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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS TO A FRACTIONAL PARABOLIC PROBLEM WITH SINGULAR WEIGHT AT THE BOUNDARY

BOUMEDIENE ABDELLAOUI, KHEIREDDINE BIROUD, EL-HAJ LAAMRI *

ABSTRACT. We consider the problem

$$(P) \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u^p}{\delta^{2s}(x)} & \text{in } \Omega_T \equiv \Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain (in the sense that $\partial\Omega$ is of class $\mathcal{C}^{0,1}$), $\delta(x) = \text{dist}(x, \partial\Omega)$, $0 < s < 1$, $p > 0$, $\lambda > 0$.

The purpose of this work is twofold.

First We analyze the interplay between the parameters s, p and λ in order to prove the existence or the nonexistence of solution to problem (P) in a suitable sense. This extends previous similar results obtained in the local case $s = 1$.

Second We will especially point out the differences between the local and nonlocal cases.

1. INTRODUCTION

Recently, great attention has been focused on the study of nonlocal diffusion equations, i.e., first-order evolution equations driven by a nonlocal operator. These operators play a crucial role in describing several phenomena as, for instance, the thin obstacle problem, anomalous diffusion, quasi-geostrophic flows... See, for instance, [49, 23, 43] and the references therein. The use of such operators reflects the need to model long-distance effects not included in the usual, local diffusion operators such as the Laplacian. Prototypical examples of a nonlocal operator are the various versions of the fractional Laplacian.

Note that the proper definition of a fractional Laplacian is not obvious and offers some choices. In the literature, several definitions, not always equivalent to one another, and different terminologies are found. However, when working in the whole \mathbb{R}^N , the alternative definitions are equivalent ; we refer the interested reader to [30, 43, 25] and the references therein.

Contrary to the case of full space \mathbb{R}^N , several different fractional Laplacians can be defined on a open subset $\Omega \neq \mathbb{R}^N$. These alternatives correspond to different ways in which the information coming from the boundary and the exterior of domain is to be taken into account. In particular, two different fractional Laplacians are widely studied in the literature, contributions include [49, 15, 16, 48, 43, 25]

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and references therein. Needless to say, these references do not exhaust the rich literature on the subject.

Here, our problem is posed on a bounded domain Ω with homogeneous Dirichlet boundary conditions, that is $u = 0$ in $\mathbb{R}^N \setminus \Omega$. In other words, the Dirichlet datum is given in $\mathbb{R}^N \setminus \Omega$ and not simply on $\partial\Omega$. Then we are working with the following fractional Laplacian operator ¹ which can be defined as :

$$(1.1) \quad (-\Delta)^s u(x) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad 0 < s < 1$$

where P.V. stands for the Cauchy principal value and

$$(1.2) \quad a_{N,s} := \frac{s 2^{2s} \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)}$$

is a normalization constant.²

Specifically, in this article, we consider the following nonlocal parabolic problem :

$$(1.3) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u^p}{\delta^{2s}} & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \end{cases}$$

where Ω is a bounded regular domain of \mathbb{R}^N , $\delta(x) = \text{dist}(x, \partial\Omega)$, $0 < s < 1$, $p > 0$, $\lambda > 0$ and $u_0 \geq 0$ is a nonnegative measurable function.

As far as we know, no works seem to have treated this problem.

The first goal of this work is to obtain “natural” conditions on the parameters s , p , λ and the initial data u_0 that allow us to prove the existence of positive solutions to Problem (1.3). By solution, we mean either energy solution or weak solution or distributional solution (for more specifics, see Subsection 2.2).

In the local case ($s = 1$), this problem corresponds to the classical heat equation; it has been studied by the first two authors in [5].

Apart from the study of the nonlocal problem itself, the second purpose of this article is to highlight the differences between the local and nonlocal cases. In fact, our results show a significant difference with respect to the local case $s = 1$. For this, see Remark 3.2, Remark 3.3 and Proposition 3.4.

In addition, let us mention that the first author et al. have been studied Problem (1.3) in the case where the weight $\delta^{-2s}(x)$ is replaced by $|x|^{-2s}$, see [7] for more details.

But, in this case, existence and nonexistence results are stable when $s \nearrow 1$. This justifies the importance and the role of the singular weight $\delta(x)^{-2s}$.

Before going further, let us mention some previous works concerning the local case and in order to compare it with the nonlocal cases.

¹This fractional Laplacian operator is sometimes called the *restricted* fractional Laplacian (see for example [49, 15, 16]), or *regional* fractional Laplacian (see for instance [48, 43]) or *Dirichlet* fractional Laplacian (see [44]).

²The choice of the constant $a_{N,s}$ is motivated, among others, by the following assertion :

$$\lim_{s \rightarrow 0^+} (-\Delta)^s u = u \text{ and } \lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u$$

where Δ is the classical Laplacian. For a proof, see for instance [30, Proposition 4.4]).

First. The case where the weight $\delta^{-2}(x)$ is replaced by $|x|^{-2}$ (i.e. $u_t - \Delta u = \frac{\lambda u^p}{|x|^2}$) and $0 \in \Omega$ has attracted³ a lot of interests by researchers. The paper [12] by Baras-Goldstein is the starting point on the existence of global solutions and blow-up for these equations. From this pioneering work, a lot of contributions have been made. Among the works dealing with inverse-square potential, we cite [12, 20, 24, 50] and references therein, with no attempt to provide an exhaustive list. Let us briefly recall these following results :

- For $p = 1$, it is proved that global existence of weak solution holds if and only if $\lambda \leq \lambda^* := \left(\frac{N-2}{2}\right)^2$.
- For $p > 1$, independently of the value of $\lambda > 0$, a nonexistence result is obtained in [20], where also an instantaneous complete blow-up result is proved.

Second. While considerable effort has been devoted to the study of the heat equation with the potential $|x|^{-2}$, relatively little is done in the case of $\delta(x)^{-2}$. As mentioned above, Problem (1.3) with $s = 1$ has been recently treated in [5] ; see also [4] for the elliptic case and their references. In that work, the authors proved a strong nonexistence result if $p < 1$. However, for $p > 1$ and under adequate assumptions on u_0 , they proved the existence of a positive solution using suitable sub- and supersolution and iteration arguments.

When $p = 1$ and Ω is a bounded convex domain, a strong nonexistence result for $\lambda > \lambda^*$ has been proved in [24]. As a consequence, a complete and instantaneous blow-up result was proved for some approximating problems.

As we shall see in more detail later on, in the nonlocal case (i.e., $0 < s < 1$), a different phenomenon appears in the cases $p < 1$ and $p = 1$. Indeed, clarifying the differences between the local and nonlocal cases is the second purpose of this work.

In order to give precise statements of the results obtained, let us introduce first the following fractional Hardy inequality (for a proof, see [32] or [34, Theorem 5]) in the form that we will use.

Theorem 1.1. *Let $s \in (0, 1)$. Assume that Ω is a bounded regular domain ($C^{0,1}$ regularity is sufficient) of \mathbb{R}^N where $N \geq 2s$. Then, there exists a positive constant $C = C(s, \Omega)$ such that for all $\phi \in C_0^\infty(\Omega)$*

$$(1.4) \quad C \int_{\Omega} \frac{\phi^2(x)}{\delta^{2s}(x)} dx \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy,$$

where $D_{\Omega} := \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^C \times \Omega^C)$.

In what follows, we denote by $\Lambda_s(\Omega)$ the *best possible* constant in (1.4), and henceforth we will use that inequality in the following form :

$$(1.5) \quad \Lambda_s(\Omega) \int_{\Omega} \frac{\phi^2(x)}{\delta^{2s}(x)} dx \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy,$$

Remark 1.2.

³This question was first posed by H. Brezis and J.-L. Lions in the early 1980s.

(i) If $s \in [1/2, 1)$ and Ω is a convex domain, $\Lambda_s(\Omega)$ is independent of Ω , is not attained by a non-zero function and it is given by

$$(1.6) \quad \Lambda_s(\Omega) = \Lambda_s^* := \frac{\Gamma^2(s + \frac{1}{2})}{\pi}.$$

(ii) In all cases, we always have $\Lambda_s(\Omega) \leq \Lambda_s^*$, see [34, Theorem 5].

(iii) The constant $\Lambda_s(\Omega)$ is attained by a non-zero function if and only if $\Lambda_s(\Omega) < \Lambda_s^*$. We refer to [14] for more details.

Remark 1.3. In general, problems with the weight $\delta^{-2s}(x)$ are more complicated to treat than the case of the classical Hardy singular potential $|x|^{-2s}$ and can generate new phenomena which are one of the main objectives of this work. Related inequalities were also treated intensively in [37] and [42] where applications to the stereometric inequality are given.

The main results of this paper can be summarized in the following points.

- In the sublinear case $p < 1$, we shall see that Problem (1.3) has a nonnegative solution, in a suitable sense, for all $u_0 \in L^1(\Omega)$. This makes a significant difference with the local case $s = 1$, where a strong nonexistence result has been proved in [5].
- In the linear case $p = 1$, we will show that the range of the parameter s , together with the validity of the Hardy inequality with optimal constant stated in (1.5) and Remark 1.2, will be decisive for existence and nonexistence of positive solutions for λ large.
- In the superlinear case $p > 1$, we are able to show the existence of a nonnegative solution for a suitable $u_0 \in L^\infty(\Omega)$.

Before ending this section, let us give the outline of the paper. In Section 2, we give some auxiliary results related to fractional Sobolev spaces and some functional inequalities. We also define the three notions of solution that we will use systematically throughout the paper : energy solution, weak solution and distributional solution.

The case where $0 < p < 1$ is treated in Section 3. Using monotonicity arguments and suitable test functions, we prove the existence and uniqueness of a global weak solution that is in a suitable Sobolev space. This seems to be quite surprising since in the local case, the corresponding problem does not have any distributional solution and a complete blow-up holds for the sequence of solutions approximating problems that will be introduced later on.

In section 4, we treat the linear case *i.e.* $p = 1$. According to the value of s , we will show the existence or the nonexistence of solution for suitable range of λ . In subsection 4.1, we prove the existence and the uniqueness of a positive weak solution for all $\lambda \leq \Lambda_s(\Omega)$ and for all $s \in (0, 1)$. In subsection 4.2, we deal with the case $s \in [1/2, 1)$ and $\Lambda_s(\Omega) = \Lambda_s^*$ which includes, in particular, the case of convex bounded domains. We are able to show that the corresponding problem does not have any nonnegative distributional solution for $\lambda > \Lambda_s^*$.

The superlinear case $p > 1$ is treated in Section 5. Under suitable assumptions on the initial data and using monotonicity arguments, we are able to get the existence of a bounded positive solution. Finally, in the last section we give some extensions to the case of the spectral fractional Laplacian.

We have tried to write this paper in an almost self-contained form. Moreover, we give precise references for all points that are not detailed in this work.

Finally, throughout this paper, we assume that

- $s \in (0, 1)$;
- Ω is a bounded open subset of \mathbb{R}^N such that $\partial\Omega$ is of class $\mathcal{C}^{0,1}$.

2. THE FUNCTIONAL SETTING AND TOOLS

In this section, we give some useful tools that will be used many times in this paper. First, we recall some results related to fractional Sobolev spaces and some useful functional inequalities respectively in subsections 2.1 and 2.2. Then, we specify the sense in which solutions are considered in subsection 2.3 and we recall an useful theorem in subsection 2.4.

2.1. Functional framework. As already announced in the introduction, we are working in a bounded domain Ω with homogeneous Dirichlet boundary conditions $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and not simply $u = 0$ in $\partial\Omega$. This makes the classical fractional Sobolev space approach not sufficient for studying our problem. Then, we need to set a natural functional framework that is inspired by (but not equivalent to) the fractional Sobolev spaces which will allow us to deal with the Dirichlet boundary condition. Of course, this subsection can be skipped by readers already familiar with these notions.

Definition 2.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain. For any $p \in [1, +\infty)$, the fractional Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) ; \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy < +\infty \right\}.$$

It is a Banach space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|_{L^p(\Omega)}^p + \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

Now we define the space $W_0^{s,p}(\Omega)$ as

$$W_0^{s,p}(\Omega) := \{ u \in W^{s,p}(\mathbb{R}^N) ; u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}.$$

$W_0^{s,p}(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}.$$

where $D_\Omega = \mathbb{R}^{2N} \setminus (\Omega^C \times \Omega^C)$.

In the particular case $p = 2$, we denote by $H^s(\Omega) = W^{s,2}(\Omega)$ and $\mathbb{H}_0^s(\Omega) = W_0^{s,2}(\Omega)$. Then, for bounded domain Ω , using Poincaré's inequality we can endow the space $\mathbb{H}_0^s(\Omega)$ with the equivalent norm

$$\|u\|_{\mathbb{H}_0^s(\Omega)}^2 := \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

where $a_{N,s}$ is the normalization constant given by (1.2).

The pair $(\mathbb{H}_0^s(\Omega), \|\cdot\|_{\mathbb{H}_0^s(\Omega)})$ is a Hilbert space (see for instance [48, Lemma 7] or [43, Lemma 1.29]). The dual space of $\mathbb{H}_0^s(\Omega)$ will be denoted by $\mathbb{H}^{-s}(\Omega)$. For more properties of the previous spaces, we refer to [30], [9] and [43, subsection 1.5]. \square

2.2. Some useful functional inequalities. In this subsection, we collect and present some functional inequalities under the form in which they will be exploited.

- The following Sobolev inequality is proved in [30] (see also [45] for a simple proof).

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$ with $2s < N$. Then, there exists a positive constant $S := S(N, s)$ such that for all $u \in \mathbb{H}_0^s(\Omega)$, we have*

$$S \left(\int_{\Omega} |u(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

where $2_s^* := \frac{2N}{N-2s}$ is the so-called critical Sobolev exponent. \square

- In order to treat the case $\lambda = \Lambda_s^* := \frac{\Gamma^2(s + \frac{1}{2})}{\pi}$, we need the following improved Hardy-Sobolev inequality obtained in [34].

Theorem 2.3. *(Improved Hardy-Sobolev inequality) Assume that $\frac{1}{2} \leq s < 1$ and $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a uniformly Lipschitz bounded convex domain. Then, there exists a positive constant $C := C(\Omega, s)$ such that*

$$(2.1) \quad C \left(\int_{\Omega} |u(x)|^{2_s^*} dx \right)^{\frac{2}{2_s^*}} \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \Lambda_s^* \int_{\Omega} \frac{u^2}{\delta^{2s}} dx$$

where $2_s^* = \frac{2N}{N-2s}$. \square

- The Kato type inequality below will be useful in this paper, we refer to [29] and [28] for the proof.

Theorem 2.4. *Let $\Phi \in C^2(\mathbb{R})$ be a convex function. Assume that $u, \Phi(u) \in W_0^{s,1}(\Omega)$ are such that $(-\Delta)^s u \in L^1(\Omega)$, $(-\Delta)^s \Phi(u) \in L^1(\Omega)$ and $\Phi'(u)(-\Delta)^s u \in L^1(\Omega)$. Then*

$$(2.2) \quad (-\Delta)^s \Phi(u) \leq \Phi'(u)(-\Delta)^s u. \square$$

- Finally, we end this subsection with the next Picone inequality. For a proof, see for instance [38] or [6].

Proposition 2.5. *Let $s \in (0, 1)$ and $u \in \mathbb{H}_0^s(\Omega)$ such that $u > 0$ in Ω . Then, for all $\phi \in C_0^\infty(\Omega)$, we have*

$$(2.3) \quad \int_{\Omega} \frac{(-\Delta)^s u}{u} \phi^2 dx \leq \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy. \square$$

2.3. Notions of solution. As already mentioned, in this paper, we will consider three notions of solution : energy solution, weak solution and distributional solution.

Let us consider the following problem

$$(2.4) \quad \begin{cases} u_t + (-\Delta)^s u &= g(x, t) & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T) \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

where u_0 and g are given functions in suitable Lebesgue spaces and T is a positive real.

- **Energy solution** Let us start by defining what we mean by energy solution.

Definition 2.6. Assume that $(g, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$. We say that u is an energy solution to Problem (2.4) if $u \in L^2((0, T), \mathbb{H}_0^s(\Omega)) \cap \mathcal{C}([0, T], L^2(\Omega))$, $u_t \in L^2((0, T), \mathbb{H}^{-s}(\Omega))$ and for all $w \in L^2((0, T), \mathbb{H}_0^s(\Omega))$, we have

$$(2.5) \quad \int_0^T \int_{\Omega} u_t w \, dx \, dt + \int_0^T \iint_{D_{\Omega}} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+2s}} \, dx \, dy \, dt = \int_0^T \int_{\Omega} g w \, dx \, dt.$$

Remark 2.7. If $(g, u_0) \in L^2(\Omega_T) \times L^2(\Omega)$, Problem (2.4) has an unique energy solution u . Moreover, $g \geq 0$ and $u_0 \geq 0$ imply $u \geq 0$. For a proof, see, e.g., [38, Theorem 26]. \square

• **Weak solution :** Since we will consider problems with general datum in L^1 , we need the concept of weak solutions. For this purpose, let us first define the space of test functions.

$$P(\Omega_T) := \{\phi : \mathbb{R}^N \times [0, T] \rightarrow \mathbb{R}, \text{ s.t. } -\phi_t + (-\Delta)^s \phi = \varphi, \varphi \in L^\infty(\Omega_T) \\ \phi(x, t) = 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0, T], \phi(x, T) = 0 \text{ in } \Omega\}.$$

It is worth to notice that if $\phi \in P(\Omega_T)$, then $\phi \in \mathcal{C}^{0,s}(\Omega_T) \cap L^\infty(\Omega_T)$, see for instance [33, Theorem 1.1 and Corollary 4.1].

Then we have the subsequent definition.

Definition 2.8. Let $(g, u_0) \in L^1(\Omega_T) \times L^1(\Omega)$. We say that $u \in \mathcal{C}([0, T], L^1(\Omega))$ is a weak solution to Problem 2.4 if for all $\phi \in P(\Omega_T)$, we have

$$\int_0^T \int_{\Omega_T} u(-\phi_t + (-\Delta)^s \phi) \, dx \, dt = \int_{\Omega_T} u \varphi \, dx \, dt = \int_0^T \int_{\Omega_T} g \phi \, dx \, dt + \int_{\Omega} u_0 \phi(x, 0) \, dx.$$

The following existence and regularity result is obtained in [38].

Theorem 2.9 (See [38, Theorem 28]). Assume that $(g, u_0) \in L^1(\Omega_T) \times L^1(\Omega)$. Then, Problem (2.4) has a unique weak solution u , that is obtained as a limit of approximations, such that $\forall k \geq 0$, $T_k(u) \in L^2((0, T), \mathbb{H}_0^s(\Omega))$, $u \in L^q(\Omega_T)$ for all $q \in [1, \frac{N+2s}{N})$ and $|(-\Delta)^{\frac{s}{2}} u| \in L^r(\Omega_T)$ for all $r \in [1, \frac{N+2s}{N+s})$.

Here, $T_k(\sigma) := \max\{-k, \min\{k, \sigma\}\}$ for $k > 0$. \square

• **Distributional solution :** Finally, we end this part by the notion of distributional solution.

Definition 2.10. Assume that $(g, u_0) \in L_{loc}^1(\Omega_T) \times L_{loc}^1(\Omega)$. We say that $u \in L^1(\Omega_T) \cap \mathcal{C}([0, T], L_{loc}^1(\Omega))$ is a distributional solution to Problem (2.4) if for all $\phi \in \mathcal{C}_0^\infty(\Omega_T)$, for all $\rho \in \mathcal{C}_0^\infty(\Omega)$ we have

$$\int_{\Omega_T} u(-\phi_t + (-\Delta)^s \phi) \, dx \, dt = \int_{\Omega_T} u \varphi \, dx \, dt = \int_{\Omega_T} g \phi \, dx \, dt,$$

and

$$\int_{\Omega} u(x, t) \rho(x) \, dx \rightarrow \int_{\Omega} u_0(x) \rho(x) \, dx \text{ as } t \rightarrow 0. \square$$

2.4. An useful Theorem. Taking into consideration the singular weight δ^{-2s} at the boundary of Ω and in order to prove a priori estimates for approximating problems, we will use the following existence result obtained in [1] (see also [10]).

Theorem 2.11 (See [1, Proposition 1.10]). *Assume that $s \in (0, 1)$ and $\beta \in (0, 2s)$. Then, the following problem*

$$(2.6) \quad \begin{cases} (-\Delta)^s \xi &= \frac{1}{\delta^\beta(x)} & \text{in } \Omega, \\ \xi &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a bounded distributional solution such that

- (i) *if $\beta < s$, then $\xi \asymp \delta^s$,*
- (ii) *if $\beta = s$, then $\xi \asymp \delta^s \log(\frac{D}{\delta(x)})$ where $D \gg \text{diam}(\Omega)$,*
- (iii) *if $\beta \in (s, 2s)$, then $\xi \asymp \delta^{2s-\beta}$.*

Note that for given nonnegative functions f and g , by $f \asymp g$, we mean that there exist two positive constants C_1 and C_2 such that $C_1 f \leq g \leq C_2 f$. \square

3. THE SUBLINEAR CASE $p < 1$: EXISTENCE RESULTS.

In this section, we are interested in the sublinear case, namely $0 < p < 1$. Contrary to the local case studied in [5], we are able here to show the existence of a solution, at least, in the distributional sense defined in Definition 2.10.

The main existence result of this section is the following.

Theorem 3.1. *Assume that $0 < s < 1$ and $0 < p < 1$. Then, for all $0 \not\leq u_0 \in L^1(\Omega)$, Problem (1.3) has a minimal positive solution $u \in L^1(\Omega_T)$ at least in the sense of distributions given in Definition 2.10. This solution is minimal in the sense that if $v \in L^1(\Omega_T)$ is an other nonnegative distributional solution to Problem (1.3), then $v \geq u$ in Ω_T .*

Moreover, if $s < \frac{1}{2}$ and $0 < p < 1 - 2s$, u is a weak solution in the sense of Definition 2.8 such that $\frac{u^p}{\delta^{2s}} \in L^1(\Omega_T)$.

Proof. : Taking into consideration that $p < 1$, then without loss of generality we can assume that $\lambda = 1$.

Now we divide the proof in five steps.

First step : We proceed by iterations. Let $n \geq 1$ and define u_n to be the unique positive energy solution to the problem

$$(3.1) \quad \begin{cases} \partial_t u_n + (-\Delta)^s u_n &= \frac{u_{n-1}^p}{(\delta(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T \\ u_n(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) &= u_{0,n}(x) & \text{in } \Omega, \end{cases}$$

where $u_{0,n} := \min(n, u_0)$ and $u_{0,0} := 0$.

The existence and the uniqueness of u_n follow using classical arguments for monotone operators. Moreover $u_n \geq 0$. On the other hand, $u_{0,n} \uparrow u_0$ as $n \rightarrow +\infty$, then the comparison principle ensures that $u_n \leq u_{n+1}$ for all integer n .

Let ξ be the solution to Problem (2.6) with $\beta \in (s, \min\{2s, 1\})$. Thanks to Theorem 2.11, $\xi \asymp \delta^{2s-\beta}$. In addition, $\beta < \min\{2s, 1\}$, then $\xi \in \mathbb{H}_0^s(\Omega)$. Hence we conclude that $\xi \in L^\infty(\Omega) \cap \mathbb{H}_0^s(\Omega)$. Thus, choosing ξ as a test function in (3.1),

taking into consideration the estimate on ξ given in Theorem 2.11 and using the fact that $u_{n-1} \leq u_n$, it holds

$$\frac{d}{dt} \int_{\Omega} u_n \xi \, dx + \int_{\Omega} \frac{u_n}{\delta^\beta} \, dx = \int_{\Omega} \frac{u_{n-1}^p \xi}{(\delta(x) + \frac{1}{n})^{2s}} \, dx \leq C \int_{\Omega} \frac{u_n^p}{\delta^\beta} \, dx.$$

Since $p < 1$, by Young's inequality, we obtain

$$(3.2) \quad \frac{d}{dt} \int_{\Omega} u_n \xi \, dx + (1 - \varepsilon) \int_{\Omega} \frac{u_n}{\delta^\beta} \, dx \leq C(\varepsilon) \int_{\Omega} \frac{1}{\delta^\beta} \, dx.$$

Recall that $\beta < 1$ implies $\int_{\Omega} \frac{1}{\delta^\beta} \, dx < \infty$. Now, by choosing ε small enough and integrating (3.2) in time, we get

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) \xi \, dx + (1 - \varepsilon) \int_0^T \int_{\Omega} \frac{u_n}{\delta^\beta} \, dx \, dt \leq TCC(\varepsilon) + \int_{\Omega} u_0 \xi \, dx < \infty.$$

Thus, there exists a measurable function $u \in L^1(\Omega_T) \cap L^\infty((0, T), L^1_{loc}(\Omega))$ such that

$$(3.3) \quad \begin{cases} \frac{u_n}{\delta^\beta} \rightarrow \frac{u}{\delta^\beta} & \text{strongly in } L^1(\Omega_T), \\ \frac{u_n^p}{\delta^\beta} \rightarrow \frac{u^p}{\delta^\beta} & \text{strongly in } L^1(\Omega_T). \end{cases}$$

Moreover

$$(3.4) \quad \frac{u_n^p}{(\delta(x) + \frac{1}{n})^{2s}} \uparrow \frac{u^p}{\delta^{2s}} \quad \text{strongly in } L^1_{loc}(\Omega_T).$$

Second step : We claim that $\{u_n \xi\}_n$ is a Cauchy sequence in the space $\mathcal{C}([0, T], L^1(\Omega))$.

To see that, let us fix $m > n > 1$ and define $w_{m,n} = u_m - u_n \geq 0$. It is clear that $w_{m,n} \geq 0$ a.e. in Ω_T and solves the following problem

$$(3.5) \quad \begin{cases} \partial_t w_{m,n} + (-\Delta)^s w_{m,n} &= \frac{u_{m-1}^p - u_{n-1}^p}{(\delta(x) + \frac{1}{n})^{2s}} + g_{m,n} & \text{in } \Omega_T \\ w_{m,n}(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ w_{m,n}(x, 0) &= u_{0,m}(x) - u_{0,n}(x) & \text{in } \Omega. \end{cases}$$

where $g_{m,n}(x, t) := u_{m-1}^p \left(\frac{1}{(\delta(x) + \frac{1}{m})^{2s}} - \frac{1}{(\delta(x) + \frac{1}{n})^{2s}} \right)$. Note that $g_{m,n} \geq 0$ in Ω_T .

Using ξ as a test function in (3.5) and the fact that $\xi(x) \asymp \delta^{2s-\beta}(x)$ yields

$$(3.6) \quad \left\{ \begin{array}{l} \sup_{t \in [0, T]} \int_{\Omega} w_{m,n}(x, t) \xi \, dx + \int_0^T \int_{\Omega} \frac{w_{m,n}}{\delta^\beta} \, dx \, dt \\ \leq \\ C \int_0^T \int_{\Omega} \frac{(u_{m-1}^p - u_{n-1}^p)}{\delta^\beta(x)} \, dx \, dt + C \int_0^T \int_{\Omega} g_{m,n}(x, t) \delta^{2s-\beta}(x) \, dx \, dt + \\ + \int_{\Omega} (u_{0,m}(x) - u_{0,n}(x)) \xi \, dx. \end{array} \right.$$

We deduce directly from (3.3) that

$$(3.7) \quad \int_0^T \int_{\Omega} \frac{(u_{m-1}^p - u_{n-1}^p)}{\delta^\beta(x)} \, dx \, dt + \int_{\Omega} (u_{0,m}(x) - u_{0,n}(x)) \xi \, dx \rightarrow 0 \text{ as } m, n \rightarrow +\infty.$$

Concerning the term $\int_0^T \int_{\Omega} g_{m,n}(x, t) \delta^{2s-\beta}(x) \, dx \, dt$, let us first remark that

$g_{m,n} \rightarrow 0$ a.e. in Ω_T as $m, n \rightarrow +\infty$ and

$$u_{m-1}^p \left(\frac{1}{(\delta(x) + \frac{1}{m})^{2s}} - \frac{1}{(\delta(x) + \frac{1}{n})^{2s}} \right) \delta^{2s-\beta}(x) \leq u^p \left(\frac{1}{(\delta(x) + \frac{1}{m})^{2s}} - \frac{1}{(\delta(x) + \frac{1}{n})^{2s}} \right) \delta^{2s-\beta}(x).$$

On the other hand, the sequence $\left\{ \frac{u \delta^{2s-\beta}}{(\delta + \frac{1}{n})^{2s}} \right\}_n$ converges in $L^1(\Omega_T)$ and then it is a Cauchy sequence. So

$$(3.8) \quad \int_0^T \int_{\Omega} u^p \left(\frac{1}{(\delta(x) + \frac{1}{m})^{2s}} - \frac{1}{(\delta(x) + \frac{1}{n})^{2s}} \right) \delta^{2s-\beta}(x) \, dx \, dt \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

In view of (3.7) and (3.8) (recall also that $w_{m,n} \geq 0$), we deduce from (3.6)

$$\sup_{t \in [0, T]} \int_{\Omega} |w_{m,n}(x, t)| \xi \, dx \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Therefore, $\{u_n \xi\}_n$ is a Cauchy sequence in the space $\mathcal{C}([0, T]), L^1(\Omega)$.

Now, let $\rho \in \mathcal{C}_0^\infty(\Omega)$ be fixed. Then, we have $\frac{|\rho|}{\xi} \leq C(\Omega, \rho)$ and

$$(3.9) \quad \left\{ \begin{array}{l} \sup_{t \in [0, T]} \int_{\Omega} |w_{m,n}(x, t)| |\rho| \, dx = \sup_{t \in [0, T]} \int_{\Omega} |w_{m,n}(x, t)| \xi \frac{|\rho|}{\xi} \, dx \\ \leq C(\Omega, \rho) \sup_{t \in [0, T]} \int_{\Omega} |w_{m,n}(x, t)| \xi \, dx. \end{array} \right.$$

So

$$\sup_{t \in [0, T]} \int_{\Omega} |w_{m,n}(x, t)| |\rho| \, dx \rightarrow 0 \text{ as } m, n \rightarrow \infty.$$

Hence, $u\rho \in \mathcal{C}([0, T]), L^1(\Omega)$ for all $\rho \in \mathcal{C}_0^\infty(\Omega)$ and

$$(3.10) \quad \int_{\Omega} u(x, t) \rho(x) \, dx \rightarrow \int_{\Omega} u_0(x) \rho(x) \, dx \text{ as } t \rightarrow 0.$$

Third step : Now, let us prove that u is a distributional solution (in the sense of Definition 2.10) to Problem (1.3). Let us fix $\phi \in \mathcal{C}_0^\infty(\Omega_T)$ and $\rho \in \mathcal{C}_0^\infty(\Omega)$. By using ϕ as a test function in (3.1), we obtain

$$\iint_{\Omega_T} u_n(-\phi_t + (-\Delta)^s \phi) dx dt = \iint_{\Omega_T} \frac{u_{n-1}^p}{(\delta + \frac{1}{n})^{2s}} \phi dx dt.$$

As $n \rightarrow \infty$, we have in view of (3.3) :

$$\iint_{\Omega_T} u_n(-\phi_t + (-\Delta)^s \phi) dx dt \rightarrow \iint_{\Omega_T} u(-\phi_t + (-\Delta)^s \phi) dx dt$$

and

$$\iint_{\Omega_T} \frac{u_{n-1}^p}{(\delta + \frac{1}{n})^{2s}} \phi dx dt \rightarrow \iint_{\Omega_T} \frac{u^p}{\delta^{2s}} \phi dx dt.$$

Thus

$$\iint_{\Omega_T} u(-\phi_t + (-\Delta)^s \phi) dx dt = \iint_{\Omega_T} \frac{u^p}{\delta^{2s}} \phi dx dt.$$

In addition, we have already proven (3.10), namely

$$\int_{\Omega} u(x, t) \rho(x) dx \rightarrow \int_{\Omega} u_0(x) \rho(x) dx \text{ as } t \rightarrow 0.$$

Therefore, $u \in \mathcal{C}([0, T], L_{loc}^1(\Omega))$ and it is a distributional solution to Problem (1.3) in the sense of Definition 2.10.

Fourth step : Now let us prove that u , the solution obtained above, is the minimal solution to problem (1.3). Assume that v is an other nonnegative distributional solution to Problem (1.3). As for all $n \geq 1$,

$$\frac{v^p}{\delta^{2s}} \geq \frac{v^p}{(\delta(x) + \frac{1}{n})^{2s}} \text{ a.e in } \Omega_T.$$

by going back to the equation in u_1 , it holds that $v \geq u_1$. Thus, by iteration and using the comparison principle, we deduce that $v \geq u_n$ in Ω_T for all $n \geq 1$. Since the sequence $\{u_n\}_n$ is increasing, we deduce that $u \leq v$ in Ω_T and then we conclude.

Fifth step : Now we deal with the case $s < \frac{1}{2}$ and $p < 1 - 2s$. Since $u \in L^1(\Omega_T)$, then by using Hölder's inequality, it follows that

$$\iint_{\Omega_T} \frac{u^p(x, t)}{\delta^{2s(x)}} dx dt \leq \left(\iint_{\Omega_T} u(x, t) dx dt \right)^p \left(\iint_{\Omega_T} \frac{1}{\delta^{\frac{2s}{1-p}}(x)} dx dt \right)^{1-p}.$$

Since $p < 1 - 2s$, then $\frac{2s}{1-p} < 1$, hence $\frac{1}{\delta^{\frac{2s}{1-p}}(x)} \in L^1(\Omega_T)$. Therefore we conclude

that $\frac{u^p}{\delta^{2s}} \in L^1(\Omega_T)$. This finishes the proof of Theorem 3.1. \square

Remark 3.2.

- In the local case (i.e. $s = 1$), the first two authors proved in [5] a strong nonexistence result to Problem (1.3) for all $p < 1$. As a consequence, they get a complete and instantaneous blow-up for the approximating problems. Hence, our existence result in Theorem 3.1 shows a significant difference between the local and the non-local cases.
- As we will see in the linear case (i.e. $p=1$), a similar phenomenon occurs as it was established in [31]. More precisely, it was proved that all functions are locally

s -harmonic up to a small error. This produces more solutions in the nonlocal case than the local case (that disappear when letting $s \rightarrow 1$).

Remark 3.3. Blow-up and non-stability as $s \nearrow 1$.

If we substitute the “Hardy” weight $\delta^{-2s}(x)$ by the “Hardy” potential $|x|^{-2s}$, existence and nonexistence results are stable if we let $s \nearrow 1$, see for instance [7]. However, comparing the existence result in Theorem 3.1 and the nonexistence result obtained in [5], we deduce that the situation is totally different when considering the weight $\delta^{-2s}(x)$.

More precisely we have the next blow up result.

Proposition 3.4. Assume that $u_0 \in L^1(\Omega)$ such that $u_0 \not\equiv 0$ and let $0 < p < 1$ be fixed. Consider a sequence $\{s_m\} \subset (0, 1)$ with $s_m \nearrow 1$ as $m \rightarrow +\infty$ and let u_m be the minimal solution to Problem (1.3) obtained in Theorem 3.1. Then

$$\left\| \frac{u_m^p}{(\delta(x) + \frac{1}{m})^{2s_m}} \right\|_{L^1(\Omega_T)} \rightarrow +\infty \text{ as } m \rightarrow +\infty.$$

Proof. Without loss of generality we can assume that $s_m > \frac{2}{3}$ and $u_0 \in L^\infty(\Omega)$. Let $p \in (0, 1)$ be fixed and consider $\{u_m\}_m$ to be the family of **minimal** solutions to Problem (1.3) obtained in Theorem 3.1 with $s = s_m$. Thanks to Theorem 3.1, $\frac{u_m^p}{\delta(x)^{2s_m}} \in L^1(\Omega_T)$ and then $\frac{u_m^p}{(\delta(x) + \frac{1}{m})^{2s_m}} \in L^1(\Omega_T)$.

To prove the main blow-up result, we will argue by contradiction. Assume that $\left\| \frac{u_m^p}{(\delta(x) + \frac{1}{m})^{2s_m}} \right\|_{L^1(\Omega_T)} \leq C$ for all m . Setting $f_m(x, t) := \frac{u_m^p}{(\delta(x) + \frac{1}{m})^{2s_m}}$, and consider v_m to be the unique weak solution to the problem

$$(3.11) \quad \begin{cases} \partial_t v_m + (-\Delta)^{s_m} v_m &= f_m(x, t) & \text{in } \Omega_T \\ v_m(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ v_m(x, 0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

Let us observe that $v_m \leq u_m$ in Ω_T for all $m \geq 1$. On the other hand, $s_m > \frac{2}{3} > \frac{1}{2}$ and $\|f_m\|_{L^1(\Omega_T)} \leq C$, then thanks to [8, Theorem 1.1], there exists $C_1 > 0$ such that $\|v_m\|_{L^\sigma(0, T; W_0^{1, \sigma}(\Omega))} \leq C_1$ for all m and for all $\sigma < \frac{N+2s_m}{N+1}$; in particular for all $\sigma_0 < \frac{N+\frac{4}{3}}{N+1}$. Hence, we get the existence of $\tilde{v} \in L^{\sigma_0}(0, T; W_0^{1, \sigma_0}(\Omega))$ such that $v_m \rightharpoonup \tilde{v}$ weakly in $L^{\sigma_0}(0, T; W_0^{1, \sigma_0}(\Omega))$ and then, up to a subsequence, $v_m \rightarrow \tilde{v}$ strongly in $L^{\sigma_0}(\Omega_T)$ and *a.e.* in Ω_T .

Notice that

$$f_m(x, t) \geq \frac{v_m^p(x, t)}{(\delta(x) + \frac{1}{m})^{2s_m}} \text{ a.e. in } \Omega_T.$$

Let $\phi \in \mathcal{C}_0^\infty(\Omega_T)$ with $\phi \geq 0$. Using ϕ as a test function in (3.11), it holds that

$$\iint_{\Omega_T} v_m(-\phi_t + (-\Delta)^{s_m} \phi) dx dt = \iint_{\Omega_T} f_m(x, t) \phi dx dt \geq \iint_{\Omega_T} \frac{v_m^p}{(\delta(x) + \frac{1}{m})^{2s_m}} \phi dx dt.$$

Notice that $|\phi_t| + |(-\Delta)^{s_m} \phi| \leq C$ in Ω_T and

$$(-\Delta)^{s_m} \phi \rightarrow -\Delta \phi \text{ in } L^\infty(\Omega_T) \text{ as } s_m \uparrow 1,$$

where we get the last convergence by using the same arguments as in [30, Proposition 4.4]. Since

$$\frac{v_m^p(x, t)}{(\delta(x) + \frac{1}{m})^{2s_m}} \rightarrow \frac{\tilde{v}^p(x, t)}{\delta^2(x)} \text{ a.e. in } \Omega_T,$$

then by Fatou's lemma, we conclude that

$$\iint_{\Omega_T} \tilde{v}(-\phi_t - \Delta\phi) dx dt \geq \iint_{\Omega_T} \frac{\tilde{v}^p}{\delta^2} \phi dx dt.$$

Thus, \tilde{v} is a non-trivial supersolution to the following problem

$$(3.12) \quad \begin{cases} z_t - \Delta z &= \frac{z^p}{\delta^2} & \text{in } \Omega_T \\ z(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ z(x, 0) &= u_0(x) & \text{in } \Omega. \end{cases}$$

Hence a contradiction with the nonexistence result proved in [5] and the result follows. \square

4. THE LINEAR CASE : EXISTENCE AND NONEXISTENCE RESULTS.

In this section, we consider the linear problem :

$$(4.1) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{u}{\delta^{2s}} & \text{in } \Omega_T, \\ u &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

where $\lambda > 0$ and u_0 is a measurable nonnegative function satisfying some assumptions that we will specify later.

This section is divided into two subsections. Subsection 4.1 is devoted to existence results and is split into two parts : in the first part, we deal with the case where $s \in (0, 1)$ and $\lambda \leq \Lambda_s(\Omega)^4$, and in the second we treat the case where $s \in (0, 1/2)$ without any restriction on λ . In Subsection 4.2, we prove that Problem 4.1 does not have any solution in the case where $s \in [1/2, 1)$ and $\lambda > \Lambda_s(\Omega) = \Lambda_s^*$ where $\Lambda_s^* := \frac{\Gamma^2(s + \frac{1}{2})}{\pi}$ (see 1.6).

4.1. Existence results.

4.1.1. Case where $s \in (0, 1)$ and $\lambda \leq \Lambda_s(\Omega)$.

The first existence result is the following.

Theorem 4.1. *Let $s \in (0, 1)$ and let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain with $N \geq 2s$. Assume that $0 \leq u_0 \in L^2(\Omega)$.*

- 1) *If $\lambda < \Lambda_s(\Omega)$, Problem (4.1) has a unique positive energy solution u such that $u \in L^2(0, T, \mathbb{H}_0^s(\Omega))$.*
- 2) *If $\Lambda_s(\Omega)$ is attained and $\lambda = \Lambda_s(\Omega)$, Problem (4.1) has a unique positive distributional solution u such that $u \in L^2(\Omega_T)$.*
- 3) *If $s \in [\frac{1}{2}, 1)$ and $\Lambda_s(\Omega) = \Lambda_s^* := \frac{\Gamma^2(s + \frac{1}{2})}{\pi}$ (recall that in this case, $\Lambda_s(\Omega)$ is not attained), then:*
 - *if $\lambda < \Lambda_s^*$, Problem (4.1) has a unique positive finite energy solution $u \in L^2((0, T), \mathbb{H}_0^s(\Omega))$;*
 - *if $\lambda = \Lambda_s^*$, Problem (4.1) has a distributional solution $u \in L^2(\Omega_T)$.*

⁴For the ease of the reader, let us recall that we denote by $\Lambda_s(\Omega)$ the best possible constant in Hardy's inequality (see Theorem 1.1 and Remark 1.2).

Proof. As in the proof of Theorem 3.1, introduce the following approximate problem,

$$(4.2) \quad \begin{cases} \partial_t u_n + (-\Delta)^s u_n &= \lambda \frac{u_{n-1}}{(\delta(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T, \\ u_n(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) &= u_{0,n}(x) & \text{in } \Omega, \end{cases}$$

where $u_{0,n} := T_n(u_0)$ and $u_{0,0} := 0$.

In the sequel, we have to distinguish two cases : case where $s \in (0, 1)$ and $\lambda \leq \Lambda_s(\Omega) < \Lambda_s^*$, and case where $s \in [\frac{1}{2}, 1)$ and $\lambda \leq \Lambda_s(\Omega) = \Lambda_s^*$.

• **First case :** $s \in (0, 1)$ and $\lambda \leq \Lambda_s(\Omega) < \Lambda_s^*$

Using u_n as a test function in (4.2), the fact that $u_{n-1} \leq u_n$, integrating over Ω_T and applying Hardy's inequality (1.5), we obtain

$$(4.3) \quad \begin{cases} \frac{1}{2} \int_{\Omega} u_n^2(x, T) dx + \\ + \frac{a_{N,s}}{2} \left(1 - \frac{\lambda}{\Lambda_s(\Omega)}\right) \int_0^T \iint_{D_{\Omega}} \frac{|u_n(x, t) - u_n(y, t)|^2}{|x - y|^{N+2s}} dx dy dt \\ \leq \frac{1}{2} \int_{\Omega} u_0^2 dx. \end{cases}$$

— If $\lambda < \Lambda_s(\Omega)$, we deduce from (4.3) that $\{u_n\}_n$ is bounded in $L^2((0, T), \mathbb{H}_0^s(\Omega))$. Thus we deduce the existence of a measurable function $u \in L^2((0, T), \mathbb{H}_0^s(\Omega))$ such that $u_n \rightharpoonup u$ weakly in $L^2((0, T), \mathbb{H}_0^s(\Omega))$, $u_n \rightarrow u$ a.e in Ω_T , and $\frac{u_n^2}{\delta^{2s}} \rightarrow \frac{u^2}{\delta^{2s}}$ strongly in $L^1(\Omega_T)$. Therefore u is an energy solution to Problem (4.1). Using $(u_n - u)$ as a test function in (4.2) and letting $n \rightarrow +\infty$, it holds that $u_n \rightarrow u$ strongly in $L^2((0, T), \mathbb{H}_0^s(\Omega))$.

— If $\lambda = \Lambda_s(\Omega)$, applying again (4.3), we obtain

$$\max_{t \in [0, T]} \int_{\Omega} u_n^2(x, t) dx \leq C.$$

Then we conclude that the sequence $\{u_n\}_n$ is bounded in $L^2(\Omega_T)$, and by monotonicity of u_n we get $u_n \rightarrow u$ in $L^2(\Omega_T)$. In addition, $\{\frac{u_n}{\delta^{2s}}\}_n$ is bounded in $L_{loc}^1(\Omega_T)$. Thus as in the proof of first part of Theorem 3.1, we conclude that u is a distributional solution to (4.1) in the sense of Definition 2.10.

• **Second case :** $s \in [\frac{1}{2}, 1)$ and $\Lambda_s(\Omega) = \Lambda_s^*$.

As above, using u_n as test function in (4.2), we get

$$\begin{aligned} \frac{1}{2} \int_{\Omega} u_n^2(x, t) dx &+ \frac{a_{N,s}}{2} \int_0^t \iint_{D_{\Omega}} \frac{|u_n(x, \sigma) - u_n(y, \sigma)|^2}{|x - y|^{N+2s}} dx dy d\sigma \\ &\leq \Lambda_s^* \int_0^t \int_{\Omega} \frac{u_n^2}{\delta^{2s}} dx d\sigma + \frac{1}{2} \int_{\Omega} u_0^2 dx. \end{aligned}$$

Now using the improved Hardy-Sobolev inequality (see Theorem 2.3), it holds that

$$\frac{1}{2} \int_{\Omega} u_n^2(x, t) dx + k(N, s) \int_0^t \left(\int_{\Omega} u_n^{2^*}(x, \sigma) dx d\sigma \right)^{\frac{2}{2^*}} \leq \frac{1}{2} \int_{\Omega} u_0^2 dx.$$

By Sobolev's inequality, we get

$$\frac{1}{2} \int_{\Omega} u_n^2(x, t) dx + C \int_0^t \int_{\Omega} u_n^2(x, \sigma) dx d\sigma \leq \frac{1}{2} \int_{\Omega} u_0^2 dx.$$

Hence

$$\sup_{t \in [0, T]} \int_{\Omega} u_n^2(x, t) dx \leq \int_{\Omega} u_0^2 dx.$$

which implies that the sequence $\{u_n\}_n$ is bounded in $L^2(\Omega_T)$.
Finally, we conclude as above. \square

Related to the asymptotic behavior of the above solution, we have the following proposition.

Proposition 4.2. *Assume that $\lambda \in (0, \Lambda_s(\Omega))$ if $s \in (0, 1)$ and $\Lambda_s(\Omega) < \Lambda_s^*$, or $\lambda \in (0, \Lambda_s^*]$ if $s \in [\frac{1}{2}, 1)$ and $\Lambda_s(\Omega) = \Lambda_s^*$. If u is the solution to Problem (4.1) obtained above, then there exists $C > 0$ such that*

$$e^{Ct} \|u(t)\|_{L^2(\Omega)} \rightarrow 0 \text{ as } t \rightarrow +\infty.$$

Proof. The first case follows easily using estimate (4.3) and Gronwall's lemma.

Suppose that $s \in [\frac{1}{2}, 1)$ and $\lambda = \Lambda_s^*$, then

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx \leq -\frac{a_{N,s}}{2} \int_0^T \iint_{D_{\Omega}} \frac{|u(x, t) - u(y, t)|^2}{|x - y|^{N+2s}} dx dy dt + \Lambda_s^* \int_{\Omega} \frac{u^2}{\delta^{2s}} dx.$$

Thus by improved Hardy-Sobolev inequality (2.1), we obtain,

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx \leq -C \left(\int_{\Omega} u^{2^*} dx \right)^{\frac{2}{2^*}}.$$

Thus Hölder's inequality allows us to get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dx \leq -C_0 \int_{\Omega} u^2 dx.$$

Thanks to Gronwall's lemma, we obtain

$$\|u(t)\|_{L^2(\Omega)} \leq e^{-C_0 t} \|u_0\|_{L^2(\Omega)}.$$

By choosing $C < C_0$, we get the desired result. \square

4.1.2. *Case where $s \in (0, 1/2)$ and $\lambda > 0$.*

In this subsection, we consider the problem

$$(4.4) \quad \begin{cases} u_t + (-\Delta)^s u &= \lambda \frac{uH(\delta(x))}{\delta^{2s}(x)} & \text{in } \Omega_T, \\ u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega. \end{cases}$$

where $\lambda > 0$ and $H : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function such that $H(0) = 0$.

If $s \in (0, \frac{1}{2})$, we are able to prove the next existence result without any restriction on the parameter λ .

Theorem 4.3. Assume that Ω is a bounded regular domain of \mathbb{R}^N with $N > 2s$, $s \in (0, \frac{1}{2})$ and $0 \not\leq u_0 \in L^1(\Omega)$. Then, for any $\lambda > 0$, Problem (4.4) has a unique weak solution $u \in \mathcal{C}((0, T), L^1(\Omega))$, in the sense of the Definition 2.8, such that

- 1) $\forall k \geq 0, T_k(u) \in L^2((0, T), \mathbb{H}_0^s(\Omega))$;
- 2) $u \in L^q(\Omega_T)$ for all $q \in [1, \frac{N+2s}{N})$;
- 3) $|(-\Delta)^{\frac{s}{2}} u| \in L^r(\Omega_T)$ for all $r \in [1, \frac{N+2s}{N+s})$.

To prove Theorem 4.3, we will need the lemma below. This lemma is more or less classic except for the fact that the solution belongs to $\mathbb{H}_0^s(\Omega)$.

Lemma 4.4. Assume that $0 < s < \frac{1}{2}$ and fix $\alpha > 0$ such that $2s + 2\alpha < 1$. Then, the following problem

$$(4.5) \quad \begin{cases} (-\Delta)^s w = \frac{1}{\delta^{2s+\alpha}} & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

has a unique weak solution w obtained as a limit of approximating problems. Moreover, $w \in \mathbb{H}_0^s(\Omega)$ and

$$(4.6) \quad w(x) \simeq \delta^{-\alpha}(x) \quad \text{in } \Omega.$$

Proof. For the sake of completeness and for the reader's convenience, let us give an idea on the proof.

Since $0 < 2s + 2\alpha < 1$, $\delta^{-2(s+\alpha)} \in L^1(\Omega)$. Therefore by [38, Theorem 23], Problem (4.5) has a unique solution w obtained as a limit of approximating problems. In addition, Estimate (4.6) comes from [3, Proposition 3].

Hence, we have just to show that $w \in \mathbb{H}_0^s(\Omega)$. Notice that w can be obtained as a limit of the approximating problem

$$(4.7) \quad \begin{cases} (-\Delta)^s w_n = \frac{1}{\delta^{2s+\alpha} + \frac{1}{n}} & \text{in } \Omega, \\ w_n > 0 & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We have $w_n \uparrow w$ strongly in $L^\sigma(\Omega)$ for all $\sigma < \frac{N}{N-2s}$. Moreover, by using w_n as a test function in (4.7) and by Young's inequality, we get

$$\begin{aligned} \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy &= \int_\Omega \frac{w_n(x)}{\delta^{2s+\alpha} + \frac{1}{n}} dx \\ &\leq \varepsilon \int_\Omega \frac{w_n^2(x)}{\delta^{2s}(x)} dx + C(\varepsilon) \int_\Omega \frac{1}{\delta^{2s+2\alpha}(x)} dx \\ &\leq \varepsilon \int_\Omega \frac{w_n^2(x)}{\delta^{2s}(x)} dx + C(\varepsilon, \Omega). \end{aligned}$$

Choosing ε small enough and applying Hardy-Sobolev inequality (2.3), we conclude that

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy \leq C(\Omega) \text{ for all } n.$$

Thus $w \in \mathbb{H}_0^s(\Omega)$. □

Now we are ready to prove Theorem 4.3.

Proof of Theorem 4.3.

We proceed by approximation. Let $n \geq 1$ and $u_n \in L^2((0, T), \mathbb{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$ be the positive solution to the approximating problem

$$(4.8) \quad \begin{cases} \partial_t u_n + (-\Delta)^s u_n &= \lambda \frac{u_{n-1} H(\delta(x))}{(\delta(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T \\ u_n(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) &= u_{0n}(x) & \text{in } \Omega, \end{cases}$$

where $u_{0n} := T_n(u_0)$ and $u_{00} := 0$.

In what follows, we denote by C any positive constant that can change from one line to the other and it is independent of n .

We claim that,

$$(4.9) \quad \forall \varepsilon > 0, \forall \lambda > 0, \exists C = C(\lambda, \varepsilon) > 0 \text{ such that } \lambda < C \delta^{2s}(x) + \frac{\varepsilon}{H(\delta(x))} \quad \forall x \in \Omega.$$

It is clear that (4.9) holds trivially in a neighborhood of the boundary. Now, far from the boundary we use the fact that for every compact set $K \subset \subset \Omega$, $\delta(x) > C(K)$ in K . Thus, by choosing C large in (4.9), the claim follows.

Now fix $\alpha \in (0, \frac{1-2s}{2})$ and consider $w \in \mathbb{H}_0^s(\Omega)$ the unique weak solution to Problem (4.5) given in Lemma 4.4. By taking w as test function in (4.8) and using the fact that $u_{n-} \leq u_n$, we obtain

$$\frac{d}{dt} \int_{\Omega} u_n w \, dx + \int_{\Omega} w (-\Delta)^s u_n \, dx \leq \lambda \int_{\Omega} \frac{u_n w H(\delta(x))}{\delta^{2s}} \, dx.$$

Thus

$$(4.10) \quad \frac{d}{dt} \int_{\Omega} u_n w \, dx + \int_{\Omega} \frac{u_n}{\delta^{2s+\alpha}} \, dx \leq \lambda \int_{\Omega} \frac{u_n w H(\delta(x))}{\delta^{2s}} \, dx.$$

Using (4.9), it holds that, for ε small enough to be chosen, we have

$$\lambda \frac{u_n w H(\delta(x))}{\delta^{2s}} \leq C u_n w H(\delta(x)) + \varepsilon \frac{u_n w}{\delta^{2s}}.$$

Taking into consideration that $H(\delta(x)) \leq C$ and $w \simeq \delta^{-\alpha}$, we get the existence of $C_1, C_2 > 0$ such that

$$\lambda \frac{u_n w H(\delta(x))}{\delta^{2s}} \leq C_1 u_n w + \varepsilon C_2 \frac{u_n}{\delta^{2s+\alpha}}.$$

Going back to (4.10) we deduce that

$$\frac{d}{dt} \int_{\Omega} u_n w \, dx + \int_{\Omega} \frac{u_n}{\delta^{2s+\alpha}} \, dx \leq C_1 \int_{\Omega} u_n w \, dx + \varepsilon C_2 \int_{\Omega} \frac{u_n}{\delta^{2s+\alpha}} \, dx.$$

Choosing ε small enough, we obtain

$$(4.11) \quad \frac{d}{dt} \int_{\Omega} u_n w \, dx + (1 - \varepsilon C_2) \int_{\Omega} \frac{u_n}{\delta^{2s+\alpha}} \, dx < C_1 \int_{\Omega} u_n w \, dx$$

which implies

$$\frac{d}{dt} \int_{\Omega} u_n w \, dx \leq C_1 \int_{\Omega} u_n w \, dx.$$

By Gronwall's Lemma and integrating in time, it holds that

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) w dx + (1 - C_2 \varepsilon) \int_0^T \int_{\Omega} \frac{u_n}{\delta^{2s+\alpha}} dx dt \leq C(T) \int_{\Omega} u_0 w dx.$$

Thus

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) w dx \leq C(T) \text{ and } \int_0^T \int_{\Omega} \frac{u_n(1 + H(\delta(x)))}{\delta^{2s}} dx dt \leq C(T).$$

Recall that the sequence $\{u_n\}_n$ is increasing. Thus, there exists a measurable function u such that

- (1) $u_n \rightarrow u$ strongly in $L^1(\Omega_T)$,
- (2) $\frac{u_n}{\delta^{2s}} \rightarrow \frac{u}{\delta^{2s}}$ strongly in $L^1(\Omega_T)$.

So, in view of the above estimates, we conclude that $u \in C([0, T], L^1(\Omega))$. Thus, u is a weak solution to problem (4.1) in the sense of Definition 2.8.

On the other hand, u is obtained as a limit of approximating problems with *monotone behavior*, then by a simple comparison principle, it follows that u is the minimal solution to Problem (4.1).

To prove the uniqueness, we argue by contradiction. For this we assume that v is another nonnegative weak solution to Problem (4.1). Since $v \geq 0$ in Ω_T , then $v \geq u_1$ defined as the unique solution of Problem (4.8) with $n = 1$. Thus by iteration it follows that $v \geq u_n$ for all $n \geq 1$. Hence $v \geq u$. Define $\mathcal{U} = v - u \geq 0$, then $\mathcal{U} \geq 0$, $\frac{\mathcal{U}}{\delta^{2s}} \in L^1(\Omega_T)$ and \mathcal{U} satisfies

$$(4.12) \quad \begin{cases} \mathcal{U}_t + (-\Delta)^s \mathcal{U} &= \lambda \frac{\mathcal{U} H(\delta(x))}{\delta^{2s}} & \text{in } \Omega_T, \\ \mathcal{U} &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \mathcal{U}(x, 0) &= 0 & \text{in } \Omega. \end{cases}$$

Taking w (the solution to Problem (4.5)) as a test function in (4.12). Then, using (4.9) and following the same arguments as in the proof of Theorem 4.3, we obtain

$$\frac{d}{dt} \int_{\Omega} \mathcal{U} w dx + (1 - \varepsilon C) \int_{\Omega} \frac{\mathcal{U}}{\delta^{2s+\alpha}(x)} dx < C \int_{\Omega} \mathcal{U} w dx,$$

where $\varepsilon \ll 1$. Since $\mathcal{U}, w \geq 0$ in Ω_T , then

$$\frac{d}{dt} \int_{\Omega} \mathcal{U} w dx \leq C \int_{\Omega} \mathcal{U} w dx.$$

Using Gronwall's lemma implies $\mathcal{U}(t, x)w(x) = 0$ a.e. in Ω for all $t > 0$. Thus $\mathcal{U}(t, x) \equiv 0$ for all $t \geq 0$ and the uniqueness follows. \square

In the case where the initial data u_0 has more regularity, we can improve the regularity of the solution u . More precisely, we have :

Theorem 4.5. *Let $s \in (0, \frac{1}{2})$. Assume that Ω is a bounded regular domain and $\lambda > 0$. Let $\sigma > 1$ and assume that $0 \leq u_0 \in L^\sigma(\Omega, w dx)$ where w is the solution to Problem (4.5).*

Then, the unique solution u to Problem (4.4) satisfies $u \in L^\infty((0, T), L^\sigma(\Omega, w dx))$.

Proof. As in the proof of Theorem 4.3, let us consider u_n the unique solution to the approximating Problem (4.8). Since $\sigma > 1$, applying Kato type inequality (2.2) to the function u_n^σ (that is a convex function), we obtain

$$(4.13) \quad \partial_t(u_n^\sigma) + (-\Delta)^s u_n^\sigma \leq \sigma u_n^{\sigma-1} \partial_t u_n + \sigma u_n^{\sigma-1} (-\Delta)^s u_n.$$

By using $u_n^{\sigma-1} w$ as a test function in (4.8) and taking into consideration estimates (4.9) and (4.13), we get

$$\begin{aligned} \frac{1}{\sigma} \frac{d}{dt} \int_{\Omega} u_n^\sigma w \, dx + \frac{1}{\sigma} \int_{\Omega} \frac{u_n^\sigma}{\delta^{2s+\alpha}} \, dx &\leq \frac{d}{dt} \int_{\Omega} u_n^{\sigma-1} u_{nt} w \, dx + \int_{\Omega} w u_n^{\sigma-1} (-\Delta)^s u_n \, dx \\ &\leq \lambda \int_{\Omega} \frac{u_n^\sigma w H(\delta(x))}{\delta^{2s}(x)} \\ &\leq C \int_{\Omega} u_n^\sigma w \, dx + \varepsilon C \int_{\Omega} \frac{u_n^\sigma}{\delta^{2s+\alpha}(x)} \, dx. \end{aligned}$$

Therefore

$$\frac{1}{\sigma} \frac{d}{dt} \int_{\Omega} u_n^\sigma w \, dx + \left(\frac{1}{\sigma} - \varepsilon C \right) \int_{\Omega} \frac{u_n^\sigma}{\delta^{2s+\alpha}} \, dx \leq C \int_{\Omega} u_n^\sigma w \, dx.$$

Hence, by choosing ε such that $\varepsilon C \ll \frac{1}{\sigma}$, we obtain that

$$\frac{d}{dt} \int_{\Omega} u_n^\sigma w \, dx \leq C \int_{\Omega} u_n^\sigma w \, dx.$$

Gronwall's inequality allows us to conclude that

$$\int_{\Omega} u_n^\sigma(x, t) w \, dx \leq \left(\int_{\Omega} u_0^\sigma w \, dx \right) e^{Ct}, \quad \forall t \geq 0.$$

Thanks to monotone convergence Theorem, we get the existence of a solution u such that $u \in L^\sigma(\Omega_T, w \, dx \, dt)$ and

$$\int_0^T \int_{\Omega} \frac{u^\sigma}{\delta^{2s+\alpha}} \, dx \, dt \leq C(T).$$

□

Remark 4.6. As a typical examples of a function H , we can take $H(\sigma) = \sigma_+^a$ for some $a > 0$, $H(\sigma) = (\log(\frac{A}{\sigma}))^{-a}$ where $a > 0$ and $A \gg \text{diam}(\Omega)$, or $H(\sigma) = \sigma_+^a (\log(\frac{A}{\sigma}))^b$ where $a, b > 0$.

Remark 4.7. In the case $s \in (0, \frac{1}{2})$, it is well-know that $\mathbb{H}_0^s(\Omega) = \mathbb{H}^s(\Omega)$ and then $\chi_\Omega \in \mathbb{H}_0^s(\Omega)$, see for instance [39] or [17]. In this case, Hardy's inequality in Theorem 1.1 holds and $\Lambda_s(\Omega) > 0$ for any regular domain. However it seems to be interesting to prove that $\Lambda_s(\Omega)$ does not depend on Ω if Ω is a convex domain as in the case $s \in [\frac{1}{2}, 1)$. Notice that in [19], the authors proved that in convex domains, and if $s \in (0, \frac{1}{2})$, then there exists a universal constant $C(s, \Omega)$ depending only on s and Ω such that $\Lambda_s(\Omega) \geq C(s, \Omega)$.

4.2. The case $s \in [\frac{1}{2}, 1)$: Nonexistence result.

Let us recall that, in this case and if Ω is a regular convex domain, then Hardy-Sobolev inequality in Theorem 1.1 holds with the optimal constant $\Lambda_s(\Omega) = \Lambda_s^*$. As in [24], if Ω is a convex domain, we are able to show the next nonexistence result.

Theorem 4.8. *Assume that $\frac{1}{2} \leq s < 1$, $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded regular domain such $\Lambda_s(\Omega) = \Lambda_s^*$ (that includes the case of convex domains). If $\lambda > \Lambda_s^*$, then Problem (4.1) does not have any weak positive solution.*

To prove Theorem 4.8, we need two useful propositions. The first one is the next version of the maximum principle.

Proposition 4.9. *Let $s \in (0, 1)$ and $\Omega \subset \mathbb{R}^N$ be a bounded regular domain. Assume that $v_0 \in L^1(\Omega)$ and consider v , the unique solution to the problem*

$$(4.14) \quad \begin{cases} v_t + (-\Delta)^s v &= 0 & \text{in } \Omega_T, \\ v(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ v(x, 0) &= v_0(x) & \text{in } \Omega. \end{cases}$$

Then, for all $s \in (0, T)$, there exist two positive functions $C_1(t)$ and $C_2(t)$ (actually, $C_1(t)$ and $C_2(t)$ depend also on N , s and Ω) such that for all $x \in \Omega$,

$$(4.15) \quad C_1(t)\delta^s(x) \leq v(x, t) \leq C_2(t)\delta^s(x).$$

Moreover, for all $t \in [t_1, t_2] \subset (0, T)$, there exist two positive constants $\widehat{C}_1, \widehat{C}_2 > 0$ independent of t such that

$$0 < \widehat{C}_1 \leq C_1(t) \leq C_2(t) \leq \widehat{C}_2.$$

Comment : It is worth to point out that Proposition 4.9 is independent of Hardy's inequality and then holds for any $s \in (0, 1)$ and any bounded regular domain of \mathbb{R}^N .

Proof. The proof follows closely the argument used in [40] for the local case. However, for the reader's convenience, we include here some details.

In what follows, we denote by $\mathcal{S}(t)$ the corresponding semi-group to (4.14), so $v(\cdot, t) = \mathcal{S}(t)v_0(\cdot)$.

Now introduce ϱ the unique solution to the problem

$$(4.16) \quad \begin{cases} (-\Delta)^s \varrho &= 1 & \text{in } \Omega, \\ \varrho &= 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then by Theorem 1.2 in [47], there exist $c_1, c_2 > 0$ such $c_1\delta^s(x) \leq \varrho(x) \leq c_2\delta^s(x)$. On the other hand, thanks Theorem 4.3 in [13], for any $t \in (0, T)$ there exists $C(t) > 0$ such that $v(x, t) \geq C(t)\varrho$. Hence, the left-hand side inequality in (4.15) follows. By applying again [13, Theorem 4.3], there exists two constants C_0 and C_2 such that for all $t \in [t_1, t_2] \subset (0, T)$, we have $0 < C_0 \leq C(t) \leq C_2$.

Let us now prove the right-hand side inequality in (4.15). For this, we will use the regularizing effect of the fractional heat semi-group $\mathcal{S}(t)$.

Fix $t > 0$, then

$$|v(x, t)| \leq \frac{C}{t^{\frac{N}{2s}}} \|v_0\|_{L^1(\Omega)}.$$

Now, let us introduce \mathcal{V}_ρ the unique solution to the problem

$$(4.17) \quad \begin{cases} \mathcal{V}_t + (-\Delta)^s \mathcal{V} &= 0 & \text{in } \Omega_T, \\ \mathcal{V}(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \mathcal{V}(x, 0) &= \rho & \text{in } \Omega. \end{cases}$$

By using Hopf's Lemma (see [33]), it follows that, for all $t > 0$ there exists $C(t) > 0$ such that

$$\left\| \frac{\mathcal{V}_\rho(\cdot, t)}{\delta^s} \right\|_{L^\infty(\Omega)} \leq C(t) \|\rho\|_{L^\infty(\Omega)}$$

where $0 < \tilde{C}_0 \leq C(t) \leq \tilde{C}_1$ uniformly if $t \in [t_1, t_2] \subset (0, T)$ with \tilde{C}_0 and \tilde{C}_1 are independent of t .

Let $\mathcal{V}(\cdot, t) := \mathcal{V}_{\chi_\Omega}(\cdot, t) = \mathcal{S}(t)\chi_\Omega(\cdot)$. Thanks to the semi-group properties and the Hopf's Lemma (see [33]), if v solves (4.14), we have

$$\left\| \frac{v(\cdot, t)}{\delta^s} \right\|_{L^\infty(\Omega)} = \left\| \delta^{-s} \mathcal{S}\left(\frac{t}{3}\right) \left[\mathcal{S}\left(\frac{2t}{3}\right) v_0(\cdot) \right] \right\|_{L^\infty(\Omega)} \leq C(t) \left\| \mathcal{S}\left(\frac{2t}{3}\right) v_0(\cdot) \right\|_{L^\infty(\Omega)},$$

and

$$\left\| \mathcal{S}\left(\frac{2t}{3}\right) v_0(\cdot) \right\|_{L^\infty(\Omega)} \leq C_2(t) \left\| \mathcal{S}\left(\frac{t}{3}\right) v_0(\cdot) \right\|_{L^1(\Omega)}.$$

Now, taking into consideration that

$$\begin{aligned} \left\| \mathcal{S}\left(\frac{t}{3}\right) v_0(\cdot) \right\|_{L^1(\Omega)} &= \int_{\Omega} v_0(x) \delta^s(x) \frac{\mathcal{S}\left(\frac{t}{3}\right) \chi_\Omega}{\delta^s(x)} dx \\ &= \int_{\Omega} v_0(x) \delta^s(x) \frac{\mathcal{V}(x, \frac{t}{3})}{\delta^s(x)} dx \leq C(t) \|\delta^s v_0\|_{L^1(\Omega)}, \end{aligned}$$

and combining with the above estimates, it follows that

$$\delta^{-s} v(\cdot, t) \leq \bar{C}(t) \|\delta^s v_0\|_{L^1(\Omega)}.$$

Hence we conclude. \square

We need also the next result that extends the one obtained in [21] in the local case.

Proposition 4.10. *Let $s \in [1/2, 1)$ and Ω be a bounded regular domain such that $\Lambda_s(\Omega) = \Lambda_s^*$. Define*

$$\hat{\Lambda} = \inf_{\phi \in \mathcal{C}_0^\infty(\Omega), \phi \neq 0} \frac{\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy - C \int_{\Omega} \frac{\phi^2}{\delta^{2s}} dx}{\int_{\Omega} \phi^2 dx},$$

where $C > \Lambda_s^*$. Then $\hat{\Lambda} = -\infty$.

Proof. We argue by contradiction. Assume that $|\hat{\Lambda}| < \infty$, then for all $\phi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy - C \int_{\Omega} \frac{\phi^2}{\delta^{2s}} dx \geq \hat{\Lambda} \int_{\Omega} \phi^2 dx.$$

Thus

(4.18)

$$J(\hat{\Lambda}) := \inf_{\phi \in \mathcal{C}_0^\infty(\Omega), \phi \neq 0} \frac{\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + |\hat{\Lambda}| \int_\Omega \phi^2 dx}{\int_\Omega \frac{\phi^2}{\delta^{2s}} dx} \geq C > \Lambda_s^*.$$

For $\beta > 0$, we define the set

$$\Omega_\beta := \left\{ x \in \Omega \text{ such that } \text{dist}(x, \partial\Omega) < \beta \right\}.$$

Then

$$J(\hat{\Lambda}) \leq \inf_{\phi \in \mathcal{C}_0^\infty(\Omega_\beta), \phi \neq 0} \frac{\frac{a_{N,s}}{2} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + |\hat{\Lambda}| \int_{\Omega_\beta} \phi^2 dx}{\int_{\Omega_\beta} \frac{\phi^2}{\delta^{2s}} dx}.$$

Notice that, using the Hardy-Sobolev inequality,

$$\begin{aligned} \int_{\Omega_\beta} \phi^2 dx &= \int_{\Omega_\beta} \frac{\phi^2}{\delta^{2s}} \delta^{2s} dx \leq \beta^{2s} \int_{\Omega_\beta} \frac{\phi^2}{\delta^{2s}} dx \\ &\leq \frac{\beta^{2s}}{\Lambda_s^*} \frac{a_{N,s}}{2} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Hence, for $\phi \in \mathcal{C}_0^\infty(\Omega_\beta)$,

$$\begin{aligned} \frac{a_{N,s}}{2} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + |\hat{\Lambda}| \int_{\Omega_\beta} \phi^2 dx &\leq \\ \frac{a_{N,s}}{2} \left(1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Therefore we conclude that

$$\begin{aligned} J(\hat{\Lambda}) &\leq \left(1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \inf_{\phi \in \mathcal{C}_0^\infty(\Omega_\beta)} \frac{\frac{a_{N,s}}{2} \iint_{D_{\Omega_\beta}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy}{\int_{\Omega_\beta} \frac{\phi^2}{\delta^{2s}} dx} \\ &\leq \left(1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \Lambda_s^*(\Omega_\beta). \end{aligned}$$

Recall that $\Lambda_s^*(\Omega) = \Lambda_s^*$, then since $\Lambda_s^*(\Omega) \leq \Lambda_s^*(\Omega_\beta) \leq \Lambda_s^*$, it follows that $\Lambda_s^*(\Omega_\beta) = \Lambda_s^*$. Now, using the fact that $C > \Lambda_s^*$, we can choose β small enough such that

$$\left(1 + \frac{\beta^{2s}}{\Lambda_s^*} |\hat{\Lambda}| \right) \Lambda_s^* < C,$$

a contradiction with (4.18). Thus $\hat{\Lambda} = -\infty$. \square

We are now ready to prove our main nonexistence result.

Proof of Theorem 4.8.

Without loss of generality, we suppose that $0 \leq u_0 \in L^\infty(\Omega)$. We argue by contradiction and suppose that problem (4.1) has a positive weak solution u for $\lambda > \Lambda_s^*$. Using an approximating argument, we get the existence of a minimal solution to Problem (4.1) obtained as a limit of approximating problems. We denote this minimal solution by u . Thus $u = \lim_{n \rightarrow +\infty} u_n$ where u_n is the unique positive solution to

$$(4.19) \quad \begin{cases} \frac{\partial}{\partial t} u_n + (-\Delta)^s u_n &= \lambda a_n(x) u_n & \text{in } \Omega_T, \\ u_n(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u_n(x, 0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

with $a_n(x) = \min\{n, \frac{1}{\delta^{2s}}\}$, $\lambda > \Lambda_s^*$. It is clear that $\{u_n\}_n$ is increasing in n and $u_n \rightarrow u$ strongly in $\mathcal{C}([0, T], L^1(\Omega))$.

It is worth to notice that $u(x, t) > 0$ for all $(x, t) \in \Omega \times (0, T)$. By applying Proposition 4.9, it follows that for all $(x, t) \in \Omega \times [t_1, t_2]$ with $t_1 > 0$, we get the existence of a positive constant $\hat{C} := \hat{C}(t_1, t_2, \Omega)$

$$u(x, t) \geq u_n(x, t) \geq \hat{C} \delta^s(x) \text{ for all } n \geq 1.$$

Let $\phi \in \mathcal{C}_0^\infty(\Omega)$ with $\int_\Omega \phi^2(x) dx = 1$, using $\frac{\phi^2}{u_n}$ as a test function in (4.19), we obtain

$$(4.20) \quad \int_\Omega \frac{\partial_t u_n}{u_n} \phi^2 dx + \int_\Omega \frac{\phi^2}{u_n} (-\Delta)^s u_n dx = \lambda \int_\Omega a_n(x) \phi^2 dx.$$

Using Picone's inequality stated in Proposition 2.5, it holds

$$(4.21) \quad \int_\Omega \frac{\phi^2}{u_n} (-\Delta)^s u_n dx \leq \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy.$$

Therefore, by combining (4.20), (4.21) and by integrating in $[t_1, t_2]$ with $t_1 > 0$, we obtain

$$\lambda \int_\Omega a_n(x) \phi^2 dx \leq \frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + \frac{1}{t_2 - t_1} \int_\Omega \ln \left(\frac{u_n(t_2)}{u_n(t_1)} \right) \phi^2 dx.$$

Notice that $\ln \left(\frac{u_n(\cdot, t_2)}{u_n(\cdot, t_1)} \right) = \ln \left(\frac{u_n(\cdot, t_2) \delta^s(\cdot)}{u_n(\cdot, t_1) \delta^s(\cdot)} \right)$. Then

$$\begin{aligned} \left| \ln \left(\frac{u_n(\cdot, t_2)}{u_n(\cdot, t_1)} \right) \right| &= \left| \ln \left(\frac{u_n(\cdot, t_2) \delta^s(\cdot)}{u_n(\cdot, t_1) \delta^s(\cdot)} \right) \right| \\ &\leq \left| \ln(u_n(\cdot, t_2) \delta^s(\cdot)) \right| + \left| \ln(u_n(\cdot, t_1) \delta^s(\cdot)) \right|. \end{aligned}$$

To estimate the behavior of the above expression as $n \rightarrow +\infty$, we need to know the regularity of the term $\log(\delta)$. Notice that, for any $\beta > 0$, $\delta^\beta(x) |\log(\delta(x))| \leq C(\beta, \Omega)$ in Ω , then $|\log(\delta)| \in L^m(\Omega_T)$ for any $m \geq 1$.

We claim that $\ln(u(\cdot, t_i) \delta^s(\cdot)) \in L^\sigma(\Omega)$ for all $\sigma \geq 1$ with $i = 1, 2$. For this purpose, we follow closely the argument used in [24], for the reader's convenience we include here some details. Let $w(x, t_i) = u(\cdot, t_i) \delta^s(\cdot)$, then

$$\ln(w(x, t_i)) = \ln(w(x, t_i) \chi_{w \geq 1}) + \ln(w(x, t_i) \chi_{w \leq 1}).$$

Since for any $a > 0$, $\ln(w(x, t_i)\chi_{w \geq 1}) \leq C_1 w^a(x, t_i) + C_2$, then by choosing $a < \frac{1}{\sigma}$, it holds that $\ln(w(\cdot, t_i)\chi_{w \geq 1}) \in L^\sigma(\Omega)$.

Now respect to $\ln(w(x, t_i)\chi_{w \leq 1})$, we know that, for all $a > 0$,

$$|\ln(w(x, t_i)\chi_{w \leq 1})| \leq \frac{C_1}{(w(x, t_i)\chi_{w \leq 1})^a} + C_2 \leq \frac{C_1(t_i)}{\delta^{2as}(x)} + C_2.$$

By choosing $a < \frac{1}{2s\sigma}$, we obtain $\frac{C_1(t_i)}{\delta^{2as}} \in L^\sigma(\Omega)$ and the claim follows.

In the same way and since $u_n \uparrow u$ strongly in $\mathcal{C}([0, T], L^1(\Omega))$, we deduce that $\ln(u_n(\cdot, t_i)\delta^s(\cdot)) \in L^\sigma(\Omega)$ for all $\sigma \geq 1$ and $\left\| \ln(u_n(\cdot, t_i)\delta^s(\cdot)) \right\|_{L^\sigma(\Omega)} \leq C(t_i, \Omega)$.

Thus, since $u_n \uparrow u$ strongly in $\mathcal{C}([0, T], L^1(\Omega))$ and using the fact that $u_n(\cdot, t) \geq \hat{C}\delta^s(\cdot)$ for all $n \geq 1$ and uniformly for $t \in [t_1, t_2]$ give

$$\ln(u_n(\cdot, t_2)\delta^s(\cdot)) \rightarrow \ln(u(\cdot, t_2)\delta^s(\cdot)) \text{ strongly in } L^\sigma(\Omega) \forall \sigma \geq 1.$$

Now, we claim that for all $\varepsilon > 0$, there exists $\check{C} := \check{C}(\varepsilon, t_1, t_2, \Omega) > 0$ such that for all $n \geq 1$:

$$\frac{1}{t_2 - t_1} \int_{\Omega} \ln \left(\frac{u_n(x, t_2)}{u_n(x, t_1)} \right) \phi^2(x) dx \leq \varepsilon \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + \check{C} \int_{\Omega} \phi^2 dx.$$

To prove the claim, let us introduce $m_n(x) := \frac{1}{t_2 - t_1} \left| \log \left(\frac{u_n(x, t_2)}{u_n(x, t_1)} \right) \right|$. By writing $m_n(x)$ under the following form $\frac{1}{(t_2 - t_1)} \left| \ln \left(\frac{u_n(\cdot, t_2)\delta^s(\cdot)}{u_n(\cdot, t_1)\delta^s(\cdot)} \right) \right|$ and using the above estimates yield $m_n \in L^\sigma(\Omega)$ for all $\sigma > 1$ and $\|m_n\|_{L^\sigma(\Omega)} \leq C(t_1, t_2, \Omega)$. Thus choosing $\sigma > \frac{N}{s}$ and applying Hölder and Young inequalities, we get

$$\begin{aligned} \frac{1}{t_2 - t_1} \int_{\Omega} \ln \left(\frac{u_n(x, t_2)}{u_n(x, t_1)} \right) \phi^2(x) dx &\leq \int_{\Omega} m_n(x) |\phi(x)| |\phi(x)| dx \\ &\leq \left(\int_{\Omega} m_n^{\frac{N}{s}}(x) dx \right)^{\frac{s}{N}} \left(\int_{\Omega} \phi^2(x) dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\phi(x)|^{2^*} dx \right)^{\frac{1}{2^*}} \\ &\leq C(t_1, t_2, \Omega) \left(\int_{\Omega} \phi^2(x) dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\phi(x)|^{2^*} dx \right)^{\frac{1}{2^*}} \\ &\leq \varepsilon \left(\int_{\Omega} |\phi(x)|^{2^*} dx \right)^{\frac{2}{2^*}} + C(\varepsilon, \Omega, t_1, t_2) \int_{\Omega} |\phi(x)|^2 dx. \end{aligned}$$

Hence using Sobolev's inequality gives

$$\frac{1}{t_2 - t_1} \int_{\Omega} \ln \left(\frac{u_n(x, t_2)}{u_n(x, t_1)} \right) \phi^2(x) dx \leq \varepsilon \frac{a_{N,s}}{2} \iint_{D_{\Omega}} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{N+2s}} dx dy + C(\varepsilon, t_1, t_2, \Omega) \int_{\Omega} \phi^2 dx.$$

and then the claim follows.

Now, letting $n \rightarrow \infty$, it follows that for all $\phi \in C_0^\infty(\Omega)$ with $\phi \neq 0$, we have

$$\frac{(1+\varepsilon)^{\frac{a_{N,s}}{2}} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy - \lambda \int_{\Omega} \frac{\phi^2}{\delta^{2s}} dx}{\int_{\Omega} \phi^2 dx} \geq -C(\varepsilon, t_1, t_2, \Omega).$$

Since $\lambda > \Lambda_s^*$, we can choose ε small enough such that $\frac{\lambda}{1+\varepsilon} > \Lambda_s^*$. Then we obtain

$$\inf_{\phi \in C_0^\infty(\Omega)} \frac{\frac{a_{N,s}}{2} \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{N+2s}} dx dy - \frac{\lambda}{1+\varepsilon} \int_{\Omega} \frac{\phi^2}{\delta^{2s}} dx}{\int_{\Omega} \phi^2 dx} \geq -\frac{C(\varepsilon, t_1, t_2, \Omega)}{1+\varepsilon} > -\infty,$$

Hence, we reach a contradiction with the result of Proposition 4.10. This ends up the proof.

5. THE SUPERLINEAR CASE $p > 1$: EXISTENCE RESULTS.

In this section, we are interested in the super-linear case, namely $p > 1$. The principal idea to get the existence result is to find a suitable supersolution and then we proceed by iteration. The main existence result of this section is the following.

Theorem 5.1. *Let $s \in (0, 1)$. Assume that Ω is a bounded regular domain of \mathbb{R}^N with $N > 2s$ and $p > 1$. Then, there exists $u_0 \in L^\infty(\Omega)$ such that the problem*

$$(5.1) \quad \begin{cases} u_t + (-\Delta)^s u &= \frac{u^p}{\delta^{2s}} & \text{in } \Omega_T, \\ u(x, t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \end{cases}$$

has a minimal positive solution $u \in L^2((0, T), \mathbb{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$.

Proof. In order to prove Theorem 5.1, we use a monotonicity argument. Fix $\beta > 0$ such that $\beta \in (0, s)$ if $p \geq 2$ and $\beta < (2-p)s$ if $p \in (1, 2)$. Let ξ be the solution to Problem (2.6), then by Theorem 2.11, it holds that $\xi \simeq \delta^s$.

Since $p > 1$, then setting $\bar{\xi} = C\xi$, we can choose C small enough such that

$$\bar{\xi}_t + (-\Delta)^s \bar{\xi} \geq \frac{\bar{\xi}^p}{\delta^{2s}}.$$

It is clear that if $u_0 \leq \bar{\xi}$, then $\bar{\xi}$ is supersolution to (5.1). Consider now $\underline{\xi}$, the unique solution to the following problem

$$(5.2) \quad \begin{cases} v_t + (-\Delta)^s v &= 0 & \text{in } \Omega_T, \\ v &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ v(x, 0) &= u_0(x) & \text{in } \Omega. \end{cases}$$

Then $\underline{\xi} \in L^2(0, T, \mathbb{H}_0^s(\Omega)) \cap L^\infty(\Omega_T)$ and $\underline{\xi}$ is a subsolution to (5.1) with $\underline{\xi} \leq \bar{\xi}$.

We define now the sequence $\{u^k\}_k$ by $u^0 := \underline{\xi}$, and for $k \geq 1$, u^k is the unique positive solution to the problem

$$(5.3) \quad \begin{cases} \frac{\partial}{\partial t} u^k + (-\Delta)^s u^k &= \frac{(u^{k-1})^p}{\delta^{2s} + \frac{1}{k}} & \text{in } \Omega_T, \\ u^k &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ u^k(x, 0) &= u_0(x) & \text{in } \Omega. \end{cases}$$

Applying the comparison principle gives $\underline{\xi} \leq u^k \leq u^{k+1} \leq \bar{\xi}$ for all $k \geq 1$.

Thus, we get the existence of $u \leq \bar{\xi}$ such that $u^k \uparrow u$ strongly in $L^\sigma(\Omega_T)$ for all $\sigma \geq 1$ and $u^k \rightarrow u$ strongly in $L^2((0, T), \mathbb{H}_0^s(\Omega))$. Then u is a bounded solution to Problem (5.1) with $\frac{u^p}{\delta^{2s}} \in L^1(\Omega_T)$. This concludes the proof. \square

Remark 5.2. *It seems to be interesting to find a natural class of initial data u_0 such that Problem (5.1) has a weak solution in the sense that $\frac{u^p}{\delta^{2s}} \in L^1(\Omega_T)$.*

6. EXTENSION AND OPEN PROBLEMS

In this section, we state some extension of the previous results to the case of the spectral fractional Laplacian. For this let us define the space

$$(6.1) \quad \widetilde{H}^s(\Omega) := \{u = \sum_{i=1}^{+\infty} a_i \phi_i \in L^2(\Omega) ; \sum_{i=1}^{+\infty} \lambda_i^s a_i^2 < +\infty\},$$

where $(\lambda_i, \phi_i)_{i \geq 1}$ are the eigenvalues and the eigenfunctions of $(-\Delta)$ with Dirichlet boundary conditions and $\|\phi_i\|_{L^2(\Omega)} = 1$. The space $\widetilde{H}^s(\Omega)$ is endowed with the norm

$$\|u\|_{\widetilde{H}^s(\Omega)}^2 = \sum_{i=1}^{+\infty} \lambda_i^s a_i^2.$$

Observe that $\widetilde{H}^s(\Omega)$ is the interpolation space $(\mathbb{H}_0^s(\Omega), L^2(\Omega))_{[1-s]}$, see [9, 43, 39]. Therefore, we obtain

$$\widetilde{H}^s(\Omega) = \begin{cases} H^s(\Omega) & \text{if } 0 < s < \frac{1}{2}, \\ H_{00}^{\frac{1}{2}}(\Omega) & \text{if } s = \frac{1}{2}, \\ \mathbb{H}_0^s(\Omega) & \text{if } \frac{1}{2} < s < 1, \end{cases}$$

where

$$H_{00}^{\frac{1}{2}}(\Omega) := \left\{ u \in H^{\frac{1}{2}}(\Omega) ; \int_{\Omega} \frac{u^2(x)}{\delta(x)} dx < +\infty \right\}.$$

For any $s \in (0, 1)$ and any $u = \sum_{i=1}^{+\infty} a_i \phi_i \in \widetilde{H}^s(\Omega)$, the *spectral fractional Laplacian*, denoted by \mathcal{A}_s , is defined by

$$(6.2) \quad \mathcal{A}_s(u) = \sum_{i=1}^{+\infty} \lambda_i^s a_i \phi_i.$$

Remark 6.1. Let $s \in (0, 1)$. The operator $(-\Delta)^s$ defined by (1.1) and \mathcal{A}_s are different. The difference seems to have been known to probabilists ; it was established later in PDEs.

— Firstly, \mathcal{A}_s depends on the domain Ω considered (its eigenfunctions and eigenvalues depend on Ω), while $(-\Delta)^s$ is independent of the domain.

— Secondly, the respective eigenfunctions of \mathcal{A}_s and $(-\Delta)^s$ are different. On one hand, the eigenfunctions of \mathcal{A}_s (i.e. of the classical Laplacian) are smooth up the boundary (if the domain is regular). On the other hand, the eigenfunctions of $(-\Delta)^s$ are, in general, no better than Hölder continuous up the boundary.

For more details, we refer the interested reader to [48], [43, chapter 5], [18] and [44].

As it was proved by Caffarelli-Silvestre in [25], by adding a new variable, it is possible to consider the spectral fractional Laplacian operator in a bounded domain by means of a local problem.

More precisely, consider the problem

$$(6.3) \quad \begin{cases} \mathcal{A}_s u &= f(x, u) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{cases}$$

Then, define $C_\Omega = \Omega \times (0, +\infty) \subset \mathbb{R}_+^{N+1}$ and denote by (x, y) a point in C_Ω . For $u \in \widetilde{H}^s(\Omega)$, we define the s -harmonic extension $w = E_s(u)$ in C_Ω as the solution to the problem

$$(6.4) \quad \begin{cases} -\operatorname{div}(y^{1-2s}\nabla w) &= 0 & \text{in } C_\Omega, \\ w &= 0 & \text{in } \partial_L\Omega, \\ w &= u & \text{on } \Omega \times \{0\}, \end{cases}$$

with $\partial_L\Omega = \partial\Omega \times (0, +\infty)$. It is obvious that w belongs to the following space

$$X_0^s(C_\Omega) := \overline{\mathcal{C}_0^\infty(C_\Omega)}^{||\cdot||_{X_0^s(C_\Omega)}} \text{ with } ||w||_{X_0^s(C_\Omega)}^2 = k_s \int_{C_\Omega} y^{1-2s} |\nabla w|^2 dx dy,$$

where k_s is a normalization constant. As consequence, for all $u \in \widetilde{H}^s(\Omega)$, we have

$$(6.5) \quad ||E_s(u)||_{X_0^s(C_\Omega)} = ||u||_{\widetilde{H}^s(\Omega)}.$$

Going back to problem (6.4), we get

$$(6.6) \quad \frac{\partial w(x, y)}{\partial \nu^s} \equiv -\frac{1}{k_s} \lim_{y \rightarrow 0^+} \frac{\partial w(x, y)}{\partial y} = \mathcal{A}^s w(x, 0) = \mathcal{A}^s u(x).$$

Now, let us recall the next regularity result obtained in [27, Lemma 2.9].

Theorem 6.2. Assume that $s \in (0, 1)$ and consider ζ to be the unique solution to problem

$$(6.7) \quad \begin{cases} \mathcal{A}_s \zeta &= 1 & \text{in } \Omega, \\ \zeta &= 0 & \text{in } \partial\Omega. \end{cases}$$

Then, $\zeta \in \mathcal{C}^\alpha(\bar{\Omega})$ for all $\alpha \in (0, \min\{2s, 1\})$. Moreover, there exists a positive constant $C \equiv C(\Omega, s)$ such that

- (i) $\zeta \leq C\delta^{2s}$ if $s < \frac{1}{2}$,
- (ii) $\zeta \leq C\delta^s$ if $\frac{1}{2} \leq s < 1$.

Set $\psi := \zeta^a$ where $a \in (0, 1)$. By the Kato type inequality (see [26]), we get

$$\mathcal{A}_s \psi \geq \frac{C}{\zeta^{1-a}}.$$

As in Theorem 3.1, we have the following existence result where additional hypothesis is needed in the case where $s < \frac{1}{2}$.

Theorem 6.3. *Let $0 < s < 1$ and consider the problem*

$$(6.8) \quad \begin{cases} u_t + \mathcal{A}_s(u) &= \lambda \frac{u^p}{\delta^{2s}} & \text{in } \Omega_T = \Omega \times (0, T), \\ u(x, 0) &= u_0(x) & \text{in } \Omega, \\ u &= 0 & \text{in } \partial\Omega \times (0, T), \end{cases}$$

where $0 \leq u_0 \in L^1(\Omega)$. Then

- (1) if $s \in (0, \frac{1}{2})$, Problem (6.8) has a positive distributional solution for all $p \in (0, 1)$;
- (2) if $s \in [\frac{1}{2}, 1)$, Problem (6.8) has a positive distributional solution for all $p \in (0, 1 - s)$.

Proof. As in the proof of Theorem 3.1, we proceed by approximation. Define u_n to be the unique solution to the problem

$$(6.9) \quad \begin{cases} u_{nt} + \mathcal{A}_s u_n &= \frac{u_{n-1}^p}{(\delta(x) + \frac{1}{n})^{2s}} & \text{in } \Omega_T \\ u_n(x, t) &= 0 & \text{in } \partial\Omega \times (0, T), \\ u_n(x, 0) &= u_{0n}(x) & \text{in } \Omega, \end{cases}$$

with $u_{0n} := \min(n, u_0)$ and $u_{00} = 0$. It is clear that $u_n \geq 0$ and $u_n \leq u_{n+1}$ for all n .

- Let us begin by the case $s \in (0, \frac{1}{2})$. Define $\psi = \zeta^a$ with $a \in (0, 1)$ where ζ is the unique solution to Problem (6.7). Using ψ as a test function in (6.9), we reach that

$$\frac{d}{dt} \int_{\Omega} u_n \psi \, dx + \int_{\Omega} \frac{u_n}{\delta^{(1-a)2s}} \, dx \leq \int_{\Omega} \frac{u_n^p \psi}{(\delta(x) + \frac{1}{n})^{2s}} \, dx \leq C \int_{\Omega} \frac{u_n^p}{\delta^{2s(1-a)}} \, dx.$$

Since $p < 1$, by Young's inequality, we obtain

$$(6.10) \quad \frac{d}{dt} \int_{\Omega} u_n \psi \, dx + C \int_{\Omega} \frac{u_n}{\delta^{2s(1-a)}} \, dx \leq \bar{C} \int_{\Omega} \frac{1}{\delta^{2s(1-a)}} \, dx.$$

Taking into consideration that $2s(1-a) < 1$, it follows that $\int_{\Omega} \frac{1}{\delta^{2s(1-a)}} \, dx < \infty$. Thus

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) \psi \, dx + C \int_0^T \int_{\Omega} \frac{u_n}{\delta^{2s(1-a)}} \, dx \, dt \leq CT + \int_{\Omega} u_0 \psi \, dx < \infty.$$

Hence the rest of the proof follows as in the proof of Theorem 3.1.

- We deal now with the case $s \in [\frac{1}{2}, 1)$. In this case, $\phi \leq C\delta^s$. Fix $a \in (0, 1)$ such that $p < (1 - \frac{s}{1-s(1-a)})$, then as above we have

$$(6.11) \quad \begin{aligned} & \frac{d}{dt} \int_{\Omega} u_n \psi \, dx + C \int_{\Omega} \frac{u_n}{\delta^{s(1-a)}} \, dx \\ & \leq \bar{C} \int_{\Omega} \frac{u^p}{\delta^{2s-a}} \, dx \leq \varepsilon \int_{\Omega} \frac{u_n}{\delta^{s(1-a)}} \, dx + C(\varepsilon) \int_{\Omega} \frac{1}{\delta^{s(\frac{1}{p-1}+1-a)}} \, dx. \end{aligned}$$

Recall that $p < 1 - s$ and $p < (1 - \frac{s}{1-s(1-a)})$, then $\int_{\Omega} \frac{1}{\delta^{s(\frac{1}{p-1}+1-a)}} \, dx < \infty$.

Hence

$$\sup_{t \in [0, T]} \int_{\Omega} u_n(x, t) \psi \, dx + C \int_0^T \int_{\Omega} \frac{u_n}{\delta^{s(1-a)}} \, dx \, dt \leq CT + \int_{\Omega} u_0 \psi \, dx < \infty.$$

Then we conclude. \square

Remark 6.4. In the case $s < \frac{1}{2}$, we believe that the condition $p < 1 - s$ is technical and a general existence result holds for any $p < 1$.

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