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EXISTENCE AND NONEXISTENCE RESULTS FOR A WEIGHTED ELLIPTIC EQUATION IN EXTERIOR DOMAINS

ZONGMING GUO, XIA HUANG, AND DONG YE

ABSTRACT. We consider positive solutions to the weighted elliptic problem

$$-\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B,$$

where B is the standard unit ball of \mathbb{R}^N . We give a complete answer for the existence question for $N' := N + \theta > 2$ and $p > 0$. In particular, for $N' > 2$ and $\tau := \ell - \theta > -2$, it is shown that for $0 < p \leq p_s := \frac{N'+2+2\tau}{N'-2}$, the only nonnegative solution to the problem is $u \equiv 0$. This nonexistence result is new, even for the classical case $\theta = \ell = 0$ and $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$, $N \geq 3$. The interesting feature here is that we do not require any behavior at infinity or any symmetry assumption.

1. INTRODUCTION

We study positive solutions to the problem

$$(1.1) \quad -\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B$$

where $B = \{x \in \mathbb{R}^N, |x| < 1\}$, $p > 0$.

For $\theta = \ell = 0$, the classical elliptic equation

$$(1.2) \quad -\Delta u = u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B$$

has been studied intensively, see for instance [1, 2, 3, 4, 7, 11, 13, 15] and the references therein.

It is well-known from [1, 2, 7, 11] that (1.2) does not admit any positive solution provided $0 < p \leq \frac{N}{N-2}$ for $N \geq 3$; or $p > 0$ for $N \leq 2$. It is also well-known from [3] that for any supercritical exponent $p > \frac{N+2}{N-2}$ and $N \geq 3$, the problem (1.2) admits a unique positive radial solution u satisfying

$$u(x) = O(|x|^{2-N}) \quad \text{as } |x| \rightarrow \infty.$$

For $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$, $N \geq 3$, Theorem 2.2 in [13] showed that (1.2) does not admit any positive radial solution. However, the following natural question remained open:

(*) Is there any *non radial* positive solution of (1.2) for $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$, $N \geq 3$?

It's worthy to mention two closely linked results, which seem to suggest that the answer for (*) could be complex. Firstly, there exist infinitely many *non radial* slow decay positive solutions to (1.2) for any supercritical exponent p . More precisely, let Ω be a bounded smooth domain in \mathbb{R}^N ($N \geq 3$), it was proved in [4] that for the homogenous Dirichlet problem $-\Delta u = u^p$ in $\mathbb{R}^N \setminus \Omega$ with $p > \frac{N+2}{N-2}$, there exist always infinitely many positive solutions such that

$$u(x) = O\left(|x|^{-\frac{2}{p-1}}\right) \quad \text{as } |x| \rightarrow \infty.$$

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Secondly, by Proposition 6.1 in [15], Zhang showed that for any bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$ ($N \geq 3$), $p > \frac{N+\ell}{N-2}$, the problem

$$-\Delta u = |x|^\ell u^p \quad \text{in } \mathbb{R}^N \setminus \Omega, \quad u = f \quad \text{on } \partial\Omega$$

admits a positive solution, if f is a *nontrivial* nonnegative function in $L^\infty(\partial\Omega)$ and $\|f\|_\infty$ is small enough.

In this paper, we study the existence of positive solutions to the more general equation (1.1), under the following basic assumption:

$$(1.3) \quad N' := N + \theta > 2, \quad \tau := \ell - \theta > -2.$$

The elliptic problem like $-\operatorname{div}(a(x)\nabla u) = b(x)f(x, u)$ can be used to modelling some physical phenomena related to the equilibrium of continuous media, see [5].

We will show that

$$p_s := \frac{N' + 2 + 2\tau}{N' - 2}$$

is effectively the critical exponent for (1.1). In particular, when $\theta = \ell = 0$, we prove that the answer to the question (*) is indeed negative.

Our main results are the follows.

Theorem 1.1. *Let $N' > 2$ and $\tau > -2$. The only nonnegative solution of (1.1) is $u_p \equiv 0$ provided $0 < p \leq p_s$.*

The interesting feature here is that we do not require any behavior at infinity or any symmetry assumption for u . Note that by scaling, our results hold obviously on $\mathbb{R}^N \setminus B_R$, $\forall R > 0$.

For $p \leq \frac{N'+\tau}{N'-2}$, we show a more general nonexistence result inspired by [1, 2], see Theorem 4.1 below. For $1 < p \leq p_s$, we apply the moving-sphere method to get a monotonicity property for the eventual positive solution, then we conclude by contradiction with integral estimate and stability argument. Comparing to Zhang's result mentioned above, the homogenous Dirichlet boundary condition plays a crucial role for our nonexistence result whenever $\frac{N'+\tau}{N'-2} < p \leq p_s$.

Theorem 1.2. *Let $N' > 2$, $\tau > -2$ and $p > p_s$. Then the problem (1.1) admits a unique positive radial solution $u_p \in C^2(\mathbb{R}^N \setminus B)$.*

Combining with the approaches in [1, 9, 12, 14], we show finally a complete picture for the existence of positive solution to (1.1), for all $N' > 2$ and $p > 0$.

Theorem 1.3. *Let $N' > 2$ and $p > 0$.*

- (1) *If $\tau > -2$, then (1.1) admits a positive solution if and only if $p > \frac{N'+2+2\tau}{N'-2}$.*
- (2) *If $\tau < -2$, then (1.1) admits a positive solution for any $p > 1$ and $p \in (0, 1)$.*
- (3) *If $\tau < -2$ and $p = 1$, then (1.1) admits a positive solution if and only if $\lambda_1(L_1) = 0$ in $H_0^{1,\theta}(B)$ where*

$$L_1(v) := -\operatorname{div}(|x|^\theta \nabla v) - |x|^{-(4+\tau-\theta)} v,$$

and $H_0^{1,\theta}(B)$ is the completion of $C_0^\infty(B)$ under the norm $\|u\| = \|\nabla u\|_{L^2(B, |x|^\theta dx)}$.

- (4) *Let $\tau = -2$. then (1.1) has a positive solution if and only if $p = 1$, $N' \geq 4$; or $p > 1$.*

In our analysis, we shall use the Kelvin transformation

$$(1.4) \quad v(y) = |x|^{N'-2} u(x) \quad \text{where } y = \frac{x}{|x|^2}.$$

If u is a solution to (1.1), then v satisfies the problem

$$(1.5) \quad -\operatorname{div}(|y|^\theta \nabla v) = |y|^\sigma v^p \quad \text{in } B \setminus \{0\}, \quad v = 0 \quad \text{on } \partial B.$$

Here

$$\sigma = (N' - 2)(p - 1) - (4 + \tau - \theta).$$

By direct calculation, there holds $\tau' := \sigma - \theta > -2$ if $p > \frac{N'+\tau}{N'-2}$ and

$$(1.6) \quad p > p_s > 1 \text{ if and only if } 1 < p < p'_s := \frac{N' + 2 + 2\tau'}{N' - 2}.$$

Theorem 1.1 implies that problem (1.5) does not admit any nontrivial nonnegative solution in $C^2(B \setminus \{0\})$ for $N' > 2$, $\tau' > -2$ and $p \geq p'_s$. A similar nonexistence result was proved in [9] with the additional assumption that $v \in C^0(\overline{B})$. We prove Theorem 1.2 by using a positive radial solution $v_p \in C^2(B \setminus \{0\}) \cap C^0(\overline{B})$ to (1.5) given in [9].

2. EXISTENCE FOR SUPERCRITICAL CASE

Here we prove Theorem 1.2 by using the Kelvin transformation (1.4), and we are looking for a solution v satisfying the equation (1.5).

Recall that $\sigma = (N' - 2)(p - 1) - (4 + \tau - \theta)$ and $\tau' = \sigma - \theta > -2$. As $1 < p < p'_s$ provided $p > p_s$, it follows from [9] that (1.5) admits a positive radial solution $v_p \in C^2(B \setminus \{0\}) \cap C^0(\overline{B})$. Hence $r \mapsto v_p(r)$ satisfies the equation

$$(2.1) \quad -(r^{N'-1}v_p')' = r^{N'+\tau'-1}v_p^p(r) \text{ in } (0, 1), \quad v_p(1) = 0.$$

As $r^{N'-1}v_p'$ is decreasing in r , $N' > 2$; v_p is positive and uniformly bounded in $(0, 1)$, v_p must satisfy the following property:

$$(2.2) \quad \lim_{r \rightarrow 0} r^{N'-1}v_p'(r) = 0.$$

Then $v_p'(r) < 0$ in $(0, 1)$, this implies that $v_p(0) > 0$.

We now prove that v_p is the unique positive radial solution of (1.5) in $C^2(B \setminus \{0\}) \cap C^0(\overline{B})$. Suppose that there is another such radial solution \underline{v} , let

$$\lambda^{\frac{2+\tau'}{p-1}} = \frac{\underline{v}(0)}{v_p(0)}, \quad \text{and } w(r) := \lambda^{\frac{2+\tau'}{p-1}} v_p(\lambda r) \text{ for all } r \in [0, \lambda^{-1}].$$

Clearly, w satisfies the equations

$$(2.3) \quad -(r^{N'-1}w')' = r^{N'+\tau'-1}w^p \text{ in } (0, \lambda^{-1}), \quad w(0) = \underline{v}(0).$$

Similarly as above,

$$\lim_{r \rightarrow 0} r^{N'-1}w'(r) = 0.$$

Consider $h = w - \underline{v}$, so $h(0) = 0$ and h satisfies the following integral equation

$$(2.4) \quad h(r) = - \int_0^r t^{1-N'} \int_0^t s^{N'+\tau'-1} [w^p(s) - \underline{v}^p(s)] ds dt, \quad \forall 0 \leq r \leq \min(1, \lambda^{-1}).$$

Fix any $0 < r_0 < \min(1, \lambda^{-1})$. By the continuity of w, \underline{v} in $[0, r_0]$, the mean value theorem and Fubini's theorem imply that there exists $C > 0$ such that

$$|h(r)| \leq C \int_0^r s^{\tau'+1} |h(s)| ds, \quad \forall r \in [0, r_0].$$

Remark that $s^{\tau'+1} \in L^1_{loc}(0, \infty)$ since $\tau' > -2$, the classical Gronwall formula implies that $h \equiv 0$ in $[0, r_0]$. Therefore $w \equiv \underline{v}$ in $[0, r_0]$, hence in $[0, \min(1, \lambda^{-1})]$. This is possible if and only if $\lambda = 1$ seeing the Dirichlet boundary conditions and $v_p, \underline{v} > 0$ in B . To conclude, $\underline{v} \equiv v_p$ in $[0, 1]$.

Conversely, let $u \in C^2(\mathbb{R}^N \setminus B)$ be a positive radial solution to (1.1), then

$$-(r^{N'-1}u')' = r^{N'+\tau-1}u^p(r) \text{ in } [1, \infty).$$

So we have $r^{N'-1}u'$ is decreasing and $u(r) = O(r^{2-N'})$ at infinity. Under the Kelvin transform (1.4), we get a bounded radial function $v \in C^2(\overline{B} \setminus \{0\})$ which satisfies (2.1). As above, the boundedness of v implies (2.2), hence v is decreasing by (2.1) and $\lim_{r \rightarrow 0^+} v(r) = \ell_0 \in \mathbb{R}$ exists,

because v is bounded. In other words, $v \in C^0(\overline{B})$. The uniqueness of solution for (2.1) in $C^2(B \setminus \{0\}) \cap C^0(\overline{B})$ completes the proof. \square

3. NONEXISTENCE FOR $1 < p \leq p_s$

Let $1 < p \leq p_s$, we show first a monotonicity property for positive solution of (1.1) if it exists.

Proposition 3.1. *Let $N' > 2$, $\tau > -2$, $1 < p \leq p_s$ and $u \in C^2(\mathbb{R}^N \setminus B)$ be a nontrivial nonnegative solution to (1.1). Then $|x|^{\frac{N'-2}{2}}u(x)$ is increasing with respect to the radius $r = |x|$.*

Proof. By the strong maximum principle, $u > 0$ in $\mathbb{R}^N \setminus \overline{B}$. We introduce the following transform:

$$(3.1) \quad v(t, \omega) = r^{\frac{N'-2}{2}}u(r, \omega), \quad t = \ln r.$$

Then, v satisfies the problem

$$(3.2) \quad \begin{cases} v_{tt} + \Delta_{\mathbb{S}^{N-1}}v + e^{p_*t}v^p - \frac{(N'-2)^2}{4}v = 0, & (t, \omega) \in (0, \infty) \times \mathbb{S}^{N-1}, \\ v(0, \cdot)|_{\mathbb{S}^{N-1}} = 0, \end{cases}$$

where

$$p_* = \frac{(N' + 2 + 2\tau) - (N' - 2)p}{2} \geq 0.$$

Let $\Sigma := (0, \infty) \times \mathbb{S}^{N-1}$. For $T > 0$, define

$$\Sigma_T := (0, T) \times \mathbb{S}^{N-1}, \quad S_T = \{T\} \times \mathbb{S}^{N-1}.$$

Furthermore, we denote by $(t, \omega)_T$ the reflection of (t, ω) with respect to S_T , namely,

$$(t, \omega)_T = (2T - t, \omega).$$

Then the functions

$$v_T(t, \omega) := v((t, \omega)_T) = v(2T - t, \omega) \quad \text{and} \quad \xi_T = v - v_T$$

are well-defined in Σ_T for $T > 0$. Moreover, ξ_T satisfies the following equation in Σ_T :

$$(\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + e^{p_*t}v^p - e^{p_*(2T-t)}v_T^p - \frac{(N' - 2)^2}{4}\xi_T = 0.$$

As $p_* \geq 0$, there holds $e^{p_*(2T-t)} \geq e^{p_*t}$ for any $t \leq T$. So we obtain

$$(\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + e^{p_*t}(v^p - v_T^p) - \frac{(N' - 2)^2}{4}\xi_T \geq 0 \quad \text{in } \Sigma_T,$$

hence

$$(3.3) \quad (\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + \eta\xi_T \geq 0 \quad \text{in } \Sigma_T, \quad \xi_T \leq 0 \quad \text{on } \partial\Sigma_T$$

where

$$\eta(t, \omega) := c(t, \omega) - \frac{(N' - 2)^2}{4} \quad \text{and} \quad c(t, \omega) = pe^{p_*t}\psi^{p-1}(t, \omega),$$

with $\psi(t, \omega)$ in the interval formed by $v(t, \omega)$ and $v_T(t, \omega)$. The fact that $\xi_T \leq 0$ on $\partial\Sigma_T$ for any $T > 0$ comes easily from $u = 0$ on ∂B .

By the continuity of v , we see that $\eta(t, \cdot)$ tends uniformly to $-\frac{(N'-2)^2}{4}$ on \mathbb{S}^{N-1} as $T \rightarrow 0^+$. This implies that there exists $\delta_1 > 0$ such that

$$\eta(t, \omega) < 0 \quad \text{for } (t, \omega) \in \Sigma_T \text{ and } 0 < T \leq \delta_1.$$

Applying the maximum principle to (3.3), it follows that $\xi_T \leq 0$ in Σ_T provided $T \leq \delta_1$. As $\xi_T(0, \omega) < 0$, the strong maximum principle (see for example [8]) yields that for $0 < T \leq \delta_1$, there hold

$$(3.4) \quad \xi_T(t, \omega) < 0 \quad \text{in } \Sigma_T, \quad \frac{\partial \xi_T}{\partial t}(t, \omega) > 0 \quad \text{on } S_T.$$

Remark also that

$$(3.5) \quad \frac{\partial \xi_T}{\partial t}(T, \omega) = 2 \frac{\partial v}{\partial t}(T, \omega), \quad \forall T > 0.$$

Define now

$$T_0 = \sup\{T_1 > 0 : (3.4) \text{ holds for } T \leq T_1\}.$$

Then $T_0 \geq \delta_1$ is well-defined. We want to claim that $T_0 = \infty$.

Suppose the contrary that $T_0 < \infty$, then $\xi_{T_0} \leq 0$ in Σ_{T_0} by continuity. Seeing the elliptic equation (3.3), as $\xi_{T_0}(0, \omega) < 0$, the strong maximum principle means that (3.4) holds true for $T = T_0$. Using (3.5), we deduce that $\partial_t v > 0$ in $(0, T_0] \times \mathbb{S}^{N-1}$. Recall that $u \in C^1(\mathbb{R}^N \setminus B)$, by the compactness of \mathbb{S}^{N-1} , there exists $\varepsilon_1 \in (0, T_0)$ such that

$$\frac{\partial v}{\partial t}(t, \omega) > 0 \quad \text{for } (t, \omega) \in (0, T_0 + 3\varepsilon_1] \times \mathbb{S}^{N-1}.$$

We get then

$$(3.6) \quad \xi_T(t, \omega) < 0 \quad \text{in } \Sigma_T \setminus \Sigma_{T_0 - \varepsilon_1}, \quad \forall T_0 \leq T \leq T_0 + \varepsilon_1.$$

On the other hand, $\max_{\bar{\Sigma}_{T_0 - \varepsilon_1}} \xi_{T_0} < 0$ by continuity and (3.4) with $T = T_0$. Using the continuity of v , there exists $\varepsilon_2 > 0$ such that

$$v(t, \omega) - v(2T - t, \omega) < 0 \quad \text{in } \bar{\Sigma}_{T_0 - \varepsilon_1}, \quad \forall T_0 \leq T \leq T_0 + \varepsilon_2.$$

Combining with (3.6), let $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$, there holds $\xi_T \leq 0$ in Σ_T for any $T \in [T_0, T_0 + \varepsilon]$.

Again, as $\xi_T < 0$ on $\{0\} \times \mathbb{S}^{N-1}$, the strong maximum principle applying to (3.3) shows that (3.4) remains valid for $T \in [T_0, T_0 + \varepsilon]$, this is a contradiction to the definition of T_0 . Therefore T_0 must be ∞ , hence (3.4) holds true for any $T > 0$.

Using (3.5), for any $\omega \in \mathbb{S}^{N-1}$, the function $r \mapsto r^{\frac{N'-2}{2}} u(r, \omega)$ is increasing in $[0, \infty)$. \square

Now we are ready to prove Theorem 1.1 for $1 < p \leq p_s$. Suppose that (1.1) admits a nontrivial nonnegative solution $u \in C^2(\mathbb{R}^N \setminus B)$. We claim that

$$\varphi(x) := \nabla u \cdot x + \frac{2 + \tau}{p - 1} u(x) > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}.$$

In fact, as $1 < p \leq p_s$, we have

$$|x|^{\frac{2+\tau}{p-1}} u(x) = |x|^\beta |x|^{\frac{N'-2}{2}} u(x), \quad \text{where } \beta := \frac{2 + \tau}{p - 1} - \frac{N' - 2}{2} \geq 0.$$

So the function $|x|^{\frac{2+\tau}{p-1}} u(x)$ is increasing in $|x|$, thanks to Proposition 3.1. By the above proof, $\partial_t v > 0$ in $(0, \infty) \times \mathbb{S}^{N-1}$, hence $\varphi > 0$ in $\mathbb{R}^N \setminus \bar{B}$.

Direct calculation via scaling yields that $\varphi \in C^1(\mathbb{R}^N \setminus B)$ is a positive weak solution to the linearized equation

$$L_u(w) := -\operatorname{div}(|x|^\theta \nabla w) - p|x|^\ell u^{p-1} w = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}.$$

By classical elliptic theory, for any bounded smooth domain $\Omega \subset \mathbb{R}^N \setminus \bar{B}$,

$$(3.7) \quad \lambda_{1, \Omega}(L_u) > 0,$$

where $\lambda_{1, \Omega}(L_u)$ stands for the first eigenvalue of the Dirichlet problem

$$L_u(h) = \lambda h \quad \text{in } \Omega, \quad h = 0 \quad \text{on } \partial\Omega.$$

So u is stable in $\mathbb{R}^N \setminus \bar{B}$.

Using Theorem 3.1 in [6], we have

$$(3.8) \quad \int_{\mathbb{R}^N \setminus \bar{B}} \left(|x|^\theta |\nabla u|^2 + |x|^\ell u^{p+1} \right) dx < \infty.$$

However, Proposition 3.1 implies then the existence of $C > 0$ such that

$$u(x) \geq C|x|^{-\frac{N'-2}{2}}, \quad \forall |x| \geq 2.$$

Recall that $1 < p \leq p_s$, there holds

$$\int_{|x|>2} |x|^\ell u^{p+1} dx \geq C \int_2^\infty r^{N-1+\ell} r^{-\frac{(p+1)(N'-2)}{2}} dr = \infty,$$

since

$$N-1+\ell - \frac{(p+1)(N'-2)}{2} \geq N'+\tau-1 - \frac{(p_s+1)(N'-2)}{2} = -1.$$

This contradiction with (3.8) means that there is no nontrivial nonnegative solution to (1.1). \square

4. FURTHER RESULTS

First, we consider the existence and nonexistence of positive solution to

$$(4.1) \quad -\operatorname{div}(|x|^\theta \nabla u) \geq |x|^\ell f(u) \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R, \quad R > 0.$$

Remark that we do not impose any boundary condition or the value of R .

A first nonexistence result has been obtained by Gidas-Spruck (see Theorem A.3 in [7]), they showed that for $\theta = \ell$, $N' = N + \theta > 2$ and $f(u) = u^p$, (4.1) admits only trivial nonnegative solution $u \equiv 0$ provided $1 < p \leq \frac{N'}{N'-2}$. Since then, many existence and nonexistence results have been established for (4.1) under suitable conditions, interested readers can look at [1, 2, 10, 12, 14, 15] and the references therein.

Inspired by Theorem 8 of [1], we show a very general result to characterize the existence of positive solution to (4.1). In particular, the following result implies Theorem 1.1 for $0 < p \leq \frac{N'+\tau}{N'-2}$.

Theorem 4.1. *Assume that $N' > 2$, $\tau > -2$ and $f : (0, \infty) \rightarrow (0, \infty)$ is continuous. Then the problem (4.1) admits a positive classical solution for some $R > 0$ if and only if*

$$(4.2) \quad \int_0^\delta f(t) t^{-\frac{2(N'-1)+\tau}{N'-2}} dt < \infty,$$

for some $\delta > 0$.

Proof. We need only to adapt the approach in [1]. Note that to prove Lemma 4 in [1], the key points are the maximum principle and the rotational invariance of the equation, which obviously remain true in our setting.

Therefore, we can claim that if (4.1) admits a positive solution, then for some $R_0 > R$, there exists a C^1 positive radial function \bar{u} verifying

$$-\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

in the weak sense. Moreover, if $\bar{u}(x) = v(|x|)$, then $v \in C^2(R_0, \infty)$ and there exists $R_1 \geq R_0$ such that v is monotone for $r > R_1$. Note that v satisfies

$$(4.3) \quad -v'' - \frac{N'-1}{r} v' = r^\tau f(v) \quad \text{in } [R_1, \infty).$$

As $N' > 2$, there holds $v(r) \leq -C_1 r^{2-N'} + C_2$. Hence v is bounded in $[R_1, \infty)$ and $\lim_{r \rightarrow \infty} v(r) = \kappa \geq 0$ exists.

If $\kappa > 0$, there exists $C > 0$ such that $(r^{N'-1} v')' \leq -C r^{N'+\tau-1}$ for r large enough. Using $N'+\tau > 0$ and integrating, we deduce that $v'(r) \leq -C r^{\tau+1}$ for large r . As $\tau > -2$, this implies that $\lim_{r \rightarrow \infty} v(r) = -\infty$ which is impossible. Hence we can claim that $v'(r) < 0$ in $[R_1, \infty)$ and $\lim_{r \rightarrow \infty} v(r) = 0$.

Similarly to [1], we introduce the change of variables

$$(4.4) \quad s := r^{2-N'}, \quad w(s) = v(r).$$

Then w is a positive solution of

$$(4.5) \quad -w''(s) = as^{-\gamma}f(w) \text{ in } (0, s_0), \quad w(0) = 0,$$

for some positive constants a and s_0 , where (as $\tau > -2$)

$$\gamma = \frac{2(N' - 1) + \tau}{N' - 2} > 2.$$

Finally, applying Theorem 6 in [1], a positive solution of (4.5) exists if and only if (4.2) holds true. \square

Remark 4.2. Let $N' > 2$, $\tau > -2$, then

$$p - \frac{2(N' - 1) + \tau}{N' - 2} \leq -1 \quad \Leftrightarrow \quad p \leq \frac{N' + \tau}{N' - 2}.$$

For $f(t) = t^p$ with $0 < p \leq \frac{N' + \tau}{N' - 2}$, Proposition 4.1 implies that (4.1) cannot have positive solution for any $R > 0$, hence the equation (1.1) does not admit positive solution.

Remark 4.3. It is well known that if $N' > 2$, $\tau > -2$ and $p > \frac{N' + \tau}{N' - 2}$, $v(x) = Cr^{-\frac{\tau+2}{p-1}}$ with suitable $C(N', \tau, p) > 0$ will satisfy $-\operatorname{div}(|x|^\theta \nabla v) = |x|^\ell v^p$ in $\mathbb{R}^N \setminus \{0\}$.

Moreover, Przeradzki and Stańczy [12, 14] obtained that if $N > 2$, $\theta = 0$ and $\ell < -2$, the equation (1.1) admits a positive solution for any $p > 1$ and $p \in (0, 1)$, see Remark 5 in [14] and Corollary 1 in [12]. It's not difficult to be convinced that their approach via radial solutions remains valid to (1.1) for any $N' > 2$ and $\tau < -2$. We get then a complete answer to the existence question of positive solution of (1.1), under the only assumption $N' := N + \theta > 2$.

Proof of Theorem 1.3. The point (1) is given by Theorems 1.1-1.2. The point (2) is ensured by [12, 14] as explained above.

Consider $\tau < -2$ and $p = 1$, assume that a positive solution exists to (1.1). As the equation is linear, the existence of a positive solution to (1.1) yields a positive *radial* solution u to (1.1), and there holds $u(r) = O(r^{2-N'})$ at infinity. By the Kelvin transform (1.4), we get a solution v to (2.1) with $p = 1$. As above, there holds $v \in C^0(\overline{B})$ and v satisfies (2.2). Using the equation, we see that $v \in H_0^{1,\theta}(B)$ is a positive solution to $L_1(v) = 0$, with $\tau' = -(4 + \tau) > -2$. It means that 0 is the principal eigenvalue to L_1 in $H_0^{1,\theta}(B)$.

Fix now $\tau = -2$.

- If $0 < p < 1$, we will prove that the problem (4.1) has no solution for any $R > 0$. Suppose the contrary, as in [1], we get a positive *radial* solution to $-\operatorname{div}(|x|^\theta \nabla v) = |x|^\ell v^p$ in $\mathbb{R}^N \setminus B_{R_1}$ for some $R_1 > R$. Under the change of variable (4.4), there exists $a, s_0 > 0$ such that

$$-w'' = as^{-2}w^p \text{ in } (0, s_0), \quad w(0) = 0.$$

However, this is impossible seeing the proof of Theorem 8 (b) in [1], where we can replace N just by $N' > 2$.

- If $p = 1$, as before, the existence of a positive solution to (1.1) is equivalent to the existence of a positive *radial* solution. Assume that u is a radial solution to (1.1), let $w(s) = u(e^s)$, then w is a solution to

$$w'' + (N' - 2)w' + w = 0 \text{ in } (0, \infty), \quad w(0) = 0, \quad \lim_{s \rightarrow \infty} w(s) = 0.$$

Clearly, such a positive solution w exists if and only if the equation $\lambda^2 + (N' - 2)\lambda + 1 = 0$ has only negative roots, or equivalently, $N' \geq 4$.

- Finally let $p > 1$. Under the Kelvin transform, we are led to consider (1.5). We check readily that $\tau' = \sigma - \theta = (N' - 2)(p - 1) - 2 > -2$ and p is subcritical, since here

$$p < \frac{N' + 2 + 2\tau'}{N' - 2} \Leftrightarrow (p - 1)(N' - 2) > 0$$

There exists a positive radial solution of (1.5) as in section 2, using the result in [9]. \square

Remark 4.4. For $N' = 2$, $\tau > -2$, similarly to [1], we can show that the problem (4.1) admits a positive solution if and only if there exists $a > 0$ such that $e^{at}f(t)$ is integrable at $+\infty$. Consequently, in this case, for any $p \in \mathbb{R}$, the equation (4.1) does not admit positive solution.

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