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TOWARDS CANONICAL REPRESENTATIONS OF FINITE HEISENBERG GROUPS

S. LYSENKO

ABSTRACT. We consider a finite abelian group M of odd exponent n with a symplectic form $\omega : M \times M \rightarrow \mu_n$ and the Heisenberg extension $1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$ with the commutator ω . According to the Stone - von Neumann theorem, H admits an irreducible representation with the tautological central character (defined up to a non-unique isomorphism). We construct such irreducible representation of H defined up to a unique isomorphism, so canonical in this sense.

1. INTRODUCTION

1.0.1. Consider a finite abelian group M of odd exponent n with a symplectic form $\omega : M \times M \rightarrow \mu_n$. It admits a unique symmetric Heisenberg extension $1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$ with the commutator ω . According to the Stone - von Neumann theorem, H admits an irreducible representation with the tautological central character (defined up to a non-unique isomorphism). We construct such irreducible representation of H defined up to a unique isomorphism, so canonical in this sense, over a suitable finite extension of \mathbb{Q} .

1.0.2. We are motivated by the following question of Dennis Gaitsgory about [6]. Let X be a smooth projective connected curve over an algebraically closed field k . Let T be a torus over k with a geometric metaplectic data \mathcal{G} as in [3]. To fix ideas, consider the sheaf-theoretic context of ℓ -adic sheaves on finite type schemes over k . Let $(H, \mathcal{G}_{Z_H}, \epsilon)$ be the metaplectic Langlands dual datum associated to (T, \mathcal{G}) in ([3], Section 6), so H is a torus over $\bar{\mathbb{Q}}_\ell$ isogenous to the Langlands dual of T . Let σ be a twisted local system on X for (H, \mathcal{G}_{Z_H}) in the sense of ([3], Section 8.4). To this data in ([3], Section 9.5.3) we attached the DG-category of Hecke eigensheaves. The question is whether this category identifies canonically with the DG-category Vect of $\bar{\mathbb{Q}}_\ell$ -vector spaces. In [6] we constructed such an irreducible Hecke eigensheaf for σ out of a given irreducible representation of certain finite Heisenberg group (denoted by Γ given by formula (33) in [6], Section 5.2.4) with the tautological central character.

This is why we are interested in constructing a canonical irreducible representation of finite Heisenberg groups as in Section 1.0.1. We do this only assuming the order of M odd, the case of even order remains open.

2. MAIN RESULT

2.0.1. Let e be an algebraically closed field of characteristic zero. Let M be a finite abelian group, $\omega : M \times M \rightarrow e^*$ a bilinear form, which is alternating, that is, $\omega(m, m) =$

0 for any $m \in M$. Assume the induced map $M \rightarrow \text{Hom}(M, e^*)$ is an isomorphism, that is, the form is nondegenerate.

If $L \subset M$ is a subgroup, $L^\perp = \{m \in M \mid \omega(m, l) = 0 \text{ for all } l \in L\}$ is its orthogonal complement, this is a subgroup. The group L is isotropic if $L \subset L^\perp$. The subgroup L is lagrangian if $L^\perp = L$. For a lagrangian subgroup we get an exact sequence

$$(1) \quad 0 \rightarrow L \rightarrow M \rightarrow L^* \rightarrow 0,$$

where $L^* = \text{Hom}(L, e^*)$. Namely, we send $m \in M$ to the character $l \mapsto \omega(m, l)$ of L . This exact sequence always admits a splitting $L^* \rightarrow M$, which is a homomorphism, see for example ([7], 4.1). For such a splitting after the obtained identification $M \xrightarrow{\sim} L \times L^*$ the form ω becomes

$$(2) \quad \omega((l_1, \chi_1), (l_2, \chi_2)) = \frac{\chi_1(l_2)}{\chi_2(l_1)}$$

for $l_i \in L, \chi_i \in L^*$.

By ([7], Theorem 2), up to an isomorphism, there is a unique central extension

$$(3) \quad 1 \rightarrow e^* \rightarrow H_{e^*} \rightarrow M \rightarrow 1$$

with the commutator ω . We are interested in understanding the category of representations of H_{e^*} with the tautological central character.

2.0.2. For a finite abelian group L its exponent is the least common multiple of the orders of the elements of L . Let n be the exponent of M , this is a divisor of $\sqrt{|M|} \in \mathbb{N}$.

Let $\mu_n = \mu_n(e)$. Let us be given a central extension

$$(4) \quad 1 \rightarrow \mu_n \rightarrow H \rightarrow M \rightarrow 1$$

together with a symmetric structure σ in the sense of ([1], Section 1.1) and commutator ω . That is, σ is an automorphism of H such that $\sigma^2 = \text{id}$, $\sigma|_{\mu_n} = \text{id}$, and $\sigma \bmod \mu_n$ is the involution $m \mapsto -m$ of M .

From now on assume n odd. Then by ([7], Section 1), there is a unique symmetric central extension (4) up to a unique isomorphism. Besides, (3) is isomorphic to the push-out of (4) under the tautological character $\iota : \mu_n \hookrightarrow e^*$.

The extension H is constructed as follows. Let $\beta : M \times M \rightarrow \mu_n$ be the unique alternating bilinear form such that $\beta^2 = \omega$. We take $H = M \times \mu_n$ with the product

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 a_2 \beta(m_1, m_2))$$

for $m_i \in M, a_i \in \mu_n$. Then $\sigma(m, a) = (-m, a)$ for $m \in M, a \in \mu_n$.

Let $G = \text{Sp}(M)$, the group of automorphisms of M preserving ω . Let $g \in G$ act on H sending (m, a) to (gm, a) . This gives the semi-direct product $H \rtimes G$.

2.0.3. The following version of the Stone - von Neumann theorem holds for H , the proof is left to a reader.

Proposition 2.0.4. *Up to an isomorphism, there is a unique irreducible representation of H over e with the tautological central character $\iota : \mu_n \hookrightarrow e^*$.*

2.0.5. Write $\mathcal{L}(M)$ for the set of lagragian subgroups in M . For $L \in \mathcal{L}(M)$ let \bar{L} be the preimage of L in H , this is a subgroup. If $\chi_L : \bar{L} \rightarrow e^*$ is a character extending the tautological character $\iota : \mu_b \hookrightarrow e^*$, set

$$\mathcal{H}_L = \{f : H \rightarrow e \mid f(\bar{l}h) = \chi_L(\bar{l})f(h), \text{ for } \bar{l} \in \bar{L}, h \in H\}$$

This is a representation of H by right translations. It is irreducible with central character ι .

2.0.6. We study the following.

Problem: Describe the category $\text{Rep}_\iota(H)$ of representations of H over e with central character $\iota : \mu_b \hookrightarrow e^*$. Is there an object of $\text{Rep}_\iota(H)$, which is irreducible and defined up to a unique isomorphism? (If yes, it would provide an equivalence between $\text{Rep}_\iota(H)$ and the category of e -vector spaces).

2.0.7. Let I be the set of primes appearing in the decomposition of n , write $n = \prod_{p \in I} p^{r(p)}$ with $r(p) > 0$. Let $K \subset e$ be the subfield generated over \mathbb{Q} by $\{\sqrt{p} \mid p \in I\}$ and μ_n .

Theorem 2.0.8. *There is an irreducible representation π of H over K with central character $\iota : \mu_n \hookrightarrow K^*$ defined up to a unique isomorphism. The H -action on π extends naturally to an action of $H \rtimes G$.*

Remark 2.0.9. *Let $K' \subset e$ be the subfield generated over \mathbb{Q} by μ_n . The field of definition of the character of π is K' . However, we do not expect that for any H with n odd Theorem 2.0.8 holds already with K replaced by K' , but we have not checked that. Our choices of \sqrt{p} for $p \in I$ are made to use the results of [5], and the formulas from [5] do not work without these choices. Note also the following. If L is an odd abelian group, and $b : L \times L \rightarrow e^*$ is a nondegenerate symmetric bilinear form then the Gauss sum of b is defined as*

$$G(L, b) = \sum_{l \in L} b(l, l)$$

Using the classification of such symmetric bilinear forms given in [4], one checks that $G(L, b)^4 = |L|^2$. Since the construction of π in Theorem 2.0.8 is related to representing the corresponding 2-cocyle (given essentially by certain Gauss sums) as a coboundary (after some minimal additional choices), we expect that our choices of \sqrt{p} for $p \in I$ are necessary.

Remark 2.0.10. *For $L \in \mathcal{L}(M)$, the H -representation \mathcal{H}_L from Section 2.0.5 is defined over K . We sometimes view it as a representation over K , the precise meaning is hopefully clear from the context.*

3. PROOF OF THEOREM 2.0.8

3.0.1. *Reduction.* For $p \in I$ let

$$H_p = \{h \in H \mid h^{(p^s)} = 1 \text{ for } s \text{ large enough}\}$$

and similarly for M_p . Then $H_p \subset H$ is a subgroup that fits into an exact sequence $1 \rightarrow \mu_{p^{r(p)}} \rightarrow H_p \rightarrow M_p \rightarrow 1$, and $H = \prod_{p \in I} H_p$, product of groups. Indeed, $\omega(H_p, H_q) = 1$

for $p, q \in I, p \neq q$. Besides, σ preserves H_p for each $p \in I$, so $(H_p, \sigma|_{H_p})$ is a symmetric Heisenberg extension of (M_p, ω_p) by $\mu_{p^{r(p)}}$. Here $\omega_p : M_p \times M_p \rightarrow \mu_{p^{r(p)}}$ is the restriction of ω . So, Problem 2.0.6 reduces to the case of a prime n . If $M_p = 0$ then take π_p be the 1-dimensional representation given by the tautological character $\mu_{p^{r(p)}} \hookrightarrow e^*$.

For $p \in I$ odd let $K_p \subset e$ be the subfield generated over \mathbb{Q} by $\mu_{p^{r(p)}}$ and \sqrt{p} . We prove Theorem 2.0.8 in the case of an odd prime n getting for $p \in I$ a representation π_p of H_p over K_p , hence over K also. Then for any odd n , $\pi = \otimes_{p \in I} \pi_p$ is the desired representation.

3.0.2. From now on we assume $n = p^r$ for an odd prime p .

3.1. **Case $r = 1$.**

3.1.1. In this section we assume M is a \mathbb{F}_p -vector space of dimension $2d$. To apply the results of [5], pick an isomorphism $\psi : \mathbb{F}_p \rightarrow \mu_p$. It allows to identify H with $M \times \mathbb{F}_p$. We then view $\mathcal{L}(M), H$ as algebraic varieties over \mathbb{F}_p . We allow the case $d = 0$ also.

3.1.2. Recall the following construction from ([5], Theorem 1).¹

Pick a prime $\ell \neq p$, and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of \mathbb{Q}_ℓ . We assume $\bar{\mathbb{Q}}_\ell$ is chosen in such a way that $K \subset \bar{\mathbb{Q}}_\ell$ is a subfield. In particular, we get $\sqrt{p} \in K \subset \bar{\mathbb{Q}}_\ell$. It gives rise to the \mathbb{Q}_ℓ -sheaf $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ over $\text{Spec } \mathbb{F}_p$.

Pick a 1-dimensional \mathbb{F}_p -vector space \mathcal{J} of parity $d \pmod 2$ as $\mathbb{Z}/2\mathbb{Z}$ -graded. Let \mathcal{A} be the line bundle (of parity zero as $\mathbb{Z}/2\mathbb{Z}$ -graded) on $\mathcal{L}(M)$ with fibre $\mathcal{J} \otimes \det L$ at $L \in \mathcal{L}(M)$. Write $\tilde{\mathcal{L}}(M)$ for the gerbe of square roots of \mathcal{A} .

In *loc.cit* we have constructed an irreducible perverse sheaf F on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$. Though in *loc.cit* we mostly worked over an algebraic closure $\bar{\mathbb{F}}_p$, F is defined over \mathbb{F}_p .

Lemma 3.1.3. *For any $i : \text{Spec } \mathbb{F}_p \rightarrow \tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$, $\text{tr}(\text{Fr}, i^*F) \in K$. Here Fr is the geometric Frobenius endomorphism.*

Proof. This follows from formula (10) in ([5], Section 3.3). Namely, after a surjective smooth localization (a choice of an additional lagrangian in M), there is an explicit formula for F as the convolution along H of two explicit rank one local systems. Their traces of Frobenius lie in K , as their definition involves only the Artin-Schreier sheaf and Tate twists. So, the same holds after the convolution along the finite group $H(\mathbb{F}_p)$. \square

3.1.4. For an algebraic stack $S \rightarrow \text{Spec } \mathbb{F}_p$ we write $S(\mathbb{F}_p)$ for the set of isomorphism classes of its \mathbb{F}_p -points. In view of the isomorphism $\psi : \mathbb{F}_p \xrightarrow{\sim} \mu_p$ fixed above, for $L \in \mathcal{L}(M)(\mathbb{F}_p)$ we identify $\bar{L} = L \times \mu_p$ with $L \times \mathbb{F}_p$. Let

$$F^{cl} : \tilde{\mathcal{L}}(M)(\mathbb{F}_p) \times \tilde{\mathcal{L}}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p) \rightarrow K$$

be the function trace of Frobenius of F .

For $L \in \mathcal{L}(M)(\mathbb{F}_p)$ its preimage in $\tilde{\mathcal{L}}(M)(\mathbb{F}_p)$ consists of two elements. We let μ_2 act on $\tilde{\mathcal{L}}(M)(\mathbb{F}_p)$ over $\mathcal{L}(M)(\mathbb{F}_p)$ permuting the elements in the preimage of each $L \in \mathcal{L}(M)(\mathbb{F}_p)$. We call a function $h : \tilde{\mathcal{L}}(M)(\mathbb{F}_p) \rightarrow K$ genuine if it changes by the

¹For this construction we adopt the conventions of *loc.cit* about $\mathbb{Z}/2\mathbb{Z}$ -gradings and étale $\bar{\mathbb{Q}}_\ell$ -sheaves on schemes over \mathbb{F}_p .

nontrivial character of μ_2 under this μ_2 -action. Recall that F^{cl} is genuine with respect to the first and the second variable.

Let us write L^0 for a point of $\tilde{\mathcal{L}}(M)(\mathbb{F}_p)$ over $L \in \mathcal{L}(M)(\mathbb{F}_p)$. As in ([5], Section 2), for $L^0, N^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$ viewing $\mathcal{H}_L, \mathcal{H}_N$ as H -representations over K , we define the canonical intertwining operator

$$F_{N^0, L^0} : \mathcal{H}_L \rightarrow \mathcal{H}_N$$

by

$$(F_{N^0, L^0} f)(h_1) = \int_{h_2 \in H} F_{N^0, L^0}(h_1 h_2^{-1}) f(h_2) dh_2,$$

where our measure dh_2 is normalized by requiring that the volume of a point is one.

Let $G = \mathrm{Sp}(M)$ viewed as an algebraic group over \mathbb{F}_p . It acts naturally on $\mathcal{L}(M), H$, and $\tilde{\mathcal{L}}(M)$. By definition, for $g \in G, (m, a) \in H, g(m, a) = (gm, a)$ for $m \in M, a \in \mathbb{F}_p$, and this action preserves the symmetric structure σ on H . If $g \in G, f : H \rightarrow K$ then $gf : H \rightarrow K$ is given by $(gf)(h) = f(g^{-1}h)$. Then $g \in G(\mathbb{F}_p)$ yields an isomorphism $\mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$. We let G act diagonally on $\tilde{\mathcal{L}}(M) \times \tilde{\mathcal{L}}(M) \times H$.

The above intertwining operators satisfy the following properties

- $F_{L^0, L^0} = \mathrm{id}$;
- $F_{R^0, N^0} \circ F_{N^0, L^0} = F_{R^0, L^0}$ for any $R^0, N^0, L^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$;
- for any $g \in G(\mathbb{F}_p)$ we have $g \circ F_{N^0, L^0} \circ g^{-1} = F_{gN^0, gL^0}$.

Definition 3.1.5. *Let π be the K -vector space of collections $f_{L^0} \in \mathcal{H}_L$ for $L^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$ satisfying the property: for $N^0, L^0 \in \tilde{\mathcal{L}}(M)(\mathbb{F}_p)$ one has*

$$F_{N^0, L^0}(f_{L^0}) = f_{N^0}$$

This is our canonical H -representation over K .

We let $G(\mathbb{F}_p)$ act on $\tilde{\mathcal{L}}(M)(\mathbb{F}_p) \times H(\mathbb{F}_p)$ diagonally. This yields a $G(\mathbb{F}_p)$ -action on π sending $\{f_{L^0}\} \in \pi$ to the collection $L^0 \mapsto g(f_{g^{-1}L^0})$.

3.2. Case $r \geq 1$.

3.2.1. Let L be a finite abelian group, p be any prime number. For $k \geq 0$ let $L[p^k] = \{l \in L \mid p^k l = 0\}$ and

$$\rho_k(L) = L[p^k]/(L[p^{k-1}] + pL[p^{k+1}])$$

Each $\rho_k(L)$ is a vector space over \mathbb{F}_p . Note that

$$\rho_k(\mathbb{Z}/p^m\mathbb{Z}) \xrightarrow{\sim} \begin{cases} \mathbb{Z}/p\mathbb{Z}, & m = k \\ 0, & \text{otherwise} \end{cases}$$

For finite abelian groups L, L' one has canonically $\rho_k(L \times L') \xrightarrow{\sim} \rho_k(L) \times \rho_k(L')$.

3.2.2. *Canonical isotropic subgroup.* Let p be any prime, M is a finite abelian p -group of exponent $n = p^r$ with an alternating nondegenerate bilinear form $\omega : M \times M \rightarrow \mu_n$. We first construct by induction a canonical isotropic subgroup $S \subset M$ such that $\text{Aut}(M)$ fixes S and S^\perp/S is a \mathbb{F}_p -vector space.

Write the set $\{r > 0 \mid \rho_r(M) \neq 0\}$ as $\{r_1, \dots, r_s\}$ with $0 < r_1 < r_2 < \dots < r_s$. There is an orthogonal direct sum $(M, \omega) \xrightarrow{\sim} \bigoplus_{i=1}^s (M_i, \omega_i)$, where $\omega_i : M_i \times M_i \rightarrow \mu_n$ is an alternating nondegenerate bilinear form, and M_i is a free \mathbb{Z}/p^{r_i} -module of finite rank.

Let

$$r' = \begin{cases} \frac{r_s}{2}, & r_s \text{ is even} \\ \frac{r_s+1}{2}, & r_s \text{ is odd} \end{cases}$$

Set $S_1 = p^{r'}M$. Since ω takes values in $\mu_{p^{r_s}}$, S_1 is isotropic and fixed by $\text{Aut}(M)$. By induction hypothesis, we have a canonical isotropic subgroup $S' \subset M_1 := S_1^\perp/S_1$ such that S'^\perp/S' is a \mathbb{F}_p -vector space, where S'^\perp denotes the orthogonal complement of S' in M_1 . Let S be the preimage of S' under $S_1^\perp \rightarrow M_1$. This is our canonical isotropic subgroup in M .

Set $M_c = S^\perp/S$, it is equipped with the induced alternating nondegenerate bilinear form $\omega_c : M_c \times M_c \rightarrow \mu_p$, the subscript c stands for ‘canonical’.

3.2.3. We keep the assumptions of Theorem 2.0.8, so p is odd. View S as a subgroup of H via $s \mapsto (s, 0) \in H$ for $s \in S$. Let $H^S = S^\perp \times \mu_n$, this is a subgroup of H . Since S lies in the kernel of $\beta : S^\perp \times S^\perp \rightarrow \mu_n$, we get the alternating nondegenerate bilinear form $\beta_c : M_c \times M_c \rightarrow \mu_p$ given by $\beta_c(m_1, m_2) = \beta(\tilde{m}_1, \tilde{m}_2)$ for $\tilde{m}_i \in S^\perp$ over m_i .

Set $H_c = M_c \times \mu_p$ with the product

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 a_2 \beta_c(m_1 m_2))$$

This is a central extension $1 \rightarrow \mu_p \rightarrow H_c \rightarrow M_c \rightarrow 1$ with the commutator ω_c and the symmetric structure $\sigma_c(m, a) = (-m, a)$ for $(m, a) \in H_c$.

Let $\alpha_S : H^S \rightarrow H_c$ be the homomorphism sending (m, a) to $(m \bmod S, a)$ for $m \in S^\perp$, its kernel is S .

As in Section 3.1, we get the algebraic stack $\tilde{\mathcal{L}}(M_c), \mathcal{L}(M_c), H_c$ over \mathbb{F}_p . Let $G = \text{Sp}(M, \omega)$ be the group of automorphisms of M preserving ω , this is a finite group. We let $g \in G$ act on H sending (m, a) to (gm, a) . Let $g \in G$ act on functions $f : H \rightarrow K$ by $(gf)(h) = f(g^{-1}h)$ for $h \in H$. For $L \in \mathcal{L}(M)$ this yields an isomorphism $g : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$ of K -vector spaces.

Since G preserves S^\perp , we have the homomorphism $G \rightarrow G_c := \text{Sp}(M_c)(\mathbb{F}_p)$. Via this map, G acts on $\mathcal{L}(M_c)(\mathbb{F}_p), \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p), H_c$.

3.2.4. We denote elements of $\mathcal{L}(M_c)$ by a capital letter with a subscript c . For $L_c \in \mathcal{L}(M_c)$ let $L \in \mathcal{L}(M)$ denote the preimage of L_c under $S^\perp \rightarrow M_c$.

For $L_c \in \mathcal{L}(M_c)$ we have the representation \mathcal{H}_{L_c} of H_c over K defined in Section 3.1.4, and the H -representation \mathcal{H}_L over K defined in Section 2.0.5.

For $L_c \in \mathcal{L}(M_c)$ any f in the space of invariants \mathcal{H}_L^S is the extension by zero under $H^S \hookrightarrow H$. The space \mathcal{H}_L^S is naturally a H_c -module. We get an isomorphism of H_c -modules $\tau_{L_c} : \mathcal{H}_{L_c} \xrightarrow{\sim} \mathcal{H}_L^S$ sending f to the composition $H^S \xrightarrow{\alpha_S} H_c \xrightarrow{f} K$ extended by zero to H .

For $g \in G$, $L_c \in \mathcal{L}(M_c)$ the isomorphism $g : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_{gL}$ yields an isomorphism $g : \mathcal{H}_L^S \xrightarrow{\sim} \mathcal{H}_{gL}^S$ of S -invariants.

3.2.5. Given $L_c^0, N_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$, we define a canonical intertwining operator

$$(5) \quad \mathcal{F}_{N_c^0, L_c^0} : \mathcal{H}_L \xrightarrow{\sim} \mathcal{H}_N$$

as the unique isomorphism of H -modules such that the diagram commutes

$$\begin{array}{ccc} \mathcal{H}_L^S & \xrightarrow{\mathcal{F}_{N_c^0, L_c^0}} & \mathcal{H}_N^S \\ \uparrow \tau_{L_c} & & \uparrow \tau_{N_c} \\ \mathcal{H}_{L_c} & \xrightarrow{F_{N_c^0, L_c^0}} & \mathcal{H}_{N_c} \end{array}$$

Here $F_{N_c^0, L_c^0}$ are the canonical intertwining operators from Section 3.1.4. The properties of the canonical intertwining operators of Section 3.1.4 imply the following properties of (5):

- $\mathcal{F}_{L_c^0, L_c^0} = \text{id}$ for $L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$;
- for $R_c^0, N_c^0, L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ one has

$$\mathcal{F}_{R_c^0, N_c^0} \circ F_{N_c^0, L_c^0} = F_{R_c^0, L_c^0}$$

- for $g \in G, N_c^0, L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ we have $g \circ F_{N_c^0, L_c^0} \circ g^{-1} = F_{gN_c^0, gL_c^0}$.

Definition 3.2.6. Let π be the K -vector space of collections $f_{L_c^0} \in \mathcal{H}_L$ for $L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ satisfying the property: for $N_c^0, L_c^0 \in \tilde{\mathcal{L}}(M_c)(\mathbb{F}_p)$ one has

$$\mathcal{F}_{N_c^0, L_c^0}(f_{L_c^0}) = f_{N_c^0}$$

The element $h \in H$ sends $\{f_{L_c^0}\} \in \pi$ to the collection $\{h(f_{L_c^0})\} \in \pi$. This is our canonical H -representation over K .

The group G acts on π sending $\{f_{L_c^0}\} \in \pi$ to the collection $L_c^0 \mapsto g(f_{g^{-1}L_c^0})$. This is a version of the Weil representation of G . (In the case when the field of coefficients is \mathbb{C} , this G -representation was also obtained in [2], however a canonical representation of H was not constructed in [2]).

The above actions of H and G on π combine to an action of the semi-direct product $H \rtimes G$ on π . Theorem 2.0.8 is proved.

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