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► **To cite this version:**

Zongming Guo, Xia Huang, Dong Ye. Existence and nonexistence results for a weighted elliptic equation in exterior domains. *Zeitschrift für Angewandte Mathematik und Physik*, Springer Verlag, 2020, 71 (4), 10.1007/s00033-020-01338-0. hal-02894451

HAL Id: hal-02894451

<https://hal.univ-lorraine.fr/hal-02894451>

Submitted on 8 Jul 2020

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EXISTENCE AND NONEXISTENCE RESULTS FOR A WEIGHTED ELLIPTIC EQUATION IN EXTERIOR DOMAINS

ZONGMING GUO, XIA HUANG, AND DONG YE

ABSTRACT. We consider positive solutions to the weighted elliptic problem

$$-\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B,$$

where B is the standard unit ball of \mathbb{R}^N . We give a complete answer for the existence question for $N' := N + \theta > 2$ and $p > 0$. In particular, for $N' > 2$ and $\tau := \ell - \theta > -2$, it is shown that for $0 < p \leq p_s := \frac{N'+2+2\tau}{N'-2}$, the only nonnegative solution to the problem is $u \equiv 0$. This nonexistence result is new, even for the classical case $\theta = \ell = 0$ and $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$, $N \geq 3$. The interesting feature here is that we do not require any behavior at infinity or any symmetry assumption.

1. INTRODUCTION

We study positive solutions to the problem

$$(1.1) \quad -\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B$$

where $B = \{x \in \mathbb{R}^N, |x| < 1\}$, $p > 0$.

For $\theta = \ell = 0$, the classical elliptic equation

$$(1.2) \quad -\Delta u = u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B$$

has been studied intensively, see for instance [1, 2, 3, 4, 7, 11, 13, 15] and the references therein.

It is well-known from [1, 2, 7, 11] that (1.2) does not admit any positive solution provided $0 < p \leq \frac{N}{N-2}$ for $N \geq 3$; or $p > 0$ for $N \leq 2$. It is also well-known from [3] that for any supercritical exponent $p > \frac{N+2}{N-2}$ and $N \geq 3$, the problem (1.2) admits a unique positive radial solution u satisfying

$$u(x) = O(|x|^{2-N}) \quad \text{as } |x| \rightarrow \infty.$$

For $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$, $N \geq 3$, Theorem 2.2 in [13] showed that (1.2) does not admit any positive radial solution. However, the following natural question remained open:

(*) Is there any *non radial* positive solution of (1.2) for $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$, $N \geq 3$?

It's worthy to mention two closely linked results, which seem to suggest that the answer for (*) could be complex. Firstly, there exist infinitely many *non radial* slow decay positive solutions to (1.2) for any supercritical exponent p . More precisely, let Ω be a bounded smooth domain in \mathbb{R}^N ($N \geq 3$), it was proved in [4] that for the homogenous Dirichlet problem $-\Delta u = u^p$ in $\mathbb{R}^N \setminus \Omega$ with $p > \frac{N+2}{N-2}$, there exist always infinitely many positive solutions such that

$$u(x) = O\left(|x|^{-\frac{2}{p-1}}\right) \quad \text{as } |x| \rightarrow \infty.$$

1991 *Mathematics Subject Classification.* 35B09, 35J60, 35J25, 35R37.

Key words and phrases. Weighted elliptic equation, Existence and non-existence, Critical exponent, Exterior domain.

Secondly, by Proposition 6.1 in [15], Zhang showed that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$), $p > \frac{N+\ell}{N-2}$, the problem

$$-\Delta u = |x|^\ell u^p \quad \text{in } \mathbb{R}^N \setminus \Omega, \quad u = f \quad \text{on } \partial\Omega$$

admits a positive solution, if f is a *nontrivial* nonnegative function in $L^\infty(\partial\Omega)$ and $\|f\|_\infty$ is small enough.

In this paper, we study the existence of positive solutions to the more general equation (1.1), under the following basic assumption:

$$(1.3) \quad N' := N + \theta > 2, \quad \tau := \ell - \theta > -2.$$

The elliptic problem like $-\operatorname{div}(a(x)\nabla u) = b(x)f(x, u)$ can be used to modelling some physical phenomena related to the equilibrium of continuous media, see [5].

We will show that

$$p_s := \frac{N' + 2 + 2\tau}{N' - 2}$$

is effectively the critical exponent for (1.1). In particular, when $\theta = \ell = 0$, we prove that the answer to the question (*) is indeed negative.

Our main results are the follows.

Theorem 1.1. *Let $N' > 2$ and $\tau > -2$. The only nonnegative solution of (1.1) is $u_p \equiv 0$ provided $0 < p \leq p_s$.*

The interesting feature here is that we do not require any behavior at infinity or any symmetry assumption for u . Note that by scaling, our results hold obviously on $\mathbb{R}^N \setminus B_R$, $\forall R > 0$.

For $p \leq \frac{N'+\tau}{N'-2}$, we show a more general nonexistence result inspired by [1, 2], see Theorem 4.1 below. For $1 < p \leq p_s$, we apply the moving-sphere method to get a monotonicity property for the eventual positive solution, then we conclude by contradiction with integral estimate and stability argument. Comparing to Zhang's result mentioned above, the homogenous Dirichlet boundary condition plays a crucial role for our nonexistence result whenever $\frac{N'+\tau}{N'-2} < p \leq p_s$.

Theorem 1.2. *Let $N' > 2$, $\tau > -2$ and $p > p_s$. Then the problem (1.1) admits a unique positive radial solution $u_p \in C^2(\mathbb{R}^N \setminus B)$.*

Combining with the approaches in [1, 9, 12, 14], we show finally a complete picture for the existence of positive solution to (1.1), for all $N' > 2$ and $p > 0$.

Theorem 1.3. *Let $N' > 2$ and $p > 0$.*

- (1) *If $\tau > -2$, then (1.1) admits a positive solution if and only if $p > \frac{N'+2+2\tau}{N'-2}$.*
- (2) *If $\tau < -2$, then (1.1) admits a positive solution for any $p > 1$ and $p \in (0, 1)$.*
- (3) *If $\tau < -2$ and $p = 1$, then (1.1) admits a positive solution if and only if $\lambda_1(L_1) = 0$ in $H_0^{1,\theta}(B)$ where*

$$L_1(v) := -\operatorname{div}(|x|^\theta \nabla v) - |x|^{-(4+\tau-\theta)} v,$$

and $H_0^{1,\theta}(B)$ is the completion of $C_0^\infty(B)$ under the norm $\|u\| = \|\nabla u\|_{L^2(B, |x|^\theta dx)}$.

- (4) *Let $\tau = -2$. then (1.1) has a positive solution if and only if $p = 1$, $N' \geq 4$; or $p > 1$.*

In our analysis, we shall use the Kelvin transformation

$$(1.4) \quad v(y) = |x|^{N'-2} u(x) \quad \text{where } y = \frac{x}{|x|^2}.$$

If u is a solution to (1.1), then v satisfies the problem

$$(1.5) \quad -\operatorname{div}(|y|^\theta \nabla v) = |y|^\sigma v^p \quad \text{in } B \setminus \{0\}, \quad v = 0 \quad \text{on } \partial B.$$

Here

$$\sigma = (N' - 2)(p - 1) - (4 + \tau - \theta).$$

By direct calculation, there holds $\tau' := \sigma - \theta > -2$ if $p > \frac{N'+\tau}{N'-2}$ and

$$(1.6) \quad p > p_s > 1 \text{ if and only if } 1 < p < p'_s := \frac{N' + 2 + 2\tau'}{N' - 2}.$$

Theorem 1.1 implies that problem (1.5) does not admit any nontrivial nonnegative solution in $C^2(B \setminus \{0\})$ for $N' > 2$, $\tau' > -2$ and $p \geq p'_s$. A similar nonexistence result was proved in [9] with the additional assumption that $v \in C^0(\overline{B})$. We prove Theorem 1.2 by using a positive radial solution $v_p \in C^2(B \setminus \{0\}) \cap C^0(\overline{B})$ to (1.5) given in [9].

2. EXISTENCE FOR SUPERCRITICAL CASE

Here we prove Theorem 1.2 by using the Kelvin transformation (1.4), and we are looking for a solution v satisfying the equation (1.5).

Recall that $\sigma = (N' - 2)(p - 1) - (4 + \tau - \theta)$ and $\tau' = \sigma - \theta > -2$. As $1 < p < p'_s$ provided $p > p_s$, it follows from [9] that (1.5) admits a positive radial solution $v_p \in C^2(B \setminus \{0\}) \cap C^0(\overline{B})$. Hence $r \mapsto v_p(r)$ satisfies the equation

$$(2.1) \quad -(r^{N'-1}v_p')' = r^{N'+\tau'-1}v_p^p(r) \text{ in } (0, 1), \quad v_p(1) = 0.$$

As $r^{N'-1}v_p'$ is decreasing in r , $N' > 2$; v_p is positive and uniformly bounded in $(0, 1)$, v_p must satisfy the following property:

$$(2.2) \quad \lim_{r \rightarrow 0} r^{N'-1}v_p'(r) = 0.$$

Then $v_p'(r) < 0$ in $(0, 1)$, this implies that $v_p(0) > 0$.

We now prove that v_p is the unique positive radial solution of (1.5) in $C^2(B \setminus \{0\}) \cap C^0(\overline{B})$. Suppose that there is another such radial solution \underline{v} , let

$$\lambda^{\frac{2+\tau'}{p-1}} = \frac{\underline{v}(0)}{v_p(0)}, \quad \text{and } w(r) := \lambda^{\frac{2+\tau'}{p-1}} v_p(\lambda r) \text{ for all } r \in [0, \lambda^{-1}].$$

Clearly, w satisfies the equations

$$(2.3) \quad -(r^{N'-1}w')' = r^{N'+\tau'-1}w^p \text{ in } (0, \lambda^{-1}), \quad w(0) = \underline{v}(0).$$

Similarly as above,

$$\lim_{r \rightarrow 0} r^{N'-1}w'(r) = 0.$$

Consider $h = w - \underline{v}$, so $h(0) = 0$ and h satisfies the following integral equation

$$(2.4) \quad h(r) = - \int_0^r t^{1-N'} \int_0^t s^{N'+\tau'-1} [w^p(s) - \underline{v}^p(s)] ds dt, \quad \forall 0 \leq r \leq \min(1, \lambda^{-1}).$$

Fix any $0 < r_0 < \min(1, \lambda^{-1})$. By the continuity of w, \underline{v} in $[0, r_0]$, the mean value theorem and Fubini's theorem imply that there exists $C > 0$ such that

$$|h(r)| \leq C \int_0^r s^{\tau'+1} |h(s)| ds, \quad \forall r \in [0, r_0].$$

Remark that $s^{\tau'+1} \in L^1_{loc}(0, \infty)$ since $\tau' > -2$, the classical Gronwall formula implies that $h \equiv 0$ in $[0, r_0]$. Therefore $w \equiv \underline{v}$ in $[0, r_0]$, hence in $[0, \min(1, \lambda^{-1})]$. This is possible if and only if $\lambda = 1$ seeing the Dirichlet boundary conditions and $v_p, \underline{v} > 0$ in B . To conclude, $\underline{v} \equiv v_p$ in $[0, 1]$.

Conversely, let $u \in C^2(\mathbb{R}^N \setminus B)$ be a positive radial solution to (1.1), then

$$-(r^{N'-1}u')' = r^{N'+\tau-1}u^p(r) \text{ in } [1, \infty).$$

So we have $r^{N'-1}u'$ is decreasing and $u(r) = O(r^{2-N'})$ at infinity. Under the Kelvin transform (1.4), we get a bounded radial function $v \in C^2(\overline{B} \setminus \{0\})$ which satisfies (2.1). As above, the boundedness of v implies (2.2), hence v is decreasing by (2.1) and $\lim_{r \rightarrow 0^+} v(r) = \ell_0 \in \mathbb{R}$ exists,

because v is bounded. In other words, $v \in C^0(\overline{B})$. The uniqueness of solution for (2.1) in $C^2(B \setminus \{0\}) \cap C^0(\overline{B})$ completes the proof. \square

3. NONEXISTENCE FOR $1 < p \leq p_s$

Let $1 < p \leq p_s$, we show first a monotonicity property for positive solution of (1.1) if it exists.

Proposition 3.1. *Let $N' > 2$, $\tau > -2$, $1 < p \leq p_s$ and $u \in C^2(\mathbb{R}^N \setminus B)$ be a nontrivial nonnegative solution to (1.1). Then $|x|^{\frac{N'-2}{2}}u(x)$ is increasing with respect to the radius $r = |x|$.*

Proof. By the strong maximum principle, $u > 0$ in $\mathbb{R}^N \setminus \overline{B}$. We introduce the following transform:

$$(3.1) \quad v(t, \omega) = r^{\frac{N'-2}{2}}u(r, \omega), \quad t = \ln r.$$

Then, v satisfies the problem

$$(3.2) \quad \begin{cases} v_{tt} + \Delta_{\mathbb{S}^{N-1}}v + e^{p_*t}v^p - \frac{(N'-2)^2}{4}v = 0, & (t, \omega) \in (0, \infty) \times \mathbb{S}^{N-1}, \\ v(0, \cdot)|_{\mathbb{S}^{N-1}} = 0, \end{cases}$$

where

$$p_* = \frac{(N' + 2 + 2\tau) - (N' - 2)p}{2} \geq 0.$$

Let $\Sigma := (0, \infty) \times \mathbb{S}^{N-1}$. For $T > 0$, define

$$\Sigma_T := (0, T) \times \mathbb{S}^{N-1}, \quad S_T = \{T\} \times \mathbb{S}^{N-1}.$$

Furthermore, we denote by $(t, \omega)_T$ the reflection of (t, ω) with respect to S_T , namely,

$$(t, \omega)_T = (2T - t, \omega).$$

Then the functions

$$v_T(t, \omega) := v((t, \omega)_T) = v(2T - t, \omega) \quad \text{and} \quad \xi_T = v - v_T$$

are well-defined in Σ_T for $T > 0$. Moreover, ξ_T satisfies the following equation in Σ_T :

$$(\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + e^{p_*t}v^p - e^{p_*(2T-t)}v_T^p - \frac{(N' - 2)^2}{4}\xi_T = 0.$$

As $p_* \geq 0$, there holds $e^{p_*(2T-t)} \geq e^{p_*t}$ for any $t \leq T$. So we obtain

$$(\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + e^{p_*t}(v^p - v_T^p) - \frac{(N' - 2)^2}{4}\xi_T \geq 0 \quad \text{in } \Sigma_T,$$

hence

$$(3.3) \quad (\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + \eta\xi_T \geq 0 \quad \text{in } \Sigma_T, \quad \xi_T \leq 0 \quad \text{on } \partial\Sigma_T$$

where

$$\eta(t, \omega) := c(t, \omega) - \frac{(N' - 2)^2}{4} \quad \text{and} \quad c(t, \omega) = pe^{p_*t}\psi^{p-1}(t, \omega),$$

with $\psi(t, \omega)$ in the interval formed by $v(t, \omega)$ and $v_T(t, \omega)$. The fact that $\xi_T \leq 0$ on $\partial\Sigma_T$ for any $T > 0$ comes easily from $u = 0$ on ∂B .

By the continuity of v , we see that $\eta(t, \cdot)$ tends uniformly to $-\frac{(N'-2)^2}{4}$ on \mathbb{S}^{N-1} as $T \rightarrow 0^+$. This implies that there exists $\delta_1 > 0$ such that

$$\eta(t, \omega) < 0 \quad \text{for } (t, \omega) \in \Sigma_T \text{ and } 0 < T \leq \delta_1.$$

Applying the maximum principle to (3.3), it follows that $\xi_T \leq 0$ in Σ_T provided $T \leq \delta_1$. As $\xi_T(0, \omega) < 0$, the strong maximum principle (see for example [8]) yields that for $0 < T \leq \delta_1$, there hold

$$(3.4) \quad \xi_T(t, \omega) < 0 \quad \text{in } \Sigma_T, \quad \frac{\partial \xi_T}{\partial t}(t, \omega) > 0 \quad \text{on } S_T.$$

Remark also that

$$(3.5) \quad \frac{\partial \xi_T}{\partial t}(T, \omega) = 2 \frac{\partial v}{\partial t}(T, \omega), \quad \forall T > 0.$$

Define now

$$T_0 = \sup\{T_1 > 0 : (3.4) \text{ holds for } T \leq T_1\}.$$

Then $T_0 \geq \delta_1$ is well-defined. We want to claim that $T_0 = \infty$.

Suppose the contrary that $T_0 < \infty$, then $\xi_{T_0} \leq 0$ in Σ_{T_0} by continuity. Seeing the elliptic equation (3.3), as $\xi_{T_0}(0, \omega) < 0$, the strong maximum principle means that (3.4) holds true for $T = T_0$. Using (3.5), we deduce that $\partial_t v > 0$ in $(0, T_0] \times \mathbb{S}^{N-1}$. Recall that $u \in C^1(\mathbb{R}^N \setminus B)$, by the compactness of \mathbb{S}^{N-1} , there exists $\varepsilon_1 \in (0, T_0)$ such that

$$\frac{\partial v}{\partial t}(t, \omega) > 0 \quad \text{for } (t, \omega) \in (0, T_0 + 3\varepsilon_1] \times \mathbb{S}^{N-1}.$$

We get then

$$(3.6) \quad \xi_T(t, \omega) < 0 \quad \text{in } \Sigma_T \setminus \Sigma_{T_0 - \varepsilon_1}, \quad \forall T_0 \leq T \leq T_0 + \varepsilon_1.$$

On the other hand, $\max_{\overline{\Sigma_{T_0 - \varepsilon_1}}} \xi_{T_0} < 0$ by continuity and (3.4) with $T = T_0$. Using the continuity of v , there exists $\varepsilon_2 > 0$ such that

$$v(t, \omega) - v(2T - t, \omega) < 0 \quad \text{in } \overline{\Sigma_{T_0 - \varepsilon_1}}, \quad \forall T_0 \leq T \leq T_0 + \varepsilon_2.$$

Combining with (3.6), let $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, there holds $\xi_T \leq 0$ in Σ_T for any $T \in [T_0, T_0 + \varepsilon]$.

Again, as $\xi_T < 0$ on $\{0\} \times \mathbb{S}^{N-1}$, the strong maximum principle applying to (3.3) shows that (3.4) remains valid for $T \in [T_0, T_0 + \varepsilon]$, this is a contradiction to the definition of T_0 . Therefore T_0 must be ∞ , hence (3.4) holds true for any $T > 0$.

Using (3.5), for any $\omega \in \mathbb{S}^{N-1}$, the function $r \mapsto r^{\frac{N'-2}{2}} u(r, \omega)$ is increasing in $[0, \infty)$. \square

Now we are ready to prove Theorem 1.1 for $1 < p \leq p_s$. Suppose that (1.1) admits a nontrivial nonnegative solution $u \in C^2(\mathbb{R}^N \setminus B)$. We claim that

$$\varphi(x) := \nabla u \cdot x + \frac{2 + \tau}{p - 1} u(x) > 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}.$$

In fact, as $1 < p \leq p_s$, we have

$$|x|^{\frac{2+\tau}{p-1}} u(x) = |x|^\beta |x|^{\frac{N'-2}{2}} u(x), \quad \text{where } \beta := \frac{2 + \tau}{p - 1} - \frac{N' - 2}{2} \geq 0.$$

So the function $|x|^{\frac{2+\tau}{p-1}} u(x)$ is increasing in $|x|$, thanks to Proposition 3.1. By the above proof, $\partial_t v > 0$ in $(0, \infty) \times \mathbb{S}^{N-1}$, hence $\varphi > 0$ in $\mathbb{R}^N \setminus \overline{B}$.

Direct calculation via scaling yields that $\varphi \in C^1(\mathbb{R}^N \setminus B)$ is a positive weak solution to the linearized equation

$$L_u(w) := -\operatorname{div}(|x|^\theta \nabla w) - p|x|^\ell u^{p-1} w = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}.$$

By classical elliptic theory, for any bounded smooth domain $\Omega \subset \mathbb{R}^N \setminus \overline{B}$,

$$(3.7) \quad \lambda_{1, \Omega}(L_u) > 0,$$

where $\lambda_{1, \Omega}(L_u)$ stands for the first eigenvalue of the Dirichlet problem

$$L_u(h) = \lambda h \quad \text{in } \Omega, \quad h = 0 \quad \text{on } \partial\Omega.$$

So u is stable in $\mathbb{R}^N \setminus \overline{B}$.

Using Theorem 3.1 in [6], we have

$$(3.8) \quad \int_{\mathbb{R}^N \setminus \overline{B}} \left(|x|^\theta |\nabla u|^2 + |x|^\ell u^{p+1} \right) dx < \infty.$$

However, Proposition 3.1 implies then the existence of $C > 0$ such that

$$u(x) \geq C|x|^{-\frac{N'-2}{2}}, \quad \forall |x| \geq 2.$$

Recall that $1 < p \leq p_s$, there holds

$$\int_{|x|>2} |x|^\ell u^{p+1} dx \geq C \int_2^\infty r^{N-1+\ell} r^{-\frac{(p+1)(N'-2)}{2}} dr = \infty,$$

since

$$N-1+\ell - \frac{(p+1)(N'-2)}{2} \geq N'+\tau-1 - \frac{(p_s+1)(N'-2)}{2} = -1.$$

This contradiction with (3.8) means that there is no nontrivial nonnegative solution to (1.1). \square

4. FURTHER RESULTS

First, we consider the existence and nonexistence of positive solution to

$$(4.1) \quad -\operatorname{div}(|x|^\theta \nabla u) \geq |x|^\ell f(u) \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R, \quad R > 0.$$

Remark that we do not impose any boundary condition or the value of R .

A first nonexistence result has been obtained by Gidas-Spruck (see Theorem A.3 in [7]), they showed that for $\theta = \ell$, $N' = N + \theta > 2$ and $f(u) = u^p$, (4.1) admits only trivial nonnegative solution $u \equiv 0$ provided $1 < p \leq \frac{N'}{N'-2}$. Since then, many existence and nonexistence results have been established for (4.1) under suitable conditions, interested readers can look at [1, 2, 10, 12, 14, 15] and the references therein.

Inspired by Theorem 8 of [1], we show a very general result to characterize the existence of positive solution to (4.1). In particular, the following result implies Theorem 1.1 for $0 < p \leq \frac{N'+\tau}{N'-2}$.

Theorem 4.1. *Assume that $N' > 2$, $\tau > -2$ and $f : (0, \infty) \rightarrow (0, \infty)$ is continuous. Then the problem (4.1) admits a positive classical solution for some $R > 0$ if and only if*

$$(4.2) \quad \int_0^\delta f(t) t^{-\frac{2(N'-1)+\tau}{N'-2}} dt < \infty,$$

for some $\delta > 0$.

Proof. We need only to adapt the approach in [1]. Note that to prove Lemma 4 in [1], the key points are the maximum principle and the rotational invariance of the equation, which obviously remain true in our setting.

Therefore, we can claim that if (4.1) admits a positive solution, then for some $R_0 > R$, there exists a C^1 positive radial function \bar{u} verifying

$$-\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

in the weak sense. Moreover, if $\bar{u}(x) = v(|x|)$, then $v \in C^2(R_0, \infty)$ and there exists $R_1 \geq R_0$ such that v is monotone for $r > R_1$. Note that v satisfies

$$(4.3) \quad -v'' - \frac{N'-1}{r} v' = r^\tau f(v) \quad \text{in } [R_1, \infty).$$

As $N' > 2$, there holds $v(r) \leq -C_1 r^{2-N'} + C_2$. Hence v is bounded in $[R_1, \infty)$ and $\lim_{r \rightarrow \infty} v(r) = \kappa \geq 0$ exists.

If $\kappa > 0$, there exists $C > 0$ such that $(r^{N'-1} v')' \leq -C r^{N'+\tau-1}$ for r large enough. Using $N'+\tau > 0$ and integrating, we deduce that $v'(r) \leq -C r^{\tau+1}$ for large r . As $\tau > -2$, this implies that $\lim_{r \rightarrow \infty} v(r) = -\infty$ which is impossible. Hence we can claim that $v'(r) < 0$ in $[R_1, \infty)$ and $\lim_{r \rightarrow \infty} v(r) = 0$.

Similarly to [1], we introduce the change of variables

$$(4.4) \quad s := r^{2-N'}, \quad w(s) = v(r).$$

Then w is a positive solution of

$$(4.5) \quad -w''(s) = as^{-\gamma}f(w) \text{ in } (0, s_0), \quad w(0) = 0,$$

for some positive constants a and s_0 , where (as $\tau > -2$)

$$\gamma = \frac{2(N' - 1) + \tau}{N' - 2} > 2.$$

Finally, applying Theorem 6 in [1], a positive solution of (4.5) exists if and only if (4.2) holds true. \square

Remark 4.2. Let $N' > 2$, $\tau > -2$, then

$$p - \frac{2(N' - 1) + \tau}{N' - 2} \leq -1 \quad \Leftrightarrow \quad p \leq \frac{N' + \tau}{N' - 2}.$$

For $f(t) = t^p$ with $0 < p \leq \frac{N' + \tau}{N' - 2}$, Proposition 4.1 implies that (4.1) cannot have positive solution for any $R > 0$, hence the equation (1.1) does not admit positive solution.

Remark 4.3. It is well known that if $N' > 2$, $\tau > -2$ and $p > \frac{N' + \tau}{N' - 2}$, $v(x) = Cr^{-\frac{\tau+2}{p-1}}$ with suitable $C(N', \tau, p) > 0$ will satisfy $-\operatorname{div}(|x|^\theta \nabla v) = |x|^\ell v^p$ in $\mathbb{R}^N \setminus \{0\}$.

Moreover, Przeradzki and Stańczy [12, 14] obtained that if $N > 2$, $\theta = 0$ and $\ell < -2$, the equation (1.1) admits a positive solution for any $p > 1$ and $p \in (0, 1)$, see Remark 5 in [14] and Corollary 1 in [12]. It's not difficult to be convinced that their approach via radial solutions remains valid to (1.1) for any $N' > 2$ and $\tau < -2$. We get then a complete answer to the existence question of positive solution of (1.1), under the only assumption $N' := N + \theta > 2$.

Proof of Theorem 1.3. The point (1) is given by Theorems 1.1-1.2. The point (2) is ensured by [12, 14] as explained above.

Consider $\tau < -2$ and $p = 1$, assume that a positive solution exists to (1.1). As the equation is linear, the existence of a positive solution to (1.1) yields a positive *radial* solution u to (1.1), and there holds $u(r) = O(r^{2-N'})$ at infinity. By the Kelvin transform (1.4), we get a solution v to (2.1) with $p = 1$. As above, there holds $v \in C^0(\overline{B})$ and v satisfies (2.2). Using the equation, we see that $v \in H_0^{1,\theta}(B)$ is a positive solution to $L_1(v) = 0$, with $\tau' = -(4 + \tau) > -2$. It means that 0 is the principal eigenvalue to L_1 in $H_0^{1,\theta}(B)$.

Fix now $\tau = -2$.

- If $0 < p < 1$, we will prove that the problem (4.1) has no solution for any $R > 0$. Suppose the contrary, as in [1], we get a positive *radial* solution to $-\operatorname{div}(|x|^\theta \nabla v) = |x|^\ell v^p$ in $\mathbb{R}^N \setminus B_{R_1}$ for some $R_1 > R$. Under the change of variable (4.4), there exists $a, s_0 > 0$ such that

$$-w'' = as^{-2}w^p \text{ in } (0, s_0), \quad w(0) = 0.$$

However, this is impossible seeing the proof of Theorem 8 (b) in [1], where we can replace N just by $N' > 2$.

- If $p = 1$, as before, the existence of a positive solution to (1.1) is equivalent to the existence of a positive *radial* solution. Assume that u is a radial solution to (1.1), let $w(s) = u(e^s)$, then w is a solution to

$$w'' + (N' - 2)w' + w = 0 \text{ in } (0, \infty), \quad w(0) = 0, \quad \lim_{s \rightarrow \infty} w(s) = 0.$$

Clearly, such a positive solution w exists if and only if the equation $\lambda^2 + (N' - 2)\lambda + 1 = 0$ has only negative roots, or equivalently, $N' \geq 4$.

- Finally let $p > 1$. Under the Kelvin transform, we are led to consider (1.5). We check readily that $\tau' = \sigma - \theta = (N' - 2)(p - 1) - 2 > -2$ and p is subcritical, since here

$$p < \frac{N' + 2 + 2\tau'}{N' - 2} \Leftrightarrow (p - 1)(N' - 2) > 0$$

There exists a positive radial solution of (1.5) as in section 2, using the result in [9]. \square

Remark 4.4. For $N' = 2$, $\tau > -2$, similarly to [1], we can show that the problem (4.1) admits a positive solution if and only if there exists $a > 0$ such that $e^{at}f(t)$ is integrable at $+\infty$. Consequently, in this case, for any $p \in \mathbb{R}$, the equation (4.1) does not admit positive solution.

Acknowledgements. Z.G is supported by NSFC (No. 11571093 and 11171092). X.H is supported by NSFC (No. 11701181). D.Y is partially supported by Science and Technology Commission of Shanghai Municipality (STCSM) (No. 18dz2271000). The authors thank the anonymous referees for their careful reading.

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