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# Partial differential equations associated to non linear backward stochastic differential equations with Gaussian Volterra processes

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## Abstract

In this paper, we generalize to Gaussian Volterra processes the existence and uniqueness of solutions for a class of non linear backward stochastic differential equations (BSDE) and we establish the relation between the non linear BSDE and the partial differential equation (PDE). A comparison theorem for the solution of the BSDE is proved and the continuity of its law is studied.

**AMS Subject Classifications.** 35K10, 60G22, 60H05, 60H07, 60H10

**Key words.** Backward stochastic differential equation, Volterra process, Itô formula, Malliavin calculus, Comparison theorem.

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## 1 Introduction

Backward stochastic differential equations (BSDE) driven by a brownian motion have been introduced by Bismut [1] in the linear case. Non linear BSDE have been studied first by Pardoux and Peng [9]. Since then BSDE have been of interest due to the connections with partial differential equations (PDE) and their applications, especially in mathematical finance, stochastic differential games and stochastic control.

In this paper, we study the BSDE

$$\begin{cases} dY_t = -f(t, N_t, Y_t, Z_t)dt - Z_t \delta X_t, & t \in [t_0, T) \text{ with } t_0 \geq 0 \\ Y_T = g(N_T) \end{cases} \quad (1)$$

where  $X = \{X_t, 0 \leq t \leq T\}$  is a zero mean continuous Gaussian process given by

$$X_t = \int_0^T K(t, s) dW_s, \quad (2)$$

where  $W = \{W_t, 0 \leq t \leq T\}$  is a standard Brownian motion and  $K : [0, T]^2 \rightarrow \mathbb{R}$  is a square integrable kernel, i.e.  $\int_{[0, T]^2} K(t, s)^2 dt ds < +\infty$ . We assume that  $K$  is of Volterra type, i.e.  $K(t, s) = 0$

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whenever  $t < s$ . Usually, the representation (2) is called a Volterra representation of  $X$ . The kernel  $K$  in (2) defines a linear operator in  $L^2([0, T])$  given by  $(K\sigma)_t = \int_0^t K(t, s)\sigma_s ds$ ,  $\sigma \in L^2([0, T])$ . The process  $(N_t, 0 \leq t \leq T)$  is given by

$$N_t := \int_0^t \sigma_s \delta X_s = \int_0^t (K_t^* \sigma)_s \delta W_s, \quad (3)$$

with  $\sigma$  being a deterministic function and  $K^*$  the adjoint operator of  $K$  ([3], lemma 1) given by (11) (see also [6]).  $f$  is called the generator of the BSDE,  $g(N_T)$  the terminal condition.

In [3]  $K$  is called *regular* if it satisfies

(H)  $\int_0^T |K|((s, T], s)^2 ds < \infty$ , where  $|K|((s, T], s)$  denotes the total variation of  $K(\cdot, s)$  on  $(s, T]$ .

We assume the following condition on  $K(t, s)$  which is more restrictive than (H) ([3], [4]):

(H1)  $K(t, s)$  is continuous for all  $0 < s \leq t < T$  and continuously differentiable in the variable  $t$  in  $0 < s < t < T$ ,

(H2) For some  $0 < \alpha, \beta < \frac{1}{2}$ , there is a finite constant  $c > 0$  such that

$$\left| \frac{\partial K}{\partial t}(t, s) \right| \leq c(t-s)^{\alpha-1} \left( \frac{t}{s} \right)^\beta, \text{ for all } 0 < s < t < T.$$

Examples of Gaussian Volterra processes that satisfy (H1) and (H2) are multi-fractional Brownian motion (mbm), multi-fractional Ornstein-Uhlenbeck process and Liouville multi-fractional Brownian motion ([6]).

The covariance function of  $X$  is given by

$$R(t, s) := \mathbb{E}X_t X_s = \int_0^{\inf(t, s)} K(t, u)K(s, u)du. \quad (4)$$

The aim of this paper is to study the nonlinear BSDE (1) and we establish the relation to the associated partial differential equation (6) that opens the possibility to solve the BSDE by means of classical or viscosity solutions of the PDE. This generalizes a result in [5] obtained for fractional brownian motion. For the existence and uniqueness for the solution of the BSDE (1), essentially two methods were applied. The existence and uniqueness of the solution of (1) is addressed in Theorem 2.1 by means of the associated PDE. Another proof will be treated in a separate paper without making reference to this PDE, but with probabilistic and functional theoretic methods. In this paper, we prove in Theorem 2.3 that  $Z_t = -\sigma_t \frac{\partial}{\partial x} u(t, x)$  for BSDE's with generators  $f \in C^{0,1}([t_0, T] \times \mathbb{R}^3)$  of polynomial growth and under the injectivity hypothesis for the adjoint operator to  $K$ . We discuss this hypothesis in Remark 4.5. This hypothesis is satisfied for the mbf (with  $H > \frac{1}{2}$ ) and comes from the preliminary Lemma 4.4 that shows a kind of orthogonality between Lebesgue and divergence integrals. The proof of this Lemma generalizes the proof given by Y. Hu and S. Peng in 2009 for fractional brownian motion ([5]). The proof of Lemma 4.4 depends itself on Proposition 4.2 where we show that, for any continuous function of exponential growth  $h$ ,  $h(N_t)$  admits a representation as a divergence integral of the heat kernel operator evaluated for  $N$ . We generalize in Theorem 2.5 a comparison theorem, known for BSDE with respect to brownian motion (see for example [10]), to the solution of the BSDE (1) and study the continuity of the law of  $Y$  in Theorem 2.6.

Here is the organisation of the paper. In Section 2 we state the main results on the solution of (1). In Section 3 we give some definitions and complements on the Skorohod integral with respect to Volterra processes. Section 4 is devoted to the proofs of the results and contains other results of independent interest, like an Itô formula for  $N$  proven in the framework of Malliavin calculus (Theorem 4.1) and a transfer formula (Proposition 4.6).

## 2 Statement of the main results

We consider  $X$  defined on the probability space  $(\Omega, \mathcal{F}, P)$  and given by (2). Let  $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F}, t \in [0, T]\}$  the filtration generated by  $X$  and augmented by the  $P$ -null sets. Let  $N$  be given by (3), where  $\sigma$  is a bounded function on  $[0, T]$  and suppose that  $\frac{d}{dt} \text{Var}(N_t) > 0$  for all  $t \in (0, T)$ .

Let  $t_0 \geq 0$  be fixed, and denote by  $\mathbb{L}^2(\mathbb{F}, \mathbb{R})$  the set of  $\mathbb{F}$ -adapted  $\mathbb{R}$ -valued processes  $Z$  such that  $\mathbb{E}(\int_{t_0}^T |Z_t|^2 dt) < \infty$ . We consider the non linear BDSE for the processes  $Y = (Y_t, t \in [t_0, T]) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R})$  and  $Z = (Z_t, t \in [t_0, T]) \in \mathbb{L}^2(\mathbb{F}, \mathbb{R})$  given by

$$Y_t = g(N_T) + \int_t^T f(s, N_s, Y_s, Z_s) ds + \int_t^T Z_s \delta X_s, \quad t \in [t_0, T], \quad (5)$$

We show that (5) is associated to the following second order PDE with terminal condition

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = -\frac{1}{2} \frac{d}{dt} \text{Var}(N_t) \frac{\partial^2 u}{\partial x^2}(t, x) - f(t, x, u(t, x), -\sigma_t \frac{\partial u}{\partial x}(t, x)), \\ u(T, x) = g(x), \quad (t, x) \in [t_0, T] \times \mathbb{R} \end{cases} \quad (6)$$

The association of this PDE to the BSDE (5) is proven by means of the Itô formula for the class of functions  $F \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$  that satisfy, together with their partial derivatives, the growth condition

$$\max \left( \left| F(t, x) \right|, \left| \frac{\partial F}{\partial t}(t, x) \right|, \left| \frac{\partial F}{\partial x}(t, x) \right|, \left| \frac{\partial^2 F}{\partial x^2}(t, x) \right| \right) \leq c e^{\lambda x^2}, \quad (7)$$

for all  $t \in [0, T]$  and  $x \in \mathbb{R}$ , where  $c, \lambda$  are positive constants such that  $\lambda < \frac{1}{4} \left( \sup_{t \in [0, T]} \text{Var}(N_t) \right)^{-1}$ .

This implies

$$\mathbb{E} \left| F(t, N_t) \right|^2 \leq c^2 \mathbb{E} \exp(2\lambda |N_t|^2) < \infty, \quad (8)$$

and the same property holds for  $\partial/\partial t F(t, x)$ ,  $\partial/\partial x F(t, x)$  and  $\partial^2/\partial x^2 F(t, x)$ .

The main results of this paper is stated below:

**Theorem 2.1.** *If (6) has a classical solution  $u(t, x)$  that satisfies (7) with  $F$  replaced by  $u$ , then  $(Y_t, Z_t) := (u(t, N_t), -\sigma_t \frac{\partial u}{\partial x}(t, N_t))$ ,  $t \in (t_0, T)$  satisfies (5) and  $Y, Z \in \mathbb{L}^2(\mathbb{F}, \mathbb{R})$ .*

**Remark 2.2.** *The mild (or evolution) solution of (6) is given by*

$$u(t, x) = \int G(T, t, x - y) g(y) dy - \int_t^T \int G(s, t, x - y) f(s, y, u(s, y), -\sigma_s \frac{\partial}{\partial x} u(s, y)) dy ds, \quad (9)$$

where

$$G(s, t, x) = (2\pi)^{-1/2} (\text{Var}(N_s) - \text{Var}(N_t))^{-1/2} \exp \left( -\frac{x^2}{2(\text{Var}(N_s) - \text{Var}(N_t))} \right).$$

*In fact, theorem 4.1 ([6]) applied to  $n = 1$  gives  $u(t, x) = \int G(T, t, x - y) g(y) dy$  in the linear case and (9) follows by classical arguments for the mild form of a nonlinear PDE.*

**Theorem 2.3.** *Suppose that  $K_T^*$  is injective. Let (5) have a solution of the form  $(Y_t = u(t, N_t), Z_t = v(t, N_t))$ , where  $u$  is as in Theorem 2.1 and  $v \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$  satisfies (7). Suppose that  $f \in \mathcal{C}^{0,1}([0, T] \times \mathbb{R}^3)$  and of polynomial growth. Then  $v(t, x) = -\sigma_t \frac{\partial}{\partial x} u(t, x)$ .*

**Remark 2.4.** If the PDE (6) has a unique classical solution, then under the hypotheses of Theorem 2.3 the BSDE (5) has a unique solution with  $Y_t = u(t, N_t)$ ,  $Z_t = v(t, N_t)$ .

In fact, let us suppose that (5) has two solutions  $(Y_t^1, Z_t^1)$  and  $(Y_t^2, Z_t^2) : Y_t^1 = u^1(t, N_t)$ ,  $Z_t^1 = -\sigma_t \frac{\partial}{\partial x} u^1(t, N_t)$  and  $Y_t^2 = u^2(t, N_t)$ ,  $Z_t^2 = -\sigma_t \frac{\partial}{\partial x} u^2(t, N_t)$ . By uniqueness of solutions of (6),  $u^1(t, \cdot) = u^2(t, \cdot)$  and therefore, for  $t \in [t_0, T]$   $Y_t^1 = Y_t^2$ ,  $Z_t^1 = Z_t^2$ .

**Theorem 2.5.** For  $i = 1, 2$  we consider the BSDE's

$$Y_t^i = g^i(N_T^i) + \int_t^T f^i(s, N_s^i, Y_s^i, Z_s^i) ds + \int_t^T Z_s^i \delta X_s^i, \quad 0 < t_0 \leq t \leq T, \quad (10)$$

where  $N_t^i = \int_0^t \sigma_s^i \delta X_s^i$  and  $c_0 < \sigma^i < C_0$  for some constants  $c_0, C_0 > 0$ . Suppose that both BSDE's satisfy the hypotheses of Theorem 2.1 and Theorem 2.3. If  $f^1(\cdot, \cdot, y, z) \geq f^2(\cdot, \cdot, y, z)$  for all  $y, z$ , and  $g^1 \geq g^2$ , then  $Y_t^1 \geq Y_t^2$   $P - a.s.$  for all  $t \in [t_0, T]$ .

**Theorem 2.6.** Suppose that  $Y$  is a solution of the BSDE (5) and  $u$  is a classical solution of the PDE (6). If  $\frac{\partial}{\partial x} u(t, N_t) \neq 0$  for all  $t \in [t_0, T]$ , the law of  $Y$  is absolutely continuous.

### 3 Preliminaries

In this section, we recall important definitions and results concerning the Malliavin calculus for Volterra process. These results will be used to study the BSDE (5).

Let  $\mathcal{E}$  be the set of step functions of  $[0, T]$ , and let  $K_T^* : \mathcal{E} \rightarrow L^2([0, T])$  be defined by

$$(K_T^* \sigma)_u := \int_u^T \sigma_s \frac{\partial K}{\partial s}(s, u) ds. \quad (11)$$

**Remarks 3.1.** a) For  $s > t$ , we have  $(K_T^* \sigma 1_{[0, t]})_s = 0$ , and we will denote  $(K_T^* \sigma 1_{[0, t]})_s$  by  $(K_t^* \sigma)_s$  where  $K_t^*$  is the adjoint of the operator  $K$  in the interval  $[0, t]$ .

b) If  $K(u, u) = 0$  for all  $u \in [0, T]$ ,  $(K_T^* 1_{[0, r]})_u = K(r, u)$  for  $u < r$ . Indeed, if  $u \leq r$ , we have

$$(K_T^* 1_{[0, r]})_u = \int_u^T 1_{[0, r]}(s) \frac{\partial K}{\partial s}(s, u) ds = \int_u^r \frac{\partial K}{\partial s}(s, u) ds.$$

Therefore

$$\begin{aligned} R(t, s) &= \mathbb{E} \left[ X_t X_s \right] = \int_0^{\inf(t, s)} (K_T^* 1_{[0, t]})_u (K_T^* 1_{[0, s]})_u du \\ &= \langle K_T^* 1_{[0, t]}, K_T^* 1_{[0, s]} \rangle_{L^2([0, T])}. \end{aligned}$$

For  $\sigma, \tilde{\sigma} \in \mathcal{E}$  this may be extended to

$$X(\sigma) := \int_0^T (K_T^* \sigma)_s dW_s \quad \text{and} \quad E \left[ X(\sigma) X(\tilde{\sigma}) \right] = \langle K_T^* \sigma, K_T^* \tilde{\sigma} \rangle_{L^2([0, T])}.$$

**Definition 3.2.** Let  $\mathcal{H}$  be the closure of the linear span of the indicator functions  $1_{[0,t]}$ ,  $t \in [0, T]$  with respect to the semi-inner product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} := \langle K_T^* 1_{[0,t]}, K_T^* 1_{[0,s]} \rangle_{L^2([0,T])}.$$

The operator  $K_T^*$  is an isometry between  $\mathcal{H}$  and a closed subspace of  $L^2([0, T])$ , and  $\|\cdot\|_{\mathcal{H}}$  is a semi-norm on  $\mathcal{H}$ . Furthermore, for  $\varphi, \psi \in \mathcal{H}$ ,

$$\begin{aligned} \langle K_T^* \varphi, K_T^* \psi \rangle_{L^2([0,T])} &= \int_0^T (K_T^* \varphi)_t (K_T^* \psi)_t dt \\ &= \int_0^T \int_t^T \varphi_r \frac{\partial K}{\partial r}(r, t) dr \int_t^T \psi_s \frac{\partial K}{\partial s}(s, t) ds dt \\ &= \int_0^T \int_0^T \left( \int_0^{\inf(r,s)} \frac{\partial K}{\partial r}(r, t) \frac{\partial K}{\partial s}(s, t) dt \right) \varphi_r \psi_s ds dr. \end{aligned}$$

For further use let

$$\phi(r, s) := \int_0^{\inf(r,s)} \frac{\partial K}{\partial r}(r, t) \frac{\partial K}{\partial s}(s, t) dt, \quad r \neq s. \quad (12)$$

$$\tilde{\phi}(r, s) := \int_0^{\inf(r,s)} \left| \frac{\partial K}{\partial r}(r, t) \right| \left| \frac{\partial K}{\partial s}(s, t) \right| dt, \quad r \neq s. \quad (13)$$

Note that  $\phi(r, s) = \partial^2 / \partial s \partial r R(r, s)$  ( $r \neq s$ ) ( $\phi$  may be infinite on the diagonal  $r = s$ ). Let  $|\mathcal{H}|$  be the closure of the linear span of indicator functions with respect to the semi-norm given by

$$\|\varphi\|_{|\mathcal{H}|}^2 = \int_0^T \left( \int_t^T |\varphi_r| \left| \frac{\partial K}{\partial r}(r, t) \right| dr \right)^2 dt = 2 \int_0^T dr \int_0^r ds \tilde{\phi}(r, s) |\varphi_r| |\varphi_s|. \quad (14)$$

We briefly recall some basic elements of the stochastic calculus of variations with respect to  $X$  given by (2). We refer to [2] and [7] for a more complete presentation. Let  $\mathcal{S}$  be the set of random variables of the form  $F = f(X(\varphi_1), \dots, X(\varphi_n))$ , where  $n \geq 1$ ,  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a  $\mathcal{C}^\infty$ -function such that  $f$  and its partial derivatives have at most polynomial growth, and  $\varphi_1, \dots, \varphi_n \in \mathcal{H}$ . The derivative of  $F$

$$D^X F := \sum_{j=1}^n \frac{\partial f}{\partial x_j}(X(\varphi_1), \dots, X(\varphi_n)) \varphi_j.$$

is an  $\mathcal{H}$ -valued random variable, and  $D^X$  is a closable operator from  $L^p(\Omega)$  to  $L^p(\Omega; \mathcal{H})$  for all  $p \geq 1$ . We denote by  $\mathbb{D}_{1,p}^X$  the closure of  $\mathcal{S}$  with respect to the norm

$$\|F\|_{1,p}^p = \mathbb{E}|F|^p + \mathbb{E}\|D^X F\|_{\mathcal{H}}^p. \quad (15)$$

We denote by  $Dom(\delta^X)$  the subset of  $L^2(\Omega, \mathcal{H})$  composed of those elements  $u$  for which there exists a positive constant  $c$  such that

$$\left| \mathbb{E} \left[ \langle D^X F, u \rangle_{\mathcal{H}} \right] \right| \leq c \sqrt{\mathbb{E}[F^2]}, \quad \text{for all } F \in \mathbb{D}_{1,2}^X \quad (16)$$

For  $u \in L^2(\Omega; \mathcal{H})$  in  $Dom(\delta^X)$ ,  $\delta^X(u)$  is the element in  $L^2(\Omega)$  defined by the duality relationship

$$\mathbb{E} \left[ F \delta^X(u) \right] = \mathbb{E} \left[ \langle D^X F, u \rangle_{\mathcal{H}} \right], \quad F \in \mathbb{D}_{1,2}^X. \quad (17)$$

We also use the notation  $\int_0^T u_t \delta X_t$  for  $\delta^X(u)$ . A class of processes that belong to the domain of  $\delta^X$  is given as follows: let  $\mathcal{S}^{|\mathcal{H}|}$  be the class of  $\mathcal{H}$ -valued random variables  $u = \sum_{j=1}^n F_j h_j$  ( $F_j \in \mathcal{S}$ ,  $h_j \in |\mathcal{H}|$ ).

In the same way  $\mathbb{D}_{1,p}^X(|\mathcal{H}|)$  is defined as the completion of  $\mathcal{S}^{|\mathcal{H}|}$  under the semi-norm  $\|u\|_{1,p,|\mathcal{H}|}^p := \mathbb{E} \|u\|_{|\mathcal{H}|}^p + \mathbb{E} \|D^X u\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^p$ , where

$$\|D^X u\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^2 = \int_{[0,T]^4} |D_s^X u_t| |D_{s'}^X u_{s'}| |\tilde{\phi}(s, s') \tilde{\phi}(t, t')| ds dt ds' dt'. \quad (18)$$

The space  $\mathbb{D}_{1,2}^X(|\mathcal{H}|)$  is included in the domain of  $\delta^X$ , and we have, for  $u \in \mathbb{D}_{1,2}^X(|\mathcal{H}|)$ ,

$$\mathbb{E}(\delta^X(u)^2) \leq \mathbb{E} \|u\|_{|\mathcal{H}|}^2 + \mathbb{E} \|D^X u\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^2.$$

**Remark 3.3.** Let  $N_t$  be given by (3), then we have:

$$\text{Var} N_t = \|K_t^* \sigma\|_{L^2([0,T])}^2 = \|\sigma 1_{[0,t]}\|_{\mathcal{H}}^2 = \int_0^t \int_0^t \phi(r, u) \sigma_r \sigma_u du.$$

## 4 Proofs of the main results

### 4.1 Proof of Theorem 2.1

Theorem 2.1 is proven by means of an Itô formula given in [6]. For the convenience of the reader we state it here.

We have the following theorem that is proved in [6].

**Theorem 4.1.** ([6]) Let  $N_t = \int_0^t \sigma_s \delta X_s$  ( $t \in [0, T]$ ), and suppose that the kernel  $K$  of  $X$  satisfies (H1) and (H2) and  $\sigma = \{\sigma_t, t \in [0, T]\}$  is a bounded function. If  $F \in \mathcal{C}^{1,2}([0, T] \times \mathbb{R})$  satisfies (7),  $\partial/\partial x F(\cdot, N_\cdot) \in \mathbb{D}_{1,2}^X(|\mathcal{H}|)$  and, for all  $t \in [0, T]$ ,

$$F(t, N_t) = F(0, 0) + \int_0^t \frac{\partial F}{\partial s}(s, N_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, N_s) \sigma_s \delta X_s + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, N_s) \frac{d}{ds} \text{Var}(N_s) ds.$$

Now, we are in position to prove Theorem 2.1.

Theorem 4.1 applied to  $u(t, N_t)$  gives

$$du(t, N_t) = \left( \frac{\partial u}{\partial t}(t, N_t) + \frac{1}{2} \frac{d}{dt} \text{Var}(N_t) \frac{\partial^2 u}{\partial x^2}(t, N_t) \right) dt + \sigma_t \frac{\partial u}{\partial x}(t, N_t) \delta X_t.$$

By plugging (6) into the first term on the right side of the equation above we get

$$du(t, N_t) = -f(t, N_t, u(t, N_t), -\sigma_t \frac{\partial u}{\partial x}(t, N_t)) dt + \sigma_t \frac{\partial u}{\partial x}(t, N_t) \delta X_t.$$

Therefore the pair  $(Y, Z)$  given by  $Y_t = u(t, N_t)$  and  $Z_t = -\sigma_t \frac{\partial u}{\partial x}(t, N_t)$  is a solution of (5). From (7) we conclude  $\mathbb{E} \int_{t_0}^T Z_s^2 ds < \infty$  and  $\mathbb{E} \int_{t_0}^T Y_s^2 ds < \infty$ . This proves the theorem. ■

## 4.2 Proof of Theorem 2.3

The proof of this theorem needs some auxiliary results.

Let

$$p_{|t|}(x) = \frac{1}{\sqrt{2\pi |t|}} e^{-x^2/2|t|}, \quad t \in \mathbb{R}^*. \quad (19)$$

Let  $k$  be a continuous function, such that the following is well defined:

$$P_{|t|}k(x) = \int_{\mathbb{R}} p_{|t|}(x-y)k(y)dy. \quad (20)$$

A straightforward calculation shows

$$\frac{\partial}{\partial t} P_{|t|}k(x) = \frac{1}{2} \text{sig}(t) \frac{\partial^2}{\partial x^2} P_{|t|}k(x). \quad (21)$$

The following proposition will be needed in the proof of Lemma 4.4 below.

**Proposition 4.2.** *Let  $\text{Var}(N_t)$  be increasing. Assume that  $h$  is a continuous function, and suppose that there exist positive constants  $c$  and  $\lambda' < (8\text{Var}(N_T))^{-1}$  such that  $|h(x)| \leq c e^{\lambda' x^2}$  for all  $x \in \mathbb{R}$ . Then*

$$h(N_s) = P_{\text{Var}(N_s)}h(0) + \int_0^s \frac{\partial}{\partial x} [P_{\text{Var}(N_s) - \text{Var}(N_t)}h](N_t) \sigma_t \delta X_t, \quad 0 < s < T.$$

*Proof.* We want to apply the Itô formula (Theorem 4.1) to  $F(t, N_t) = [P_{\text{Var}(N_s) - \text{Var}(N_t)}h](N_t)$ ,  $t < s$ . We begin by verifying the hypotheses of the Itô formula:

$$\begin{aligned} \bullet |F(t, x)| &= |P_{\text{Var}N_s - \text{Var}N_t}h(x)| = \int_{\mathbb{R}} p_{\text{Var}N_s - \text{Var}N_t}(x-y) |h(y)| dy \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} e^{-\frac{(x-y)^2}{2(\text{Var}N_s - \text{Var}N_t)}} |h(y)| dy \\ &\leq c \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} e^{-\frac{(x-y)^2}{2(\text{Var}N_s - \text{Var}N_t)}} e^{\lambda' y^2} dy \\ &= \frac{1}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2(\text{Var}N_s - \text{Var}N_t)}} e^{\lambda' y^2} dy \\ &= \frac{c}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} \int_{\mathbb{R}} e^{-\frac{z^2}{2(\text{Var}N_s - \text{Var}N_t)}} e^{\lambda'(x+z)^2} dz \\ &= \frac{c e^{2\lambda' x^2}}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} \int_{\mathbb{R}} e^{-\frac{z^2}{2(\text{Var}N_s - \text{Var}N_t)}} e^{2\lambda' z^2} dz \\ &\leq \frac{c e^{2\lambda' x^2}}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} \int_{\mathbb{R}} e^{-\frac{z^2}{2(\text{Var}N_s - \text{Var}N_t)}} e^{2\lambda' \frac{\text{Var}N_s}{(\text{Var}N_s - \text{Var}N_t)} z^2} dz \\ &= \frac{c e^{2\lambda' x^2}}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} \int_{\mathbb{R}} e^{-\frac{z^2}{(\text{Var}N_s - \text{Var}N_t)} (\frac{1}{2} - 2\lambda' \text{Var}N_s)} dz \\ &\leq \frac{c e^{2\lambda' x^2}}{\sqrt{2\pi(\text{Var}N_s - \text{Var}N_t)}} \sqrt{\frac{2\pi(\text{Var}N_s - \text{Var}N_t)}{1 - 4\lambda' \text{Var}N_s}} = \frac{c e^{2\lambda' x^2}}{\sqrt{1 - 4\lambda' \text{Var}N_s}} \leq M e^{\lambda x^2}, \end{aligned}$$



where the last inequality follows by choosing  $M = \frac{c}{\sqrt{1-4\lambda VarN_T}}$  and  $\lambda = 2\lambda'$ .

The proofs for the upper bounds of  $\left| \frac{\partial}{\partial x} F(t, x) \right|$  and  $\left| \frac{\partial^2}{\partial x^2} F(t, x) \right|$  are similar.

$$\begin{aligned} \bullet \left| \frac{\partial}{\partial x} F(t, x) \right| &\leq \frac{2\pi c}{(2\pi(VarN_T - VarN_t))^{\frac{3}{2}}} \int_{\mathbb{R}} |x - y| e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}} e^{\lambda' y^2} dy \\ &= \frac{2\pi c}{[2\pi(VarN_s - VarN_t)]^{\frac{3}{2}}} \int_{\mathbb{R}} |z'| e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{\lambda'(x-z')^2} dz' \\ &\leq \frac{ce^{2\lambda' x^2}}{2\pi(VarN_s - VarN_t)} \int_{\mathbb{R}} \frac{|z'|}{\sqrt{VarN_s - VarN_t}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{\frac{2\lambda' VarN_s}{(VarN_s - VarN_t)}(z')^2} dz', \quad (t < s). \end{aligned}$$

Moreover, for any  $\epsilon > 0$ , there is a constant  $K_\epsilon$  such that  $\frac{|z'|}{\sqrt{VarN_s - VarN_t}} \leq K_\epsilon e^{\frac{\epsilon}{VarN_s - VarN_t}(z')^2}$ .

Then, we get

$$\begin{aligned} \left| \frac{\partial}{\partial x} F(t, x) \right| &\leq cK_\epsilon \frac{e^{2\lambda' x^2}}{2\pi(VarN_s - VarN_t)} \int_{\mathbb{R}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}(1-2\epsilon-4\lambda' VarN_s)} dz' \\ &= cK_\epsilon \frac{e^{2\lambda' x^2}}{\sqrt{2\pi(VarN_s - VarN_t)}} \frac{1}{\sqrt{1-2\epsilon-4\lambda' VarN_s}} \leq M e^{\lambda x^2}, \quad (\text{for fixed } t, t < s) \end{aligned}$$

with a suitable constant  $M$  and  $2\lambda' = \lambda$ .

$$\begin{aligned} \bullet \left| \frac{\partial^2}{\partial x^2} F(t, x) \right| &\leq \int_{\mathbb{R}} \frac{2\pi}{[2\pi(VarN_s - VarN_t)]^{\frac{3}{2}}} e^{-\frac{(x-y)^2}{2|VarN_T - VarN_t|}} |h(y)| dy \\ &\quad + \int_{\mathbb{R}} \frac{2\pi |x - y|^2}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} e^{-\frac{(x-y)^2}{2(VarN_s - VarN_t)}} |h(y)| dy \\ &\leq 2Me^{\lambda x^2} + 2\pi c \int_{\mathbb{R}} \frac{(z')^2}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} e^{-\frac{(z')^2}{2(VarN_s - VarN_t)}} e^{\lambda'(x-z')^2} dz' \\ &\leq 2Me^{\lambda x^2} + 2\pi c \frac{e^{2\lambda' x^2}}{[2\pi(VarN_s - VarN_t)]^{\frac{5}{2}}} \int_{\mathbb{R}} (z')^2 e^{-\frac{(z')^2}{VarN_s - VarN_t}(\frac{1-4\lambda' VarN_s}{2})} dz' \\ &= 2Me^{\lambda x^2} + 2 \frac{ce^{2\lambda' x^2}}{(2\pi)^{\frac{3}{2}}(VarN_s - VarN_t)(1-4\lambda' VarN_s)^{\frac{3}{2}}} \leq M' e^{\lambda x^2}, \quad (\text{for fixed } t, t < s) \end{aligned}$$

with a suitable constant  $M'$  and  $2\lambda' = \lambda$ .

Moreover, for fixed  $t, t < s$  we have  $\frac{\partial}{\partial t} P_{VarN_s - VarN_t} h(x) = \frac{1}{2} \frac{\partial^2}{\partial x^2} P_{VarN_s - VarN_t} h(x)$ , and we conclude  $\left| \frac{\partial}{\partial t} F(t, x) \right| \leq M'' e^{\lambda x^2}$  with a suitable constant  $M''$ .

Applying the Itô formula (Theorem 4.1) to  $F(t, N_t) = [P_{Var(N_s) - Var(N_t)} h](N_t)$  on  $[0, s - \epsilon]$ , we obtain

$$\begin{aligned} F(s - \epsilon, N_{s-\epsilon}) = F(0, 0) &+ \int_0^{s-\epsilon} \frac{\partial}{\partial x} [P_{Var(N_s) - Var(N_t)} h](N_t) \sigma_t \delta X_t + \int_0^{s-\epsilon} \frac{\partial}{\partial t} [P_{Var(N_s) - Var(N_t)} h](N_t) dt \\ &+ \frac{1}{2} \int_0^{s-\epsilon} \frac{\partial^2}{\partial x^2} [P_{Var(N_s) - Var(N_t)} h](N_t) \frac{d}{dt} Var(N_t) dt. \end{aligned}$$

Furthermore, we have

$$\frac{\partial}{\partial t} P_{Var(N_s) - Var(N_t)} h(x) = \frac{\partial}{\partial \tau} P_\tau h(x) \Big|_{\tau=VarN_s - VarN_t} \frac{\partial \tau}{\partial t} \Big|_{\tau=VarN_s - VarN_t}$$

$$\begin{aligned}
&= \frac{1}{2} \frac{\partial^2}{\partial x^2} P_\tau h(x) \Big|_{\tau=Var N_s - Var N_t} \left( -\frac{d}{dt} Var(N_t) \right) \\
&= -\frac{1}{2} \frac{d}{dt} Var(N_t) \frac{\partial^2}{\partial x^2} P_{Var N_s - Var N_t} h(x). \tag{22}
\end{aligned}$$

Therefore

$$h(N_{s-\epsilon}) - P_{Var(N_s)} h(0) = \int_0^{s-\epsilon} \frac{\partial}{\partial x} [P_{Var(N_s) - Var(N_t)} h](N_t) \sigma_t \delta X_t.$$

with  $\epsilon \rightarrow 0$  we get

$$h(N_s) = P_{Var(N_s)} h(0) + \int_0^s \frac{\partial}{\partial x} [P_{Var(N_s) - Var(N_t)} h](N_t) \sigma_t \delta X_t, \tag{23}$$

since  $h$  is a continuous function and  $\frac{\partial}{\partial x} [P_{Var(N_s) - Var(N_t)} h](N_t) \in \mathbb{D}_{1,2}^X(\mathcal{H}) \subset Dom(\delta^X)$ . ■

**Remarks 4.3.** If  $Var N_t$  is decreasing, we have for all  $0 \leq t < s < T$

$$\frac{\partial}{\partial t} P_{|Var(N_s) - Var(N_t)|} h(x) = -\frac{1}{2} \frac{d}{dt} Var(N_t) \frac{\partial^2}{\partial x^2} P_{|Var(N_s) - Var(N_t)|} h(x).$$

In this case we must add the following hypothesis in the previous proposition  $\frac{Var(N_s)}{|Var(N_s) - Var(N_t)|} \geq 1$ , for all  $0 \leq t < s < T$ . We need this hypothesis to verify (7) and to apply the Itô formula.

The following lemma will play an important role in this paper.

**Lemma 4.4.** Suppose that  $K_T^*$  is injective. Let  $b(s, x)$  and  $a(s, x)$ ,  $s \in (t_0, T)$ ,  $x \in \mathbb{R}$  be continuous functions,  $b$  continuously differentiable with respect to  $x$  and of a polynomial growth. Let  $a(\cdot, N_\cdot) \in Dom(\delta^X)$ . If

$$\int_{t_0}^t b(s, N_s) ds + \int_{t_0}^t a(s, N_s) \delta X_s = 0, \text{ for all } t \in [t_0, T], \tag{24}$$

then

$$b(s, x) = a(s, x) = 0 \text{ for all } s \in (t_0, T), x \in \mathbb{R}. \tag{25}$$

*Proof.* First we show that

$$b(s, N_s) = \mathbb{E}[b(s, N_s)] + \int_0^s \left( \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \right) \sigma_u \delta X_u. \tag{26}$$

In fact, on the one hand, for all  $u \in (0, s)$ , we have

$$\begin{aligned}
\int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy &= \frac{\partial}{\partial x} \int_{\mathbb{R}} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \\
&= \frac{\partial}{\partial x} P_{Var(N_s) - Var(N_u)} b(s, N_u).
\end{aligned}$$

Moreover, we have

$$\mathbb{E}[b(s, N_s)] = \int_{\mathbb{R}} b(s, x) \frac{e^{-\frac{x^2}{2Var(N_s)}}}{\sqrt{2\pi Var(N_s)}} dx, \text{ since } N_s \sim \mathcal{N}(0, Var(N_s)).$$

Therefore,

$$\mathbb{E}[b(s, N_s)] = \int_{\mathbb{R}} b(s, x) p_{Var(N_s)}(x) dx = \int_{\mathbb{R}} b(s, x) p_{Var(N_s)}(-x) dx = P_{Var(N_s)} b(s, 0).$$

On the other hand, we apply Proposition 4.2 to  $h(x) = b(s, x)$  (for fixed  $s$ ) and

$$b(s, N_s) = P_{Var(N_s)} b(s, 0) + \int_0^s \frac{\partial}{\partial x} [P_{Var(N_s) - Var(N_u)} b(s, \cdot)] (N_u) \sigma_u \delta X_u.$$

Therefore,

$$b(s, N_s) = \mathbb{E}[b(s, N_s)] + \int_0^s \left( \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \right) \sigma_u \delta X_u.$$

Now, we show that

$$\begin{aligned} & \int_{t_0}^t \left[ \int_0^s \left( \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \right) \sigma_u \delta X_u \right] ds \\ &= \int_{t_0}^t \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy ds \delta X_u. \end{aligned} \quad (27)$$

In fact, for  $F \in L^2(\Omega, \mathcal{H})$

$$\begin{aligned} & \mathbb{E} \left[ \int_{t_0}^t \left[ \int_0^s \left( \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \right) \sigma_u \delta X_u \right] ds F \right] \\ &= \int_{t_0}^t \mathbb{E} \left[ \int_0^s \left( \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \right) \sigma_u \delta X_u F \right] ds \\ &= \int_{t_0}^t \mathbb{E} \left[ \left\langle \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \sigma_u, D^X F \right\rangle_{\mathcal{H}} \right] ds \\ &= \mathbb{E} \left[ \left\langle \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy ds \sigma_u, D^X F \right\rangle_{\mathcal{H}} \right] \\ &= \mathbb{E} \left[ \int_{t_0}^t \sigma_u \left( \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy ds \right) \delta X_u F \right]. \end{aligned}$$

From (26) and (27) we get

$$\begin{aligned} \int_{t_0}^t b(s, N_s) ds &= \int_{t_0}^t \mathbb{E}[b(s, N_s)] ds + \int_{t_0}^t \left[ \int_0^s \left( \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \right) \sigma_u \delta X_u \right] ds. \\ &= \int_{t_0}^t \mathbb{E}[b(s, N_s)] ds + \int_{t_0}^t \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy ds \delta X_u. \end{aligned}$$

Then from (24) we get

$$\begin{aligned} 0 &= \int_{t_0}^t b(s, N_s) ds + \int_{t_0}^t a(s, N_s) \delta X_s \\ &= \int_{t_0}^t \mathbb{E}[b(s, N_s)] ds + \int_{t_0}^t a(s, N_s) \delta X_s + \int_{t_0}^t \sigma_u \left[ \int_u^t \left( \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy \right) ds \right] \delta X_u \\ &= \int_{t_0}^t \left[ a(u, N_u) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{Var(N_s) - Var(N_u)} (N_u - y) b(s, y) dy ds \right] \delta X_u + \int_{t_0}^t \mathbb{E}[b(s, N_s)] ds. \end{aligned}$$

Thus

$$\int_{t_0}^t \mathbb{E}[b(s, N_s)] ds = 0, \quad (28)$$

and

$$\int_{t_0}^t \left[ a(u, N_u) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{\text{Var}(N_s) - \text{Var}(N_u)}(N_u - y) b(s, y) dy ds \right] \delta X_u = 0. \quad (29)$$

Let

$$Z(u, N_u) := a(u, N_u) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{\text{Var}(N_s) - \text{Var}(N_u)}(N_u - y) b(s, y) dy ds$$

(29) implies that  $\text{Var}(\int_{t_0}^t Z(u, N_u) \delta X_u) = 0$ . Therefore  $K_t^* Z(\cdot, N_\cdot)_u = 0$   $u$ -a.e. in  $[t_0, t]$ . By injectivity of  $K_t^*$ , we get

$$Z(u, N_u) = 0, \text{ for all } u \in [t_0, t].$$

Therefore

$$a(u, z) + \sigma_u \int_u^t \int_{\mathbb{R}} \frac{\partial}{\partial x} p_{\text{Var}(N_s) - \text{Var}(N_u)}(z - y) b(s, y) dy ds = 0,$$

for all  $z \in \mathbb{R}$  ([5]). Now by differentiating with respect to  $t$ , we get

$$\int_{\mathbb{R}} \frac{\partial}{\partial x} p_{\text{Var}(N_t) - \text{Var}(N_u)}(z - y) b(t, y) dy = 0.$$

for all  $t > u$  and  $z \in \mathbb{R}$ .

An integration by parts formula yields

$$\int_{\mathbb{R}} p_{\text{Var}(N_t) - \text{Var}(N_u)}(z - y) \frac{\partial}{\partial y} b(t, y) dy = 0.$$

Let  $u \rightarrow t$  then we see that  $\frac{\partial}{\partial y} b(t, y) = 0$  for all  $t \in (t_0, T - \epsilon)$  and  $y \in \mathbb{R}$ . This means that there is a  $b_1(t)$  such that  $b(t, y) = b_1(t)$ . Now from (28) we have

$$\int_{t_0}^t b_1(s) ds = 0, \quad t_0 \leq t \leq T - \epsilon.$$

This implies that  $b_1(t) = 0$  for all  $t \in (t_0, T)$  and accordingly  $b(t, y) = 0$  for all  $t \in (t_0, T)$  and  $y \in \mathbb{R}$ . Thus,  $a(t, y) = 0$ . ■

**Remarks 4.5.** In the proof of Lemma 4.4 we have

$$\text{Var} \int_{t_0}^t Z(u, N_u) \delta X_u = \text{Var} \int_{t_0}^t K_t^* Z(\cdot, N_\cdot)_u \delta W_u = 0$$

which implies that  $K_t^* Z(\cdot, N_\cdot)_u = \int_u^t \frac{\partial}{\partial v} K(v, u) Z(v, N_v) dv = 0$   $(u, \omega)$  - a.e. on  $[t_0, t] \times \Omega$ , for all  $t \in (t_0, T]$ . We would like to conclude that  $Z(v, N_v) = 0$   $(v, \omega)$  - a.e. on  $[t_0, t] \times \Omega$ .

An evident hypothesis is the injectivity of  $K_t^* Z(\cdot, N)_u$  as a function of  $u \in [t_0, t]$ .  
Let us look for a sufficient condition for injectivity:

$$0 = \int_{t_0}^t K_t^* Z(\cdot, N)_u du = \int_{t_0}^t Z(s, N_s, \omega) \int_{t_0}^s \frac{\partial}{\partial s} K(s, u) duds.$$

Suppose that  $K_t^* Z(\cdot, N)_u = 0$   $u - a.e.$  on  $[t_0, t]$  for all  $t \in (t_0, T]$  and  $Z > 0$  on  $(a, b) \subset [t_0, T]$ .  
Then

$$\begin{aligned} 0 &= \int_{t_0}^b K_b^* Z(\cdot, N)_u du - \int_{t_0}^a K_a^* Z(\cdot, N)_u du \\ &= \int_{t_0}^b Z(s, N_s, \omega) \int_{t_0}^s \frac{\partial}{\partial s} K(s, u) duds - \int_{t_0}^a Z(s, N_s, \omega) \int_{t_0}^s \frac{\partial}{\partial s} K(s, u) duds \\ &= \int_a^b Z(s, N_s, \omega) \int_{t_0}^s \frac{\partial}{\partial s} K(s, u) duds \end{aligned}$$

Let  $\tilde{K}_{t_0}(s) = \int_{t_0}^s \frac{\partial}{\partial s} K(s, u) du$ . If we suppose that  $\tilde{K}_{t_0}(s) \neq 0$  on  $[t_0, T]$  (and therefore does not change sign on  $[t_0, T]$ ), we obtain a contradiction. Therefore, a sufficient condition for  $K_T^*$  to be injective is  $\tilde{K}_{t_0}(s) \neq 0$  on  $[t_0, T]$ . This last hypothesis is satisfied in particular if  $\frac{\partial}{\partial s} K(s, u) > 0$  (or  $< 0$ ) for  $u \in (t_0, s)$  for all  $s \in (t_0, T)$ .

We are now ready to prove Theorem 2.3. By Theorem 4.1 we get

$$du(t, N_t) = \left[ \frac{\partial}{\partial t} u(t, N_t) + \frac{1}{2} \frac{d}{dt} \text{Var}(N_t) \frac{\partial^2}{\partial x^2} u(t, N_t) \right] dt + \sigma_t \frac{\partial}{\partial x} u(t, N_t) \delta X_t.$$

Moreover, we can write

$$u(t, N_t) - g(N_T) = - \int_t^T \left[ \frac{\partial}{\partial s} u(s, N_s) + \frac{1}{2} \frac{d}{ds} \text{Var}(N_s) \frac{\partial^2}{\partial x^2} u(s, N_s) \right] ds - \int_t^T \sigma_s \frac{\partial}{\partial x} u(s, N_s) \delta X_s.$$

Using (5), we get

$$\begin{aligned} &\int_t^T f(s, N_s, u(s, N_s), v(s, N_s)) ds + \int_t^T v(s, N_s) \delta X_s \\ &= - \int_t^T \left[ \frac{\partial}{\partial s} u(s, N_s) + \frac{1}{2} \frac{d}{ds} \text{Var}(N_s) \frac{\partial^2}{\partial x^2} u(s, N_s) \right] ds - \int_t^T \sigma_s \frac{\partial}{\partial x} u(s, N_s) \delta X_s. \end{aligned}$$

We evaluate for  $t = t_0$  and we make the subtracting with the above equation. We obtain for all  $t_0 \leq t \leq T - \epsilon$

$$\begin{aligned} &\int_{t_0}^t f(s, N_s, u(s, N_s), v(s, N_s)) ds + \int_{t_0}^t v(s, N_s) \delta X_s \\ &= - \int_{t_0}^t \left[ \frac{\partial}{\partial s} u(s, N_s) + \frac{1}{2} \frac{d}{ds} \text{Var}(N_s) \frac{\partial^2}{\partial x^2} u(s, N_s) \right] ds - \int_{t_0}^t \sigma_s \frac{\partial}{\partial x} u(s, N_s) \delta X_s. \end{aligned}$$

Using (6), we obtain

$$\int_{t_0}^t \left[ f(s, N_s, u(s, N_s), v(s, N_s)) - f(s, N_s, u(s, N_s), -\sigma_s \frac{\partial}{\partial x} u(s, N_s)) \right] ds$$

$$+ \int_{t_0}^t \left[ v(s, N_s) + \sigma_s \frac{\partial}{\partial x} u(s, N_s) \right] \delta X_s = 0.$$

Since  $u, v$  satisfy (7) and  $f \in \mathcal{C}^{0,1}([0, T] \times \mathbb{R}^3)$  is of polynomial growth, we apply now Lemma 4.4 and obtain

$$v(t, x) = -\sigma_t \frac{\partial}{\partial x} u(t, x), \quad \forall t \in (t_0, T), x \in \mathbb{R}. \quad \blacksquare$$

### 4.3 Proof of Theorem 2.5 and Theorem 2.6

We start with proving this preliminary result:

**Proposition 4.6.** (Transfer formula) *Let  $X$  be given by (2). Let  $D^W$  be the Malliavin derivative with respect to the Brownian motion. Then,  $K_T^* D^X = D^W$  on  $\mathbb{D}_{1,2}^X$ .*

Proof. We have  $F = f(X(\varphi_1), \dots, X(\varphi_n)) = f(W(K_T^* \varphi_1), \dots, W(K_T^* \varphi_n))$ ,  $n \geq 1$ .

Recall that  $X(\varphi_i) = \int_0^T \varphi_i(s) \delta X_s = \int_0^T (K_T^* \varphi_i)_s \delta W_s = W(K_T^* \varphi_i)$ ,  $i = 1, \dots, n$ ,  $\varphi_1, \dots, \varphi_n \in \mathcal{H}$ . Furthermore

$$\begin{aligned} (K_T^* D^X F)_t &= \int_t^T D_u^X F \frac{\partial K}{\partial u}(u, t) du \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\varphi_1), \dots, X(\varphi_n)) \int_t^T \varphi_i(u) \frac{\partial K}{\partial u}(u, t) du \\ &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(X(\varphi_1), \dots, X(\varphi_n)) (K_T^* \varphi_i)_t = D_t^W F, \quad t \in [0, T]. \end{aligned}$$

We extend the equality  $(K_T^* D^X F)_t = D_t^W F$  to the closure of the linear combinations of  $\mathcal{S}$  by means of the norm

$$\|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}\|D^X F\|_{\mathcal{H}}^2.$$

We have  $F_1, F_2 \in \mathcal{S}$ ,  $a_1, a_2 \in \mathbb{R}$  :  $(K_T^* D^X (a_1 F_1 + a_2 F_2))_t = D_t^W (a_1 F_1 + a_2 F_2)$  because  $K^*$ ,  $D^X$  and  $D^W$  are linear operators.

Let  $(F_n)_{n \in \mathbb{N}} \subset \text{span}(\mathcal{S})$  such that  $(F_n)_{n \in \mathbb{N}}$  converges in norm  $\|\cdot\|_{1,2}$  to  $F$ . Then,  $F \in \mathbb{D}_{1,2}^X$ . Thus,  $\|D^X F - D^X F_n\|_{\mathcal{H}} \rightarrow 0$ , with  $n \mapsto \infty$ . Thus,  $\|K_T^* D^X F - K_T^* D^X F_n\|_{L^2(\Omega, L^2(0, T))} \rightarrow 0$ , with  $n \mapsto \infty$  by isometry.

Since  $K_T^* D^X F_n = D^W F_n$ ,  $(D^W F_n)_{n \in \mathbb{N}}$  is a convergent sequence in  $L^2(\Omega, L^2(0, T))$ . By proposition 1.2.1 ([7]) the limit in  $L^2(\Omega, L^2(0, T))$  is  $D^W F$ . Therefore,  $K_T^* D^X = D^W$ .  $\blacksquare$

#### 4.3.1 Proof of Theorem 2.5

Let  $u^i(t, x)$  be the solution to (6) with  $f$  replaced by  $f^i$ ,  $g$  replaced by  $g^i$ . Then the solution to (10) is given by  $Y_t^i = u^i(t, N_t)$ . It suffices to prove  $u^1(t, x) \geq u^2(t, x)$ .

Denote  $\rho_t = \sqrt{\frac{d}{dt} \text{Var}(N_t)}$ , and  $\zeta_t = \int_0^t \rho_s \delta W_s$ , where  $W$  is standard Brownian motion.

Applying Itô's formula with respect to  $W$ , we have

$$du^i(t, \zeta_t) = \rho_t \frac{\partial u^i}{\partial x}(t, \zeta_t) dW_t + \frac{\partial u^i}{\partial t}(t, \zeta_t) dt + \frac{1}{2} \frac{\partial^2 u^i}{\partial x^2}(t, \zeta_t) \frac{d}{dt} \text{Var}(N_t) dt$$

$$= -f^i(t, \zeta_t, u^i(t, \zeta_t), -\sigma_t \frac{\partial u^i}{\partial x}(t, \zeta_t))dt + \rho_t \frac{\partial u^i}{\partial x}(t, \zeta_t)dW_t.$$

Thus,  $(\tilde{Y}_t^i, \tilde{Z}_t^i) = (u^i(t, \zeta_t), \rho_t \frac{\partial u^i}{\partial x}(t, \zeta_t))$  is a solution to the following BSDE

$$\begin{cases} d\tilde{Y}_t^i = -f^i(t, \zeta_t, \tilde{Y}_t^i, -\sigma_t \rho_t^{-1} \tilde{Z}_t^i)dt + \tilde{Z}_t^i dW_t, & 0 < t_0 \leq t \leq T \\ \tilde{Y}_T^i = g^i(\zeta_T) \end{cases}$$

By the classical comparison theorem ([10]),  $\tilde{Y}_t^1 \geq \tilde{Y}_t^2$  almost surely. Thus,  $u^1(t, \zeta_t) \geq u^2(t, \zeta_t)$ . Since  $\zeta_t$  is a gaussian random variable with positive variance, from lemma 3.7 ([5]), we conclude  $u^1(t, x) \geq u^2(t, x)$ . ■

### 4.3.2 Proof of Theorem 2.6

The proof of this theorem is based on an application of theorem 5.1 in [8]. In fact, we have

$$\begin{aligned} \| D^X Y_t \|_{\mathcal{H}} &= \| (K_T^* D^X Y)_t \|_{L^2[0, T]} \\ &= \| D^W Y_t \|_{L^2[0, T]}, \quad (\text{by Proposition 4.6}). \end{aligned}$$

Moreover

$$D_s^W Y_t = D_s^W u(t, N_t) = \frac{\partial}{\partial x} u(t, N_t) (K_t^* \sigma)_s.$$

Therefore

$$\begin{aligned} \| D^X Y_t \|_{\mathcal{H}} &= \left\| \frac{\partial}{\partial x} u(t, N_t) (K_t^* \sigma) \right\|_{L^2[0, T]} \\ &= \int_0^T \left( \frac{\partial}{\partial x} u(t, N_t) \right)^2 (K_t^* \sigma)_s^2 ds \\ &= \left( \frac{\partial}{\partial x} u(t, N_t) \right)^2 \text{Var} N_t. \end{aligned}$$

Under the hypothesis  $\text{Var} N_t > 0$  and  $\frac{\partial}{\partial x} u(t, N_t) \neq 0$ , we apply theorem 5.1 in [8], we get  $\| D^X Y_t \|_{\mathcal{H}} > 0$ . ■

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## References

- [1] J. M. BISMUT. Linear quadratic stochastic control with random coefficients. *SIAM. J. Control and Optimization*, 14(3):419–444, 1976.

- [2] I. NOURDIN AND GIOVANNI PECCATI. *Normal approximations with Malliavin calculus*. From Stein's method to universality. Cambridge Tracts in Mathematics 192, Cambridge University Press, 2012.
- [3] E. ALÒS, O. MAZET AND D. NUALART. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, 29:766 – 801, 2001.
- [4] P. ČOUPEK, B. MASŁOWSKI. Stochastic evolution equations with Volterra noise. *Stochastic Processes and their Applications*, 127:877 – 900, 2017.
- [5] Y. HU, S. PENG. Backward stochastic differential equation driven by fractional brownian motion. *SIAM J. Control Optim.*, 48:1675 – 1700, 2009.
- [6] H. KNANI, M. DOZZI. Linear backward stochastic differential equations with gaussian volterra processes, <https://arxiv.org/abs/1912.00054>. submitted.
- [7] D. NUALART . *The Malliavin calculus and related topics*. 2nd edn. Springer, 2006.
- [8] D. NUALART. *Malliavin calculus and its applications*. CBMS 110, American mathematical society, 2009.
- [9] E. PARDOUX, S. PENG. Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.*, 14:55 – 61, 1990.
- [10] J. ZHANG. *Backward Stochastic differential equations, probability theory and stochastic modelling*. Springer Verlag, 2017.