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# GEOMETRIC THETA-LIFTING FOR UNITARY GROUPS

SERGEY LYSENKO

ABSTRACT. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality.

## 1. INTRODUCTION

**1.1.** In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality. At level of functions the theta-lifting for unitary groups was studied by many authors ([1, 3, 4, 13, 15]).

The definition uses a splitting of the metaplectic extension over the unitary groups. The splitting existing in the literature at the level of functions ([6, 4, 14, 15]) are not entirely satisfactory. Namely, given a local field  $F$  and its degree two extension  $E$ , for the corresponding unitary group the splitting in *loc.cit.* is defined up to a multiplication by a character of  $E^*/F^*$  with values in  $\mathbb{C}^1$  (the group of complex numbers of absolute value one). We give a more canonical geometric construction of this splitting.

**1.2. Notations.** We follow the conventions of [10]. So, we work over an algebraically closed field  $k$  of characteristic  $p \neq 2$ . We fix a prime  $\ell \neq p$ , write  $\bar{\mathbb{Q}}_\ell$  for the algebraic closure of  $\mathbb{Q}_\ell$ . All our stacks are defined over  $k$ . For an algebraic stack locally of finite type  $S$  we have the derived categories  $D(S), D^-(S), D^{\prec}(S)$  of complexes of  $\bar{\mathbb{Q}}_\ell$ -sheaves on  $S$  as in *loc.cit.*

Let  $X$  be a smooth projective connected curve. Write  $\Omega$  for the canonical line bundle on  $X$ . Let  $\pi : Y \rightarrow X$  be an étale degree 2 cover with  $Y$  connected. Write  $\sigma$  for the nontrivial automorphism of  $Y$  over  $X$ ,  $\pi_*\mathcal{O} = \mathcal{O} \oplus \mathcal{E}$ , where  $\mathcal{E}$  is the sheaf of  $\sigma$ -antiinvariants, so  $\mathcal{E}^2 \xrightarrow{\sim} \mathcal{O}$ . Once and for all we pick a line bundles  $\mathcal{E}', \Omega^{\frac{1}{2}}$  on  $X$  with isomorphisms  $\mathcal{E}'^2 \xrightarrow{\sim} \mathcal{E}, (\Omega^{\frac{1}{2}})^2 \xrightarrow{\sim} \Omega$ .

For an algebraic group  $G$  write  $\text{Bun}_G$  for the stack of  $G$ -torsors on  $X$ . For  $n \geq 1$  write  $\text{Bun}_n$  (resp.,  $\text{Bun}_{n,Y}$ ) for the stack of  $G$ -torsors on  $X$  (resp., on  $Y$ ).

## 2. MODULI OF UNITARY BUNDLES AND GEOMETRIC THETA-LIFTING

**2.1.** Pick  $\epsilon = \pm 1$ . For  $n \geq 1$  an  $\epsilon$ -hermitian vector bundle on  $Y$  (with respect to  $\pi$ ) is a datum of  $V \in \text{Bun}_{n,Y}$  with a nondegenerate form  $\phi : V \otimes \sigma^*V \rightarrow \mathcal{O}_Y$  such that the

diagram commutes

$$\begin{array}{ccc} \phi^*(V \otimes \sigma^*V) & \xrightarrow{\sigma^*\phi} & \phi^*\mathcal{O} = \mathcal{O} \\ \downarrow & & \downarrow \epsilon \\ V \otimes \sigma^*V & \xrightarrow{\phi} & \mathcal{O}, \end{array}$$

the left vertical arrow being a canonical isomorphism. Write  $\text{Bun}_{U_n}$  (resp.,  $\text{Bun}_{U_n^-}$ ) for the stack of hermitian (resp., skew-hermitian) vector bundles on  $Y$ .

Here  $U_n$  is the group scheme on  $X$ , the quotient of  $(\text{GL}_n \times Y)/\{\text{id}, \sigma\}$  by the diagonal action. Here  $\sigma(g) = ({}^t g)^{-1}$  for  $g \in \text{GL}_n$ . The stacks like  $\text{Bun}_{U_n}$  have been studied for example in [5, 12].

**2.1.1.** For an  $\epsilon$ -hermitian vector bundle  $V$  on  $Y$  viewing  $\phi$  as an isomorphism  $\phi : \sigma^*V \xrightarrow{\sim} V^*$ , one gets  $(\sigma^*\phi)^* = \epsilon\phi$ . Consider the isomorphism

$$(1) \quad (\sigma^*\phi, \phi) : V \oplus \sigma^*V \rightarrow \sigma^*V^* \oplus V^*$$

on  $Y$ . Let  $M = \pi_*C$ . Let  $\sigma$  act on  $V \oplus \sigma^*V$  such that the natural isomorphism  $V \oplus \sigma^*V \xrightarrow{\sim} \pi^*M$  is  $\sigma$ -invariant. Then (1) is  $\sigma$ -equivariant, so descends to an isomorphism  $\bar{\phi} : M \xrightarrow{\sim} M^*$  such that  $\bar{\phi}^* = \epsilon\bar{\phi}$ . We also view the latter as a map  $\bar{\phi} : M \otimes M \rightarrow \mathcal{O}_X$ . So,  $\bar{\phi}$  is symmetric for  $\epsilon = 1$  (resp., anti-symmetric for  $\epsilon = -1$ ).

Consider the isomorphism

$$(2) \quad (-\sigma^*\phi, \phi) : V \oplus \sigma^*V \rightarrow \sigma^*V^* \oplus V^*$$

As above, there is a unique isomorphism  $\phi' : M \xrightarrow{\sim} M^* \otimes \mathcal{E}$  such that  $\pi^*(\phi') = (-\sigma^*\phi, \phi)$ . Besides,  $\phi'$  is symmetric for  $\epsilon = -1$  (resp., antisymmetric for  $\epsilon = 1$ ).

Let  $\text{Bun}_{\mathbb{O}_{2n}}$  be the stack classifying  $M \in \text{Bun}_{2n}$  with a nondegenerate symmetric form  $\text{Sym}^2 M \rightarrow \mathcal{O}_X$ . Let  $\mathfrak{q}_n : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\mathbb{O}_{2n}}$  be the map sending  $(V, \phi)$  to  $(M = \pi_*V, \bar{\phi})$ . Let  $\mathfrak{q}_n^- : \text{Bun}_{U_n^-} \rightarrow \text{Bun}_{\mathbb{O}_{2n}}$  be the map sending  $(V, \phi)$  to  $(M \otimes \mathcal{E}', \phi')$ . Here we have viewed  $\phi'$  as a map  $\text{Sym}^2(M \otimes \mathcal{E}') \rightarrow \mathcal{O}_X$ .

The stack  $\text{Bun}_{\text{Sp}_{2n}}$  classifies  $M \in \text{Bun}_{2n}$  with a symplectic form  $\wedge^2 M \rightarrow \mathcal{O}_X$ . Let  $\mathfrak{p}_n : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\text{Sp}_{2n}}$  be the map sending  $(V, \phi)$  to  $(M \otimes \mathcal{E}', \phi')$ . Here  $M = \pi_*V$ . Let also  $\mathfrak{p}_n^- : \text{Bun}_{U_n^-} \rightarrow \text{Bun}_{\text{Sp}_{2n}}$  be the map sending  $(V, \phi)$  to  $(M, \bar{\phi})$ .

**2.1.2.** We have a canonical identification  $\pi^*\mathcal{E} \xrightarrow{\sim} \mathcal{O}$  such that the descent data for  $\pi^*\mathcal{E}$  are given by the action of  $\sigma$  on  $\mathcal{O}$  as  $-1$ . This gives an isomorphism  $\delta : \text{Bun}_{U_n} \xrightarrow{\sim} \text{Bun}_{U_n^-}$  sending  $(V, \phi)$  to  $(V \otimes \pi^*\mathcal{E}', \phi)$ . The diagram commutes

$$\begin{array}{ccccc} & & \text{Bun}_{U_n} & & \\ & \swarrow \mathfrak{q}_n & \downarrow \delta & \searrow \mathfrak{p}_n & \\ \text{Bun}_{\mathbb{O}_{2n}} & \xleftarrow{\mathfrak{q}_n^-} & \text{Bun}_{U_n^-} & \xrightarrow{\mathfrak{p}_n^-} & \text{Bun}_{\text{Sp}_{2n}} \end{array}$$

**2.1.3.** The stack  $\text{Bun}_{U_1}$  is a group stack described in ([8], Appendix A). We have the norm map  $N : \text{Bun}_{1,Y} \rightarrow \text{Bun}_1$  sending  $\mathcal{L}$  to  $\mathcal{E} \otimes \det(\pi_*\mathcal{L})$ , this is a homomorphism of group stacks. The stack  $\text{Bun}_{U_1}$  classifies  $V \in \text{Bun}_{1,Y}$  together with a trivialization  $N(V) \xrightarrow{\sim} \mathcal{O}_X$ . The stack  $\text{Bun}_{U_1}$  has connected components  $\text{Bun}_{U_1}^a$  indexed by  $a \in \mathbb{Z}/2\mathbb{Z}$ ,  $\text{Bun}_{U_1}^0$  being the connected component of unity. The stack  $\text{Bun}_{U_1^-}$  classifies  $V \in \text{Bun}_{1,Y}$  with an isomorphism  $N(V) \xrightarrow{\sim} \mathcal{E}$ .

The group stack  $\mathrm{Bun}_{U_1}$  acts on  $\mathrm{Bun}_{U_n^-}$  sending  $(\mathcal{L}, \phi_{\mathcal{L}}) \in \mathrm{Bun}_{U_1}, (V, \phi) \in \mathrm{Bun}_{U_n^-}$  to  $(V \otimes \mathcal{L}, \phi \otimes \phi_{\mathcal{L}}) \in \mathrm{Bun}_{U_n^-}$ . Let  $\rho_n : \mathrm{Bun}_{U_n} \rightarrow \mathrm{Bun}_{U_1}$  be the map sending  $(V, \phi)$  to  $(\det V, \det \phi)$ .

**2.1.4.** For  $n > 1$  let  $(V, \phi) \in \mathrm{Bun}_{U_n}$ . Then  $\det \phi : \sigma^* \det V \xrightarrow{\sim} \det V^*$  can be seen as a trivialization  $\xi : N(\det V) \xrightarrow{\sim} \mathcal{O}$ . Let  $\mathrm{Bun}_{SU_n}$  be the stack classifying  $V \in \mathrm{Bun}_{U_n}$  with an isomorphism  $\det V \xrightarrow{\sim} \mathcal{O}_Y$  such that the induced isomorphism  $N(\det V) \xrightarrow{\sim} N(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X$  is  $\xi$ . So,  $\mathrm{Bun}_{SU_n}$  is the fibre of the map  $\rho_n : \mathrm{Bun}_{U_n} \rightarrow \mathrm{Bun}_{U_1}$  over the unit  $\mathcal{O}_Y \in \mathrm{Bun}_{U_1}$  of this group stack.

By ([5], Theorem 2) the stack  $\mathrm{Bun}_{SU_n}$  is connected. By ([5], Theorem 3), if  $n > 1$  then  $\mathrm{Pic}(\mathrm{Bun}_{SU_n}) \xrightarrow{\sim} \mathbb{Z}$ . The line bundle on  $\mathrm{Bun}_{SU_n}$  sending  $V \in \mathrm{Bun}_{U_n}$  to  $\det \mathrm{R}\Gamma(Y, V)$  is twice a generator of  $\mathrm{Pic}(\mathrm{Bun}_{SU_n})$  by ([12], Remark 3.6(2)).

**2.2. Twist by  $\Omega$ .** As in [8] write  $\mathrm{Bun}_{G_n}$  for the stack classifying  $M \in \mathrm{Bun}_{2n}$  with a symplectic form  $\wedge^2 M \rightarrow \Omega_X$ . Write  $\mathrm{Bun}_{\mathbb{O}_{2n}, \Omega}$  for the stack classifying  $M \in \mathrm{Bun}_{2n}$  with a nondegenerate symmetric bilinear form  $\mathrm{Sym}^2 M \rightarrow \Omega$ .

Write  $\mathrm{Bun}_{U_{n,s}}$  for the stack classifying  $V \in \mathrm{Bun}_{n,Y}$  together with an isomorphism

$$\phi : \sigma^* V \xrightarrow{\sim} V^* \otimes \pi^* \Omega$$

such that  $\phi$  is skew-hermitian. In other words,  $(\sigma^* \phi)^* = -\phi$ . Here  $s$  stands for ‘skew-hermitian’. Write  $\Omega_Y$  for the canonical line bundle on  $Y$ , one has  $\pi^* \Omega \xrightarrow{\sim} \Omega_Y$ .

As in Section 2.1.1, for  $(V, \phi) \in \mathrm{Bun}_{U_{n,s}}$  let  $M = \pi_* V$ . There is a unique symplectic form  $\bar{\phi} : M \xrightarrow{\sim} M^* \otimes \Omega$  such that

$$\pi^* \bar{\phi} = (\sigma^* \phi, \phi) : V \oplus \sigma^* V \rightarrow (\sigma^* V^* \oplus V^*) \otimes \pi^* \Omega$$

There is also a unique symmetric bilinear form  $\phi' : M \rightarrow M^* \otimes \Omega \otimes \mathcal{E}$  such that

$$\pi^* \phi' = (-\sigma^* \phi, \phi) : V \oplus \sigma^* V \rightarrow (\sigma^* V^* \oplus V^*) \otimes \pi^* (\Omega \otimes \mathcal{E})$$

This defines a diagram

$$\mathrm{Bun}_{\mathbb{O}_{2n}, \Omega} \xleftarrow{\mathfrak{q}_{n,s}} \mathrm{Bun}_{U_{n,s}} \xrightarrow{\mathfrak{p}_{n,s}} \mathrm{Bun}_{G_n},$$

where  $\mathfrak{p}_{n,s}$  sends  $(V, \phi)$  to  $(M = \pi_* V, \bar{\phi})$ , and  $\mathfrak{q}_{n,s}$  sends  $(V, \phi)$  to  $(M \otimes \mathcal{E}', \phi')$ .

**2.3. Square root.** Denote by  $\mathcal{A}_n$  the line bundle on  $\mathrm{Bun}_{G_n}$  whose fibre at  $M$  is  $\det \mathrm{R}\Gamma(X, M)$ . Write  $\widetilde{\mathrm{Bun}}_{G_n}$  for the gerbe of square roots of  $\mathcal{A}_n$  over  $\mathrm{Bun}_{G_n}$ .

Our immediate purpose is to construct a distinguished square root of the line bundle  $\mathfrak{p}_{n,s}^* \mathcal{A}_n$ . Our construction will depend only on the choices of  $\Omega^{\frac{1}{2}}, \mathcal{E}'$  that we made in Section 1.2, and also on a choice of  $i \in k$  with  $i^2 = -1$ .

**2.3.1.** For  $W \in \mathrm{Bun}_n$  write for brevity  $d(W) = \det \mathrm{R}\Gamma(X, W)$ , we view it as a  $\mathbb{Z}/2\mathbb{Z}$ -graded line.

For  $\mathcal{A}_i \in \mathrm{Bun}_1$  set

$$K(\mathcal{A}_1, \mathcal{A}_2) = \frac{d(\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes d(\mathcal{O})}{d(\mathcal{A}_1) \otimes d(\mathcal{A}_2)}$$

We view this line as a  $\mathbb{Z}/2\mathbb{Z}$ -graded. Then  $K$  is bilinear up to a canonical isomorphism by ([9], Section 4.2.1-4.2.2).

One has a canonical Pfaffian line bundle  $Pf$  on  $\text{Bun}_{\mathbb{O}_{2n}, \Omega}$  defined in ([2], Section 4.2.1). Pick  $i \in k$  with  $i^2 = -1$ . This choice yields a canonical  $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism  $Pf(W)^2 \xrightarrow{\sim} d(W)$  for  $W \in \text{Bun}_{\mathbb{O}_{2n}, \Omega}$  as in *loc.cit.* An alternative construction of  $Pf$  is given in [7].

Given  $(V, \phi) \in \text{Bun}_{U_{n,s}}$  let  $M = \phi_* V$ . The symplectic form  $\bar{\phi} : \wedge^2 M \rightarrow \Omega$  induces an isomorphism  $\det M \xrightarrow{\sim} \Omega^n$ . By ([8], Lemma 1) one has canonically for  $(V, \phi) \in \text{Bun}_{U_{n,s}}$  and  $M = \pi_* V$

$$(3) \quad d(M \otimes \mathcal{E}') \xrightarrow{\sim} d(M) \otimes K(\Omega^n, \mathcal{E}') \otimes \frac{d(\mathcal{E}')^{2n}}{d(\mathcal{O})^{2n}}$$

Our choice of  $\Omega^{\frac{1}{2}}$ , bilinearity of  $K$ , and (3) yield an isomorphism

$$(4) \quad Pf(M \otimes \mathcal{E}')^2 \xrightarrow{\sim} d(M) \otimes \left( K(\Omega^{\frac{1}{2}}, \mathcal{E}') \otimes \frac{d_X(\mathcal{E}')}{d_X(\mathcal{O})} \right)^{2n}$$

Denote by  $\mathcal{L}_n$  the line bundle on  $\text{Bun}_{U_{n,s}}$  whose fibre at  $(V, \phi)$  is

$$Pf(M \otimes \mathcal{E}) \otimes \left( \frac{d_X(\mathcal{O})}{K_X(\Omega^{\frac{1}{2}}, \mathcal{E}) \otimes d_X(\mathcal{E}')} \right)^n$$

Then (4) yields the desired isomorphism over  $\text{Bun}_{U_{n,s}}$

$$\mathcal{L}_n^2 \xrightarrow{\sim} \mathfrak{p}_{n,s}^* \mathcal{A}_n$$

Let  $\tilde{\mathfrak{p}}_{n,s} : \text{Bun}_{U_{n,s}} \rightarrow \widetilde{\text{Bun}}_{G_n}$  be the map sending  $(V, \psi)$  to  $\mathfrak{p}_{n,s}(V, \phi) = (M, \bar{\phi})$  and a line  $\mathcal{L}_n(V, \phi)$  equipped with the above isomorphism  $\mathcal{L}_n(V, \phi)^2 \xrightarrow{\sim} d(M)$ .

**2.4. Dual pair  $U_n, U_m$ .** Let  $n, m \geq 1$ . Write

$$\tau : \text{Bun}_{U_n} \times \text{Bun}_{U_{m,s}} \rightarrow \text{Bun}_{U_{nm,s}}$$

for the map sending  $(V_1, \phi_1) \in \text{Bun}_{U_n}$ ,  $(V_2, \phi_2) \in \text{Bun}_{U_{m,s}}$  to  $V_1 \otimes V_2$  with the induced isomorphism

$$\phi_1 \otimes \phi_2 : \sigma^*(V_1 \otimes V_2) \xrightarrow{\sim} (V_1 \otimes V_2)^* \otimes \pi^* \Omega$$

The groups  $U_n, U_m$  form a dual pair in  $G_{nm}$  essentially via the map  $\mathfrak{p}_{nm,s} \tau$ . Let  $\text{Aut}$  be the theta-sheaf on  $\widetilde{\text{Bun}}_{G_n}$  given in ([11], Definition 1). Define the theta-lifting functors

$$F_s : D^-(\text{Bun}_{U_n})! \rightarrow D^{\prec}(\text{Bun}_{U_{m,s}}), \quad F : D^-(\text{Bun}_{U_{m,s}})! \rightarrow D^{\prec}(\text{Bun}_{U_n})$$

following the framework of the geometric Langlands functoriality proposed in ([10], Section 2) for the kernel

$$\mathcal{M} = \tau^* \tilde{\mathfrak{p}}_{nm,s}^* \text{Aut}[\dim \text{Bun}_{U_n} \times \text{Bun}_{U_{m,s}} - \dim \text{Bun}_{G_{nm}}]$$

That is, for  $K \in D^-(\text{Bun}_{U_n})!$  and  $K' \in D^-(\text{Bun}_{U_{m,s}})!$  we let

$$F_s(K) = (q_s)_!(q_s^* K \otimes \mathcal{M})[-\dim \text{Bun}_{U_n}] \quad \text{and} \quad F(K') = q_!(q_s^* K' \otimes \mathcal{M})[-\dim \text{Bun}_{U_{m,s}}]$$

for the diagram of projections

$$\text{Bun}_{U_n} \xleftarrow{q} \text{Bun}_{U_n} \times \text{Bun}_{U_{m,s}} \xrightarrow{q_s} \text{Bun}_{U_{m,s}}$$

## REFERENCES

- [1] J. Adams, L-functoriality for dual pairs, *Astérisque*, 171-172, *Orbites Unipotentes et Représentations*, 2 (1989), 85-129.
- [2] A. Beilinson, V. Drinfeld, Quantization of Hitchin integrable system and Hecke eigen-sheaves, preprint available at <http://math.uchicago.edu/~drinfeld/langlands/hitchin/BD-hitchin.pdf>
- [3] St. Gelbart, On theta-series lifting for unitary groups, in: M. Ram Murty ed., *Theta Functions from the Classical to the Modern*, American Mathematical Society, 1993
- [4] M. Harris, St. Kudla, W. J. Sweet, Theta dichotomy for unitary groups, *Journal of AMS*, Vol. 9, Nu. 4 (1996), 941-1004
- [5] J. Heinloth, Uniformization of  $\mathcal{G}$ -bundles, *Math. Annalen* (2010), Vol. 347, Issue 3, 499528
- [6] St. Kudla, Splitting metaplectic covers of dual reductive pairs, *Israel J. Math.* (1994), Vol. 87, Issue 13, 361401
- [7] Y. Laszlo and C. Sorger. The line bundles on the moduli of parabolic  $G$ -bundles over curves and their sections. *Ann. Sci. ENS.*, ser. 4, vol. 30 (1997), 499525.
- [8] S. Lysenko, Geometric Waldspurger periods, *Compositio Math.* 144 (2008), 377438
- [9] S. Lysenko, Twisted geometric Langlands correspondence for a torus, *Int. Math. Res. Notices* (2015) (18): 8680-8723
- [10] S. Lysenko, On the automorphic sheaves for  $GSp_4$ , arXiv:1901.04447
- [11] S. Lysenko, Moduli of metaplectic bundles on curves and Theta-sheaves, *Ann. Scient. ENS*, 4 série, t. 39 (2006), p. 415-466
- [12] G. Pappas, M. Rapoport, Some questions about  $G$ -bundles on curves, available at <https://www.math.uni-bonn.de/ag/algeom/preprints/G-bundles6.pdf>
- [13] D. Prasad, Theta correspondence for unitary groups, *Pacific J. Math.*, Vol.194, No.2 (2000), 427-438
- [14] Shu-Yen Pan, Splittings of the metaplectic covers of some reductive dual pairs, *Pacific J. Math.*, Vol. 199, No. 1 (2001), 163-226
- [15] Ch. Wu, Irreducibility of theta lifting for unitary groups, *J. Number Theory* 133 (2013), 3296 - 3318

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