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Multifractional processes and nonlinear functionals of Gaussian random fields

Renaud Marty*

Abstract

In this paper we consider sequences of nonlinear functionals of Gaussian random fields. We prove their convergence to multifractional processes which generalize Hermite processes.

Keywords : Limit theorems, multifractional processes, Gaussian random fields

1 Introduction

This paper deals with sequences of processes defined from functionals of Gaussian random fields. Many works state that such sequences converge to various types of limits which depend on the form of the functionals and on statistical properties of the random fields. Consider for instance a stationary and Gaussian sequence $(X_n)_{n \in \mathbb{N}}$ of random variables with mean zero and variance 1. We assume that there exist $m \in \mathbb{N}^*$, $C > 0$, and $\alpha \in (0, \frac{1}{m})$ such that

$$\mathbb{E}(X_0 X_n) \sim \frac{C}{n^\alpha} \quad (1)$$

as $n \rightarrow \infty$. For all $N \in \mathbb{N}$ we define the process $S_{\phi, H}^N$ by

$$(S_{\phi, H}^N(t))_{t \geq 0} = \left(\frac{1}{N^H} \sum_{n=1}^{\lfloor Nt \rfloor} \phi(X_n) \right)_{t \geq 0} \quad (2)$$

where $H \in (\frac{1}{2}, 1)$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that

$$\int_{\mathbb{R}} (\phi(x))^2 e^{-x^2/2} dx < \infty \text{ and } \int_{\mathbb{R}} \phi(x) e^{-x^2/2} dx = 0.$$

Assume that $\alpha = \frac{2-2H}{m}$ and that the Hermite rank of ϕ is equal to m . By [4] and [9], as $N \rightarrow \infty$, $S_{\phi, H}^N$ converges in distribution to a Hermite process $B_{m, H}$ defined by

$$(B_{m, H}(t))_{t \geq 0} = \left(c(m, H) \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t \prod_{k=1}^m (\theta - x_k)_+^{\tilde{H}-3/2} d\theta \right)_{t \geq 0} \quad (3)$$

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where $\tilde{H} = 1 + \frac{H-1}{m}$, $D_m = \{x_1 < x_2 < \dots < x_m\}$, W is a Brownian motion, and $c(m, H)$ is a constant. The process $B_{m,H}$ can also be written as

$$(B_{m,H}(t))_{t \geq 0} = \left(\hat{c}(m, H) \int_{\mathbb{R}^m} d\widehat{W}_{\xi_1} \cdots d\widehat{W}_{\xi_m} \frac{\exp(it(\xi_1 + \dots + \xi_m)) - 1}{i|\xi_1 \cdots \xi_m|^{\tilde{H}-1/2}(\xi_1 + \dots + \xi_m)} \right)_{t \geq 0} \quad (4)$$

where \widehat{W} is the Fourier transform of a Gaussian measure and $\hat{c}(m, H)$ is a constant. It is a self-similar process with index H and its local Hölder exponent is H at every point. Moreover it is not Gaussian when $m > 1$, and $B_{1,H}$ is a fractional Brownian motion with Hurst index H .

The above result is extended in a multifractional framework in [3] and [6]. Let $(X_n(H))_{(n,H) \in \mathbb{N} \times (1/2, 1)}$ be a Gaussian sequence of random fields with mean zero. We assume that for all $(H_1, H_2) \in (\frac{1}{2}, 1)^2$ there exists $R(H_1, H_2) > 0$ such that

$$\mathbb{E}(X_0(H_1)X_n(H_2)) \sim \frac{R(H_1, H_2)}{n^{2-H_1-H_2}} \quad (5)$$

as $n \rightarrow \infty$. Notice that if $H_1 = H_2$, then (5) is similar to (1). We consider a continuous function $h : \mathbb{R}_+ \rightarrow (\frac{1}{2}, 1)$ and, for all $N \in \mathbb{N}$, we define the process S_h^N by

$$(S_h^N(t))_{t \geq 0} = \left(\sum_{n=1}^{\lfloor Nt \rfloor} \frac{X_n(h(n/N))}{N^{h(n/N)}} \right)_{t \geq 0} \quad (6)$$

By [3] and under additional assumptions, as $N \rightarrow \infty$ the finite-dimensional distributions of S_h^N converge to those of a Gaussian process S_h with mean zero and such that for all t_1 and t_2 ,

$$\mathbb{E}(S_h(t_1)S_h(t_2)) = \int_0^{t_1} \int_0^{t_2} \frac{R(h(\max\{\theta, \sigma\}), h(\min\{\theta, \sigma\}))}{|\theta - \sigma|^{2-h(\theta)-h(\sigma)}} d\theta d\sigma.$$

This is a multifractional extension of the result of [4] and [9] in the special case where $\phi(x) = x$ for all $x \in \mathbb{R}$. If h is constant, then S_h is a fractional Brownian motion, namely a Hermite process with $m = 1$. If h is non-constant, then S_h is multifractional like, for instance, the multifractional Brownian motions (defined in [1, 8]).

Now consider for all $N \in \mathbb{N}^*$ the process $S_{\phi,h}^N$ defined by

$$(S_{\phi,h}^N(t))_{t \geq 0} = \left(\sum_{n=1}^{\lfloor Nt \rfloor} \frac{\phi\left(X_n\left(\tilde{h}(n/N)\right)\right)}{N^{h(n/N)}} \right)_{t \geq 0} \quad (7)$$

where

$$\tilde{h} = 1 + \frac{h-1}{m}.$$

We assume that there exist a function $g : (H, \xi) \mapsto g(H, \xi)$ and a symmetric compact set $K \subset \mathbb{R}$ such that the field $X = (X_n(H))_{(n,H) \in \mathbb{N} \times (1/2,1)}$ of (7) is

$$X_n(H) = \int_K e^{in\xi} \frac{g(H, \xi)}{|\xi|^{H-1/2}} d\widehat{W}(\xi). \quad (8)$$

Under additional assumptions on g , X satisfies (5) with

$$R(H_1, H_2) = g(H_1, 0)g(H_2, 0) \int_{\mathbb{R}} \frac{e^{i\xi}}{|\xi|^{H_1+H_2-1}} d\xi$$

Notice that in this case,

$$R(H_1, H_2) = R(H_2, H_1). \quad (9)$$

The main result of [6] states that, as $N \rightarrow \infty$, the process $S_{\phi, h}^N$ converges in distribution to a multifractional process $S_{m, h}$ defined by

$$(S_{m, h}(t))_{t \geq 0} = \left(\int_{\mathbb{R}^m} d\widehat{W}_{\xi_1} \cdots d\widehat{W}_{\xi_m} \int_0^t \tilde{g}(\theta) \frac{\exp(i\theta(\xi_1 + \cdots + \xi_m))}{|\xi_1 \cdots \xi_m|^{\tilde{h}(\theta)-1/2}} d\theta \right)_{t \geq 0} \quad (10)$$

where \tilde{g} is a deterministic function. This is an extension in a non-Gaussian setting of the result of [3]. Moreover, the process $S_{m, h}$ generalizes $B_{m, H}$, namely, if there exists $H \in (\frac{1}{2}, 1)$ such that $h(x) = H$ for all x , then $S_{m, h} = B_{m, H}$.

In this paper we extend the results mentioned above. We consider fields X such that for all $H_1 \neq H_2$,

$$R(H_1, H_2) \neq R(H_2, H_1),$$

in contrast with (9) and [6]. With such fields X , we prove the convergence of sequences defined as in (7) to a process of the form

$$\left(\int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t f(\theta) \prod_{k=1}^m (\theta - x_k)_+^{\tilde{h}(\theta)-3/2} d\theta \right)_{t \geq 0} \quad (11)$$

where f is a deterministic function. The limit is multifractional and generalizes Hermite processes, even though it generally differs from $S_{m, h}$ defined by (10) and studied in [6]. In addition, we prove the convergence when $N \rightarrow \infty$ of

$$\left(\sum_{n=1}^{\lfloor Nt \rfloor} \frac{\phi \left(X_n \left(\tilde{h} \left(n/N^\beta \right) \right) \right)}{N^{h(n/N^\beta)}} \right)_{t \geq 0} \quad (12)$$

for all $\beta \in \mathbb{R} \setminus \{1\}$. In this case, the limit is a Hermite process with Hurst index depending on the sign of $\beta - 1$.

The paper is organized as follows. In Section 2 we present the setting and state the main result of the paper. In Section 3 we analyze a multifractional process of the form (11). Section 4 is devoted to the proof of the main result.

2 Setting and main result

Our main result is the convergence of a sequence defined from a functional of a Gaussian field with long-range dependence. This random field is presented in Section 2.1 and two examples are given in Section 2.2. The main result is stated in Section 2.3.

2.1 Gaussian fields with long-range dependence

We consider a random field $X = \{X(t, H)\}_{(t, H) \in \mathbb{R}_+ \times (\frac{1}{2}, 1)}$ such that for all $(t, H) \in \mathbb{R}_+ \times (\frac{1}{2}, 1)$,

$$X(t, H) = \int_{-\infty}^t a(t-x, H) dW_x \quad (13)$$

where W is a Brownian motion and $a : \mathbb{R}_+ \times (\frac{1}{2}, 1) \rightarrow \mathbb{R}_+^*$ is a continuous function such that $\int_0^\infty a(x, H)^2 dx = 1$ for all H . As a consequence, X is a Gaussian field with mean zero and covariance function r which can be written as

$$\begin{aligned} r(t_1, H_1, t_2, H_2) &= \mathbb{E}(X(t_1, H_1)X(t_2, H_2)) \\ &= \int_{-\infty}^{\min\{t_1, t_2\}} a(t_1-x, H_1)a(t_2-x, H_2)dx \end{aligned} \quad (14)$$

for all $(t_1, H_1, t_2, H_2) \in (\mathbb{R}_+ \times (\frac{1}{2}, 1))^2$, and which satisfies $r(t, H, t, H) = \mathbb{E}(X(t, H)^2) = 1$ for all (t, H) . Moreover, we assume that there exists a continuous function $A : (\frac{1}{2}, 1) \rightarrow \mathbb{R}_+^*$ such that for all compact set $K \subset (\frac{1}{2}, 1)$,

$$\lim_{t \rightarrow \infty} \sup_{H \in K} |t^{\frac{3}{2}-H} a(t, H) - A(H)| = 0. \quad (15)$$

We deduce the following property of the covariance which implies that X satisfies the assumptions considered in [3].

Lemma 1. *For all compact set $K \subset (\frac{1}{2}, 1)^2$,*

$$\lim_{|t_1-t_2| \rightarrow \infty} \sup_{(H_1, H_2) \in K} ||t_1 - t_2|^{2-H_1-H_2} r(t_1, H_1, t_2, H_2) - R(t_1, H_1, t_2, H_2)| = 0. \quad (16)$$

where $R : (\mathbb{R}_+ \times (\frac{1}{2}, 1))^2 \rightarrow \mathbb{R}_+^*$ is a continuous function such that for all $(t_1, H_1, t_2, H_2) \in (\mathbb{R}_+ \times (\frac{1}{2}, 1))^2$,

$$\begin{aligned} R(t_1, H_1, t_2, H_2) &= 1_{t_1 \geq t_2} A(H_1)A(H_2) \int_0^\infty (1+x)^{H_1-\frac{3}{2}} x^{H_2-\frac{3}{2}} dx \\ &\quad + 1_{t_1 < t_2} A(H_1)A(H_2) \int_0^\infty (1+x)^{H_2-\frac{3}{2}} x^{H_1-\frac{3}{2}} dx. \end{aligned} \quad (17)$$

Notice that if we set for all (H_1, H_2) ,

$$C(H_1, H_2) = A(H_1)A(H_2) \int_0^\infty (1+x)^{H_1-\frac{3}{2}} x^{H_2-\frac{3}{2}} dx$$

then

$$C(H_1, H_2) \neq C(H_2, H_1)$$

when $H_1 \neq H_2$. Indeed, if we assume that $H_1 > H_2$, then

$$\begin{aligned} \int_0^\infty (1+x)^{H_1-3/2} x^{H_2-3/2} dx &= \int_0^\infty \left(1 + \frac{1}{x}\right)^{H_1-3/2} x^{H_1+H_2-3} dx \\ &> \int_0^\infty \left(1 + \frac{1}{x}\right)^{H_2-3/2} x^{H_1+H_2-3} dx \\ &= \int_0^\infty (1+x)^{H_2-3/2} x^{H_1-3/2} dx. \end{aligned} \quad (18)$$

Hence, the framework of this paper is different from that of [6].

Proof. (Lemma 1) We assume $t_1 > t_2$. We have

$$\begin{aligned} &r(t_1, H_1, t_2, H_2) \quad (19) \\ &= \int_{-\infty}^{t_2} (a(t_1-x, H_1) - (t_1-x)^{H_1-\frac{3}{2}} A(H_1))(a(t_2-x, H_2) - (t_2-x)^{H_2-\frac{3}{2}} A(H_2)) dx \\ &\quad + \int_{-\infty}^{t_2} (a(t_1-x, H_1) - (t_1-x)^{H_1-\frac{3}{2}} A(H_1))(t_2-x)^{H_2-\frac{3}{2}} A(H_2) dx \\ &\quad + \int_{-\infty}^{t_2} (t_1-x)^{H_1-\frac{3}{2}} A(H_1)(a(t_2-x, H_2) - (t_2-x)^{H_2-\frac{3}{2}} A(H_2)) dx \\ &\quad + \int_{-\infty}^{t_2} (t_1-x)^{H_1-\frac{3}{2}} A(H_1)(t_2-x)^{H_2-\frac{3}{2}} A(H_2) dx \end{aligned} \quad (20)$$

We set $\delta = t_1 - t_2$. We have

$$\begin{aligned} &r(t_1, H_1, t_2, H_2) \\ &= \int_0^\infty (a(\delta+x, H_1) - (\delta+x)^{H_1-\frac{3}{2}} A(H_1))(a(x, H_2) - x^{H_2-\frac{3}{2}} A(H_2)) dx \\ &\quad + \int_0^\infty (a(\delta+x, H_1) - (\delta+x)^{H_1-\frac{3}{2}} A(H_1)) x^{H_2-\frac{3}{2}} A(H_2) dx \\ &\quad + \int_0^\infty (\delta+x)^{H_1-\frac{3}{2}} A(H_1)(a(x, H_2) - x^{H_2-\frac{3}{2}} A(H_2)) dx \\ &\quad + \int_0^\infty (\delta+x)^{H_1-\frac{3}{2}} A(H_1) x^{H_2-\frac{3}{2}} A(H_2) dx \end{aligned} \quad (21)$$

As a consequence, by the change of variable $x \rightarrow \delta x$,

$$\begin{aligned} \delta^{2-H_1-H_2} r(t_1, H_1, t_2, H_2) - A(H_1)A(H_2) \int_0^\infty (1+x)^{H_1-\frac{3}{2}} x^{H_2-\frac{3}{2}} dx \\ = R_1(\delta, H_1, H_2) + R_2(\delta, H_1, H_2) + R_3(\delta, H_1, H_2) \end{aligned} \quad (22)$$

where

$$\begin{aligned} R_1(\delta, H_1, H_2) = \delta^{3-H_1-H_2} \int_0^\infty (a(\delta + \delta x, H_1) - (\delta + \delta x)^{H_1-\frac{3}{2}} A(H_1)) \\ \times (a(\delta x, H_2) - (\delta x)^{H_2-\frac{3}{2}} A(H_2)) dx, \end{aligned}$$

$$R_2(\delta, H_1, H_2) = \delta^{3/2-H_1} A(H_2) \int_0^\infty (a(\delta + \delta x, H_1) - (\delta + \delta x)^{H_1-\frac{3}{2}} A(H_1)) x^{H_2-\frac{3}{2}} dx$$

and

$$R_3(\delta, H_1, H_2) = \delta^{3/2-H_2} A(H_1) \int_0^\infty (1+x)^{H_1-\frac{3}{2}} (a(\delta x, H_2) - (\delta x)^{H_2-\frac{3}{2}} A(H_2)) dx$$

We have

$$R_3(\delta, H_1, H_2) = A(H_1) \int_0^\infty (1+x)^{H_1-\frac{3}{2}} x^{H_2-\frac{3}{2}} ((\delta x)^{3/2-H_2} a(\delta x, H_2) - A(H_2)) dx \quad (23)$$

Let K be a compact set. Because of (15), for all $\nu > 0$ there exist two positive constants μ and c such that

$$\begin{aligned} \sup_{(H_1, H_2) \in K} |R_3(\delta, H_1, H_2)| \\ \leq \nu \sup_{(H_1, H_2)} |A(H_1) \int_{\mu/\delta}^\infty (1+x)^{H_1-\frac{3}{2}} x^{H_2-\frac{3}{2}} dx| \\ + c \sup_{(H_1, H_2)} |A(H_1) \int_0^{\mu/\delta} (1+x)^{H_1-\frac{3}{2}} x^{H_2-\frac{3}{2}} dx| \end{aligned} \quad (24)$$

so that

$$\lim_{\delta \rightarrow \infty} \sup_{(H_1, H_2) \in K} |R_3(\delta, H_1, H_2)| = 0 \quad (25)$$

Similarly, we prove that

$$\lim_{\delta \rightarrow \infty} \sup_{(H_1, H_2) \in K} |R_1(\delta, H_1, H_2)| = \lim_{\delta \rightarrow \infty} \sup_{(H_1, H_2) \in K} |R_2(\delta, H_1, H_2)| = 0. \quad (26)$$

This completes the proof. \square

2.2 Examples

We give two examples of fields satisfying the assumption presented in Section 2.1.

Consider that X_1 is defined for all t and H by

$$X_1(t, H) = \frac{1}{C_1(H)} \int_{-\infty}^t \left((t-x)_+^{H-1/2} - (t-1-x)_+^{H-1/2} \right) dW_x \quad (27)$$

where

$$C_1(H) = \sqrt{\int_0^\infty \left((x)_+^{H-1/2} - (x-1)_+^{H-1/2} \right)^2 dx}.$$

For all H , $X_1(\cdot, H) : t \mapsto X_1(t, H)$ is the fractional Gaussian noise (see [9] for instance), namely the process of the increments $B_H(\cdot) - B_H(\cdot - 1)$ of the fractional Brownian motion B_H such that for all t ,

$$B_H(t) = \frac{1}{C_1(H)} \int_{-\infty}^t \left((t-x)_+^{H-1/2} - (-x)_+^{H-1/2} \right) dW_x. \quad (28)$$

Then for all (t, H) ,

$$X_1(t, H) = \int_{-\infty}^t a_1(t-x, H) dW_x$$

where for all $x \geq 0$,

$$a_1(x, H) = \frac{1}{C_1(H)} \left((x)_+^{H-1/2} - (x-1)_+^{H-1/2} \right)$$

so that for all $x > 1$,

$$a_1(x, H) = x^{H-3/2} \frac{(H-1/2)}{C_1(H)} + \rho_1(x, H)$$

with

$$|\rho_1(x, H)| \leq \frac{(H-1/2)(3/2-H)}{C_1(H)x_+^{5/2-H}}.$$

Hence, the field defined X_1 by (27) satisfies (15) with the function $A(H) = (H-1/2)/C_1(H)$.

We give a second example. Consider the field X_2 defined for all t and H by

$$X_2(t, H) = \frac{1}{C_2(H)} \int_{-\infty}^t \left((t-x)_+^{H-1/2} - e^{-(t-x)} \int_0^{t-x} e^{\xi} \xi^{H-1/2} d\xi \right) dW_x \quad (29)$$

where

$$C_2(H) = \sqrt{\int_0^\infty \left(x_+^{H-1/2} - e^{-x} \int_0^x e^{\xi} \xi^{H-1/2} d\xi \right)^2 dx}. \quad (30)$$

For all H , $t \mapsto X_2(t, H)$ is the stationary fractional Ornstein-Uhlenbeck process (see [2] for instance) such that

$$X_2(t, H) = B_H(t) - e^{-t} \int_{-\infty}^t e^\theta B_H(\theta) d\theta$$

where B_H is the fractional Brownian motion defined by (28) with the constant $C_2(H)$ instead of $C_1(H)$. In this case, for all (t, H) ,

$$X_2(t, H) = \int_{-\infty}^t a_2(t-x, H) dW_x$$

where for all $x \geq 0$,

$$\begin{aligned} a_2(x, H) &= \frac{1}{C_2(H)} \left(x_+^{H-1/2} - e^{-x} \int_0^x e^\xi \xi^{H-1/2} d\xi \right) \\ &= \frac{(H-1/2)}{C_2(H) x_+^{3/2-H}} e^{-x} \int_0^x e^\xi \xi^{H-1/2} d\xi \\ &= \frac{(H-1/2)}{C_2(H) x_+^{3/2-H}} - \rho_2(x, H) \end{aligned}$$

with

$$\rho_2(x, H) = \frac{(H-1/2)e^{-x}}{C_2(H)} \left(x_+^{H-3/2} + (H-3/2) \int_0^x (e^\xi - 1) \xi^{H-5/2} d\xi \right),$$

which satisfies

$$|\rho_2(x, H)| \leq \frac{(H-1/2)}{C_2(H)} \left(e^{-x} x_+^{H-3/2} + (3/2-H) \left(e^2 x_+^{H-3/2} + e^{-x} \int_0^2 \frac{e^\xi - 1}{\xi^{5/2-H}} d\xi \right) \right).$$

Then we deduce that the field X_2 defined by (29) satisfies (15).

2.3 Main result

Let $\Phi \in L^2(\mathbb{R}, e^{-x^2/2} dx)$ be a function with Hermite rank $m \in \mathbb{N}^*$, which means that

$$\Phi = \sum_{k=m}^{\infty} \frac{\phi_k}{k!} P_k \tag{31}$$

where the convergence of $\sum_{k=m}^{\infty}$ is defined from the norm $\|\cdot\|_{L^2(\mathbb{R}, e^{-x^2/2} dx)}$ and for all k , P_k is the Hermite polynomial of degree k , namely

$$P_k : x \mapsto P_k(x) = (-1)^k e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2/2}$$

and

$$\phi_k = \langle \Phi, P_k \rangle_{L^2(\mathbb{R}, e^{-x^2/2} dx)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \Phi(x) P_k(x) e^{-x^2/2} dx.$$

Let $h : \mathbb{R}_+ \rightarrow (\frac{1}{2}, 1)$ be a continuous function and for all $\theta \in \mathbb{R}_+$,

$$\tilde{h}(\theta) = 1 + \frac{h(\theta) - 1}{m} \in \left(1 - \frac{1}{2m}, 1\right). \quad (32)$$

We set $h^- = \min h$ and $h^+ = \max h$, and assume that $[h^-, h^+] \subset (\frac{1}{2}, 1)$.

Let X be a Gaussian field defined as in Section 2.1. For all $N \in \mathbb{N}^*$ and $t \in \mathbb{R}_+^*$, we consider

$$\Sigma_{\Phi, h}^N(t) = \sum_{n=1}^{\lfloor Nt \rfloor} \frac{\Phi(X(n, \tilde{h}(n/N^\beta)))}{N h(n/N^\beta)} \quad (33)$$

where $\beta \in \mathbb{R}$. Moreover, in the case $\beta < 1$, we assume that $\lim_{\theta \rightarrow \infty} h(\theta)$ exists and belongs to $(\frac{1}{2}, 1)$. We denote this limit by $h(\infty)$.

The main result states the convergence of the process $\Sigma_{\Phi, h}^N$ as $N \rightarrow \infty$. The proof is postponed to Section 4.

Theorem 1. *As $N \rightarrow \infty$, the process $\Sigma_{\Phi, h}^N$ converges in distribution in $\mathcal{D}([0, \infty), \mathbb{R})$ to the process $\Sigma_{m, h}$ defined by*

$$(\Sigma_{m, h}(t))_{t \geq 0} = \left(\int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t G(\theta, x_1, \dots, x_m) d\theta \right)_{t \geq 0} \quad (34)$$

where $D_m = \{x_1 < x_2 < \dots < x_m\}$, W is a Brownian motion, and for all $(\theta, x_1, \dots, x_m) \in \mathbb{R}_+ \times D_m$,

$$G(\theta, x_1, \dots, x_m) = \begin{cases} \phi_m A(\tilde{h}(0))^m \prod_{k=1}^m (\theta - x_k)_+^{\tilde{h}(0) - \frac{3}{2}} & \text{if } \beta > 1, \\ \phi_m A(\tilde{h}(\theta))^m \prod_{k=1}^m (\theta - x_k)_+^{\tilde{h}(\theta) - \frac{3}{2}} & \text{if } \beta = 1, \\ \phi_m A(\tilde{h}(\infty))^m \prod_{k=1}^m (\theta - x_k)_+^{\tilde{h}(\infty) - \frac{3}{2}} & \text{if } \beta < 1. \end{cases} \quad (35)$$

If $\beta \neq 1$, then the limit $\Sigma_{m, h}$ is a Hermite process as defined in [4, 9] with Hurst index equal to $h(0)$ or $h(\infty)$ depending on the sign of $\beta - 1$.

If $\beta = 1$ and h is a non-constant function, then the limit $\Sigma_{m, h}$ is a multifractional process as those we analyze in Section 3.

We conclude this section with a continuous version of Theorem 1. For all $\varepsilon > 0$ we define the process $\tilde{\Sigma}_{\Phi, h}^\varepsilon$ by

$$\begin{aligned}\tilde{\Sigma}_{\Phi, h}^\varepsilon(t) &= \int_0^t \varepsilon^{h(\varepsilon^{\beta-1}\theta)-1} \Phi(X(\varepsilon^{-1}\theta, \tilde{h}(\varepsilon^{\beta-1}\theta))) d\theta \\ &= \int_0^{t/\varepsilon} \varepsilon^{h(\varepsilon^\beta\theta)} \Phi(X(\theta, \tilde{h}(\varepsilon^\beta\theta))) d\theta.\end{aligned}$$

The following theorem states the convergence of $\tilde{\Sigma}_{\Phi, h}^\varepsilon$ as $\varepsilon \rightarrow 0$.

Theorem 2. *As $\varepsilon \rightarrow 0$, the process $\tilde{\Sigma}_{\Phi, h}^\varepsilon$ converges in distribution in $\mathcal{C}([0, \infty), \mathbb{R})$ to the process $\Sigma_{m, h}$ defined as in Theorem 1.*

The proof of Theorem 2 is omitted because it is similar to that of Theorem 1 in a continuous setting.

3 A multifractional process

In this section we analyze a stochastic process which has the form of $\Sigma_{m, h}$ (see Theorem 1). In particular we prove that it satisfies multifractional properties.

We fix $m \in \mathbb{N}^*$ and for all $t \geq 0$ we define

$$Y(t) = \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t f(\theta) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta)-3/2} d\theta \quad (36)$$

where $D_m = \{x_1 < x_2 < \cdots < x_m\}$, W is a Brownian motion (see [5] for the definition of the multiple Wiener integrals), and $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\varphi : \mathbb{R}_+ \rightarrow (1 - \frac{1}{2m}, 1)$ are continuous functions. The process Y is well defined by Lemma 2 which is stated below and is used throughout the paper.

Lemma 2. *For every $(t, \gamma_1, \dots, \gamma_m) \in \mathbb{R}_+ \times (1 - \frac{1}{2m}, 1)^m$,*

$$\int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t \prod_{k=1}^m (\theta - x_k)_+^{\gamma_k-3/2} d\theta \right)^2 < \infty. \quad (37)$$

Proof. By the Fubini theorem and changes of variables,

$$\begin{aligned}& \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t (\theta - x_k)_+^{\gamma_k-3/2} d\theta \right)^2 \\ &= \int_0^t \int_0^t d\theta d\sigma \int_{\mathbb{R}^m} dx_1 \cdots dx_m \prod_{k=1}^m ((\theta - x_k)_+ (\sigma - x_k)_+)^{\gamma_k-3/2} \\ &\leq \int_0^t \int_0^t d\theta d\sigma |\theta - \sigma|^{2 \sum_j \gamma_j - 2} \prod_{k=1}^m \int_0^\infty dx ((1+x)x)^{\gamma_k-3/2}.\end{aligned} \quad (38)$$

Since $\gamma_k \in (1 - \frac{1}{2m}, 1)$ for all k , the upper bound of (38) is finite. \square

The following lemma states the continuity of Y .

Lemma 3. *The sample paths of Y are continuous almost surely.*

Proof. For all $s < t$ such that $|t - s|$ is small enough,

$$\begin{aligned} & \mathbb{E}((Y(t) - Y(s))^2) \\ &= \int_{D_m} dx_1 \cdots dx_m \left(\int_s^t f(\theta) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta)-3/2} d\theta \right)^2 \\ &\leq \int_s^t d\theta \int_s^t d\sigma \int_{\mathbb{R}^m} dx_1 \cdots dx_m f(\theta) f(\sigma) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta)-3/2} (\sigma - x_k)_+^{\varphi(\sigma)-3/2} \end{aligned}$$

By changes of variables,

$$\begin{aligned} & \mathbb{E}((Y(t) - Y(s))^2) \\ &\leq \sup_{[s,t]} |f|^2 \int_0^1 d\theta \int_0^1 d\sigma (t-s)^{2+m(\varphi((t-s)\theta+s)+\varphi((t-s)\sigma+s)-2)} \\ &\quad \times |\theta - \sigma|^{m(\varphi((t-s)\theta+s)+\varphi((t-s)\sigma+s)-2)} \\ &\quad \times \left(\int_0^\infty dx (1+x)^{\varphi((t-s)\theta+s)-3/2} x^{\varphi((t-s)\sigma+s)-3/2} \right)^m \end{aligned} \quad (39)$$

We deduce that there exists a constant $C > 0$ independent of (s, t) such that

$$\mathbb{E}((Y(t) - Y(s))^2) \leq C \sup_{[s,t]} |f|^2 (t-s)^{2(1+m(\inf_{[s,t]} \varphi - 1))}. \quad (40)$$

Remark that $2(1 + m(\inf_{[s,t]} \varphi - 1)) > 1$. By the Kolmogorov lemma, we deduce that Y has a modification with almost surely continuous sample paths. \square

The multifractional properties of Y are stated in the following two theorems. The first one establishes that Y is locally self-similar.

Theorem 3. *Let $t \geq 0$. If φ is Hölder-continuous then, as $\varepsilon \rightarrow 0$, the process*

$$\left(\frac{Y(t + \varepsilon u) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} \right)_{u \geq 0} \quad (41)$$

converges in distribution to

$$\left(f(t) \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^u \prod_{k=1}^m (\theta - x_k)_+^{\varphi(t)-\frac{3}{2}} d\theta \right)_{u \geq 0} \quad (42)$$

in $\mathcal{C}([0, \infty), \mathbb{R})$.

Proof. For all t and u ,

$$\begin{aligned}
& \frac{Y(t + \varepsilon u) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} \\
&= \varepsilon^{-1-m(\varphi(t)-1)} \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_t^{t+\varepsilon u} f(\theta) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta)-\frac{3}{2}} d\theta \\
&= \varepsilon^{-m(\varphi(t)-1)} \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^u f(\varepsilon\theta + t) \prod_{k=1}^m (\varepsilon\theta + t - x_k)_+^{\varphi(\varepsilon\theta+t)-\frac{3}{2}} d\theta.
\end{aligned}$$

Then, by a change of variable in the stochastic integral, for all t the process defined by (41) is equal in distribution to

$$\left(\int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^u \psi^\varepsilon(t, x, \theta) d\theta \right)_{u \geq 0} \quad (43)$$

where

$$\psi^\varepsilon(t, x, \theta) = \varepsilon^{m(\varphi(\varepsilon\theta+t)-\varphi(t))} f(\varepsilon\theta + t) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\varepsilon\theta+t)-\frac{3}{2}} \quad (44)$$

To prove the convergence of the finite-dimensional distributions, it suffices to show

$$\lim_{\varepsilon \rightarrow 0} \int_{D_m} dx_1 \cdots dx_m \left(\int_0^u d\theta \left(\psi^\varepsilon(t, x, \theta) - f(t) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(t)-\frac{3}{2}} \right) \right)^2 \quad (45)$$

For all (t, x, θ) , by the continuity of f and φ ,

$$\lim_{\varepsilon \rightarrow 0} \left(\psi^\varepsilon(t, x, \theta) - f(t) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(t)-\frac{3}{2}} \right) = 0.$$

Moreover, there exists a constant $C > 0$ such that for all (t, x, θ) ,

$$\begin{aligned}
& \left| \psi^\varepsilon(t, x, \theta) - f(t) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(t)-\frac{3}{2}} \right| \\
& \leq C \prod_{k=1}^m \left((\theta - x_k)_+^{\min \varphi - \frac{3}{2}} - (\theta - x_k)_+^{\max \varphi - \frac{3}{2}} \right)
\end{aligned}$$

Hence, by the bounded convergence theorem and Lemma 2, we obtain (45). It remains

to prove the tightness of the family of processes defined by (41). For all u and v ,

$$\begin{aligned}
& \mathbb{E} \left(\left(\frac{Y(t + \varepsilon u) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} - \frac{Y(t + \varepsilon v) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} \right)^2 \right) \\
&= \varepsilon^{-2-2m(\varphi(t)-1)} \int_{D_m} dx_1 \cdots dx_m \left(\int_{t+\varepsilon v}^{t+\varepsilon u} f(\theta) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta)-\frac{3}{2}} d\theta \right)^2 \\
&= \int_{D_m} dx_1 \cdots dx_m \left(\int_v^u \psi^\varepsilon(t, x, \theta) d\theta \right)^2 \\
&\leq \|f\|_\infty^2 \int_{D_m} dx_1 \cdots dx_m \left(\int_v^u d\theta \prod_{k=1}^m \left((\theta - x_k)_+^{\min \varphi - \frac{3}{2}} + (\theta - x_k)_+^{\max \varphi - \frac{3}{2}} \right) \right)^2.
\end{aligned}$$

Then there exists a constant $C > 0$ independent of (s, t) such that

$$\mathbb{E} \left(\left(\frac{Y(t + \varepsilon u) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} - \frac{Y(t + \varepsilon v) - Y(t)}{\varepsilon^{1+m(\varphi(t)-1)}} \right)^2 \right) \leq C |u - v|^{2(1+m(\inf \varphi - 1))}.$$

The tightness is proved, then Y is locally self-similar. \square

The following results gives the local Hölder exponent.

Theorem 4. *Let $t \geq 0$ and assume that $f(t) \neq 0$. The local Hölder exponent of Y at t is $1 + m(\varphi(t) - 1)$.*

Proof. Since Theorem 3 and by similar arguments to those of the proof of Proposition 10 of [8], it suffices to show that for all $p \in \mathbb{N}^*$ there exists $C_p > 0$ such that for all $s \leq t$ in a neighborhood of t_0 ,

$$\mathbb{E}((Y(t) - Y(s))^{2p}) \leq C_p (t - s)^{2p \inf_{[s, t]} (1+m(\varphi-1))}. \quad (46)$$

This is obtained from (40) and [7]. \square

We conclude this analysis of Y with an important remark. Assume that $f = \alpha \circ \varphi$ where α is a strictly positive function, so that

$$Y(t) = \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t \alpha(\varphi(\theta)) \prod_{k=1}^m (\theta - x_k)_+^{\varphi(\theta)-3/2} d\theta \quad (47)$$

which is the form of the limit $\Sigma_{m,h}$ of Theorem 1 (Section 2.3). For non-constant functions φ , the processes defined by (47) are generally different from those of [6]. This can be shown as follows. As in the proof of Theorem 3 (see also the proof of Proposition 1 of [3]), we can establish that for all $s \neq t$,

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mathbb{E}((Y(t + \varepsilon) - Y(t))(Y(s + \varepsilon) - Y(s))) \\
&= |t - s|^{\varphi(t)+\varphi(s)-2} \alpha(\varphi(s)) \alpha(\varphi(t)) R_Y(\varphi(\max\{s, t\}), \varphi(\min\{s, t\}))
\end{aligned}$$

with for all (H_1, H_2) ,

$$R_Y(H_1, H_2) = \frac{1}{m!} \left(\int_0^\infty (1+x)^{H_1-3/2} x^{H_2-3/2} dx \right)^m.$$

Notice that $R_Y(H_1, H_2) \neq R_Y(H_2, H_1)$ if $H_1 \neq H_2$ (see Section 2.1).

Now consider the process such that for all t ,

$$Z(t) = \int_{\mathbb{R}^m} d\widehat{W}_{\xi_1} \cdots d\widehat{W}_{\xi_m} \int_0^t \beta(\varphi(\theta)) \frac{\exp(i\theta(\xi_1 + \cdots + \xi_m))}{|\xi_1 \cdots \xi_m|^{\varphi(\theta)-1/2}} d\theta \quad (48)$$

where β is a continuous function and \widehat{W} the Fourier transform of W . This is the form of the processes studied in [6]. We can prove that for all $s \neq t$,

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \varepsilon^{-2} \mathbb{E}((Z(t+\varepsilon) - Z(t))(Z(s+\varepsilon) - Z(s))) \\ &= |t-s|^{\varphi(t)+\varphi(s)-2} \beta(\varphi(s))\beta(\varphi(t)) R_Z(\varphi(\max\{s, t\}), \varphi(\min\{s, t\})) \end{aligned}$$

with for all (H_1, H_2) ,

$$R_Z(H_1, H_2) = \left(\int_{-\infty}^\infty \exp(i\xi) |\xi|^{1-H_1-H_2} d\xi \right)^m.$$

The function R_Z is symmetric, in contrast with R_Y . Consider three points t_1, t_2 and $t_3 \in \mathbb{R}_+$ such that $t_1 < t_2 < t_3$, and a function φ such that $\varphi(t_1) = \varphi(t_3) > \varphi(t_2)$. Since R_Z is symmetric,

$$R_Z(\varphi(\max\{t_1, t_2\}), \varphi(\min\{t_1, t_2\})) = R_Z(\varphi(\max\{t_3, t_2\}), \varphi(\min\{t_3, t_2\})). \quad (49)$$

As a consequence of (18),

$$\begin{aligned} R_Y(\varphi(\max\{t_1, t_2\}), \varphi(\min\{t_1, t_2\})) &= R_Y(\varphi(t_2), \varphi(t_1)) \\ &< R_Y(\varphi(t_1), \varphi(t_2)) \\ &= R_Y(\varphi(t_3), \varphi(t_2)) \\ &= R_Y(\varphi(\max\{t_3, t_2\}), \varphi(\min\{t_3, t_2\})), \end{aligned}$$

which differs from (49). Finally, this proves the following result.

Theorem 5. *There exist functions $\varphi \in \mathcal{C}(\mathbb{R}_+, (\frac{1}{2}, 1))$ such that for all $\alpha \in \mathcal{C}((\frac{1}{2}, 1), \mathbb{R}_+^*)$ and $\beta \in \mathcal{C}((\frac{1}{2}, 1), \mathbb{R})$ the processes Y and Z defined by (47) and (48) have different distributions.*

Hence, the processes given by (47) are generally different from those studied in [6].

4 Proof of Theorem 1

In this section we prove Theorem 1. It is a consequence of the tightness (Section 4.1) and of the convergence of the finite-dimensional distributions of $\Sigma_{\Phi, h}^N$ as $N \rightarrow \infty$ (Section 4.2). Technical lemmas are postponed to Section 4.3.

4.1 Tightness

The following lemma establishes the tightness of $(\Sigma_{\Phi,h}^N)_N$ in $\mathcal{D}([0, \infty), \mathbb{R})$.

Lemma 4. *The sequence of processes $(\Sigma_{\Phi,h}^N)_N$ is tight in $\mathcal{D}([0, \infty), \mathbb{R})$.*

Proof. Proceeding as in Section 4.4 of [6] and by Lemma 1, we prove the tightness of $(\Sigma_{\Phi,h}^N)_N$ in $\mathcal{D}([0, \infty), \mathbb{R})$. \square

4.2 Convergence of the finite-dimensional distributions

In this subsection we prove the convergence of the finite-dimensional distributions of $\Sigma_{\Phi,h}^N$ as $N \rightarrow \infty$. For all n and N we set

$$h_n^N := h \left(\frac{n}{N^\beta} \right) \quad \text{and} \quad \tilde{h}_n^N := \tilde{h} \left(\frac{n}{N^\beta} \right).$$

Lemma 5. *Let $d \in \mathbb{N}^*$ and $(t_1, \dots, t_d) \in [0, \infty)^d$. As $N \rightarrow \infty$, $(\Sigma_{\Phi,h}^N(t_1), \dots, \Sigma_{\Phi,h}^N(t_d))$ converges in distribution to $(\Sigma_{m,h}(t_1), \dots, \Sigma_{m,h}(t_d))$ in \mathbb{R}^d .*

Proof. Following Lemma 6, it suffices to prove the convergence of the finite-dimensional distributions of the process Θ_1^N defined for all $t \geq 0$ by

$$\Theta_1^N(t) = \frac{\phi_m}{m!} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} P_m(X(n, \tilde{h}_n^N)) \quad (50)$$

Since P_m is the Hermite polynomial of degree m and $\mathbb{E}(X(t, H)^2) = 1$ for all (t, H) ,

$$P_m(X(t, H)) = m! \int_{D_m(t)} \prod_{k=1}^m a(t - x_k, H) dW_{x_1} \cdots dW_{x_m} \quad (51)$$

where

$$D_m(t) = \{x_1 < x_2 < \cdots < x_m \leq t\}.$$

As a consequence,

$$\begin{aligned} \Theta_1^N(t) &= \frac{\phi_m}{m!} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} P_m(X(n, \tilde{h}_n^N)) \\ &= \phi_m \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} \int_{D_m(n)} \prod_{k=1}^m a(n - x_k, \tilde{h}_n^N) dW_{x_1} \cdots dW_{x_m} \\ &= \phi_m \int_{D_m(n)} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N} \prod_{k=1}^m a(n - x_k, \tilde{h}_n^N). \end{aligned} \quad (52)$$

By a change of variable in the stochastic integral and the self-similarity property of W , Θ_1^N is equal in distribution to Θ_2^N defined for all t by

$$\begin{aligned}\Theta_2^N(t) &= \phi_m \int_{D_m} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N + m/2} 1_{Nx_m < n} \prod_{k=1}^m a(n - Nx_k, \tilde{h}_n^N) \\ &= \phi_m \int_{D_m} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-h_n^N + m/2} \prod_{k=1}^m 1_{Nx_k < n} a(n - Nx_k, \tilde{h}_n^N).\end{aligned}\quad (53)$$

Because of Lemma 10, the convergence of the finite-dimensional distributions of Θ_2^N is equivalent to the convergence of those of Θ_3^N which is defined for all t by

$$\begin{aligned}\Theta_3^N(t) &= \phi_m \int_{D_m} dW_{x_1} \cdots dW_{x_m} \sum_{n=1}^{\lfloor Nt \rfloor} N^{-1} A(\tilde{h}_n^N)^m \prod_{k=1}^m \left(\frac{n}{N} - x_k\right)_+^{\tilde{h}_n^N - \frac{3}{2}} \\ &= \int_{D_m} dW_{x_1} \cdots dW_{x_m} \int_0^t G^N(\theta, x_1, \dots, x_m) d\theta\end{aligned}\quad (54)$$

with for all $\theta \in [0, t)$,

$$\begin{aligned}G^N(\theta, x_1, \dots, x_m) &= \phi_m \sum_{n=1}^{\lfloor Nt \rfloor} 1_{((n-1)/N, n/N]}(\theta) A(\tilde{h}_n^N)^m \prod_{k=1}^m \left(\frac{n}{N} - x_k\right)_+^{\tilde{h}_n^N - \frac{3}{2}} \\ &= \phi_m A(\tilde{h}_{\lceil N\theta \rceil}^N)^m \prod_{k=1}^m \left(\frac{\lceil N\theta \rceil}{N} - x_k\right)_+^{\tilde{h}_{\lceil N\theta \rceil}^N - 3/2}\end{aligned}\quad (55)$$

where $\lceil \cdot \rceil$ is the ceiling function. Finally, by Lemma 11, the proof is completed. \square

4.3 Technical lemmas

In this section we prove Lemmas 6, 10, and 11 which are used in the proof of the convergence of the finite-dimensional distributions of $\Sigma_{\Phi, h}^N$ (Lemma 5). The norm $\|\cdot\|_{L^2(\Omega, \mathbb{P})}$ is denoted by $\|\cdot\|$.

Lemma 6. *For all t , $\lim_{N \rightarrow \infty} \left\| \Sigma_{\Phi, h}^N(t) - \Theta_1^N(t) \right\| = 0$.*

Proof. As a consequence of (16), the proof is similar to that of Lemma 4.5 of [6]. \square

Lemmas 7, 8, and 9 which follow are used in the proof of Lemma 10.

Lemma 7. *For all $\eta > 0$ there exist two positive constants $C(\eta) > 0$ and $K(\eta) > 0$ such that for all $t > K(\eta)$ and $H \in [h^-, h^+]$,*

$$\begin{aligned}& |a(t_1, H) \cdots a(t_m, H) - A(H)^m (t_1 \cdots t_m)^{H - \frac{3}{2}}| \\ & \leq B(t_1 \cdots t_m)^{H - \frac{3}{2}} \sum_{k=1}^m (\eta 1_{t_k > K(\eta)} + C(\eta) 1_{t_k \leq K(\eta)})\end{aligned}\quad (56)$$

with

$$B = \max_{k=1, \dots, m} \left(\max_{(t, H) \in \mathbb{R}_+ \times [h^-, h^+]} |t^{\frac{3}{2}-H} a(t, H)|^{k-1} \times \max_{H \in [h^-, h^+]} |A(H)|^{m-k-1} \right) < \infty.$$

Proof. Since (15), for all $\eta > 0$ there exists $K(\eta) > 0$ such that for all $t > K(\eta)$ and $H \in [h^-, h^+]$,

$$|a(t, H) - A(H)t^{H-\frac{3}{2}}| \leq t^{H-\frac{3}{2}}\eta. \quad (57)$$

Moreover, for all $t \in [0, K(\eta)]$,

$$\begin{aligned} & |a(t, H) - A(H)t^{H-\frac{3}{2}}| \\ & \leq \max_{(t, H) \in [0, K(\eta)] \times [h^-, h^+]} |a(t, H)| + t^{H-\frac{3}{2}} \max_{H \in [h^-, h^+]} |A(H)| \\ & \leq t^{H-\frac{3}{2}}(1 + t^{\frac{3}{2}-H}) \left(\max_{(t, H) \in [0, K(\eta)] \times [h^-, h^+]} |a(t, H)| + \max_{H \in [h^-, h^+]} |A(H)| \right) \\ & \leq C(\eta)t^{H-\frac{3}{2}} \end{aligned} \quad (58)$$

with

$$C(\eta) = (1 + K(\eta)^{\frac{3}{2}-H}) \left(\max_{(t, H) \in [0, K(\eta)] \times [h^-, h^+]} |a(t, H)| + \max_{H \in [h^-, h^+]} |A(H)| \right).$$

From (57) and (58) we deduce that for all $t \geq 0$ and $H \in [h^-, h^+]$,

$$|a(t, H) - A(H)t^{H-\frac{3}{2}}| \leq t^{H-\frac{3}{2}}(\eta 1_{t > K(\eta)} + C(\eta)1_{t \leq K(\eta)}). \quad (59)$$

Hence, for all (t_1, \dots, t_m, H) ,

$$\begin{aligned} & |a(t_1, H) \cdots a(t_m, H) - A(H)^m (t_1 \cdots t_m)^{H-\frac{3}{2}}| \\ & \leq \sum_{k=1}^m |a(t_1, H) \cdots a(t_{k-1}, H)(a(t_k, H) - A(H)t_k^{H-\frac{3}{2}})A(H)^{m-k-1}(t_{k+1} \cdots t_m)^{H-\frac{3}{2}}| \\ & \leq (t_1 \cdots t_m)^{H-\frac{3}{2}} \sum_{k=1}^m \left| \left(\prod_{j=1}^{k-1} t_j^{H-\frac{3}{2}} a(t_j, H) \right) (\eta 1_{t_k > K(\eta)} + C(\eta)1_{t_k \leq K(\eta)}) A(H)^{m-k-1} \right| \\ & \leq B(t_1 \cdots t_m)^{H-\frac{3}{2}} \sum_{k=1}^m (\eta 1_{t_k > K(\eta)} + C(\eta)1_{t_k \leq K(\eta)}) \end{aligned}$$

which completes the proof. \square

Lemma 8. Let $t \geq 0$. For all $(\theta, x) \in [0, t] \times \mathbb{R}$,

$$\lim_{N \rightarrow \infty} \left(\frac{[N\theta]}{N} - x \right)_+^{\tilde{h}_{[N\theta]}^N - 3/2} = \begin{cases} (\theta - x)_+^{\tilde{h}^{(\theta)} - 3/2} & \text{if } \beta = 1, \\ (\theta - x)_+^{\tilde{h}^{(\infty)} - 3/2} & \text{if } \beta < 1, \\ (\theta - x)_+^{\tilde{h}^{(0)} - 3/2} & \text{if } \beta > 1. \end{cases} \quad (60)$$

Moreover, for all $(N, \theta, x) \in \mathbb{N}^* \times [0, t] \times \mathbb{R}$,

$$\left(\frac{[N\theta]}{N} - x \right)_+^{\tilde{h}_{[N\theta]}^N - \frac{3}{2}} \leq \begin{cases} (\theta - x)_+^{\sup_{[0,t]} \tilde{h} - \frac{3}{2}} + (\theta - x)_+^{\inf_{[0,t]} \tilde{h} - \frac{3}{2}} & \text{if } \beta \geq 1, \\ (\theta - x)_+^{\sup_{[0,\infty)} \tilde{h} - \frac{3}{2}} + (\theta - x)_+^{\inf_{[0,\infty)} \tilde{h} - \frac{3}{2}} & \text{if } \beta < 1. \end{cases} \quad (61)$$

Proof. For all θ and N ,

$$\theta \leq \frac{[N\theta]}{N} < \theta + \frac{1}{N}.$$

Then (60) follows from the continuity of h . Moreover, for all x ,

$$(\theta - x)_+ \leq \left(\frac{[N\theta]}{N} - x \right)_+.$$

Since $\tilde{h}_{[N\theta]}^N - \frac{3}{2} < \frac{1}{2} < 0$,

$$\left(\frac{[N\theta]}{N} - x \right)_+^{\tilde{h}_{[N\theta]}^N - \frac{3}{2}} \leq (\theta - x)_+^{\tilde{h}_{[N\theta]}^N - \frac{3}{2}}$$

and then we get (61). \square

Lemma 9. Let $t \geq 0$ and F_t be the function such that for all $(\theta, x_1, \dots, x_m) \in [0, t] \times \mathbb{R}^m$,

$$F_t(\theta, x_1, \dots, x_m) = \begin{cases} \prod_{k=1}^m ((\theta - x_k)_+^{\sup_{[0,t]} \tilde{h} - \frac{3}{2}} + (\theta - x_k)_+^{\inf_{[0,t]} \tilde{h} - \frac{3}{2}}) & \text{if } \beta \geq 1, \\ \prod_{k=1}^m ((\theta - x_k)_+^{\sup_{[0,\infty)} \tilde{h} - \frac{3}{2}} + (\theta - x_k)_+^{\inf_{[0,\infty)} \tilde{h} - \frac{3}{2}}) & \text{if } \beta < 1. \end{cases} \quad (62)$$

We have

$$\int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t F_t(\theta, x_1, \dots, x_m) d\theta \right)^2 < \infty. \quad (63)$$

Proof. It is a consequence of Lemma 2. \square

Lemma 10. For all t , $\lim_{N \rightarrow \infty} \|\Theta_2^N(t) - \Theta_3^N(t)\| = 0$.

Proof. Fix $\eta > 0$. By Lemma 7,

$$\begin{aligned} & \sum_{n=1}^{[Nt]} \left| N^{-h_n^N + m/2} \prod_{k=1}^m a((n - Nx_k)_+, \tilde{h}_n^N) - \frac{1}{N} A(\tilde{h}_n^N)^m \prod_{k=1}^m \left(\frac{n}{N} - x_k \right)_+^{\tilde{h}_n^N - \frac{3}{2}} \right| \\ & \leq \frac{1}{N} \sum_{n=1}^{[Nt]} \prod_{k=1}^m \left(\frac{n}{N} - x_k \right)_+^{\tilde{h}_n^N - \frac{3}{2}} \sum_{j=1}^m (\eta 1_{n - Nx_j \geq K(\eta)} + C(\eta) 1_{n - Nx_j < K(\eta)}) \\ & = \int_0^t d\theta \prod_{k=1}^m \left(\frac{[N\theta]}{N} - x_k \right)_+^{\tilde{h}_{[N\theta]}^N - \frac{3}{2}} \sum_{j=1}^m (\eta 1_{[N\theta]/N - x_j \geq K(\eta)/N} + C(\eta) 1_{[N\theta]/N - x_j < K(\eta)/N}) \end{aligned}$$

As a consequence,

$$\|\Theta_2^N(t) - \Theta_3^N(t)\|^2 \leq 2(\phi_m)^2 (B_1^N(t) + B_2^N(t))$$

with

$$B_1^N(t) = \eta^2 \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t d\theta \sum_{j=1}^m 1_{\lceil N\theta \rceil / N - x_j \geq K(\eta)/N} \prod_{k=1}^m \left(\frac{\lceil N\theta \rceil}{N} - x_k \right)_+^{\tilde{h}_{\lceil N\theta \rceil}^N - \frac{3}{2}} \right)^2$$

and

$$B_2^N(t) = C(\eta)^2 \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t d\theta \sum_{j=1}^m 1_{\lceil N\theta \rceil / N - x_j < K(\eta)/N} \prod_{k=1}^m \left(\frac{\lceil N\theta \rceil}{N} - x_k \right)_+^{\tilde{h}_{\lceil N\theta \rceil}^N - \frac{3}{2}} \right)^2$$

By Lemmas 8 and 9 and the bounded convergence theorem, for all $\eta > 0$,

$$\limsup_{N \rightarrow \infty} \|\Theta_2^N(t) - \Theta_3^N(t)\|^2 \leq 2\eta^2 (\phi_m)^2 \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t F_t(\theta, x_1, \dots, x_m) d\theta \right)^2.$$

This completes the proof. \square

Lemma 11. For all t , $\lim_{N \rightarrow \infty} \|\Theta_3^N(t) - \Sigma_{m,h}(t)\| = 0$.

Proof. For all t ,

$$\|\Theta_3^N(t) - \Sigma_{m,h}(t)\| \leq \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t |G^N(\theta, x_1, \dots, x_m) - G(\theta, x_1, \dots, x_m)| d\theta \right)^2. \quad (64)$$

Since (60), for almost every $(\theta, x_1, \dots, x_m)$,

$$\lim_{N \rightarrow \infty} |G^N(\theta, x_1, \dots, x_m) - G(\theta, x_1, \dots, x_m)| = 0.$$

Moreover, by (61), there exists $C > 0$ such that for almost every $(\theta, x_1, \dots, x_m)$,

$$|G^N(\theta, x_1, \dots, x_m) - G(\theta, x_1, \dots, x_m)| \leq CF_t(\theta, x_1, \dots, x_m)$$

where F_t is defined by (62). Because of Lemma 9, by the bounded convergence theorem, we deduce that

$$\lim_{N \rightarrow \infty} \int_{\mathbb{R}^m} dx_1 \cdots dx_m \left(\int_0^t |G^N(\theta, x_1, \dots, x_m) - G(\theta, x_1, \dots, x_m)| d\theta \right)^2 = 0,$$

which completes the proof with (64). \square

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