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# The Twelve Triangular Matrix Forms of the Pascal Triangle: a Systematic Approach with the Set of Circulant Operators

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*Abstract:* This work is devoted to a systematic investigation of triangular matrix forms of the Pascal Triangle. To start, the twelve matrix forms (collectively referred to as G-matrices) are presented. To highlight one way in which the G-matrices relate to each other, a set of four operators named circulant operators is introduced. These operators provide a new insight into the structure of the space of square matrices.

*Key-Words:* Pascal Matrices, Pascal Triangle, Circulant Operators, Square Matrix bipartition, cobweb Partition, G-matrices,

## 1 Introduction

This section presents the definition of the twelve G-matrices,  $G_{k,n}$   $1 \leq k \leq 12$  as in [2]. In the sequel, we write  $[G_{k,n}]_{ij}$  to denote the coefficient at the intersection of row  $i$  and column  $j$  and denote by  $\mathcal{G}_n$  the set of the twelve G-matrices. Generic binomial coefficients  $\frac{a!}{(a-b)!b!}$  will be denoted  $\binom{a}{b}$ . The Pascal triangle is assumed to comprise levels  $0 \dots n$ .

**Definition 1 :**  $G_{1,n}$

$$[G_{1,n}]_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases}$$

**Definition 2 :**  $G_{2,n}$

$$[G_{2,n}]_{ij} = \binom{j}{n-i} \quad i, j = 0, \dots, n$$

**Definition 3 :**  $G_{3,n}$

$$[G_{3,n}]_{ij} = \begin{cases} \binom{n+j-i}{j} & \text{if } i \geq j \\ 0 & \text{otherwise} \end{cases}$$

**Definition 4 :**  $G_{4,n}$

$$[G_{4,n}]_{ij} = \binom{n-i}{j} \quad i, j = 0, \dots, n$$

**Definition 5 :**  $G_{5,n}$

$$[G_{5,n}]_{ij} = \binom{n-i}{n-j} \quad i, j = 0, \dots, n$$

**Definition 6 :**  $G_{6,n}$

$$[G_{6,n}]_{ij} = \begin{cases} \binom{2n-i-j}{n-j} & \text{if } i+j \geq n \\ 0 & \text{otherwise} \end{cases}$$

**Definition 7 :**  $G_{7,n}$

$$[G_{7,n}]_{ij} = \binom{n-j}{n-i} \quad i, j = 0, \dots, n$$

**Definition 8 :**  $G_{8,n}$

$$[G_{8,n}]_{ij} = \binom{n-j}{i} \quad i, j = 0, \dots, n$$

**Definition 9 :**  $G_{9,n}$

$$[G_{9,n}]_{ij} = \begin{cases} \binom{n+i-j}{i} & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases}$$

**Definition 10 :**  $G_{10,n}$

$$[G_{10,n}]_{ij} = \binom{i}{n-j} \quad i, j = 0, \dots, n$$

**Definition 11** :  $G_{11,n}$

$$[G_{11,n}]_{ij} = \binom{i}{j} \quad i, j = 0, \dots, n$$

**Definition 12** :  $G_{12,n}$

$$[G_{12,n}]_{ij} = \begin{cases} \binom{i+j}{i} & \text{if } i+j \leq n \\ 0 & \text{otherwise} \end{cases}$$

Matrices like the above and other matrix forms derived from the Pascal triangle are encountered in [3, 9, 10, 1, 4, 16, 8]. Every author simply refers to the particular form encountered as the Pascal matrix. Some authors refer to  $G_{1,n}$  and  $G_{11,n}$  respectively as upper and lower triangular Pascal matrices [5]. In [7, 8], the fourth G-matrix,  $G_{4,n}$  is called binomial matrix. But, as it can be seen from the above definitions, this is just one of the three possible upper-left triangular forms. In [11] one recognizes  $G_{12,n}$ , the third northwest triangular matrix form. [12] refers to  $G_{7,n}$  as the reflection of  $G_{11,n}$  about the main antidiagonal.

Considering the matrix formulation (1) of the well-known binomial theorem as proposed in [3],

$$(1 \ t \ t^2 \ \dots \ t^n) \underbrace{\begin{pmatrix} (1+t) & (1+t)^2 & \dots & (1+t)^n \\ \binom{0}{0} & \binom{1}{0} & \dots & \dots & \binom{n}{0} \\ 0 & \binom{1}{1} & \dots & \dots & \binom{n}{1} \\ \vdots & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \binom{n}{n} \end{pmatrix}}_{G_{1,n}} \quad (1)$$

these matrices can be seen as some reordering of the components of the polynomial power basis vector, and hence of the polynomial space [3].

To our knowledge, this is the first systematic attempt to investigate matrix forms of the Pascal triangle as mathematical objects in their own right.

Section 2 presents the set of four operators named circulant operators and the way they intervene in generating the set of the twelve Pascal matrices, starting with one of them. Section 3 generalize to the case of any matrix.

## 2 The circulant operators

This section is dedicated to the presentation of the set of circulant operators.

These operators are presented as transformations of

the generic matrix subscript vector  $(i, j)$  ( $0 \leq i, j \leq n$ ).

**Definition 13** The  $\alpha$ -circulant operator

$$(i, j) \xrightarrow{\alpha} (i, i+j) \pmod{n+1}$$

**Definition 14** The  $\beta$ -circulant operator

$$(i, j) \xrightarrow{\beta} (-1+i-j, j) \pmod{n+1}$$

**Definition 15** The  $\gamma$ -circulant operator

$$(i, j) \xrightarrow{\gamma} (i-j, j) \pmod{n+1}$$

**Definition 16** The  $\delta$ -circulant operator

$$(i, j) \xrightarrow{\delta} (i, 1+i+j) \pmod{n+1}$$

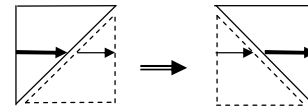


Figure 1: The  $\alpha$ -circulant operator

As illustration, figure 1 shows how  $\alpha$  transforms a square matrix.

Circulant transformations carry circulant matrices to associated row- or column-constant matrices. To see this, consider the circulant matrix  $C$  given below and its associated row-constant matrix denoted  $\tilde{C}$ :

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ c_n & c_1 & c_2 & \dots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \dots & c_1 \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} c_n & c_n & c_n & \dots & c_n \\ c_{n-1} & c_{n-1} & c_{n-1} & \dots & c_{n-1} \\ c_{n-2} & c_{n-2} & c_{n-2} & \dots & c_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_1 & c_1 & \dots & c_1 \end{pmatrix}$$

It is readily seen for example that:  $\tilde{C} = \beta C$ .

**Theorem 17** Action of circulant operators on  $\mathcal{G}_n$

- If  $i \in \{1, 5, 9\}$  then  $\beta G_{i,n} = G_{i+1[12],n}$
- If  $i \in \{2, 6, 10\}$  then  $\delta G_{i,n} = G_{i+1[12],n}$
- If  $i \in \{3, 7, 11\}$  then  $\gamma G_{i,n} = G_{i+1[12],n}$
- If  $i \in \{4, 8, 12\}$  then  $\alpha G_{i,n} = G_{i+1[12],n}$

where  $i + 1[12] \equiv i + 1 \pmod{12}$ .

The stage is now set to derive all the twelve G-matrices starting with any particular one. Thus starting with  $G_{1,n}$ , the twelve G-matrices can be derived in their natural order by the following sequence:

$$G_{1,n} \xrightarrow{\beta} G_{2,n} \xrightarrow{\delta} G_{3,n} \xrightarrow{\gamma} G_{4,n} \xrightarrow{\alpha} G_{5,n}$$

$$G_{5,n} \xrightarrow{\beta} G_{6,n} \xrightarrow{\delta} G_{7,n} \xrightarrow{\gamma} G_{8,n} \xrightarrow{\alpha} G_{9,n}$$

$$G_{9,n} \xrightarrow{\beta} G_{10,n} \xrightarrow{\delta} G_{11,n} \xrightarrow{\gamma} G_{12,n} \xrightarrow{\alpha} G_{1,n}$$

This is a cycle since  $(\alpha\gamma\delta\beta)^3 G_{1,n} = G_{1,n}$ . Moreover, it is readily verified that  $\{G_{1,n}, G_{5,n}, G_{9,n}\}$  is globally  $(\alpha\gamma\delta\beta)$ -invariant.

Similar inspections show that:  $(\gamma\delta\beta\alpha)^3 G_{4,n} = G_{4,n}$ ,  $(\delta\beta\alpha\gamma)^3 G_{3,n} = G_{3,n}$ ,  $(\beta\alpha\gamma\delta)^3 G_{2,n} = G_{2,n}$ .

The circulant transformations provide a useful framework for investigating the G-matrices. Section 3 extends their application to square matrices to provide new insights into the structure of the space of square matrices.

### 3 Applications and generalization

#### 3.1 Triangular bipartition of a square matrix

Now, the four triangular bipartitions of any square matrix are illustrated in figure 2. The triangular sub-block in full line indicates the block embracing the entries of the main- or anti- diagonal.

**Definition 18** The North-East/south-west bipartition  
Let  $A$  be a  $(n + 1) \times (n + 1)$  matrix with subscripts  $i$  and  $j$  ranging from 0 to  $n$ . The matrices two  $A_{NE}$  and  $A_{sw}$ :

$$[A_{NE}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

and

$$[A_{sw}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$$

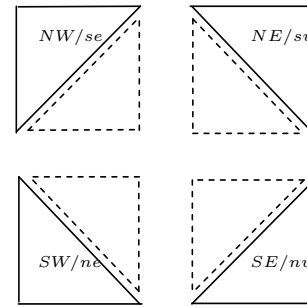


Figure 2: Four bipartitions of a square matrix

define the North-East/south-west bipartition of  $A$ . Symbolically, we write:

$$A = A_{NE} + A_{sw} \equiv A_{NE/sw}$$

**Definition 19** The South-West/north-east bipartition  
Let  $A$  be a  $(n + 1) \times (n + 1)$  matrix with subscripts  $i$  and  $j$  ranging from 0 to  $n$ . The two matrices  $A_{SW}$  and  $A_{ne}$ :

$$[A_{SW}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}$$

and

$$[A_{ne}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases}$$

define the South-West/north-east bipartition of  $A$ . Symbolically, we write:

$$A = A_{SW} + A_{ne} \equiv A_{SW/ne}$$

**Definition 20** The North-West/south-east bipartition  
let  $A$  be a  $(n + 1) \times (n + 1)$  matrix with subscripts  $i$  and  $j$  ranging from 0 to  $n$ . The two matrices  $A_{NW}$  and  $A_{se}$ :

$$[A_{NW}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j \leq n \\ 0 & \text{if } i + j > n \end{cases}$$

and

$$[A_{se}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j > n \\ 0 & \text{if } i + j \leq n \end{cases}$$

define the North-West/south-east bipartition of  $A$ . Symbolically, we write:

$$A = A_{NW} + A_{se} \equiv A_{NW/se}$$

**Definition 21** The South-East/north-west bipartition  
Let  $A$  be a  $(n + 1) \times (n + 1)$  matrix with subscripts  $i$  and  $j$  ranging from 0 to  $n$ . The matrices  $A_{SE}$  and  $A_{nw}$ , such that:

$$[A_{SE}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j \geq n \\ 0 & \text{if } i + j < n \end{cases}$$

and

$$[A_{nw}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j < n \\ 0 & \text{if } i + j \geq n \end{cases}$$

Subsets	Matrices
$\mathcal{G}_n^{NE/sw}$	$G_{1,n} \ G_{5,n} \ G_{9,n}$
$\mathcal{G}_n^{SE/nw}$	$G_{2,n} \ G_{6,n} \ G_{10,n}$
$\mathcal{G}_n^{SW/ne}$	$G_{3,n} \ G_{7,n} \ G_{11,n}$
$\mathcal{G}_n^{NW/se}$	$G_{4,n} \ G_{8,n} \ G_{12,n}$

Table 1: Partition of the set  $\mathcal{G}_n$

define the **South-East/north-west bipartition** of  $A$ . Symbolically, we write:

$$A = A_{SE} + A_{nw} \equiv A_{SE/nw}$$

### 3.2 Partition of the set $\mathcal{G}_n$ induced by the circulant operators and generalization

As it can be seen, each of the twelve G-matrices presents a natural bipartition: a major sub-block with non-zero entries and a minor sub-block with zero entries.

This leads to the natural partition of  $\mathcal{G}_n$  as presented in Table 1.

It is easily verified that the set  $\mathcal{G}_n^{NE/sw}$  is invariant (globally) under the action of  $\alpha\gamma\delta\beta$ .

So are  $\mathcal{G}_n^{SE/nw}$ ,  $\mathcal{G}_n^{SW/ne}$  and  $\mathcal{G}_n^{NW/se}$  respectively under  $\beta\alpha\gamma\delta$ ,  $\delta\beta\alpha\gamma$  and  $\gamma\delta\beta\alpha$ .

It follows that in additions, it can be shown that:

$$\begin{array}{ccccc} \mathcal{G}_n^{NE/sw} & \xrightarrow{\beta} & \mathcal{G}_n^{SE/nw} & \xrightarrow{\delta} & \mathcal{G}_n^{SW/ne} & \xrightarrow{\gamma} & \mathcal{G}_n^{NW/se} \\ \mathcal{G}_n^{NW/se} & \xrightarrow{\alpha} & \mathcal{G}_n^{NE/sw} & & & & \end{array}$$

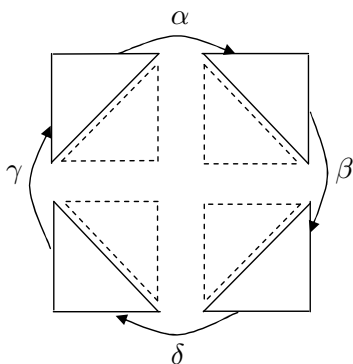


Figure 3: Four bipartitions linked by circulant operators

In what precedes,  $\xrightarrow{\mathcal{O}}$  denotes the transformation by the operator  $\mathcal{O}$  as shown in figure 3 which gives the generalization to the case of any square matrix  $A$ . Table 2 summarizes the general case by analogy to  $\mathcal{G}_n$ .

More explicitly,

Subsets	Matrices
$\mathcal{A}_n^{NE/sw}$	$A_{1,n} \ A_{5,n} \ A_{9,n}$
$\mathcal{A}_n^{SE/nw}$	$A_{2,n} \ A_{6,n} \ A_{10,n}$
$\mathcal{A}_n^{SW/ne}$	$A_{3,n} \ A_{7,n} \ A_{11,n}$
$\mathcal{A}_n^{NW/se}$	$A_{4,n} \ A_{8,n} \ A_{12,n}$

Table 2: Partition of the set  $\mathcal{A}_n$

$\mathcal{O}A_{i,n} = A_{i+1 [12],n}$  and  $A_{1,n} = A$  with:

$$i + 1 [12] = i + 1 \text{ modulus } 12,$$

$$\text{and } \mathcal{O} = \begin{cases} \beta & \text{if } i \in \{1, 5, 9\} \\ \delta & \text{if } i \in \{2, 6, 10\} \\ \gamma & \text{if } i \in \{3, 7, 11\} \\ \alpha & \text{if } i \in \{4, 8, 12\} \end{cases}$$

In fine:

$$\begin{array}{ccccccc} \mathbf{A}_{1,n} & \xrightarrow{\beta} & A_{2,n} & \xrightarrow{\delta} & A_{3,n} & \xrightarrow{\gamma} & A_{4,n} & \xrightarrow{\alpha} & A_{5,n} \\ A_{5,n} & \xrightarrow{\beta} & A_{6,n} & \xrightarrow{\delta} & A_{7,n} & \xrightarrow{\gamma} & A_{8,n} & \xrightarrow{\alpha} & A_{9,n} \\ A_{9,n} & \xrightarrow{\beta} & A_{10,n} & \xrightarrow{\delta} & A_{11,n} & \xrightarrow{\gamma} & A_{12,n} & \xrightarrow{\alpha} & \mathbf{A}_{1,n} \end{array}$$

This leads to the set  $\mathcal{A}_n$  of twelve new matrices from the single matrix  $A$ . Figure 4 shows a graph with  $\mathcal{A}_n$  as set of vertices, and the circulant operators as transitions labels.

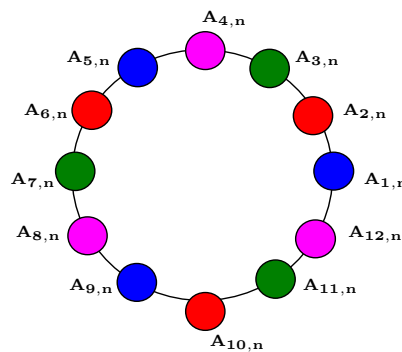


Figure 4: Ring behavior of  $\mathcal{A}_n$  with alphabet in  $\{\alpha, \beta, \gamma, \delta\}$

## 4 New structure of the space of square matrices

Let  $A = ([A]_{i,j})_{0 \leq i,j \leq n}$  be a square matrix:

- the set of the coefficients of  $A$  will be denoted by

$Coef(A)$ ,

- the permutation group on  $\{(i, j), 0 \leq i, j \leq n\}$  will be denoted by  $Perm(n)$ ,

- For  $\sigma \in Perm(n)$ :

$$\sigma((i, j)) \equiv (i_\sigma, j_\sigma)$$

$$[A]_{i_\sigma, j_\sigma} \equiv [\sigma A]_{i, j}, \quad 0 \leq i, j \leq n,$$

-  $\sigma \mathcal{A}_n = \{\sigma A, A \in \mathcal{A}_n\}$

-  $\mathcal{A}_{k,n} = \{\sigma A_{k,n}, \sigma \in Perm(n)\}$  where  $1 \leq k \leq 12$

**Definition 22** A permutation  $\sigma \in Perm(n)$  preserves a partition  $A_{B/b}$  if there exists  $\sigma_B$  and  $\sigma_b$  such that

$$\sigma = \sigma_B \sigma_b$$

where :

$$[A_b]_{i_{\sigma_B}, j_{\sigma_B}} = [A_b]_{i, j} \quad \forall i, j$$

$$[A_B]_{i_{\sigma_b}, j_{\sigma_b}} = [A_B]_{i, j} \quad \forall i, j$$

In the sequel this property will be symbolized by the identity:  $\sigma(A_{B/b}) = A_{B/b}$ .

**Theorem 23** Partition of  $\sigma \mathcal{A}_n$

If  $\sigma \in Perm(n)$  and  $\sigma(A_{NE/sw}) = A_{NE/sw}$  then

$$\sigma \mathcal{A}_n = \bigcup_{k=1}^{12} \mathcal{A}_{k,n} \quad \text{and} \quad \mathcal{A}_{r,n} \cap \mathcal{A}_{s,n} = \emptyset \quad \forall r \neq s$$

**Proof:**

It follows directly from the definition of the set  $\mathcal{A}_{r,n}$ .  
□

The above theorem leads to a partitioning of the set  $\mathcal{A}_n$  into orbits  $\sigma \mathcal{A}_{r,n}$  as represented in figure 5.

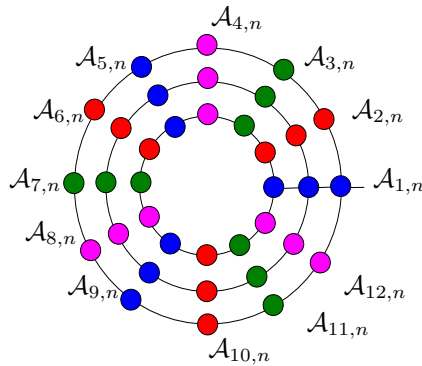


Figure 5: Cobweb partitioning of  $\{\sigma \mathcal{A}_n\}_\sigma$

## 5 Conclusion and discussions

In this work we present an investigation of twelve triangular matrix forms of the Pascal Triangle (collectively referred to as G-matrices).

To highlight one way in which the G-matrices relate to each other, the set of circulant operators is introduced. These operators turn out to provide a new insight into the structure of the space of square matrices. It is established that, beginning with a single matrix, one can derive in this space concentric orbits describing a cobweb partition graph. Exploring the cobweb cyclically, one can reach the twelve matrices (which can be seen as state of a particular system) through the four bipartitions. On the other hand, a radial trajectory enables access "states" in which the two sets of coefficients in the bipartition are globally.

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