



HAL
open science

Fractional Laplacians: a short survey

Maha Daoud, El Haj Laamri

► **To cite this version:**

Maha Daoud, El Haj Laamri. Fractional Laplacians: a short survey. Discrete and Continuous Dynamical Systems - Series S, In press, 10.3934/dcdss.2021027 . hal-03276152

HAL Id: hal-03276152

<https://hal.univ-lorraine.fr/hal-03276152v1>

Submitted on 1 Jul 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

1 **FRACTIONAL LAPLACIANS : A SHORT SURVEY**

MAHA DAUD

Department of Mathematics and Computer Science
Faculty of sciences Ain Chock
B.P. 5366 Maarif, Casablanca, Morocco

EL HAJ LAAMRI*

Institut Elie Cartan de Lorraine
Université de Lorraine
B.P. 239, Vandoeuvre-lès-Nancy, France

(Communicated by the associate editor name)

ABSTRACT. This paper describes the state of the art and gives a survey of the wide literature published in the last years on the fractional Laplacian. We will first recall some definitions of this operator in \mathbb{R}^N and its main properties. Then, we will introduce the four main operators often used in the case of a bounded domain ; and we will give several simple and significant examples to highlight the difference between these four operators. Also we give a rather long list of references : it is certainly not exhaustive but hopefully rich enough to track most connected results. We hope that this short survey will be useful for young researchers of all ages who wish to have a quick idea of the fractional Laplacian(s).

2 1. **Introduction.** The theory of fractional powers of operators on Banach spaces
3 is a classical topic in mathematical analysis and Probability. It was developed over
4 the last century, we refer the interested reader for instance to [9, 12, 56, 57] and the
5 references therein. In the last years, this theory has gained a higher interest due
6 to the increasing need for modeling in many scientific fields. In fact, many models
7 issued from Biology, Ecology, Finance, etc, lead to nonlinear problems driven by
8 fractional Laplace-type operators.

9 The basic example of these operators is the fractional Laplacian. In contrast to
10 the Laplacian which is a local operator, the fractional Laplacian is a paradigm of
11 the vast family of nonlocal operators, and this has immediate consequence in the
12 formulation of basic questions such as the Poisson problem. There is an extensive
13 literature devoted to this topic, contributions include [38, 62, 87, 21, 80, 24, 25,
14 42, 41, 51, 61] and the references therein. Needless to say, these references do not
15 exhaust the rich literature on the subject.

2020 *Mathematics Subject Classification.* nonlocal operator, fractional laplacian, regional
laplacian, restricted Laplacian, spectral Laplacian, peridynamic Laplacian.

Key words and phrases. Fractional Laplacian, nonlocal operators, regional fractional Laplacian,
spectral fractional Laplacian, fractional Poisson equations.

* Corresponding author : El Haj Laamri.

1 From a probabilistic point of view, the fractional Laplacian is related to anomalous diffusion, that employs fractional order derivatives ; it represents the infinitesimal generator of Lévy stable diffusion process, as it is well known in the theory of probability. For more details, see for instance [13, 72, 51, 78, 30] and the references therein. On the other hand, it is possible to obtain a fractional heat equation as a limit of a random walk with long jumps, we refer the interested reader to the excellent note [86].

8 As is well known, there are several ways of defining this operator in the whole space \mathbb{R}^N . As far as we know, there are ten definitions in the literature. It turns out that all these definitions are equivalent (see for instance [58]). On the other hand, several different fractional Laplacians can be defined on an open subset $\Omega \neq \mathbb{R}^N$. These alternatives correspond to different ways of taking into account the information from the boundary and the exterior of domain.

14 The first goal of this work is to present an up-to-date survey on the fractional Laplacian in \mathbb{R}^N , and on the different fractional Laplacians on an open subset $\Omega \neq \mathbb{R}^N$. More precisely, we will present *four fractional Laplacians*. Two of them are well known and have been widely studied, while the other two have only recently been introduced in the literature. The second goal is to highlight the difference between these four operators. To this end, we will first give some simple and significant examples to illustrate this difference; then we will investigate it numerically. For the sake of simplicity, we will only consider homogeneous Dirichlet boundary-value problems, which will allow us to focus on the main features and subtleties of the theory.

24 Let us emphasize that the originality of this work does not lie so much in the content, for which we have drawn inspiration from existing literature — notably that cited in the bibliography — as in the presentation and choice of the themes. We do not claim to make significant additions to a theory that has changed little for several years; but we justify our presentation as a new and quick insight for new readers who wish to have a global and quick understanding of the subject before going deeper.

31 The rest of paper is composed of four sections. Section 2 is devoted to a review of several (and equivalent) definitions of the fractional Laplacian $(-\Delta)^s$ in \mathbb{R}^N and its main properties. For the sake of brevity and clarity of exposition, we have chosen to treat only the case $0 < s < 1$. In section 3, we briefly recall some properties of fractional Sobolev spaces on an open bounded subset Ω of \mathbb{R}^N . In Section 4, we expose in some detail four different “fractional Laplacians” : *regional* (in Subsection 37 4.1), *spectral* (in Subsection 4.2), *restricted* (in Subsection 4.4) and *peridynamic* (in Subsection 38 4.5). Moreover, Subsections 4.3 and 4.6 are devoted to highlighting the differences between these four operators by suitable one-dimensional examples.

40 Needless to say, it is never pleasant for a mathematician not to give the demonstrations. Nevertheless, in order to keep this paper reasonably short, we will mainly state and comment results without reproducing the proofs. Accordingly, we have tried to give precise references that allow the reader to find them. Hence our rather long bibliography : it is certainly not exhaustive, but hopefully rich enough to track 45 most relevant results.

1 **Notations :**

2 Let N be a positive integer, $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ and
 3 $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{N}^N$.

4 Throughout this paper, we denote by :

5 • $\Omega \subset \mathbb{R}^N$ an open, bounded with smooth boundary ;

6 • $D^\alpha := \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}$ where $|\alpha| := \alpha_1 + \dots + \alpha_N$;

7 • $\mathbf{x}^\alpha := x_1^{\alpha_1} \dots x_N^{\alpha_N}$;

8 • $(\mathbf{x} | \xi) := \sum_{k=1}^N x_k \xi_k$ the euclidean inner product of \mathbb{R}^N ;

9 • $\|\mathbf{x}\|^2 := \sum_{k=1}^N x_k^2$;

10 • $\Delta u = \sum_{k=1}^N \frac{\partial^2 u}{\partial x_k^2}$;

11 • $\|u\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}$ where $p \in [1, +\infty)$;

12 • $B(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^N ; \|\mathbf{x} - \mathbf{y}\| < r\}$ where $r \in (0, +\infty)$;

13 • $\mathcal{C}_0(\mathbb{R}^N) := \{u \in \mathcal{C}(\mathbb{R}^N) ; u(\mathbf{x}) \rightarrow 0 \text{ as } \|\mathbf{x}\| \rightarrow +\infty\}$;

14 • $\mathcal{C}_c^\infty(\mathbb{R}^N) := \{u \in \mathcal{C}^\infty(\mathbb{R}^N) ; u \text{ with compact support}\}$;

15 • $\mathcal{C}_c^\infty(\Omega) := \{u \in \mathcal{C}^\infty(\Omega) ; u \text{ with compact support in } \Omega\}$;

16 • $\mathcal{S}(\mathbb{R}^N) := \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^N) ; \forall \alpha, \beta \in \mathbb{N}^N, \sup_{\mathbf{x} \in \mathbb{R}^N} |\mathbf{x}^\beta D^\alpha \varphi(\mathbf{x})| < +\infty\}$;

17 • $\mathcal{F}u(\xi) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-i(\xi|\mathbf{x})} u(\mathbf{x}) d\mathbf{x}$ the Fourier transform of u ;

18 • $\mathcal{F}^{-1}u(\mathbf{x}) := \frac{1}{(2\pi)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{i(\xi|\mathbf{x})} u(\xi) d\xi$;

19 • $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$, $z \in (0, +\infty)$, the Gamma function.

20 **2. Fractional Laplacian operator in \mathbb{R}^N .**

21 Let $s \in (0, 1)$. In this section, we will introduce the fractional Laplacian $(-\Delta)^s$
 22 in the whole-space \mathbb{R}^N and its basic properties. To the best of our knowledge, there
 23 are ten equivalent definitions of the fractional Laplacian in the whole-space \mathbb{R}^N , see
 24 e.g. [58]. Here, we will give only five definitions, namely the most frequently used.

1 **2.1. Fractional Sobolev spaces** $H^s(\mathbb{R}^N)$. As will appear later (see Proposition
 2 **2.6** below), the fractional Sobolev space $H^s(\mathbb{R}^N)$ is the right framework to define
 3 the fractional Laplacian in \mathbb{R}^N . For the sake of completeness and the ease of reader,
 4 we now recall two equivalent definitions of this space.

First, we define the fractional Sobolev space $H^s(\mathbb{R}^N)$ through the so-called Gagliardo semi-norm :

$$[u]_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x}d\mathbf{y} \right)^{\frac{1}{2}}$$

Then,

$$H^s(\mathbb{R}^N) := \{u \in L^2(\mathbb{R}^N) ; [u]_{H^s(\mathbb{R}^N)} < +\infty\}.$$

For $u, v \in H^s(\mathbb{R}^N)$, we set

$$\langle u, v \rangle_{H^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} u(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x}d\mathbf{y}.$$

5 One can easily check that $\langle \cdot, \cdot \rangle_{H^s(\mathbb{R}^N)}$ is an inner product on $H^s(\mathbb{R}^N)$ and that the
 6 space $H^s(\mathbb{R}^N)$ endowed with this inner product is a Hilbert space.

Second, the space $H^s(\mathbb{R}^N)$ can be defined alternatively via the Fourier transform. We denote

$$\widehat{H}^s(\mathbb{R}^N) := \left\{ u \in L^2(\mathbb{R}^N) ; \int_{\mathbb{R}^N} (1 + \|\xi\|^{2s}) |\mathcal{F}u(\xi)|^2 d\xi < +\infty \right\}.$$

7 The fractional Sobolev space $H^s(\mathbb{R}^N)$ coincides with $\widehat{H}^s(\mathbb{R}^N)$ (see for instance [38,
 8 Proposition 3.4] or [62, Corollary 1.15]). \square

9 2.2. Definitions of fractional Laplacian and main properties.

10 In this paragraph, we will give some equivalent definitions of fractional Laplacian
 11 in \mathbb{R}^N and its main properties.

12 2.2.1. *First definition.*

13 The first definition of the fractional Laplacian $(-\Delta)^s$ ($s \in (0, 1)$) is expressed via
 14 the Fourier transform.

15 Let $u \in \mathcal{S}(\mathbb{R}^N)$. We have that $\mathcal{F}u \in \mathcal{S}(\mathbb{R}^N)$, but the function $\xi \mapsto \|\xi\|^{2s}\mathcal{F}u(\xi) \notin$
 16 $\mathcal{S}(\mathbb{R}^N)$ because $\|\xi\|^{2s}$ creates a singularity at $\xi = 0$. On the other hand, $\|\xi\|^{2s}\mathcal{F}u(\xi) \in$
 17 $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, so we can use the inverse Fourier transform. Therefore, we have
 18 the following definition :

19 For any $u \in \mathcal{S}(\mathbb{R}^N)$, we define the operator $(-\Delta)^s : \mathcal{S}(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ by

$$(-\Delta)^s u(\mathbf{x}) = \mathcal{F}^{-1}(\|\xi\|^{2s}(\mathcal{F}u)(\xi))(\mathbf{x}) \quad \forall \mathbf{x} \in \mathbb{R}^N, \quad (2.1)$$

20 This operator is called *the fractional Laplacian* of order s .

21 **First properties :** Now, we give some elementary properties of the fractional
 22 Laplacian (see for instance [80]). Before this, let us recall that $\mathcal{S}(\mathbb{R}^N) \subset \mathcal{C}_0(\mathbb{R}^N)$
 23 and the inclusion is strict.

24 Let $u \in \mathcal{S}(\mathbb{R}^N)$, we have :

- 25 • $(-\Delta)^0 u = u$;
- 26 • $(-\Delta)^1 u = -\Delta u$;
- 27 • for any $0 < s_1, s_2 < 1$,

$$(-\Delta)^{s_1} \circ (-\Delta)^{s_2} u = (-\Delta)^{s_2} \circ (-\Delta)^{s_1} u ;$$

- for any multi-index $\alpha \in \mathbb{N}^N$, $D^\alpha(-\Delta)^s u = (-\Delta)^s D^\alpha u$. In particular,

$$(-\Delta)^s u \in \mathcal{C}^\infty(\mathbb{R}^N). \square$$

1 **2.2.2. Second definition.**

2 The second definition of the fractional Laplacian is often used in the literature. It
 3 is also the most adequate to define the *regional* fractional Laplacian (see Definition
 4 4.1) and the *restricted* fractional Laplacian (see Definition 4.6) in the case of a
 5 bounded subset of \mathbb{R}^N .

6 For any $u \in \mathcal{S}(\mathbb{R}^N)$,

$$(-\Delta)^s u(\mathbf{x}) := C(N, s) \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B(\mathbf{x}, \varepsilon)} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \quad \forall \mathbf{x} \in \mathbb{R}^N, \quad (2.2)$$

7 where

$$C(N, s) := \frac{s 4^s \Gamma(s + \frac{N}{2})}{\pi^{\frac{N}{2}} \Gamma(1 - s)}. \quad (2.3)$$

Remark 2.1. Let us give some light on the normalizing constant $C(N, s)$ given by
 (2.3).

(i) $C(N, s)$ is chosen so that the four definitions (2.1), (2.2), (2.7) and (2.8) are
 equivalent and the following two identities :

$$\lim_{s \rightarrow 0^+} (-\Delta)^s u = u \text{ and } \lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u$$

8 hold (see Proposition 2.1).

9 (ii) Moreover

$$C(N, s) \sim s(1 - s) \text{ as } s \rightarrow 0^+ \text{ and } s \rightarrow 1^-. \quad (2.4)$$

10 For more details on the asymptotic behavior of $C(N, s)$, the interested reader is
 11 referred to [38, Section 4] (see also [62, Section 1.3.1]).

We recall that by definition :

$$P.V. \int_{\mathbb{R}^N} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} := \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B(\mathbf{x}, \varepsilon)} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y}.$$

12 In other words, P.V. stands for the “Cauchy principal value”. Since then, we can
 13 write

$$(-\Delta)^s u(\mathbf{x}) := C(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \quad \forall \mathbf{x} \in \mathbb{R}^N. \quad (2.5)$$

14 The following theorem (see for instance [80, Theorem 1]) shows that we can do
 15 without using the “Cauchy principal value” in (2.5) if $s \in (0, \frac{1}{2})$.

16 **Theorem 2.1.** *Let $u \in \mathcal{S}(\mathbb{R}^N)$ and $\mathbf{x} \in \mathbb{R}^N$. The integral*

$$\int_{\mathbb{R}^N} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \quad (2.6)$$

17 *is absolutely convergent if and only if $s \in (0, \frac{1}{2})$.*

18 *So we have*

$$(-\Delta)^s u(\mathbf{x}) = C(N, s) \int_{\mathbb{R}^N} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y},$$

19 *for $s \in (0, \frac{1}{2})$.*

1 **2.2.3. Third definition.**

2 The third definition is given in the form of a well-defined integral on \mathbb{R}^N as
 3 follows :

4 For any $u \in \mathcal{S}(\mathbb{R}^N)$, we have

$$(-\Delta)^s u(\mathbf{x}) := -\frac{1}{2}C(N, s) \int_{\mathbb{R}^N} \frac{u(\mathbf{x} + \mathbf{y}) + u(\mathbf{x} - \mathbf{y}) - 2u(\mathbf{x})}{\|\mathbf{y}\|^{N+2s}} d\mathbf{y}, \quad \forall \mathbf{x} \in \mathbb{R}^N. \quad (2.7)$$

5 It is worth to point out that the Definitions 2.1, 2.2 and 2.7 are equivalent. For
 6 a proof, see for instance [38, section 3] or [62, subsection 1.3].

7 **2.2.4. Fourth definition.**

8 Another definition of the fractional Laplacian comes from the theory of semi-
 9 groups in $\mathbb{R}^N \times [0, +\infty)$. This definition is classical (see, e.g., [9, 56, 57, 89]) and it
 10 is more useful to study partial differential equations.

11 For any $u \in \mathcal{S}(\mathbb{R}^N)$, we have

$$(-\Delta)^s u(\mathbf{x}) = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} (u(\mathbf{x}) - w(\mathbf{x}, t)) \frac{dt}{t^{1+s}}, \quad (2.8)$$

where $w(\mathbf{x}, t)$ is the solution to the following problem

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{on } \mathbb{R}^N \times [0, +\infty), \\ w(\mathbf{x}, 0) = u(\mathbf{x}). \end{cases}$$

12 All the definitions above are equivalent. For a proof of the equivalence between
 13 2.2 and 2.8, see for instance [81, Lemma 2.1]. \square

14 **2.2.5. Relation between the “classical” Laplacian and the fractional Lapla-**
 15 **cian.**

16 The following proposition describes the relation between the “classical” Laplacian
 17 and the fractional Laplacian (see, e.g., [21, Lemma 3.11] and [38, Proposition 4.4]).

18 **Proposition 2.1.** *Let $u \in \mathcal{S}(\mathbb{R}^N)$. Then*

19 (i) $\lim_{s \rightarrow 0^+} (-\Delta)^s u = u,$

20 (ii) $\lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u.$

21 As we will see later, Proposition 2.1 holds for less regular functions, see Theorem
 22 2.2 and Theorem 2.3 below. \square

23 **2.2.6. Extensions for less regular functions.**

24 Definitions (2.1), (2.2), (2.7) and (2.8) are valid for any $u \in \mathcal{S}(\mathbb{R}^N)$. Actually,
 25 we can extend these definitions to $\mathcal{C}^{0,\beta}(\mathbb{R}^N)$ and even to $H^s(\mathbb{R}^N)$.

26 **• Extension to $\mathcal{C}^{0,\beta}(\mathbb{R}^N)$ and some regularity results**

Let V be an open subset of \mathbb{R}^N , $k \in \mathbb{N}$ and $\beta \in (0, 1]$. We denote by $\mathcal{C}^{0,\beta}(V)$ the
 space of β -Hölder continuous functions on V , and by $\mathcal{C}^{k,\beta}(V)$ the space of functions
 of $\mathcal{C}^k(V)$ whose k th partial derivatives are Hölder continuous with exponent β . Let
 us recall the seminorms :

$$[u]_{\mathcal{C}^{0,\beta}(V)} := \sup_{\substack{\mathbf{x}, \mathbf{y} \in V \\ \mathbf{x} \neq \mathbf{y}}} \left(\frac{|u(\mathbf{x}) - u(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|^\beta} \right)$$

and

$$[u]_{\mathcal{C}^{k,\beta}(V)} := \sum_{|\alpha|=k} \sup_{\substack{\mathbf{x}, \mathbf{y} \in V \\ \mathbf{x} \neq \mathbf{y}}} \frac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|}{\|\mathbf{x} - \mathbf{y}\|^\beta}, \quad k > 0.$$

Furthermore, let introduce the following space

$$L_s^1(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} ; \int_{\mathbb{R}^N} \frac{|u(\mathbf{x})|}{1 + \|\mathbf{x}\|^{N+2s}} d\mathbf{x} < +\infty \right\}.$$

1 We have the following results whose proofs can be found in [77, Propositions 2.4,
2 2.5, 2.6 and 2.7] (see also [76] and [80, section 7]).

3 **Proposition 2.2.** *Let $u \in L_s^1(\mathbb{R}^N)$. Assume that $u \in \mathcal{C}^{0,2s+\varepsilon}(V)$ if $s \in (0, \frac{1}{2})$ or
4 $u \in \mathcal{C}^{1,2s+\varepsilon-1}(V)$ if $s \in [\frac{1}{2}, 1)$, for some $\varepsilon > 0$. Then, $(-\Delta)^s u$ is a continuous
5 function in V and $(-\Delta)^s u(\mathbf{x})$ is given by the definitions above for every $\mathbf{x} \in V$.*

Proposition 2.3. *Let $u \in \mathcal{C}^{0,\beta}(\mathbb{R}^N)$ for $\beta > 2s > 0$. Then, $(-\Delta)^s u \in \mathcal{C}^{0,\beta-2s}(\mathbb{R}^N)$
and there exists $C > 0$ such that*

$$[(-\Delta)^s u]_{\mathcal{C}^{0,\beta-2s}(\mathbb{R}^N)} \leq C[u]_{\mathcal{C}^{0,\beta}(\mathbb{R}^N)},$$

6 where C depends only on β , s and N .

7 **Proposition 2.4.** *Let $u \in \mathcal{C}^{1,\beta}(\mathbb{R}^N)$.*

(i) *If $\beta > 2s$, then $(-\Delta)^s u \in \mathcal{C}^{1,\beta-2s}(\mathbb{R}^N)$ and there exists $C > 0$ such that*

$$[(-\Delta)^s u]_{\mathcal{C}^{1,\beta-2s}(\mathbb{R}^N)} \leq C[u]_{\mathcal{C}^{1,\beta}(\mathbb{R}^N)},$$

8 where C depends only on β , s and N .

(ii) *If $\min(0, 2s - 1) < \beta < 2s$, then $(-\Delta)^s u \in \mathcal{C}^{0,\beta-2s+1}(\mathbb{R}^N)$ and there exists
 $C > 0$ such that*

$$[(-\Delta)^s u]_{\mathcal{C}^{0,\beta-2s+1}(\mathbb{R}^N)} \leq C[u]_{\mathcal{C}^{1,\beta}(\mathbb{R}^N)},$$

9 where C depends only on β , s , and N .

10 **Proposition 2.5.** *Let $u \in \mathcal{C}^{k,\beta}(\mathbb{R}^N)$ such that $k + \beta - 2s$ is not an integer. Then,
11 $(-\Delta)^s u \in \mathcal{C}^{\ell,\alpha}(\mathbb{R}^N)$ where ℓ is the integer part of $k + \beta - 2s$ and $\alpha = k + \beta - 2s - \ell$.*

12 Before ending this paragraph, it is worth to point out that Proposition 2.1 can
13 be extended for less regular functions. More precisely, we have the following two
14 results (see [80, Theorem 3] and [80, Theorem 4]).

Theorem 2.2. *Let $\mathbf{x} \in \mathbb{R}^N$ and $u \in \mathcal{C}^2(B(\mathbf{x}, 1)) \cap L^\infty(\mathbb{R}^N)$. Then*

$$\lim_{s \rightarrow 1^-} (-\Delta)^s u(\mathbf{x}) = -\Delta u(\mathbf{x}).$$

Theorem 2.3. *Let $\mathbf{x} \in \mathbb{R}^N$ and $u \in \mathcal{C}^{0,\beta}(B(\mathbf{x}, 1)) \cap L_0(\mathbb{R}^N)$ where*

$$L_0^1(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable} ; \int_{\mathbb{R}^N} \frac{|u(\mathbf{x})|}{1 + \|\mathbf{x}\|^N} d\mathbf{x} < +\infty \right\}.$$

Then

$$\lim_{s \rightarrow 0^+} (-\Delta)^s u(\mathbf{x}) = u(\mathbf{x}).$$

15 **•• Extension to $H^s(\mathbb{R}^N)$**

16 Since the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ is dense in $H^s(\mathbb{R}^N)$, the fractional Laplacian
17 operator $(-\Delta)^s$ can be extended by density to $u \in H^s(\mathbb{R}^N)$. More precisely, we
18 have the following relation (see, e.g., [38, Proposition 3.6], or [62, Proposition 1.18]).

19 **Proposition 2.6.** *For any $u \in H^s(\mathbb{R}^N)$,*

$$[u]_{H^s(\mathbb{R}^N)}^2 = 2C(N, s)^{-1} \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}^2. \quad (2.9)$$

20 As mentioned before, the identity (2.9) shows that the space $H^s(\mathbb{R}^N)$ is the right
21 framework to define the fractional Laplacian $(-\Delta)^s$. \square

1 **2.2.7. Fifth definition.**

2 An important tool in some works on nonlinear problems driven by fractional
 3 Laplacians (see, e.g., [76, 24, 77, 25]) is the following transformation : any fractional
 4 Laplacian can be characterized as an operator that maps a Dirichlet boundary
 5 condition to a Neumann-type condition via an extension problem. More precisely,
 6 the fractional Laplacian can be realized in a local manner through the boundary
 7 value problem

$$\begin{cases} \operatorname{div}(y^a \nabla U) = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ U(\mathbf{x}, 0) = u(\mathbf{x}) & \text{on } \partial \mathbb{R}_+^{N+1}, \end{cases} \quad (2.10)$$

where $\mathbb{R}_+^{N+1} := \{(\mathbf{x}, y) \in \mathbb{R}^N \times \mathbb{R} ; y > 0\}$ is the upper half-space and $\partial \mathbb{R}_+^{N+1} := \mathbb{R}^N$.

The parameter a belongs to $(-1, 1)$ and is related to the power s of the fractional Laplacian $(-\Delta)^s$ by the formula $a = 1 - 2s \in (-1, 1)$. We set

$$\frac{\partial U}{\partial \nu^a} := - \lim_{y \rightarrow 0^+} y^a \partial_y U.$$

The following formula relating the fractional Laplacian to the Dirichlet-to-Neumann operator for (2.10) has been proven in [24] (cf. also [18, 27, 82]) :

$$(-\Delta)^s u = d_s \frac{\partial U}{\partial \nu^a} \quad \text{in } \mathbb{R}^N,$$

8 where $d_s = 2^{2s-1} \frac{\Gamma(s)}{\Gamma(1-s)}$ (see e.g. [23] for more details). \square

9 **3. Fractional Sobolev spaces on an open bounded subset of \mathbb{R}^N .**

10 In what follows, Ω is an open bounded subset of \mathbb{R}^N with smooth boundary, and
 11 $s \in (0, 1)$.

12 As in the case of \mathbb{R}^N , we start this paragraph by recalling one of the possible
 13 definitions of the classical fractional Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$. Then we
 14 define new spaces $\mathbb{X}_0^s(\Omega)$, which are the appropriate functional framework to study
 15 Dirichlet boundary-value problems governed by fractional Laplacians.

16 **3.1. Classical fractional Sobolev spaces $H^s(\Omega)$ and $H_0^s(\Omega)$.**

17 — First, we define the fractional Sobolev space $H^s(\Omega)$ through the Gagliardo
 18 semi-norm :

$$[u]_{H^s(\Omega)} := \left(\int_{\Omega} \int_{\Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x} d\mathbf{y} \right)^{\frac{1}{2}}. \quad (3.1)$$

Then,

$$H^s(\Omega) := \{u \in L^2(\Omega) ; [u]_{H^s(\Omega)} < +\infty\}.$$

For $u, v \in H^s(\Omega)$, we set

$$\langle u, v \rangle_{H^s(\Omega)} := \int_{\Omega} u(\mathbf{x})v(\mathbf{x})d\mathbf{x} + \int_{\Omega} \int_{\Omega} \frac{(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x} d\mathbf{y}.$$

19 It is easy to check that $\langle \cdot, \cdot \rangle_{H^s(\Omega)}$ is an inner product and the space $H^s(\Omega)$ endowed
 20 with this inner product is a Hilbert space.

21 From now on, we denote

$$\|u\|_{H^s(\Omega)} := \left(\int_{\Omega} |u(\mathbf{x})|^2 d\mathbf{x} + \int_{\Omega} \int_{\Omega} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x} d\mathbf{y} \right)^{\frac{1}{2}}. \quad (3.2)$$

22 — Furthermore, we define $H_0^s(\Omega)$ as the closure of $\mathcal{C}_c^\infty(\Omega)$ in $H^s(\Omega)$.

1 — It is worth to notice that the spaces $H^s(\Omega)$ have no trace where $s \in (0, \frac{1}{2}]$. More
 2 precisely, we have the following result :

3 **Theorem 3.1.** [59, Theorem 11.1]

- 4 – If $s \in (0, \frac{1}{2}]$, then $H_0^s(\Omega) = H^s(\Omega)$.
- 5 – If $s \in (\frac{1}{2}, 1)$, the inclusion $H_0^s(\Omega) \subset H^s(\Omega)$ is strict.

6 The previous result holds for any $p \in (1, +\infty)$, see [52, Theorem 1.4.2.4].□

7 3.2. Other type of fractional Sobolev spaces $\mathbb{X}_0^s(\Omega)$.

8 In order to study nonlocal problems governed by fractional Laplacians, we need
 9 appropriate fractional Sobolev spaces. These new spaces are inspired by (but not
 10 equivalent to) the fractional Sobolev spaces $H^s(\Omega)$, in order to correctly encode the
 11 data of the homogeneous Dirichlet condition in the weak formulation.

For this purpose, let us introduce the following space

$$\mathbb{X}_0^s(\Omega) := \{u \in H^s(\mathbb{R}^N) ; u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

12 endowed with the norm induced by $\|\cdot\|_{H^s(\mathbb{R}^N)}$.

13 Let $u \in \mathbb{X}_0^s(\Omega)$, we have

$$\begin{aligned} \|u\|_{\mathbb{X}_0^s(\Omega)}^2 &= \|u\|_{H^s(\mathbb{R}^N)}^2 \\ &= \|u\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x}d\mathbf{y}. \end{aligned}$$

14 As $\|u\|_{L^2(\mathbb{R}^N)}^2 = \|u\|_{L^2(\Omega)}^2$, then

$$\|u\|_{\mathbb{X}_0^s(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x}d\mathbf{y}. \quad (3.3)$$

15 We deduce from (3.3) that the norms $\|\cdot\|_{\mathbb{X}_0^s(\Omega)}$ and $\|\cdot\|_{H^s(\Omega)}$ are not the same.
 16 Moreover, the norm $\|\cdot\|_{\mathbb{X}_0^s(\Omega)}$ takes into account interaction between Ω and $\mathbb{R}^N \setminus \Omega$.
 17 So, the space $\mathbb{X}_0^s(\Omega)$ is the appropriate space to deal with elliptic partial differential
 18 equations driven by fractional laplacian operators.

Now, for $u \in \mathbb{X}_0^s(\Omega)$, we set

$$\| |u| \|_{\mathbb{X}_0^s(\Omega)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(\mathbf{x}) - u(\mathbf{y})|^2}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x}d\mathbf{y} \right)^{\frac{1}{2}}.$$

19 We have the following result (see, e.g., [62, Lemma 1.29]) :

Proposition 3.1. *The space $(\mathbb{X}_0^s(\Omega), \langle \cdot, \cdot \rangle_{\mathbb{X}_0^s(\Omega)})$ is a Hilbert space with the inner product*

$$\langle u, v \rangle_{\mathbb{X}_0^s(\Omega)} := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(\mathbf{x}) - u(\mathbf{y}))(v(\mathbf{x}) - v(\mathbf{y}))}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{x}d\mathbf{y} \quad \forall u, v \in \mathbb{X}_0^s(\Omega). \square$$

20 Moreover , the norms $\| |\cdot| \|_{\mathbb{X}_0^s(\Omega)}$ and $\|\cdot\|_{\mathbb{X}_0^s(\Omega)}$ are equivalent.

Remark 3.1. This space is also denoted $\widetilde{H}^s(\Omega)$ or $H_{00}^s(\Omega)$ in the literature. As is well-known (see, e.g., [59, 52, 6]), it holds that

$$\mathbb{X}_0^s(\Omega) = \begin{cases} H^s(\Omega) & \text{if } 0 < s < \frac{1}{2}, \\ H_0^s(\Omega) & \text{if } \frac{1}{2} < s < 1, \end{cases}$$

while in the limiting case $s = \frac{1}{2}$, it holds that :

$$\mathbb{X}_0^{\frac{1}{2}}(\Omega) := \left\{ u \in H^{\frac{1}{2}}(\Omega) ; \int_{\Omega} \frac{u^2(\mathbf{x})}{\rho(\mathbf{x})} d\mathbf{x} < +\infty \right\},$$

1 with $\rho(x) = \text{distance}(x, \partial\Omega)$.

2 In all cases, the canonical norm induced by the alternative definition is equivalent
3 to the previous ones.

4 **4. Fractional Laplacians on an open bounded domain.** Throughout this para-
5 graph, Ω is a bounded open subset of \mathbb{R}^N with smooth boundary.

6 As we have seen in Section 2, the different definitions of the fractional Laplacian
7 $(-\Delta)^s$ are equivalent when the operator acts in \mathbb{R}^N . Contrary to the case of whole
8 space, several different fractional Laplacians ¹ can be defined on a open bounded
9 subset of \mathbb{R}^N . These alternatives correspond to different ways in which the infor-
10 mation coming from the boundary and the exterior of domain is to be taken into
11 account.

12 In this section, we will present four fractional Laplacians and compare their
13 definitions, their probabilistic interpretations and their analytical properties.

14 For pedagogical purposes, this section is composed of six subsections. In the first
15 one, we introduce the *regional* fractional Laplacian (FL_{Reg} for short) and give some
16 examples. Subsection 2 is devoted to the *spectral* fractional Laplacian (FL_{Spec}). In
17 subsection 3, we highlight the difference between these two operators by comparing,
18 in particular, their respective spectra. In Subsection 4, we introduce the so-called
19 *restricted* fractional Laplacian (FL_{Rest}), and we compare it with FL_{Reg}. Then, we
20 introduce the so-called *peridynamic* fractional Laplacian (FL_{Pery}) in Subsection 5.
21 In the last subsection, we give three examples in dimension 1 in order to illustrate
22 the difference between these four operators.

23 In order to go deeper into the study of fractional Laplacian operators, we refer
24 the reader, for instance, to [60, 10, 11]. Those papers give basic estimates and some
25 regularity results for solutions to problems governed by $(-\Delta)^s$.

26 On the other hand, the paper [31] gives, through the theory of semigroups, a char-
27 acterization of the realization in $L^2(\Omega)$ of the fractional Laplacian with different
28 exterior conditions. \square

29 4.1. Regional Fractional Laplacian (FL_{Reg}).

30 In this subsection, we give the definition of the so-called regional fractional Lapla-
31 cian which we denote by $(-\Delta)_{Reg}^s$. We also give a few examples for illustration
32 purposes.

33 **• Definition** This operator acts on functions u defined in Ω and extended by zero
34 to $\mathbb{R}^N \setminus \Omega$. So we can use Definition 2.2.

35 Let $u \in \mathbb{X}_0^s(\Omega)$. The *regional fractional Laplacian*² is defined by :

$$(-\Delta)_{Reg}^s u(\mathbf{x}) := C(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y}. \quad (4.1)$$

¹Unfortunately, the mathematical community has not yet reached a unified terminology and standard notations for the fractional Laplacians on a bounded domain. The reader is strongly advised to check the definitions carefully in order to avoid any confusion.

²This operator is also called the restricted fractional Laplacian (see e.g. [87, 16, 15]), or hypersingular fractional Laplacian (see for instance [42, 50]), or standard fractional Laplacian (see for example [26]), or Riesz fractional Laplacian see, e.g., [61], or simply fractional Laplacian, ...

Remark 4.1. From (4.1), we deduce

$$(-\Delta)_{Reg}^s u(\mathbf{x}) := C(N, s) \left[P.V. \int_{\Omega} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} + \int_{\mathbb{R}^N \setminus \Omega} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \right].$$

1 As $u = 0 \in \mathbb{R}^N \setminus \Omega$, we obtain

$$(-\Delta)_{Reg}^s u(\mathbf{x}) := C(N, s) \left[P.V. \int_{\Omega} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} + \int_{\mathbb{R}^N \setminus \Omega} \frac{u(\mathbf{x})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \right]. \tag{4.2}$$

2 Formula (4.2) shows that $(-\Delta)_{Reg}^s$ depends on Ω and $\mathbb{R}^N \setminus \Omega$.

3 **Probabilistic interpretation :** The regional fractional Laplacian is the infinitesimal generator of jump-type processes killed upon leaving the domain Ω . In this
 4 respect, the operator generates a killed subordinate Brownian motion (see for instance [13, 30, 50, 51, 79] and references therein). Indeed, this process describes
 5 trajectories of particles which are allowed to jump outside Ω and then one subordi-
 6 nates these paths in Ω , *i.e.*, one leaves marks only on the parts of paths inside the
 7 domain. The homogeneous Dirichlet exterior condition in $\mathbb{R}^N \setminus \Omega$ corresponds to
 8 this operation.

11 • **Examples** For the reader's convenience, let us give some examples, (see [43]).

12 **Example 1.** *One-dimensional case :* $\Omega = (-1, 1)$.

$u(x)$	$(-\Delta)_{Reg}^s u(x)$
$(1 - x^2)_+^{s-1}$	0
$(1 - x^2)_+^s$	$\Gamma(2s + 1)$
$(1 - x^2)_+^{s+1}$	$(s + 1)\Gamma(2s + 1)(1 - (2s + 1)x^2)$
$(1 - x^2)_+^{s+2}$	$\frac{(s+1)(s+2)}{2}\Gamma(2s + 1)(1 - (4s + 2)x^2 + (\frac{2s}{3} + 1)(2s + 1)x^4)$
$x(1 - x^2)_+^{s-1}$	0
$x(1 - x^2)_+^s$	$\Gamma(2s + 2)x$
$x(1 - x^2)_+^{s+1}$	$\frac{\Gamma(2s+3)}{6}(3 - (2s + 3)x^2)x$
$x(1 - x^2)_+^{s+2}$	$\frac{s+2}{60}\Gamma(2s + 3)(15 - (20s + 30)x^2 + (2s + 3)(2s + 5)x^4)x$

13 **Table 1.** The regional fractional Laplacian of some functions that vanish in $\mathbb{R} \setminus (-1, 1)$.
 14

15 **Example 2. Multidimensional case :** Let $N \geq 2$, $\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$ and
 16 $\Omega = B(0, 1)$ be the ball of center the origin and radius 1.

$u(\mathbf{x})$	$(-\Delta)_{Reg}^s u(\mathbf{x})$
$(1 - \ \mathbf{x}\ ^2)_+^s$	$4^s \Gamma(s + 1) \Gamma(\frac{2s+N}{2}) \Gamma(\frac{N}{2})^{-1}$
$(1 - \ \mathbf{x}\ ^2)_+^{s+1}$	$4^s \Gamma(s + 2) \Gamma(\frac{2s+N}{2}) \Gamma(\frac{N}{2})^{-1} (1 - (1 + \frac{2s}{N})\ \mathbf{x}\ ^2)$
$(1 - \ \mathbf{x}\ ^2)_+^s x_N$	$4^s \Gamma(s + 1) \Gamma(\frac{2s+N}{2} + 1) \Gamma(\frac{N}{2} + 1)^{-1} x_N$
$(1 - \ \mathbf{x}\ ^2)_+^{s+1} x_N$	$4^s \Gamma(s + 2) \Gamma(\frac{2s+N}{2} + 1) \Gamma(\frac{N}{2} + 1)^{-1} (1 - (1 + \frac{2s}{N+2})\ \mathbf{x}\ ^2)$

17 **Table 2.** The regional fractional Laplacian of some functions that vanish in $\mathbb{R}^N \setminus B(0, 1)$.
 18

19 • **Spectrum of FL_{Reg}**

It is shown (see, e.g., [73], [62, Chapter 3]) that, thus defined, $(-\Delta)_{Reg}^s : \mathbb{X}_0^s(\Omega) \rightarrow L^2(\Omega)$ is self-adjoint operator with a discrete spectrum and positive eigenvalues,

more precisely

$$0 < \mu_{1,s} \leq \mu_{2,s} \leq \cdots \leq \mu_{k,s} \leq \mu_{k+1,s} \leq \cdots$$

1 From now on, we denote by $\{e_{k,s}^R\}_{k \geq 1}$ the corresponding eigenfunctions normalized
 2 in $L^2(\Omega)$ (i.e. $\|e_{k,s}^R\|_{L^2(\Omega)} = 1$).

3 It is worth to point out that the $\{e_{k,s}^R\}_{k \geq 1}$ are (only) Hölder continuous up the
 4 boundary, namely $e_{k,s}^R \in C^{0,s}(\bar{\Omega})$ (see, e.g., [30], [69]). \square

5 4.2. Spectral Fractional Laplacian (FL_{Spec}).

6 In this subsection, we will give two (equivalent) definitions of the *spectral* frac-
 7 tional Laplacian and we denote it simply by $(-\Delta)_{\text{Spec}}^s$. The first one is given via the
 8 spectrum of the “classical” Laplacian on the same domain, see for instance [73], [62,
 9 Chapter 5],[30], [2], [25], [16] and the references therein. The second one is given by
 10 an integral formula via the semigroups, see e.g. [81], [87] and the references therein.

• First definition :

Let $\{\lambda_k\}_{k \in \mathbb{N}^*}$ with $0 < \lambda_1 < \lambda_2 \leq \dots$ be the divergent sequence of eigenvalues of the
 “classical” Laplacian $-\Delta$ in Ω with homogeneous Dirichlet boundary data on $\partial\Omega$,
 and $\{e_k\}_{k \in \mathbb{N}^*}$ the corresponding eigenfunctions, that is,

$$\begin{cases} -\Delta e_k = \lambda_k e_k & \text{in } \Omega, \\ e_k = 0 & \text{on } \partial\Omega, \end{cases}$$

11 normalized in such a way that $\|e_k\|_{L^2(\Omega)} = 1$ (see, e.g., [55]).

12 Now let us introduce the following space defined by

$$\tilde{H}^s(\Omega) := \left\{ u = \sum_{k=1}^{+\infty} u_k e_k \in L^2(\Omega) ; \sum_{k=1}^{+\infty} \lambda_k^s u_k^2 < +\infty \right\}. \quad (4.3)$$

We easily check that $(\tilde{H}^s(\Omega), \langle \cdot, \cdot \rangle_{\tilde{H}^s(\Omega)})$ is a Hilbert space, where

$$\langle u, v \rangle_{\tilde{H}^s(\Omega)} := \sum_{k=1}^{+\infty} \lambda_k^s u_k v_k,$$

13 for $u = \sum_{k=1}^{+\infty} u_k e_k$ and $v = \sum_{k=1}^{+\infty} v_k e_k$ belong $\tilde{H}^s(\Omega)$.

14 It turns out that $\tilde{H}^s(\Omega) = \mathbb{X}_0^s(\Omega)$, see for instance [16, section 3.1.3] and the
 15 references therein.

16 Now we are ready to state the definition. More precisely :

17 for $u = \sum_{k=1}^{+\infty} u_k e_k \in \mathbb{X}_0^s(\Omega)$, the *spectral* fractional Laplacian, denoted by $(-\Delta)_{\text{Spec}}^s$,
 18 is defined by

$$(-\Delta)_{\text{Spec}}^s u(\mathbf{x}) := \sum_{k=1}^{+\infty} u_k \lambda_k^s e_k(\mathbf{x}), \quad (4.4)$$

19 • Spectrum of FL_{Spec}

20 Thanks to (4.4), the eigenvalues and the eigenfunctions of FL_{Spec} are, respectively,
 21 λ_k^s and e_k for all $k \in \mathbb{N}^*$.

22 In order to avoid any confusion, from here on we denote by $\{e_{k,s}^S, \lambda_{k,s}\}_{k \in \mathbb{N}^*}$ the
 23 eigenfunctions and the eigenvalues of $(-\Delta)_{\text{Spec}}^s$ on Ω in the rest of this paper.

1 **Example 3.**

- Let $\Omega = (0, 1)$ and $u(x) = \sin(\pi x)$. Then,

$$(-\Delta)_{Spec}^s u(x) = \pi^{2s} \sin(\pi x).$$

- Let $\Omega = (-1, 1)$ and $u(x) = \sin\left(\frac{\pi(1+x)}{2}\right)$. Then,

$$(-\Delta)_{Spec}^s u(x) = \left(\frac{\pi}{2}\right)^{2s} \sin\left(\frac{\pi(1+x)}{2}\right).$$

2 • **Second definition**

3 As in the case of \mathbb{R}^N (see Definition 2.8), the spectral fractional Laplacian can also
 4 be defined through the heat semigroup (see, e.g., [81] and the references therein).
 5 This technique has been efficiently used in recent research papers (see for instance
 6 [32, 33, 66]). More precisely, we have :

$$(-\Delta)_{Spec}^s u(\mathbf{x}) = \frac{s}{\Gamma(1-s)} \int_0^{+\infty} (u(\mathbf{x}) - e^{t\Delta} u(\mathbf{x})) \frac{dt}{t^{1+s}}, \quad \mathbf{x} \in \Omega, \quad (4.5)$$

7 where $e^{t\Delta} u$ denotes the solution to the heat equation in $\Omega \times [0, +\infty)$ with initial
 8 datum u .

9 Definitions 4.4 and 4.5 are equivalent (see, e.g., [25, Lemma 2.2]).

10 • **Probabilistic interpretation :** Roughly speaking, the spectral fractional
 11 Laplacian generates a subordinate killed Brownian motion, *i.e.* the process that
 12 first kills Brownian motion in Ω and then subordinates it via a s -stable subordina-
 13 tor. The killed process is generated by the classical Laplacian $-\Delta$ with homogeneous
 14 Dirichlet boundary conditions, and the subordination provides its fractional power
 15 $(-\Delta)_{Spec}^s$. See, e.g., [13, 30, 50, 51, 53, 79] and the references therein. \square

16 **4.3. Comparison of the operators FL_{Spec} and FL_{Reg} .**

17 The main goal of this paragraph is to show that the operators FL_{Spec} and FL_{Reg}
 18 are different.

19 **First :** As we have just seen in the previous two paragraphs, the definition itself
 20 of $(-\Delta)_{Spec}^s$ only depends on the domain Ω considered, while that of $(-\Delta)_{Reg}^s$
 21 depends on the domain Ω and on $\mathbb{R}^N \setminus \Omega$ (see (4.2)).

22 **Second :** We state below three results showing that the spectra of $(-\Delta)_{Spec}^s$
 23 and $(-\Delta)_{Reg}^s$ are indeed different. For more details, the interested reader is re-
 24 ferred to [73] or [62, Chapter 5] and the references therein.

25 **Theorem 4.1.** [73, Theorem 1.1 and Theorem 1.2] *The eigenfunctions $e_{k,s}^R$ are*
 26 *only Hölder continuous on $\bar{\Omega}$. However, the eigenfunctions $e_{k,s}^S$ are smooth on*
 27 *$\bar{\Omega}$ (at least $e_{k,s}^S \in C^\infty(\Omega) \cap C^2(\bar{\Omega})$).*

28 Furthermore, the operators $(-\Delta)_{Reg}^s$ and $(-\Delta)_{Spec}^s$ have different eigenvalues.
 29 More precisely, we have these two following results :

Theorem 4.2. [73, Theorem 1] *We recall that $\mu_{1,s}$ (resp. $\lambda_{1,s}$) is the first eigen-*
value of FL_{Reg} (resp. FL_{Spec}). Then,

$$\mu_{1,s} < \lambda_{1,s}.$$

Theorem 4.3. [30, Theorem 4.5] *There exists a constant $C = C(\Omega) > 0$ such that:*

$$c\lambda_{k,s} \leq \mu_{k,s} \leq \lambda_{k,s}, \quad \forall k \in \mathbb{N}^*, k \geq 2.$$

1 *In particular, $C = \frac{1}{2}$ when Ω is convex.*

2 We refer also to [63] and [64] . \square

3 4.4. Restricted Fractional Laplacian (FL_{Rest}).

4 In this subsection, we will introduce another fractional Laplacian operator called
5 *restricted* fractional Laplacian,³ which we denote $(-\Delta)_{\text{Rest}}^s$. It is defined for any
6 $u \in \mathbb{X}_0^s(\Omega)$ by the formula

$$(-\Delta)_{\text{Rest}}^s u(\mathbf{x}) = C(N, s) P.V. \int_{\Omega} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y}, \quad \mathbf{x} \in \Omega. \quad (4.6)$$

7 It goes without saying that the operators FL_{Rest} and FL_{Reg} are different from
8 each other, even though their names and notations may be interchanged in some
9 works. In what follows, we will highlight these differences :

10 **First:** Unlike the regional fractional Laplacian, the operator FL_{Rest} is generated
11 by symmetric stable processes describing motions of random particles in Ω which
12 are allowed to jump inside Ω , but are either reflected in Ω or killed when they reach
13 the boundary $\partial\Omega$ (see for instance [13, 29, 53, 79]).

14 **Second:** Now, we will illustrate their difference analytically.

15 Let u be a smooth function such that $u(x) = 0$ for $x \in \mathbb{R} \setminus \Omega$. Let $x \in \Omega$. Thanks
16 to formula (4.2), we have

$$\begin{aligned} (-\Delta)_{\text{Reg}}^s u(x) &= C(N, s) P.V. \int_{\mathbb{R}^N} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \\ &= C(N, s) \left[P.V. \int_{\Omega} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} + \int_{\mathbb{R}^N \setminus \Omega} \frac{u(\mathbf{x})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \right] \end{aligned}$$

17 So

$$D_1 u(\mathbf{x}) = [(-\Delta)_{\text{Reg}}^s - (-\Delta)_{\text{Rest}}^s] u(\mathbf{x}) = u(\mathbf{x}) \int_{\mathbb{R}^N \setminus \Omega} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y} \quad (4.7)$$

Now, for the reader's comfort, we explicitly calculate the difference (4.7) in the
case $\Omega = (-L, L)$ with $L > 0$. Then we have

$$D_1 u(x) = C(1, s) \left(\int_{-\infty}^{-L} \frac{1}{|x - y|^{1+2s}} dy + \int_L^{+\infty} \frac{1}{|x - y|^{1+2s}} dy \right) u(x).$$

18 Thus, for any $x \in \Omega$,

$$D_1 u(x) = \frac{C(1, s)}{2s} \left(\frac{1}{(x + L)^{2s}} + \frac{1}{(L - x)^{2s}} \right) u(x). \quad (4.8)$$

19 As $\lim_{s \rightarrow 1^-} C(1, s) = 0$ (see (2.4)), we deduce from (4.8) that $\lim_{s \rightarrow 1^-} D_1 u(x) = 0$. Con-
20 sequently, FL_{Rest} can be used to approximate FL_{Reg} as $s \rightarrow 1^-$.

³As already mentioned, there is no consensus in terminology. Indeed, the operator FL_{Rest} is
sometimes called regional (see, e.g., [41, 61, 46]) or *censored*. We have opted for 'restricted' because
the values of $(-\Delta)_{\text{Rest}}^s u$ only depend on the values of u in Ω , not in the complementary.

21 Applying again (2.4), then $\lim_{s \rightarrow 0^+} \frac{C(1, s)}{2s} = \frac{1}{2}$. So $\lim_{s \rightarrow 0^+} D_1 u(x) = u(x)$. Thus, when
 22 $s \rightarrow 0^+$, FL_{Reg} can be written as the sum of FL_{Rest} and the identity operator.

1 To end this sub-paragraph, it is worth pointing out that unlike the regional and
 2 spectral fractional Laplacians, the operator FL_{Rest} has not been studied much. For
 3 example, nothing is known about its eigenvalues and eigenfunctions. \square

4 **4.5. Peridynamic Fractional Laplacian (FL_{Pery}).**

5 The peridynamic model was originally proposed as a reformulation of the classical
 6 solid mechanics. We refer the interested reader to the pioneering article [75], as well
 7 as to [34, 54, 41, 7, 61] and references therein.

8 **Probabilistic interpretation :** Let $\delta \in (0, +\infty)$. The peridynamic fractional
 9 Laplacian (FL_{Pery} in short) of horizon δ can be seen as the infinitesimal generator
 10 of a symmetric $2s$ -stable Lévy process restricted to $B(\mathbf{x}, \delta)$ ⁴ for $\mathbf{x} \in \Omega$. For more
 11 details, see, e.g., [61], [41] and the references therein. The real number δ is called the
 12 horizon parameter, and it is often used to measure the range of nonlocal interactions
 13 in many nonlocal models. Thence, this operator is sometimes called the “horizon-
 14 based nonlocal” operator.

15 **Notation :** In what follows, we will use the following notation $(-\Delta)_{\text{Pery}}^{s, \delta}$ or
 16 $(-\Delta)_{\text{Pery}}^s$ if there is no risk of confusion.

Definition :

Before stating the definition, we need to define the so-called *interaction domain* of
 horizon δ , defined by

$$\Omega_{I, \delta} := \{ \mathbf{y} \in \mathbb{R}^N \setminus \Omega ; \|\mathbf{x} - \mathbf{y}\| < \delta, \text{ for } \mathbf{x} \in \Omega \}.$$

17 In other words, the domain $\Omega_{I, \delta}$ consists of those points outside of Ω that interact
 18 with points in Ω . In fact, $(-\Delta)_{\text{Pery}}^{s, \delta}$ acts on functions u defined in Ω extended by 0
 19 to $\Omega_{I, \delta}$. More precisely, $(-\Delta)_{\text{Pery}}^{s, \delta}$ is defined by

$$(-\Delta)_{\text{Pery}}^s u(\mathbf{x}) = C(N, s) P.V. \int_{B(\mathbf{x}, \delta)} \frac{u(\mathbf{x}) - u(\mathbf{y})}{\|\mathbf{x} - \mathbf{y}\|^{N+2s}} d\mathbf{y}. \quad (4.9)$$

20 **Comparison between FL_{Reg} and FL_{Pery}**

21 1) In the limiting case of $\delta \rightarrow \infty$, FL_{Pery} coincides with FL_{Reg} , and thus the latter
 22 is often used to approximate FL_{Pery} if the parameter δ is sufficiently large (see for
 23 example [34, 54]).

24 2) In order to illustrate the difference between these two operators, we will proceed
 25 as in the previous paragraph. For this, we choose a smooth function u such that
 26 $u(x) = 0$ for $x \in \mathbb{R} \setminus \Omega$ with $\Omega = (-L, L)$. Here, we assume that the horizon size
 27 δ in (4.9) is large enough such that $\delta > 2L$. As is done in the previous paragraph,

⁴Let us recall that $B(\mathbf{x}, \delta)$ denotes the ball centered at point \mathbf{x} with radius δ

28 we compute their difference :

$$\begin{aligned}
D_2u(x) &= ((-\Delta)_{Reg}^s - (-\Delta)_{Pery}^s) u(x) \\
&= C(1, s) \left(P.V. \int_{\mathbb{R}} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy - P.V. \int_{x-\delta}^{x+\delta} \frac{u(x) - u(y)}{|x - y|^{1+2s}} dy \right) \\
&= C(1, s) \left(\int_{-\infty}^{x-\delta} \frac{1}{|x - y|^{1+2s}} dy + \int_{x+\delta}^{+\infty} \frac{1}{|x - y|^{1+2s}} dy \right) u(x)
\end{aligned}$$

1 So, for all $x \in \Omega$

$$D_2u(x) = ((-\Delta)_{Reg}^s - (-\Delta)_{Pery}^s) u(x) = \frac{C(1, s)}{s\delta^{2s}} u(x), \quad (4.10)$$

2 We conclude that the difference of these two operators is of order $\mathcal{O}(\frac{1}{\delta^{2s}})$ when
3 $u(x)$ is uniformly bounded on Ω , hence their difference vanishes as $\delta \rightarrow \infty$. On the
4 other hand, the convergence of FL_{Pery} to FL_{Reg} as $\delta \rightarrow \infty$ depends on the power
5 s , and it may degenerate rapidly for small s . Additionally, in the limiting case of
6 $s \rightarrow 1^-$, the difference $D_2u(x) \rightarrow 0$, because the coefficient $C(1, s) \rightarrow 0$ (see (2.4)). \square

7 4.6. Examples.

8 After this brief review of pairwise differences between them, let us compare all
9 four operators by applying them to the same functions on $\Omega = (-1, 1)$.

10 **Example 1 :** Consider the function (see [41]) defined by

$$u(x) = \begin{cases} \sin\left(\frac{\pi(1+x)}{2}\right) & \text{if } x \in (-1, 1), \\ 0 & \text{otherwise.} \end{cases} \quad (4.11)$$

11 As already noticed (see Example 3, Paragraph 4.2), we have

$$(-\Delta)_{Spec}^s u(x) = \left(\frac{\pi}{2}\right)^{2s} \sin\left(\frac{\pi(1+x)}{2}\right) \quad \text{for } x \in \Omega,$$

12 However, it is not easy to calculate $(-\Delta)_{Reg}^s u$, $(-\Delta)_{Rest}^s u$ and $(-\Delta)_{Pery}^s u$, thus
13 they are numerically computed using the finite difference method proposed in [39].

14 Figure 1 below shows the curves of the four functions $(-\Delta)_{Reg}^s u$, $(-\Delta)_{Rest}^s u$, $(-\Delta)_{Spec}^s u$
15 and $(-\Delta)_{Pery}^s u$. We observe that :

- 16 — for all $s \in (0, 1)$, $(-\Delta)_{Spec}^s u$ is proportional to the function u in Ω ;
- 17 — as $s \rightarrow 1^-$, $(-\Delta)_{Reg}^s u$, $(-\Delta)_{Rest}^s u$ and $(-\Delta)_{Pery}^s u$ converge to $-\Delta u$;
- 18 — both $(-\Delta)_{Rest}^s$ and $(-\Delta)_{Pery}^s$ can be used to approximate $(-\Delta)_{Reg}^s$, if s is
19 close to 1 ;
- 20 — for small s , the values of $(-\Delta)_{Rest}^s$ are at variance with those of $(-\Delta)_{Reg}^s$.
21 However, $(-\Delta)_{Pery}^s$ can still provide a good approximation to $(-\Delta)_{Reg}^s$ by
22 enlarging the horizon size δ .

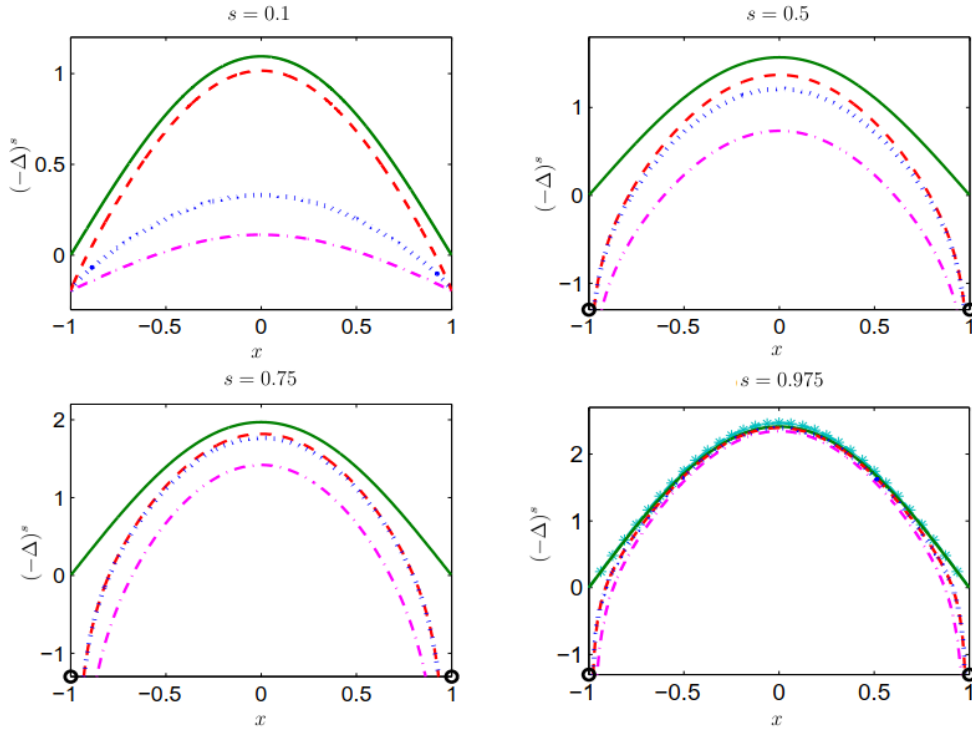


Figure 1. Comparison of the functions $(-\Delta)^s u$ where u is defined in (4.11) and $(-\Delta)^s$ represents $(-\Delta)^s_{Spec}$ (—), $(-\Delta)^s_{Reg}$ (- - -), $(-\Delta)^s_{Rest}$ (- · - ·) or $(-\Delta)^s_{Pery}$ (····) with $\delta = 4$. The result for $-\Delta$ (* * *) is included in the plot of $s = 0.975$.

1 **Example 2 :** Consider the function (see [41]) defined by

$$u(x) = \begin{cases} (1-x^2)^{q+s} & \text{if } x \in (-1, 1), \\ 0 & \text{otherwise,} \end{cases} \quad (4.12)$$

where $q \in \mathbb{N}$. In this case, $(-\Delta)^s_{Reg} u$ can be computed explicitly

$$(-\Delta)^s_{Reg} u(x) = \frac{2^{2s} \Gamma(\frac{1+2s}{2}) \Gamma(s+q+1)}{\sqrt{\pi} \Gamma(q+1)} {}_2F_1\left(\frac{1+2s}{2}, -q; \frac{1}{2}; x^2\right) \quad \text{for } x \in \Omega,$$

2 where ${}_2F_1$ denotes the Gauss hypergeometric function. Moreover, we can obtain
 3 the exact values of $(-\Delta)^s_{Rest} u$ and $(-\Delta)^s_{Pery} u$ by using their relation to $(-\Delta)^s_{Reg} u$
 4 in (4.8) and (4.10), respectively. For $(-\Delta)^s_{Spec}$, $(-\Delta)^s_{Spec} u$ is numerically computed
 5 by the finite difference method proposed in [40].

6 In Figure 2, the four functions $(-\Delta)^s_{Spec} u$, $(-\Delta)^s_{Reg} u$, $(-\Delta)^s_{Pery} u$ and $(-\Delta)^s_{Rest} u$
 7 are compared. The function u is defined by (4.12) with $q = 2$.

8 Figure 2 shows that the functions $(-\Delta)^s_{Spec} u$, $(-\Delta)^s_{Rest} u$, $(-\Delta)^s_{Pery} u$ and $(-\Delta)^s_{Reg} u$
 9 exist on the closed domain $\bar{\Omega}$ for any $s \in (0, 1)$, but their values are very different,
 10 especially for small s . The values of $(-\Delta)^s_{Spec} u$ are always zero at boundary points.
 11 Also, both $(-\Delta)^s_{Rest}$ and $(-\Delta)^s_{Pery}$ with relatively small δ can provide a good ap-
 1 proximation to $(-\Delta)^s_{Reg}$, if s is large. When s is small, however, $(-\Delta)^s_{Pery}$ can be

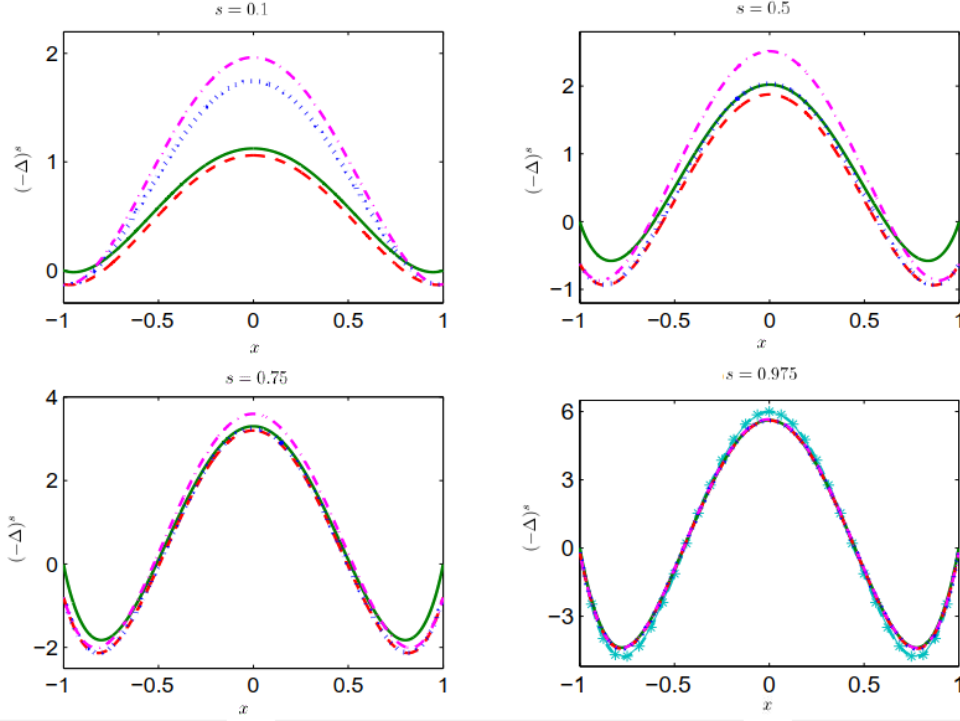


Figure 2. Comparison of the functions $(-\Delta)^s u$ where u is defined in (4.12) and $(-\Delta)^s$ represents $(-\Delta)^s_{Spec}$ (—), $(-\Delta)^s_{Reg}$ (- - -), $(-\Delta)^s_{Rest}$ (- · - ·) or $(-\Delta)^s_{Pery}$ (····) with $\delta = 4$. The result for $-\Delta$ (***) is included in the plot of $s = 0.975$.

2 still used to approximate $(-\Delta)^s_{Reg}$ with a large δ , but $(-\Delta)^s_{Rest}$ gives results starkly
 3 different from $(-\Delta)^s_{Reg}$. Finally, as $s \rightarrow 1^-$, the differences between the four oper-
 4 ators become negligible (see the right-bottom square), and they all converge to the
 5 Laplacian $-\Delta$.

6 **Example 3 :** We consider the following Poisson problem (see[41]) :

$$\begin{cases} -\mathcal{L}u(x) = 1 & \text{if } x \in (-1, 1), \\ u(x) = 0 & \text{if } x \in \Sigma, \end{cases} \quad (4.13)$$

7 where $\mathcal{L} = (-\Delta)^s_{Reg}$ and $\Sigma = (-\infty, -1] \cup [1, +\infty)$; $\mathcal{L} = (-\Delta)^s_{Spec}$ and $\Sigma = \{-1, 1\}$;
 8 $\mathcal{L} = (-\Delta)^s_{Pery}$ and $\Sigma = \Omega_{I,\delta} := \{y \in \mathbb{R} \setminus (-1, 1); |x - y| < \delta, \text{ for } x \in (-1, 1)\}$;
 9 or $\mathcal{L} = (-\Delta)^s_{Rest}$ and $\Sigma = \{-1, 1\}$.

10 One can check that:

- if $\mathcal{L} = (-\Delta)^s_{Reg}$,

$$u(x) = \frac{2^{-2s} \sqrt{\pi}}{\Gamma(\frac{1+2s}{2}) \Gamma(1+s)} (1-x^2)^s \quad \text{for } x \in (-1, 1);$$

- if $\mathcal{L} = (-\Delta)_{Spec}^s$,

$$u(x) = \sum_{k=0}^{+\infty} \frac{2(1 - (-1)^k)}{k\pi} \left(\frac{k\pi}{2}\right)^{-2s} \sin\left(\frac{k\pi}{2}(1+x)\right) \quad \text{for } x \in (-1, 1).$$

11 On the other hand, the solutions of the nonlocal problem (4.13) with $\mathcal{L} = (-\Delta)_{Rest}^s$
 1 or $(-\Delta)_{Pery}^s$ are not analytically known and will be numerically computed.

Figure 3 shows the solutions to (4.13) with all four operators. In the case of FL_{Rest} , since the solution does not exist for $s \leq \frac{1}{2}$, we only present it for $s > \frac{1}{2}$. Generally speaking, these solutions are significantly different. However, as $s \rightarrow 1^-$ they all converge to the function $u(x) = \frac{1}{2}(1 - x^2)$, the solution to the classical Poisson equation :

$$-u''(x) = 1, \quad \text{if } x \in (-1, 1); \quad u(-1) = u(1) = 0.$$

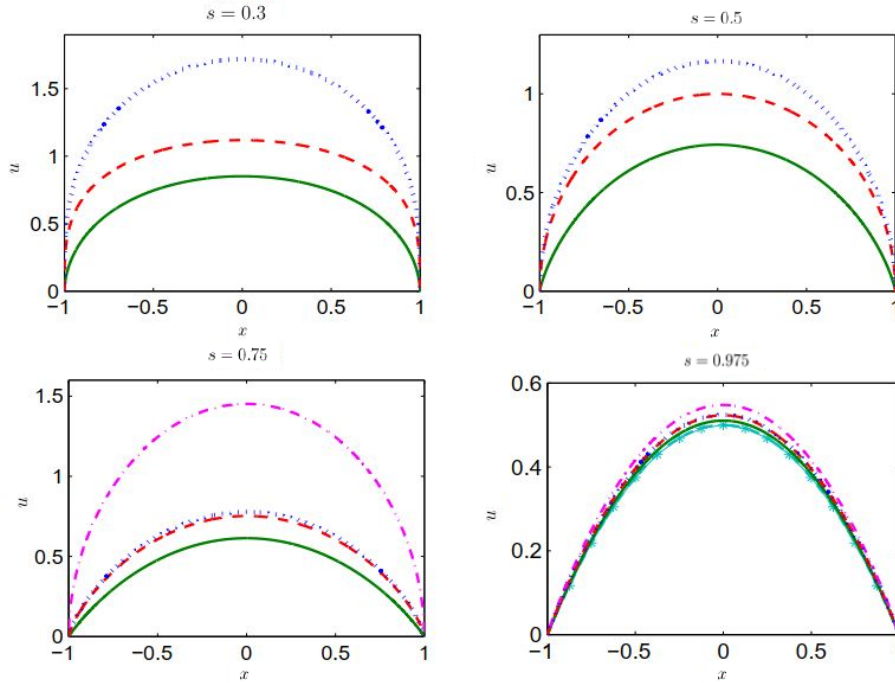


Figure 3. Comparison of the solution to (4.13) with $(-\Delta)_{Spec}^s$ (—), $(-\Delta)_{Reg}^s$ (---), $(-\Delta)_{Rest}^s$ (- · - ·) or $(-\Delta)_{Pery}^s$ (····) with $\delta = 4$. The result for $-\Delta$ (* * * *) is included in the plot of $s = 0.975$.

2

3

REFERENCES

4 [1] N. Abatangelo : *Large solutions for fractional Laplacian Operators*. PhD thesis, 2015.
 5 [2] N. Abatangelo ; L. Dupaigne : *Nonhomogeneous boundary conditions for the spectral frac-*
 6 *tional Laplacian*. Ann. Inst. H. Poincaré Anal. Non Linéaire 34(2) (2017), 439-467.
 7 [3] B. Abdellaoui ; K. Biroud ; E.-H. Laamri : *Existence et nonexistence of positive solutions to*
 8 *a fractional parabolic problem with singular weight at the boundary*. To appear In Journal of
 9 Evolution Equations.

- 10 [4] G. Acosta ; J. P. Borthagaray ; O. Bruno ; M. Maas : *Regularity Theory and High Order*
 1 *Numerical Methods for the (1d)-Fractional Laplacian*. Mathematics of Computation 87(312)
 2 (2018), 1821-1857.
- 3 [5] G. Acosta ; J. P. Borthagaray : *A fractional Laplace equation: regularity of solutions and*
 4 *Finite Element approximations*. SIAM Journal on Numerical Analysis 55(2) (2017), 472-495.
- 5 [6] R. A. Adams ; J. F. Fournier : *Sobolev spaces*. Second edition, Pure and Applied Mathematics
 6 (Amsterdam), 140, Elsevier/Academic Press, Amsterdam, 2003.
- 7 [7] B. Alali ; N. Albin : *Fourier multipliers for nonlocal Laplace operators*. Applicable Analysis
 8 (2019) DOI: 10.1080/00036811.2019.1692134
- 9 [8] D. Applebaum : *Lévy Processes and Stochastic Calculus*. 2nd edition, Cambridge Studies in
 10 Advanced Mathematics, 116, Cambridge University Press, Cambridge, 2009.
- 11 [9] A. V. Balakrishnan : *Fractional powers of closed operators and the semigroups generated by*
 12 *them*. Pacific J. Math. 10 (1960), 419-437.
- 13 [10] U. Biccari ; M. Warma ; E. Zuazua : *Local elliptic regularity for the Dirichlet fractional*
 14 *Laplacian*. Adv. Nonlinear Stud. 17 (2017), 387-409.
- 15 [11] U. Biccari ; M. Warma ; E. Zuazua : *Local regularity for fractional heat equations*. Recent
 16 Advances in PDEs: Analysis, Numerics and Control, 233-249, SEMA SIMAI Springer Ser.,
 17 Springer, Cham, 2018.
- 18 [12] S. Bochner : *Diffusion equation and stochastic processes*. Proc. Nat. Acad. Sci. U. S. A. 35
 19 (1949), 368-370.
- 20 [13] K. Bogdan ; K. Burdzy ; Z.-Q. Chen : *Censored stable processes*. Probab. Theory Rel. 127
 21 (2003), 89-152.
- 22 [14] K. Bogdan ; T. Grzywny ; M. Ryznar : *Heat kernel estimates for the fractional Laplacian*
 23 *with Dirichlet conditions*. The Annals of Probability 38(5) (2010), 1901-1923.
- 24 [15] M. Bonforte ; A. Figalli ; J. L. Vázquez : *Sharp global estimates for local and nonlocal porous*
 25 *medium-type equations in bounded domains*. Anal. PDE 11(4) (2018), 945-982.
- 26 [16] M. Bonforte ; Y. Sire ; J. L. Vázquez : *Existence, uniqueness and asymptotic behavior for*
 27 *fractional porous medium equations on bounded domains*. Discrete Contin. Dyn. Syst. 35(12)
 28 (2015), 5725-5767.
- 29 [17] M. Bonforte ; J. L. Vázquez : *A Priori Estimates for Fractional Nonlinear Degenerate Dif-*
 30 *fusion Equations on Bounded Domains*. Archive for Rational Mechanics and Analysis 218
 31 (2015), 317-362.
- 32 [18] C. Brändle ; E. Colorado ; A. De Pablo ; U. Sánchez : *A concave convex elliptic problem*
 33 *involving the fractional Laplacian*. Proc. R. Soc. Edinb. 143(01) (2013), 39-71.
- 34 [19] B. Barrios ; E. Colorado ; A. de Pablo ; U. Sánchez : *On some critical problems for the*
 35 *fractional Laplacian operator*. Journal of Differential Equations 252(11) (2012), 6133-6162.
- 36 [20] J. P. Borthagaray ; P. Ciarlet : *On the convergence in H^1 -norm for the fractional Laplacian*.
 37 hal-01912092 (2018). Submitted
- 38 [21] C. Bucur ; E. Valdinoci : *Nonlocal diffusion and applications*. ISBN 978-3-319-28738-6. Lec-
 39 ture Notes of the Unione Matematica Italiana, 2016.
- 40 [22] X. Cabré ; J.-M. Roquejoffre : *The influence of fractional diffusion in Fisher-KPP equations*.
 41 Comm. Math. Physics 320 (2013), 679-722.
- 42 [23] X. Cabré ; Y. Sire : *Nonlinear equations for fractional Laplacians I : Regularity, maximum*
 43 *principles, and Hamiltonian estimates*. Ann. Inst. H. Poincaré Anal. Non Linéaire 31 (2014),
 44 23-53.
- 45 [24] L. Caffarelli ; L. Silvestre : *An extension problem related to the fractional Laplacian*. Comm.
 46 Partial Differential Equations 32(8) (2007), 1245-1260.
- 47 [25] L. Caffarelli ; P. R. Stinga : *Fractional elliptic equations, Caccioppoli estimates and regularity*.
 48 Ann. Inst. H. Poincaré Anal. Non Linéaire 33(3) (2014), 767-807.
- 49 [26] X. Cao ; Y.-H. Lin ; H. Liu : *Simultaneously recovering potentials and embedded obstacles*
 50 *for anisotropic fractional Schrödinger operators*. Inverse Problems & Imaging 13(1) (2019),
 51 197-210.
- 52 [27] A. Capella ; J. Dávila ; L. Dupaigne ; Y. Sire : *Regularity of radial extremal solutions for*
 53 *some non-local semilinear equations*. Commun. Part. Differ. Equ. 36(8) (2011), 1353-1384.
- 54 [28] W. Chen ; S. Holm : *Fractional Laplacian time-space models for linear and nonlinear lossy*
 55 *media exhibiting arbitrary frequency power-law dependency*. J. Acoust. Soc. Am. 115 (2004),
 56 1424-1430.
- 57 [29] Z.-Q. Chen ; P. Kim ; R. Song : *Two-sided heat kernel estimates for censored stable-like*
 58 *processes*. Probab. Theory Rel. 146 (2010), 361-399.

- 59 [30] Z.-Q. Chen ; R. Song : *Two-sided eigenvalue estimates for subordinate processes in domains.*
 1 Journal of Functional Analysis 226 (2005), 90-113.
- 2 [31] B. Claus ; M. Warm : *Realization of the fractional Laplacian with nonlocal exterior conditions*
 3 *via forms method.* Journal of Evolution Equations 20 (2020), 1597-16313.
- 4 [32] N. Cusimano ; F. Del Teso ; L. Gerardo-Giorda ; G. Pagnini : *Discretizations of the spec-*
 5 *tral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary*
 6 *conditions.* SIAM J. Numer. Anal. 56(3) (2018), 1243-1272.
- 7 [33] N. Cusimano ; F. Del Teso ; L. Gerardo-Giorda ; G. Pagnini : *Numerical approximations*
 8 *for fractional elliptic equations via the method of semigroups.* ESAIM Math. Model. Numer.
 9 Anal. 54(3) (2020), 751-774.
- 10 [34] M. Délia ; M. Gunzburger : *The fractional Laplacian operator on bounded domains as a*
 11 *special case of the nonlocal diffusion operator.* Comput. Math. Appl. 66 (2013), 1245-1260.
- 12 [35] M. Daoud ; E.-H. Laamri : *Reaction-diffusion systems with nonlocal fractional diffusion.* In
 13 preparation.
- 14 [36] A. De Pablo ; F. Quirós ; A. Rodríguez ; J. L. Vázquez : *A fractional porous medium equation.*
 15 Adv. Math. 226 (2011), 1378-1409.
- 16 [37] A. De Pablo ; F. Quirós ; A. Rodríguez ; J. L. Vázquez : *A general fractional porous medium*
 17 *equation.* Commun. Pure Appl. Math. 65 (2012), 1242-1284.
- 18 [38] E. Di Nezza ; G. Palatucci ; E. Valdinoci : *Hitchhiker's guide to the fractional Sobolev spaces.*
 19 Bull. Sci. Math. 136(5) (2012), 521-573.
- 20 [39] S. Duo ; H. W. Van Wyk ; Y. Zhang : *A novel and accurate weighted trapezoidal finite*
 21 *difference method for the fractional Laplacian.* Journal of Computational Physics 355 (2018),
 22 233-252.
- 23 [40] S. Duo ; L. Ju ; Y. Zhang : *A fast algorithm for solving the space-time fractional diffusion*
 24 *equation.* Comput. Math. Appl. 75(6) (2017), 1929-1941.
- 25 [41] S. Duo ; H. Wang ; Y. Zhang : *A comparative study on nonlocal diffusion operators related*
 26 *to the fractional Laplacian.* Discrete and Continuous Dynamical Systems B 24(1) (2019),
 27 231-256.
- 28 [42] S. Duo ; Y. Zhang : *Accurate numerical methods for two and three dimensional integral frac-*
 29 *tional Laplacian with applications.* Computer Methods in Applied Mechanics and Engineering
 30 355 (2019), 639-662.
- 31 [43] B. Dyda : *Fractional calculus for power functions and eigenvalues of the fractional Laplacian.*
 32 Fract. Calc. Appl. Anal. 15(4) (2012), 536-555.
- 33 [44] B. Dyda ; A. Kuznetsov ; M. Kwaśnicki : *Eigenvalues of the fractional Laplace operator in*
 34 *the unit ball.* J. Lond. Math. Soc. 95(2) (2017), 500-518.
- 35 [45] B. Dyda ; A. Kuznetsov ; M. Kwaśnicki : *Fractional Laplace operator and Meijer G-function.*
 36 Constructive Approximation 45(3) (2017), 427-448.
- 37 [46] M. M. Fall : *Regional Fractional Laplacians: boundary regularity.* arXiv:2007.04808v2
 38 [math.AP] 21 Aug 2020.
- 39 [47] M. M. Fall ; T. Weth : *Nonexistence results for a class of fractional elliptic boundary value*
 40 *problems.* Journal of Functional Analysis 263(15) (2012), 2205-2227.
- 41 [48] A. Fiscella ; R. Servadei ; E. Valdinoci : *Density properties for fractional Sobolev spaces.*
 42 Annales Academiae Scientiarum Fennicae. Mathematica 40(1) (2015), 235-253.
- 43 [49] R. L. Frank ; L. Geisinger : *Refined semiclassical asymptotic for fractional powers of the*
 44 *Laplace operator.* J. Reine Angew. Math.(Crelles Journal) (2014).
- 45 [50] P. Garbaczewski : *Fractional Laplacians and Levy flights in bounded domains.* Acta Physica
 46 Polonica Series B 49(5) (2018), 921-942 .
- 47 [51] P. Garbaczewski ; V. Stephanovich : *Fractional Laplacians in bounded domains : Killed,*
 48 *reflected, censored and taboo Lévy flights.* Phys. Rev. E99 (2019), 042126.
- 49 [52] P. Grisvard : *Elliptic Problems in Nonsmooth Domains.* Monographs and Studies in Math-
 50 ematics, vol. 24. (1985). Boston, MA: Pitman (Advanced Publishing Program), pp. xiv+410.
- 51 [53] Q.-Y. Guan ; Z.-M. Ma : *Boundary problems for fractional Laplacians.* Stochastics and
 52 Dynamics 5(3) (2005), 385-424.
- 53 [54] Q. Guan ; M. Gunzburger : *Analysis and approximation of a nonlocal obstacle problem.*
 54 Comput. Appl. Math. 313 (2017), 102-118.
- 55 [55] A. Henrot : *Extremum problems for eigenvalues of Elliptic operators.* Frontiers in Mathe-
 56 matics, Birkhäuser Verlag, Basel, Switzerland (2006).
- 57 [56] T. Kato : *Fractional powers of dissipative operators.* J. Math. Soc. Japan 13 (1961), 246-274.
- 58 [57] H. Komatsu : *Fractional powers of operators.* Pacific J. Math. 19 (1966), 285-346.

- 59 [58] M. Kwaśnicki : *Ten equivalent definitions of the fractional Laplace operator*. Fract. Calc.
1 Appl. Anal. 20(1) (2015), 7-51.
- 2 [59] J.-L. Lions ; E. Magenes : *Non-homogeneous Problems and Applications*. Vol. I, Translated
3 from the French by P. Kenneth, Band 181, Springer-Verlag, New York-Heidelberg, 1972.
- 4 [60] T. Leonori ; I. Peral ; A. Primo ; F. Soria : *Basic estimates for solutions of a class of nonlocal
5 elliptic and parabolic equations*. Discrete Contin. Dyn. Syst. 35(12) (2015), 6031-6068.
- 6 [61] A. Lischke ; G. Pang ; M. Gulian ; F. Song ; C. Glusa ; X. Zheng ; Z. Mao ; W. Cai ; M.
7 M. Meerschaert ; M. Ainsworth ; G. E Karniadakis : *What is the fractionnal Laplacian ?*.
8 Journal of computational Physics 404 (2020) 109009
- 9 [62] G. Molica Bisci ; V. D. Radulescu ; R. Servadei :
10 *Variational methods for nonlocal fractional problems*. Cambridge University Press, 2016.
- 11 [63] R. Musina ; A. I. Nazarov : *On fractional Laplacians*. Communications in Partial Differential
12 Equations 39(9) (2014), 1780-1790.
- 13 [64] R. Musina ; A. I. Nazarov : *On fractional Laplacians - 2*. Ann. Inst. H. Poincaré Anal. Non
14 Linéaire 33(6) (2016), 1667-1673.
- 15 [65] R. Musina ; A. I. Nazarov : *On fractional Laplacians - 3*. ESAIM: Control, Optimisation and
16 Calculus of Variations 22(3) (2016), 832-841.
- 17 [66] R. H. Nochetto ; E. Otárola ; A. J. Salgado : *A PDE approach to space-time fractional
18 parabolic problems*. SIAM J. Numer. Anal. 54(2) (2016), 848-873.
- 19 [67] R. H. Nochetto ; E. Otárola ; A. J. Salgado : *A PDE approach to fractional diffusion in
20 general domains : a priori error analysis*. Found. Comput. Math. 15(3) (2014), 733-791.
- 21 [68] C. Pozrikidis : *The fractional Laplacian*. University of Massachusetts Amherst, USA. Taylor
22 & Francis Group, LLC 2016.
- 23 [69] X. Ros-Oton ; J. Serra : *The Dirichlet problem for the fractional Laplacian: regularity up
24 to the boundary*. J. Math. Pures Appl. 101(3) (2014), 275-302.
- 25 [70] M. P. Saborid : *The Coordinate-Free Approach to Spherical Harmonics*. arXiv preprint
26 arXiv:0806.3367, 2008.
- 27 [71] S. G. Samko ; A. A. Kilbas ; O. I. Marichev : *Fractional integrals and derivatives*. Gordon
28 and Breach Science Publishers, Yverdon, 1993.
- 29 [72] K. Sato : *Lévy Processes and Infinitely Divisible Distributions*. Cambridge Stud. Adv. Math.
30 vol. 68, Cambridge University Press, Cambridge, 1999, translated from the 1990 Japanese
31 original, revised by the author.
- 32 [73] R. Servadei ; E. Valdinoci : *On the spectrum of two different fractional operators*. Proceedings
33 of the Royal Society of Edinburgh 144A (2014), 831-855.
- 34 [74] R. Servadei ; E. Valdinoci : *Variational methods for non-local operators of elliptic type*.
35 Discrete Contin. Dyn. Syst. 33(5) (2013), 2105-2137.
- 36 [75] S. A. Silling : *Reformulation of elasticity theory for discontinuities and long-range forces*.
37 Mech. Phys. Solids 48 (2000), 175-209.
- 38 [76] L. Silvestre : *Regularity of the obstacle problem for a fractional power of the Laplace operator*.
39 PhD thesis, The University of Texas at Austin, 2005.
- 40 [77] L. Silvestre : *Regularity of the obstacle problem for a fractional power of the Laplace operator*.
41 Comm. Pure Appl. Math. 60 (2007), 67-112.
- 42 [78] R. Song ; Z. Vondraček : *Potential theory of subordinate killed Brownian motion in a domain*.
43 Probab. Theory Relat. Fields 125 (2003), 578-592.
- 44 [79] R. Song ; Z. Vondraček : *On the relationship between subordinate killed and killed subordinate
45 processes*. Electron. Commun. Probab. 13 (2008), 325-336.
- 46 [80] P. R. Stinga : *User's guide to the fractional Laplacian and the method of semigroups*. Hand-
47 book of Fractional Calculus with Applications vol.2, 2019.
- 48 [81] P. R. Stinga : *Fractional powers of second order partial differential operators: extension
49 problem and regularity theory*. Ph.D.-thesis, Universidad Autónoma de Madrid, 2010.
- 50 [82] P. R. Stinga ; J. L. Torrea : *Extension problem and Harnack's inequality for some fractional
51 operators*. Commun. Part. Differ. Equ. 35(11) (2010), 2092-2122.
- 52 [83] J. Tan : *Positive solutions for non local elliptic problems*. Discrete Contin. Dyn. Syst. 33(2)
53 (2013), 837-859.
- 54 [84] S. Terracini ; G. Tortone ; S. Vita : *On s-harmonic functions on cones*. Analysis & PDE
55 11(7) (2018), 1653-1691.
- 56 [85] S. Terracini, G. Verzini, A. Zilio : *Uniform Hölder regularity with small exponent in
57 competition-fractional diffusion systems*. Discret. Contin. Dyn. Syst. 34(6) (2014), 2669-2691.

- 58 [86] E. Valdinoci : *From the long jump random walk to the fractional Laplacian*. Bol. Soc. Esp.
1 Mat. Apl. 49 (2009),33-44.
- 2 [87] J. L. Vázquez : *The mathematical theories of diffusion. Nonlinear and fractional diffusion*.
3 Springer Lecture Notes in Mathematics, CIME Subseries, 2017.
- 4 [88] S. Y. Yolcu ; T. Yolcu : *Refined eigenvalue bounds on the Dirichlet fractional Laplacian*. J.
5 of Math. Phys. 56(7) (2015), 073506.
- 6 [89] K. Yosida : *Functional Analysis*. Classics in Mathematics, Springer-Verlag, Berlin Heidelberg,
7 1995.

8 Received xxxx 20xx; revised xxxx 20xx.

9 *E-mail address:* mahaadaoud@gmail.com

10 *E-mail address:* el-haj.laamri@univ-lorraine.fr