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Exploring the Set of Square Matrices with Circulant Operators

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Abstract: Circulant operators are useful tools to highlight the way in which the twelve triangular matrix forms of Pascal triangle (collectively referred to as G-matrices) relate to each other. In this paper, we recall the notions of G-matrices and circulant operators, and then we present some main algebraic properties of these operators. We generalize these operators to the case of any square matrix.

Key-Words: Pascal Triangle, Pascal Matrices, k-Circulant Operators, Square Matrix bipartition, cobweb Partition, G-matrices.

1 Introduction

This section recalls the definition of the twelve G-matrices, $G_{k,n}$ $1 \leq k \leq 12$ as in [2, 22]. In the sequel, we write $[G_{k,n}]_{ij}$ to denote the coefficient at the intersection of row i and column j and denote by \mathcal{G}_n the set of the twelve G-matrices. Generic binomial coefficients $\frac{a!}{(a-b)!b!}$ will be denoted $\binom{a}{b}$. The Pascal triangle is assumed to comprise levels $0 \dots n$.

1.1 The set \mathcal{G}_n of G-matrices

The following definitions present the twelve triangular matrix forms of the Pascal triangle. They are organized into four subsets.

Definition 1 - *The NE/sw G-matrices*

$G_{1,n}$, $G_{5,n}$ and $G_{9,n}$

$$[G_{1,n}]_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$[G_{5,n}]_{ij} = \binom{n-i}{n-j} \quad i, j = 0, \dots, n \quad (2)$$

$$[G_{9,n}]_{ij} = \begin{cases} \binom{n+i-j}{i} & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

$$\mathcal{G}_n^{NE/sw} = \{G_{1,n}, G_{5,n}, G_{9,n}\}$$

Definition 2 - *The SE/nw G-matrices*

$G_{2,n}$, $G_{6,n}$ and $G_{10,n}$

$$[G_{2,n}]_{ij} = \binom{j}{n-i} \quad i, j = 0, \dots, n \quad (4)$$

$$[G_{6,n}]_{ij} = \begin{cases} \binom{2n-i-j}{n-j} & \text{if } i+j \geq n \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

$$[G_{10,n}]_{ij} = \binom{i}{n-j} \quad i, j = 0, \dots, n \quad (6)$$

$$\mathcal{G}_n^{SE/nw} = \{G_{2,n}, G_{6,n}, G_{10,n}\}$$

Definition 3 - *The SW/ne G-matrices*

$G_{3,n}$, $G_{7,n}$ and $G_{11,n}$

$$[G_{3,n}]_{ij} = \begin{cases} \binom{n+j-i}{j} & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

$$[G_{7,n}]_{ij} = \binom{n-j}{n-i} \quad i, j = 0, \dots, n \quad (8)$$

$$[G_{11,n}]_{ij} = \binom{i}{j} \quad i, j = 0, \dots, n \quad (9)$$

$$\mathcal{G}_n^{SW/ne} = \{G_{3,n}, G_{7,n}, G_{11,n}\}$$

Definition 4 - The NW/se G-matrices

$G_{4,n}$, $G_{8,n}$ and $G_{12,n}$

$$[G_{4,n}]_{ij} = \binom{n-i}{j} \quad i, j = 0, \dots, n \quad (10)$$

$$[G_{8,n}]_{ij} = \binom{n-j}{i} \quad i, j = 0, \dots, n \quad (11)$$

$$[G_{12,n}]_{ij} = \begin{cases} \binom{i+j}{i} & \text{if } i+j \leq n \\ 0 & \text{otherwise} \end{cases} \quad (12)$$

$$\mathcal{G}_n^{NW/se} = \{G_{4,n}, G_{8,n}, G_{12,n}\}$$

Notation:

$$\mathcal{G}_n = \mathcal{G}_n^{NW/sw} \cup \mathcal{G}_n^{SE/nw} \cup \mathcal{G}_n^{SW/ne} \cup \mathcal{G}_n^{NW/se} \quad (13)$$

It is easy to observe that:

Theorem 5 If $B \neq B'$ and $b \neq b'$ then

$$\mathcal{G}_n^{B/b} \cap \mathcal{G}_n^{B'/b'} = \emptyset$$

Proof:

Considering definitions 1-4, one can easily establish this statement.

Matrices like the above and other matrix forms derived from the Pascal triangle are encountered in [1, 3, 4, 9, 10, 16, 8]. Every author simply refers to a particular form encountered as the Pascal matrix.

Some authors refer to $G_{1,n}$ and $G_{11,n}$ respectively as upper and lower triangular Pascal matrices [5, 19, 17, 18, 20, 21]. In [7, 8], the fourth G-matrix, $G_{4,n}$ is called binomial matrix. But, as it can be seen from the above definitions, this is just one of the three possible upper-left triangular forms. In some works, one recognizes $G_{12,n}$, the third northwest triangular matrix form. [12] refers to $G_{7,n}$ as the reflection of $G_{11,n}$ about the main anti-diagonal.

Considering the matrix formulation (14) of the well-known binomial theorem as proposed in [3],

$$(1+t) (1+t)^2 \dots (1+t)^n = \underbrace{\begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \dots & \dots & \binom{n}{0} \\ 0 & \binom{1}{1} & \dots & \dots & \binom{n}{1} \\ \vdots & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \binom{n}{n} \end{pmatrix}}_{G_{1,n}} \quad (14)$$

these matrices can be seen as some reordering of the components of the polynomial power basis vector, and hence of the polynomial space [3].

To our knowledge, this is the first attempt to systematically investigate matrix forms of the Pascal triangle as mathematical objects in their own right.

Section 1.2 presents the four circulant operators and the way they intervene in generating the set of the twelve Pascal matrices, starting with one of them. Section 3 generalize to the case of any square matrix.

1.2 The circulant operators

In this section, the circulant operators are presented as transformations of the generic matrix subscript vector (i, j) ($0 \leq i, j \leq n$).

Definition 6 - Circulant operator

The α -circulant operator

$$(i, j) \xrightarrow{\alpha} (i, i+j) \pmod{n+1}$$

The β -circulant operator

$$(i, j) \xrightarrow{\beta} (-1+i-j, j) \pmod{n+1}$$

The γ -circulant operator

$$(i, j) \xrightarrow{\gamma} (i-j, j) \pmod{n+1}$$

The δ -circulant operator

$$(i, j) \xrightarrow{\delta} (i, 1+i+j) \pmod{n+1}$$

Figure 1 illustrates how α transforms a square matrix.

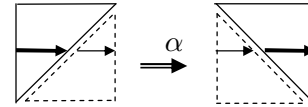


Figure 1: The α -circulant operator

Circulant transformations carry circulant matrices to associated row- or column-constant matrices. To see this, consider the circulant matrix C given below and its associated row-constant matrix denoted \tilde{C} :

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ c_n & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_1 \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} c_n & c_n & c_n & \cdots & c_n \\ c_{n-1} & c_{n-1} & c_{n-1} & \cdots & c_{n-1} \\ c_{n-2} & c_{n-2} & c_{n-2} & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_1 & c_1 & \cdots & c_1 \end{pmatrix}$$

One readily sees for example that: $\tilde{C} = \beta C$.

1.3 Circulant Operators action on \mathcal{G}_n

Theorem 7 Action of circulant operators on \mathcal{G}_n

If $i \in \{1, 5, 9\}$ then $\beta G_{i,n} = G_{i+1[12],n}$

If $i \in \{2, 6, 10\}$ then $\delta G_{i,n} = G_{i+1[12],n}$

If $i \in \{3, 7, 11\}$ then $\gamma G_{i,n} = G_{i+1[12],n}$

If $i \in \{4, 8, 12\}$ then $\alpha G_{i,n} = G_{i+1[12],n}$

where $i + 1[12] \equiv i + 1 \pmod{12}$.

From this stage one can now derived all the twelve G-matrices starting with any particular one. Thus starting with $G_{1,n}$, the twelve G-matrices can be derived in their natural order by the following sequence:

$$\begin{aligned} \mathbf{G}_{1,n} &\xrightarrow{\beta} G_{2,n} \xrightarrow{\delta} G_{3,n} \xrightarrow{\gamma} G_{4,n} \xrightarrow{\alpha} G_{5,n} \\ G_{5,n} &\xrightarrow{\beta} G_{6,n} \xrightarrow{\delta} G_{7,n} \xrightarrow{\gamma} G_{8,n} \xrightarrow{\alpha} G_{9,n} \\ G_{9,n} &\xrightarrow{\beta} G_{10,n} \xrightarrow{\delta} G_{11,n} \xrightarrow{\gamma} G_{12,n} \xrightarrow{\alpha} \mathbf{G}_{1,n} \end{aligned}$$

This is a cycle since $(\alpha\gamma\delta\beta)^3 G_{1,n} = G_{1,n}$. Moreover, it is readily verified that $\{G_{1,n}, G_{5,n}, G_{9,n}\}$ is globally $(\alpha\gamma\delta\beta)$ -invariant.

Similar inspections show that: $(\gamma\delta\beta\alpha)^3 G_{4,n} = G_{4,n}$, $(\delta\beta\alpha\gamma)^3 G_{3,n} = G_{3,n}$, $(\beta\alpha\gamma\delta)^3 G_{2,n} = G_{2,n}$,

The circulant transformations provide a useful framework for investigating the G-matrices. Section 2 gives some basic properties of these operators which arise in the p -factor compositions. In section 3 we will extend their application to square matrices to provide new insights into the structure of the space of square matrices.

The set of circulant operators will be denoted by \mathcal{C}_n in dimension $n + 1$.

2 Main algebraic properties of the circulant operators

In this section the set \mathcal{C}_n is consider as an *alphabet*. The aim here is to give a complete list of identities

which arise in the p -factor products (composition) of circulant operators ($p = 1, 2, 3$).

2.1 Notations and Definitions

Let's put forward these notations first:

$$\langle \alpha \rangle = \{\epsilon, \alpha, \alpha^2, \alpha^3, \dots, \alpha^n\}$$

$$\langle \beta \rangle = \{\epsilon, \beta, \beta^2, \beta^3, \dots, \beta^n\}$$

$$\langle \delta \rangle = \{\epsilon, \delta, \delta^2, \delta^3, \dots, \delta^n\}$$

$$\langle \gamma \rangle = \{\epsilon, \gamma, \gamma^2, \gamma^3, \dots, \gamma^n\}$$

$$\langle \xi \rangle^\diamond = \{\xi, \xi^2, \xi^3, \dots, \xi^n\} \quad \forall \xi \in \mathcal{C}_n$$

with $\epsilon = \alpha^{n+1} = \beta^{n+1} = \delta^{n+1} = \gamma^{n+1}$ and $\epsilon M = M$ for all $(n + 1) \times (n + 1)$ matrix.

Definition 8 - Empty string

ϵ is called empty string and $\forall \xi \in \mathcal{C}_n, \xi^0 \equiv \epsilon$

Definition 9 - k -circulant operator

An operator ξ is k -circulant if:

$$\exists \omega \in \mathcal{C}_n, \text{ such that } \xi = \omega^k$$

Theorem 10 For all $\xi, \psi \in \mathcal{C}_n$,

$$\langle \xi \rangle^\diamond \cap \langle \psi \rangle^\diamond = \emptyset \quad (15)$$

2.2 Identities without k -circulants

This section present all identities of exactly p -factors strings for $p = 2, 3, 4$ without k -circulant as substring.

2.3 Two-factors identities

Theorem 11

$$\alpha\delta = \delta\alpha \quad \beta\gamma = \gamma\beta \quad (16)$$

Proof:

It is easy to see that:

$$\begin{aligned} \alpha\delta(i, j) &= \alpha[\delta(i, j)] \\ &= \alpha[(i, 1 + i + j) \pmod{(n + 1)}] \\ &= \alpha[(i, 1 + i + j) \pmod{(n + 1)}] \\ &= (i, 1 + 2i + j) \pmod{(n + 1)} \end{aligned}$$

and

$$\begin{aligned}
\delta\alpha(i, j) &= \delta[\alpha(i, j)] \\
&= \delta[(i, i + j) \pmod{(n + 1)}] \\
&= \delta[(i, i + j)] \pmod{(n + 1)} \\
&= (i, 1 + 2i + j) \pmod{(n + 1)}
\end{aligned}$$

This leads to $\alpha\delta = \delta\alpha$.

The second identity is also established in the same way.

2.4 Three-factors identities

We give above a collection of twelve identities.

$$\alpha\beta\alpha = \beta\alpha\beta \quad \alpha\gamma\alpha = \gamma\alpha\gamma \quad (17)$$

$$\alpha\delta\beta = \delta\alpha\beta \quad \alpha\delta\gamma = \delta\alpha\gamma \quad (18)$$

$$\alpha\gamma\delta = \beta\alpha\gamma \quad \alpha\gamma\delta = \delta\beta\alpha \quad (19)$$

$$\alpha\gamma\delta = \gamma\delta\beta \quad \beta\alpha\gamma = \delta\beta\alpha \quad (20)$$

$$\beta\alpha\gamma = \gamma\delta\beta \quad \beta\delta\beta = \delta\beta\delta \quad (21)$$

$$\delta\beta\alpha = \gamma\delta\beta \quad \delta\gamma\delta = \gamma\delta\gamma \quad (22)$$

One of the main results deriving from these identities is:

Theorem 12 -

row-column-row=column-row-column

For all $r \in \{\alpha, \gamma\}$ and all $c \in \{\beta, \delta\}$,

$$rcr = crc$$

Proof:

Using the same technic as for equations 16, it is easy to verify that:

$\alpha\beta\alpha = \beta\alpha\beta$ and $\alpha\gamma\alpha = \gamma\alpha\gamma$ on the one hand

$\beta\delta\beta = \delta\beta\delta$ and $\delta\gamma\delta = \gamma\delta\gamma$ on the other hand.

Another result is provided by:

Theorem 13

$$\alpha\gamma\delta = \beta\alpha\gamma = \delta\beta\alpha = \gamma\delta\beta$$

Proof:

It directly follows using the definitions.

These identities give the base on which all other p -factor identities ($p > 3$) will be derived. From this stage we now give some relevant identities with k -circulant operators for $k = 1, 2, 3$.

2.5 k -circulants identities

2.5.1 Two-factors for $k = 2$

$$\alpha^2\delta^2 = \delta^2\alpha^2 \quad \beta^2\gamma^2 = \gamma^2\beta^2 \quad (23)$$

2.5.2 Three-factors for $k = 2$

$$\alpha^2\delta^2\beta^2 = \delta^2\alpha^2\beta^2 \quad \alpha^2\delta^2\gamma^2 = \delta^2\alpha^2\gamma^2 \quad (24)$$

$$\beta^2\gamma^2\alpha^2 = \gamma^2\beta^2\alpha^2 \quad \beta^2\gamma^2\delta^2 = \gamma^2\beta^2\delta^2 \quad (25)$$

2.5.3 Two-factors for $k = 3$

$$\alpha^3\delta^3 = \delta^3\alpha^3 \quad \beta^3\gamma^3 = \gamma^3\beta^3 \quad (26)$$

2.5.4 Three-factors for $k = 3$

$$\alpha^3\delta^3\beta^3 = \delta^3\alpha^3\beta^3 \quad \alpha^3\delta^3\gamma^3 = \delta^3\alpha^3\gamma^3 \quad (27)$$

$$\beta^3\gamma^3\alpha^3 = \gamma^3\beta^3\alpha^3 \quad \beta^3\gamma^3\delta^3 = \gamma^3\beta^3\delta^3 \quad (28)$$

2.6 Identities involving k -circulant and circulant operators ($k \geq 2$)

We give here a theorem which can be considered as a basic tool to derive identities involving both k -circulant and circulant operators.

Theorem 14

$$\alpha^k\delta^k = \delta^k\alpha^k \quad \beta^k\gamma^k = \gamma^k\beta^k \quad (29)$$

$$\alpha\delta^k = \delta^k\alpha \quad \beta\gamma^k = \gamma^k\beta \quad (30)$$

$$\alpha^k\delta = \delta\alpha^k \quad \beta^k\gamma = \gamma\beta^k \quad (31)$$

Proof:

One can prove the first one (for example) by recurrence.

Assuming that $\alpha^k\delta^k = \delta^k\alpha^k$ for a $k \in \mathbb{N}$,

$$\begin{aligned}
\alpha^{n+1}\delta^{k+1} &= \alpha^k\alpha\delta\delta^k \\
&= \alpha^k\delta\alpha\delta^{k-1} \\
&= \alpha^k\delta^2\alpha\delta^{k-1} \\
&= \dots \\
&= \alpha^k\delta^p\alpha\delta^{k-p+1} \\
&= \dots \\
&= \alpha^k\delta^k\alpha\delta \\
&= \delta^k\alpha^k\delta\alpha \\
&= \delta^k\alpha^{k-1}\alpha\delta\alpha \\
&= \delta^k\alpha^{k-1}\delta\alpha^2 \\
&= \dots \\
&= \delta^k\alpha^{k-p+1}\delta\alpha^p \\
&= \dots \\
&= \delta^k\alpha\delta\alpha^k \\
&= \delta^{k+1}\alpha^{k+1}
\end{aligned}$$

or:

$$\begin{aligned}
\alpha^{k+1}\delta^{k+1} &= \alpha\alpha^k\delta^k\delta \\
&= \alpha\delta^k\alpha^k\delta \\
&= \alpha\delta\delta^{k-1}\alpha^{k-1}\alpha\delta \\
&= \delta\alpha\delta^{k-1}\alpha^{k-1}\delta\alpha \\
&= \dots \\
&= \delta^p\alpha\delta^{k-p}\alpha^{k-p}\delta\alpha^p \\
&= \dots \\
&= \delta^k\alpha\delta\alpha^k \\
&= \delta^k\delta\alpha\alpha^k \\
&= \delta^{k+1}\alpha^{k+1}
\end{aligned}$$

On the other side:

$$\begin{aligned}
\alpha\delta^k &= (\alpha\delta)\delta^{k-1} \\
&= \delta\alpha\delta^{k-1} \\
&= \delta(\alpha\delta)\delta^{k-2} \\
&= \delta^2\alpha\delta^{k-2} \\
&\dots \\
&= \delta^p\alpha\delta^{k-p} \\
&\dots \\
&= \delta^k\alpha
\end{aligned}$$

This ends the proof.

3 Applications and generalizations

3.1 Triangular bipartition of a square matrix

Now, the four triangular bipartitions of any square matrix as illustrated in figure 2 are presented. The triangular sub-block in full line indicate which block includes the main or anti diagonal.

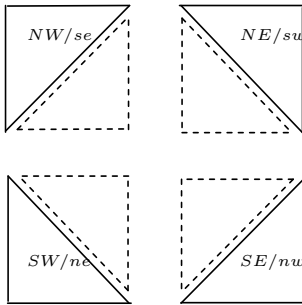


Figure 2: Four bipartitions of a square matrix

Definition 15 - Triangular bipartitions

Given A , a $(n+1) \times (n+1)$ matrix with subscripts i and j ranging from 0 to n .

(i) The North-East/south-west bipartition

The matrices A_{NE} and A_{sw} , such that:

$$[A_{NE}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

and

$$[A_{sw}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$$

define the North-East/south-west bipartition of A and

$$A = A_{NE} + A_{sw} \equiv A_{NE/sw}$$

(ii) The South-West/north-east bipartition

The matrices A_{SW} and A_{ne} , such that:

$$[A_{SW}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}$$

and

$$[A_{ne}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases}$$

define the South-West/north-east bipartition of A and

$$A = A_{SW} + A_{ne} \equiv A_{SW/ne}$$

(iii) The North-West/south-east bipartition

The matrices A_{NW} and A_{se} , such that:

$$[A_{NW}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i+j \leq n \\ 0 & \text{if } i+j > n \end{cases}$$

and

$$[A_{se}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i+j > n \\ 0 & \text{if } i+j \leq n \end{cases}$$

define the North-West/south-east bipartition of A and

$$A = A_{NW} + A_{se} \equiv A_{NW/se}$$

(iv) The South-East/north-west bipartition

The matrices A_{SE} and A_{nw} , such that:

$$[A_{SE}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i+j \geq n \\ 0 & \text{if } i+j < n \end{cases}$$

and

$$[A_{nw}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i+j < n \\ 0 & \text{if } i+j \geq n \end{cases}$$

define the South-East/north-west bipartition of A and

$$A = A_{SE} + A_{nw} \equiv A_{SE/nw}$$

Theorem 16

$$\alpha(A_{NW/se}) = \alpha(A_{NW}) + \alpha(A_{se}) \quad (32)$$

$$\beta(A_{NE/sw}) = \beta(A_{NE}) + \beta(A_{sw}) \quad (33)$$

$$\delta(A_{SE/nw}) = \delta(A_{SE}) + \delta(A_{nw}) \quad (34)$$

$$\gamma(A_{SW/ne}) = \gamma(A_{SW}) + \gamma(A_{ne}) \quad (35)$$

| Subsets | Matrices |
|-------------------------|--------------------------------|
| $\mathcal{G}_n^{NE/sw}$ | $G_{1,n} \ G_{5,n} \ G_{9,n}$ |
| $\mathcal{G}_n^{SE/nw}$ | $G_{2,n} \ G_{6,n} \ G_{10,n}$ |
| $\mathcal{G}_n^{SW/ne}$ | $G_{3,n} \ G_{7,n} \ G_{11,n}$ |
| $\mathcal{G}_n^{NW/se}$ | $G_{4,n} \ G_{8,n} \ G_{12,n}$ |

Table 1: Partition of the set \mathcal{G}_n

Proof:

It is derived from the fact that:

- A_B and A_b are triangular matrices (juste as G-matrices),
- circulant operators, if judiciously applied, doesn't mixe two triangular blocks in a square matrix as we have seen it in the case of G-matrices.

3.2 Partition of the set \mathcal{G}_n with the circulant operators and generalization

The four bipartitions of the square matrix leads to a partition of \mathcal{G}_n into four subsets as it is described in table 1. One can observe that the subsets in the first column of this table (up to down) are globally invariant by the action of $\alpha\gamma\delta\beta$, $\beta\alpha\gamma\delta$, $\delta\beta\alpha\gamma$ and $\gamma\delta\beta\alpha$ respectively.

Globally, it follows that:

$$\begin{array}{ccccccc} \mathcal{G}_n^{NE/sw} & \xrightarrow{\beta} & \mathcal{G}_n^{SE/nw} & \xrightarrow{\delta} & \mathcal{G}_n^{SW/ne} & \xrightarrow{\gamma} & \\ \mathcal{G}_n^{NW/se} & \xrightarrow{\alpha} & \mathcal{G}_n^{NE/sw} & & & & \end{array}$$

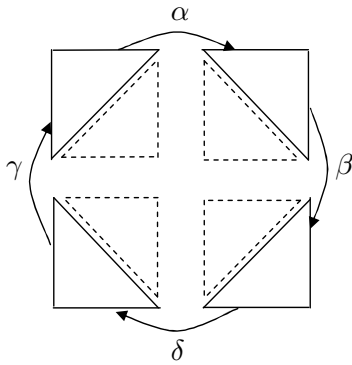


Figure 3: Four bipartitions linked by circulant operators

where $\xrightarrow{\mathcal{O}}$ denotes the transformation by the operator \mathcal{O} as shown in figure 3 which summarizes the generalization to the case of any square matrix A shown in table 2, where $\mathcal{O}A_{i,n} = A_{i+1[12],n}$ and $A_{1,n} = A$, with:
 $i + 1 [12] = i + 1 \text{ modulus } 12$,

| Subsets | Matrices |
|-------------------------|--------------------------------|
| $\mathcal{A}_n^{NE/sw}$ | $A_{1,n} \ A_{5,n} \ A_{9,n}$ |
| $\mathcal{A}_n^{SE/nw}$ | $A_{2,n} \ A_{6,n} \ A_{10,n}$ |
| $\mathcal{A}_n^{SW/ne}$ | $A_{3,n} \ A_{7,n} \ A_{11,n}$ |
| $\mathcal{A}_n^{NW/se}$ | $A_{4,n} \ A_{8,n} \ A_{12,n}$ |

Table 2: Partition of the set \mathcal{A}_n

$$\text{and } \mathcal{O} = \begin{cases} \beta & \text{if } i \in \{1, 5, 9\} \\ \delta & \text{if } i \in \{2, 6, 10\} \\ \gamma & \text{if } i \in \{3, 7, 11\} \\ \alpha & \text{if } i \in \{4, 8, 12\} \end{cases}$$

Thus:

$$\begin{array}{ccccccc} \mathbf{A}_{1,n} & \xrightarrow{\beta} & A_{2,n} & \xrightarrow{\delta} & A_{3,n} & \xrightarrow{\gamma} & A_{4,n} & \xrightarrow{\alpha} & A_{5,n} \\ A_{5,n} & \xrightarrow{\beta} & A_{6,n} & \xrightarrow{\delta} & A_{7,n} & \xrightarrow{\gamma} & A_{8,n} & \xrightarrow{\alpha} & A_{9,n} \\ A_{9,n} & \xrightarrow{\beta} & A_{10,n} & \xrightarrow{\delta} & A_{11,n} & \xrightarrow{\gamma} & A_{12,n} & \xrightarrow{\alpha} & \mathbf{A}_{1,n} \end{array}$$

This leads to the set \mathcal{A}_n of twelve new matrices from the single matrix A . Figure 4 shows a graph with \mathcal{A}_n as set of vertices, and the circulant operators as transitions labels.

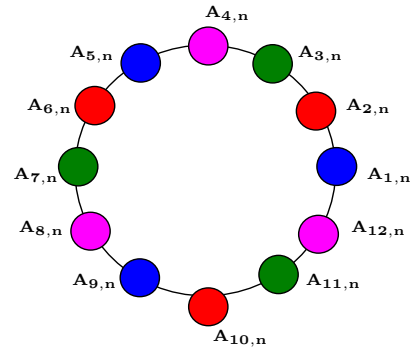


Figure 4: Ring behavior of \mathcal{A}_n with alphabet in $\{\alpha, \beta, \gamma, \delta\}$

4 New structure of the space of square matrices

Let $A = ([A]_{i,j})_{0 \leq i,j \leq n}$ be a square matrix:

- the set of the coefficients of A will be denoted by $\text{Coef}(A)$,
- the permutation group on $\{(i, j), 0 \leq i, j \leq n\}$ will be denoted by $\text{Perm}(n)$,
- For $\sigma \in \text{Perm}(n)$:

$$\begin{aligned} \sigma((i, j)) &\equiv (i_\sigma, j_\sigma) \\ [A]_{i_\sigma, j_\sigma} &\equiv [\sigma A]_{i, j}, \quad 0 \leq i, j \leq n, \\ \sigma \mathcal{A}_n &= \{\sigma A, A \in \mathcal{A}_n\} \\ \mathcal{A}_{k, n} &= \{\sigma A_{k, n}, \sigma \in \text{Perm}(n)\} \text{ where } 1 \leq k \leq 12 \end{aligned}$$

Definition 17 - A permutation $\sigma \in \text{Perm}(n)$ preserves a partition $A_{B/b}$ if there exists σ_B and σ_b such that

$$\sigma = \sigma_B \sigma_b$$

where :

$$\begin{aligned} [A_b]_{i_{\sigma_B}, j_{\sigma_B}} &= [A_b]_{i, j} \quad \forall i, j \\ [A_B]_{i_{\sigma_b}, j_{\sigma_b}} &= [A_B]_{i, j} \quad \forall i, j \end{aligned}$$

In the sequel this property will be denoted by the following equality: $\sigma(A_{B/b}) = A_{B/b}$.

As example of such a permutation, one can easily take the permutation derived from the composition $\beta\delta\gamma\alpha$

Theorem 18 -

Partition and exploration in $\bigcup_{\sigma} \{\sigma \mathcal{A}_n\}$

If $\sigma_i \in \text{Perm}(n)$ and $\sigma(A_{NE/sw}) = A_{NE/sw}$ then

$$\sigma_p \mathcal{A}_n \cap \sigma_q \mathcal{A}_n = \emptyset \quad \forall p \neq q \quad (36)$$

$$A_{r, n} \cap A_{s, n} = \emptyset \quad \forall r \neq s \quad (37)$$

Proof:

It follows directly from the definition of the set $A_{r, n}$.

The above theorem leads to a partitioning of the set \mathcal{A}_n into orbits $\sigma \mathcal{A}_{r, n}$ as represented in figure 5.

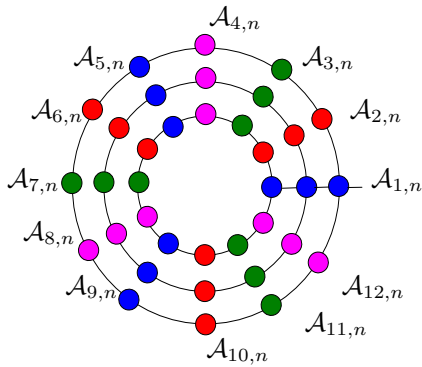


Figure 5: Cobweb partitioning of $\bigcup_{\sigma} \{\sigma \mathcal{A}_n\}$

5 Conclusion and discussions

In this work we have presented an investigation of twelve triangular matrix forms of the Pascal Triangle. We have then presented one way in which the G-matrices relate to each other by introducing a set of circulant operators. As algebraic tools, these operators have several properties among which we gave some concerning their composition. Besides these properties, these operators turn out to provide a new insight into the structure of the space of square matrices. We established that, beginning with a single matrix, one can derive in this space concentric orbits describing a cobweb partition graph. Exploring the cobweb cyclically, one can reach the twelve matrices (which can be seen as state of a grid system) through the four bipartitions. On the other hand, a radial trajectory enable to access *states* in which the two sets of coefficients in the bipartition are globally invariants. As matter of applications, this study can provide some basic tools to effective simulations in the area of grid networks.

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