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The Twelve Triangular Matrix Forms of the Pascal Triangle: a Systematic Approach with the Set of Circulant Operators

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Abstract: This work is devoted to a systematic investigation of triangular matrix forms of the Pascal Triangle. To start, the twelve matrix forms (collectively referred to as G-matrices) are presented. To highlight one way in which the G-matrices relate to each other, a set of four operators named circulant operators is introduced. These operators provide a new insight into the structure of the space of square matrices.

Key–Words: Pascal Matrices, Pascal Triangle, Circulant Operators, Square Matrix bipartition, cobweb Partition, G-matrices,

1 Introduction

This section presents the definition of the twelve G-matrices, $G_{k,n} \quad 1 \leq k \leq 12$ as in [2]. In the sequel, we write $[G_{k,n}]_{ij}$ to denote the coefficient at the intersection of row i and column j and denote by $\mathcal{G}_n$ the set of the twelve G-matrices. Generic binomial coefficients $\frac{a!}{(a-b)!b!}$ will be denoted $\binom{a}{b}$. The Pascal triangle is assumed to comprise levels $0 \ldots n$.

Definition 1 *:* $G_{1,n}$

$$[G_{1,n}]_{ij} = \begin{cases} \begin{pmatrix} i \\ j \end{pmatrix} & if\ i \leq j \\ 0 & otherwise \end{cases}$$

Definition 2 *:* $G_{2,n}$

$$[G_{2,n}]_{ij} = \begin{pmatrix} j \\ n-i \end{pmatrix} \quad i,\, j = 0,\, ...,\, n$$

Definition 3 *:* $G_{3,n}$

$$[G_{3,n}]_{ij} = \begin{cases} \begin{pmatrix} n+j-i \\ j \end{pmatrix} & if\ i \geq j \\ 0 & otherwise \end{cases}$$

Definition 4 *:* $G_{4,n}$

$$[G_{4,n}]_{ij} = \begin{pmatrix} n-i \\ j \end{pmatrix} \quad i,\, j = 0,\, ...,\, n$$

Definition 5 *:* $G_{5,n}$

$$[G_{5,n}]_{ij} = \begin{pmatrix} n-i \\ n-j \end{pmatrix} \quad i,\, j = 0,\, ...,\, n$$

Definition 6 *:* $G_{6,n}$

$$[G_{6,n}]_{ij} = \begin{cases} \begin{pmatrix} 2n-i-j \\ n-j \end{pmatrix} & if\ i+j \geq n \\ 0 & otherwise \end{cases}$$

Definition 7 *:* $G_{7,n}$

$$[G_{7,n}]_{ij} = \begin{pmatrix} n-j \\ n-i \end{pmatrix} \quad i,\, j = 0,\, ...,\, n$$

Definition 8 *:* $G_{8,n}$

$$[G_{8,n}]_{ij} = \begin{pmatrix} n-j \\ i \end{pmatrix} \quad i,\, j = 0,\, ...,\, n$$

Definition 9 *:* $G_{9,n}$

$$[G_{9,n}]_{ij} = \begin{cases} \begin{pmatrix} n+i-j \\ i \end{pmatrix} & if\ i \leq j \\ 0 & otherwise \end{cases}$$

Definition 10 *:* $G_{10,n}$

$$[G_{10,n}]_{ij} = \begin{pmatrix} i \\ n-j \end{pmatrix} \quad i,\, j = 0,\, ...,\, n$$

Definition 11 *:* $G_{11,n}$

$$[G_{11,n}]_{ij} = \binom{i}{j} \quad i,\, j = 0,\, ...,\, n$$

Definition 12 *:* $G_{12,n}$

$$[G_{12,n}]_{ij} = \begin{cases} \binom{i+j}{i} & if\ i+j \leq n \\ 0 & otherwise \end{cases}$$

Matrices like the above and other matrix forms derived from the Pascal triangle are encountered in [3, 9, 10, 1, 4, 16, 8]. Every author simply refers to the particular form encountered as the Pascal matrix.
Some authors refer to $G_{1,n}$ and $G_{11,n}$ respectively as upper and lower triangular Pascal matrices [5]. In [7, 8], the fourth G-matrix, $G_{4,n}$ is called binomial matrix. But, as it can be seen from the above definitions, this is just one of the three possible upper-left triangular forms. In [11] one recognizes $G_{12,n}$, the third northwest triangular matrix form. [12] refers to $G_{7,n}$ as the reflection of $G_{11,n}$ about the main antidiagonal.
Considering the matrix formulation (1) of the well-known binomial theorem as proposed in [3],

$$\begin{pmatrix} 1 & (1+t) & (1+t)^2 & \dots & (1+t)^n \end{pmatrix} = \begin{pmatrix} 1 & t & t^2 & \dots & t^n \end{pmatrix} \underbrace{\begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \dots & \dots & \binom{n}{0} \\ 0 & \binom{1}{1} & \dots & \dots & \binom{n}{1} \\ \vdots & \ddots & \ddots & \dots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \binom{n}{n} \end{pmatrix}}_{G_{1,n}} \quad (1)$$

these matrices can be seen as some reordering of the components of the polynomial power basis vector, and hence of the polynomial space [3].
To our knowledge, this is the first systematic attempt to investigate matrix forms of the Pascal triangle as mathematical objects in their own right.
Section 2 presents the set of four operators named circulant operators and the way they intervene in generating the set of the twelve Pascal matrices, starting with one of them. Section 3 generalize to the case of any matrix.

2 The circulant operators

This section is dedicated to the presentation of the set of circulant operators.
These operators are presented as transformations of the generic matrix subscript vector (i, j) $(0 \leq i,\, j \leq n)$.

Definition 13 *The* α*-circulant operator*

$$(\ i\,,\, j\ ) \xrightarrow{\alpha} (\ i\,,\, i+j\ ) \quad \bmod n+1$$

Definition 14 *The* β*-circulant operator*

$$(\ i\,,\, j\ ) \xrightarrow{\beta} (\ -1+i-j\,,\, j\ ) \quad \bmod n+1$$

Definition 15 *The* γ*-circulant operator*

$$(\ i\,,\, j\ ) \xrightarrow{\gamma} (\ i-j\,,\, j\ ) \quad \bmod n+1$$

Definition 16 *The* δ*-circulant operator*

$$(\ i\,,\, j\ ) \xrightarrow{\delta} (\ i\,,\, 1+i+j\ ) \quad \bmod n+1$$

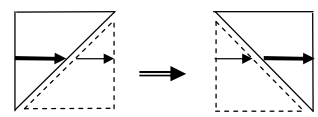

Figure 1: The α-circulant operator

As illustration, figure 1 shows how α transforms a square matrix.

Circulant transformations carry circulant matrices to associated row- or column-constant matrices. To see this, consider the circulant matrix C given below and its associated row-constant matrix denoted $\tilde{C}$:

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_n \\ c_n & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_n & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{pmatrix},$$

$$\tilde{C} = \begin{pmatrix} c_n & c_n & c_n & \cdots & c_n \\ c_{n-1} & c_{n-1} & c_{n-1} & \cdots & c_{n-1} \\ c_{n-2} & c_{n-2} & c_{n-2} & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_1 & c_1 & c_1 & \cdots & c_1 \end{pmatrix}$$

It is readily seen for example that: $\tilde{C} = \beta C$.

Theorem 17 *Action of circulant operators on $\mathcal{G}_n$*

$$\begin{array}{rcl} \textit{If } i \in \{1,\,5,\,9\} & \textit{then} & \beta G_{i,n} = G_{i+1\,[12],n} \\ \textit{If } i \in \{2,\,6,\,10\} & \textit{then} & \delta G_{i,n} = G_{i+1\,[12],n} \\ \textit{If } i \in \{3,\,7,\,11\} & \textit{then} & \gamma G_{i,n} = G_{i+1\,[12],n} \\ \textit{If } i \in \{4,\,8,\,12\} & \textit{then} & \alpha G_{i,n} = G_{i+1\,[12],n} \end{array}$$

where $i+1\,[12] \equiv i+1 \mod 12$.

The stage is now set to derive all the twelve G-matrices starting with any particular one. Thus starting with $G_{1,n}$, the twelve G-matrices can be derived in their natural order by the following sequence:

$$\mathbf{G_{1,n}} \xrightarrow{\beta} G_{2,n} \xrightarrow{\delta} G_{3,n} \xrightarrow{\gamma} G_{4,n} \xrightarrow{\alpha} G_{5,n}$$

$$G_{5,n} \xrightarrow{\beta} G_{6,n} \xrightarrow{\delta} G_{7,n} \xrightarrow{\gamma} G_{8,n} \xrightarrow{\alpha} G_{9,n}$$

$$G_{9,n} \xrightarrow{\beta} G_{10,n} \xrightarrow{\delta} G_{11,n} \xrightarrow{\gamma} G_{12,n} \xrightarrow{\alpha} \mathbf{G_{1,n}}$$

This is a cycle since $(\alpha\gamma\delta\beta)^3 G_{1,n} = G_{1,n}$. Moreover, it is readily verified that $\{G_{1,n},\, G_{5,n},\, G_{9,n}\}$ is globally $(\alpha\gamma\delta\beta)$-invariant.
Similar inspections show that: $(\gamma\delta\beta\alpha)^3 G_{4,n} = G_{4,n}$, $(\delta\beta\alpha\gamma)^3 G_{3,n} = G_{3,n}$, $(\beta\alpha\gamma\delta)^3 G_{2,n} = G_{2,n}$,
The circulant transformations provide a useful framework for investigating the G-matrices. Section 3 extends their application to square matrices to provide new insights into the structure of the space of square matrices.

3 Applications and generalization

3.1 Triangular bipartition of a square matrix

Now, the four triangular bipartitions of any square matrix are illustrated in figure 2. The triangular sub-block in full line indicates the block embracing the entries of the main- or anti- diagonal.

Definition 18 *The* **N***orth-***E***ast/***s***outh-***w***est bipartition*
Let A be a $(n+1)\times(n+1)$ matrix with subscripts i and j ranging from 0 to n. The matrices two A_{NE} and A_{sw}:

$$[A_{NE}]_{ij} = \begin{cases} [A]_{ij} & if \quad i \le j \\ 0 & if \quad i > j \end{cases}$$

and

$$[A_{sw}]_{ij} = \begin{cases} [A]_{ij} & if \quad i > j \\ 0 & if \quad i \le j \end{cases}$$

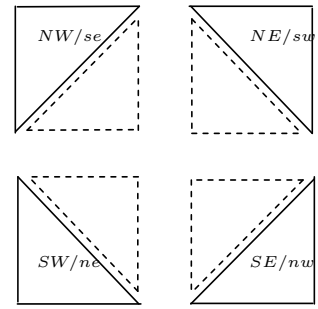

Figure 2: Four bipartitions of a square matrix

define the **N***orth-***E***ast/***s***outh-***w***est bipartition of A. Symbolically, we write:*

$$A = A_{NE} + A_{sw} \equiv A_{NE/sw}$$

Definition 19 *The* **S***outh-***W***est/***n***orth-***e***ast bipartition*
Let A be a $(n+1)\times(n+1)$ matrix with subscripts i and j ranging from 0 to n. The two matrices A_{SW} and A_{ne}:

$$[A_{SW}]_{ij} = \begin{cases} [A]_{ij} & if \quad i \ge j \\ 0 & if \quad i < j \end{cases}$$

and

$$[A_{ne}]_{ij} = \begin{cases} [A]_{ij} & if \quad i < j \\ 0 & if \quad i \ge j \end{cases}$$

define the **S***outh-***W***est/***n***orth-***e***ast bipartition of A. Symbolically, we write:*

$$A = A_{SW} + A_{ne} \equiv A_{SW/ne}$$

Definition 20 *The* **N***orth-***W***est/***s***outh-***e***ast bipartition*
let A be a $(n+1)\times(n+1)$ matrix with subscripts i end j ranging from 0 to n. The two matrices A_{NW} and A_{se}:

$$[A_{NW}]_{ij} = \begin{cases} [A]_{ij} & if \quad i+j \le n \\ 0 & if \quad i+j > n \end{cases}$$

and

$$[A_{se}]_{ij} = \begin{cases} [A]_{ij} & if \quad i+j > n \\ 0 & if \quad i+j \le n \end{cases}$$

define the **N***orth-***W***est/***s***outh-***e***ast bipartition of A. Symbolically, we write:*

$$A = A_{NW} + A_{se} \equiv A_{NW/se}$$

Definition 21 *The* **S***outh-***E***ast/***n***orth-***w***est bipartition*
Let A be a $(n+1)\times(n+1)$ matrix with subscripts i end j ranging from 0 to n. The matrices A_{SE} and A_{nw}, such that:

$$[A_{SE}]_{ij} = \begin{cases} [A]_{ij} & if \quad i+j \ge n \\ 0 & if \quad i+j < n \end{cases}$$

and

$$[A_{nw}]_{ij} = \begin{cases} [A]_{ij} & if \quad i+j < n \\ 0 & if \quad i+j \ge n \end{cases}$$

$Subsets$	$Matrices$
${\mathcal{G}_n}^{NE/sw}$	$G_{1,n}$ $G_{5,n}$ $G_{9,n}$
${\mathcal{G}_n}^{SE/nw}$	$G_{2,n}$ $G_{6,n}$ $G_{10,n}$
${\mathcal{G}_n}^{SW/ne}$	$G_{3,n}$ $G_{7,n}$ $G_{11,n}$
${\mathcal{G}_n}^{NW/se}$	$G_{4,n}$ $G_{8,n}$ $G_{12,n}$

Table 1: Partition of the set $\mathcal{G}_n$

define the **S***outh-***E***ast/***n***orth-***w***est bipartition of* A*. Symbolically, we write:*

$$A = A_{SE} + A_{nw} \equiv A_{SE/nw}$$

3.2 Partition of the set $\mathcal{G}_n$ induced by the circulant operators and generalization

As it can be seen, each of the twelves G-matrices presents a natural bipartition: a major sub-block with non-zero entries and a minor sub-block with zero entries.
This leads to the natural partition of $\mathcal{G}_n$ as presented in Table 1.
It is easily verified that the set ${\mathcal{G}_n}^{NE/sw}$ is invariant (globally) under the action of $\alpha\gamma\delta\beta$.
So are ${\mathcal{G}_n}^{SE/nw}$, ${\mathcal{G}_n}^{SW/ne}$ and ${\mathcal{G}_n}^{NW/se}$ respectively under $\beta\alpha\gamma\delta$, $\delta\beta\alpha\gamma$ and $\gamma\delta\beta\alpha$.
It follows that in additions, it can be shown that:
${\mathcal{G}_n}^{NE/sw} \xrightarrow{\beta} {\mathcal{G}_n}^{SE/nw} \xrightarrow{\delta} {\mathcal{G}_n}^{SW/ne} \xrightarrow{\gamma}$
${\mathcal{G}_n}^{NW/se} \xrightarrow{\alpha} {\mathcal{G}_n}^{NE/sw}$

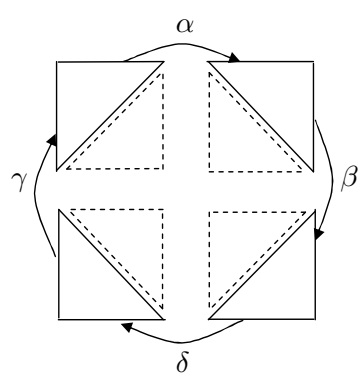

Figure 3: Four bipartitions linked by circulant operators

In what precedes, $\xrightarrow{\mathcal{O}}$ denotes the transformation by the operator $\mathcal{O}$ as shown in figure 3 which gives the generalization to the case of any square matrix A.
Table 2 summarizes the general case by analogy to $\mathcal{G}_n$.
More explicitly,

$Subsets$	$Matrices$
$\mathcal{A}_n^{NE/sw}$	$A_{1,n}$ $A_{5,n}$ $A_{9,n}$
$\mathcal{A}_n^{SE/nw}$	$A_{2,n}$ $A_{6,n}$ $A_{10,n}$
$\mathcal{A}_n^{SW/ne}$	$A_{3,n}$ $A_{7,n}$ $A_{11,n}$
$\mathcal{A}_n^{NW/se}$	$A_{4,n}$ $A_{8,n}$ $A_{12,n}$

Table 2: Partition of the set $\mathcal{A}_n$

$\mathcal{O}A_{i,n} = A_{i+1\,[12],n}$ and $A_{1,n} = A$
with:
$i + 1\,[12] = i + 1\, modulus\ 12$,

and $\quad \mathcal{O} = \begin{cases} \beta & \text{if } i \in \{1,\ 5,\ 9\} \\ \delta & \text{if } i \in \{2,\ 6,\ 10\} \\ \gamma & \text{if } i \in \{3,\ 7,\ 11\} \\ \alpha & \text{if } i \in \{4,\ 8,\ 12\} \end{cases}$

In fine:

$$\mathbf{A_{1,n}} \xrightarrow{\beta} A_{2,n} \xrightarrow{\delta} A_{3,n} \xrightarrow{\gamma} A_{4,n} \xrightarrow{\alpha} A_{5,n}$$

$$A_{5,n} \xrightarrow{\beta} A_{6,n} \xrightarrow{\delta} A_{7,n} \xrightarrow{\gamma} A_{8,n} \xrightarrow{\alpha} A_{9,n}$$

$$A_{9,n} \xrightarrow{\beta} A_{10,n} \xrightarrow{\delta} A_{11,n} \xrightarrow{\gamma} A_{12,n} \xrightarrow{\alpha} \mathbf{A_{1,n}}$$

This leads to the set $\mathcal{A}_n$ of twelve new matrices from the single matrix A. Figure 4 shows a graph with $\mathcal{A}_n$ as set of vertices, and the circulant operators as transitions labels.

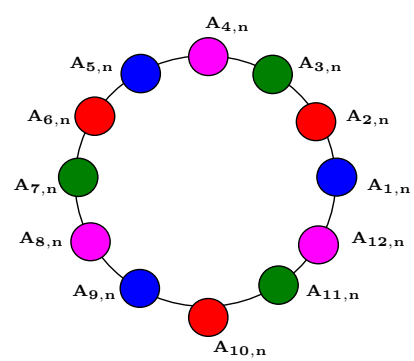

Figure 4: Ring behavior of $\mathcal{A}_n$ with alphabet in $\{\alpha,\ \beta,\ \gamma,\ \delta\}$

4 New structure of the space of square matrices

Let $A = \left([A]_{i,j}\right)_{0 \le i,j \le n}$ be a square matrix:
- the set of the coefficients of A will be denoted by

$Coef(A)$,
- the permutation group on $\{(i,j),\ 0 \leq i,j \leq n\}$ will be denoted by $Perm(n)$,
- For $\sigma \in Perm(n)$:
$\sigma((i,j)) \equiv (i_\sigma\ ,\ j_\sigma)$
$[A]_{i_\sigma,j_\sigma} \equiv [\sigma A]_{i,j}\ ,\quad 0 \leq i,j \leq n,$
- $\sigma\mathcal{A}_n = \{\sigma A,\ A \in \mathcal{A}_n\}$
- $\mathcal{A}_{k,n} = \{\sigma A_{k,n},\ \sigma \in Perm\,(n)\}$ where $1 \leq k \leq 12$

Definition 22 *A permutation $\sigma \in Perm(n)$ preserves a partition $A_{B/b}$ if there exists σ_B and σ_b such that*

$$\sigma = \sigma_B\,\sigma_b$$

where :
$[A_b]_{i_{\sigma_B},j_{\sigma_B}} = [A_b]_{i,j}\ \ \forall\, i,\ j$
$[A_B]_{i_{\sigma_b},j_{\sigma_b}} = [A_B]_{i,j}\ \ \forall\, i,\ j$

In the sequel this property will be symbolized by the identity: $\sigma\left(A_{B/b}\right) = A_{B/b}$.

Theorem 23 *Partition of $\sigma\mathcal{A}_n$*
If $\sigma \in Perm(n)$ and $\sigma\left(A_{NE/sw}\right) = A_{NE/sw}$ then

$$\sigma\mathcal{A}_n = \bigcup_{k=1}^{12} \mathcal{A}_{k,n} \quad and \quad \mathcal{A}_{r,n} \cap \mathcal{A}_{s,n} = \emptyset\ \forall\, r \neq s$$

Proof:
It follows directly from the definition of the set $\mathcal{A}_{r,n}$. □

The above theorem leads to a partitioning of the set $\mathcal{A}_n$ into orbits $\sigma\mathcal{A}_{r,n}$ as represented in figure 5.

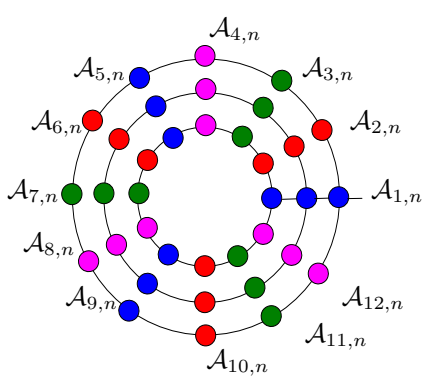

Figure 5: Cobweb partitioning of $\{\sigma\mathcal{A}_n\}_\sigma$

5 Conclusion and discussions

In this work we present an investigation of twelve triangular matrix forms of the Pascal Triangle (collectively referred to as G-matrices).

To highlight one way in which the G-matrices relate to each other, the set of circulant operators is introduce. These operators turn out to provide a new insight into the structure of the space of square matrices. It is establish that, beginning with a single matrix, one can derive in this space concentric orbits describing a cobweb partition graph. Exploring the cobweb cyclically, one can reach the twelve matrices (which can be seen as state of a particular system) through the four bipartitions. On the other hand, a radial trajectory enable to access "states" in which the two sets of coefficients in the bipartition are globally.

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