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GEOMETRIC THETA-LIFTING FOR UNITARY GROUPS

SERGEY LYSENKO

Abstract. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality.

1. Introduction

1.1. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality. At level of functions the theta-lifting for unitary groups was studied by many authors ([1, 3, 4, 13, 15]).

The definition uses a splitting of the metaplectic extension over the unitary groups. The splitting existing in the literature at the level of functions ([6, 4, 14, 15]) are not entirely satisfactory. Namely, given a local field $F$ and its degree two extension $E$, for the corresponding unitary group the splitting in loc. cit. is defined up to a multiplication by a character of $E^*/F^*$ with values in $\mathbb{C}^1$ (the group of complex numbers of absolute value one). We give a more canonical geometric construction of this splitting.

1.2. Notations. We follow the conventions of [10]. So, we work over an algebraically closed field $k$ of characteristic $p \neq 2$. We fix a prime $\ell \neq p$, write $\overline{\mathbb{Q}}_\ell$ for the algebraic closure of $\mathbb{Q}_\ell$. All our stacks are defined over $k$. For an algebraic stack locally of finite type $S$ we have the derived categories $D(S)$, $D^{-}(S)$!, $D^{\prec}(S)$ of complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on $S$ as in loc. cit.

Let $X$ be a smooth projective connected curve. Write $\Omega$ for the canonical line bundle on $X$. Let $\pi : Y \to X$ be an étale degree 2 cover with $Y$ connected. Write $\sigma$ for the nontrivial automorphism of $Y$ over $X$, $\pi_* \mathcal{O} = \mathcal{O} \oplus \mathcal{E}$, where $\mathcal{E}$ is the sheaf of $\sigma$-antiinvariants, so $\mathcal{E}^2 \to \mathcal{O}$. Once and for all we pick a line bundles $\mathcal{E}', \Omega^{\frac{1}{2}}$ on $X$ with isomorphisms $\mathcal{E}'^2 \to \mathcal{E}$, $(\Omega^{\frac{1}{2}})^2 \to \Omega$.

For an algebraic group $G$ write $\text{Bun}_G$ for the stack of $G$-torsors on $X$. For $n \geq 1$ write $\text{Bun}_n$ (resp., $\text{Bun}_{n,Y}$) for the stack of $G$-torsors on $X$ (resp., on $Y$).

2. Moduli of unitary bundles and geometric theta-lifting

2.1. Pick $\epsilon = \pm 1$. For $n \geq 1$ an $\epsilon$-hermitian vector bundle on $Y$ (with respect to $\pi$) is a datum of $V \in \text{Bun}_{n,Y}$ with a nondegenerate form $\phi : V \otimes \sigma^*V \to \mathcal{O}_Y$ such that the
diagram commutes
\[
\begin{array}{ccc}
\phi^* (V \otimes \sigma^* V) & \phi^* \phi = \phi^\circ & \phi^\circ \phi = \phi^* \phi = \phi^\circ \phi = \phi^* \\
\downarrow & \downarrow & \downarrow \\
V \otimes \sigma^* V & \phi & \phi, \phi
\end{array}
\]
the left vertical arrow being a canonical isomorphism. Write Bun\(_{U_n}\) (resp., Bun\(_{U_n}^-\)) for the stack of hermitian (resp., skew-hermitian) vector bundles on \(Y\).

Here \(U_n\) is the group scheme on \(X\), the quotient of \((\GL_n \times Y)/\{\id, \sigma\}\) by the diagonal action. Here \(\sigma(g) = (g^t)^{-1}\) for \(g \in \GL_n\). The stacks like Bun\(_{U_n}\) have been studied for example in [5, 12].

**2.1.1.** For an \(\epsilon\)-hermitian vector bundle \(V\) on \(Y\) viewing \(\phi\) as an isomorphism \(\phi : \sigma^* V \xrightarrow{\sim} V^*\), one gets \((\sigma^* \phi)^\circ = \epsilon \phi\). Consider the isomorphism
\[
(1) \quad (\sigma^* \phi, \phi) : V \oplus \sigma^* V \rightarrow \sigma^* V^* \oplus V^*
\]
on \(Y\). Let \(M = \pi_s C\). Let \(\sigma\) act on \(V \oplus \sigma^* V\) such that the natural isomorphism \(V \oplus \sigma^* V \xrightarrow{\sim} \pi^* M\) is \(\sigma\)-invariant. Then (1) is \(\sigma\)-equivariant, so descends to an isomorphism \(\tilde{\phi} : M \rightarrow M^*\) such that \(\tilde{\phi}^* = \epsilon \tilde{\phi}\). We also view the latter as a map \(\tilde{\phi} : M \otimes M \rightarrow \mathcal{O}_X\).

So, \(\phi\) is symmetric for \(\epsilon = 1\) (resp., anti-symmetric for \(\epsilon = -1\)).

Consider the isomorphism
\[
(2) \quad (-\sigma^* \phi, \phi) : V \oplus \sigma^* V \rightarrow \sigma^* V^* \oplus V^*
\]
As above, there is a unique isomorphism \(\phi' : M \rightarrow M^* \otimes \mathcal{E}\) such that \(\pi^* (\phi') = (-\sigma^* \phi, \phi)\).

Besides, \(\phi'\) is symmetric for \(\epsilon = -1\) (resp., antisymmetric for \(\epsilon = 1\)).

Let Bun\(_{\mathcal{O}_{2n}}\) be the stack classifying \(M \in \text{Bun}_{2n}\) with a nondegenerate symmetric form \(\text{Sym}^2 M \rightarrow \mathcal{O}_X\). Let \(q_n : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\mathcal{O}_{2n}}\) be the map sending \((V, \phi)\) to \((M = \pi_s V, \tilde{\phi})\).

Let \(q_n : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\mathcal{O}_{2n}}\) be the map sending \((V, \phi)\) to \((M \otimes \mathcal{E}', \phi')\). Here we have viewed \(\phi'\) as a map \(\text{Sym}^2 (M \otimes \mathcal{E}') \rightarrow \mathcal{O}_X\).

The stack Bun\(_{\mathcal{O}_{2n}}\) classifies \(M \in \text{Bun}_{2n}\) with a symplectic form \(\wedge^2 M \rightarrow \mathcal{O}_X\). Let \(p_n : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\mathcal{O}_{2n}}\) be the map sending \((V, \phi)\) to \((M \otimes \mathcal{E}', \phi')\). Here \(M = \pi_s V\). Let also \(p_n^- : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\mathcal{O}_{2n}}\) be the map sending \((V, \phi)\) to \((M, \tilde{\phi})\).

**2.1.2.** We have a canonical identification \(\pi^* \mathcal{E} \xrightarrow{\sim} \mathcal{O}\) such that the descent data for \(\pi^* \mathcal{E}\) are given by the action of \(\sigma\) on \(\mathcal{O}\) as \(-1\). This gives an isomorphism \(\delta : \text{Bun}_{U_n} \xrightarrow{\sim} \text{Bun}_{U_n^-}\) sending \((V, \phi)\) to \((V \otimes \pi^* \mathcal{E}, \phi)\). The diagram commutes
\[
\begin{array}{ccc}
\text{Bun}_{U_n} & \xrightarrow{q_n} & \text{Bun}_{\mathcal{O}_{2n}} \\
\downarrow \delta & & \downarrow \delta \\
\text{Bun}_{U_n^-} & \xleftarrow{p_n^-} & \text{Bun}_{\mathcal{O}_{2n}}
\end{array}
\]

**2.1.3.** The stack Bun\(_{U_1}\) is a group stack described in ([8], Appendix A). We have the norm map \(N : \text{Bun}_{1,Y} \rightarrow \text{Bun}_1\) sending \(\mathcal{L}\) to \(\mathcal{E} \otimes \det(\pi_* \mathcal{L})\), this is a homomorphism of group stacks. The stack Bun\(_{U_1}\) classifies \(V \in \text{Bun}_{1,Y}\) together with a trivialization \(N(V) \rightarrow \mathcal{O}_X\). The stack Bun\(_{U_1}\) has connected components \(\text{Bun}_{U_1}^a\) indexed by \(a \in \mathbb{Z}/2\mathbb{Z}\), \(\text{Bun}_{U_1}^0\) being the connected component of unity. The stack Bun\(_{U_1^-}\) classifies \(V \in \text{Bun}_{1,Y}\) with an isomorphism \(N(V) \xrightarrow{\sim} \mathcal{E}\).
The group stack $\text{Bun}_{U_n}$ acts on $\text{Bun}_U$ sending $(\mathcal{L}, \phi_{\mathcal{L}}) \in \text{Bun}_{U_1}, (V, \phi) \in \text{Bun}_{U_n}$ to $(V \otimes \mathcal{L}, \phi \otimes \phi_{\mathcal{L}}) \in \text{Bun}_{U_n}$. Let $\rho_n : \text{Bun}_{U_n} \to \text{Bun}_{U_1}$ be the map sending $(V, \phi)$ to $(\det V, \det \phi)$.

2.1.4. For $n > 1$ let $(V, \phi) \in \text{Bun}_{U_n}$. Then $\det \phi : \sigma^* \det V \to \det V^*$ can be seen as a trivialization $\xi : N(\det V) \to \mathcal{O}$. Let $\text{Bun}_{SU_2}$ be the stack classifying $V \in \text{Bun}_{U_n}$ with an isomorphism $\det V \to \mathcal{O}_Y$ such that the induced isomorphism $N(\det V) \to N(\mathcal{O}_Y) \to \mathcal{O}_X$ is $\xi$. So, $\text{Bun}_{SU_2}$ is the fibre of the map $\rho_n : \text{Bun}_{U_n} \to \text{Bun}_{U_1}$ over the unit $\mathcal{O}_Y \in \text{Bun}_{U_1}$ of this group stack.

By ([5], Theorem 2) the stack $\text{Bun}_{SU_2}$ is connected. By ([5], Theorem 3), if $n > 1$ then $\text{Pic}(\text{Bun}_{SU_2}) \to \mathbb{Z}$. The line bundle on $\text{Bun}_{SU_2}$ sending $V \in \text{Bun}_{U_n}$ to $\det R\Gamma(Y, V)$ is twice a generator of $\text{Pic}(\text{Bun}_{SU_2})$ by ([12], Remark 3.6(2)).

2.2. **Twist by $\Omega$.** As in [8] write $\text{Bun}_{G_2}$ for the stack classifying $M \in \text{Bun}_{2n}$ with a symplectic form $\wedge^2 M \to \Omega_X$. Write $\text{Bun}_{G_2,\Omega}$ for the stack classifying $M \in \text{Bun}_{2n}$ with a nondegenerate symmetric bilinear form $\text{Sym}^2 M \to \Omega$.

Write $\text{Bun}_{U_n,s}$ for the stack classifying $V \in \text{Bun}_{n,Y}$ together with an isomorphism $\phi : \sigma^* V \to V^* \otimes \pi^* \Omega$ such that $\phi$ is skew-hermitian. In other words, $(\sigma^* \phi)^* = -\phi$. Here $s$ stands for ‘skew-hermitian’. Write $\Omega_Y$ for the canonical line bundle on $Y$, one has $\pi^* \Omega \to \Omega_Y$.

As in Section 2.1.1, for $(V, \phi) \in \text{Bun}_{U_n,s}$ let $M = \pi_* V$. There is a unique symplectic form $\phi : M \to M^* \otimes \Omega$ such that

$$\pi^* \phi = (\sigma^* \phi, \phi) : V \oplus \sigma^* V \to (\sigma^* V^* \oplus V^*) \otimes \pi^* \Omega$$

There is also a unique symmetric bilinear form $\phi' : M \to M^* \otimes \Omega \otimes \mathcal{E}$ such that

$$\pi^* \phi' = (\sigma^* \phi, \phi) : V \oplus \sigma^* V \to (\sigma^* V^* \oplus V^*) \otimes \pi^* (\Omega \otimes \mathcal{E})$$

This defines a diagram

$$\text{Bun}_{G_2,\Omega} \xrightarrow{q_{n,s}} \text{Bun}_{U_n,s} \xrightarrow{p_{n,s}} \text{Bun}_{G_2},$$

where $p_{n,s}$ sends $(V, \phi)$ to $(M = \pi_* V, \phi)$, and $q_{n,s}$ sends $(V, \phi)$ to $(M \otimes \mathcal{E}', \phi')$.

2.3. **Square root.** Denote by $\mathcal{A}_n$ the line bundle on $\text{Bun}_{G_2}$ whose fibre at $M$ is $\det R\Gamma(X, M)$. Write $\text{Bun}_{G_2}$ for the gerbe of square roots of $\mathcal{A}_n$ over $\text{Bun}_{G_2}$.

Our immediate purpose is to construct a distinguished square root of the line bundle $p_{n,s}^* \mathcal{A}_n$. Our construction will depend only on the choices of $\Omega_X^1, \mathcal{E}'$ that we made in Section 1.2, and also on a choice of $i \in k$ with $i^2 = -1$.

2.3.1. For $W \in \text{Bun}_n$ write for brevity $d(W) = \det R\Gamma(X, W)$, we view it as a $\mathbb{Z}/2\mathbb{Z}$-graded line.

For $\mathcal{A}_i \in \text{Bun}_1$ set

$$K(\mathcal{A}_1, \mathcal{A}_2) = \frac{d(\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes d(\mathcal{O})}{d(\mathcal{A}_1) \otimes d(\mathcal{A}_2)}$$

We view this line as a $\mathbb{Z}/2\mathbb{Z}$-graded. Then $K$ is bilinear up to a canonical isomorphism by ([9], Section 4.2.1-4.2.2).
One has a canonical Pfaffian line bundle $Pf$ on $\text{Bun}_{\Omega,2n}$ defined in ([2], Section 4.2.1). Pick $i \in k$ with $i^2 = -1$. This choice yields a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $Pf(W)^2 \cong d(W)$ for $W \in \text{Bun}_{\Omega,2n}$ as in loc.cit. An alternative construction of $Pf$ is given in [7].

Given $(V, \phi) \in \text{Bun}_{U, n}$ let $M = \phi_* V$. The symplectic form $\tilde{\phi} : \wedge^2 M \to \Omega$ induces an isomorphism $\det M \cong \tilde{\Omega}$. By ([8], Lemma 1) one has canonically for $(V, \phi) \in \text{Bun}_{U, n}$ and $M = \pi_* V$

$$(3) \quad d(M \otimes \mathcal{E}') \cong d(M) \otimes K(\Omega^n, \mathcal{E}') \otimes \frac{d(\mathcal{E}')^{2n}}{d(\Omega)^{2n}}$$

Our choice of $\Omega^1_\mathbb{Z}$, bilinearity of $K$, and (3) yield an isomorphism

$$(4) \quad Pf(M \otimes \mathcal{E}')^2 \cong d(M) \otimes \left(K(\Omega^1_\mathbb{Z}, \mathcal{E}') \otimes \frac{dX(\mathcal{E}')}{dX(\Omega)}\right)^{2n}$$

Denote by $\mathcal{L}_n$ the line bundle on $\text{Bun}_{U, n}$ whose fibre at $(V, \phi)$ is

$$Pf(M \otimes \mathcal{E}) \otimes \left(\frac{dX(\Omega)}{KX(\Omega^1_\mathbb{Z}, \mathcal{E}') \otimes dX(\mathcal{E}')}\right)^{n}$$

Then (4) yields the desired isomorphism over $\text{Bun}_{U, n}$

$$\mathcal{L}_n^2 \cong p^*_{n,s} \mathcal{A}_n$$

Let $\tilde{p}_{n,s} : \text{Bun}_{U, n} \to \text{Bun}_{G, n}$ be the map sending $(V, \psi)$ to $p_{n,s}(V, \phi) = (M, \tilde{\phi})$ and a line $\mathcal{L}_n(V, \phi)$ equipped with the above isomorphism $\mathcal{L}_n(V, \phi)^2 \cong d(M)$.

2.4. Dual pair $U_n, U_m$. Let $n, m \geq 1$. Write

$$\tau : \text{Bun}_{U, n} \times \text{Bun}_{U, m, s} \to \text{Bun}_{U, n, m, s}$$

for the map sending $(V_1, \phi_1) \in \text{Bun}_{U, n}$, $(V_2, \phi_2) \in \text{Bun}_{U, m, s}$ to $V_1 \otimes V_2$ with the induced isomorphism

$$\phi_1 \otimes \phi_2 : \sigma^*(V_1 \otimes V_2) \cong (V_1 \otimes V_2)^* \otimes \pi^* \Omega$$

The groups $U_n, U_m$ form a dual pair in $G_{nm}$ essentially via the map $p_{nm, s} \tau$. Let $\text{Aut}$ be the theta-sheaf on $\text{Bun}_{G, n}$ given in ([11], Definition 1). Define the theta-lifting functors

$$F_s : D^- (\text{Bun}_{U, n})! \to D^\omega (\text{Bun}_{U, m, s}), \quad F : D^- (\text{Bun}_{U, m, s})! \to D^\omega (\text{Bun}_{U, n})$$

following the framework of the geometric Langlands functoriality proposed in ([10], Section 2) for the kernel

$$M = \tau^* \tilde{p}_{nm, s}^* \text{Aut}[\dim \text{Bun}_{U, n} \times \text{Bun}_{U, m, s} - \dim \text{Bun}_{G, nm}]$$

That is, for $K \in D^- (\text{Bun}_{U, n})!$ and $K' \in D^- (\text{Bun}_{U, m, s})!$ we let

$$F_s(K) = (q_* s)_! (q^* s K \otimes M)[- \dim \text{Bun}_{U, n}] \quad \text{and} \quad F(K') = q_! (q^* s K' \otimes M)[- \dim \text{Bun}_{U, m, s}]$$

for the diagram of projections

$$\text{Bun}_{U, n} \xleftarrow{q_s} \text{Bun}_{U, n} \times \text{Bun}_{U, m, s} \xrightarrow{q_s} \text{Bun}_{U, m, s}$$
References


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