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# EXISTENCE AND NONEXISTENCE RESULTS FOR A WEIGHTED ELLIPTIC EQUATION IN EXTERIOR DOMAINS

ZONGMING GUO, XIA HUANG, AND DONG YE

ABSTRACT. We consider positive solutions to the weighted elliptic problem

$$-\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B,$$

where  $B$  is the standard unit ball of  $\mathbb{R}^N$ . We give a complete answer for the existence question for  $N' := N + \theta > 2$  and  $p > 0$ . In particular, for  $N' > 2$  and  $\tau := \ell - \theta > -2$ , it is shown that for  $0 < p \leq p_s := \frac{N'+2+2\tau}{N'-2}$ , the only nonnegative solution to the problem is  $u \equiv 0$ . This nonexistence result is new, even for the classical case  $\theta = \ell = 0$  and  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$ ,  $N \geq 3$ . The interesting feature here is that we do not require any behavior at infinity or any symmetry assumption.

## 1. INTRODUCTION

We study positive solutions to the problem

$$(1.1) \quad -\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B$$

where  $B = \{x \in \mathbb{R}^N, |x| < 1\}$ ,  $p > 0$ .

For  $\theta = \ell = 0$ , the classical elliptic equation

$$(1.2) \quad -\Delta u = u^p \text{ in } \mathbb{R}^N \setminus \overline{B}, \quad u = 0 \text{ on } \partial B$$

has been studied intensively, see for instance [1, 2, 3, 4, 7, 11, 13, 15] and the references therein.

It is well-known from [1, 2, 7, 11] that (1.2) does not admit any positive solution provided  $0 < p \leq \frac{N}{N-2}$  for  $N \geq 3$ ; or  $p > 0$  for  $N \leq 2$ . It is also well-known from [3] that for any supercritical exponent  $p > \frac{N+2}{N-2}$  and  $N \geq 3$ , the problem (1.2) admits a unique positive radial solution  $u$  satisfying

$$u(x) = O(|x|^{2-N}) \quad \text{as } |x| \rightarrow \infty.$$

For  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$ ,  $N \geq 3$ , Theorem 2.2 in [13] showed that (1.2) does not admit any positive radial solution. However, the following natural question remained open:

(\*) Is there any *non radial* positive solution of (1.2) for  $\frac{N}{N-2} < p \leq \frac{N+2}{N-2}$ ,  $N \geq 3$ ?

It's worthy to mention two closely linked results, which seem to suggest that the answer for (\*) could be complex. Firstly, there exist infinitely many *non radial* slow decay positive solutions to (1.2) for any supercritical exponent  $p$ . More precisely, let  $\Omega$  be a bounded smooth domain in  $\mathbb{R}^N$  ( $N \geq 3$ ), it was proved in [4] that for the homogenous Dirichlet problem  $-\Delta u = u^p$  in  $\mathbb{R}^N \setminus \Omega$  with  $p > \frac{N+2}{N-2}$ , there exist always infinitely many positive solutions such that

$$u(x) = O\left(|x|^{-\frac{2}{p-1}}\right) \quad \text{as } |x| \rightarrow \infty.$$

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Secondly, by Proposition 6.1 in [15], Zhang showed that for any bounded Lipschitz domain  $\Omega \subset \mathbb{R}^N$  ( $N \geq 3$ ),  $p > \frac{N+\ell}{N-2}$ , the problem

$$-\Delta u = |x|^\ell u^p \quad \text{in } \mathbb{R}^N \setminus \Omega, \quad u = f \quad \text{on } \partial\Omega$$

admits a positive solution, if  $f$  is a *nontrivial* nonnegative function in  $L^\infty(\partial\Omega)$  and  $\|f\|_\infty$  is small enough.

In this paper, we study the existence of positive solutions to the more general equation (1.1), under the following basic assumption:

$$(1.3) \quad N' := N + \theta > 2, \quad \tau := \ell - \theta > -2.$$

The elliptic problem like  $-\operatorname{div}(a(x)\nabla u) = b(x)f(x, u)$  can be used to modelling some physical phenomena related to the equilibrium of continuous media, see [5].

We will show that

$$p_s := \frac{N' + 2 + 2\tau}{N' - 2}$$

is effectively the critical exponent for (1.1). In particular, when  $\theta = \ell = 0$ , we prove that the answer to the question (\*) is indeed negative.

Our main results are the follows.

**Theorem 1.1.** *Let  $N' > 2$  and  $\tau > -2$ . The only nonnegative solution of (1.1) is  $u_p \equiv 0$  provided  $0 < p \leq p_s$ .*

The interesting feature here is that we do not require any behavior at infinity or any symmetry assumption for  $u$ . Note that by scaling, our results hold obviously on  $\mathbb{R}^N \setminus B_R$ ,  $\forall R > 0$ .

For  $p \leq \frac{N'+\tau}{N'-2}$ , we show a more general nonexistence result inspired by [1, 2], see Theorem 4.1 below. For  $1 < p \leq p_s$ , we apply the moving-sphere method to get a monotonicity property for the eventual positive solution, then we conclude by contradiction with integral estimate and stability argument. Comparing to Zhang's result mentioned above, the homogenous Dirichlet boundary condition plays a crucial role for our nonexistence result whenever  $\frac{N'+\tau}{N'-2} < p \leq p_s$ .

**Theorem 1.2.** *Let  $N' > 2$ ,  $\tau > -2$  and  $p > p_s$ . Then the problem (1.1) admits a unique positive radial solution  $u_p \in C^2(\mathbb{R}^N \setminus B)$ .*

Combining with the approaches in [1, 9, 12, 14], we show finally a complete picture for the existence of positive solution to (1.1), for all  $N' > 2$  and  $p > 0$ .

**Theorem 1.3.** *Let  $N' > 2$  and  $p > 0$ .*

- (1) *If  $\tau > -2$ , then (1.1) admits a positive solution if and only if  $p > \frac{N'+2+2\tau}{N'-2}$ .*
- (2) *If  $\tau < -2$ , then (1.1) admits a positive solution for any  $p > 1$  and  $p \in (0, 1)$ .*
- (3) *If  $\tau < -2$  and  $p = 1$ , then (1.1) admits a positive solution if and only if  $\lambda_1(L_1) = 0$  in  $H_0^{1,\theta}(B)$  where*

$$L_1(v) := -\operatorname{div}(|x|^\theta \nabla v) - |x|^{-(4+\tau-\theta)} v,$$

*and  $H_0^{1,\theta}(B)$  is the completion of  $C_0^\infty(B)$  under the norm  $\|u\| = \|\nabla u\|_{L^2(B, |x|^\theta dx)}$ .*

- (4) *Let  $\tau = -2$ . then (1.1) has a positive solution if and only if  $p = 1$ ,  $N' \geq 4$ ; or  $p > 1$ .*

In our analysis, we shall use the Kelvin transformation

$$(1.4) \quad v(y) = |x|^{N'-2} u(x) \quad \text{where } y = \frac{x}{|x|^2}.$$

If  $u$  is a solution to (1.1), then  $v$  satisfies the problem

$$(1.5) \quad -\operatorname{div}(|y|^\theta \nabla v) = |y|^\sigma v^p \quad \text{in } B \setminus \{0\}, \quad v = 0 \quad \text{on } \partial B.$$

Here

$$\sigma = (N' - 2)(p - 1) - (4 + \tau - \theta).$$

By direct calculation, there holds  $\tau' := \sigma - \theta > -2$  if  $p > \frac{N'+\tau}{N'-2}$  and

$$(1.6) \quad p > p_s > 1 \text{ if and only if } 1 < p < p'_s := \frac{N' + 2 + 2\tau'}{N' - 2}.$$

Theorem 1.1 implies that problem (1.5) does not admit any nontrivial nonnegative solution in  $C^2(B \setminus \{0\})$  for  $N' > 2$ ,  $\tau' > -2$  and  $p \geq p'_s$ . A similar nonexistence result was proved in [9] with the additional assumption that  $v \in C^0(\overline{B})$ . We prove Theorem 1.2 by using a positive radial solution  $v_p \in C^2(B \setminus \{0\}) \cap C^0(\overline{B})$  to (1.5) given in [9].

## 2. EXISTENCE FOR SUPERCRITICAL CASE

Here we prove Theorem 1.2 by using the Kelvin transformation (1.4), and we are looking for a solution  $v$  satisfying the equation (1.5).

Recall that  $\sigma = (N' - 2)(p - 1) - (4 + \tau - \theta)$  and  $\tau' = \sigma - \theta > -2$ . As  $1 < p < p'_s$  provided  $p > p_s$ , it follows from [9] that (1.5) admits a positive radial solution  $v_p \in C^2(B \setminus \{0\}) \cap C^0(\overline{B})$ . Hence  $r \mapsto v_p(r)$  satisfies the equation

$$(2.1) \quad -(r^{N'-1}v_p')' = r^{N'+\tau'-1}v_p^p(r) \text{ in } (0, 1), \quad v_p(1) = 0.$$

As  $r^{N'-1}v_p'$  is decreasing in  $r$ ,  $N' > 2$ ;  $v_p$  is positive and uniformly bounded in  $(0, 1)$ ,  $v_p$  must satisfy the following property:

$$(2.2) \quad \lim_{r \rightarrow 0} r^{N'-1}v_p'(r) = 0.$$

Then  $v_p'(r) < 0$  in  $(0, 1)$ , this implies that  $v_p(0) > 0$ .

We now prove that  $v_p$  is the unique positive radial solution of (1.5) in  $C^2(B \setminus \{0\}) \cap C^0(\overline{B})$ . Suppose that there is another such radial solution  $\underline{v}$ , let

$$\lambda^{\frac{2+\tau'}{p-1}} = \frac{\underline{v}(0)}{v_p(0)}, \quad \text{and } w(r) := \lambda^{\frac{2+\tau'}{p-1}} v_p(\lambda r) \text{ for all } r \in [0, \lambda^{-1}].$$

Clearly,  $w$  satisfies the equations

$$(2.3) \quad -(r^{N'-1}w')' = r^{N'+\tau'-1}w^p \text{ in } (0, \lambda^{-1}), \quad w(0) = \underline{v}(0).$$

Similarly as above,

$$\lim_{r \rightarrow 0} r^{N'-1}w'(r) = 0.$$

Consider  $h = w - \underline{v}$ , so  $h(0) = 0$  and  $h$  satisfies the following integral equation

$$(2.4) \quad h(r) = - \int_0^r t^{1-N'} \int_0^t s^{N'+\tau'-1} [w^p(s) - \underline{v}^p(s)] ds dt, \quad \forall 0 \leq r \leq \min(1, \lambda^{-1}).$$

Fix any  $0 < r_0 < \min(1, \lambda^{-1})$ . By the continuity of  $w, \underline{v}$  in  $[0, r_0]$ , the mean value theorem and Fubini's theorem imply that there exists  $C > 0$  such that

$$|h(r)| \leq C \int_0^r s^{\tau'+1} |h(s)| ds, \quad \forall r \in [0, r_0].$$

Remark that  $s^{\tau'+1} \in L^1_{loc}(0, \infty)$  since  $\tau' > -2$ , the classical Gronwall formula implies that  $h \equiv 0$  in  $[0, r_0]$ . Therefore  $w \equiv \underline{v}$  in  $[0, r_0]$ , hence in  $[0, \min(1, \lambda^{-1})]$ . This is possible if and only if  $\lambda = 1$  seeing the Dirichlet boundary conditions and  $v_p, \underline{v} > 0$  in  $B$ . To conclude,  $\underline{v} \equiv v_p$  in  $[0, 1]$ .

Conversely, let  $u \in C^2(\mathbb{R}^N \setminus B)$  be a positive radial solution to (1.1), then

$$-(r^{N'-1}u')' = r^{N'+\tau-1}u^p(r) \text{ in } [1, \infty).$$

So we have  $r^{N'-1}u'$  is decreasing and  $u(r) = O(r^{2-N'})$  at infinity. Under the Kelvin transform (1.4), we get a bounded radial function  $v \in C^2(\overline{B} \setminus \{0\})$  which satisfies (2.1). As above, the boundedness of  $v$  implies (2.2), hence  $v$  is decreasing by (2.1) and  $\lim_{r \rightarrow 0^+} v(r) = \ell_0 \in \mathbb{R}$  exists,

because  $v$  is bounded. In other words,  $v \in C^0(\overline{B})$ . The uniqueness of solution for (2.1) in  $C^2(B \setminus \{0\}) \cap C^0(\overline{B})$  completes the proof.  $\square$

### 3. NONEXISTENCE FOR $1 < p \leq p_s$

Let  $1 < p \leq p_s$ , we show first a monotonicity property for positive solution of (1.1) if it exists.

**Proposition 3.1.** *Let  $N' > 2$ ,  $\tau > -2$ ,  $1 < p \leq p_s$  and  $u \in C^2(\mathbb{R}^N \setminus B)$  be a nontrivial nonnegative solution to (1.1). Then  $|x|^{\frac{N'-2}{2}}u(x)$  is increasing with respect to the radius  $r = |x|$ .*

**Proof.** By the strong maximum principle,  $u > 0$  in  $\mathbb{R}^N \setminus \overline{B}$ . We introduce the following transform:

$$(3.1) \quad v(t, \omega) = r^{\frac{N'-2}{2}}u(r, \omega), \quad t = \ln r.$$

Then,  $v$  satisfies the problem

$$(3.2) \quad \begin{cases} v_{tt} + \Delta_{\mathbb{S}^{N-1}}v + e^{p_*t}v^p - \frac{(N'-2)^2}{4}v = 0, & (t, \omega) \in (0, \infty) \times \mathbb{S}^{N-1}, \\ v(0, \cdot)|_{\mathbb{S}^{N-1}} = 0, \end{cases}$$

where

$$p_* = \frac{(N' + 2 + 2\tau) - (N' - 2)p}{2} \geq 0.$$

Let  $\Sigma := (0, \infty) \times \mathbb{S}^{N-1}$ . For  $T > 0$ , define

$$\Sigma_T := (0, T) \times \mathbb{S}^{N-1}, \quad S_T = \{T\} \times \mathbb{S}^{N-1}.$$

Furthermore, we denote by  $(t, \omega)_T$  the reflection of  $(t, \omega)$  with respect to  $S_T$ , namely,

$$(t, \omega)_T = (2T - t, \omega).$$

Then the functions

$$v_T(t, \omega) := v((t, \omega)_T) = v(2T - t, \omega) \quad \text{and} \quad \xi_T = v - v_T$$

are well-defined in  $\Sigma_T$  for  $T > 0$ . Moreover,  $\xi_T$  satisfies the following equation in  $\Sigma_T$ :

$$(\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + e^{p_*t}v^p - e^{p_*(2T-t)}v_T^p - \frac{(N' - 2)^2}{4}\xi_T = 0.$$

As  $p_* \geq 0$ , there holds  $e^{p_*(2T-t)} \geq e^{p_*t}$  for any  $t \leq T$ . So we obtain

$$(\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + e^{p_*t}(v^p - v_T^p) - \frac{(N' - 2)^2}{4}\xi_T \geq 0 \quad \text{in } \Sigma_T,$$

hence

$$(3.3) \quad (\xi_T)_{tt} + \Delta_{\mathbb{S}^{N-1}}\xi_T + \eta\xi_T \geq 0 \quad \text{in } \Sigma_T, \quad \xi_T \leq 0 \quad \text{on } \partial\Sigma_T$$

where

$$\eta(t, \omega) := c(t, \omega) - \frac{(N' - 2)^2}{4} \quad \text{and} \quad c(t, \omega) = pe^{p_*t}\psi^{p-1}(t, \omega),$$

with  $\psi(t, \omega)$  in the interval formed by  $v(t, \omega)$  and  $v_T(t, \omega)$ . The fact that  $\xi_T \leq 0$  on  $\partial\Sigma_T$  for any  $T > 0$  comes easily from  $u = 0$  on  $\partial B$ .

By the continuity of  $v$ , we see that  $\eta(t, \cdot)$  tends uniformly to  $-\frac{(N'-2)^2}{4}$  on  $\mathbb{S}^{N-1}$  as  $T \rightarrow 0^+$ . This implies that there exists  $\delta_1 > 0$  such that

$$\eta(t, \omega) < 0 \quad \text{for } (t, \omega) \in \Sigma_T \text{ and } 0 < T \leq \delta_1.$$

Applying the maximum principle to (3.3), it follows that  $\xi_T \leq 0$  in  $\Sigma_T$  provided  $T \leq \delta_1$ . As  $\xi_T(0, \omega) < 0$ , the strong maximum principle (see for example [8]) yields that for  $0 < T \leq \delta_1$ , there hold

$$(3.4) \quad \xi_T(t, \omega) < 0 \quad \text{in } \Sigma_T, \quad \frac{\partial \xi_T}{\partial t}(t, \omega) > 0 \quad \text{on } S_T.$$

Remark also that

$$(3.5) \quad \frac{\partial \xi_T}{\partial t}(T, \omega) = 2 \frac{\partial v}{\partial t}(T, \omega), \quad \forall T > 0.$$

Define now

$$T_0 = \sup\{T_1 > 0 : (3.4) \text{ holds for } T \leq T_1\}.$$

Then  $T_0 \geq \delta_1$  is well-defined. We want to claim that  $T_0 = \infty$ .

Suppose the contrary that  $T_0 < \infty$ , then  $\xi_{T_0} \leq 0$  in  $\Sigma_{T_0}$  by continuity. Seeing the elliptic equation (3.3), as  $\xi_{T_0}(0, \omega) < 0$ , the strong maximum principle means that (3.4) holds true for  $T = T_0$ . Using (3.5), we deduce that  $\partial_t v > 0$  in  $(0, T_0] \times \mathbb{S}^{N-1}$ . Recall that  $u \in C^1(\mathbb{R}^N \setminus B)$ , by the compactness of  $\mathbb{S}^{N-1}$ , there exists  $\varepsilon_1 \in (0, T_0)$  such that

$$\frac{\partial v}{\partial t}(t, \omega) > 0 \quad \text{for } (t, \omega) \in (0, T_0 + 3\varepsilon_1] \times \mathbb{S}^{N-1}.$$

We get then

$$(3.6) \quad \xi_T(t, \omega) < 0 \quad \text{in } \Sigma_T \setminus \Sigma_{T_0 - \varepsilon_1}, \quad \forall T_0 \leq T \leq T_0 + \varepsilon_1.$$

On the other hand,  $\max_{\bar{\Sigma}_{T_0 - \varepsilon_1}} \xi_{T_0} < 0$  by continuity and (3.4) with  $T = T_0$ . Using the continuity of  $v$ , there exists  $\varepsilon_2 > 0$  such that

$$v(t, \omega) - v(2T - t, \omega) < 0 \quad \text{in } \bar{\Sigma}_{T_0 - \varepsilon_1}, \quad \forall T_0 \leq T \leq T_0 + \varepsilon_2.$$

Combining with (3.6), let  $\varepsilon = \min(\varepsilon_1, \varepsilon_2) > 0$ , there holds  $\xi_T \leq 0$  in  $\Sigma_T$  for any  $T \in [T_0, T_0 + \varepsilon]$ .

Again, as  $\xi_T < 0$  on  $\{0\} \times \mathbb{S}^{N-1}$ , the strong maximum principle applying to (3.3) shows that (3.4) remains valid for  $T \in [T_0, T_0 + \varepsilon]$ , this is a contradiction to the definition of  $T_0$ . Therefore  $T_0$  must be  $\infty$ , hence (3.4) holds true for any  $T > 0$ .

Using (3.5), for any  $\omega \in \mathbb{S}^{N-1}$ , the function  $r \mapsto r^{\frac{N'-2}{2}} u(r, \omega)$  is increasing in  $[0, \infty)$ .  $\square$

Now we are ready to prove Theorem 1.1 for  $1 < p \leq p_s$ . Suppose that (1.1) admits a nontrivial nonnegative solution  $u \in C^2(\mathbb{R}^N \setminus B)$ . We claim that

$$\varphi(x) := \nabla u \cdot x + \frac{2 + \tau}{p - 1} u(x) > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}.$$

In fact, as  $1 < p \leq p_s$ , we have

$$|x|^{\frac{2+\tau}{p-1}} u(x) = |x|^\beta |x|^{\frac{N'-2}{2}} u(x), \quad \text{where } \beta := \frac{2 + \tau}{p - 1} - \frac{N' - 2}{2} \geq 0.$$

So the function  $|x|^{\frac{2+\tau}{p-1}} u(x)$  is increasing in  $|x|$ , thanks to Proposition 3.1. By the above proof,  $\partial_t v > 0$  in  $(0, \infty) \times \mathbb{S}^{N-1}$ , hence  $\varphi > 0$  in  $\mathbb{R}^N \setminus \bar{B}$ .

Direct calculation via scaling yields that  $\varphi \in C^1(\mathbb{R}^N \setminus B)$  is a positive weak solution to the linearized equation

$$L_u(w) := -\operatorname{div}(|x|^\theta \nabla w) - p|x|^\ell u^{p-1} w = 0 \quad \text{in } \mathbb{R}^N \setminus \bar{B}.$$

By classical elliptic theory, for any bounded smooth domain  $\Omega \subset \mathbb{R}^N \setminus \bar{B}$ ,

$$(3.7) \quad \lambda_{1, \Omega}(L_u) > 0,$$

where  $\lambda_{1, \Omega}(L_u)$  stands for the first eigenvalue of the Dirichlet problem

$$L_u(h) = \lambda h \quad \text{in } \Omega, \quad h = 0 \quad \text{on } \partial\Omega.$$

So  $u$  is stable in  $\mathbb{R}^N \setminus \bar{B}$ .

Using Theorem 3.1 in [6], we have

$$(3.8) \quad \int_{\mathbb{R}^N \setminus \bar{B}} \left( |x|^\theta |\nabla u|^2 + |x|^\ell u^{p+1} \right) dx < \infty.$$

However, Proposition 3.1 implies then the existence of  $C > 0$  such that

$$u(x) \geq C|x|^{-\frac{N'-2}{2}}, \quad \forall |x| \geq 2.$$

Recall that  $1 < p \leq p_s$ , there holds

$$\int_{|x|>2} |x|^\ell u^{p+1} dx \geq C \int_2^\infty r^{N-1+\ell} r^{-\frac{(p+1)(N'-2)}{2}} dr = \infty,$$

since

$$N-1+\ell - \frac{(p+1)(N'-2)}{2} \geq N'+\tau-1 - \frac{(p_s+1)(N'-2)}{2} = -1.$$

This contradiction with (3.8) means that there is no nontrivial nonnegative solution to (1.1).  $\square$

#### 4. FURTHER RESULTS

First, we consider the existence and nonexistence of positive solution to

$$(4.1) \quad -\operatorname{div}(|x|^\theta \nabla u) \geq |x|^\ell f(u) \quad \text{in } \mathbb{R}^N \setminus \overline{B}_R, \quad R > 0.$$

Remark that we do not impose any boundary condition or the value of  $R$ .

A first nonexistence result has been obtained by Gidas-Spruck (see Theorem A.3 in [7]), they showed that for  $\theta = \ell$ ,  $N' = N + \theta > 2$  and  $f(u) = u^p$ , (4.1) admits only trivial nonnegative solution  $u \equiv 0$  provided  $1 < p \leq \frac{N'}{N'-2}$ . Since then, many existence and nonexistence results have been established for (4.1) under suitable conditions, interested readers can look at [1, 2, 10, 12, 14, 15] and the references therein.

Inspired by Theorem 8 of [1], we show a very general result to characterize the existence of positive solution to (4.1). In particular, the following result implies Theorem 1.1 for  $0 < p \leq \frac{N'+\tau}{N'-2}$ .

**Theorem 4.1.** *Assume that  $N' > 2$ ,  $\tau > -2$  and  $f : (0, \infty) \rightarrow (0, \infty)$  is continuous. Then the problem (4.1) admits a positive classical solution for some  $R > 0$  if and only if*

$$(4.2) \quad \int_0^\delta f(t) t^{-\frac{2(N'-1)+\tau}{N'-2}} dt < \infty,$$

for some  $\delta > 0$ .

*Proof.* We need only to adapt the approach in [1]. Note that to prove Lemma 4 in [1], the key points are the maximum principle and the rotational invariance of the equation, which obviously remain true in our setting.

Therefore, we can claim that if (4.1) admits a positive solution, then for some  $R_0 > R$ , there exists a  $C^1$  positive radial function  $\bar{u}$  verifying

$$-\operatorname{div}(|x|^\theta \nabla u) = |x|^\ell f(u) \quad \text{in } \mathbb{R}^N \setminus B_{R_0}$$

in the weak sense. Moreover, if  $\bar{u}(x) = v(|x|)$ , then  $v \in C^2(R_0, \infty)$  and there exists  $R_1 \geq R_0$  such that  $v$  is monotone for  $r > R_1$ . Note that  $v$  satisfies

$$(4.3) \quad -v'' - \frac{N'-1}{r} v' = r^\tau f(v) \quad \text{in } [R_1, \infty).$$

As  $N' > 2$ , there holds  $v(r) \leq -C_1 r^{2-N'} + C_2$ . Hence  $v$  is bounded in  $[R_1, \infty)$  and  $\lim_{r \rightarrow \infty} v(r) = \kappa \geq 0$  exists.

If  $\kappa > 0$ , there exists  $C > 0$  such that  $(r^{N'-1} v')' \leq -C r^{N'+\tau-1}$  for  $r$  large enough. Using  $N'+\tau > 0$  and integrating, we deduce that  $v'(r) \leq -C r^{\tau+1}$  for large  $r$ . As  $\tau > -2$ , this implies that  $\lim_{r \rightarrow \infty} v(r) = -\infty$  which is impossible. Hence we can claim that  $v'(r) < 0$  in  $[R_1, \infty)$  and  $\lim_{r \rightarrow \infty} v(r) = 0$ .

Similarly to [1], we introduce the change of variables

$$(4.4) \quad s := r^{2-N'}, \quad w(s) = v(r).$$

Then  $w$  is a positive solution of

$$(4.5) \quad -w''(s) = as^{-\gamma}f(w) \text{ in } (0, s_0), \quad w(0) = 0,$$

for some positive constants  $a$  and  $s_0$ , where (as  $\tau > -2$ )

$$\gamma = \frac{2(N' - 1) + \tau}{N' - 2} > 2.$$

Finally, applying Theorem 6 in [1], a positive solution of (4.5) exists if and only if (4.2) holds true.  $\square$

**Remark 4.2.** Let  $N' > 2$ ,  $\tau > -2$ , then

$$p - \frac{2(N' - 1) + \tau}{N' - 2} \leq -1 \quad \Leftrightarrow \quad p \leq \frac{N' + \tau}{N' - 2}.$$

For  $f(t) = t^p$  with  $0 < p \leq \frac{N' + \tau}{N' - 2}$ , Proposition 4.1 implies that (4.1) cannot have positive solution for any  $R > 0$ , hence the equation (1.1) does not admit positive solution.

**Remark 4.3.** It is well known that if  $N' > 2$ ,  $\tau > -2$  and  $p > \frac{N' + \tau}{N' - 2}$ ,  $v(x) = Cr^{-\frac{\tau+2}{p-1}}$  with suitable  $C(N', \tau, p) > 0$  will satisfy  $-\operatorname{div}(|x|^\theta \nabla v) = |x|^\ell v^p$  in  $\mathbb{R}^N \setminus \{0\}$ .

Moreover, Przeradzki and Stańczy [12, 14] obtained that if  $N > 2$ ,  $\theta = 0$  and  $\ell < -2$ , the equation (1.1) admits a positive solution for any  $p > 1$  and  $p \in (0, 1)$ , see Remark 5 in [14] and Corollary 1 in [12]. It's not difficult to be convinced that their approach via radial solutions remains valid to (1.1) for any  $N' > 2$  and  $\tau < -2$ . We get then a complete answer to the existence question of positive solution of (1.1), under the only assumption  $N' := N + \theta > 2$ .

**Proof of Theorem 1.3.** The point (1) is given by Theorems 1.1-1.2. The point (2) is ensured by [12, 14] as explained above.

Consider  $\tau < -2$  and  $p = 1$ , assume that a positive solution exists to (1.1). As the equation is linear, the existence of a positive solution to (1.1) yields a positive *radial* solution  $u$  to (1.1), and there holds  $u(r) = O(r^{2-N'})$  at infinity. By the Kelvin transform (1.4), we get a solution  $v$  to (2.1) with  $p = 1$ . As above, there holds  $v \in C^0(\overline{B})$  and  $v$  satisfies (2.2). Using the equation, we see that  $v \in H_0^{1,\theta}(B)$  is a positive solution to  $L_1(v) = 0$ , with  $\tau' = -(4 + \tau) > -2$ . It means that 0 is the principal eigenvalue to  $L_1$  in  $H_0^{1,\theta}(B)$ .

Fix now  $\tau = -2$ .

- If  $0 < p < 1$ , we will prove that the problem (4.1) has no solution for any  $R > 0$ . Suppose the contrary, as in [1], we get a positive *radial* solution to  $-\operatorname{div}(|x|^\theta \nabla v) = |x|^\ell v^p$  in  $\mathbb{R}^N \setminus B_{R_1}$  for some  $R_1 > R$ . Under the change of variable (4.4), there exists  $a, s_0 > 0$  such that

$$-w'' = as^{-2}w^p \text{ in } (0, s_0), \quad w(0) = 0.$$

However, this is impossible seeing the proof of Theorem 8 (b) in [1], where we can replace  $N$  just by  $N' > 2$ .

- If  $p = 1$ , as before, the existence of a positive solution to (1.1) is equivalent to the existence of a positive *radial* solution. Assume that  $u$  is a radial solution to (1.1), let  $w(s) = u(e^s)$ , then  $w$  is a solution to

$$w'' + (N' - 2)w' + w = 0 \text{ in } (0, \infty), \quad w(0) = 0, \quad \lim_{s \rightarrow \infty} w(s) = 0.$$

Clearly, such a positive solution  $w$  exists if and only if the equation  $\lambda^2 + (N' - 2)\lambda + 1 = 0$  has only negative roots, or equivalently,  $N' \geq 4$ .



- Finally let  $p > 1$ . Under the Kelvin transform, we are led to consider (1.5). We check readily that  $\tau' = \sigma - \theta = (N' - 2)(p - 1) - 2 > -2$  and  $p$  is subcritical, since here

$$p < \frac{N' + 2 + 2\tau'}{N' - 2} \Leftrightarrow (p - 1)(N' - 2) > 0$$

There exists a positive radial solution of (1.5) as in section 2, using the result in [9].  $\square$

**Remark 4.4.** For  $N' = 2$ ,  $\tau > -2$ , similarly to [1], we can show that the problem (4.1) admits a positive solution if and only if there exists  $a > 0$  such that  $e^{at}f(t)$  is integrable at  $+\infty$ . Consequently, in this case, for any  $p \in \mathbb{R}$ , the equation (4.1) does not admit positive solution.

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