

STATIONARY REACTION-DIFFUSION SYSTEMS IN L^1 REVISITED

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ABSTRACT. We prove existence of L^1 -weak solutions to the reaction-diffusion system obtained as a stationary version of the system arising for the evolution of concentrations in a reversible chemical reaction, coupled with space diffusion. This extends a previous [result](#) by the same authors where restrictive assumptions on the number of chemical species are removed.

1. Introduction and main result. The goal of this note is to give a complete answer to the following question which was left open in [\[3\]](#).

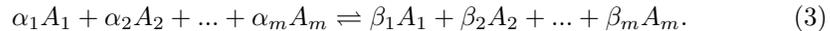
Let us consider the stationary reaction-diffusion system

$$(CHS) \begin{cases} \text{For all } k = 1, \dots, m, \\ u_k - d_k \Delta u_k = (\beta_k - \alpha_k) \left(K_1 \prod_{\ell=1}^m u_\ell^{\alpha_\ell} - K_2 \prod_{\ell=1}^m u_\ell^{\beta_\ell} \right) + f_k, \\ \partial_\nu u_k = 0 \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is a bounded regular open subset of \mathbb{R}^N , $K_1, K_2 \in (0, +\infty)$ and for all $k = 1, \dots, m$, $d_k \in (0, +\infty)$, $\alpha_k, \beta_k \in \{0\} \cup [1, +\infty)$, $f_k \in L^1(\Omega)^+$. Moreover

$$\begin{cases} I := \{i \in \{1, \dots, m\}; \alpha_i - \beta_i > 0\}, \\ \text{and } J := \{j \in \{1, \dots, m\}; \beta_j - \alpha_j > 0\} \\ \text{satisfy:} \\ I \neq \emptyset, J \neq \emptyset, I \cup J = \{1, \dots, m\}. \end{cases} \quad (2)$$

This system is a stationary version of the evolution reaction-diffusion system satisfied by the concentrations of the chemical species $A_i, i = 1, \dots, m$ in the following reversible reaction, under the classical mass action law and with Fick's spatial diffusion :



The question discussed here is the following : does the stationary system (CHS) have a weak solution ? By this we mean :

$$\begin{cases} \forall k = 1, \dots, m, u_k, \Delta u_k \in L^1(\Omega), \\ h(u) := K_1 \prod_{\ell=1}^m u_\ell^{\alpha_\ell} - K_2 \prod_{\ell=1}^m u_\ell^{\beta_\ell} \in L^1(\Omega), \\ \text{and the } u_k \text{'s satisfy the equations in (CHS)}. \end{cases} \quad (4)$$

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We gave a partial positive answer to this question in [3], at least when also $f_k \log f_k \in L^1(\Omega)$, but with restrictions on the number of elements of I or of J . Here we are able to prove the existence of weak solutions without any restriction on the number of elements of I and J .

Theorem 1.1. *Assume $f_k \in L^1(\Omega)^+$, $f_k \log f_k \in L^1(\Omega)$ for all $k = 1, \dots, m$. Then there exists a nonnegative weak solution to (CHS) (in the sense of (4)). If moreover $m = 2$, then the same result holds if only $f_k \in L^1(\Omega)^+$ for $k = 1, 2$.*

Remark 1. *About the condition $f_k \log f_k \in L^1(\Omega)$, $k = 1, \dots, m$. It is not surprising since the entropy inequality (see (2.4.3)) is strongly used in the proof of Theorem 1.1 and naturally involves these quantities (see (24), (25)). Note that for the corresponding evolution system – i.e. when $u_k = u_k(t, x)$ and $u_k - d_k \Delta u_k$ is replaced by $\partial_t u_k - d_k \Delta u_k$ – global existence of renormalized solutions is proved in [2] under the assumption that the initial data satisfy $u_k(0, \cdot) \in L^1(\Omega)^+$, $u_k(0, \cdot) \log u_k(0, \cdot) \in L^1(\Omega)$. The entropy inequality is also a main ingredient in [2]. Note that it is not known whether the nonlinear reactive terms are in $L^1((0, T) \times \Omega)$ or not in the corresponding evolution system. It may be that Theorem 1.1 will help understanding the evolution case through a time-discretization process (see more comments on this in [3]).*

Remark 2. *About the boundary conditions.* Here we work with Neumann boundary conditions. The same result would follow as well for Dirichlet (or also Robin) boundary conditions. An important feature though is that these boundary conditions have to be the same for all k equations, $k = 1, \dots, m$. This implies that all equations are governed by the same operator (here the Laplace operator with Neumann boundary conditions). It allows to have expressions like in (6), (7) coming out by summing two different equations. Actually this adding of equations is a bit complicated here by the fact that the diffusion coefficients d_k are different from each other. We encourage the reader to assume in a first reading that all d_k are equal to 1 (and all γ_k as well). This helps to better understand the main ideas and the steps of the proof without being lost in the sometimes technical formulas due to the fact that the d_k are different from each other.

Note that surprising facts appear in the corresponding evolution reaction-diffusion systems when the boundary conditions are different from one equation to the other (see for instance the analysis in [4]). Indeed, exploiting the sum of two equations is then not so easy since the governing operators are different. We leave this as an open question for the above stationary systems : is it possible to obtain similar L^1 results with a mixture of Neumann and Dirichlet boundary conditions, including the nonhomogeneous case? Same open question for nonlinear boundary conditions of the type $\partial_\nu u_k + \beta(u_k) = 0$ on $\partial\Omega$ where $\beta : [0, +\infty) \rightarrow \mathbb{R}$ is increasing.

We could also easily generalize Theorem 1.1 to more general elliptic operators, but again only if the same operator appears in all equations in order to be able to exploit adding equations. The case of operators varying with k is open.

Remark 3. *About the regularity of the solutions in Theorem 1.1.* As stated in Theorem 1.1, the solutions u_k are such that $u_k, \Delta u_k \in L^1(\Omega)$. This implies (see e.g. [1]) that $\nabla u_k \in L^p(\Omega)$ for all $p \in [1, N/(N-1))$. As a consequence, the trace of ∇u_k on $\partial\Omega$ is well defined in $L^1(\partial\Omega)^N$ (at least) and therefore the trace of $\partial_\nu u_k = \nabla u_k \cdot \nu$ is well defined as well ($\nu =$ unit exterior normal to $\partial\Omega$). In [3], we wrote that $u_k \in W^{2,1}(\Omega)$, without actually redefining this Sobolev space, while we

meant that u_k and $\Delta u_k \in L^1(\Omega)$ “only”. It is well-known that this does not imply that $u_k \in W^{2,1}(\Omega)$ if we define, for all $1 \leq p \leq +\infty$,

$$W^{2,p}(\Omega) = \{v \in L^p(\Omega) ; \partial_{x_i} v, \partial_{x_i x_j} v \in L^p(\Omega), \forall i, j = 1, \dots, N\}.$$

By L^p -regularity theory, it is well-known that u_k and $\Delta u_k \in L^p(\Omega)$ implies $u_k \in W^{2,p}(\Omega)$ if $1 < p < +\infty$, but this is false for $p = 1$.

Remark 4. A simple change of variable allows to reduce all systems of the type

$$u_k - d_k \Delta u_k = \lambda_k h(u) + f_k, \quad k = 1, \dots, m,$$

to (CHS) if $\lambda_k(\beta_k - \alpha_k) > 0$ for all k (see [3]). But this does not include a 2×2 system like

$$u_k - d_k \Delta u_k = (-1)^k u_1^{\sigma_1} u_2^{\sigma_2} [u_2^{\gamma_2} - u_1^{\gamma_1}] + f_k, \quad \sigma_k, \gamma_k \in [1, \infty), \quad k = 1, 2.$$

Here $(-1)^k(\beta_k - \alpha_k) = -\gamma_k < 0$. It is not known whether this system has weak- L^1 solutions.

2. Proof of Theorem 1.1. The particular case $m = 2$ is proved in [3]. We only consider here the case with the extra assumption $f_k \log f_k \in L^1(\Omega)$.

2.1. Approximate problem. Existence in Theorem 1.1 follows by passing to the limit in the next approximate system where $f_k^n := \inf\{f_k, n\}$ for all $k = 1, \dots, m$.

Lemma 2.1. *There exists a nonnegative solution $u^n \in \cap_{p \in [1, \infty)} W^{2,p}(\Omega)^{+m}$ of*

$$(CHSn) \begin{cases} \text{for } k = 1, \dots, m, \\ u_k^n - d_k \Delta u_k^n = (\beta_k - \alpha_k) h(u^n) + f_k^n \text{ in } \Omega, \\ \partial_\nu u_k^n = 0 \text{ on } \partial\Omega. \end{cases} \quad (5)$$

Moreover, we have, with $\gamma_k := |\beta_k - \alpha_k|$,

$$\gamma_j u_i^n + \gamma_i u_j^n - \Delta(\gamma_j d_i u_i^n + \gamma_i d_j u_j^n) = \gamma_j f_i^n + \gamma_i f_j^n, \quad \forall i \in I, j \in J, \quad (6)$$

and u^n is bounded independently of n in $L^{1+\eta}(\Omega)^m$ for some $\eta > 0$.

This approximation lemma is proved in [3] as a consequence of a more general abstract lemma. For completeness, and for simplicity, we give a direct and shorter proof in Section 3.

The proof of Theorem 1.1 consists in passing to the limit as $n \rightarrow +\infty$ in this Lemma 2.1. The main point is to prove that $h(u^n)$ is bounded in $L^1(\Omega)$ independently of n . As explained in [3] and recalled below for completeness, this L^1 -estimate is a direct consequence of the existence of the vectorial function $\theta^n = (\theta_k^n)_{1 \leq k \leq m}$ as stated in the main Lemma 2.2 below.

In order to state this lemma, let us introduce these notations :

$$\sigma_k := \min\{\alpha_k, \beta_k\}, \quad \gamma_k := |\alpha_k - \beta_k|, \quad c := [\min_{1 \leq k \leq m} d_k]^{-1}.$$

We may now write (see the definitions of I and J in (2))

$$h(u) = (\Pi_{\ell=1}^m u_\ell^{\sigma_\ell}) B(u), \quad B(u) := K_1 \Pi_{i \in I} u_i^{\gamma_i} - K_2 \Pi_{j \in J} u_j^{\gamma_j},$$

and we rewrite the main equations in (5) and in (6) as follows :

$$\begin{cases} cd_k u_k^n - \Delta(d_k u_k^n) = (\beta_k - \alpha_k) h(u^n) + g_k^n, \quad g_k^n := f_k^n + u_k^n (cd_k - 1) \geq 0, \\ (c\mathcal{I} - \Delta)(\gamma_j d_i u_i^n + \gamma_i d_j u_j^n) = \gamma_j g_i^n + \gamma_i g_j^n, \quad \forall i \in I, j \in J, \\ \Rightarrow \gamma_j d_i u_i^n + \gamma_i d_j u_j^n = \gamma_j G_i^n + \gamma_i G_j^n, \\ \text{where } G_k^n := (c\mathcal{I} - \Delta)^{-1}(g_k^n), \quad \partial_\nu G_k^n = 0 \text{ on } \partial\Omega \quad \forall k = 1, \dots, m, \end{cases} \quad (7)$$

where \mathcal{I} denotes the identity in $L^1(\Omega)$.

Remark 5. As already mentioned in Remark 2, a first reading of the proof may be made by assuming that $d_k = 1 = \gamma_k$ for all $k = 1, \dots, m$. This helps to make the main ideas more clear and to first avoid some technicalities. With this simplification, above relations simply become

$$u_k^n - \Delta u_k^n = \text{sign}(\beta_k - \alpha_k)h(u^n) + f_k^n, \quad (\mathcal{I} - \Delta)(u_i^n + u_j^n) = f_i^n + f_j^n, \quad \text{and so on.}$$

When the diffusion coefficients d_k are different (which is the general case), the diffusion operators are different from each other in the m equations. A slight change of writing allows us to deal with the only operator $c\mathcal{I} - \Delta$: this is the reason why the equations are written as they are in (7). Adding equations is then efficient as such.

2.2. Statement of the main lemma.

Lemma 2.2. *Under the assumptions of Theorem 1.1, there exists $\theta^n = (\theta_k^n)_{1 \leq k \leq m} \in L^1(\Omega)^{+m}$ with $\Delta \theta_k^n \in L^1(\Omega)$, $k = 1, \dots, m$ such that*

$$\begin{cases} \gamma_j d_i \theta_i^n + \gamma_i d_j \theta_j^n = \gamma_j G_i^n + \gamma_i G_j^n, \quad \forall i \in I, j \in J, \\ K_1 \prod_{i \in I} (\theta_i^n)^{\gamma_i} = K_2 \prod_{j \in J} (\theta_j^n)^{\gamma_j} \quad (\text{or } B(\theta^n) = 0), \\ \partial_\nu \theta_k^n = 0 \text{ on } \partial\Omega, \quad \forall k = 1, \dots, m. \end{cases} \quad (8)$$

Moreover

$$\sup_n \left\{ \max_{1 \leq k \leq m} \|\theta_k^n\|_{L^1(\Omega)} + \|\Delta \theta_k^n\|_{L^1(\Omega)} \right\} < +\infty. \quad (9)$$

We postpone the proof of this main lemma and we recall why it indeed implies Theorem 1.1.

2.3. Lemma 2.2 implies Theorem 1.1. The main point is that the existence of θ^n as in Lemma 2.2 implies that

$$h(u^n) = (\prod_{k=1}^m (u_k^n)^{\sigma_k}) B(u^n) \text{ is bounded in } L^1(\Omega) \text{ independently of } n. \quad (10)$$

Then, it can be deduced that a subsequence of $\{u^n\}$ converges to a weak solution of (1). We skip this part of the proof here and we refer to [3] or also to [5] where an even more involved proof is given for *evolution* reaction-diffusion systems.

Let us prove the L^1 -bound on $h(u^n)$. Without loss of generality, *we assume that* $1 \in I$. Using the first equation ($k = 1$) in (5) and using $B(\theta^n) = 0$, we may write

$$\begin{cases} u_1^n - \theta_1^n - d_1 \Delta (u_1^n - \theta_1^n) + \gamma_1 \prod_{k=1}^m (u_k^n)^{\sigma_k} [B(u^n) - B(\theta^n)] = \rho^n, \\ \rho^n := f_1^n - \theta_1^n + d_1 \Delta \theta_1^n. \end{cases} \quad (11)$$

But, replacing all u_i^n (resp. θ_i^n), $i \in I \setminus \{1\}$ and u_j^n (resp. θ_j^n), $j \in J$ in terms of u_1^n (resp. θ_1^n), we have

$$\begin{cases} B(u^n) = b^n(\cdot, u_1^n), \quad B(\theta^n) = b^n(\cdot, \theta_1^n), \\ \text{where } r \rightarrow b^n(\cdot, r) \text{ is an increasing function.} \end{cases} \quad (12)$$

This increasing property is the main point in the structure of System (1). *We will prove it below*, but let us assume (12) and end the proof of the L^1 -estimate (10). Let us multiply the equation (11) by $\text{sign}(u_1^n - \theta_1^n)$ and integrate on Ω . Using that

$$\begin{cases} \text{sign}(u_1^n - \theta_1^n)[B(u^n) - B(\theta^n)] &= \text{sign}(u_1^n - \theta_1^n)[b^n(\cdot, u_1^n) - b^n(\cdot, \theta_1^n)] \\ &= |b^n(\cdot, u_1^n) - b^n(\cdot, \theta_1^n)|, \end{cases}$$

and that

$$\int_{\Omega} \text{sign}(u_1^n - \theta_1^n)[u_1^n - \theta_1^n - d_1 \Delta(u_1^n - \theta_1^n)] \geq 0,$$

we deduce

$$\gamma_1 \int_{\Omega} \prod_{k=1}^m (u_k^n)^{\sigma_k} |b^n(\cdot, u_1^n) - b^n(\cdot, \theta_1^n)| \leq \int_{\Omega} |\rho^n|.$$

But since again $b(\cdot, \theta_1^n) = B(\theta_1^n) = 0$, the integral on the left-hand side is exactly $\gamma_1 \int_{\Omega} |h(u^n)|$ whence the expected L^1 -bound (10) thanks to (9).

Let us now come back to the details of the proof of (12). Before that, let us introduce

$$F_i^n := d_i^{-1} G_i^n - \delta_i G_1^n, \quad \forall i \in I, \quad F_j^n := d_j^{-1} G_j^n + \delta_j G_1^n, \quad \forall j \in J, \quad (13)$$

where we denote $\delta_k := \gamma_k / (\gamma_1 d_k) \forall k = 1, \dots, m$.

By using (7), we can write all u_k^n in terms of u_1^n as follows

$$u_i^n = F_i^n + d_1 \delta_i u_1^n, \quad \forall i \in I, \quad u_j^n = F_j^n - d_1 \delta_j u_1^n, \quad \forall j \in J.$$

By (8) in Lemma 2.2, we also have in a similar way

$$\theta_i^n = F_i^n + d_1 \delta_i \theta_1^n, \quad \forall i \in I, \quad \theta_j^n = F_j^n - d_1 \delta_j \theta_1^n, \quad \forall j \in J.$$

It follows that

$$\begin{cases} B(u^n) &= K_1 \prod_{i \in I} (u_i^n)^{\gamma_i} - K_2 \prod_{j \in J} (u_j^n)^{\gamma_j}, \\ &= K_1 \prod_{i \in I} (F_i^n + d_1 \delta_i u_1^n)^{\gamma_i} - K_2 \prod_{j \in J} (F_j^n - d_1 \delta_j u_1^n)^{\gamma_j}. \end{cases}$$

From now on, we will use the following notations :

$$\begin{cases} b^n(x, r) := K_1 \prod_{i \in I} (F_i^n(x) + d_1 \delta_i r)^{\gamma_i} - K_2 \prod_{j \in J} (F_j^n(x) - d_1 \delta_j r)^{\gamma_j}, \\ \text{for all } x \in \Omega, r \in [r_-^n(x), r_+^n(x)] \text{ where} \\ r_-^n(x) := \max_{i \in I} [F_i^n(x)]^- / (d_1 \delta_i), \quad r_+^n(x) := \min_{j \in J} F_j^n(x) / (d_1 \delta_j). \end{cases} \quad (14)$$

We check that $r_-^n(x) \leq r_+^n(x)$ and that $r \in [r_-^n(x), r_+^n(x)] \rightarrow b^n(x, r)$ is increasing. We similarly have that $B(\theta^n) = b^n(\cdot, \theta_1^n)$ thanks to (8). This ends the proof of (12) and of the fact that **Lemma 2.2 implies Theorem 1.1**.

2.4. Proof of the main Lemma 2.2. Let us first notice that **in the trivial case** when there exists $(i_0, j_0) \in I \times J$ such that $\gamma_{j_0} G_{i_0}^n + \gamma_{i_0} G_{j_0}^n \equiv 0$ (i.e. $G_{i_0}^n \equiv 0 \equiv G_{j_0}^n$), then θ^n is simply given by $\theta_k^n = G_k^n / d_k$ for all $k = 1, \dots, m$. We will assume that it is not the case, that is $\gamma_j G_i^n + \gamma_i G_j^n \not\equiv 0$ for all $(i, j) \in I \times J$ which implies by strict maximum principle applied to the equations defining G_k^n in (7) that

$$\min\{\gamma_j G_i^n + \gamma_i G_j^n\} > 0, \quad \forall (i, j) \in I \times J. \quad (15)$$

The new idea here (compared to [3]) is to prove the existence of θ^n as the limit of the solution of an adequate approximate problem to which we are able to apply the *entropy estimate*.

For technical reasons, we extend the function $r \in [r_-^n(x), r_+^n(x)] \rightarrow b^n(x, r)$ defined in (14) to the whole set \mathbb{R} as follows: $\forall (x, r) \in \Omega \times \mathbb{R}$,

$$b^n(x, r) := K_1 \prod_{i \in I} [(F_i^n(x) + d_1 \delta_i r)^+]^{\gamma_i} - K_2 [\prod_{j \in J} (F_j^n(x) - d_1 \delta_j r)^+]^{\gamma_j}. \quad (16)$$

We check that $r \in \mathbb{R} \rightarrow b^n(\cdot, r)$ is nondecreasing.

2.4.1. *Introduction of the approximation ζ^η of θ^n .* In what follows, and up to the formula (27), we drop the indexing by n for simplicity (but everything depends on n which is fixed all along the following computations). For all $\eta > 0$, we define $\zeta^\eta = (\zeta_k^\eta)_{1 \leq k \leq m} \in L^1(\Omega)^{+m}$ which is meant to converge to θ^n as $\eta \rightarrow 0$. We first define ζ_1^η as the solution of

$$\begin{cases} \zeta_1^\eta \in L^1(\Omega)^+, \Delta \zeta_1^\eta \in L^1(\Omega), \\ \eta d_1(c\zeta_1^\eta - \Delta \zeta_1^\eta) + \gamma_1 b(\cdot, \zeta_1^\eta) = \eta g_1 \text{ in } \Omega, \\ \partial_\nu \zeta_1^\eta = 0 \text{ on } \partial\Omega. \end{cases} \quad (17)$$

This problem has a (unique) solution since $r \in \mathbb{R} \rightarrow b^n(\cdot, r)$ is nondecreasing and $\sup_{|r| \leq R} |b(\cdot, r)| < +\infty$ for all R (see e.g. [1, Section 4], for a proof). Then we complete the vector $\zeta^\eta = (\zeta_k^\eta)_{1 \leq k \leq m}$ as follows :

$$\begin{cases} \forall i \in I \setminus \{1\}, \zeta_i^\eta := F_i + d_1 \delta_i \zeta_1^\eta, \\ \forall j \in J, \zeta_j^\eta := F_j - d_1 \delta_j \zeta_1^\eta, \\ \zeta^\eta := (\zeta_k^\eta)_{1 \leq k \leq m}. \end{cases} \quad (18)$$

We may rewrite this definition in the more condensed form (including $k = 1$) :

$$\begin{cases} \forall k = 1, \dots, m, \zeta_k^\eta = F_k - \eta_k d_1 \delta_k \zeta_1^\eta, \\ \eta_k := -1 \text{ if } k \in I, \eta_k := 1 \text{ if } k \in J. \end{cases} \quad (19)$$

2.4.2. *Some properties of the approximation ζ^η .* With these definitions and according to (16), we have :

$$\begin{aligned} b(x, \zeta_1^\eta) &:= K_1 \Pi_{i \in I} [(F_i(x) + d_1 \delta_i \zeta_1^\eta)^+]^{\gamma_i} - K_2 \Pi_{j \in J} [(F_j(x) - d_1 \delta_j \zeta_1^\eta)^+]^{\gamma_j}, \\ &\Rightarrow b(\cdot, \zeta_1^\eta) = K_1 \Pi_{i \in I} [(\zeta_i^\eta)^+]^{\gamma_i} - K_2 \Pi_{j \in J} [(\zeta_j^\eta)^+]^{\gamma_j} =: B(\zeta^\eta). \end{aligned}$$

Note that this is an extension to \mathbb{R}^m of the previous function $B(\cdot)$ which was defined on $[0, +\infty)^m$. We will use that it satisfies

$$\eta_k B(\zeta^\eta) (\zeta_k^\eta)^- \geq 0, \text{ for all } k = 1, \dots, m, \text{ where } (\zeta_k^\eta)^- := \sup\{0, -\zeta_k^\eta\}. \quad (20)$$

Let us now write the equations satisfied by $\zeta_k^\eta, k = 2, \dots, m$. Recall that by the definitions (13) and (7)

$$\forall k = 1, \dots, m, cF_k - \Delta F_k = d_k^{-1} g_k + \eta_k \delta_k g_1. \quad (21)$$

Using (19), (21) and the definition (17) of ζ_1^η , we deduce

$$\begin{aligned} c\zeta_k^\eta - \Delta \zeta_k^\eta &= d_k^{-1} g_k + \eta_k \delta_k g_1 - \eta_k d_1 \delta_k [c\zeta_1^\eta - \Delta \zeta_1^\eta], \\ &\Rightarrow \eta d_k (c\zeta_k^\eta - \Delta \zeta_k^\eta) = \eta_k \gamma_k B(\zeta^\eta) + \eta g_k. \end{aligned} \quad (22)$$

Let us multiply the equation (22) by $(\zeta_k^\eta)^- := \sup\{0, -\zeta_k^\eta\}$ and integrate on Ω . Using $\int_\Omega (\zeta_k^\eta)^- \Delta \zeta_k^\eta \geq 0$ and (20), we obtain

$$-\eta d_k c \int_\Omega (\zeta_k^\eta)^- \geq \int_\Omega (\zeta_k^\eta)^- [\eta_k \gamma_k B(\zeta^\eta) + \eta g_k] \geq 0.$$

We deduce that $\zeta_k^\eta \geq 0$. This also implies that $r_- \leq \zeta_1^\eta \leq r_+$ (see (14)) so that we will not need the extension (16) of $b(\cdot, \cdot)$ any more.

2.4.3. *The entropy inequality.* We will now use the *entropy inequality*. Let us introduce

$$\begin{cases} \forall s \in [0, +\infty), L_k(s) := s(\log s - 1 + \mu_k) + e^{-\mu_k}, k = 1, \dots, m, \\ \mu_k := [\log K_2/K_1][m\eta_k\gamma_k]^{-1}. \end{cases}$$

Note that $L'_k(s) = \log s + \mu_k$, $L_k \geq 0$. Moreover, the choice of the μ_k 's implies that for all $r = (r_1, \dots, r_m) \in (0, +\infty)^m$

$$\sum_k \eta_k \gamma_k B(r) [\log r_k + \mu_k] = -B(r) [\log (K_1 \prod_{i \in I} r_i^{\gamma_i}) - \log (K_2 \prod_{j \in J} r_j^{\gamma_j})] \leq 0. \quad (23)$$

We now consider the functions $w_k^\eta := L_k(\zeta_k^\eta)$. We have

$$\nabla w_k^\eta = [\log \zeta_k^\eta + \mu_k] \nabla \zeta_k^\eta, \quad \Delta w_k^\eta = [\log \zeta_k^\eta + \mu_k] \Delta \zeta_k^\eta + \frac{|\nabla \zeta_k^\eta|^2}{\zeta_k^\eta},$$

so that, using (22), we deduce

$$\begin{cases} \eta d_k [c(\log \zeta_k^\eta + \mu_k) \zeta_k^\eta - \Delta w_k^\eta] + \eta d_k \frac{|\nabla \zeta_k^\eta|^2}{\zeta_k^\eta} \\ = \eta d_k (\log \zeta_k^\eta + \mu_k) [c \zeta_k^\eta - \Delta \zeta_k^\eta] \\ = (\log \zeta_k^\eta + \mu_k) [\eta_k \gamma_k B(\zeta^\eta) + \eta g_k]. \end{cases}$$

Dividing by η and using (23), we obtain after summing over k that :

$$\sum_k c d_k (\log \zeta_k^\eta + \mu_k) \zeta_k^\eta - \Delta \sum_k d_k w_k^\eta + \sum_k d_k \frac{|\nabla \zeta_k^\eta|^2}{\zeta_k^\eta} \leq \sum_k [\log \zeta_k^\eta + \mu_k] g_k. \quad (24)$$

Using the Young's inequality

$$\forall r \in [0, +\infty), \forall s \in \mathbb{R}, rs \leq (r \log r - r) + e^s,$$

applied with $r := g_k$ and $s := \log \zeta_k^\eta$, we have

$$g_k \log \zeta_k^\eta \leq (g_k \log g_k - g_k) + \zeta_k^\eta.$$

Plugging this into (24) and integrating over Ω lead to

$$\begin{cases} \sum_k \int_\Omega c d_k [\log \zeta_k^\eta + \mu_k] \zeta_k^\eta + d_k \frac{|\nabla \zeta_k^\eta|^2}{\zeta_k^\eta} \\ \leq \int_\Omega \sum_k (g_k \log g_k - g_k + \zeta_k^\eta + \mu_k g_k). \end{cases} \quad (25)$$

2.4.4. *Estimates on the ζ_k^η .* In the estimates below, $\Lambda \in (0, +\infty)$ denotes a constant which does not depend on η , neither on n .

From the definition (18) of ζ^η , we have

$$\gamma_j d_i \zeta_i^\eta + \gamma_i d_j \zeta_j^\eta = \gamma_j G_i + \gamma_i G_j, \quad \forall i \in I, j \in J. \quad (26)$$

Recall that $\zeta_k^\eta \geq 0$. Since G_1, \dots, G_m do not depend on η and are bounded in $L^1(\Omega)$ independently of n (see (7) and Lemma 2.1), we have

$$\max_{1 \leq k \leq m} \int_\Omega \zeta_k^\eta \leq \Lambda. \quad (27)$$

We here go back to the indexing by n . As seen in (7), $g_k^n = f_k^n + u_k^n (cd_k - 1)$. Under the assumption $f_k, f_k \log f_k \in L^1(\Omega)$ (recall that $f_k^n = \inf\{f_k, n\}$) and thanks to the end of Lemma 2.1, it follows that

$$\max_{1 \leq k \leq m} \{ \|g_k^n\|_{L^1(\Omega)}, \|g_k^n \log g_k^n\|_{L^1(\Omega)} \} \leq \Lambda. \quad (28)$$

Going back to (25) and noticing that $|\nabla\zeta_k^\eta|^2/|\zeta_k^\eta| = 4\left|\nabla\sqrt{\zeta_k^\eta}\right|^2$, we deduce

$$\max_{1 \leq k \leq m} \left\{ \int_{\Omega} \zeta_k^\eta \log \zeta_k^\eta, \int_{\Omega} \left| \nabla \sqrt{\zeta_k^\eta} \right|^2 \right\} \leq \Lambda. \quad (29)$$

2.4.5. *Passing to the limit as $\eta \rightarrow 0$ and building the function θ^n of the main Lemma 2.2.* It follows from (29) that $\sqrt{\zeta_k^\eta}$ is bounded for all $k = 1, \dots, m$ in $H^1(\Omega)$ independently of η and n . For n fixed, up to a subsequence as $\eta \rightarrow 0$, $\sqrt{\zeta_k^\eta}$ converges for all k in $L^2(\Omega)$, a.e. and also weakly in $H^1(\Omega)$ to some $v_k^n \in H^1(\Omega)$. Thanks to (26) and to their nonnegativity, the ζ_k^η are bounded in $L^\infty(\Omega)$ and therefore $v_k^n \in L^\infty(\Omega)^+$ (not independently of n though). Let $\theta_k^n := (v_k^n)^2$, $k = 1, \dots, m$ so that $\nabla\theta_k^n = 2v_k^n \nabla v_k^n$. Then $\theta_k^n \in L^\infty(\Omega)^+ \cap H^1(\Omega)$ and by lower semicontinuity for the weak convergence and (29), we have

$$\forall k = 1, \dots, m, \int_{\Omega} \frac{|\nabla\theta_k^n|^2}{\theta_k^n} = 4 \int_{\Omega} |\nabla\sqrt{\theta_k^n}|^2 \leq 4 \liminf_{\eta \rightarrow 0} \int_{\Omega} |\nabla\sqrt{\zeta_k^\eta}|^2 \leq \Lambda. \quad (30)$$

By (27), we also have

$$\forall k = 1, \dots, m, \int_{\Omega} \theta_k^n \leq \Lambda.$$

Passing to the limit as $\eta \rightarrow 0$ in the definition (18) of ζ^η , we obtain

$$\forall i \in I, \theta_i^n = F_i^n + d_1 \delta_i \theta_1^n, \quad \forall j \in J, \theta_j^n = F_j^n - d_1 \delta_j \theta_1^n. \quad (31)$$

Now for n fixed, $b^n(x, \zeta_1^\eta(x))$ converges a.e. to $b^n(x, \theta_1^n(x))$ as $\eta \rightarrow 0$. Moreover it stays uniformly bounded. Thus the convergence holds in the sense of distributions. Letting η tend to 0 in the equation (17) (which defines ζ_1^η) leads to

$$b^n(x, \theta_1^n(x)) = 0 = B(\theta^n), \quad \forall x \in \Omega. \quad (32)$$

Note that $\theta_1^n(x)$ is uniquely determined by (32) since $r \rightarrow b^n(x, r)$ is increasing. Moreover $x \rightarrow \theta_1^n(x)$ is regular by the implicit function theorem since $(x, r) \rightarrow b^n(x, r)$ is regular thanks to the fact that $F_k^n \in \cap_{p \in [1, \infty)} W^{2,p}(\Omega)$ and to the definition of b^n , namely (see (14))

$$b^n(x, r) := K_1 \prod_{i \in I} (F_i^n(x) + d_1 \delta_i r)^{\gamma_i} - K_2 \prod_{j \in J} (F_j^n(x) - d_1 \delta_j r)^{\gamma_j}.$$

Using (15), we check that $F_i^n(x) + d_1 \delta_i \theta_1^n(x) = \theta_i^n(x) > 0$ for all $i \in I$ and $F_j^n(x) - d_1 \delta_j \theta_1^n(x) = \theta_j^n(x) > 0$ for all $j \in J$. Thus, we have the $W^{2,p}$ -regularity for all $p \in [1, \infty)$ of $b^n(\cdot, \cdot)$ around all points $(x, \theta_1^n(x))$. Whence the same regularity for $x \rightarrow \theta_1^n(x)$ and, by (31), for all $x \rightarrow \theta_k^n(x)$.

2.4.6. *More estimates on θ^n ; end of the proof of the main Lemma 2.2.* Let us prove that $\partial_\nu \theta_k^n = 0$ on $\partial\Omega$ for all k . We start by taking a logarithmic differentiation of : $0 = B(\theta^n) = K_1 \prod_{i \in I} (\theta_i^n)^{\gamma_i} - K_2 \prod_{j \in J} (\theta_j^n)^{\gamma_j}$, namely

$$\sum_{i \in I} \gamma_i \frac{\nabla \theta_i^n}{\theta_i^n} = \sum_{j \in J} \gamma_j \frac{\nabla \theta_j^n}{\theta_j^n}. \quad (33)$$

Inserting (31), we deduce

$$d_1 \nabla \theta_1^n A^n = \sum_{j \in J} \frac{\gamma_j \nabla F_j^n}{\theta_j^n} - \sum_{i \in I} \frac{\gamma_i \nabla F_i^n}{\theta_i^n}, \quad A^n := \sum_{k=1}^m \frac{\gamma_k \delta_k}{\theta_k^n}.$$

Since $\nabla F_k^n \cdot \nu = 0$ on $\partial\Omega$ for all $k = 1, \dots, m$, we deduce from this equality that $\nabla \theta_1^n \cdot \nu = 0$ on $\partial\Omega$ as well. And by (31), it also follows that $\nabla \theta_k^n \cdot \nu = 0$ on $\partial\Omega$.

It remains to prove the estimate on $\Delta\theta_k^n$ in $L^1(\Omega)$ independently of n . For this, we use the same computations as in [3]. We differentiate (33) to obtain

$$\sum_{i \in I} \gamma_i \frac{\Delta\theta_i^n}{\theta_i^n} - \gamma_i \frac{|\nabla\theta_i^n|^2}{(\theta_i^n)^2} = \sum_{j \in J} \gamma_j \frac{\Delta\theta_j^n}{\theta_j^n} - \gamma_j \frac{|\nabla\theta_j^n|^2}{(\theta_j^n)^2}. \quad (34)$$

Using (31), we may rewrite (34) as

$$d_1 A^n \Delta\theta_1^n = \sum_{i \in I} -\frac{\gamma_i \Delta F_i^n}{\theta_i^n} + \frac{\gamma_i |\nabla\theta_i^n|^2}{(\theta_i^n)^2} + \sum_{j \in J} \frac{\gamma_j \Delta F_j^n}{\theta_j^n} - \frac{\gamma_j |\nabla\theta_j^n|^2}{(\theta_j^n)^2}. \quad (35)$$

If we denote $\alpha_k^n := \gamma_k \delta_k / A^n \theta_k^n$ for $k = 1, \dots, m$, then

$$0 \leq \alpha_k^n \leq 1 \text{ and } \sum_{k=1}^m \alpha_k^n = 1.$$

Then we rewrite again (35) as

$$d_1 \Delta\theta_1^n = \sum_{i \in I} \alpha_i \left[-\frac{\Delta F_i^n}{\delta_i} + \frac{|\nabla\theta_i^n|^2}{\delta_i \theta_i^n} \right] + \sum_{j \in J} \alpha_j \left[\frac{\Delta F_j^n}{\delta_j} - \frac{|\nabla\theta_j^n|^2}{\delta_j \theta_j^n} \right].$$

We know that ΔF_k^n is bounded in $L^1(\Omega)$ uniformly in n (see the definition of the F_k^n in (13) together with (7) and (28)). Moreover, $|\nabla\theta_k^n|^2/\theta_k^n$ is bounded in $L^1(\Omega)$ by (30). Therefore this last identity proves that $\Delta\theta_1^n$ is bounded in $L^1(\Omega)$ uniformly in n . Going back to the relations (31), we deduce that

$$\max_{1 \leq k \leq m} \|\Delta\theta_k^n\|_{L^1(\Omega)} \leq \Lambda.$$

And this ends the proof of the main Lemma 2.2.

3. Proof of Lemma 2.1. Let n be fixed and let $\epsilon \in (0, 1)$. We introduce

$$h^\epsilon(v) := \frac{h(v)}{1 + \epsilon|h(v)|}, \text{ for all } v \in \mathbb{R}^m.$$

Note that $|h^\epsilon(v)| \leq \epsilon^{-1}$ for all $v \in \mathbb{R}^m$. We consider the mapping $\mathcal{T} : v \in L^\infty(\Omega)^m \rightarrow u^\epsilon \in L^\infty(\Omega)^m$ where $u^\epsilon = (u_k^\epsilon)_{1 \leq k \leq m}$ is solution of

$$\begin{cases} u^\epsilon \in \cap_{p \in [1, \infty)} W^{2,p}(\Omega)^m, & \text{and for } k = 1, \dots, m, \\ u_k^\epsilon - d_k \Delta u_k^\epsilon = (\beta_k - \alpha_k) h^\epsilon(v^+) + f_k^n, & \text{in } \Omega, \partial_\nu u_k^\epsilon = 0 \text{ on } \partial\Omega, \end{cases}$$

where $v^+ := ((v_k)^+)_{1 \leq k \leq m}$, $(v_k)^+ := \max\{v_k, 0\}$. Obviously

$$\|u_k^\epsilon - d_k \Delta u_k^\epsilon\|_{L^\infty(\Omega)} \leq \epsilon^{-1} \gamma_k + n, \quad \forall k = 1, \dots, m.$$

It follows that u^ϵ is bounded in all $W^{2,p}(\Omega)^m$, $p < +\infty$ independently of $v \in L^\infty(\Omega)^m$ (ϵ, n being fixed). Therefore, \mathcal{T} maps $L^\infty(\Omega)^m$ into a compact subset of itself. Since \mathcal{T} is obviously continuous, by the Schauder fixed-point theorem, \mathcal{T} has a fixed point in $\cap_{p \in [1, +\infty)} W^{2,p}(\Omega)^m$.

Let us check that the *quasipositivity* of the nonlinearity $((\beta_k - \alpha_k)h(v^+))_{1 \leq k \leq m}$ imply that $u_k^\epsilon \geq 0$. By quasipositivity, we mean (and this is easily checked) that

$$\text{for all } k = 1, \dots, m, \quad (\beta_k - \alpha_k)h(v^+) \geq 0 \text{ for all } v \in \mathbb{R}^m \text{ with } v_k = 0.$$

Thus let us multiply the equation in u_k^ϵ namely

$$u_k^\epsilon - d_k \Delta u_k^\epsilon = (\beta_k - \alpha_k) h^\epsilon((u^\epsilon)^+) + f_k^n, \quad \partial_\nu(u_k^\epsilon) = 0, \quad (36)$$

by $(u_k^\epsilon)^- := \sup\{-u_k^\epsilon, 0\}$. Using $\int_\Omega (u_k^\epsilon)^- \Delta u_k^\epsilon \geq 0$ and also the quasipositivity property, we obtain

$$-\int_\Omega (u_k^\epsilon)^- \geq \int_\Omega (u_k^\epsilon)^- [(\beta_k - \alpha_k)h^\epsilon((u^\epsilon)^+) + f_k^n] \geq 0.$$

Whence $(u_k^\epsilon)^- \equiv 0$ or $u_k^\epsilon \geq 0$.

Now we use that

$$h^\epsilon((u^\epsilon)^+) [\gamma_j(\beta_i - \alpha_i) + \gamma_i(\beta_j - \alpha_j)] \equiv 0, \quad \forall i \in I, j \in J,$$

to deduce from (36) that

$$\gamma_j u_i^\epsilon + \gamma_i u_j^\epsilon - \Delta(\gamma_j d_i u_i^\epsilon + \gamma_i d_j u_j^\epsilon) = \gamma_j f_i^n + \gamma_i f_j^n, \quad \forall i \in I, j \in J. \quad (37)$$

If $c := [\max_{1 \leq k \leq m} d_k]^{-1}$ as above, thanks to the nonnegativity of the u_k^ϵ , this implies

$$\begin{aligned} c[\gamma_j d_i u_i^\epsilon + \gamma_i d_j u_j^\epsilon] - \Delta(\gamma_j d_i u_i^\epsilon + \gamma_i d_j u_j^\epsilon) &\leq \gamma_j f_i^n + \gamma_i f_j^n, \\ \Rightarrow 0 \leq \gamma_j d_i u_i^\epsilon + \gamma_i d_j u_j^\epsilon &\leq (c\mathcal{I} - \Delta)^{-1}(\gamma_j f_i^n + \gamma_i f_j^n). \end{aligned} \quad (38)$$

This implies that, for n fixed, the u_k^ϵ are bounded in $L^\infty(\Omega)$ independently of ϵ . Going back to the equations (37), we deduce that the u_k^ϵ are also bounded in $\cap_{p \in [1, \infty)} W^{2,p}(\Omega)$. Whence their compactness in $L^\infty(\Omega)$. Let us denote the limit as $\epsilon \rightarrow 0$ (along a subsequence) by $u^n = (u_k^n)_{1 \leq k \leq m}$. An easy passing to the limit in (36) and (37) proves that u^n satisfies the system (CHSn) in (5) and the identities (6) as well.

Finally, the last point of Lemma 2.1 is obtained thanks to the inequality (38) which is also valid at the limit for $\gamma_j d_i u_i^n + \gamma_i d_j u_j^n$. Since the f_k^n are bounded in $L^1(\Omega)$ independently of n , this implies that $\gamma_j d_i u_i^n + \gamma_i d_j u_j^n$ is bounded in any $L^p(\Omega)$ for all $p \in [1, N/(N-2)^+)$ (see e.g. [1]).

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