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# TOWARDS CANONICAL REPRESENTATIONS OF FINITE HEISENBERG GROUPS

S. LYSENKO

ABSTRACT. We consider a finite abelian group  $M$  of odd exponent  $n$  with a symplectic form  $\omega : M \times M \rightarrow \mu_n$  and the Heisenberg extension  $1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$  with the commutator  $\omega$ . According to the Stone - von Neumann theorem,  $H$  admits an irreducible representation with the tautological central character (defined up to a non-unique isomorphism). We construct such irreducible representation of  $H$  defined up to a unique isomorphism, so canonical in this sense.

## 1. INTRODUCTION

1.0.1. Consider a finite abelian group  $M$  of odd exponent  $n$  with a symplectic form  $\omega : M \times M \rightarrow \mu_n$ . It admits a unique symmetric Heisenberg extension  $1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$  with the commutator  $\omega$ . According to the Stone - von Neumann theorem,  $H$  admits an irreducible representation with the tautological central character (defined up to a non-unique isomorphism). We construct such irreducible representation of  $H$  defined up to a unique isomorphism, so canonical in this sense, over a suitable finite extension of  $\mathbb{Q}$ .

1.0.2. We are motivated by the following question of Dennis Gaitsgory about [6]. Let  $X$  be a smooth projective connected curve over an algebraically closed field  $k$ . Let  $T$  be a torus over  $k$  with a geometric metaplectic data  $\mathcal{G}$  as in [3]. To fix ideas, consider the sheaf-theoretic context of  $\ell$ -adic sheaves on finite type schemes over  $k$ . Let  $(H, \mathcal{G}_{Z_H}, \epsilon)$  be the metaplectic Langlands dual datum associated to  $(T, \mathcal{G})$  in ([3], Section 6), so  $H$  is a torus over  $\bar{\mathbb{Q}}_\ell$  isogenous to the Langlands dual of  $T$ . Let  $\sigma$  be a twisted local system on  $X$  for  $(H, \mathcal{G}_{Z_H})$  in the sense of ([3], Section 8.4). To this data in ([3], Section 9.5.3) we attached the DG-category of Hecke eigensheaves. The question is whether this category identifies canonically with the DG-category  $\text{Vect}$  of  $\bar{\mathbb{Q}}_\ell$ -vector spaces. In [6] we constructed such an irreducible Hecke eigensheaf for  $\sigma$  out of a given irreducible representation of certain finite Heisenberg group (denoted by  $\Gamma$  given by formula (33) in [6], Section 5.2.4) with the tautological central character.

This is why we are interested in constructing a canonical irreducible representation of finite Heisenberg groups as in Section 1.0.1. We do this only assuming the order of  $M$  odd, the case of even order remains open.

## 2. MAIN RESULT

2.0.1. Let  $e$  be an algebraically closed field of characteristic zero. Let  $M$  be a finite abelian group,  $\omega : M \times M \rightarrow e^*$  a bilinear form, which is alternating, that is,  $\omega(m, m) =$

0 for any  $m \in M$ . Assume the induced map  $M \rightarrow \text{Hom}(M, e^*)$  is an isomorphism, that is, the form is nondegenerate.

If  $L \subset M$  is a subgroup,  $L^\perp = \{m \in M \mid \omega(m, l) = 0 \text{ for all } l \in L\}$  is its orthogonal complement, this is a subgroup. The group  $L$  is isotropic if  $L \subset L^\perp$ . The subgroup  $L$  is lagrangian if  $L^\perp = L$ . For a lagrangian subgroup we get an exact sequence

$$(1) \quad 0 \rightarrow L \rightarrow M \rightarrow L^* \rightarrow 0,$$

where  $L^* = \text{Hom}(L, e^*)$ . Namely, we send  $m \in M$  to the character  $l \mapsto \omega(m, l)$  of  $L$ . This exact sequence always admits a splitting  $L^* \rightarrow M$ , which is a homomorphism, see for example ([7], 4.1). For such a splitting after the obtained identification  $M \xrightarrow{\sim} L \times L^*$  the form  $\omega$  becomes

$$(2) \quad \omega((l_1, \chi_1), (l_2, \chi_2)) = \frac{\chi_1(l_2)}{\chi_2(l_1)}$$

for  $l_i \in L, \chi_i \in L^*$ .

By ([7], Theorem 2), up to an isomorphism, there is a unique central extension

$$(3) \quad 1 \rightarrow e^* \rightarrow H_{e^*} \rightarrow M \rightarrow 1$$

with the commutator  $\omega$ . We are interested in understanding the category of representations of  $H_{e^*}$  with the tautological central character.

2.0.2. For a finite abelian group  $L$  its exponent is the least common multiple of the orders of the elements of  $L$ . Let  $n$  be the exponent of  $M$ , this is a divisor of  $\sqrt{|M|} \in \mathbb{N}$ .

Let  $\mu_n = \mu_n(e)$ . Let us be given a central extension

$$(4) \quad 1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$$

together with a symmetric structure  $\sigma$  in the sense of ([1], Section 1.1) and commutator  $\omega$ . That is,  $\sigma$  is an automorphism of  $H$  such that  $\sigma^2 = \text{id}$ ,  $\sigma|_{\mu_n} = \text{id}$ , and  $\sigma \bmod \mu_n$  is the involution  $m \mapsto -m$  of  $M$ .

From now on assume  $n$  odd. Then by ([7], Section 1), there is a unique symmetric central extension (4) up to a unique isomorphism. Besides, (3) is isomorphic to the push-out of (4) under the tautological character  $\iota : \mu_n \hookrightarrow e^*$ .

The extension  $H$  is constructed as follows. Let  $\beta : M \times M \rightarrow \mu_n$  be the unique alternating bilinear form such that  $\beta^2 = \omega$ . We take  $H = M \times \mu_n$  with the product

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 a_2 \beta(m_1, m_2))$$

for  $m_i \in M, a_i \in \mu_n$ . Then  $\sigma(m, a) = (-m, a)$  for  $m \in M, a \in \mu_n$ .

Let  $G = \text{Sp}(M)$ , the group of automorphisms of  $M$  preserving  $\omega$ . Let  $g \in G$  act on  $H$  sending  $(m, a)$  to  $(gm, a)$ . This gives the semi-direct product  $H \rtimes G$ .

2.0.3. The following version of the Stone - von Neumann theorem holds for  $H$ , the proof is left to a reader.

**Proposition 2.0.4.** *Up to an isomorphism, there is a unique irreducible representation of  $H$  over  $e$  with the tautological central character  $\iota : \mu_n \hookrightarrow e^*$ .*

2.0.5. Write  $\mathcal{L}(M)$  for the set of lagragian subgroups in  $M$ . For  $L \in \mathcal{L}(M)$  let  $\bar{L}$  be the preimage of  $L$  in  $H$ , this is a subgroup. If  $\chi_L : \bar{L} \rightarrow e^*$  is a character extending the tautological character  $\iota : \mu_b \hookrightarrow e^*$ , set

$$\mathcal{H}_L = \{f : H \rightarrow e \mid f(\bar{l}h) = \chi_L(\bar{l})f(h), \text{ for } \bar{l} \in \bar{L}, h \in H\}$$

This is a representation of  $H$  by right translations. It is irreducible with central character  $\iota$ .

2.0.6. We study the following.

**Problem:** Describe the category  $\text{Rep}_\iota(H)$  of representations of  $H$  over  $e$  with central character  $\iota : \mu_b \hookrightarrow e^*$ . Is there an object of  $\text{Rep}_\iota(H)$ , which is irreducible and defined up to a unique isomorphism? (If yes, it would provide an equivalence between  $\text{Rep}_\iota(H)$  and the category of  $e$ -vector spaces).

2.0.7. Let  $I$  be the set of primes appearing in the decomposition of  $n$ , write  $n = \prod_{p \in I} p^{r(p)}$  with  $r(p) > 0$ . Let  $K \subset e$  be the subfield generated over  $\mathbb{Q}$  by  $\{\sqrt{p} \mid p \in I\}$  and  $\mu_n$ .

**Theorem 2.0.8.** *There is an irreducible representation  $\pi$  of  $H$  over  $K$  with central character  $\iota : \mu_n \hookrightarrow K^*$  defined up to a unique isomorphism. The  $H$ -action on  $\pi$  extends naturally to an action of  $H \rtimes G$ .*

**Remark 2.0.9.** *Let  $K' \subset e$  be the subfield generated over  $\mathbb{Q}$  by  $\mu_n$ . The field of definition of the character of  $\pi$  is  $K'$ . However, we do not expect that for any  $H$  with  $n$  odd Theorem 2.0.8 holds already with  $K$  replaced by  $K'$ , but we have not checked that. Our choices of  $\sqrt{p}$  for  $p \in I$  are made to use the results of [5], and the formulas from [5] do not work without these choices. Note also the following. If  $L$  is an odd abelian group, and  $b : L \times L \rightarrow e^*$  is a nondegenerate symmetric bilinear form then the Gauss sum of  $b$  is defined as*

$$G(L, b) = \sum_{l \in L} b(l, l)$$

*Using the classification of such symmetric bilinear forms given in [4], one checks that  $G(L, b)^4 = |L|^2$ . Since the construction of  $\pi$  in Theorem 2.0.8 is related to representing the corresponding 2-cocyle (given essentially by certain Gauss sums) as a coboundary (after some minimal additional choices), we expect that our choices of  $\sqrt{p}$  for  $p \in I$  are necessary.*

**Remark 2.0.10.** *For  $L \in \mathcal{L}(M)$ , the  $H$ -representation  $\mathcal{H}_L$  from Section 2.0.5 is defined over  $K$ . We sometimes view it as a representation over  $K$ , the precise meaning is hopefully clear from the context.*

### 3. PROOF OF THEOREM 2.0.8

3.0.1. *Reduction.* For  $p \in I$  let

$$H_p = \{h \in H \mid h^{(p^s)} = 1 \text{ for } s \text{ large enough}\}$$

and similarly for  $M_p$ . Then  $H_p \subset H$  is a subgroup that fits into an exact sequence  $1 \rightarrow \mu_{p^{r(p)}} \rightarrow H_p \rightarrow M_p \rightarrow 1$ , and  $H = \prod_{p \in I} H_p$ , product of groups. Indeed,  $\omega(H_p, H_q) = 1$

for  $p, q \in I, p \neq q$ . Besides,  $\sigma$  preserves  $H_p$  for each  $p \in I$ , so  $(H_p, \sigma|_{H_p})$  is a symmetric Heisenberg extension of  $(M_p, \omega_p)$  by  $\mu_{p^{r(p)}}$ . Here  $\omega_p : M_p \times M_p \rightarrow \mu_{p^{r(p)}}$  is the restriction of  $\omega$ . So, Problem 2.0.6 reduces to the case of a prime  $n$ . If  $M_p = 0$  then take  $\pi_p$  be the 1-dimensional representation given by the tautological character  $\mu_{p^{r(p)}} \hookrightarrow e^*$ .

For  $p \in I$  odd let  $K_p \subset e$  be the subfield generated over  $\mathbb{Q}$  by  $\mu_{p^{r(p)}}$  and  $\sqrt{p}$ . We prove Theorem 2.0.8 in the case of an odd prime  $n$  getting for  $p \in I$  a representation  $\pi_p$  of  $H_p$  over  $K_p$ , hence over  $K$  also. Then for any odd  $n$ ,  $\pi = \otimes_{p \in I} \pi_p$  is the desired representation.

3.0.2. From now on we assume  $n = p^r$  for an odd prime  $p$ .

3.1. **Case  $r = 1$ .**

3.1.1. In this section we assume  $M$  is a  $\mathbb{F}_p$ -vector space of dimension  $2d$ . To apply the results of [5], pick an isomorphism  $\psi : \mathbb{F}_p \rightarrow \mu_p$ . It allows to identify  $H$  with  $M \times \mathbb{F}_p$ . We then view  $\mathcal{L}(M), H$  as algebraic varieties over  $\mathbb{F}_p$ . We allow the case  $d = 0$  also.

3.1.2. Recall the following construction from ([5], Theorem 1).<sup>1</sup>

Pick a prime  $\ell \neq p$ , and an algebraic closure  $\bar{\mathbb{Q}}_\ell$  of  $\mathbb{Q}_\ell$ . We assume  $\bar{\mathbb{Q}}_\ell$  is chosen in such a way that  $K \subset \bar{\mathbb{Q}}_\ell$  is a subfield. In particular, we get  $\sqrt{p} \in K \subset \bar{\mathbb{Q}}_\ell$ . It gives rise to the  $\mathbb{Q}_\ell$ -sheaf  $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$  over  $\text{Spec } \mathbb{F}_p$ .

Pick a 1-dimensional  $\mathbb{F}_p$ -vector space  $\mathcal{J}$  of parity  $d \pmod 2$  as  $\mathbb{Z}/2\mathbb{Z}$ -graded. Let  $\mathcal{A}$  be the line bundle (of parity zero as  $\mathbb{Z}/2\mathbb{Z}$ -graded) on  $\mathcal{L}(M)$  with fibre  $\mathcal{J} \otimes \det L$  at  $L \in \mathcal{L}(M)$ . Write  $\tilde{\mathcal{L}}(M)$  for the gerbe of square roots of  $\mathcal{A}$ .

In *loc.cit* we have constructed an irreducible perverse sheaf  $F$  on  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ . Though in *loc.cit* we mostly worked over an algebraic closure  $\bar{\mathbb{F}}_p$ ,  $F$  is defined over  $\mathbb{F}_p$ .

**Lemma 3.1.3.** *For any  $i : \text{Spec } \mathbb{F}_p \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ ,  $\text{tr}(\text{Fr}, i^*F) \in K$ . Here  $\text{Fr}$  is the geometric Frobenius endomorphism.*

*Proof.* This follows from formula (10) in ([5], Section 3.3). Namely, after a surjective smooth localization (a choice of an additional lagrangian in  $M$ ), there is an explicit formula for  $F$  as the convolution along  $H$  of two explicit rank one local systems. Their traces of Frobenius lie in  $K$ , as their definition involves only the Artin-Schreier sheaf and Tate twists. So, the same holds after the convolution along the finite group  $H(\mathbb{F}_p)$ .  $\square$

3.1.4. For an algebraic stack  $S \rightarrow \text{Spec } \mathbb{F}_p$  we write  $S(\mathbb{F}_p)$  for the set of isomorphism classes of its  $\mathbb{F}_p$ -points. In view of the isomorphism  $\psi : \mathbb{F}_p \xrightarrow{\sim} \mu_p$  fixed above, for  $L \in \mathcal{L}(M)(\mathbb{F}_p)$  we identify  $\bar{L} = L \times \mu_p$  with  $L \times \mathbb{F}_p$ . Let

$$F^{cl} : \tilde{\mathcal{L}}(M)(\mathbb{F}_p) \times \tilde{\mathcal{L}}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p) \rightarrow K$$

be the function trace of Frobenius of  $F$ .

For  $L \in \mathcal{L}(M)(\mathbb{F}_p)$  its preimage in  $\tilde{\mathcal{L}}(M)(\mathbb{F}_p)$  consists of two elements. We let  $\mu_2$  act on  $\tilde{\mathcal{L}}(M)(\mathbb{F}_p)$  over  $\mathcal{L}(M)(\mathbb{F}_p)$  permuting the elements in the preimage of each  $L \in \mathcal{L}(M)(\mathbb{F}_p)$ . We call a function  $h : \tilde{\mathcal{L}}(M)(\mathbb{F}_p) \rightarrow K$  genuine if it changes by the

<sup>1</sup>For this construction we adopt the conventions of *loc.cit* about  $\mathbb{Z}/2\mathbb{Z}$ -gradings and étale  $\bar{\mathbb{Q}}_\ell$ -sheaves on schemes over  $\mathbb{F}_p$ .

nontrivial character of  $\mu_2$  under this  $\mu_2$ -action. Recall that  $F^{cl}$  is genuine with respect to the first and the second variable.

Let us write  $L^0$  for a point of  $\tilde{\mathcal{L}}(M)(\mathbb{F}_p)$  over  $L \in \mathcal{L}(M)(\mathbb{F}_p)$ . As in ([5], Section 2), for  $L^0, N^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$  viewing  $\mathcal{H}_L, \mathcal{H}_N$  as  $H$ -representations over  $K$ , we define the canonical intertwining operator

$$F_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$$

by

$$(F_{N^0, L^0} f)(h_1) = \int_{h_2 \in H} F_{N^0, L^0}(h_1 h_2^{-1}) f(h_2) dh_2,$$

where our measure  $dh_2$  is normalized by requiring that the volume of a point is one.

Let  $G = \mathrm{Sp}(M)$  viewed as an algebraic group over  $\mathbb{F}_p$ . It acts naturally on  $\mathcal{L}(M), H$ , and  $\tilde{\mathcal{L}}(M)$ . By definition, for  $g \in G, (m, a) \in H, g(m, a) = (gm, a)$  for  $m \in M, a \in \mathbb{F}_p$ , and this action preserves the symmetric structure  $\sigma$  on  $H$ . If  $g \in G, f : H \rightarrow K$  then  $gf : H \rightarrow K$  is given by  $(gf)(h) = f(g^{-1}h)$ . Then  $g \in G(\mathbb{F}_p)$  yields an isomorphism  $\mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$ . We let  $G$  act diagonally on  $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$ .

The above intertwining operators satisfy the following properties

- $F_{L^0, L^0} = \mathrm{id}$ ;
- $F_{R^0, N^0} \circ F_{N^0, L^0} = F_{R^0, L^0}$  for any  $R^0, N^0, L^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$ ;
- for any  $g \in G(\mathbb{F}_p)$  we have  $g \circ F_{N^0, L^0} \circ g^{-1} = F_{gN^0, gL^0}$ .

**Definition 3.1.5.** Let  $\pi$  be the  $K$ -vector space of collections  $f_{L^0} \in \mathcal{H}_L$  for  $L^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$  satisfying the property: for  $N^0, L^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$  one has

$$F_{N^0, L^0}(f_{L^0}) = f_{N^0}$$

This is our canonical  $H$ -representation over  $K$ .

We let  $G(\mathbb{F}_p)$  act on  $\tilde{\mathcal{L}}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p)$  diagonally. This yields a  $G(\mathbb{F}_p)$ -action on  $\pi$  sending  $\{f_{L^0}\} \in \pi$  to the collection  $L^0 \mapsto g(f_{g^{-1}L^0})$ .

### 3.2. Case $r \geq 1$ .

3.2.1. Let  $L$  be a finite abelian group,  $p$  be any prime number. For  $k \geq 0$  let  $L[p^k] = \{l \in L \mid p^k l = 0\}$  and

$$\rho_k(L) = L[p^k]/(L[p^{k-1}] + pL[p^{k+1}])$$

Each  $\rho_k(L)$  is a vector space over  $\mathbb{F}_p$ . Note that

$$\rho_k(\mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbb{Z}/p\mathbb{Z}, & m = k \\ 0, & \text{otherwise} \end{cases}$$

For finite abelian groups  $L, L'$  one has canonically  $\rho_k(L \times L') \xrightarrow{\sim} \rho_k(L) \times \rho_k(L')$ .

3.2.2. *Canonical isotropic subgroup.* Let  $p$  be any prime,  $M$  is a finite abelian  $p$ -group of exponent  $n = p^r$  with an alternating nondegenerate bilinear form  $\omega : M \times M \rightarrow \mu_n$ . We first construct by induction a canonical isotropic subgroup  $S \subset M$  such that  $\text{Aut}(M)$  fixes  $S$  and  $S^\perp/S$  is a  $\mathbb{F}_p$ -vector space.

Write the set  $\{r > 0 \mid \rho_r(M) \neq 0\}$  as  $\{r_1, \dots, r_s\}$  with  $0 < r_1 < r_2 < \dots < r_s$ . There is an orthogonal direct sum  $(M, \omega) \xrightarrow{\sim} \bigoplus_{i=1}^s (M_i, \omega_i)$ , where  $\omega_i : M_i \times M_i \rightarrow \mu_n$  is an alternating nondegenerate bilinear form, and  $M_i$  is a free  $\mathbb{Z}/p^{r_i}$ -module of finite rank.

Let

$$r' = \begin{cases} \frac{r_s}{2}, & r_s \text{ is even} \\ \frac{r_s+1}{2}, & r_s \text{ is odd} \end{cases}$$

Set  $S_1 = p^{r'}M$ . Since  $\omega$  takes values in  $\mu_{p^{r_s}}$ ,  $S_1$  is isotropic and fixed by  $\text{Aut}(M)$ . By induction hypothesis, we have a canonical isotropic subgroup  $S' \subset M_1 := S_1^\perp/S_1$  such that  $S'^\perp/S'$  is a  $\mathbb{F}_p$ -vector space, where  $S'^\perp$  denotes the orthogonal complement of  $S'$  in  $M_1$ . Let  $S$  be the preimage of  $S'$  under  $S_1^\perp \rightarrow M_1$ . This is our canonical isotropic subgroup in  $M$ .

Set  $M_c = S^\perp/S$ , it is equipped with the induced alternating nondegenerate bilinear form  $\omega_c : M_c \times M_c \rightarrow \mu_p$ , the subscript  $c$  stands for ‘canonical’.

3.2.3. We keep the assumptions of Theorem 2.0.8, so  $p$  is odd. View  $S$  as a subgroup of  $H$  via  $s \mapsto (s, 0) \in H$  for  $s \in S$ . Let  $H^S = S^\perp \times \mu_n$ , this is a subgroup of  $H$ . Since  $S$  lies in the kernel of  $\beta : S^\perp \times S^\perp \rightarrow \mu_n$ , we get the alternating nondegenerate bilinear form  $\beta_c : M_c \times M_c \rightarrow \mu_p$  given by  $\beta_c(m_1, m_2) = \beta(\tilde{m}_1, \tilde{m}_2)$  for  $\tilde{m}_i \in S^\perp$  over  $m_i$ .

Set  $H_c = M_c \times \mu_p$  with the product

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 a_2 \beta_c(m_1 m_2))$$

This is a central extension  $1 \rightarrow \mu_p \rightarrow H_c \rightarrow M_c \rightarrow 1$  with the commutator  $\omega_c$  and the symmetric structure  $\sigma_c(m, a) = (-m, a)$  for  $(m, a) \in H_c$ .

Let  $\alpha_S : H^S \rightarrow H_c$  be the homomorphism sending  $(m, a)$  to  $(m \bmod S, a)$  for  $m \in S^\perp$ , its kernel is  $S$ .

As in Section 3.1, we get the algebraic stack  $\tilde{\mathcal{L}}(M_c), \mathcal{L}(M_c), H_c$  over  $\mathbb{F}_p$ . Let  $G = \text{Sp}(M, \omega)$  be the group of automorphisms of  $M$  preserving  $\omega$ , this is a finite group. We let  $g \in G$  act on  $H$  sending  $(m, a)$  to  $(gm, a)$ . Let  $g \in G$  act on functions  $f : H \rightarrow K$  by  $(gf)(h) = f(g^{-1}h)$  for  $h \in H$ . For  $L \in \mathcal{L}(M)$  this yields an isomorphism  $g : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$  of  $K$ -vector spaces.

Since  $G$  preserves  $S^\perp$ , we have the homomorphism  $G \rightarrow G_c := \text{Sp}(M_c)(\mathbb{F}_p)$ . Via this map,  $G$  acts on  $\mathcal{L}(M_c)(\mathbb{F}_p), \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p), H_c$ .

3.2.4. We denote elements of  $\mathcal{L}(M_c)$  by a capital letter with a subscript  $c$ . For  $L_c \in \mathcal{L}(M_c)$  let  $L \in \mathcal{L}(M)$  denote the preimage of  $L_c$  under  $S^\perp \rightarrow M_c$ .

For  $L_c \in \mathcal{L}(M_c)$  we have the representation  $\mathcal{H}_{L_c}$  of  $H_c$  over  $K$  defined in Section 3.1.4, and the  $H$ -representation  $\mathcal{H}_L$  over  $K$  defined in Section 2.0.5.

For  $L_c \in \mathcal{L}(M_c)$  any  $f$  in the space of invariants  $\mathcal{H}_L^S$  is the extension by zero under  $H^S \hookrightarrow H$ . The space  $\mathcal{H}_L^S$  is naturally a  $H_c$ -module. We get an isomorphism of  $H_c$ -modules  $\tau_{L_c} : \mathcal{H}_{L_c} \xrightarrow{\sim} \mathcal{H}_L^S$  sending  $f$  to the composition  $H^S \xrightarrow{\alpha_S} H_c \xrightarrow{f} K$  extended by zero to  $H$ .

For  $g \in G$ ,  $L_c \in \mathcal{L}(M_c)$  the isomorphism  $g : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$  yields an isomorphism  $g : \mathcal{H}_L^S \xrightarrow{\sim} \mathcal{H}_{gL}^S$  of  $S$ -invariants.

3.2.5. Given  $L_c^0, N_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ , we define a canonical intertwining operator

$$(5) \quad \mathcal{F}_{N_c^0, L_c^0} : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_N$$

as the unique isomorphism of  $H$ -modules such that the diagram commutes

$$\begin{array}{ccc} \mathcal{H}_L^S & \xrightarrow{\mathcal{F}_{N_c^0, L_c^0}} & \mathcal{H}_N^S \\ \uparrow \tau_{L_c} & & \uparrow \tau_{N_c} \\ \mathcal{H}_{L_c} & \xrightarrow{F_{N_c^0, L_c^0}} & \mathcal{H}_{N_c} \end{array}$$

Here  $F_{N_c^0, L_c^0}$  are the canonical intertwining operators from Section 3.1.4. The properties of the canonical intertwining operators of Section 3.1.4 imply the following properties of (5):

- $\mathcal{F}_{L_c^0, L_c^0} = \text{id}$  for  $L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ ;
- for  $R_c^0, N_c^0, L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$  one has

$$\mathcal{F}_{R_c^0, N_c^0} \circ F_{N_c^0, L_c^0} = F_{R_c^0, L_c^0}$$

- for  $g \in G, N_c^0, L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$  we have  $g \circ F_{N_c^0, L_c^0} \circ g^{-1} = F_{gN_c^0, gL_c^0}$ .

**Definition 3.2.6.** Let  $\pi$  be the  $K$ -vector space of collections  $f_{L_c^0} \in \mathcal{H}_L$  for  $L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$  satisfying the property: for  $N_c^0, L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$  one has

$$\mathcal{F}_{N_c^0, L_c^0}(f_{L_c^0}) = f_{N_c^0}$$

The element  $h \in H$  sends  $\{f_{L_c^0}\} \in \pi$  to the collection  $\{h(f_{L_c^0})\} \in \pi$ . This is our canonical  $H$ -representation over  $K$ .

The group  $G$  acts on  $\pi$  sending  $\{f_{L_c^0}\} \in \pi$  to the collection  $L_c^0 \mapsto g(f_{g^{-1}L_c^0})$ . This is a version of the Weil representation of  $G$ . (In the case when the field of coefficients is  $\mathbb{C}$ , this  $G$ -representation was also obtained in [2], however a canonical representation of  $H$  was not constructed in [2]).

The above actions of  $H$  and  $G$  on  $\pi$  combine to an action of the semi-direct product  $H \rtimes G$  on  $\pi$ . Theorem 2.0.8 is proved.

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