} 0\bar{1}\bar{1}\}$
$\{2_{110} \overline{1/2 1/2 1/2}\}$	$\{2_{110} 00\bar{1}\}$
$\{2_{\bar{1}\bar{1}0} \overline{1/2 1/2 1/2}\}$	$\{2_{\bar{1}\bar{1}0} \bar{1}\bar{1}\bar{1}\}$
$\{\bar{1} \overline{1/2 1/2 1/2}\}$	$\{\bar{1} \bar{1}\bar{1}\bar{1}\}$
$\{m_{001} \overline{1/2 1/2 1/2}\}$	$\{m_{001} 00\bar{1}\}$
$\{4^+_{001} \overline{1/2 1/2 1/2}\}$	$\{4^+_{001} 0\bar{1}\bar{1}\}$



$\{\bar{4}_{001} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$	$\{\bar{4}_{001} \bar{1}0\bar{1}\}$
$\{m_{010} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$	$\{m_{010} 0\bar{1}0\}$
$\{m_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$	$\{m_{100} \bar{1}00\}$
$\{m_{110} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$	$\{m_{110} \bar{1}\bar{1}0\}$
$\{m_{1\bar{1}0} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$	$\{m_{1\bar{1}0} 000\}$

**Table 3.** Comparison of the atomic coordinates of zoisite (right) and of the model of zoisite obtained from the structure of clinozoisite (left), expressed in the maximal common  $P12_1/m$  subgroup (hydrogen atoms are not included because of the uncertainty on their position). Same conventions as in Table 1.

WP	Atom Pairings				Atomic Displacements							
	Atomic coordinates of the model of zoisite obtained from clinozoisite				Atomic coordinates of zoisite				$\Delta x$	$\Delta y$	$\Delta z$	$d(\text{\AA})$
2e	Al	0.6054	3/4	0.1708	Al	0.6055	3/4	0.1996	0.0001	0.0000	0.0289	0.2930
4f	Al	0.7500	0.0000	0.2500	Al	0.7497	0.9970	0.3103	-0.0003	-0.0030	0.0603	0.6112
2e	Al	0.8947	1/4	0.6708	Al	0.8945	1/4	0.6996	-0.0001	0.0000	0.0289	0.2923
4f	Al	0.7500	0.0000	0.7500	Al	0.7503	0.0030	0.8103	0.0003	0.0030	0.0603	0.6127
2e	Ca	0.0534	3/4	0.6299	Ca	0.0482	3/4	0.6150	-0.0052	0.0000	-0.0149	0.1842
2e	Ca	0.1303	3/4	0.9757	Ca	0.1332	3/4	0.9373	0.0029	0.0000	-0.0384	0.3850
2e	Ca	0.3697	1/4	0.4757	Ca	0.3668	1/4	0.4373	-0.0029	0.0000	-0.0384	0.3996
2e	Ca	0.4466	1/4	0.1299	Ca	0.4518	1/4	0.1150	0.0052	0.0000	-0.0149	0.1612
2e	O	0.0084	3/4	0.8302	O	0.0084	3/4	0.8361	0.0001	0.0000	0.0060	0.0605
2e	O	0.0061	1/4	0.7085	O	0.0040	1/4	0.7048	-0.0021	0.0000	-0.0037	0.0545
2e	O	0.0692	1/4	0.9659	O	0.0789	1/4	0.9431	0.0097	0.0000	-0.0228	0.2586
4f	O	0.1447	0.0129	0.7993	O	0.1413	0.0103	0.7450	-0.0034	-0.0026	-0.0542	0.5616
4f	O	0.3554	0.5129	0.2993	O	0.3587	0.5103	0.2450	0.0034	-0.0026	-0.0542	0.5447
2e	O	0.4308	3/4	0.4659	O	0.4211	3/4	0.4431	-0.0097	0.0000	-0.0228	0.2994
2e	O	0.4917	1/4	0.3302	O	0.4916	1/4	0.3361	-0.0001	0.0000	0.0060	0.0602
2e	O	0.4939	3/4	0.2085	O	0.4960	3/4	0.2048	0.0021	0.0000	-0.0037	0.0466
2e	O	0.7115	3/4	0.8639	O	0.7318	3/4	0.9246	0.0203	0.0000	0.0607	0.7423
2e	O	0.7229	3/4	0.1456	O	0.7193	3/4	0.1996	-0.0036	0.0000	0.0541	0.5423
4f	O	0.6328	0.4964	0.3226	O	0.6307	0.5006	0.3547	-0.0021	0.0042	0.0321	0.3229

4f	O	0.5995	0.4852	0.0484	O	0.6011	0.4863	0.0691	0.0016	0.0011	0.0208	0.2164
2e	O	0.7195	1/4	0.8790	O	0.7282	1/4	0.9400	0.0087	0.0000	0.0610	0.6558
2e	O	0.7885	1/4	0.3639	O	0.7682	1/4	0.4246	-0.0203	0.0000	0.0607	0.6513
2e	O	0.7805	3/4	0.3790	O	0.7718	3/4	0.4400	-0.0087	0.0000	0.0610	0.6128
2e	O	0.7302	1/4	0.1266	O	0.7275	1/4	0.1881	-0.0027	0.0000	0.0615	0.6185
4f	O	0.9006	0.9852	0.5484	O	0.8989	0.9863	0.5691	-0.0016	0.0011	0.0208	0.2081
2e	O	0.7772	1/4	0.6456	O	0.7807	1/4	0.6996	0.0036	0.0000	0.0541	0.5602
4f	O	0.8672	0.9964	0.8226	O	0.8693	0.0006	0.8547	0.0021	0.0042	0.0321	0.3335
2e	O	0.7698	3/4	0.6266	O	0.7725	3/4	0.6881	0.0027	0.0000	0.0615	0.6322
2e	Si	0.0896	1/4	0.8144	Si	0.0895	1/4	0.7824	-0.0001	0.0000	-0.0319	0.3242
2e	Si	0.5808	1/4	0.3712	Si	0.5813	1/4	0.3945	0.0006	0.0000	0.0233	0.2384
2e	Si	0.4105	3/4	0.3144	Si	0.4105	3/4	0.2824	0.0001	0.0000	-0.0319	0.3240
2e	Si	0.6587	1/4	0.0249	Si	0.6600	1/4	0.0643	0.0013	0.0000	0.0394	0.4035
2e	Si	0.9193	3/4	0.8712	Si	0.9187	3/4	0.8945	-0.0006	0.0000	0.0233	0.2356
2e	Si	0.8413	3/4	0.5249	Si	0.8400	3/4	0.5643	-0.0013	0.0000	0.0394	0.3969

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**Table 4.** The space groupoid of zoisite obtained by applying the partial operation  $\{m_{100}|\frac{1}{2}\frac{1}{2}\frac{1}{2}\}$  to the nucleus  $P(1)2_1/m1$  expressing the diperiodic symmetry of layer common to clinozoisite and zoisite.

$P(1)2_1/m1$	$P(1)2_1/m1 \{m_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$
$\{1 000\}$	$\{m_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$
$\{2_{010} 0\frac{1}{2}0\}$	$\{m_{001} \frac{1}{2}0\frac{1}{2}\}$
$\{\bar{1} 000\}$	$\{2_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$
$\{m_{010} 0\frac{1}{2}0\}$	$\{2_{001} \frac{1}{2}1\frac{1}{2}\}$
$\{m_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\} P(1)2_1/m1$	$\{m_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\} P(1)2_1/m1 \{m_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$
$\{m_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$	$\{1 000\}$
$\{m_{001} \frac{1}{2}1\frac{1}{2}\}$	$\{2_{010} 1\frac{1}{2}1\}$
$\{2_{100} \frac{1}{2}\frac{1}{2}\frac{1}{2}\}$	$\{\bar{1} 111\}$
$\{2_{001} \frac{1}{2}1\frac{1}{2}\}$	$\{m_{010} 03/20\}$

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**Table 5.** The local operations in the nucleus  $N_B$  of the layer B the of AB ( $2H$ ) polytype obtained by conjugation  $t(\frac{2}{3}\frac{1}{3}\frac{1}{2})N_A t(\frac{2}{3}\frac{1}{3}\frac{1}{2})$  of nucleus of the A layer. In bold the operations that differ by a full lattice translation and have a continuation from one layer to other, being therefore prolonged to global operations of the polytype. The infinitely many operations of  $N_A$  are represented by the 24 operations with a reduced translational component (zero in this case). The operations are represented by their Seitz symbol.

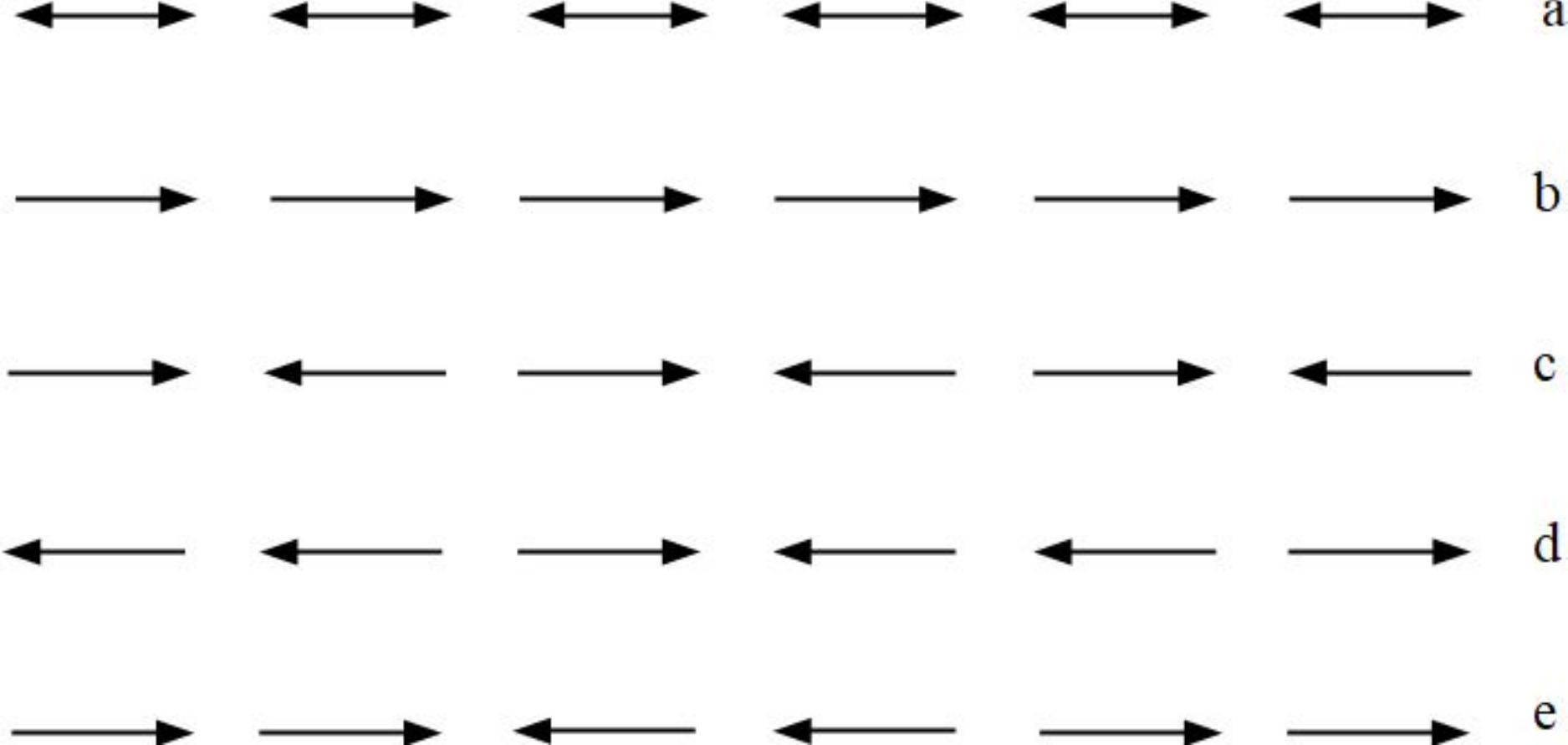
No.	Operation	No.	Operation
<b>1</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{1 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{1 000\}$	13	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{\bar{1} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{\bar{1} 1\frac{1}{3},\frac{2}{3},1\}$
<b>2</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{3^+_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{3^+_{001} 100\}$	14	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{3^+_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{3^+_{001} \frac{1}{3}\frac{2}{3}1\}$
<b>3</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{3^-_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{3^-_{001} 110\}$	15	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{3^-_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{3^-_{001} \frac{1}{3}\bar{1}31\}$
4	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{2_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{2_{001} 1\frac{1}{3},\frac{2}{3},0\}$	<b>16</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{m_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{m_{001} 001\}$
5	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{6^-_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{6^-_{001} \frac{1}{3}\frac{2}{3}0\}$	<b>17</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{\bar{6}^-_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{\bar{6}^-_{001} 101\}$
6	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{6^+_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{6^+_{001} \frac{1}{3}\bar{1}30\}$	<b>18</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{\bar{6}^+_{001} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{\bar{6}^+_{001} 111\}$
7	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{2_{110} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{2_{110} \frac{1}{3}\bar{1}31\}$	<b>19</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{m_{110} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{m_{110} 110\}$
8	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{2_{100} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{2_{100} \frac{1}{3}\frac{2}{3}1\}$	<b>20</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{m_{100} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{m_{100} 100\}$
9	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{2_{010} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{2_{010} 1\frac{1}{3},\frac{2}{3},1\}$	<b>21</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{m_{010} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{m_{010} 000\}$
<b>10</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{2_{\bar{1}\bar{0}} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{2_{\bar{1}\bar{0}} 111\}$	22	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{m_{\bar{1}\bar{0}} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{m_{\bar{1}\bar{0}} \frac{1}{3}\bar{1}30\}$
<b>11</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{2_{120} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{2_{120} 101\}$	23	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{m_{120} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{m_{120} \frac{1}{3}\frac{2}{3}0\}$
<b>12</b>	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{2_{210} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{2_{210} 001\}$	24	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) \{m_{210} 000\} t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) = \{m_{210} 1\frac{1}{3},\frac{2}{3},0\}$

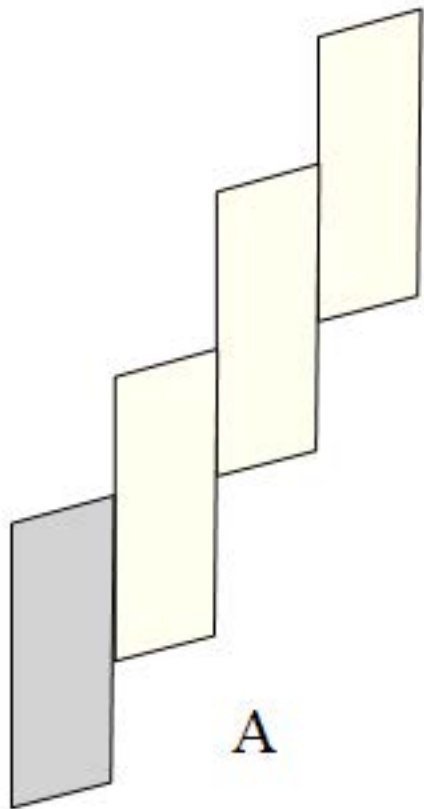
**Table 6.** The partial operations of the groupoid of the AB polytype that are prolonged to global operations. This occurs when the operations  $\Phi(B \rightarrow A)$  and  $\Phi(A \rightarrow B)$  differ by a full lattice translation. The infinitely many operations of are represented by the 12 operations with a reduced translational component. The operations are represented by their Seitz symbol.

No.	Global operations	First row of $\mathcal{D}$ : $\Phi(B \rightarrow A)$	Second row of $\mathcal{D}$ : $\Phi(A \rightarrow B)$
		$P(6/m)mm \ t(\overline{2/3^1/3^1/2})$	$t(2/3^1/3^1/2) \ P(6/m)mm$
1	$\{2_{001} 000\}$	$2/3^1/3^1/2$	$2/3^1/3^1/2$
2	$\{6^-_{001} 000\}$	$\overline{1/3^1/3^1/2}$	$2/3^1/3^1/2$
3	$\{6^+_{001} 000\}$	$\overline{1/3^2/3^1/2}$	$2/3^1/3^1/2$
4	$\{2_{110} 000\}$	$\overline{1/3^2/3^1/2}$	$2/3^1/3^1/2$
5	$\{2_{100} 000\}$	$\overline{1/3^1/3^1/2}$	$2/3^1/3^1/2$
6	$\{2_{010} 000\}$	$2/3^1/3^1/2$	$2/3^1/3^1/2$
7	$\{\overline{1} 000\}$	$2/3^1/3^1/2$	$2/3^1/3^1/2$
8	$\{\overline{3}^+_{001} 000\}$	$\overline{1/3^1/3^1/2}$	$2/3^1/3^1/2$
9	$\{\overline{3}^-_{001} 000\}$	$\overline{1/3^2/3^1/2}$	$2/3^1/3^1/2$
10	$\{m_{1\overline{1}0} 000\}$	$\overline{1/3^2/3^1/2}$	$2/3^1/3^1/2$
11	$\{m_{120} 000\}$	$\overline{1/3^1/3^1/2}$	$2/3^1/3^1/2$
12	$\{m_{210} 000\}$	$2/3^1/3^1/2$	$2/3^1/3^1/2$

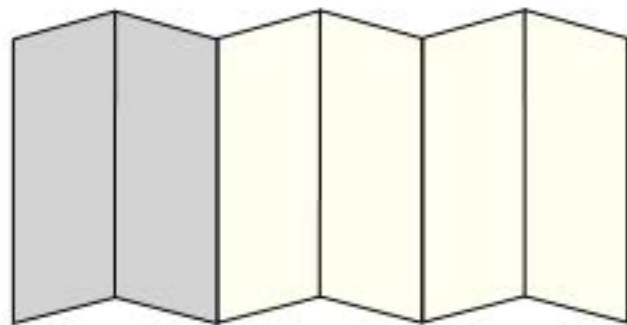
**Table 7.** The partial operations of the groupoid of the AB polytype that are not prolonged to global operations. This occurs when the operations  $\Phi(B \rightarrow A)$  and  $\Phi(A \rightarrow B)$  do not differ by a full lattice translation. The infinitely many operations of are represented by the 12 operations with a reduced translational component. The operations are represented by their Seitz symbol.

$\Phi(B \rightarrow A)$	$\Phi(A \rightarrow B)$
$P(6/m)mm \overline{t(\frac{2}{3}\frac{1}{3}\frac{1}{2})}$	$t(\frac{2}{3}\frac{1}{3}\frac{1}{2}) P(6/m)mm$
$\{\overline{t \frac{2}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{t \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{3^+_{001} \frac{1}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{3^+_{001} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{3^-_{001} \frac{1}{3}\frac{2}{3}\frac{1}{2}}\}$	$\{3^-_{001} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{2_{1\bar{1}0} \frac{1}{3}\frac{2}{3}\frac{1}{2}}\}$	$\{2_{1\bar{1}0} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{2_{120} \frac{1}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{2_{120} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{2_{210} \frac{2}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{2_{210} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{m_{001} \frac{2}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{m_{001} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{6^-_{001} \frac{1}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{\overline{6^-_{001} \frac{2}{3}\frac{1}{3}\frac{1}{2}}\}$
$\{\overline{6^+_{001} \frac{1}{3}\frac{2}{3}\frac{1}{2}}\}$	$\{\overline{6^+_{001} \frac{2}{3}\frac{1}{3}\frac{1}{2}}\}$
$\{\overline{m_{110} \frac{1}{3}\frac{2}{3}\frac{1}{2}}\}$	$\{m_{110} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{m_{100} \frac{1}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{m_{100} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$
$\{\overline{m_{010} \frac{2}{3}\frac{1}{3}\frac{1}{2}}\}$	$\{m_{010} \frac{2}{3}\frac{1}{3}\frac{1}{2}\}$





A



C

