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# NONLINEAR ELLIPTIC SYSTEMS WITH COUPLED GRADIENT TERMS

AHMED ATTAR, RACHID BENTIFOUR, EL-HAJ LAAMRI

ABSTRACT. In this paper, we analyze the existence and non-existence of nonnegative solutions to a class of nonlinear elliptic systems of type :

$$\begin{cases} -\Delta u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ -\Delta v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u, v \geq 0 & \text{in } \Omega, \end{cases}$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  and  $p, q \geq 1$ .  $f, g$  are nonnegative measurable functions with additional hypotheses and  $\lambda, \mu \geq 0$ .

This extends previous similar results obtained in the case where the right-hand sides are potential and gradient terms, see [2, 4].

## 1. INTRODUCTION

The main goal of this work is to analyze the issue of existence and non-existence of nonnegative solutions to a class of nonlinear elliptic systems with gradient terms. More precisely, we deal with the following model :

$$(1.1) \quad \begin{cases} -\Delta u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ -\Delta v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u, v \geq 0 & \text{in } \Omega, \end{cases}$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$ ,  $(p, q) \in [1, +\infty]^2$ ,  $(\lambda, \mu) \in (0, +\infty)^2$  and  $f, g$  are nonnegative measurable functions that satisfy additional assumptions that will be specified later.

The purpose of this paper is to find the conditions between  $(f, g)$  and  $(\lambda, \mu)$  which allow us to establish existence of solutions to System (1.1). Moreover, we will prove some non-existence results that show, in some sense, the optimality of our assumptions. In this article, we focus our attention on the proof of the existence of weak solutions ; by weak solution, we mean solution in the sense of distributions (see Definition 2.5).

Let us mention that a class of nonlinear elliptic systems with gradient terms has come to the fore in some electrochemical models of engineering, and in some other models in fluid dynamics. More recently, another class was introduced in [29, 30] in the theory of mean-field games ; see also [16] for an in-depth analysis.

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Û Let us recall some previously known results related to the present study.

*First Case : One Equation.* In the case where the gradient term acts as source term, the problem

$$(1.2) \quad -\Delta w = |\nabla w|^q + \lambda h \quad \text{in } \Omega, \quad w \in W_0^{1,q}(\Omega),$$

is known as a stationary Hamilton–Jacobi equation, or the stationary Kardar–Parisi–Zhang equation. This problem has been extensively investigated. Contributions in this topic include [12, 13, 26, 28, 3, 24] and the references therein. Needless to say, these references do not exhaust the rich literature on the subject. To be more precise, if  $q \leq 2$  (sub-quadratic and quadratic growth), existence of solutions is obtained using comparison arguments and *a priori* estimates. The super-quadratic case ( $q > 2$ ) has been studied, for instance, in the references [28], [35] and [34] where the existence of solutions is obtained using potential theory and some fixed-point arguments.

Concerning the case of one equation where the gradient term acts as an absorption term, the associated problem appears in theory of mean-field games, we refer to the pioneering works [32], [29] and [30] for more details.

*Second Case : Systems.* In contrast to single equations, only few results are available in the literature for systems where the source terms depend on the gradient. Moreover, only partial results have been obtained, see for instance [2, 4, 17, 15, 22]. We recall the results of [22] and [15] where the dependence on the gradient appears in one equation. In [4] and [2], nonlinear systems with potential and gradient terms are studied. Existence of solutions is obtained under some optimal conditions on the data.

The main objective of this work is to analyze System (1.1), where the dependence on the gradient appears in both equations, without any restriction on the value of  $p$  and  $q$ . To the best of our knowledge, no result was previously obtained under these general assumptions.

Moreover, in order to show that (1.1) has a solution, it is necessary to have (minimal) regularity assumptions on the data  $f$ ,  $g$  and smallness conditions on  $\lambda$  and  $\mu$ . Indeed, such conditions are necessary even in the case of a single equation. More precisely, for the problem (1.2), we have the following existence result proved in [28],

**Theorem 1.1.** *Assume that  $h \in L^1(\Omega)$  is a nonnegative function. Define*

$$(1.3) \quad m(h) := \inf_{\{\phi \in C_0^\infty(\Omega); \phi \neq 0\}} \frac{\int_{\Omega} |\nabla \phi|^{q'} dx}{\int_{\Omega} h |\phi|^{q'} dx}.$$

*Then, Problem (1.2) has a nonnegative solution  $u \in W_0^{1,q}(\Omega)$  for  $\lambda > 0$  if and only if  $m(h) > 0$ . Moreover, there exists  $\lambda^*(m(h)) > 0$  such that Problem (1.2) has a solution for all  $\lambda < \lambda^*(m(h))$ .*

In term of elliptic capacity, the positivity of  $m(h)$  is equivalent to the following fact : denoting by  $\text{cap}_{1,q'}(E)$ , the elliptic capacity defined by

$$\text{cap}_{1,q'}(E) := \inf \left\{ \int_{\Omega} |\nabla \phi|^{q'} dx \text{ with } \phi \in C_0^\infty(\Omega) \text{ and } \phi \geq \chi_E \right\},$$

then  $m(h) > 0$  if and only if

$$(1.4) \quad |E|_h \leq C \text{cap}_{1,q'}(E),$$

where  $E \subset \Omega$  is a measurable set and  $|E|_h = \int_E h(y)dy$ . We refer to [28, Theorems 1.1, 1.2, 3.1] for more details.

The paper is organized as follows. In section 2, we introduce some useful tools such as regularity results for elliptic problems, the sense in which solutions are defined, and the Schauder fixed-point Theorem used in this paper. In Section 3, we state and prove the main existence results of this work. Under suitable conditions on the data, we prove the existence of a nonnegative solution in suitable Sobolev spaces. Some examples and non-existence results showing the optimality of our conditions are also presented in Section 4. Finally, in the last section we treat more general systems where the interaction between  $u$  and  $v$  appears in both gradient and potential terms.

Throughout this paper, for  $a, b \geq 0$ , we use the notation  $a \simeq b$ , to say that,

$$C_1 a \leq b \leq C_2 a,$$

with  $C_1, C_2 > 0$  are independent of  $a, b$ .

## 2. PRELIMINARIES AND USEFUL RESULTS

For the convenience of the reader and for the sake of completeness, we recall in this short section some classical results we will use in our proofs.

In order to prove the existence results of this paper we will invoke the following Proposition and the famous Schauder fixed-point Theorem.

**Proposition 2.1.** *Let  $h \in L^1(\Omega)$  and  $\{h_n\}_n \subset L^r(\Omega)$  with  $1 < r < +\infty$ . Assume that  $h_n \rightarrow h$  strongly in  $L^1(\Omega)$  and  $\{h_n\}_n$  is bounded in  $L^r(\Omega)$ . Then,  $h \in L^r(\Omega)$  and for any  $a \in [1, r)$ , we have the following interpolation inequality*

$$(2.1) \quad \|h_n - h\|_{L^a(\Omega)} \leq \|h_n - h\|_{L^1(\Omega)}^\theta \|h_n - h\|_{L^r(\Omega)}^{1-\theta} \text{ where } \theta := \frac{r-a}{a(r-1)}$$

*In particular,  $h_n \rightarrow h$  strongly in  $L^a(\Omega)$ .*

**Theorem 2.2 (Schauder fixed-point Theorem).** *Assume that  $E$  is a closed convex set of a Banach space  $X$ . Let  $T$  be a continuous and compact mapping from  $E$  into itself. Then,  $T$  has a fixed point in  $E$ .*

Systematically, we will use the following existence and compactness result proved in [8, Appendix].

**Theorem 2.3.** *Assume that  $h \in L^1(\Omega)$ , then the problem*

$$(2.2) \quad \begin{cases} -\Delta z = h & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega, \end{cases}$$

*has a unique weak solution  $z \in W_0^{1,s}(\Omega)$  for all  $s \in [1, \frac{N}{N-1})$ .*

*Moreover, for  $s \in [1, \frac{N}{N-1})$  fixed, there exists a positive constant  $C = C(\Omega, N, s)$  such that*

$$(2.3) \quad \|\nabla z\|_{L^s(\Omega)} \leq C \|h\|_{L^1(\Omega)}.$$

*and the operator  $\Phi : h \mapsto z$  is compact from  $L^1(\Omega)$  into  $W_0^{1,s}(\Omega)$ .*

Finally, let us recall the following classical regularity result that we will use in several proofs below.

**Theorem 2.4.** *Let  $z$  be the unique weak solution to Problem (2.2) with  $h \in L^m(\Omega)$ ,  $m > 1$ . Then, there exists a positive constant  $C = C(\Omega, N, m)$  independent of  $h$  such that*

(1) *If  $1 < m < N$ , then*

$$(2.4) \quad \|\nabla z\|_{L^{m^*}(\Omega)} \leq C \|h\|_{L^m(\Omega)} \text{ where } m^* = \frac{mN}{N-m}.$$

(2) *If  $m = N$ ,  $|\nabla z| \in L^s(\Omega)$  for all  $s \in [1, +\infty)$ .*

(3) *If  $m > N$ ,  $z \in C^{1,\gamma}(\Omega)$  for some  $\gamma \in (0, 1)$ .*

For a proof see for instance [18] or [10].

We now define what we mean by solution to our system (1.1).

**Definition 2.5.** *Let  $(f, g) \in (L^1(\Omega)^+)^2$  be nonnegative functions. We say that  $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$  is a weak solution to System (1.1) if  $u, v \geq 0$  and for all  $\varphi, \psi \in C_0^\infty(\Omega)$ , we have*

$$(2.5) \quad \int_{\Omega} \nabla \varphi \nabla u \, dx = \int_{\Omega} |\nabla v|^q \varphi \, dx + \lambda \int_{\Omega} f \varphi \, dx, \text{ and } \int_{\Omega} \nabla \psi \nabla v \, dx = \int_{\Omega} |\nabla u|^p \psi \, dx + \mu \int_{\Omega} g \psi \, dx.$$

*In what follows, we denote by  $C_1, C_2, \dots$  any positive constants that depend only on the datum of the problem and that can be changed from one line to next one.*

### 3. EXISTENCE RESULTS

Recall that we are considering the following nonlinear system

$$(3.1) \quad \begin{cases} -\Delta u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ -\Delta v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \\ u, v \geq 0 & \text{in } \Omega, \end{cases}$$

where  $\Omega$  is a bounded domain of  $\mathbb{R}^N$  and  $p, q \geq 1$ .

Let us begin by stating the first existence result of this paper.

**Theorem 3.1.** *Assume that  $p, q \geq 1$  are such that  $pq > 1$ . Let  $(m, \sigma) \in [1, +\infty)^2$ . Suppose that  $(f, g) \in L^m(\Omega)^+ \times L^\sigma(\Omega)^+$  where  $(m, \sigma)$  satisfies one of the following conditions :*

$$(3.2) \quad m, \sigma \in (1, N), \quad p\sigma < m^* = \frac{mN}{N-m} \text{ and } qm < \frac{\sigma N}{N-\sigma} = \sigma^* ;$$

$$(3.3) \quad \text{or} \quad m \geq N \text{ and } \sigma > \frac{qmN}{N+qm} ;$$

$$(3.4) \quad \text{or} \quad \sigma \geq N \text{ and } m > \frac{p\sigma N}{N+p\sigma}.$$

Then, there exists  $\Lambda^* > 0$  such that for all  $(\lambda, \mu) \in \Pi$  where

$$(3.5) \quad \Pi := \{(\lambda, \mu) \in [0, +\infty) \times [0, +\infty) \mid \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq \Lambda^*\},$$

System (3.1) has a nonnegative solution  $(u, v)$ . Moreover  $(u, v) \in W_0^{1, \theta_1}(\Omega) \times W_0^{1, \theta_2}(\Omega)$  for all  $\theta_1 < \frac{mN}{(N-m)_+}$  and  $\theta_2 < \frac{\sigma N}{(N-\sigma)_+}$ .

*Proof.* We will give the proof under the assumption (3.2). The other cases follow using similar approach.

The proof will be given in several steps.

**Step 1 :** For  $s \geq 0$ , we define the function

$$\Upsilon(s) := s^{\frac{1}{pq}} - \tilde{C}s,$$

where  $\tilde{C}$  is a positive constant depending only on the data, which we will specify later.

As  $pq > 1$ , then there exists  $s_0 > 0$  such that  $\Upsilon(s_0) = 0$ ,  $\Upsilon(s) > 0$ ,  $\forall s \in (0, s_0)$ ,  $\Upsilon(s) < 0$ ,  $\forall s \in (s_0, +\infty)$ . Hence, there exist two positive constants  $\ell > 0$  and  $\Lambda^* > 0$  such that

$$\max_{s \geq 0} \Upsilon(s) = \Upsilon(\ell) = \Lambda^*.$$

Thus

$$\ell^{\frac{1}{pq}} = \tilde{C} \left( \ell + \frac{\Lambda^*}{\tilde{C}} \right).$$

Fix  $\ell > 0$  as above, and define the set

$$\Pi := \left\{ (\lambda, \mu) \in [0, +\infty) \times [0, +\infty) ; \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq \frac{\Lambda^*}{\tilde{C}} \right\}.$$

Obviously,  $\Pi$  is a non-empty, bounded and closed set of  $\mathbb{R}^2$ . Moreover, for all  $(\lambda, \mu) \in \Pi$ , we have

$$(3.6) \quad \tilde{C} \left( \ell + \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \right) \leq \ell^{\frac{1}{pq}}.$$

For a fixed  $r \in (qm, \frac{\sigma N}{N-\sigma})$ , let

$$(3.7) \quad H := \left\{ \varphi \in W_0^{1,r}(\Omega) \text{ and } \|\nabla \varphi\|_{L^r(\Omega)} \leq \ell^{\frac{1}{pq}} \right\}.$$

It is clear that  $H$  is a convex set. We claim that  $H$  is a closed subset of  $W_0^{1,1}(\Omega)$ . To see that, let  $\{u_n\}_n \subset H$  be such that  $u_n \rightarrow u$  strongly in  $W_0^{1,1}(\Omega)$ . Then  $\|\nabla u_n\|_{L^r(\Omega)} \leq \ell^{\frac{1}{pq}}$  for all  $n$  and  $\|\nabla u_n - \nabla u\|_{L^1(\Omega)} \rightarrow 0$ . Thus, up to a subsequence,  $\nabla u_n \rightarrow \nabla u$  a.e in  $\Omega$ . Using Fatou's lemma we obtain that  $|\nabla u| \in L^r(\Omega)$  and

$$\|\nabla u\|_{L^r(\Omega)} \leq \liminf_{n \rightarrow \infty} \|\nabla u_n\|_{L^r(\Omega)} \leq \ell^{\frac{1}{pq}}.$$

Thus  $u \in H$  and the claim follows.

Let  $\varphi \in H$ , then  $|\nabla \varphi|^a \in L^1(\Omega)$  for all  $a \leq r$ ; in particular  $|\nabla \varphi|^q \in L^1(\Omega)$ . Thus, we can define  $u$  as the unique weak solution to the problem

$$(3.8) \quad \begin{cases} -\Delta u &= |\nabla \varphi|^q + \lambda f & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases}$$

In addition  $|\nabla\varphi|^q \in L^m(\Omega)$ , thus using Theorem 2.4, it holds that

$$\|\nabla u\|_{L^\beta(\Omega)} \leq C_1 \left\| |\nabla\varphi|^q + \lambda f \right\|_{L^m(\Omega)}, \quad \text{for all } \beta \leq m^* = \frac{mN}{N-m},$$

and then

$$(3.9) \quad \|\nabla u\|_{L^\beta(\Omega)} \leq C_2 \left( \|\nabla\varphi\|_{L^{qm}(\Omega)}^q + \lambda \|f\|_{L^m(\Omega)} \right), \quad \text{for all } \beta \leq m^*.$$

Recall that  $qm \leq r$ , hence setting  $\beta = p \leq m^*$ , it follows that

$$(3.10) \quad \|\nabla u\|_{L^p(\Omega)} \leq C_3 \|\nabla\varphi\|_{L^r(\Omega)}^q + C_4 \|f\|_{L^m(\Omega)}.$$

Thanks to Theorem 2.3, we define  $v$  as the unique weak solution to the problem

$$(3.11) \quad \begin{cases} -\Delta v &= |\nabla u|^p + \mu g & \text{in } \Omega, \\ v &= 0 & \text{on } \partial\Omega. \end{cases}$$

By the same theorem,  $v \in W_0^{1,a}(\Omega)$  for all  $a < \frac{N}{N-1}$ .

Therefore the operator

$$\begin{aligned} T : H &\longrightarrow W_0^{1,1}(\Omega) \\ \varphi &\longmapsto T(\varphi) = v, \end{aligned}$$

is well defined. Moreover, if  $v$  is a fixed point of  $T$ , then  $(u, v)$  is a weak solution to System (3.1). So we just have to show that  $T$  has a fixed point in  $H$ .

**Step 2 :** We show that  $T(H) \subset H$ .

According to (3.9) and since  $p\sigma < m^*$ , then  $|\nabla u|^p \in L^\sigma(\Omega)$ . Hence by using Theorem 2.4, we obtain that for all  $\theta \leq \sigma^* = \frac{\sigma N}{N-\sigma}$ ,

$$(3.12) \quad \begin{aligned} \|\nabla v\|_{L^\theta(\Omega)} &\leq C_5 \left\| |\nabla u|^p + \mu g \right\|_{L^\sigma(\Omega)} \\ &\leq C_6 \left( \|\nabla u\|_{L^{p\sigma}(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \right). \end{aligned}$$

Using again  $p\sigma < m^*$ ,  $qm < r$  and (3.9), we get

$$\|\nabla v\|_{L^\theta(\Omega)} \leq C_7 \left( \|\nabla\varphi\|_{L^r(\Omega)}^{pq} + \mu \|g\|_{L^\sigma(\Omega)} + \lambda^p \|f\|_{L^m(\Omega)}^p \right).$$

Since  $r < \sigma^*$ , by choosing  $\theta = r$  in the previous inequality it holds that

$$\|\nabla v\|_{L^r(\Omega)} \leq C_8 \left( \|\nabla\varphi\|_{L^r(\Omega)}^{pq} + \mu \|g\|_{L^\sigma(\Omega)} + \lambda^p \|f\|_{L^m(\Omega)}^p \right).$$

Recall that  $\varphi \in H$ , thus

$$\|\nabla v\|_{L^r(\Omega)} \leq C_9 \left( \ell + \mu \|g\|_{L^\sigma(\Omega)} + \lambda^p \|f\|_{L^m(\Omega)}^p \right).$$

By choosing  $\tilde{C} = C_9$  and taking into consideration the definition of  $\ell$ , we conclude that  $\|\nabla v\|_{L^r(\Omega)} \leq \ell^{\frac{1}{p^q}}$ . Thus  $v \in H$  and then  $T(H) \subset H$ .

**Step 3 :**  $T$  is a continuous and compact operator on  $H$  endowed with the topology of  $W_0^{1,1}(\Omega)$ .

• We begin by proving the continuity of  $T$ . Let  $\{\varphi_n\}_n \subset H, \varphi \in H$  be such that  $\varphi_n \rightarrow \varphi$  in  $W_0^{1,1}(\Omega)$  and define  $v_n = T(\varphi_n), v = T(\varphi)$ . Then  $(u_n, v_n)$  and  $(u, v)$  satisfy

$$(3.13) \quad \begin{cases} -\Delta u_n = |\nabla \varphi_n|^q + \lambda f & \text{in } \Omega, \\ -\Delta u = |\nabla \varphi|^q + \lambda f & \text{in } \Omega, \\ u_n = u = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(3.14) \quad \begin{cases} -\Delta v_n = |\nabla u_n|^p + \mu g & \text{in } \Omega, \\ -\Delta v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ v_n = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Thanks to Theorem 2.3, for all  $s < \frac{N}{N-1}$ ,

$$\|\nabla u_n - \nabla u\|_{L^s(\Omega)} \leq C \left\| |\nabla \varphi_n|^q - |\nabla \varphi|^q \right\|_{L^1(\Omega)}.$$

Recall that  $\varphi_n \rightarrow \varphi$  in  $W_0^{1,1}(\Omega)$  and  $\{\varphi_n\}_n \subset H$ , thus  $\|\nabla \varphi_n\|_{L^r(\Omega)} \leq \ell^{\frac{1}{p}}$  for all  $n$ . Hence for  $a \in (1, r)$  fixed, setting  $\theta = \frac{r-a}{a(r-1)}$  and using interpolation inequality (2.1), we obtain

$$\begin{aligned} \left\| \nabla \varphi_n - \nabla \varphi \right\|_{L^a(\Omega)} &\leq \left\| \nabla \varphi_n - \nabla \varphi \right\|_{L^1(\Omega)}^\theta \left\| \nabla \varphi_n - \nabla \varphi \right\|_{L^r(\Omega)}^{1-\theta} \\ &\leq C \left\| \nabla \varphi_n - \nabla \varphi \right\|_{L^1(\Omega)}^\theta \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Thus  $\nabla \varphi_n \rightarrow \nabla \varphi$  strongly in  $(L^a(\Omega))^N$  for all  $a < r$ . Since  $r > qm$ , we can choose  $a = q$  to conclude that

$$\left\| |\nabla \varphi_n|^q - |\nabla \varphi|^q \right\|_{L^1(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, for all  $s < \frac{N}{N-1}$ ,

$$\|\nabla u_n - \nabla u\|_{L^s(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

In particular  $u_n \rightarrow u$  strongly in  $W_0^{1,1}(\Omega)$ .

Now going back to (3.9), we deduce

$$(3.15) \quad \|\nabla u_n\|_{L^{p\sigma}(\Omega)} \leq C_{10} \left( \ell^{\frac{1}{p}} + \lambda \|f\|_{L^m(\Omega)} \right).$$

Thus  $\{u_n\}_n$  is bounded in  $W_0^{1,p\sigma}(\Omega)$ . Since  $p < p\sigma$ , we again apply Proposition 2.1 to obtain

$$\|\nabla u_n - \nabla u\|_{L^p(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, we deduce from Theorem 2.3 that  $v_n \rightarrow v$  strongly in  $W_0^{1,1}(\Omega)$  and then the continuity of  $T$  follows.

• Now, we prove the compactness of  $T$ . Let  $\{\varphi_n\}_n \subset H$  be such that  $\|\varphi_n\|_{W_0^{1,1}(\Omega)} \leq C_{11}$  and define  $v_n = T(\varphi_n)$ . Since  $\{\varphi_n\}_n \subset H$ , then  $\|\nabla \varphi_n\|_{L^r(\Omega)} \leq C_{12}$ . So,  $|\nabla \varphi_n|^q + \lambda f$  is bounded in  $L^1(\Omega)$ , then according to Theorem 2.3, we deduce that, up to a subsequence,  $u_n \rightarrow u$  strongly in  $W_0^{1,s}(\Omega)$  for all  $s < \frac{N}{N-1}$ . On the other hand,  $\|\nabla u_n\|_{L^{p\sigma}(\Omega)} \leq C_{13}$  for all  $n$ . Hence, by using



again Proposition 2.1, we conclude that  $u_n \rightarrow u$  strongly in  $W_0^{1,a}(\Omega)$  for all  $a < p\sigma$ . In particular,  $|\nabla u_n|^p + \mu g \rightarrow |\nabla u|^p + \mu g$  strongly in  $L^1(\Omega)$ . Thus, by setting  $v$  as the unique solution to the problem

$$\begin{cases} -\Delta v &= |\nabla u|^p + \mu g & \text{in } \Omega, \\ v &= 0 & \text{on } \partial\Omega, \end{cases}$$

we deduce from Theorem 2.3 that  $v_n \rightarrow v$  strongly in  $W_0^{1,a}(\Omega)$  for all  $a < \frac{N}{N-1}$ . Hence we conclude.

Therefore using Schauder fixed-point Theorem, we get the existence of  $v \in H$  such that  $T(v) = v$ . Thus  $(u, v)$  is a solution to System (3.1). Moreover,  $(u, v) \in W_0^{1,\theta_1}(\Omega) \times W_0^{1,\theta_2}(\Omega)$  for all  $\theta_1 < \frac{mN}{N-m}$  and  $\theta_2 < \frac{\sigma N}{N-\sigma}$ .  $\square$

**Remark 1.** *Using the same arguments as above and under the hypotheses of Theorem 3.1, we can treat the more general system*

$$(3.16) \quad \begin{cases} -\Delta u &= \alpha |\nabla v|^q + \lambda f & \text{in } \Omega, \\ -\Delta v &= \beta |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega, \\ u, v &\geq 0 & \text{in } \Omega, \end{cases}$$

where  $\alpha, \beta \in \{-1, 1\}$ . Existence is proved under suitable smallness condition on  $\lambda$  and  $\mu$ . Notice that for  $\alpha = \beta = -1$ , we obtain a generalization of the well-known system appearing in mean-field game theory. See for instance [16] and the references therein.  $\square$

It is clear that the above concavity argument does not hold if  $p = q = 1$ . In this case, we have to take into consideration the loss of homogeneity ; then, we will consider the following modified system

$$(3.17) \quad \begin{cases} u - \Delta u &= |\nabla v| + f & \text{in } \Omega, \\ v - \Delta v &= |\nabla u| + g & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega, \\ u, v &\geq 0 & \text{in } \Omega. \end{cases}$$

Hence we have the next existence result.

**Theorem 3.2.** *Let  $N \geq 2$ . Assume that  $f, g$  are nonnegative functions such that  $(f, g) \in L^{\frac{2N}{N+2}}(\Omega) \times L^{\frac{2N}{N+2}}(\Omega)$ . Then, System (3.17) has a nonnegative solution  $(u, v)$  such that  $u, v \in W_0^{1,2}(\Omega)$ .*

*Proof.* We proceed by approximation. By [31], the following system

$$(3.18) \quad \begin{cases} u_n - \Delta u_n &= \frac{|\nabla v_n|}{1 + \frac{1}{n}|\nabla v_n|} + \frac{f}{1 + \frac{1}{n}f} & \text{in } \Omega, \\ v_n - \Delta v_n &= \frac{|\nabla u_n|}{1 + \frac{1}{n}|\nabla u_n|} + \frac{g}{1 + \frac{1}{n}g} & \text{in } \Omega, \\ u_n = v_n &= 0 & \text{on } \partial\Omega, \\ u_n, v_n &\geq 0 & \text{in } \Omega, \end{cases}$$

has a nonnegative solution  $(u_n, v_n) \in (W_0^{1,2}(\Omega))^2$ .

Using  $u_n$  as a test function in the first equation,  $v_n$  as a test function in the second equation and applying Young's inequality, we obtain

$$(3.19) \quad \int_{\Omega} u_n^2 dx + \int_{\Omega} |\nabla u_n|^2 dx \leq \int_{\Omega} |\nabla v_n| u_n dx + \int_{\Omega} f u_n dx \leq \int_{\Omega} |\nabla v_n| u_n dx + \varepsilon \int_{\Omega} |\nabla u_n|^2 dx + C(\varepsilon) \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}^2,$$

and

$$(3.20) \quad \int_{\Omega} v_n^2 dx + \int_{\Omega} |\nabla v_n|^2 dx \leq \int_{\Omega} |\nabla u_n| v_n dx + \int_{\Omega} g v_n dx \leq \int_{\Omega} |\nabla v_n| u_n dx + \varepsilon \int_{\Omega} |\nabla u_n|^2 dx + C(\varepsilon) \|g\|_{L^{\frac{2N}{N+2}}(\Omega)}^2,$$

where  $\varepsilon > 0$  will be chosen later.

Using again Young's inequality, with a particular choice of the constant, we get

$$\int_{\Omega} |\nabla v_n| u_n dx \leq \int_{\Omega} u_n^2 dx + \frac{1}{4} \int_{\Omega} |\nabla v_n|^2 dx \quad \text{and} \quad \int_{\Omega} |\nabla u_n| v_n dx \leq \int_{\Omega} v_n^2 dx + \frac{1}{4} \int_{\Omega} |\nabla u_n|^2 dx.$$

Now, going back to (3.19) and (3.20), we obtain

$$(1 - \varepsilon) \int_{\Omega} |\nabla u_n|^2 dx \leq \frac{1}{4} \int_{\Omega} |\nabla v_n|^2 dx + C(\varepsilon) \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}^2,$$

and

$$(1 - \varepsilon) \int_{\Omega} |\nabla v_n|^2 dx \leq \frac{1}{4} \int_{\Omega} |\nabla u_n|^2 dx + C(\varepsilon) \|g\|_{L^{\frac{2N}{N+2}}(\Omega)}^2.$$

Thus

$$(3.21) \quad (1 - \varepsilon) \int_{\Omega} |\nabla u_n|^2 dx \leq \frac{1}{16(1 - \varepsilon)} \int_{\Omega} |\nabla u_n|^2 dx + C(\varepsilon) \left( \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}^2 + \|g\|_{L^{\frac{2N}{N+2}}(\Omega)}^2 \right).$$

Choosing  $\varepsilon$  small enough such that  $(1 - \varepsilon) > \frac{1}{16(1 - \varepsilon)}$ , it follows

$$(3.22) \quad \int_{\Omega} |\nabla u_n|^2 dx \leq C(\varepsilon) \left( \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}^2 + \|g\|_{L^{\frac{2N}{N+2}}(\Omega)}^2 \right).$$

In the same way we obtain that

$$\int_{\Omega} |\nabla v_n|^2 dx \leq C(\varepsilon) \left( \|f\|_{L^{\frac{2N}{N+2}}(\Omega)}^2 + \|g\|_{L^{\frac{2N}{N+2}}(\Omega)}^2 \right).$$

Hence setting

$$D_n(x) := \frac{|\nabla v_n|}{1 + \frac{1}{n} |\nabla v_n|} + \frac{f}{1 + \frac{1}{n} f} \quad \text{and} \quad \hat{D}_n(x) := \frac{|\nabla u_n|}{1 + \frac{1}{n} |\nabla u_n|} + \frac{g}{1 + \frac{1}{n} g},$$

it holds that  $\{D_n\}_n, \{\hat{D}_n\}_n$  are bounded in  $L^{\frac{2N}{N+2}}(\Omega)$ . By the compactness result of Theorem 2.3, we get that, up to a subsequence,  $u_n \rightarrow u$  and  $v_n \rightarrow v$  strongly in  $W_0^{1,s}(\Omega)$  for all  $s < \frac{N}{N-1}$ . Thus,  $(u, v)$  is a weak solution to System (3.17). From (3.21) and (3.22) and by Fatou's Lemma, we conclude that  $u, v \in W_0^{1,2}(\Omega)$  and then the proof is finished.  $\square$

**Remark 2.** *It would be interesting to consider the case  $m = \sigma = 1$ . However, it seems complicated to adapt the argument based on the choice of test functions.  $\square$*

#### 4. NON-EXISTENCE RESULTS

In this section, we show that the size of  $\lambda$ ,  $\mu$  and regularity conditions on the data found in the previous section are optimal in order to get the existence results.

Let us begin with the following result.

**Theorem 4.1.** *Let  $p, q > 1$ . Assume that  $f, g$  are nonnegative functions such that  $(f, g) \neq (0, 0)$  and  $(f, g) \in L^m(\Omega) \times L^\sigma(\Omega)$  where  $m, \sigma$  satisfy the conditions of Theorem 3.1. For  $\phi \in C_0^\infty(\Omega)$  with  $\phi \geq 0$ , we define*

$$F(\phi) = \int_{\Omega} \phi^{1-q'} |\nabla \phi|^{q'} dx + \int_{\Omega} \phi^{1-p'} |\nabla \phi|^{p'} dx.$$

Let  $(\lambda, \mu) \in (0, +\infty)^2$  be such that System (3.1) has a nonnegative solution  $(u, v)$ . Then there exists a positive constant  $C(p, q)$  (depending only) on  $p$  and  $q$  such that  $\lambda \leq \Sigma(f) = C(p, q)\lambda^*(f)$  and  $\mu \leq \Sigma(g) = C(p, q)\mu^*(g)$  where

$$\lambda^*(f) := \inf \left\{ F(\phi) ; 0 \leq \phi \in C_0^\infty(\Omega) \text{ and } \int_{\Omega} f\phi dx = 1 \right\},$$

and

$$\mu^*(g) := \inf \left\{ F(\phi) ; 0 \leq \phi \in C_0^\infty(\Omega) \text{ and } \int_{\Omega} g\phi dx = 1 \right\}.$$

*Proof.* Fix  $\lambda, \mu > 0$  be such that System (3.1) has a nonnegative solution  $(u, v)$ . Without loss of generality, we can assume that  $m, \sigma < N$ . Let  $\phi \in C_0^\infty(\Omega)$  such that  $\phi \geq 0$ . Using  $\phi$  as a test function in System (3.1), we obtain

$$\int_{\Omega} (|\nabla v|^q + \lambda f) \phi dx = \int_{\Omega} \nabla u \nabla \phi dx \text{ and } \int_{\Omega} (|\nabla u|^p + \mu g) \phi dx = \int_{\Omega} \nabla v \nabla \phi dx.$$

Thanks to Young's inequality, we get

$$\begin{aligned} \int_{\Omega} (|\nabla v|^q + \lambda f) \phi dx &\leq \int_{\Omega} |\nabla \phi| |\nabla u| \phi^{1/p} \phi^{-1/p} dx \\ &\leq \int_{\Omega} \phi |\nabla u|^p dx + C(p) \int_{\Omega} \phi^{1-p'} |\nabla \phi|^{p'} dx. \end{aligned}$$

In the same way, we obtain

$$\int_{\Omega} (|\nabla u|^p + \mu g) \phi dx \leq \int_{\Omega} \phi |\nabla v|^q dx + C(q) \int_{\Omega} \phi^{1-q'} |\nabla \phi|^{q'} dx.$$

Thus, combining the previous inequalities gives

$$(4.1) \quad \int_{\Omega} (|\nabla v|^q + \lambda f) \phi dx \leq \int_{\Omega} \phi |\nabla v|^q dx + C(q) \int_{\Omega} \phi^{1-q'} |\nabla \phi|^{q'} dx + C(p) \int_{\Omega} \phi^{1-p'} |\nabla \phi|^{p'} dx.$$

Hence

$$(4.2) \quad \lambda \int_{\Omega} f\phi dx \leq C(q) \int_{\Omega} \phi^{1-q'} |\nabla \phi|^{q'} dx + C(p) \int_{\Omega} \phi^{1-p'} |\nabla \phi|^{p'} dx.$$

Setting

$$F(\phi) := \int_{\Omega} \phi^{1-q'} |\nabla \phi|^{q'} dx + \int_{\Omega} \phi^{1-p'} |\nabla \phi|^{p'} dx,$$

then  $\lambda \leq \Sigma(f) = C(p, q)\lambda^*(f)$  where

$$(4.3) \quad \lambda^*(f) := \inf \left\{ F(\phi) ; 0 \leq \phi \in C_0^\infty(\Omega) \text{ and } \int_{\Omega} f\phi dx = 1 \right\}.$$

Conversely, if  $\lambda > \Sigma(f)$ , System (3.1) does not have any nonnegative solution.

In a similar way, we obtain that  $\mu \leq \Sigma(g) = C(p, q)\mu^*(g)$  with

$$\mu^*(g) := \inf \left\{ F(\phi) ; 0 \leq \phi \in C_0^\infty(\Omega) \text{ and } \int_{\Omega} g\phi dx = 1 \right\}.$$

Therefore, if  $\mu > \Sigma(g)$ , System (3.1) does not have any nonnegative solution.

To complete our proof, we just have to show that  $\lambda^*(f), \mu^*(g) > 0$ .

Notice that for  $\phi \in C_0^\infty(\Omega)$  with  $\phi \not\equiv 0$ , we have

$$F(\phi) = (p')^{p'} \int_{\Omega} |\nabla \phi^{\frac{1}{p'}}|^{p'} dx + (q')^{q'} \int_{\Omega} |\nabla \phi^{\frac{1}{q'}}|^{q'} dx.$$

Hence, using Sobolev's inequality to obtain

$$F(\phi) \geq C_{16} \left[ \left( \int_{\Omega} \phi^{\frac{N}{N-p'}} dx \right)^{\frac{N-p'}{N}} + \left( \int_{\Omega} \phi^{\frac{N}{N-q'}} dx \right)^{\frac{N-q'}{N}} \right].$$

Now, according to the values of  $m$  and  $\sigma$ , we will distinguish two cases  $m > \sigma$  and  $m \leq \sigma$ .

• *Case  $\sigma < m$ .* According to condition (3.2) in Theorem 3.1, we have  $qm < \sigma^*$ . Thus  $qm < m^*$  and then  $m' < \frac{N}{N-q'}$ . By using Hölder and Sobolev inequalities, we get

$$\begin{aligned} \int_{\Omega} f\phi dx &\leq \|f\|_{L^m(\Omega)} \|\phi\|_{L^{m'}(\Omega)} \leq C_{17} \|f\|_{L^m(\Omega)} \left( \int_{\Omega} \phi^{\frac{N}{N-q'}} dx \right)^{\frac{N-q'}{N}} \\ &\leq C_{18} \|f\|_{L^m(\Omega)} \int_{\Omega} |\nabla \phi^{\frac{1}{q'}}|^{q'} dx \leq C_{19} \|f\|_{L^m(\Omega)} F(\phi). \end{aligned}$$

• *Case  $\sigma \geq m$ .* In this case and since  $p\sigma < m^*$ , it holds that  $m' < \frac{N}{N-p'}$ . Thus

$$\int_{\Omega} f\phi dx \leq \|f\|_{L^m(\Omega)} \|\phi\|_{L^{m'}(\Omega)} \leq C_{20} \|f\|_{L^m(\Omega)} \int_{\Omega} |\nabla \phi^{\frac{1}{p'}}|^{p'} dx \leq C_{21} \|f\|_{L^m(\Omega)} F(\phi).$$

In the two cases, we obtain that  $\frac{F(\phi)}{\int_{\Omega} f\phi} \geq \frac{C_{22}}{\|f\|_{L^m(\Omega)}}$  and then  $\lambda^*(f) \geq \frac{C_{22}}{\|f\|_{L^m(\Omega)}} > 0$ .

By the same argument, we show that  $\frac{F(\phi)}{\int_{\Omega} g\phi} \geq \frac{C_{23}}{\|g\|_{L^\sigma(\Omega)}}$ , thus  $\mu^*(g) > 0$ . □

To assess the optimality of the conditions on  $f$  and  $g$ , we prove the following Theorem.

**Theorem 4.2.** *Let  $p, q > 1$ . Assume that  $f, g$  are positive functions such that  $(f, g) \in L^m(\Omega) \times L^\sigma(\Omega)$  where  $m, \sigma \geq 1$ . Define*

$$\Lambda(f) := \inf_{\phi \in \mathcal{C}_0^\infty(\Omega), \phi \geq 0} \frac{\int_{\Omega} |\nabla \phi|^{p'} dx + \int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx}{\int_{\Omega} f |\phi|^{p'} dx},$$

and

$$\Gamma(g) := \inf_{\phi \in \mathcal{C}_0^\infty(\Omega), \phi \geq 0} \frac{\int_{\Omega} |\nabla \phi|^{q'} dx + \int_{\Omega} |\nabla \phi|^{p'} |\phi|^{q'-p'} dx}{\int_{\Omega} g |\phi|^{q'} dx}.$$

Assume that System (3.1) has a nonnegative solution for some  $\lambda, \mu > 0$ . Then  $\Lambda(f), \Gamma(g) > 0$ .

*Proof.* We follow closely the arguments used in [7]. Let  $(u, v)$  be a nonnegative solution to (3.1). Let  $\phi \in \mathcal{C}_0^\infty(\Omega)$  such that  $\phi \geq 0$ . Using  $|\phi|^{p'}$  as a test function in the equation for  $v$ , it follows that

$$(4.4) \quad \int_{\Omega} \nabla v \nabla |\phi|^{p'} dx = p' \int_{\Omega} \nabla v \nabla \phi |\phi|^{p'-2} \phi dx = \int_{\Omega} |\nabla u|^p |\phi|^{p'} dx + \mu \int_{\Omega} g |\phi|^{p'} dx.$$

Using Young's inequality we get

$$\begin{aligned} p' \left| \int_{\Omega} \nabla v \nabla \phi |\phi|^{p'-2} \phi dx \right| &\leq p' \int_{\Omega} |\nabla v| \frac{|\nabla \phi|}{|\phi|} |\phi|^{p'} dx \\ &\leq \varepsilon \int_{\Omega} |\nabla v|^q |\phi|^{p'} dx + C(\varepsilon) \int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx. \end{aligned}$$

Thus

$$(4.5) \quad \int_{\Omega} |\nabla u|^p |\phi|^{p'} dx + \mu \int_{\Omega} g |\phi|^{p'} dx \leq \varepsilon \int_{\Omega} |\nabla v|^q |\phi|^{p'} dx + C(\varepsilon) \int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx.$$

On the other hand, taking  $|\phi|^{p'}$  as a test function in the equation in  $u$ , it holds

$$(4.6) \quad \int_{\Omega} \nabla u \nabla |\phi|^{p'} dx = p' \int_{\Omega} \nabla u \nabla \phi |\phi|^{p'-2} \phi dx = \int_{\Omega} |\nabla v|^q |\phi|^{p'} dx + \lambda \int_{\Omega} f |\phi|^{p'} dx.$$

As above, using again Young's inequality to obtain

$$p' \left| \int_{\Omega} \nabla u \nabla \phi |\phi|^{p'-2} \phi dx \right| \leq \varepsilon \int_{\Omega} |\nabla u|^p |\phi|^{p'} dx + C(\varepsilon) \int_{\Omega} |\nabla \phi|^{p'} dx.$$

Hence

$$(4.7) \quad \int_{\Omega} |\nabla v|^q |\phi|^{p'} dx + \lambda \int_{\Omega} f |\phi|^{p'} dx \leq \varepsilon \int_{\Omega} |\nabla u|^p |\phi|^{p'} dx + C(\varepsilon) \int_{\Omega} |\nabla \phi|^{p'} dx.$$

Now combining (4.5), (4.7) and choosing  $\varepsilon$  small enough, we obtain :

$$(1 - \varepsilon^2) \int_{\Omega} |\nabla v|^q |\phi|^{p'} dx + \lambda \int_{\Omega} f |\phi|^{p'} dx \leq C(\varepsilon) \int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx + C(\varepsilon) \int_{\Omega} |\nabla \phi|^{p'} dx.$$

So, for  $\varepsilon < 1$ ,

$$\lambda \int_{\Omega} f |\phi|^{p'} dx \leq C'(\varepsilon) \int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx + C(\varepsilon) \int_{\Omega} |\nabla \phi|^{p'} dx,$$

and then  $\Lambda(f) > 0$ . Now, using  $|\phi|^{q'}$  as a test function and following the same computations as above, we obtain :

$$(4.8) \quad \mu \int_{\Omega} g |\phi|^{q'} dx \leq C'(\varepsilon) \int_{\Omega} |\nabla \phi|^{p'} |\phi|^{q'-p'} dx + C(\varepsilon) \int_{\Omega} |\nabla \phi|^{q'} dx.$$

Then the result follows.  $\square$

**Remark 3.**

- (1) *The result of Theorem 4.2 means that we have the same kind of conditions as in the case of one equation. See condition (1.3) in Theorem 1.1.*
- (2) *For  $\phi \geq 0$ , setting*

$$Q(\phi) := \frac{\int_{\Omega} |\nabla \phi|^{p'} dx + \int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx}{\int_{\Omega} f |\phi|^{p'} dx}, \quad L(\phi) := \frac{\int_{\Omega} |\nabla \phi|^{q'} dx + \int_{\Omega} |\nabla \phi|^{p'} |\phi|^{q'-p'} dx}{\int_{\Omega} g |\phi|^{q'} dx}.$$

*Then,  $Q(\eta\phi) = Q(\phi)$  and  $L(\eta\phi) = L(\phi)$  for all  $\eta \neq 0$ .*

- (3) *Notice that  $\Sigma(f) \simeq \Lambda(f)$ . This follows from the fact that if we set  $\varphi = |\phi|^{p'-1} \phi$ , then*

$$\int_{\Omega} |\nabla \phi|^{p'} dx + \int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx \simeq \int_{\Omega} |\varphi|^{1-p'} |\nabla \varphi|^{p'} dx + \int_{\Omega} |\varphi|^{1-q'} |\nabla \varphi|^{q'} dx = F(\varphi).$$

*In the same way, we have  $\Sigma(g) \simeq \Gamma(g)$ .  $\square$*

We are now able to prove the next non-existence result.

**Theorem 4.3.** *Suppose that  $p, q > 1$  are such that  $qm > \sigma^*$  and  $p\sigma > m^*$ . Then, there exist  $f \in L^m(\Omega)$  and  $g \in L^\sigma(\Omega)$  with  $f, g \geq 0$  such that System (3.1) does not have any nonnegative solution for any  $\lambda, \mu > 0$ .*

*Proof.* To get the non-existence result, we will construct  $f, g$  satisfying the hypotheses of the Theorem and such that  $\Lambda(f) = 0$  or  $\Gamma(g) = 0$ .

Without loss of generality, we assume that  $m, \sigma < N$ . In what follows, we will assume that  $\Omega = B_1(0)$ . For technical reasons, we distinguish two cases.

**Case 1 :**  $m \geq \sigma$  and  $p' \geq q'$ . Using Hölder's inequality, it holds that

$$\int_{\Omega} |\nabla \phi|^{q'} |\phi|^{p'-q'} dx \leq \left( \int_{\Omega} |\nabla \phi|^{p'} dx \right)^{\frac{q'}{p'}} \left( \int_{\Omega} |\phi|^{p'} dx \right)^{\frac{p'-q'}{p'}}.$$

According to Poincaré's inequality, we obtain that

$$\int_{\Omega} |\nabla\phi|^{q'} |\phi|^{p'-q'} dx \leq C_P \int_{\Omega} |\nabla\phi|^{p'} dx.$$

Thus

$$\int_{\Omega} |\nabla\phi|^{p'} dx + \int_{\Omega} |\nabla\phi|^{q'} \phi^{p'-q'} dx \simeq \int_{\Omega} |\nabla\phi|^{p'} dx.$$

Then

$$Q(\phi) \simeq \frac{\int_{\Omega} |\nabla\phi|^{p'} dx}{\int_{\Omega} f|\phi|^{p'} dx}.$$

Notice that  $pm \geq p\sigma > m^*$ , then  $p > \frac{N}{N-m}$ . Thus, we can fix  $\alpha > 0$  such that  $p' < \alpha < \frac{N}{m}$ . Define  $f(x) = \frac{1}{|x|^\alpha}$ , then  $f \in L^m(\Omega)$ . Now, introduce the function  $\phi$  defined by

$$\phi(x) = \begin{cases} \frac{1}{|x|^\theta} & \text{if } |x| \leq \frac{1}{4} \\ (1-|x|)^\gamma & \text{if } \frac{1}{2} \leq |x| \leq 1, \end{cases}$$

where  $\frac{N-\alpha}{p'} < \theta < \frac{N-p'}{p'}$  and  $\gamma > 1$  and such that  $\phi \in \mathcal{C}^1(B_1(0) \setminus \{0\})$  and  $\phi > 0$  in  $B_1(0)$ .

Since  $\theta < \frac{N-p'}{p'}$ , then  $\int_{\Omega} |\nabla\phi|^{p'} dx < +\infty$ . However

$$\int_{\Omega} f|\phi|^{p'} dx \simeq \int_{\Omega} \frac{1}{|x|^{p'\theta+\alpha}} dx = +\infty.$$

Hence, using a suitable approximation argument, we conclude that  $\Lambda(f) = 0$  and the non-existence result follows in this case.

**Case 2 :**  $m \geq \sigma$  and  $p' < q'$ . In the same way as above, we obtain  $q\sigma > m^* > \sigma^*$ . Thus  $q > \frac{N}{N-\sigma}$ , hence  $q' < \frac{N}{\sigma}$ . As in the proof of the first case, since  $p' < q'$ , by Hölder's inequality, we obtain

$$\int_{\Omega} |\nabla\phi|^{q'} dx + \int_{\Omega} |\nabla\phi|^{p'} \phi^{q'-p'} dx \simeq \int_{\Omega} |\nabla\phi|^{q'} dx.$$

Thus

$$L(\phi) \simeq \frac{\int_{\Omega} |\nabla\phi|^{q'} dx}{\int_{\Omega} g|\phi|^{q'} dx}.$$

Setting  $g(x) = \frac{1}{|x|^\beta}$  where  $q' < \beta < \frac{N}{\sigma}$ , then  $g \in L^\sigma(\Omega)$ . Choosing  $\phi$  as in the first case with  $\frac{N-\beta}{q'} < \theta < \frac{N-q'}{q'}$ , then  $\int_{\Omega} |\nabla \phi|^{p'} dx < +\infty$  and

$$\int_{\Omega} g|\phi|^{q'} dx \simeq \int_{\Omega} \frac{1}{|x|^{q'\theta+\beta}} dx = +\infty.$$

Hence  $\Gamma(g) = 0$  and then we conclude.

In a similar way, we can treat the case  $m < \sigma$ . Hence the result follows.  $\square$

### 5. A SYSTEM WITH COUPLED GRADIENT-POTENTIAL TERMS

This section is devoted to the slightly different system given as :

$$(5.1) \quad \begin{cases} -\Delta u &= u|\nabla v|^q + \lambda f & \text{in } \Omega, \\ -\Delta v &= v|\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v &= 0 & \text{on } \partial\Omega, \\ u, v &\geq 0 & \text{in } \Omega. \end{cases}$$

In this case, we have the following existence result.

**Theorem 5.1.** *Let  $p, q \geq 1$  and  $m, \sigma \geq 1$ . Assume that  $f, g$  are nonnegative functions such that  $(f, g) \in L^m(\Omega) \times L^\sigma(\Omega)$  where  $m, \sigma$  satisfy*

$$(5.2) \quad m > \frac{pN}{p+2}, \quad \text{and} \quad \sigma > \frac{qN}{q+2}.$$

*Then, there exists  $A^* > 0$  such that, if  $\lambda + \mu < A^*$ , System (5.1) has a nonnegative solution  $(u, v) \in W_0^{1,\alpha}(\Omega) \times W_0^{1,\beta}(\Omega)$  for all  $\alpha < \frac{Nm}{(N-m)_+}$  and  $\beta < \frac{N\sigma}{(N-\sigma)_+}$ .*

*Proof.* We will give the proof in the case  $1 \leq m, \sigma < N$ . The other cases follow using similar approach.

As in the proof of Theorem 3.1, we define the function

$$\Upsilon_1(s) := s - \tilde{C}(s^{1+q} + s^{1+p}),$$

with  $\tilde{C}$  being a positive constant depending only on the data. Since  $p, q \geq 1$ , we deduce the existence of a unique value  $\ell$  such that

$$\max_{s \geq 0} \Upsilon_1(s) = \Upsilon_1(\ell) = \Lambda^*.$$

Thus,

$$(5.3) \quad \ell = \tilde{C}(\ell^{1+q} + \ell^{1+p} + \frac{\Lambda^*}{\tilde{C}}).$$

As  $\ell > 0$ , there exists  $A^* > 0$  such that, if  $\lambda + \mu < A^*$ , then

$$(5.4) \quad \lambda \|f\|_{L^m(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)} \leq \frac{\Lambda^*}{\tilde{C}}.$$



Hence

$$(5.5) \quad \tilde{C} \left( \ell^{1+q} + \ell^{1+p} + \lambda \|f\|_{L^m(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)} \right) \leq \ell.$$

Now, we define the set  $H_{r,\theta} \subset W_0^{1,1}(\Omega) \times W_0^{1,1}(\Omega)$  as :

$$H_{r,\theta} = \left\{ (\varphi, \psi) \in W_0^{1,r}(\Omega) \times W_0^{1,\theta}(\Omega) \text{ and } \|\varphi\|_{W_0^{1,r}(\Omega)} + \|\psi\|_{W_0^{1,\theta}(\Omega)} \leq \ell \right\},$$

where  $r$  and  $\theta$  are chosen such that

$$(5.6) \quad \begin{cases} \frac{p\sigma N\theta}{\theta(N+\sigma) - N\sigma} < r < \frac{mN}{N-m}, \\ \frac{qmNr}{r(N+m) - Nm} < \theta < \frac{N\sigma}{N-\sigma}. \end{cases}$$

Notice that, the existence of  $(r, \theta)$  follows from (5.2). As above,  $H_{r,\theta}$  is a closed convex subset of  $W_0^{1,1}(\Omega) \times W_0^{1,1}(\Omega)$ .

We claim that if  $(\varphi, \psi) \in H_{r,\theta}(\Omega)$ , then

$$(5.7) \quad \left\| |\varphi_+| |\nabla \psi|^q \right\|_{L^m(\Omega)} \leq C_{24} \|\varphi\|_{W_0^{1,r}(\Omega)} \times \|\psi\|_{W_0^{1,r}(\Omega)}^q.$$

Indeed, by definition:

$$\left\| |\varphi_+| |\nabla \psi|^q \right\|_{L^m(\Omega)}^m = \int_{\Omega} \varphi_+^m |\nabla \psi|^{qm} dx \leq \int_{\Omega} |\varphi|^m |\nabla \psi|^{qm} dx.$$

Hence, using the Sobolev and Hölder inequalities, it follows that

$$(5.8) \quad \left\| |\varphi_+| |\nabla \psi|^q \right\|_{L^m(\Omega)}^m \leq C_{25} \left( \int_{\Omega} |\varphi|^{r^*} dx \right)^{\frac{m}{r^*}} \left( \int_{\Omega} |\nabla \psi|^{qm \frac{r^*}{r^* - m}} dx \right)^{\frac{r^* - m}{r^*}}.$$

On the other hand,  $qm \frac{r^*}{r^* - m} = qm \frac{rN}{r(N+m) - mN}$ . Hence, by the definition of  $r$  and  $\theta$ , it holds that  $qm \frac{r^*}{r^* - m} < \theta$ . Finally,

$$\left\| |\varphi_+| |\nabla \psi|^q \right\|_{L^m(\Omega)}^m \leq C_{26} \|\varphi\|_{W_0^{1,r}(\Omega)}^m \times \|\psi\|_{W_0^{1,\theta}(\Omega)}^{qm},$$

and the claim (5.7) follows.

In the same way, using  $p\sigma \frac{N\theta}{\theta(N+\sigma) - \sigma N} < r$ , we obtain

$$(5.9) \quad \left\| |\psi_+| |\nabla \varphi|^p \right\|_{L^\sigma(\Omega)} \leq C_{27} \|\psi\|_{W_0^{1,\theta}(\Omega)} \times \|\varphi\|_{W_0^{1,r}(\Omega)}^p.$$

Therefore, we can define  $u$  and  $v$  as the unique weak solutions to the problems

$$(5.10) \quad \begin{cases} -\Delta u & = \varphi_+ |\nabla \psi|^q + \lambda f & \text{in } \Omega, \\ u & = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$(5.11) \quad \begin{cases} -\Delta v &= \psi_+ |\nabla \varphi|^p + \mu g & \text{in } \Omega, \\ v &= 0 & \text{on } \partial\Omega. \end{cases}$$

According to Theorem 2.4 and estimates (5.7), (5.9), it holds that

$$(5.12) \quad \|\nabla u\|_{L^{m^*}(\Omega)} \leq C_{28} \left( \|\varphi\|_{W_0^{1,r}(\Omega)} \times \|\psi\|_{W_0^{1,\theta}(\Omega)}^q + \lambda \|f\|_{L^m(\Omega)} \right),$$

and

$$(5.13) \quad \|\nabla v\|_{L^{\sigma^*}(\Omega)} \leq C_{29} \left( \|\psi\|_{W_0^{1,\theta}(\Omega)} \times \|\varphi\|_{W_0^{1,r}(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \right).$$

So the operator

$$\begin{aligned} T : H_{r,\theta} &\longmapsto W_0^{1,1}(\Omega) \times W_0^{1,1}(\Omega) \\ (\varphi, \psi) &\longmapsto L(\varphi, \psi) = (u, v) \end{aligned}$$

is well defined. Moreover, any fixed point of  $T$  in  $H_{r,\theta}$  is a solution to System (5.1). Since  $r < m^*$  and  $\theta < \sigma^*$ , then  $(u, v) \in W_0^{1,r}(\Omega) \times W_0^{1,\theta}(\Omega)$  and

$$\begin{aligned} \|u\|_{W_0^{1,r}(\Omega)} + \|v\|_{W_0^{1,\theta}(\Omega)} &\leq C_{30} \left( \|\varphi\|_{W_0^{1,r}(\Omega)} \|\psi\|_{W_0^{1,\theta}(\Omega)}^q \right. \\ &\quad \left. + \|\psi\|_{W_0^{1,\theta}(\Omega)} \|\varphi\|_{W_0^{1,r}(\Omega)}^p + \lambda \|f\|_{L^m(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)} \right) \\ &\leq C_{30} \left( \ell^{1+q} + \ell^{1+p} + \lambda \|f\|_{L^m(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)} \right). \end{aligned}$$

By choosing  $\tilde{C} = C_{30}$ , we deduce from (5.5) that

$$\|u\|_{W_0^{1,r}(\Omega)} + \|v\|_{W_0^{1,\theta}(\Omega)} \leq \ell$$

So we conclude that  $T(H_{r,\theta}) \subset H_{r,\theta}$ .

The continuity and compactness of  $T$  follow as in the proof of Theorem 3.1. We conclude thanks to the Schauder fixed-point Theorem.  $\square$

**Remark 4.** *Let us review some particular, explicit cases, where the existence conditions hold.*

- *If  $m, \sigma > N$ , the existence result holds for all  $p, q > 1$ .*
- *If  $p = q = 1$  and  $m = \sigma$ , a sufficient condition on  $m$  is that  $m = \sigma > \frac{N}{3}$ .*
- *If  $p = q = 2$  and  $m = \sigma$ , condition (5.2) implies that  $m = \sigma > \frac{N}{2}$ .*
- *If  $m = \sigma = 2$ , condition (5.2) implies that*

$$p < \frac{4}{N-2}, \quad q < \frac{4}{N-2}.$$

*which imply  $p, q < 1$ , if  $N \geq 6$ . In this case, Theorem 5.1 fails.  $\square$*

To show that condition (5.2) is in some sense optimal, we state and prove the following non-existence result.

**Proposition 5.2.** *Assume that  $m = \sigma = 2$ . For any  $p, q > 1$ , we can find  $f, g \in L^2(\Omega)$  and  $n_0 \in \mathbb{N}$  such that if  $N > n_0$ , System (5.1) does not have any nonnegative solution.*

*Proof.* Without loss of generality, we assume that  $0 \in \Omega$ . Set  $f(x) = g(x) = \frac{1}{|x|^\alpha}$  where  $2 < \alpha < \frac{N}{2}$ . If  $(u, v)$  is a solution to System (5.1), then

$$-\Delta u \geq \frac{\lambda}{|x|^\alpha}, \quad -\Delta v \geq \frac{\mu}{|x|^\alpha} \text{ in } \Omega.$$

Then, using a suitable comparison principle it holds that

$$u(x), v(x) \geq \frac{C_{31}}{|x|^{\alpha-2}} \text{ in } B_R(0) \subset \subset \Omega, \quad \text{for a suitable } R.$$

Thus :

$$\int_{\Omega} |x|^{-(\alpha-2)} |\nabla u|^p dx + \int_{\Omega} |x|^{-(\alpha-2)} |\nabla v|^q dx < \infty.$$

According to Caffarelli–Kohn–Nirenberg inequality ([19]), we obtain

$$\left( \int_{\Omega} \frac{u^{p^*}}{|x|^{\frac{p^*}{p}(\alpha-2)}} dx \right)^{\frac{p}{p^*}} + \left( \int_{\Omega} \frac{v^{q^*}}{|x|^{\frac{q^*}{q}(\alpha-2)}} dx \right)^{\frac{q}{q^*}} < \infty.$$

Now, taking into account the behavior of  $u, v$  near the origin, we find :

$$\int_{B_R(0)} \frac{1}{|x|^{p^*(\alpha-2) + \frac{p^*}{p}(\alpha-2)}} dx + \int_{B_R(0)} \frac{1}{|x|^{q^*(\alpha-2) + \frac{q^*}{q}(\alpha-2)}} dx \leq C(\Omega).$$

Thus

$$p^*(\alpha-2) + \frac{p^*}{p}(\alpha-2) < N \text{ and } q^*(\alpha-2) + \frac{q^*}{q}(\alpha-2) < N.$$

For  $p, q > 1$  fixed, choosing  $N > n_0 := \max\{\frac{2p+4}{p-1}, \frac{2q+4}{q-1}\}$ , we get the existence of  $\alpha \in (2, \frac{N}{2})$  such that

$$p^*(\alpha-2) + \frac{p^*}{p}(\alpha-2) \geq N \text{ and } q^*(\alpha-2) + \frac{q^*}{q}(\alpha-2) \geq N,$$

and we get a contradiction with the existence hypothesis.  $\square$

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