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GEOMETRIC THETA-LIFTING FOR UNITARY GROUPS

SERGEY LYSENKO

ABSTRACT. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality.

1. INTRODUCTION

1.1. In this note we define the geometric theta-lifting functors in the global nonramified setting. They are expected to provide new cases of the geometric Langlands functoriality. At level of functions the theta-lifting for unitary groups was studied by many authors ([1, 3, 4, 13, 15]).

The definition uses a splitting of the metaplectic extension over the unitary groups. The splitting existing in the literature at the level of functions ([6, 4, 14, 15]) are not entirely satisfactory. Namely, given a local field F and its degree two extension E , for the corresponding unitary group the splitting in *loc.cit.* is defined up to a multiplication by a character of E^*/F^* with values in \mathbb{C}^1 (the group of complex numbers of absolute value one). We give a more canonical geometric construction of this splitting.

1.2. Notations. We follow the conventions of [10]. So, we work over an algebraically closed field k of characteristic $p \neq 2$. We fix a prime $\ell \neq p$, write $\bar{\mathbb{Q}}_\ell$ for the algebraic closure of \mathbb{Q}_ℓ . All our stacks are defined over k . For an algebraic stack locally of finite type S we have the derived categories $D(S), D^-(S), D^\prec(S)$ of complexes of $\bar{\mathbb{Q}}_\ell$ -sheaves on S as in *loc.cit.*

Let X be a smooth projective connected curve. Write Ω for the canonical line bundle on X . Let $\pi : Y \rightarrow X$ be an étale degree 2 cover with Y connected. Write σ for the nontrivial automorphism of Y over X , $\pi_*\mathcal{O} = \mathcal{O} \oplus \mathcal{E}$, where \mathcal{E} is the sheaf of σ -antiinvariants, so $\mathcal{E}^2 \xrightarrow{\sim} \mathcal{O}$. Once and for all we pick a line bundles $\mathcal{E}', \Omega^{\frac{1}{2}}$ on X with isomorphisms $\mathcal{E}'^2 \xrightarrow{\sim} \mathcal{E}, (\Omega^{\frac{1}{2}})^2 \xrightarrow{\sim} \Omega$.

For an algebraic group G write Bun_G for the stack of G -torsors on X . For $n \geq 1$ write Bun_n (resp., $\text{Bun}_{n,Y}$) for the stack of G -torsors on X (resp., on Y).

2. MODULI OF UNITARY BUNDLES AND GEOMETRIC THETA-LIFTING

2.1. Pick $\epsilon = \pm 1$. For $n \geq 1$ an ϵ -hermitian vector bundle on Y (with respect to π) is a datum of $V \in \text{Bun}_{n,Y}$ with a nondegenerate form $\phi : V \otimes \sigma^*V \rightarrow \mathcal{O}_Y$ such that the

diagram commutes

$$\begin{array}{ccc} \phi^*(V \otimes \sigma^*V) & \xrightarrow{\sigma^*\phi} & \phi^*\mathcal{O} = \mathcal{O} \\ \downarrow & & \downarrow \epsilon \\ V \otimes \sigma^*V & \xrightarrow{\phi} & \mathcal{O}, \end{array}$$

the left vertical arrow being a canonical isomorphism. Write Bun_{U_n} (resp., $\text{Bun}_{U_n^-}$) for the stack of hermitian (resp., skew-hermitian) vector bundles on Y .

Here U_n is the group scheme on X , the quotient of $(\text{GL}_n \times Y)/\{\text{id}, \sigma\}$ by the diagonal action. Here $\sigma(g) = ({}^t g)^{-1}$ for $g \in \text{GL}_n$. The stacks like Bun_{U_n} have been studied for example in [5, 12].

2.1.1. For an ϵ -hermitian vector bundle V on Y viewing ϕ as an isomorphism $\phi : \sigma^*V \xrightarrow{\sim} V^*$, one gets $(\sigma^*\phi)^* = \epsilon\phi$. Consider the isomorphism

$$(1) \quad (\sigma^*\phi, \phi) : V \oplus \sigma^*V \rightarrow \sigma^*V^* \oplus V^*$$

on Y . Let $M = \pi_*C$. Let σ act on $V \oplus \sigma^*V$ such that the natural isomorphism $V \oplus \sigma^*V \xrightarrow{\sim} \pi^*M$ is σ -invariant. Then (1) is σ -equivariant, so descends to an isomorphism $\bar{\phi} : M \xrightarrow{\sim} M^*$ such that $\bar{\phi}^* = \epsilon\bar{\phi}$. We also view the latter as a map $\bar{\phi} : M \otimes M \rightarrow \mathcal{O}_X$. So, $\bar{\phi}$ is symmetric for $\epsilon = 1$ (resp., anti-symmetric for $\epsilon = -1$).

Consider the isomorphism

$$(2) \quad (-\sigma^*\phi, \phi) : V \oplus \sigma^*V \rightarrow \sigma^*V^* \oplus V^*$$

As above, there is a unique isomorphism $\phi' : M \xrightarrow{\sim} M^* \otimes \mathcal{E}$ such that $\pi^*(\phi') = (-\sigma^*\phi, \phi)$. Besides, ϕ' is symmetric for $\epsilon = -1$ (resp., antisymmetric for $\epsilon = 1$).

Let $\text{Bun}_{\mathbb{O}_{2n}}$ be the stack classifying $M \in \text{Bun}_{2n}$ with a nondegenerate symmetric form $\text{Sym}^2 M \rightarrow \mathcal{O}_X$. Let $\mathfrak{q}_n : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\mathbb{O}_{2n}}$ be the map sending (V, ϕ) to $(M = \pi_*V, \bar{\phi})$. Let $\mathfrak{q}_n^- : \text{Bun}_{U_n^-} \rightarrow \text{Bun}_{\mathbb{O}_{2n}}$ be the map sending (V, ϕ) to $(M \otimes \mathcal{E}', \phi')$. Here we have viewed ϕ' as a map $\text{Sym}^2(M \otimes \mathcal{E}') \rightarrow \mathcal{O}_X$.

The stack $\text{Bun}_{\text{Sp}_{2n}}$ classifies $M \in \text{Bun}_{2n}$ with a symplectic form $\wedge^2 M \rightarrow \mathcal{O}_X$. Let $\mathfrak{p}_n : \text{Bun}_{U_n} \rightarrow \text{Bun}_{\text{Sp}_{2n}}$ be the map sending (V, ϕ) to $(M \otimes \mathcal{E}', \phi')$. Here $M = \pi_*V$. Let also $\mathfrak{p}_n^- : \text{Bun}_{U_n^-} \rightarrow \text{Bun}_{\text{Sp}_{2n}}$ be the map sending (V, ϕ) to $(M, \bar{\phi})$.

2.1.2. We have a canonical identification $\pi^*\mathcal{E} \xrightarrow{\sim} \mathcal{O}$ such that the descent data for $\pi^*\mathcal{E}$ are given by the action of σ on \mathcal{O} as -1 . This gives an isomorphism $\delta : \text{Bun}_{U_n} \xrightarrow{\sim} \text{Bun}_{U_n^-}$ sending (V, ϕ) to $(V \otimes \pi^*\mathcal{E}', \phi)$. The diagram commutes

$$\begin{array}{ccccc} & & \text{Bun}_{U_n} & & \\ & \swarrow \mathfrak{q}_n & \downarrow \delta & \searrow \mathfrak{p}_n & \\ \text{Bun}_{\mathbb{O}_{2n}} & \xleftarrow{\mathfrak{q}_n^-} & \text{Bun}_{U_n^-} & \xrightarrow{\mathfrak{p}_n^-} & \text{Bun}_{\text{Sp}_{2n}} \end{array}$$

2.1.3. The stack Bun_{U_1} is a group stack described in ([8], Appendix A). We have the norm map $N : \text{Bun}_{1,Y} \rightarrow \text{Bun}_1$ sending \mathcal{L} to $\mathcal{E} \otimes \det(\pi_*\mathcal{L})$, this is a homomorphism of group stacks. The stack Bun_{U_1} classifies $V \in \text{Bun}_{1,Y}$ together with a trivialization $N(V) \xrightarrow{\sim} \mathcal{O}_X$. The stack Bun_{U_1} has connected components $\text{Bun}_{U_1}^a$ indexed by $a \in \mathbb{Z}/2\mathbb{Z}$, $\text{Bun}_{U_1}^0$ being the connected component of unity. The stack $\text{Bun}_{U_1^-}$ classifies $V \in \text{Bun}_{1,Y}$ with an isomorphism $N(V) \xrightarrow{\sim} \mathcal{E}$.

The group stack Bun_{U_1} acts on $\mathrm{Bun}_{U_n^-}$ sending $(\mathcal{L}, \phi_{\mathcal{L}}) \in \mathrm{Bun}_{U_1}, (V, \phi) \in \mathrm{Bun}_{U_n^-}$ to $(V \otimes \mathcal{L}, \phi \otimes \phi_{\mathcal{L}}) \in \mathrm{Bun}_{U_n^-}$. Let $\rho_n : \mathrm{Bun}_{U_n} \rightarrow \mathrm{Bun}_{U_1}$ be the map sending (V, ϕ) to $(\det V, \det \phi)$.

2.1.4. For $n > 1$ let $(V, \phi) \in \mathrm{Bun}_{U_n}$. Then $\det \phi : \sigma^* \det V \xrightarrow{\sim} \det V^*$ can be seen as a trivialization $\xi : N(\det V) \xrightarrow{\sim} \mathcal{O}$. Let Bun_{SU_n} be the stack classifying $V \in \mathrm{Bun}_{U_n}$ with an isomorphism $\det V \xrightarrow{\sim} \mathcal{O}_Y$ such that the induced isomorphism $N(\det V) \xrightarrow{\sim} N(\mathcal{O}_Y) \xrightarrow{\sim} \mathcal{O}_X$ is ξ . So, Bun_{SU_n} is the fibre of the map $\rho_n : \mathrm{Bun}_{U_n} \rightarrow \mathrm{Bun}_{U_1}$ over the unit $\mathcal{O}_Y \in \mathrm{Bun}_{U_1}$ of this group stack.

By ([5], Theorem 2) the stack Bun_{SU_n} is connected. By ([5], Theorem 3), if $n > 1$ then $\mathrm{Pic}(\mathrm{Bun}_{SU_n}) \xrightarrow{\sim} \mathbb{Z}$. The line bundle on Bun_{SU_n} sending $V \in \mathrm{Bun}_{U_n}$ to $\det \mathrm{R}\Gamma(Y, V)$ is twice a generator of $\mathrm{Pic}(\mathrm{Bun}_{SU_n})$ by ([12], Remark 3.6(2)).

2.2. Twist by Ω . As in [8] write Bun_{G_n} for the stack classifying $M \in \mathrm{Bun}_{2n}$ with a symplectic form $\wedge^2 M \rightarrow \Omega_X$. Write $\mathrm{Bun}_{\mathbb{O}_{2n}, \Omega}$ for the stack classifying $M \in \mathrm{Bun}_{2n}$ with a nondegenerate symmetric bilinear form $\mathrm{Sym}^2 M \rightarrow \Omega$.

Write $\mathrm{Bun}_{U_{n,s}}$ for the stack classifying $V \in \mathrm{Bun}_{n,Y}$ together with an isomorphism

$$\phi : \sigma^* V \xrightarrow{\sim} V^* \otimes \pi^* \Omega$$

such that ϕ is skew-hermitian. In other words, $(\sigma^* \phi)^* = -\phi$. Here s stands for ‘skew-hermitian’. Write Ω_Y for the canonical line bundle on Y , one has $\pi^* \Omega \xrightarrow{\sim} \Omega_Y$.

As in Section 2.1.1, for $(V, \phi) \in \mathrm{Bun}_{U_{n,s}}$ let $M = \pi_* V$. There is a unique symplectic form $\bar{\phi} : M \xrightarrow{\sim} M^* \otimes \Omega$ such that

$$\pi^* \bar{\phi} = (\sigma^* \phi, \phi) : V \oplus \sigma^* V \rightarrow (\sigma^* V^* \oplus V^*) \otimes \pi^* \Omega$$

There is also a unique symmetric bilinear form $\phi' : M \rightarrow M^* \otimes \Omega \otimes \mathcal{E}$ such that

$$\pi^* \phi' = (-\sigma^* \phi, \phi) : V \oplus \sigma^* V \rightarrow (\sigma^* V^* \oplus V^*) \otimes \pi^* (\Omega \otimes \mathcal{E})$$

This defines a diagram

$$\mathrm{Bun}_{\mathbb{O}_{2n}, \Omega} \xleftarrow{\mathfrak{q}_{n,s}} \mathrm{Bun}_{U_{n,s}} \xrightarrow{\mathfrak{p}_{n,s}} \mathrm{Bun}_{G_n},$$

where $\mathfrak{p}_{n,s}$ sends (V, ϕ) to $(M = \pi_* V, \bar{\phi})$, and $\mathfrak{q}_{n,s}$ sends (V, ϕ) to $(M \otimes \mathcal{E}', \phi')$.

2.3. Square root. Denote by \mathcal{A}_n the line bundle on Bun_{G_n} whose fibre at M is $\det \mathrm{R}\Gamma(X, M)$. Write $\widetilde{\mathrm{Bun}}_{G_n}$ for the gerbe of square roots of \mathcal{A}_n over Bun_{G_n} .

Our immediate purpose is to construct a distinguished square root of the line bundle $\mathfrak{p}_{n,s}^* \mathcal{A}_n$. Our construction will depend only on the choices of $\Omega^{\frac{1}{2}}$, \mathcal{E}' that we made in Section 1.2, and also on a choice of $i \in k$ with $i^2 = -1$.

2.3.1. For $W \in \mathrm{Bun}_n$ write for brevity $d(W) = \det \mathrm{R}\Gamma(X, W)$, we view it as a $\mathbb{Z}/2\mathbb{Z}$ -graded line.

For $\mathcal{A}_i \in \mathrm{Bun}_1$ set

$$K(\mathcal{A}_1, \mathcal{A}_2) = \frac{d(\mathcal{A}_1 \otimes \mathcal{A}_2) \otimes d(\mathcal{O})}{d(\mathcal{A}_1) \otimes d(\mathcal{A}_2)}$$

We view this line as a $\mathbb{Z}/2\mathbb{Z}$ -graded. Then K is bilinear up to a canonical isomorphism by ([9], Section 4.2.1-4.2.2).

One has a canonical Pfaffian line bundle Pf on $\text{Bun}_{\mathbb{O}_{2n}, \Omega}$ defined in ([2], Section 4.2.1). Pick $i \in k$ with $i^2 = -1$. This choice yields a canonical $\mathbb{Z}/2\mathbb{Z}$ -graded isomorphism $Pf(W)^2 \xrightarrow{\sim} d(W)$ for $W \in \text{Bun}_{\mathbb{O}_{2n}, \Omega}$ as in *loc.cit.* An alternative construction of Pf is given in [7].

Given $(V, \phi) \in \text{Bun}_{U_n, s}$ let $M = \phi_* V$. The symplectic form $\bar{\phi} : \wedge^2 M \rightarrow \Omega$ induces an isomorphism $\det M \xrightarrow{\sim} \Omega^n$. By ([8], Lemma 1) one has canonically for $(V, \phi) \in \text{Bun}_{U_n, s}$ and $M = \pi_* V$

$$(3) \quad d(M \otimes \mathcal{E}') \xrightarrow{\sim} d(M) \otimes K(\Omega^n, \mathcal{E}') \otimes \frac{d(\mathcal{E}')^{2n}}{d(\mathcal{O})^{2n}}$$

Our choice of $\Omega^{\frac{1}{2}}$, bilinearity of K , and (3) yield an isomorphism

$$(4) \quad Pf(M \otimes \mathcal{E}')^2 \xrightarrow{\sim} d(M) \otimes \left(K(\Omega^{\frac{1}{2}}, \mathcal{E}') \otimes \frac{d_X(\mathcal{E}')}{d_X(\mathcal{O})} \right)^{2n}$$

Denote by \mathcal{L}_n the line bundle on $\text{Bun}_{U_n, s}$ whose fibre at (V, ϕ) is

$$Pf(M \otimes \mathcal{E}) \otimes \left(\frac{d_X(\mathcal{O})}{K_X(\Omega^{\frac{1}{2}}, \mathcal{E}) \otimes d_X(\mathcal{E}')} \right)^n$$

Then (4) yields the desired isomorphism over $\text{Bun}_{U_n, s}$

$$\mathcal{L}_n^2 \xrightarrow{\sim} \mathfrak{p}_{n, s}^* \mathcal{A}_n$$

Let $\tilde{\mathfrak{p}}_{n, s} : \text{Bun}_{U_n, s} \rightarrow \widetilde{\text{Bun}}_{G_n}$ be the map sending (V, ψ) to $\mathfrak{p}_{n, s}(V, \phi) = (M, \bar{\phi})$ and a line $\mathcal{L}_n(V, \phi)$ equipped with the above isomorphism $\mathcal{L}_n(V, \phi)^2 \xrightarrow{\sim} d(M)$.

2.4. Dual pair U_n, U_m . Let $n, m \geq 1$. Write

$$\tau : \text{Bun}_{U_n} \times \text{Bun}_{U_m, s} \rightarrow \text{Bun}_{U_{nm}, s}$$

for the map sending $(V_1, \phi_1) \in \text{Bun}_{U_n}$, $(V_2, \phi_2) \in \text{Bun}_{U_m, s}$ to $V_1 \otimes V_2$ with the induced isomorphism

$$\phi_1 \otimes \phi_2 : \sigma^*(V_1 \otimes V_2) \xrightarrow{\sim} (V_1 \otimes V_2)^* \otimes \pi^* \Omega$$

The groups U_n, U_m form a dual pair in G_{nm} essentially via the map $\mathfrak{p}_{nm, s} \tau$. Let Aut be the theta-sheaf on $\widetilde{\text{Bun}}_{G_n}$ given in ([11], Definition 1). Define the theta-lifting functors

$$F_s : D^-(\text{Bun}_{U_n})! \rightarrow D^{\prec}(\text{Bun}_{U_m, s}), \quad F : D^-(\text{Bun}_{U_m, s})! \rightarrow D^{\prec}(\text{Bun}_{U_n})$$

following the framework of the geometric Langlands functoriality proposed in ([10], Section 2) for the kernel

$$\mathcal{M} = \tau^* \tilde{\mathfrak{p}}_{nm, s}^* \text{Aut}[\dim \text{Bun}_{U_n} \times \text{Bun}_{U_m, s} - \dim \text{Bun}_{G_{nm}}]$$

That is, for $K \in D^-(\text{Bun}_{U_n})!$ and $K' \in D^-(\text{Bun}_{U_m, s})!$ we let

$$F_s(K) = (q_s)_!(q_s^* K \otimes \mathcal{M})[-\dim \text{Bun}_{U_n}] \quad \text{and} \quad F(K') = q'_!(q_s^* K' \otimes \mathcal{M})[-\dim \text{Bun}_{U_m, s}]$$

for the diagram of projections

$$\text{Bun}_{U_n} \xleftarrow{q} \text{Bun}_{U_n} \times \text{Bun}_{U_m, s} \xrightarrow{q_s} \text{Bun}_{U_m, s}$$

REFERENCES

- [1] J. Adams, L-functoriality for dual pairs, *Astérisque*, 171-172, *Orbites Unipotentes et Représentations*, 2 (1989), 85-129.
- [2] A. Beilinson, V. Drinfeld, Quantization of Hitchin integrable system and Hecke eigen-sheaves, preprint available at <http://math.uchicago.edu/~drinfeld/langlands/hitchin/BD-hitchin.pdf>
- [3] St. Gelbart, On theta-series lifting for unitary groups, in: M. Ram Murty ed., *Theta Functions from the Classical to the Modern*, American Mathematical Society, 1993
- [4] M. Harris, St. Kudla, W. J. Sweet, Theta dichotomy for unitary groups, *Journal of AMS*, Vol. 9, Nu. 4 (1996), 941-1004
- [5] J. Heinloth, Uniformization of \mathcal{G} -bundles, *Math. Annalen* (2010), Vol. 347, Issue 3, 499528
- [6] St. Kudla, Splitting metaplectic covers of dual reductive pairs, *Israel J. Math.* (1994), Vol. 87, Issue 13, 361401
- [7] Y. Laszlo and C. Sorger. The line bundles on the moduli of parabolic G -bundles over curves and their sections. *Ann. Sci. ENS.*, ser. 4, vol. 30 (1997), 499525.
- [8] S. Lysenko, Geometric Waldspurger periods, *Compositio Math.* 144 (2008), 377438
- [9] S. Lysenko, Twisted geometric Langlands correspondence for a torus, *Int. Math. Res. Notices* (2015) (18): 8680-8723
- [10] S. Lysenko, On the automorphic sheaves for GSp_4 , arXiv:1901.04447
- [11] S. Lysenko, Moduli of metaplectic bundles on curves and Theta-sheaves, *Ann. Scient. ENS*, 4 série, t. 39 (2006), p. 415-466
- [12] G. Pappas, M. Rapoport, Some questions about G -bundles on curves, available at <https://www.math.uni-bonn.de/ag/algeom/preprints/G-bundles6.pdf>
- [13] D. Prasad, Theta correspondence for unitary groups, *Pacific J. Math.*, Vol.194, No.2 (2000), 427-438
- [14] Shu-Yen Pan, Splittings of the metaplectic covers of some reductive dual pairs, *Pacific J. Math.*, Vol. 199, No. 1 (2001), 163-226
- [15] Ch. Wu, Irreducibility of theta lifting for unitary groups, *J. Number Theory* 133 (2013), 3296 - 3318

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