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ON THE AUTOMORPHIC SHEAVES FOR GSp_4

S. LYSENKO

ABSTRACT. In this paper we first review the setting for the geometric Langlands functoriality and establish a result for the ‘backward’ functoriality functor. We illustrate this by known examples of the geometric theta-lifting. We then apply the above result to obtain new Hecke eigen-sheaves. The most important application is a construction of the automorphic sheaf for $G = \mathrm{GSp}_4$ attached to a \check{G} -local system on a curve X such that its standard representation is an irreducible local system of rank 4 on X .

1. INTRODUCTION

1.1. Let X be a smooth projective curve over an algebraically closed field k of characteristic $p \geq 0$. The purpose of this paper is twofold:

- i) We formulate a setting for the geometric Langlands functoriality (at the non-ramified level) and prove an (easy) result on the ‘backward’ functoriality functor. We illustrate our setting with known examples of the geometric theta-lifting and Eisenstein series.
- ii) We apply this result combined with previously established results on the geometric theta-lifting to obtain new examples of automorphic sheaves. The most important application is a construction of new automorphic sheaves for GSp_4 .

1.1.1. Let G, H be split connected reductive groups over k , write \check{G}, \check{H} for the Langlands dual groups over k . Write Bun_G for the stack of G -torsors on X , $\mathrm{D}(\mathrm{Bun}_G)$ for the derived category of $\bar{\mathbb{Q}}_\ell$ -sheaves on Bun_G . Given a homomorphism $\bar{\kappa} : \check{G} \times \mathrm{SL}_2 \rightarrow \check{H}$, let $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$ be its restriction to the standard maximal torus of SL_2 . According to the Langlands philosophy, one may ask about the corresponding geometric Langlands functoriality functor $F_H : \mathrm{D}(\mathrm{Bun}_G) \rightarrow \mathrm{D}(\mathrm{Bun}_H)$ commuting with the actions of $\mathrm{Rep}(\check{H})$. It is understood that $\mathrm{Rep}(\check{H})$ acts on $\mathrm{D}(\mathrm{Bun}_G)$ via its restriction through κ . The functor F_H is expected to be given by some kernel \mathcal{M} on $\mathrm{Bun}_G \times \mathrm{Bun}_H$. Here SL_2 is the SL_2 of *Arthur*. We discuss the corresponding setting in Section 2.

Consider the ‘backward’ functor $F_G : \mathrm{D}(\mathrm{Bun}_H) \rightarrow \mathrm{D}(\mathrm{Bun}_G)$ given by the same kernel \mathcal{M} . Our first result is Theorem 2.1.11 describing the natural relation between F_G and the Hecke functors on $\mathrm{D}(\mathrm{Bun}_H)$ and $\mathrm{D}(\mathrm{Bun}_G)$. This naturally leads to a notion of a partial Hecke property of $K \in \mathrm{D}(\mathrm{Bun}_G)$ with respect to a given \check{H} -local system $E_{\check{H}}$ on X (cf. Definition 2.1.1). We illustrate our setting of the geometric Langlands functoriality with known cases of the geometric theta-lifting and geometric Eisenstein series in Sections 2.2-2.4.

1.1.2. In Section 3.1 we obtain some results related to a possible extension of the partial Hecke property of $K \in D(\mathrm{Bun}_G)$ to the whole Hecke property, provided that the corresponding local system $E_{\check{G}}$ is a \check{G} -local system. This question can be thought of in terms of the seminal recent paper [1], where a spectral decomposition of the derived category $D(\mathrm{Bun}_G)$ is established over the stack $\mathrm{LocSys}_{\check{G}}^{\mathrm{restr}}$ of *restricted* \check{G} -local systems.

We establish Proposition 3.1.4, which gives examples where a partial Hecke property can be extended to the whole Hecke property. Our proof of parts 3),4) of Proposition 3.1.4 uses ([1], Theorem 10.5.2). It is used only to obtain some properties of our automorphic sheaves, not an additional structure. Our construction of the Hecke eigen-sheaves under consideration is algorithmic.

1.1.3. We apply the above to construct new automorphic sheaves for $G = \mathrm{GSp}_4$. Namely, assume k of characteristic $p > 2$ (this is needed as we apply the results on the geometric theta-lifting). Let $E_{\check{G}}$ be a \check{G} -local system on X such that its standard representation is an irreducible (rank 4) local system on X .

We construct an object \mathcal{K} of the derived category $D(\mathrm{Bun}_G)$ of $\bar{\mathbb{Q}}_\ell$ -sheaves on Bun_G , which is a $E_{\check{G}}$ -Hecke eigen-sheaf. It is obtained via the geometric theta-lifting and confirms a conjecture proposed twelve years ago ([30], Conjecture 6(ii)). The non-vanishing of \mathcal{K} is established in [32]. In the case when both the curve and $E_{\check{G}}$ are defined over a finite subfield $k_0 \subset k$, we also give also an alternative argument showing that \mathcal{K} does not vanish. The complex \mathcal{K} is bounded from above over each open substack of Bun_G of finite type, and all its perverse cohomology sheaves are constructible.

In Appendix B we discuss some general properties of abelian categories over stacks related to the above question of the extending the partial Hecke property.

1.1.4. Organization. In Section 2 we propose a setting for the geometric Langlands functoriality and prove Theorem 2.1.11. We also illustrate our setting with known examples of the geometric theta-lifting and geometric Eisenstein series. In Section 3 we formulate our results that have not appeared in Section 2. They are related to the extension of a partial Hecke property and the construction of new automorphic sheaves. The proofs are collected in Section 4.

1.2. Notation.

1.2.1. Work over an algebraically closed field k of characteristic $p > 2$. The case $p = 2$ is excluded as we are using the results of [28, 29, 31, 23] on the the geometric Weil representation (they could possibly be extended to the $p = 2$ case using [20]). All our stacks are defined over k . Let $\ell \neq p$ be a prime, $\bar{\mathbb{Q}}_\ell$ the algebraic closure of \mathbb{Q}_ℓ .

We work with etale $\bar{\mathbb{Q}}_\ell$ -sheaves on algebraic stacks locally of finite type (cf. [25]). We ignore the Tate twists everywhere (they are easy to recover if necessary). Let X be a smooth projective connected curve, Ω the canonical line bundle on X .

We use the following conventions from ([28], Section 2.1). For an algebraic stack locally of finite type S we denote by $D(S)$ the derived category of unbounded $\bar{\mathbb{Q}}_\ell$ -complexes on S with constructible cohomologies defined in ([25], Remark 3.21) and denoted $D_c(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.* For $* = +, -, b$ we have the corresponding full subcategory $D^*(S) \subset D(S)$ denoted $D_c^*(S, \bar{\mathbb{Q}}_\ell)$ in *loc.cit.*

Write $D^-(S)_! \subset D^-(S)$ for the full subcategory of objects, which are extensions by zero from some open substack of finite type. Write $D^\sphericalcap(S) \subset D(S)$ for the full subcategory of complexes $K \in D(S)$ such that for any open substack of finite type $U \subset S$, $K|_U \in D^-(U)$.

For an algebraic stack locally of finite type S we also consider the version $D_{nc}(S)$ of $D(S)$, where we do not assume that the cohomologies are constructible. It is defined as the homotopy category of the corresponding DG-category of $\bar{\mathbb{Q}}_\ell$ -sheaves in étale topology on S , cf. [12, 11, 1, 14]. Then $D(S) \subset D_{nc}(S)$ is a full subcategory. For any morphism $f : S \rightarrow S'$ of algebraic stacks locally of finite type the direct image with compact support $f_! : D_{nc}(S) \rightarrow D_{nc}(S')$ is defined by passing to the homotopy for the corresponding functor between the DG-categories ([11], Corollary 1.4.2). Similarly, we have the functors $f^!, f^* : D_{nc}(S') \rightarrow D_{nc}(S)$ and $f_* : D_{nc}(S) \rightarrow D_{ns}(S')$.

For an algebraic stack S locally of finite type $D_{nc}(S)$ is equipped with a perverse t-structure. The corresponding DG-category of is left complete for this t-structure by ([1], Th. 1.1.4), so by ([27], 1.2.1.19)

(P) If $K \in D^\sphericalcap(S)$ is such that $H^i(K) = 0$ for all i then $K = 0$.

We denote by $H^i(K)$ the i -cohomology sheaf of K on S .

In our proof of Proposition 3.1.4 we apply ([1], Theorem 10.5.2), which requires the whole formalism of the DG-categories of $\bar{\mathbb{Q}}_\ell$ -sheaves on prestacks from [12, 1]. However, we work mostly in the above setting of triangulated categories, not the DG-setting of a theory of sheaves from [11, 1]. The reason is that the results of [28, 31] that we are using are for the moment established only on the level of triangulated categories.

1.2.2. For a connected reductive group G over k write Bun_G for the stack of G -torsors on X . Denote by \check{G} the Langlands dual group to G over $\bar{\mathbb{Q}}_\ell$. Denote by $\mathrm{Rep}(\check{G})$ the abelian category of finite-dimensional representations of \check{G} over $\bar{\mathbb{Q}}_\ell$. The trivial G -torsor on some base is denoted \mathcal{F}_G^0 .

For $x \in X$ let $\mathrm{Gr}_{G,x}$ denote the affine Grassmanian classifying (\mathcal{F}_G, β) , where \mathcal{F}_G is a G -torsor on X , $\beta : \mathcal{F}_G \xrightarrow{\sim} \mathcal{F}_G^0$ is a trivialization over $X - x$. Let \mathcal{O}_x be the completed local ring of X at x , $D_x = \mathrm{Spec} \mathcal{O}_x$. The spherical Hecke category Sph_G is defined as the category of $G(\mathcal{O}_x)$ -equivariant perverse sheaves on $\mathrm{Gr}_{G,x}$. The Satake equivalence provides an equivalence of symmetric monoidal categories $\mathrm{Loc} : \mathrm{Rep}(\check{G}) \xrightarrow{\sim} \mathrm{Sph}_G$.

Write \mathcal{H}_G for the Hecke stack classifying $(x \in X, \mathcal{F}_G \xrightarrow{\sim} \mathcal{F}'_G|_{X-x})$ with $\mathcal{F}_G, \mathcal{F}'_G \in \mathrm{Bun}_G$. It fits into the diagram

$$\mathrm{Bun}_G \times X \begin{array}{c} \xleftarrow{h_G^\leftarrow} \\ \xrightarrow{\mathrm{supp}} \\ \xrightarrow{h_G^\rightarrow} \end{array} \mathcal{H}_G \begin{array}{c} \xrightarrow{h_G^\rightarrow} \\ \xleftarrow{h_G^\leftarrow} \end{array} \mathrm{Bun}_G,$$

where h_G^\leftarrow (resp., h_G^\rightarrow) sends the above point to \mathcal{F}_G (resp., \mathcal{F}'_G). The map supp sends the above point to x . The Hecke functors

$$(1) \quad H_G^\leftarrow, H_G^\rightarrow : \mathrm{Sph}_G \times D^\sphericalcap(S \times \mathrm{Bun}_G) \rightarrow D^\sphericalcap(S \times \mathrm{Bun}_G \times X)$$

are defined in ([28], Section 2.1.1). Namely, for $x \in X$ write ${}_x\mathcal{H}_G$ for the base change of \mathcal{H}_G by $\mathrm{Spec} k \xrightarrow{x} X$. Write $\mathrm{Bun}_{G,x}$ for the stack classifying $\mathcal{F}_G \in \mathrm{Bun}_G$ with a trivialization $\mathcal{F}_G \xrightarrow{\sim} \mathcal{F}_G^0|_{D_x}$. Write $\mathrm{id}^l, \mathrm{id}^r$ for the isomorphisms

$${}_x\mathcal{H}_G \xrightarrow{\sim} \mathrm{Bun}_{G,x} \times^{G(\mathcal{O}_x)} \mathrm{Gr}_{G,x}$$

such that the projection on the first factor corresponds to $h_G^\leftarrow, h_G^\rightarrow$ respectively. To $\mathcal{S} \in \text{Sph}_G$, $K \in D^\leftarrow(S \times \text{Bun}_G)$ one attaches their twisted external products $(K \boxtimes \mathcal{S})^l, (K \boxtimes \mathcal{S})^r$ on ${}_x\mathcal{H}_G$. They are normalized to be perverse for K perverse. Then

$${}_x\text{H}_G^\leftarrow, {}_x\text{H}_G^\rightarrow : \text{Sph}_G \times D^\leftarrow(S \times \text{Bun}_G) \rightarrow D^\leftarrow(S \times \text{Bun}_G)$$

are given by

$${}_x\text{H}_G^\leftarrow(\mathcal{S}, K) = (h_G^\leftarrow)_!(K \boxtimes \mathcal{S})^r \quad \text{and} \quad {}_x\text{H}_G^\rightarrow(\mathcal{S}, K) = (h_G^\rightarrow)_!(K \boxtimes \mathcal{S})^l$$

We have denoted by $*$: $\text{Sph}_G \xrightarrow{\sim} \text{Sph}_G$ the covariant equivalence induced by the map $G(F_x) \xrightarrow{\sim} G(F_x), g \mapsto g^{-1}$. Letting x vary in X , one similarly gets the functors (1). Along the same line one also defines

$$\text{H}_G^\leftarrow, \text{H}_G^\rightarrow : \text{Sph}_G \times D_{nc}(S \times \text{Bun}_G) \rightarrow D_{nc}(S \times \text{Bun}_G \times X)$$

In view of the Satake equivalence, we also view $*$ as a functor $*$: $\text{Rep}(\check{G}) \rightarrow \text{Rep}(\check{G})$. It is induced by a Chevalley involution of \check{G} .

1.2.3. For $\theta \in \pi_1(G)$ and $x \in X$ let Gr_G^θ denote the connected component of Gr_G classifying (\mathcal{F}_G, β) , where \mathcal{F}_G is a G -torsor on X , $\beta : \mathcal{F}_G \xrightarrow{\sim} \mathcal{F}_G^0$ is a trivialization over $X - x$ such that $V_{\mathcal{F}_G^0} \xrightarrow{\sim} V_{\mathcal{F}_G}(\langle \theta, \check{\lambda} \rangle x)$ for a one dimensional G -module of weight $\check{\lambda}$. Here $\pi_1(G)$ is the algebraic fundamental group of G (the quotient of the coweights lattice by the coroots lattice). Write Bun_n for the stack of rank n vector bundles on X .

1.2.4. As in ([28], Section 2.1.2), write Loc_X for the category of local systems on X and set $\text{DLoc}_X = \bigoplus_{i \in \mathbb{Z}} \text{Loc}_X[i] \subset D(X)$. Then DLoc_X as a symmetric monoidal category naturally. If $x \in X$ then a datum of a symmetric monoidal functor $E : \text{Rep}(\check{G}) \rightarrow \text{DLoc}_X$ is equivalent to a datum of a homomorphism $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{G}$ algebraic along \mathbb{G}_m and continuous along $\pi_1(X, x)$.

As in ([28], Section 2.1.2) we set $\text{DSph}_G = \bigoplus_{i \in \mathbb{Z}} \text{Sph}_G[i] \subset D(\text{Gr}_G)$. The above Satake equivalence is extended to an equivalence of symmetric monoidal categories

$$\text{Loc}^\tau : \text{Rep}(\check{G} \times \mathbb{G}_m) \xrightarrow{\sim} \text{DSph}_G,$$

so that $z \in \mathbb{G}_m$ acts on $V \in \text{Rep}(\check{G} \times \mathbb{G}_m)$ by $z \mapsto z^{-r}$ if and only if $\text{Loc}^\tau(V) \in \text{Sph}_G[r]$. The above Hecke action is extended to an action of DSph_G on $D^\leftarrow(\text{Bun}_G)$, so that $\text{H}_G^\leftarrow(\mathcal{S}[r], \cdot) = \text{H}_G^\leftarrow(\mathcal{S}, \cdot)[r]$ for $\mathcal{S} \in \text{Sph}_G$. We extend $*$ to a functor $*$: $\text{DSph}_G \rightarrow \text{DSph}_G$ by $*(\mathcal{S}[r]) \xrightarrow{\sim} (*\mathcal{S})[r]$.

2. GEOMETRIC LANGLANDS FUNCTORIALITY AND THE THETA-LIFTING

In this section we formulate our results and definitions related to the geometric Langlands functoriality. We prove Theorem 2.1.11 consisting of two parts. The 'straight direction' part is known (cf. [28], Theorem 3), and the 'backward direction' part is new to the best of our knowledge. We review the geometric theta-lifting (cf. [28, 15]) and the geometric Eisenstein series (cf. [4]) giving examples of the geometric Langlands functoriality.

2.1. Geometric Langlands functoriality. We use the following version of the notion of a Hecke eigen-sheaf from ([9], Section 2.8). Write s for the involution of X^2 permuting the two copies of X . The diagonal in X^2 is sometimes denoted $\Delta(X)$.

Definition 2.1.1. (*Partial Hecke property*). Let G, H be connected reductive groups over k , $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$ be a homomorphism over $\bar{\mathbb{Q}}_\ell$. Pick $x \in X$. Let $E : \mathrm{Rep}(\check{H}) \rightarrow \mathrm{DLoc}_X$, $V \mapsto E^V$ be a symmetric monoidal functor. It yields the corresponding homomorphism $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{H}$.

We say that $K \in D^\prec(\mathrm{Bun}_G)$ is equipped with a E -Hecke (or σ -Hecke) property with respect to κ if for $V \in \mathrm{Rep}(\check{H})$ we are given a functorial isomorphism

$$(2) \quad \alpha_V : H_G^\leftarrow(V, K) \xrightarrow{\sim} K \boxtimes E^V[1]$$

on $\mathrm{Bun}_G \times X$ such that H1) and H2) below are satisfied. First, for $V_1, \dots, V_n \in \mathrm{Rep}(\check{H})$ iterating α , one gets an isomorphism on $\mathrm{Bun}_G \times X^n$

$$\alpha_{V_1, \dots, V_n} : H_G^\leftarrow(V_1 \boxtimes \dots \boxtimes V_n, K) \xrightarrow{\sim} K \boxtimes E^{V_1}[1] \boxtimes \dots \boxtimes E^{V_n}[1]$$

We require that for $V_1, V_2 \in \mathrm{Rep}(\check{H})$ the following two diagrams commute H1)

$$\begin{array}{ccc} H_G^\leftarrow(V_1 \boxtimes V_2, K) & \xrightarrow{\alpha_{V_1, V_2}} & K \boxtimes E^{V_1}[1] \boxtimes E^{V_2}[1] \\ \downarrow & & \downarrow \\ (\mathrm{id} \times s)^* H_G^\leftarrow(V_2 \boxtimes V_1, K) & \xrightarrow{(\mathrm{id} \times s)^* \alpha_{V_2, V_1}} & K \boxtimes s^*(E^{V_2}[1] \boxtimes E^{V_1}[1]), \end{array}$$

H2)

$$\begin{array}{ccc} H_G^\leftarrow(V_1 \boxtimes V_2, K) |_{\mathrm{Bun}_G \times \Delta(X)} & \xrightarrow{\alpha_{V_1, V_2}} & K \boxtimes E^{V_1}[1] \boxtimes E^{V_2}[1] |_{\mathrm{Bun}_G \times \Delta(X)} \\ \downarrow & & \downarrow \\ H_G^\leftarrow(V_1 \otimes V_2, K)[1] & \xrightarrow{\alpha_{V_1 \otimes V_2}} & K \boxtimes E^{V_1 \otimes V_2}[2] \end{array}$$

Here the vertical arrows are the natural isomorphisms. It is understood that $\mathrm{Rep}(\check{H})$ acts on $D^\prec(\mathrm{Bun}_G)$ via the restriction by κ .

If moreover $G = H$ and $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{G}$ is the projection then we say that K is a E -Hecke eigen-sheaf.

We insist that $K \in D^\prec(\mathrm{Bun}_G)$ in Definition 2.1.1 has only a ‘partial Hecke property’, the isomorphisms α_V are not given for all $V \in \mathrm{Rep}(\check{G})$, but only for a part of them.

Remark 2.1.2. The existence of the left vertical isomorphism in the diagram H1) of Definition 2.1.1 is evident over $\mathrm{Bun}_G \times (X^2 - \Delta(X))$. The fact that it extends to the whole of $\mathrm{Bun}_G \times X^2$ follows from ([9], Proposition 2.8).

Remark 2.1.3. In the situation of Definition 2.1.1 let $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{G}$ be a homomorphism, and $K \in D^\prec(\mathrm{Bun}_G)$ be a nonzero σ -Hecke eigen-sheaf. According to the Arthur-Langlands philosophy, σ should be a composition

$$\pi_1(X, x) \times \mathbb{G}_m \xrightarrow{\mathrm{id} \times t} \pi_1(X, x) \times \mathrm{SL}_2 \rightarrow \check{G},$$

where $t : \mathbb{G}_m \hookrightarrow \mathrm{SL}_2$ is the torus of diagonal matrices ([5], Section 4.3).

Remark 2.1.4. *The notion of a Hecke eigen-sheaf is better formulated (from the point of view of the formalism of [14]) on a DG-level, for example as ([13], Section 9.5.3). Our Definition 2.1.1 on the level of triangulated categories is a compromise as we will apply the results of [28, 31], which are for the moment not established on the DG-level.*

2.1.5. Let G, H be connected reductive groups over k . Set for brevity $\text{Bun}_{G,H} = \text{Bun}_G \times \text{Bun}_H$. Assume given a complex $\mathcal{M} \in \text{D}^{\prec}(\text{Bun}_{G,H})$. Consider the diagram of projections

$$\text{Bun}_H \xleftarrow{p_H} \text{Bun}_{G,H} \xrightarrow{p_G} \text{Bun}_G$$

Define the functors $F_H : \text{D}^-(\text{Bun}_G)! \rightarrow \text{D}^{\prec}(\text{Bun}_H)$ and $F_G : \text{D}^-(\text{Bun}_H)! \rightarrow \text{D}^{\prec}(\text{Bun}_G)$ by

$$F_H(K) = (p_H)_!((p_G^*K) \otimes \mathcal{M})[-\dim \text{Bun}_G] \quad \text{and} \quad F_G(K) = (p_G)_!((p_H^*K) \otimes \mathcal{M})[-\dim \text{Bun}_H]$$

The finiteness assumptions on the corresponding derived categories are imposed in order for the definition to make sense in the formalism of [25].

Removing the constructibility assumptions, one gets the functors $F_H : \text{D}_{nc}(\text{Bun}_G) \rightarrow \text{D}_{nc}(\text{Bun}_H)$ and $F_G : \text{D}_{nc}(\text{Bun}_H) \rightarrow \text{D}_{nc}(\text{Bun}_G)$ defined by the same formulas. They extend the corresponding lifting functors for constructible sheaves.

For a scheme of finite type S by abuse of notations we still denote by

$$F_H : \text{D}_{nc}(\text{Bun}_G \times S) \rightarrow \text{D}_{nc}(\text{Bun}_H \times S)$$

the functor defined as above with \mathcal{M} replaced by $\text{pr}^* \mathcal{M}$, where $\text{pr} : \text{Bun}_H \times \text{Bun}_G \times S \rightarrow \text{Bun}_H \times \text{Bun}_G$ is the projection (and similarly for F_G).

Let $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$ be a homomorphism. We assume that the composition

$$\text{Rep}(\check{H}) \xrightarrow{*} \text{Rep}(\check{H}) \xrightarrow{\text{Res}^{\kappa}} \text{Rep}(\check{G} \times \mathbb{G}_m)$$

is isomorphic to the composition $\text{Rep}(\check{H}) \xrightarrow{\text{Res}^{\kappa}} \text{Rep}(\check{G} \times \mathbb{G}_m) \xrightarrow{*} \text{Rep}(\check{G} \times \mathbb{G}_m)$ (cf. Remark 2.1.7 below). This holds in all the examples we are interested in. Denote by $\kappa_{ex} : \check{G} \times \mathbb{G}_m \rightarrow \check{H} \times \mathbb{G}_m$ the map (κ, pr) , where $\text{pr} : \check{G} \times \mathbb{G}_m \rightarrow \mathbb{G}_m$ is the projection.

Definition 2.1.6. (*Straight direction*). *Assume given isomorphisms on $\text{Bun}_H \times X$*

$$\alpha_V : \text{H}_H^{\leftarrow}(V, F_H(K)) \xrightarrow{\sim} F_H \text{H}_G^{\leftarrow}(V, K)$$

functorial in $K \in \text{D}_{nc}(\text{Bun}_G)$ and $V \in \text{Rep}(\check{H})$. It is understood that $\text{Rep}(\check{H})$ acts on $\text{D}_{nc}(\text{Bun}_G)$ via the restriction through κ . We require the properties H'1) and H'2) below. First, for $V_1, \dots, V_n \in \text{Rep}(\check{H})$ iterating α_V , one gets the isomorphism over $\text{Bun}_H \times X^n$

$$\alpha_{V_1, \dots, V_n} : \text{H}_H^{\leftarrow}(V_1 \boxtimes \dots \boxtimes V_n, F_H(K)) \xrightarrow{\sim} F_H \text{H}_G^{\leftarrow}(V_1 \boxtimes \dots \boxtimes V_n, K)$$

We require that for $V_1, V_2 \in \text{Rep}(\check{H})$ the following diagrams commute H'1)

$$\begin{array}{ccc} \text{H}_H^{\leftarrow}(V_1 \boxtimes V_2, F_H(K)) & \xrightarrow{\alpha_{V_1, V_2}} & F_H \text{H}_G^{\leftarrow}(V_1 \boxtimes V_2, K) \\ \downarrow & & \downarrow \\ (\text{id} \times s)^* \text{H}_H^{\leftarrow}(V_2 \boxtimes V_1, F_H(K)) & \xrightarrow{(\text{id} \times s)^* \alpha_{V_2, V_1}} & F_H(\text{id} \times s)^* \text{H}_G^{\leftarrow}(V_2 \boxtimes V_1, K), \end{array}$$

$H'2)$

$$\begin{array}{ccc} \mathrm{H}_H^{\leftarrow}(V_1 \boxtimes V_2, F_H(K)) \big|_{\mathrm{Bun}_H \times \Delta(X)} & \xrightarrow{\alpha_{V_1, V_2}} & F_H \mathrm{H}_G^{\leftarrow}(V_1 \boxtimes V_2, K) \big|_{\mathrm{Bun}_H \times \Delta(X)} \\ \downarrow & & \downarrow \\ \mathrm{H}_H^{\leftarrow}(V_1 \otimes V_2, F_H(K))[1] & \xrightarrow{\alpha_{V_1 \otimes V_2}} & F_H \mathrm{H}_G^{\leftarrow}(V_1 \otimes V_2, K)[1] \end{array}$$

Here the vertical arrows are natural isomorphisms. In this case we say that $F_H : \mathrm{D}_{nc}(\mathrm{Bun}_G) \rightarrow \mathrm{D}_{nc}(\mathrm{Bun}_H)$ commutes with the Hecke actions along κ (so, realizes the geometric Langlands functoriality for κ).

Remark 2.1.7. Assume $F_H : \mathrm{D}_{nc}(\mathrm{Bun}_G) \rightarrow \mathrm{D}_{nc}(\mathrm{Bun}_H)$ commutes with Hecke actions along κ . According to Arthur-Langlands philosophy, κ is expected always to be a composition

$$\check{G} \times \mathbb{G}_m \xrightarrow{\mathrm{id} \times t} \check{G} \times \mathrm{SL}_2 \rightarrow \check{H},$$

where $t : \mathbb{G}_m \rightarrow \mathrm{SL}_2$ is the torus of diagonal matrices.

Definition 2.1.8. (Backward direction). Assume we are in the situation of Section 2.1.5, so $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$, and F_G is the lifting functor in the backward direction. Assume given isomorphisms on $\mathrm{Bun}_G \times X$

$$\alpha_V : \mathrm{H}_G^{\leftarrow}(V, F_G(K)) \xrightarrow{\sim} F_G \mathrm{H}_H^{\leftarrow}(V, K)$$

functorial in $K \in \mathrm{D}_{nc}(\mathrm{Bun}_H)$ and $V \in \mathrm{Rep}(\check{H})$. It is understood that $\mathrm{Rep}(\check{H})$ acts on $\mathrm{D}_{nc}(\mathrm{Bun}_G)$ via the restriction through κ . We require the properties $H''1)$ and $H''2)$ below. First, for $V_1, \dots, V_n \in \mathrm{Rep}(\check{H})$ iterating α_V , one gets the isomorphism over $\mathrm{Bun}_G \times X^n$

$$\alpha_{V_1, \dots, V_n} : \mathrm{H}_G^{\leftarrow}(V_1 \boxtimes \dots \boxtimes V_n, F_G(K)) \xrightarrow{\sim} F_G \mathrm{H}_H^{\leftarrow}(V_1 \boxtimes \dots \boxtimes V_n, K)$$

We require that for $V_1, V_2 \in \mathrm{Rep}(\check{H})$ the following diagrams commute $H''1)$

$$\begin{array}{ccc} \mathrm{H}_G^{\leftarrow}(V_1 \boxtimes V_2, F_G(K)) & \xrightarrow{\alpha_{V_1, V_2}} & F_G \mathrm{H}_H^{\leftarrow}(V_1 \boxtimes V_2, K) \\ \downarrow & & \downarrow \\ (\mathrm{id} \times s)^* \mathrm{H}_G^{\leftarrow}(V_2 \boxtimes V_1, F_G(K)) & \xrightarrow{(\mathrm{id} \times s)^* \alpha_{V_2, V_1}} & F_G(\mathrm{id} \times s)^* \mathrm{H}_H^{\leftarrow}(V_2 \boxtimes V_1, K) \end{array}$$

$H''2)$

$$\begin{array}{ccc} \mathrm{H}_G^{\leftarrow}(V_1 \boxtimes V_2, F_G(K)) \big|_{\mathrm{Bun}_G \times \Delta(X)} & \xrightarrow{\alpha_{V_1, V_2}} & F_G \mathrm{H}_H^{\leftarrow}(V_1 \boxtimes V_2, K) \big|_{\mathrm{Bun}_G \times \Delta(X)} \\ \downarrow & & \downarrow \\ \mathrm{H}_G^{\leftarrow}(V_1 \otimes V_2, F_G(K))[1] & \xrightarrow{\alpha_{V_1 \otimes V_2}} & F_G \mathrm{H}_H^{\leftarrow}(V_1 \otimes V_2, K)[1] \end{array}$$

Here the vertical arrows are natural isomorphisms. In this case we say that F_G commutes with the Hecke actions backward κ .

Corollary 2.1.9. *i)* Assume $F_H : \mathrm{D}_{nc}(\mathrm{Bun}_G) \rightarrow \mathrm{D}_{nc}(\mathrm{Bun}_H)$ commutes with Hecke functors along κ . Let $x \in X$, $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{G}$ be a homomorphism. Write

$$\sigma^{ex} : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{G} \times \mathbb{G}_m$$

for the extension of σ , whose second component is the projection on \mathbb{G}_m . Let $\sigma_H = \kappa \circ \sigma^{ex}$. If $K \in \mathcal{D}^-(\mathrm{Bun}_G)_!$ is a σ -Hecke eigen-sheaf then $F_H(K) \in \mathcal{D}^-(\mathrm{Bun}_H)$ is naturally a σ_H -Hecke eigen-sheaf.

ii) Assume $F_G : \mathcal{D}_{nc}(\mathrm{Bun}_H) \rightarrow \mathcal{D}_{nc}(\mathrm{Bun}_G)$ commutes with Hecke functors backward κ . Let $x \in X$, $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{H}$ be a homomorphism. If $K \in \mathcal{D}^-(\mathrm{Bun}_H)_!$ is a σ -Hecke eigen-sheaf then $F_G(K) \in \mathcal{D}^-(\mathrm{Bun}_G)$ is naturally equipped with a σ -Hecke property with respect to κ . So, the Hecke property of $F_G(K)$ is only partial.

Definition 2.1.10. Assume we are in the situation of Section 2.1.5. Assume given isomorphisms

$$\beta_V : \mathbb{H}_H^{\leftarrow}(V, \mathcal{M}) \xrightarrow{\sim} \mathbb{H}_G^{\rightarrow}(V, \mathcal{M})$$

in $\mathcal{D}^-(\mathrm{Bun}_{G,H} \times X)$ functorial in $V \in \mathrm{Rep}(\check{H})$. We require properties $\mathcal{H}1)$, $\mathcal{H}2)$ below. First, iterating β_V , one gets isomorphisms over $\mathrm{Bun}_{G,H} \times X^n$

$$\beta_{V_1, \dots, V_n} : \mathbb{H}_H^{\leftarrow}(V_1 \boxtimes \dots \boxtimes V_n, \mathcal{M}) \xrightarrow{\sim} \mathbb{H}_G^{\rightarrow}(V_1 \boxtimes \dots \boxtimes V_n, \mathcal{M})$$

Here by $\mathbb{H}_G^{\rightarrow}(V, \mathcal{M})$ we mean $\mathbb{H}_G^{\rightarrow}(\mathrm{Res}^{\kappa}(V), \mathcal{M})$, and similarly for the iterated isomorphisms. It is required that for $V_1, V_2 \in \mathrm{Rep}(\check{H})$ the following diagrams commute $\mathcal{H}1)$

$$\begin{array}{ccc} \mathbb{H}_H^{\leftarrow}(V_1 \boxtimes V_2, \mathcal{M}) & \xrightarrow{\beta_{V_1, V_2}} & \mathbb{H}_G^{\rightarrow}(V_1 \boxtimes V_2, \mathcal{M}) \\ \downarrow & & \downarrow \\ (\mathrm{id} \times s)^* \mathbb{H}_H^{\leftarrow}(V_2 \boxtimes V_1, \mathcal{M}) & \xrightarrow{(\mathrm{id} \times s)^* \beta_{V_2, V_1}} & (\mathrm{id} \times s)^* \mathbb{H}_G^{\rightarrow}(V_2 \boxtimes V_1, \mathcal{M}) \end{array}$$

$\mathcal{H}2)$

$$\begin{array}{ccc} \mathbb{H}_H^{\leftarrow}(V_1 \boxtimes V_2, \mathcal{M}) |_{\mathrm{Bun}_{G,H} \times \Delta(X)} & \xrightarrow{\beta_{V_1, V_2}} & \mathbb{H}_G^{\rightarrow}(V_1 \boxtimes V_2, \mathcal{M}) |_{\mathrm{Bun}_{G,H} \times \Delta(X)} \\ \downarrow & & \downarrow \\ \mathbb{H}_H^{\leftarrow}(V_1 \otimes V_2, \mathcal{M})[1] & \xrightarrow{\beta_{V_1 \otimes V_2}} & \mathbb{H}_G^{\rightarrow}(V_1 \otimes V_2, \mathcal{M})[1] \end{array}$$

Here the vertical arrows are natural isomorphisms. In this case we say that \mathcal{M} satisfies the Hecke property for κ .

Theorem 2.1.11. Assume $\mathcal{M} \in \mathcal{D}^-(\mathrm{Bun}_{G,H})$ satisfies the Hecke property for κ . Then F_H (resp., F_G) commutes with the Hecke functors along κ (resp., backward κ).

Proof. i) The case of F_H is established precisely as in ([28], the derivation of Theorem 3 part 1) from Theorem 4 part 1)).

ii) To establish the desired structure for F_G , consider the diagram

$$(3) \quad \begin{array}{ccccc} \mathrm{Bun}_H \times X & \xleftarrow{h_H^{\leftarrow \times \mathrm{supp}}} & \mathcal{H}_H & \xrightarrow{h_H^{\rightarrow}} & \mathrm{Bun}_H \\ \uparrow q \times \mathrm{id} & & \uparrow & & \uparrow q \\ \mathrm{Bun}_{G,H} \times X & \xleftarrow{h_H^{\leftarrow \times \mathrm{supp}}} & \mathcal{H}_H \times \mathrm{Bun}_G & \xrightarrow{h_H^{\rightarrow}} & \mathrm{Bun}_{G,H} \\ \downarrow p_G \times \mathrm{id} & & & & \\ \mathrm{Bun}_G \times X & & & & \end{array}$$

similar to that of ([28], Section 8.1). From definitions for $V \in \mathrm{Rep}(\check{H})$, $K \in \mathrm{D}_{nc}(\mathrm{Bun}_H)$ we get

$$(4) \quad F_G \mathbf{H}_H^\leftarrow(V, K) \xrightarrow{\sim} (p_G \times \mathrm{id})_!(\mathbf{H}_H^\rightarrow(V, \mathcal{M}) \otimes p_H^* K)[- \dim \mathrm{Bun}_H]$$

We used the fact that

$$(\mathrm{IC}(\mathrm{Bun}_H) \boxtimes * \mathcal{S})^r \xrightarrow{\sim} (\mathrm{IC}(\mathrm{Bun}_H) \boxtimes \mathcal{S})^l$$

on \mathcal{H}_H for $\mathcal{S} \in \mathrm{Sph}_H$, cf. Section 1.2.2. By our assumptions, (4) identifies with

$$(p_G \times \mathrm{id})_!(\mathbf{H}_G^\leftarrow(V, \mathcal{M}) \otimes p_H^* K)[- \dim \mathrm{Bun}_H]$$

By the base change and the projection formula, the latter complex identifies with $\mathbf{H}_G^\leftarrow(V, F_G(K))$. \square

2.2. Theta-lifting.

2.2.1. For $r \geq 1$ let G_r be the group scheme on X of automorphisms of $\mathcal{O}_X^r \oplus \Omega^r$ preserving the natural symplectic form $\wedge^2(\mathcal{O}_X^r \oplus \Omega^r) \rightarrow \Omega$. The stack Bun_{G_r} classifies $M \in \mathrm{Bun}_{2r}$ with a symplectic form $\wedge^2 M \rightarrow \Omega$. Let \mathcal{A}_r be the line bundle on Bun_{G_r} with fibre $\det \mathrm{R}\Gamma(X, M)$ at M . Write $\widetilde{\mathrm{Bun}}_{G_r} \rightarrow \mathrm{Bun}_{G_r}$ for the μ_2 -gerbe of square roots of \mathcal{A}_r . The theta-sheaf Aut on $\widetilde{\mathrm{Bun}}_{G_r}$ is defined in [29].

2.2.2. *Orthogonal-symplectic dual pair.* Let $n, m \geq 1$. Let $G = G_n$ and $H = \mathrm{SO}_{2m}$ split. The stack Bun_H classifies $V \in \mathrm{Bun}_{2m}$, a nondegenerate symmetric form $\mathrm{Sym}^2 V \rightarrow \mathcal{O}$, and a compatible trivialization $\det V \xrightarrow{\sim} \mathcal{O}_X$. The theta-lifting functors

$$(5) \quad F_G : \mathrm{D}^-(\mathrm{Bun}_H)! \rightarrow \mathrm{D}^\leftarrow(\mathrm{Bun}_G), \quad F_H : \mathrm{D}^-(\mathrm{Bun}_G)! \rightarrow \mathrm{D}^\leftarrow(\mathrm{Bun}_H)$$

from ([28], Section 2.3) are defined as follows.

Let $\tau : \mathrm{Bun}_G \times \mathrm{Bun}_H \rightarrow \mathrm{Bun}_{G_{2nm}}$ be the map sending (M, V) to $M \otimes V$ with the induced symplectic form $\wedge^2(M \otimes V) \rightarrow \Omega$. There is a canonical lift

$$\tilde{\tau} : \mathrm{Bun}_G \times \mathrm{Bun}_H \rightarrow \widetilde{\mathrm{Bun}}_{G_{2nm}}$$

extending τ (cf. [28], Section 2.3.1). Let $\mathcal{M} = \tilde{\tau}^* \mathrm{Aut}[\dim \mathrm{Bun}_G \times \mathrm{Bun}_H - \dim \mathrm{Bun}_{G_{2nm}}]$. Using the kernel \mathcal{M} , define functors (5) as in Section 2.1.5.

2.2.3. For $m > n$ define $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$ as in ([28], Section 2.3.2). Namely, let $\mathcal{W} = \mathbb{Q}_\ell^{2m}$ be the standard representation of $\check{H} = \mathrm{SO}_{2m}$. Pick an orthogonal decomposition $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, where $\dim \mathcal{W}_1 = 2n + 1$. Identify $\check{G} \xrightarrow{\sim} \mathrm{SO}(\mathcal{W}_1)$, this fixes the inclusion $\check{G} \hookrightarrow \check{H}$. The connected centralizer of \check{G} in \check{H} is $\mathrm{SO}(\mathcal{W}_2)$. Let $\mathrm{prin} : \mathrm{SL}_2 \rightarrow \mathrm{SO}(\mathcal{W}_2)$ be the homomorphism corresponding to a principal (irreducible) representation of SL_2 on \mathcal{W}_2 . Then the restriction of κ to \mathbb{G}_m is the composition $\mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{\mathrm{prin}} \mathrm{SO}(\mathcal{W}_2) \hookrightarrow \check{H}$, where the first map is the maximal torus of diagonal matrices in SL_2 .

For $m \leq n$ define $\kappa : \check{H} \times \mathbb{G}_m \rightarrow \check{G}$ as in *loc.cit.* Namely, let $\mathcal{W} = \mathbb{Q}_\ell^{2n+1}$ be the standard representation of $\check{G} = \mathrm{SO}_{2n+1}$. Pick an orthogonal decomposition $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$, where $\dim \mathcal{W}_1 = 2m$ and identify $\check{H} \xrightarrow{\sim} \mathrm{SO}(\mathcal{W}_1)$. This fixes the inclusion $\check{H} \hookrightarrow \check{G}$. The connected centralizer of \check{H} in \check{G} is $\mathrm{SO}(\mathcal{W}_2)$. Let $\mathrm{prin} : \mathrm{SL}_2 \rightarrow \mathrm{SO}(\mathcal{W}_2)$ be the homomorphism corresponding to a principal (irreducible) representation of SL_2 on

W_2 . Then the restriction of κ to \mathbb{G}_m is the composition $\mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{\text{prin}} \mathrm{SO}(W_2) \hookrightarrow \check{G}$, where the first map is the maximal torus of diagonal matrices in SL_2 .

Theorem 2.2.4 ([28], Theorem 4). *Assume we are in the situation of Section 2.2.2. For $n, m \geq 1$ the complex \mathcal{M} satisfies the Hecke property for κ .*

Corollary 2.2.5. *Assume we are in the situation of Section 2.2.2.*

- If $m \leq n$ then F_G (resp., F_H) commutes with Hecke actions along κ (resp., backward κ).
- If $m > n$ then F_H (resp., F_G) commutes with Hecke actions along κ (resp., backward κ).

Proof. This follows from Theorem 2.1.11. The straight direction is also established in ([28], Theorem 3). \square

2.2.6. Theta-lifting for similitudes. Let $n, m \geq 1$. Let $\mathbb{G} = \mathrm{GSp}_{2n}$, $\mathbb{H} = \mathrm{GSO}_{2m} = (\mathbb{G}_m \times \mathrm{SO}_{2m})/(-1, -1)$ be as in ([31], Section 2.4.4). Here \mathbb{G}, \mathbb{H} are split. The theta-lifting functors

$$(6) \quad F_{\mathbb{G}} : D^-(\mathrm{Bun}_{\mathbb{H}})! \rightarrow D^{\prec}(\mathrm{Bun}_{\mathbb{G}}), \quad F_{\mathbb{H}} : D^-(\mathrm{Bun}_{\mathbb{G}})! \rightarrow D^{\prec}(\mathrm{Bun}_{\mathbb{H}})$$

from ([31], Definition 2.4.8) are as follows.

The stack $\mathrm{Bun}_{\mathbb{G}}$ classifies $(M \in \mathrm{Bun}_{2n}, \mathcal{A} \in \mathrm{Bun}_1)$ with a symplectic form $\wedge^2 M \rightarrow \mathcal{A}$ on X . The stack $\mathrm{Bun}_{\mathbb{H}}$ classifies $(V \in \mathrm{Bun}_{2m}, \mathcal{C} \in \mathrm{Bun}_1)$, a nondegenerate symmetric form $\mathrm{Sym}^2 V \rightarrow \mathcal{C}$, and a compatible trivialization $\gamma : \mathcal{C}^{-m} \otimes \det V \xrightarrow{\sim} \mathcal{O}_X$.

Let $\mathrm{Bun}_{\mathbb{G}\mathbb{H}} = \mathrm{Bun}_{\mathbb{G}} \times_{\mathrm{Bun}_1} \mathrm{Bun}_{\mathbb{H}}$, where the map $\mathrm{Bun}_{\mathbb{G}} \rightarrow \mathrm{Bun}_1$ sends (M, \mathcal{A}) to \mathcal{A} , and $\mathrm{Bun}_{\mathbb{H}} \rightarrow \mathrm{Bun}_1$ sends (V, \mathcal{C}, γ) to $\mathcal{C}^{-1} \otimes \Omega$. The notation $\mathrm{Bun}_{\mathbb{G}\mathbb{H}}$ is not to be confused with $\mathrm{Bun}_{\mathbb{G}, \mathbb{H}} = \mathrm{Bun}_{\mathbb{G}} \times \mathrm{Bun}_{\mathbb{H}}$.

Let $\tau : \mathrm{Bun}_{\mathbb{G}\mathbb{H}} \rightarrow \mathrm{Bun}_{G_{2nm}}$ be the map sending a collection $(M, \mathcal{A}, V, \mathcal{C}, \gamma)$ to $M \otimes V$ with the induced symplectic form $\wedge^2(M \otimes V) \rightarrow \Omega$. The map τ admits a canonical lift

$$\tilde{\tau} : \mathrm{Bun}_{\mathbb{G}\mathbb{H}} \rightarrow \widetilde{\mathrm{Bun}}_{G_{2nm}}$$

defined in ([30], Section 2.4.7). Define the complex

$$\mathrm{Aut}_{\mathbb{G}\mathbb{H}} = \tilde{\tau}^* \mathrm{Aut}[\dim \mathrm{Bun}_{\mathbb{G}\mathbb{H}} - \dim \mathrm{Bun}_{G_{2nm}}]$$

on $\mathrm{Bun}_{\mathbb{G}\mathbb{H}}$. Let $q : \mathrm{Bun}_{\mathbb{G}\mathbb{H}} \rightarrow \mathrm{Bun}_{\mathbb{G}, \mathbb{H}}$ be the natural map. Set $\mathcal{M} = q_! \mathrm{Aut}_{\mathbb{G}\mathbb{H}}$. Define the functors (6) as in Section 2.1.5.

2.2.7. For $m \geq 3$ consider the group Spin_m defined in ([21], Section 6.3.3). By ([21], Theorem 6.3.5), it is equipped with a distinguished surjection $\mathbb{S}\mathrm{pin}_m \rightarrow \mathrm{SO}_m$ given by the standard representation, whose kernel is denoted $\{1, \iota\} \xrightarrow{\sim} \mu_2$. For $m \geq 3$ set $\mathrm{GSpin}_m = \mathbb{G}_m \times \mathrm{Spin}_m / \{(-1, \iota)\}$. We convent that $\mathrm{GSpin}_2 = \mathbb{G}_m \times \mathbb{G}_m$. Recall the Langlands dual groups are $\check{\mathbb{H}} \xrightarrow{\sim} \mathrm{GSpin}_{2m}$, $\check{\mathbb{G}} \xrightarrow{\sim} \mathrm{GSpin}_{2n+1}$ (cf. [31], Section 3).

2.2.8. For $m > n$ define $\bar{\kappa} : \check{\mathbb{G}} \times \mathbb{G}_m \rightarrow \check{\mathbb{H}}$ as the map $(i_\kappa, \delta_\kappa)$ from ([31], Sections 2.5 and 5.12.5). For $m \leq n$ define $\bar{\kappa} : \check{\mathbb{H}} \times \mathbb{G}_m \rightarrow \check{\mathbb{G}}$ as $(i_\kappa, \delta_\kappa)$ from ([31], Sections 2.5 and 5.12.6).

For example, for $m = 3, n = 2$ the map $i_\kappa : \check{\mathbb{G}} \rightarrow \check{\mathbb{H}}$ identifies with the quotient of the map $\mathbb{G}_m \times \mathrm{Sp}_4 \rightarrow \mathbb{G}_m \times \mathrm{SL}_4, (x, y) \mapsto (x^{-1}, y)$ by the diagonally embedded $(-1, -1)$. We will especially need this case.

Consider the surjections $st : \check{\mathbb{G}} \rightarrow \check{G} = \mathrm{SO}_{2n+1}$ and $st : \check{\mathbb{H}} \rightarrow \check{H} = \mathrm{SO}_{2m}$ given by the standard representations. For $m \leq n$ consider the map $\kappa : \check{H} \times \mathbb{G}_m \rightarrow \check{G}$ of Section 2.2.3. Then the diagram commutes

$$\begin{array}{ccc} \check{\mathbb{H}} \times \mathbb{G}_m & \xrightarrow{\bar{\kappa}} & \check{\mathbb{G}} \\ \downarrow st \times \mathrm{id} & & \downarrow st \\ \check{H} \times \mathbb{G}_m & \xrightarrow{\kappa} & \check{G} \end{array}$$

Let $C(\check{\mathbb{H}}) \subset \check{\mathbb{G}}$ be the connected centralizer of $i_\kappa(\check{\mathbb{H}})$ in $\check{\mathbb{G}}$, $\mathrm{SL}_2 \xrightarrow{prin} C(\check{\mathbb{H}})$ be the homomorphism corresponding to the principal unipotent orbit. Then $\delta_\kappa : \mathbb{G}_m \rightarrow \check{\mathbb{G}}$ is the composition $\mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{prin} C(\check{\mathbb{H}}) \hookrightarrow \check{\mathbb{G}}$.

For $m > n$ consider the map $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$ from Section 2.2.3. Then the diagram commutes

$$\begin{array}{ccc} \check{\mathbb{G}} \times \mathbb{G}_m & \xrightarrow{\bar{\kappa}} & \check{\mathbb{H}} \\ \downarrow st \times \mathrm{id} & & \downarrow st \\ \check{G} \times \mathbb{G}_m & \xrightarrow{\kappa} & \check{H} \end{array}$$

Let $C(\check{\mathbb{G}})$ be the connected centralizer of $i_\kappa(\check{\mathbb{G}})$ in $\check{\mathbb{H}}$. Let $\mathrm{SL}_2 \xrightarrow{prin} C(\check{\mathbb{G}})$ be the homomorphism corresponding to the principal unipotent orbit. Then $\delta_\kappa : \mathbb{G}_m \rightarrow \check{\mathbb{H}}$ is the composition $\mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{prin} C(\check{\mathbb{G}}) \hookrightarrow \check{\mathbb{H}}$.

If $m = n$ or $m = n + 1$ then δ_κ is trivial.

Theorem 2.2.9. *Assume we are in the situation of Section 2.2.6. Then for any n, m the complex $\mathcal{M} \in \mathrm{D}^{\check{}}(\mathrm{Bun}_{\mathbb{G}, \mathbb{H}})$ satisfies the Hecke property for κ .*

Proof. We derive this from ([31], Theorem 2.5.8). We give the proof only in the case $m > n$, the case $m \leq n$ is similar.

As in ([31], Section 2.5.7), for $a \in \mathbb{Z}$ write ${}^a \mathrm{Bun}_{\mathbb{G}, \mathbb{H}}$ for the stack classifying $(M, \mathcal{A}) \in \mathrm{Bun}_{\mathbb{G}}, (V, \mathcal{C}, \gamma) \in \mathrm{Bun}_{\mathbb{H}}, x \in X$, and an isomorphism $\mathcal{A} \otimes \mathcal{C} \xrightarrow{\sim} \Omega(ax)$. We have the commutative diagram

$$\begin{array}{ccccc} \bigsqcup_{a \in \mathbb{Z}} {}^a \mathrm{Bun}_{\mathbb{G}, \mathbb{H}} & \xleftarrow{\bar{h}_{\mathbb{H}}^{\check{}}} & \mathcal{H}_{\mathbb{H}} \times_{\mathrm{Bun}_{\mathbb{H}}} \mathrm{Bun}_{\mathbb{G}, \mathbb{H}} & \xrightarrow{h_{\mathbb{H}}^{\check{}} \times \mathrm{id}} & \mathrm{Bun}_{\mathbb{G}, \mathbb{H}} \\ \downarrow \bar{q} & & \downarrow & & \downarrow q \\ \mathrm{Bun}_{\mathbb{G}, \mathbb{H}} \times X & \xleftarrow{h_{\mathbb{H}}^{\check{}} \times \mathrm{id} \times \mathrm{supp}} & \mathcal{H}_{\mathbb{H}} \times \mathrm{Bun}_{\mathbb{G}} & \xrightarrow{h_{\mathbb{H}}^{\check{}} \times \mathrm{id}} & \mathrm{Bun}_{\mathbb{G}, \mathbb{H}}, \end{array}$$

where we used the map $h_{\mathbb{H}}^{\check{}} : \mathcal{H}_{\mathbb{H}} \rightarrow \mathrm{Bun}_{\mathbb{H}}$ to define the fibred product $\mathcal{H}_{\mathbb{H}} \times_{\mathrm{Bun}_{\mathbb{H}}} \mathrm{Bun}_{\mathbb{G}, \mathbb{H}}$. We denoted by \bar{q} the natural projection. Here a point of $\mathcal{H}_{\mathbb{H}} \times_{\mathrm{Bun}_{\mathbb{H}}} \mathrm{Bun}_{\mathbb{G}, \mathbb{H}}$ is given by a collection $(x, V, \mathcal{C}, V', \mathcal{C}') \in \mathcal{H}_{\mathbb{H}}$ and $(M, \mathcal{A}, V', \mathcal{C}') \in \mathrm{Bun}_{\mathbb{G}, \mathbb{H}}$. The map $\bar{h}_{\mathbb{H}}^{\check{}}$ sends this collection to $(x, M, \mathcal{A}, V, \mathcal{C})$. This gives an isomorphism for $V \in \mathrm{Rep}(\check{\mathbb{H}})$

$$(7) \quad \mathrm{H}_{\mathbb{H}}^{\check{}}(V, \mathcal{M}) \xrightarrow{\sim} \bar{q}_! \mathrm{H}_{\mathbb{H}}^{\check{}}(V, \mathrm{Aut}_{\mathbb{G}, \mathbb{H}})$$

By ([31], Theorem 2.5.8),

$$H_{\mathbb{H}}^{\leftarrow}(V, \text{Aut}_{\mathbb{G}\mathbb{H}}) \xrightarrow{\sim} H_{\mathbb{G}}^{\rightarrow}(V, \text{Aut}_{\mathbb{G}\mathbb{H}})$$

over $\bigsqcup_{a \in \mathbb{Z}}^a \text{Bun}_{\mathbb{G}\mathbb{H}}$. So, (7) identifies with $\bar{q}_! H_{\mathbb{G}}^{\rightarrow}(V, \text{Aut}_{\mathbb{G}\mathbb{H}})$. The diagram commutes

$$\begin{array}{ccccc} \text{Bun}_{\mathbb{G}\mathbb{H}} & \xleftarrow{h_{\mathbb{G}}^{\leftarrow} \times \text{id}} & \mathcal{H}_{\mathbb{G}} \times_{\text{Bun}_{\mathbb{G}}} \text{Bun}_{\mathbb{G}\mathbb{H}} & \xrightarrow{\bar{h}_{\mathbb{G}}^{\rightarrow}} & \bigsqcup_{a \in \mathbb{Z}}^a \text{Bun}_{\mathbb{G}\mathbb{H}} \\ \downarrow q & & \downarrow & & \downarrow \bar{q} \\ \text{Bun}_{\mathbb{G}, \mathbb{H}} & \xleftarrow{h_{\mathbb{G}}^{\leftarrow}} & \mathcal{H}_{\mathbb{G}} \times \text{Bun}_{\mathbb{H}} & \xrightarrow{h_{\mathbb{G}}^{\rightarrow} \times \text{id} \times \text{supp}} & \text{Bun}_{\mathbb{G}, \mathbb{H}} \times X, \end{array}$$

where we used the map $h_{\mathbb{G}}^{\leftarrow} : \mathcal{H}_{\mathbb{G}} \rightarrow \text{Bun}_{\mathbb{G}}$ to define the fibred product $\mathcal{H}_{\mathbb{G}} \times_{\text{Bun}_{\mathbb{G}}} \text{Bun}_{\mathbb{G}\mathbb{H}}$. So, a point of $\mathcal{H}_{\mathbb{G}} \times_{\text{Bun}_{\mathbb{G}}} \text{Bun}_{\mathbb{G}\mathbb{H}}$ is given by $(x, M, \mathcal{A}, M', \mathcal{A}') \in \mathcal{H}_{\mathbb{G}}, (M, \mathcal{A}, V, \mathcal{C}) \in \text{Bun}_{\mathbb{G}\mathbb{H}}$. The map $\bar{h}_{\mathbb{G}}^{\rightarrow}$ sends this collection to $(x, M', \mathcal{A}', V, \mathcal{C})$. This gives an isomorphism

$$\bar{q}_! H_{\mathbb{G}}^{\rightarrow}(V, \text{Aut}_{\mathbb{G}\mathbb{H}}) \xrightarrow{\sim} H_{\mathbb{G}}^{\rightarrow}(V, \mathcal{M})$$

Our claim follows. \square

Corollary 2.2.10. *Assume we are in the situation of Section 2.2.6.*

- If $m \leq n$ then $F_{\mathbb{G}}$ (resp., $F_{\mathbb{H}}$) commutes with Hecke actions along $\bar{\kappa}$ (resp., backward κ).
- If $m > n$ then $F_{\mathbb{H}}$ (resp., $F_{\mathbb{G}}$) commutes with Hecke actions along $\bar{\kappa}$ (resp., backward κ).

Proof. This follows from Theorem 2.1.11. The straight direction is also established in ([31], Theorem 2.5.5). \square

Remark 2.2.11. *For future references, we record the following. Assume we are in the situation of Section 2.2.8. Write $st : \text{Spin}_{2n+1} \rightarrow \text{SO}_{2n+1}$ and $st : \text{Spin}_{2m} \rightarrow \text{SO}_{2m}$ for the standard representations of these groups. If $m \leq n$ then the preimage of $\check{H} \hookrightarrow \check{G}$ under $st : \text{Spin}_{2n+1} \rightarrow \text{SO}_{2n+1} = \check{G}$ identifies with Spin_{2n} . This gives the embedding $\kappa : \text{Spin}_{2m} \hookrightarrow \text{Spin}_{2n+1}$.*

If $m > n$ then the preimage of $\check{G} \hookrightarrow \check{H}$ under $st : \text{Spin}_{2m} \rightarrow \text{SO}_{2m}$ identifies with Spin_{2n+1} . This gives the embedding $\kappa : \text{Spin}_{2n+1} \hookrightarrow \text{Spin}_{2m}$.

2.2.12. Theta-lifting for general linear groups. Let $n, m \geq 1$. Set $G = \text{GL}_n, H = \text{GL}_m$. In this case the theta-lifting functors (5) are defined by $F_G = F_{m,n}, F_H = F_{n,m}$, where $F_{n,m}$ are given in ([28], Definition 3). Recall this definition.

Let $\mathcal{W}_{n,m}$ be the stack classifying $L \in \text{Bun}_n, U \in \text{Bun}_m$ and a section $\mathcal{O}_X \rightarrow L \otimes U$ on X . Let $q : \mathcal{W}_{n,m} \rightarrow \text{Bun}_n \times \text{Bun}_m$ be the projection sending the above point to (L, U) . Let

$$\mathcal{J} = \bar{\mathbb{Q}}_{\ell}[\dim(\text{Bun}_n \times \text{Bun}_m) + a_{n,m}] \in D^b(\mathcal{W}_{n,m}),$$

where $a_{n,m}$ is the function of a connected component of $\text{Bun}_n \times \text{Bun}_m$ sending (L, U) to the Euler characteristics $\chi(X, L \otimes U)$. Set $\mathcal{M} = q_! \mathcal{J} \in D^b(\text{Bun}_{G,H})$. Define the functors (5) as in Section 2.1.5 using the kernel \mathcal{M} .

2.2.13. Assume $m \geq n$. Define $\kappa = (i_\kappa, \delta_\kappa) : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$ as follows. Write \mathcal{W} for the standard representation of $\check{H} = \mathrm{GL}_m$. Pick a decomposition of vector spaces $\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2$ with $\dim \mathcal{W}_1 = n$. So, $\mathrm{GL}(\mathcal{W}_1) \times \mathrm{GL}(\mathcal{W}_2) \subset \check{H}$ is a Levi subgroup. Define $i_\kappa : \mathrm{GL}_n \rightarrow \mathrm{GL}_m$ as the composition $\mathrm{GL}(\mathcal{W}_1) \xrightarrow{\sigma} \mathrm{GL}(\mathcal{W}_1) \hookrightarrow \mathrm{GL}(\mathcal{W})$, where $\sigma(g) = ({}^t g)^{-1}$. We have denoted by ${}^t g$ the transpose of a matrix g .

Let $\mathrm{prin} : \mathrm{SL}_2 \rightarrow \mathrm{GL}(\mathcal{W}_2)$ be a principal (irreducible) representation of SL_2 . Set $\kappa = (i_\kappa, \delta_\kappa)$, where δ_κ the composition

$$\mathbb{G}_m \hookrightarrow \mathrm{SL}_2 \xrightarrow{\mathrm{prin}} \mathrm{GL}(\mathcal{W}_2) \hookrightarrow \mathrm{GL}(\mathcal{W})$$

Here the first map is the standard maximal torus of diagonal matrices.

Theorem 2.2.14. *Assume we are in the situation of Section 2.2.12. Then \mathcal{M} satisfies the Hecke property for κ .*

Proof. We derive this from ([28], Theorem 6). As in ([28], Section 2.4), let ${}_\infty \mathcal{W}_{n,m}$ be the stack classifying $x \in X$, $L \in \mathrm{Bun}_n, U \in \mathrm{Bun}_m$ and a section $t : \mathcal{O}_X \rightarrow L \otimes U({}_\infty x)$, which is allowed to have any pole at x . This is an ind-algebraic stack. Let $\bar{q} : {}_\infty \mathcal{W}_{n,m} \rightarrow \mathrm{Bun}_{G,H} \times X$ be the map forgetting t . The Hecke functors

$$\mathrm{H}_G^\leftarrow, \mathrm{H}_G^\rightarrow : \mathrm{Rep}(\check{G} \times \mathbb{G}_m) \times \mathrm{D}^\leftarrow(\mathcal{W}_{n,m}) \rightarrow \mathrm{D}^\leftarrow({}_\infty \mathcal{W}_{n,m})$$

and

$$\mathrm{H}_H^\leftarrow, \mathrm{H}_H^\rightarrow : \mathrm{Rep}(\check{H} \times \mathbb{G}_m) \times \mathrm{D}^\leftarrow(\mathcal{W}_{n,m}) \rightarrow \mathrm{D}^\leftarrow({}_\infty \mathcal{W}_{n,m})$$

are defined in ([28], Section 7). By ([28], Theorem 6), for $V \in \mathrm{Rep}(\check{H})$ one has

$$\mathrm{H}_H^\leftarrow(V, \mathcal{J}) \xrightarrow{\sim} \mathrm{H}_G^\rightarrow(V, \mathcal{J})$$

in $\mathrm{D}^\leftarrow({}_\infty \mathcal{W}_{n,m})$. Now the argument analogous to our proof of Theorem 2.2.9 gives the desired claim. \square

Corollary 2.2.15. *Assume we are in the situation of Section 2.2.12 with $m \geq n$. Then F_H (resp., F_G) commutes with Hecke actions along κ (resp., backward κ).*

Proof. This follows from Theorem 2.1.11. The straight direction is also established in ([28], Theorem 5). \square

2.3. Geometric Eisenstein series.

2.3.1. Let H be a connected reductive group over k with a given maximal torus and Borel subgroup. Let G be a standard Levi subgroup of H , so we have the corresponding inclusion of dual groups $\check{G} \hookrightarrow \check{H}$. Extend it to a map $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$ trivial on \mathbb{G}_m . Let $P \subset G$ be a standard parabolic subgroup with Levi quotient M . Consider the Drinfeld compactification $\widetilde{\mathrm{Bun}}_P$ defined in ([4], Section 1.3.6). We have the diagram of projections $\mathrm{Bun}_M \xleftarrow{q} \widetilde{\mathrm{Bun}}_P \xrightarrow{p} \mathrm{Bun}_G$. Let $q : \mathrm{Bun}_P \rightarrow \mathrm{Bun}_M \times \mathrm{Bun}_G$ be the map $q \times p$, set $\mathcal{M} = q! \mathrm{IC}_{\widetilde{\mathrm{Bun}}_P}$. Define the functor F_H using \mathcal{M} as in Section 2.1.5.

Proposition 2.3.2. *The complex \mathcal{M} satisfies the Hecke property for κ .*

Proof. This is derived from ([4], Corollary 4.1.7) as in our proof of Theorems 2.2.9 and 2.2.14. \square

2.4. Liftings of Hecke eigen-sheaves. In all the above cases of theta-lifting and geometric Eisenstein series we can derive from Corollary 2.1.9 the results like the following.

Corollary 2.4.1. *Assume we are in the situation of Section 2.2.2. So, $n, m \geq 1$, $G = G_n$, $H = \mathrm{SO}_{2m}$.*

i) *Let $m > n$, $x \in X$. Let $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{H}$ be a homomorphism. Let $K \in D^-(\mathrm{Bun}_H)_!$ be a σ -Hecke eigen-sheaf. Then $F_G(K)$ satisfies the σ -Hecke property with respect to $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$.*

ii) *Let $m \leq n$, $x \in X$. Let $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{G}$ be a homomorphism. Let $K \in D^-(\mathrm{Bun}_G)_!$ be a σ -Hecke eigen-sheaf. Then $F_H(K)$ satisfies the σ -Hecke property with respect to $\kappa : \check{H} \times \mathbb{G}_m \rightarrow \check{G}$.*

Corollary 2.4.2. *Assume we are in the situation of Section 2.2.6. So, $n, m \geq 1$ and $G = \mathrm{GSp}_{2n}$, $H = \mathrm{GSO}_{2m}$.*

i) *Let $m > n$, $x \in X$. Let $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{H}$ be a homomorphism. Let $K \in D^-(\mathrm{Bun}_H)_!$ be a σ -Hecke eigen-sheaf. Then $F_G(K)$ satisfies the σ -Hecke property with respect to $\kappa : \check{G} \times \mathbb{G}_m \rightarrow \check{H}$.*

ii) *Let $m \leq n$, $x \in X$. Let $\sigma : \pi_1(X, x) \times \mathbb{G}_m \rightarrow \check{G}$ be a homomorphism. Let $K \in D^-(\mathrm{Bun}_G)_!$ be a σ -Hecke eigen-sheaf. Then $F_H(K)$ satisfies the σ -Hecke property with respect to $\kappa : \check{H} \times \mathbb{G}_m \rightarrow \check{G}$.*

3. MORE RESULTS

In Section 3 we formulate our main results that have not appeared in Section 2. They are related to the extension of a partial Hecke property and the construction of new automorphic sheaves. Their proofs are collected in Section 4.

3.1. Extending the Hecke property. \hat{E}

3.1.1. In Appendix B we study the following question. Given an inclusion of connected reductive groups $\kappa : \check{G} \hookrightarrow \check{H}$ over \mathbb{Q}_ℓ , an abelian category \mathcal{C} with a $\mathrm{Rep}(\check{G})$ -action, and a $\mathrm{Rep}(\check{H})$ -Hecke eigen-object c of \mathcal{C} , can one extend this partial Hecke property to a $\mathrm{Rep}(\check{G})$ -Hecke property of c ? The answer in some sense is given by Proposition B.2.4. The results of this section are inspired by Proposition B.2.4.

Proposition 3.1.2. *Let G, H be connected reductive groups over k , $\kappa : \check{G} \hookrightarrow \check{H}$ be a closed subgroup, $E_{\check{G}}$ be a \check{G} -local system on X , E be the \check{H} -local system on X induced via κ . Assume for any irreducible representation V^λ of \check{G} there is $W \in \mathrm{Rep}(\check{H})$ such that V^λ appears in $\mathrm{Res}^\kappa(W)$ with multiplicity one. Let $K \in D^-(\mathrm{Bun}_G)$ be equipped with a E -Hecke property with respect to κ . (It is understood that κ is extended to a morphism $\check{G} \times \mathbb{G}_m \rightarrow \check{H}$ trivial on \mathbb{G}_m). Then there could exist at most one extension of this structure to a structure of a $E_{\check{G}}$ -Hecke eigen-sheaf on K .*

Remark 3.1.3. *The assumptions of Proposition 3.1.2 are satisfied for the following embeddings κ :*

$$\text{A1) } \mathrm{GL}_{n-1} \hookrightarrow \mathrm{GL}_n \text{ given as the subgroup of matrices of the form } \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}.$$

- A2) for $n \geq 2$ the inclusion $\mathrm{Spin}_{2n-1} \hookrightarrow \mathrm{Spin}_{2n}$ of Remark 2.2.11.
A3) for $n \geq 2$ the inclusion $\mathrm{Spin}_{2n} \hookrightarrow \mathrm{Spin}_{2n+1}$ of Remark 2.2.11.
A4) for $n \geq 2$ the inclusion $\mathrm{GSpin}_{2n-1} \hookrightarrow \mathrm{GSpin}_{2n}$ given in Section 2.2.8.
A5) for $n \geq 1$ the natural inclusion $\mathrm{Sp}_{2n} \hookrightarrow \mathrm{SL}_{2n}$.
A6) let \check{G}, \check{G}_1 be connected reductive groups over $\bar{\mathbb{Q}}_\ell$ with a given homomorphism $\check{G} \rightarrow \check{G}_1$, write $\kappa : \check{G} \hookrightarrow \check{G} \times \check{G}_1$ for its graph.

We give an example where the E -Hecke property of K with respect to κ in Proposition 3.1.2 does not extend to a structure of a $E_{\check{G}}$ -Hecke property (cf. Remark 4.1.2). One can always twist the E -Hecke property of K by an element of the center of \check{H} as in Remark 4.1.2. If the original E -Hecke property of K extends to a $E_{\check{G}}$ -Hecke property, this is not necessarily the case after this twisting.

Proposition 3.1.4. *In the situation of Proposition 3.1.2 assume one of the following:*

- 1) $\kappa : \check{G} \hookrightarrow \check{H}$ is the inclusion A1) with $n \geq 2$. View $E_{\check{G}}$ as a rank $n - 1$ local system E_0 on X . Assume

$$H^0(X, E_0) = H^0(X, E_0^*) = 0$$

- 2) $\kappa : \check{G} \hookrightarrow \check{H}$ is the inclusion A6).
3) $\kappa : \check{G} \hookrightarrow \check{H}$ is the inclusion $\mathrm{Sp}_{2n} \hookrightarrow \mathrm{SL}_{2n}$. View $E_{\check{G}}$ as a rank $2n$ local system E on X with a symplectic form $\omega : \wedge^2 E \rightarrow \bar{\mathbb{Q}}_\ell$. Let $\mathcal{W} = \mathrm{Ker} \omega$. Assume

$$H^0(X, \mathcal{W}) = H^0(X, \mathcal{W}^*) = 0$$

- 4) Let $G = \mathrm{GSp}_4$, $H = \mathrm{GSO}_6$ split, and $\kappa : \check{G} \hookrightarrow \check{H}$ be the inclusion A4) for $n = 3$. View $E_{\check{G}}$ as a pair (E, χ) , where E (resp., χ) is a rank 4 (resp., rank one) local system on X with a symplectic form $\omega : \wedge^2 E \rightarrow \chi$. Set $\mathcal{W} = \mathrm{Ker} \omega$. Assume

$$H^0(X, \mathcal{W} \otimes \chi^{-1}) = H^0(X, \mathcal{W}^* \otimes \chi) = 0$$

In cases 1), 2) the complex $K \in \mathrm{D}^\vee(\mathrm{Bun}_G)$ admits a natural $E_{\check{G}}$ -Hecke property (which does not necessarily extend the E -Hecke property of $K \in \mathrm{D}^\vee(\mathrm{Bun}_G)$ with respect to κ).

In case 3) there is a direct sum decomposition

$$K \xrightarrow{\sim} \bigoplus_{a \in \mu_n} K_a$$

in $\mathrm{D}^\vee(\mathrm{Bun}_G)$ compatible with the E -Hecke property of K with respect to κ . For each $a \in \mu_n$, K_a can be naturally equipped with a $E_{\check{G}}$ -Hecke property.

In case 4) there is a direct sum decomposition

$$K \xrightarrow{\sim} \bigoplus_{a \in \mu_2} K_a$$

in $\mathrm{D}^\vee(\mathrm{Bun}_G)$ compatible with the E -Hecke property of K with respect to κ . For each $a \in \mu_2$, K_a can be naturally equipped with a $E_{\check{G}}$ -Hecke property.

Remark 3.1.5. i) Assume that we are in the situation of Proposition 3.1.4 1). Then the set of liftings of $E_{\check{H}}$ to a \check{G} -local system identifies with $\bar{\mathbb{Q}}_\ell^*$.

ii) Assume that we are in the situation of Proposition 3.1.4 3). So, E is a SL_{2n} -local system on X equipped with a distinguished lifting to a \check{G} -local system. This lifting is given by the symplectic form $\omega : \wedge^2 E \rightarrow \bar{\mathbb{Q}}_\ell$. Then any other lifting of E to a \check{G} -local

system on X is given by the symplectic form $a\omega : \wedge^2 E \rightarrow \bar{\mathbb{Q}}_\ell$ with $a \in \mu_n$. Besides, for any $a \in \mu_n$ the \check{G} -local systems (E, ω) and $(E, a\omega)$ are isomorphic. Namely, a choice of $b \in \mu_{2n} \subset \bar{\mathbb{Q}}_\ell^\times$ with $b^2 = a$ provides such an isomorphism $(E, \omega) \rightarrow (E, a\omega)$ given by the multiplication by b .

Assume for a moment that we work in the setting of \mathcal{D} -modules, so that we have the corresponding algebraic stacks $\text{LocSys}_{\check{G}}, \text{LocSys}_{\check{H}}$ of local systems on X and the induction map $\text{LocSys}_{\check{G}} \rightarrow \text{LocSys}_{\check{H}}$. Then the fibre of the latter map over $E_{\check{H}}$ identifies with the scheme μ_n .

3.2. Applications.

3.2.1. Let $m \geq 2$, $n = m - 1$. Let (G, H) and $\kappa : \check{G} \hookrightarrow \check{H}$ be as in Section 2.2.12 (the factor \mathbb{G}_m , on which κ is trivial, is omitted). So, $\kappa : \text{GL}_{m-1} \hookrightarrow \text{GL}_m$ is the inclusion A1).

Corollary 3.2.2. *Let $x \in X$. Let E_0 be a rank $m - 1$ local system on X , $E = E_0 \oplus \bar{\mathbb{Q}}_\ell$ be the induced \check{H} -local system on X . Assume $H^0(X, E_0) = H^0(X, E_0^*) = 0$. Let $K \in D^-(\text{Bun}_{\check{H}})_!$ be equipped with a structure of a E -Hecke eigen-sheaf. Then $F_G(K)$ is naturally equipped with a E_0 -Hecke property.*

Proof. Combine Proposition 3.1.4 1), Theorem 2.2.14, and Theorem 2.1.11. \square

Corollary 3.2.3. *Assume we are in the situation of Section 2.2.6 with $m = 2, n = 1$. So, $\mathbb{G} = \text{GL}_2, \mathbb{H} = \text{GSO}_4$, and $\kappa : \check{G} \hookrightarrow \check{H}$ is trivial on the factor \mathbb{G}_m , which is omitted. Let $E_{\check{G}}$ be a \check{G} -local system on X , E be the \check{H} -local system on X induced via κ . Let $K \in D^-(\text{Bun}_{\check{H}})_!$ be equipped with a structure of a E -Hecke eigen-sheaf. Then $F_{\mathbb{G}}(K)$ is naturally equipped with a $E_{\check{G}}$ -Hecke property.*

Proof. The argument is similar to Proposition 3.1.4 2) using Corollary 2.4.2. There is a homomorphism $\nu : \check{H} \rightarrow \check{G}$ such that $\nu\kappa = \text{id}$. \square

3.3. Automorphic sheaves for GSp_4 .

3.3.1. Use notations of Section 2.2.6 with $m = 3, n = 2$. So, $\mathbb{H} = \text{GSO}_6, \mathbb{G} = \text{GSp}_4$. Let $E_{\check{G}}$ be a \check{G} -local system on X viewed as a pair (E, χ) , where E (resp., χ) is a rank 4 (resp., rank 1) local system on X with a symplectic form $\omega : \wedge^2 E \rightarrow \chi$. Let $\kappa : \check{G} \hookrightarrow \check{H}$ be the map denoted by i_κ in Section 2.2.8. Let $E_{\check{H}}$ be the \check{H} -local system on X obtained from $E_{\check{G}}$ by the extension of scalars via $\kappa : \check{G} \hookrightarrow \check{H}$. Set $\mathcal{W} = \text{Ker } \omega$.

For the convenience of the reader recall that $\check{H} \xrightarrow{\sim} \{(c, b) \in \mathbb{G}_m \times \text{GL}_4 \mid \det b = c^2\}$. We may view $E_{\check{H}}$ as the pair (E, χ) , where we forget the symplectic form but keep the induced isomorphism $\det E \xrightarrow{\sim} \chi^2$ of local systems on X .

Corollary 3.3.2. *Assume in the situation of Section 3.3.1 that $H^0(X, \mathcal{W} \otimes \chi^{-1}) = H^0(X, \mathcal{W}^* \otimes \chi) = 0$. Let $K \in D^-(\text{Bun}_{\check{H}})_!$ be equipped with a structure of a E -Hecke eigen-sheaf. Then there is a decomposition $F_{\mathbb{G}}(K) \xrightarrow{\sim} \bigoplus_{a \in \mu_2} \mathcal{K}_a$ in $D^-(\text{Bun}_{\mathbb{G}})$ such that for each $a \in \mu_2$, \mathcal{K}_a is naturally equipped with a $E_{\check{G}}$ -Hecke property. \square*

Proof. Combine Proposition 3.1.4 4) and Corollary 2.4.2 i). \square

3.3.3. In the situation of Section 3.3.1 suppose in addition that E is an irreducible rank 4 local system on X . Under this assumption we have constructed a perverse sheaf denoted $K_{E,\chi,\mathbb{H}}$ on $\mathrm{Bun}_{\mathbb{H}}$ in ([30], Lemma 17). This is a $E_{\mathbb{H}}$ -Hecke eigen-sheaf.

The following is our main result, it essentially establishes ([30], Conjecture 6(ii)).

Theorem 3.3.4. *i) Under the assumptions of Section 3.3.3, there is a decomposition*

$$(8) \quad F_G(K_{E,\chi,\mathbb{H}}) \xrightarrow{\sim} \bigoplus_{a \in \mu_2} \mathcal{K}_a$$

in $D^{\prec}(\mathrm{Bun}_{\mathbb{G}})$. For $a \in \mu_2$, \mathcal{K}_a is naturally equipped with a $E_{\mathbb{G}}$ -Hecke property.

ii) The complex (8) is nonzero. So, there is a nonzero $E_{\mathbb{G}}$ -Hecke eigen-sheaf in $D^{\prec}(\mathrm{Bun}_{\mathbb{G}})$.

Proof. i) By construction, $K_{E,\chi,\mathbb{H}}$ is a $E_{\mathbb{H}}$ -Hecke eigen-sheaf on $\mathrm{Bun}_{\mathbb{H}}$ in the sense of Definition 2.1.1. An irreducible local system on X may admit at most a unique (up to a scalar) nondegenerate bilinear form with values in a rank one local system on X . This implies $H^0(X, \mathcal{W} \otimes \chi^{-1}) = H^0(X, \mathcal{W}^* \otimes \chi) = 0$. Part i) follows now from Corollary 3.3.2.

ii) In Section A we check that (8) is nonzero provided that X comes from a curve X_0 defined over a finite subfield $k_0 \subset k$, and $E_{\mathbb{G}}$ comes from a \mathbb{G} -local system $E_{0,\mathbb{G}}$ over X_0 . Actually (8) is always nonzero, as is shown in ([32], Theorem 2.6.2). \square

Remark 3.3.5. *i) For $G = \mathrm{GSp}_4$ the geometric Bessel periods of an object of $D(\mathrm{Bun}_G)$ with a given central character are introduced in ([30], Definition 11). A conjectural description of these Bessel periods for any Hecke eigen-sheaf in $D(\mathrm{Bun}_G)$ was proposed in ([30], Conjecture 4). The geometric Bessel periods of $F_G(K_{E,\chi,\mathbb{H}})$ from Theorem 3.3.4 are described in terms of the generalized Waldspurger periods of $K_{E,\chi,\mathbb{H}}$ in ([30], Proposition 11) and further studied in [32].*

ii) For the construction of $K_{E,\chi,\mathbb{H}}$ on $\mathrm{Bun}_{\mathbb{H}}$ in ([30], Lemma 17 and Definition 8) we used the perverse sheaf Aut_E on Bun_4 normalized as in [6]. However, Aut_E is a E^* -Hecke eigen-sheaf in the sense of our Definition 2.1.1. This agrees with the fact that the geometric theta-lifting functors for the pair of split groups $(\mathrm{GSp}_{2n}, \mathrm{GSO}_{2m})$ send a complex with a given central character to a complex with the opposite central character, cf ([30], Remark 2).

The first Whittaker coefficients functor $\mathrm{Whit} : D^{\prec}(\mathrm{Bun}_{\mathbb{G}}) \rightarrow D^-(\mathrm{Spec} k)$ for \mathbb{G} is defined in ([24], Definition 1).

Conjecture 3.3.6. *In the situation of Theorem 3.3.4 the complex $F_G(K_{E,\chi,\mathbb{H}})$ on $\mathrm{Bun}_{\mathbb{G}}$ is of the form $\mathcal{K} \otimes \mathcal{E}$, where \mathcal{E} is a constant complex, and \mathcal{K} is a perverse sheaf irreducible of each connected component of $\mathrm{Bun}_{\mathbb{G}}$. Moreover, the first Whittaker coefficient $\mathrm{Whit}(\mathcal{K})$ identifies with \mathbb{Q}_{ℓ} (up to a cohomological shift).*

4. PROOFS

4.0.1. Given complexes $K_1, K_2, K'_1, K'_2 \in D^{\prec}(S)$ for some algebraic stack S and a map $f : K_1 \oplus K_2 \rightarrow K'_1 \oplus K'_2$ in $D^{\prec}(S)$, say that it is diagonal with respect to this decomposition if it is a sum of maps $K_i \rightarrow K'_i$ in $D^{\prec}(S)$. This is a property of f , not an additional structure.

Lemma 4.0.2. *Let S be an algebraic stack locally of finite type, $K_i, K'_i \in \mathcal{D}^\prec(S)$. Let $g : K_1 \oplus K_2 \xrightarrow{\sim} K'_1 \oplus K'_2$ be an isomorphism. Write $g_{12} : K_1 \rightarrow K'_2$ for the corresponding component of g . Assume $H^m(g_{12}) : H^m(K_1) \rightarrow H^m(K'_2)$ vanishes for all m . Then the components $g_{ii} : K_i \rightarrow K'_i$ of g for $i = 1, 2$ are isomorphisms in $\mathcal{D}^\prec(S)$.*

Proof. For $m \in \mathbb{Z}$ the isomorphism $H^m(g) : H^m(K_1) \oplus H^m(K_2) \xrightarrow{\sim} H^m(K'_1) \oplus H^m(K'_2)$ is triangular, so $H^m(g_{ii})$ are isomorphisms for $i = 1, 2$. Our claim follows from the property (P) given in Section 1.2. \square

Lemma 4.0.3. *Let S be an algebraic stack locally of finite type, $K \in \mathcal{D}^\prec(S)$, $q : S \times X \rightarrow S$ the projection. Let $f : q^*K \rightarrow q^*K$ be a map in $\mathcal{D}^\prec(S \times X)$, which is an idempotent. Then there is an idempotent $\bar{f} : K \rightarrow K$ in $\mathcal{D}^\prec(S)$ such that $q^*\bar{f} = f$.*

Proof. 1) The category $\mathcal{D}^\prec(S)$ is idempotent complete. Indeed, the DG-category underlying $\mathcal{D}_{nc}(S)$ is cocomplete, so a retract of an object $F \in \mathcal{D}^\prec(S)$ is given by a direct summand of F in $\mathcal{D}_{nc}(S)$. The latter lies in $\mathcal{D}^\prec(S)$ by its definition.

For a morphism $g : q^*K \rightarrow q^*K$ in $\mathcal{D}^\prec(S \times X)$ let $\tilde{g} : K \rightarrow q_*q^*K = K \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell)$ be the morphism in $\mathcal{D}^\prec(S)$ corresponding to g by adjointness. Given $f_i : q^*K \rightarrow q^*K$, the map $\bar{f}_1 \bar{f}_2 : K \rightarrow K \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell)$ is the composition

$$K \xrightarrow{\bar{f}_2} K \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \xrightarrow{\bar{f}_1} K \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \xrightarrow{\text{id} \times m} K \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell),$$

where m is the product in $\mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell)$ given as the composition $\mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \rightarrow \mathbf{R}\Gamma(X^2, \bar{\mathcal{Q}}_\ell) \xrightarrow{\Delta^*} \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell)$. Here $\Delta : X \rightarrow X^2$ is the diagonal. To see this, we used the following. For the map $\bar{q} : X \rightarrow \text{Spec } k$ let $\epsilon : \bar{q}^* \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \rightarrow \bar{\mathcal{Q}}_\ell$ be the map on X corresponding by adjointness to $\text{id} : \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \rightarrow \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell)$. Then $\bar{q}_* \epsilon : \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell) \rightarrow \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell)$ equals m .

So, our f corresponds to $\tilde{f} : K \rightarrow K \otimes \mathbf{R}\Gamma(X, \bar{\mathcal{Q}}_\ell)$ by adjointness. Write $\tilde{f} = \sum_{i=0}^2 h_i$, where $h_i : K \rightarrow K \otimes H^i(X, \bar{\mathcal{Q}}_\ell)[-i]$ is the corresponding component. It is clear that $h_0 : K \rightarrow K$ is an idempotent. Let $K = K_0 \oplus K_1$ be the decomposition of K such that h_0 acts by 0 (resp., by 1) on K_0 (resp., on K_1). Similarly, we have the decomposition $q^*K = F_0 \oplus F_1$ of q^*K under f . Let $p_i : q^*K \rightarrow F_i$ be the corresponding projection. We claim that the composition $q^*K_i \hookrightarrow q^*K \xrightarrow{p_i} F_i$ is an isomorphism in $\mathcal{D}^\prec(S \times X)$. Indeed, this is an isomorphism after passing to any cohomology sheaf on X . Our claim follows now from (P) in Section 1.2. \square

4.1. Proof of Proposition 3.1.2. For an irreducible $V^\lambda \in \text{Rep}(\check{G})$ pick $W \in \text{Rep}(\check{H})$ such that $\text{Res}^\kappa(W) \xrightarrow{\sim} V^\lambda \oplus V'$ in $\text{Rep}(\check{G})$, and V^λ does not appear in V' . There could exist at most one isomorphism α_V given by (2) for $V = V^\lambda$ such that the corresponding isomorphism α_W is diagonal with respect to the above decomposition of $\text{Res}(W)$. This defines uniquely the desired isomorphism α_V for any $V \in \text{Rep}(\check{G})$. The commutations of diagrams H1), H2) is a property, not an additional structure. So, the E -Hecke property admits at most a unique extension to a $E_{\check{G}}$ -Hecke property of K .

The necessary conditions for this extension to exist are: i) for V^λ , W as above the isomorphism α_W is diagonal with respect to the decomposition $\text{Res}^\kappa(W) \xrightarrow{\sim} V^\lambda \oplus V'$; ii) the diagrams H1), H2) commute. \square

4.1.1. Proof of Remark 3.1.3. For the embeddings A1-A3) this follows from the branching rules ([21], Section 8.1.1). For A4) this follows from A2). For A5) this follows from [35] (cf. also [33]). The case A6) is easy, as the composition $\check{G} \xrightarrow{\kappa} \check{G} \times \check{G}_1 \xrightarrow{\mathrm{pr}} \check{G}$ is the identity. \square

Remark 4.1.2. *In the situation of Proposition 3.1.2, the E -Hecke property of $K \in \mathrm{D}^\prec(\mathrm{Bun}_G)$ with respect to κ does not always extend to a $E_{\check{G}}$ -Hecke property. Consider $\kappa : \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_2$ given by A1). Assume $E_{\check{G}}$ trivial, $K = \mathbb{Q}_\ell$ on $\mathrm{Bun}_{\mathrm{G}_m}$. Pick $h \in \check{H}(\mathbb{Q}_\ell)$. Equip K with the isomorphisms α_V for each $V \in \mathrm{Rep}(\check{H})$ obtained as the compositions*

$$\mathrm{H}_G^+(V, K) \xrightarrow{\sim} V[1] \boxtimes K \xrightarrow{h \boxtimes \mathrm{id}} V[1] \boxtimes K,$$

where the first map is the tautological E -Hecke property of K , and the second one comes from the action of h on V . This E -Hecke property does not extend to a $E_{\check{G}}$ -Hecke property unless h is diagonal.

4.2. Reformulated Hecke property for GL_n . Let $G = H = \mathrm{GL}_n$. Let E be a GL_n -local system on X . In [10, 6] it was shown that the following reformulation of the E -Hecke property of $K \in \mathrm{D}(\mathrm{Bun}_{\mathrm{GL}_n})$ is equivalent to Definition 2.1.1:

Definition 4.2.1. *An object $K \in \mathrm{D}^\prec(\mathrm{Bun}_G)$ is equipped with a E -Hecke property if we are given an isomorphism α_V as in Definition 2.1.1 only for the standard representation V_0 of \check{H} , for which H1) of Definition 2.1.1 holds with $V_1 = V_2 = V_0$.*

4.3. Weyl's construction for symplectic groups. For the convenience of the reader, we recall some facts about Weyl's construction for Sp_{2n} given in ([7], Section 17.3) and ([21], Chapter 10). It is used in our proof of Proposition 3.1.4 below.

4.3.1. Pick $n \geq 1$. Let $\kappa : \check{G} \hookrightarrow \check{H}$ be the inclusion $\mathrm{Sp}_{2n} \hookrightarrow \mathrm{SL}_{2n}$. Let V be the standard representation of \check{H} . Pick a maximal torus and a system of positive roots in Sp_{2n} as in ([7], Section 16.1). Write Λ_G for the lattice of weights of \check{G} , Λ_G^+ for dominant weights of \check{G} . So, $\Lambda_G = \mathbb{Z}^n$ and

$$\Lambda_G^+ = \{(a_1, \dots, a_n) \in \Lambda_G \mid a_1 \geq \dots \geq a_n \geq 0\}$$

For $\lambda \in \Lambda_G^+$ write V^λ for the irreducible representation of \check{G} with highest weight λ .

For $d \geq 0$ let $\mathrm{Par}(d, n) = \{\lambda \in \Lambda_G^+ \mid \sum_i a_i = d\}$. The notation Par stands for *partition*. For $\lambda \in \mathrm{Par}(d, n)$ let W^λ denote an irreducible S_d -module associated to λ by the Schur-Weyl duality normalized as in ([7], Theorem 4.3). For example, if $\lambda = (d, 0, \dots, 0)$ then W^λ is trivial. If $\lambda = (1, \dots, 1)$ then W^λ is the sign representation. It is understood that S_0 is the trivial group.

For $d \geq 0$ the subspace $\mathcal{H}(V^{\otimes d}) \subset V^{\otimes d}$ of harmonic tensors is defined as the intersection of kernels of the operators C_{ij} for $1 \leq i < j \leq d$. Here $C_{ij} : V^{\otimes d} \rightarrow V^{\otimes d-2}$ is given by

$$C_{ij}(v_1 \otimes \dots \otimes v_d) = \omega(v_i, v_j)v_1 \otimes \dots \otimes \hat{v}_i \otimes \dots \otimes \hat{v}_j \otimes \dots \otimes v_d$$

for $v_i \in V$. Here $\omega : \wedge^2 V \rightarrow \mathbb{Q}_\ell$ is the symplectic form. By ([21], Theorem 10.2.7), one has a canonical decomposition as a $\check{G} \times S_d$ -module

$$(9) \quad \mathcal{H}(V^{\otimes d}) \xrightarrow{\sim} \bigoplus_{\lambda \in \mathrm{Par}(d, n)} V^\lambda \otimes W^\lambda$$

Let $\mathcal{B}_d = \{x \in \text{End}(V^{\otimes d}) \mid xg = gx \text{ for } g \in \check{G}\}$ be the centralizer algebra of the \check{G} -action on $V^{\otimes d}$. By ([21], Theorem 4.2.1 and Section 10.1.1) one has a canonical decomposition

$$(10) \quad V^{\otimes d} \simeq \bigoplus_{0 \leq r \leq \frac{d}{2}} \left(\bigoplus_{\lambda \in \text{Par}(d-2r, n)} V^\lambda \otimes F_d^\lambda \right),$$

where each F_d^λ is a nonzero irreducible representation of \mathcal{B}_d . If $F_d^\lambda \simeq F_d^{\lambda'}$ as \mathcal{B}_d -modules in this decomposition then $\lambda = \lambda'$, that is, (10) is the isotypic decomposition under the action of \mathcal{B}_d . The summand in (10) corresponding to $r = 0$ is $\mathcal{H}(V^{\otimes d})$. So, if $\lambda \in \text{Par}(d, n)$ then $F_d^\lambda \simeq W^\lambda$ as S_d -modules.

Let $\rho_d : S_d \rightarrow \text{GL}(V^{\otimes d})$ be the natural representation. Write P^0 for the composition $V^{\otimes 2} \xrightarrow{\omega} \bar{\mathcal{Q}}_\ell \hookrightarrow V^{\otimes 2}$, the second map being the canonical \check{G} -invariant inclusion. For $d \geq 2$ let P denote the map $\text{id} \otimes P^0 : V^{\otimes d-2} \otimes V^{\otimes 2} \rightarrow V^{\otimes d-2} \otimes V^{\otimes 2}$.

Proposition 4.3.2 ([21], Theorem 10.1.6). *If $d \geq 2$ then \mathcal{B}_d is generated, as an associative k -algebra, by $\rho_d(g), g \in S_d$ and P .*

4.4. Proof of Proposition 3.1.4. 1) Write V_0 for the standard representation of \check{G} . Let $V = V_0 \oplus \bar{\mathcal{Q}}_\ell$, this is the standard representation of \check{H} . Consider the isomorphism α_V of Definition 2.1.1 for this particular representation of \check{H} . So,

$$\alpha_V : \text{H}_G^{\check{G}}(V_0, K) \oplus (K \boxtimes \bar{\mathcal{Q}}_\ell[1]) \simeq (K \boxtimes E_0[1]) \oplus (K \boxtimes \bar{\mathcal{Q}}_\ell[1])$$

on $\text{Bun}_G \times X$.

Consider the component $\alpha_{12} : K \boxtimes \bar{\mathcal{Q}}_\ell[1] \rightarrow K \boxtimes E_0[1]$ of α_V . For any $m \in \mathbb{Z}$, the map $\text{H}^m(\alpha_V) : \text{H}^m(K \boxtimes \bar{\mathcal{Q}}_\ell[1]) \rightarrow \text{H}^m(K \boxtimes E_0[1])$ on $\text{Bun}_G \times X$ is zero, because $\text{H}^0(X, E_0) = 0$. By Lemma 4.0.2, the components

$$\alpha_{V_0} : \text{H}_G^{\check{G}}(V_0, K) \rightarrow K \boxtimes E_0[1]$$

and $K \boxtimes \bar{\mathcal{Q}}_\ell[1] \rightarrow K \boxtimes \bar{\mathcal{Q}}_\ell[1]$ of α_V are isomorphisms.

Any map $\text{H}^m(K \boxtimes E_0) \rightarrow \text{H}^m(K \boxtimes \bar{\mathcal{Q}}_\ell)$ on $\text{Bun}_G \times X$ is zero, because $\text{H}^0(X, E_0^*) = 0$. So, for any $m \in \mathbb{Z}$ the map $\text{H}^m(\alpha_V)$ is diagonal with respect to the decomposition $V = V_0 \oplus \bar{\mathcal{Q}}_\ell$.

We obtained the isomorphism α_{V_0} for the standard representation V_0 of \check{G} . By H1) of Definition 2.1.1, $\alpha_{V, V}$ is S_2 -equivariant. Since $V_0 \otimes V_0 \subset V \otimes V$ is S_2 -stable, the corresponding map α_{V_0, V_0} is also S_2 -equivariant. In view of Section 4.2, we are done.

2) The composition $\check{G} \xrightarrow{\kappa} \check{H} \xrightarrow{\text{pr}} \check{G}$ is the identity, where pr is the projection. For $V \in \text{Rep}(\check{G})$ define α_V as $\alpha_{\text{Res}^{\text{pr}}(V)}$, where $\text{Res}^{\text{pr}}(V)$ denotes the restriction of V via pr. This gives the required $E_{\check{G}}$ -Hecke property.

3) We derive our result from ([1], Theorem 10.5.2). Let us work for a moment on the level of DG-categories on prestacks as in *loc.cit.* First, we show that $K \in \text{D}^{\check{G}}(\text{Bun}_G)$ has nilpotent singular support in the sense of [1]. For any $\lambda \in \Lambda_G^+$, the irreducible G -module V^λ appears in $V^{\otimes d}$, where d is such that $\lambda \in \text{Par}(d, n)$. So, $\text{H}_G^{\check{G}}(V^\lambda, K)$ is a direct summand of $\text{H}_G^{\check{G}}(V^{\otimes d}, K) \simeq K \boxtimes E^{\otimes d}[1]$. By ([1], 10.3.7), K has nilpotent singular support.

Let $Lift_E$ be the scheme of lifting of E to a \check{G} -local system on X . Then $Lift_E$ identifies with μ_n . Namely, any symplectic form on E equals $a\omega$ for $a \in \mu_n$. The element $a = 1$ corresponds to the \check{G} -local system $E_{\check{G}}$.

In the notations of ([1], Theorem 10.5.2), $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}^{restr}(X))$ acts on $Shv_{\mathrm{Nilp}}(\mathrm{Bun}_G)$. So, the fibre of $Shv_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ over $E \in \mathrm{Bun}_H$ is a DG-category over $Lift_E$. Since K is equipped with a E -Hecke property with respect to κ , it decomposes canonically as $K \xrightarrow{\sim} \bigoplus_{a \in \mu_n} K_a$ in a way compatible with the E -Hecke eigen-property with respect to κ , here K_a is the summand lying in the fibre of $Shv_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ over $(E, a\omega)$. Thus, each K_a is compatibly equipped with a Hecke property for the local system $(E, a\omega) \in Lift_E$.

Note that $(E, a\omega)$ and (E, ω) are isomorphic as \check{G} -local systems on X , a choice of $b \in \mu_{2n}$ with $b^2 = a$ provides an isomorphism $(E, a\omega) \rightarrow (E, \omega)$ given by the multiplication by b . Thus, K_a is naturally a $E_{\check{G}}$ -Hecke eigen-sheaf. This completes the proof, which is however, not explicit.

We present below an argument allowing to obtain the above decomposition explicitly as well as the Hecke property of each K_a . Keep notations of Section 4.3. We obtain the isomorphisms (2) using Weyl's construction for Sp_{2n} recalled in Section 4.3.

Step 1 For $d > 0$ consider the isomorphism on $\mathrm{Bun}_G \times X$

$$(11) \quad \alpha_{V^{\otimes d}} : \mathrm{H}_G^{\leftarrow}(V^{\otimes d}, K) \xrightarrow{\sim} K \boxtimes E^{\otimes d}[1]$$

given by the E -Hecke property of K with respect to κ . Let S_d act on $E^{\otimes d}$ naturally. By H1) and H2) of Definition 2.1.1, (11) is S_d -equivariant. We claim that $\alpha_{V^{\otimes d}}$ is diagonal with respect to the decomposition (10).

First let $d = 2$. One has $\mathcal{H}(V^{\otimes 2}) = (\mathrm{Sym}^2 V) \oplus V^\lambda$ with $\lambda = (1, 1, 0, \dots, 0)$. Besides, $F_2^0 = \mathbb{Q}_\ell$ on which S_2 acts by the sign character. By the above arguments based on [1],

$$\alpha_{\wedge^2 V} : \mathrm{H}_G^{\leftarrow}(\wedge^2 V, K) \xrightarrow{\sim} K \boxtimes (\wedge^2 E)[1]$$

is diagonal with respect to the decomposition $\wedge^2 V \xrightarrow{\sim} V^\lambda \oplus V^0$ for $\lambda = (1, 1, 0, \dots, 0)$. This is the only place where we appeal to *loc.cit.* in this algorithmic part of the proof. Thus, $\alpha_{V^{\otimes 2}}$ is diagonal with respect to (10).

For $d \geq 2$ first restrict the isomorphism $\alpha_{V, \dots, V}$ to a diagonal of codimension one in X^d and use the above diagonal decomposition of $\alpha_{V^{\otimes V}}$ for this diagonal. Then further restrict to the main diagonal in X^d . From the description of \mathcal{B}_d given in Proposition 4.3.2 we conclude that for all $m \in \mathbb{Z}$ the map $\mathrm{H}^m(\alpha_{V^{\otimes d}})$ is \mathcal{B}_d -equivariant. Since (10) is the isotypic decomposition under the action of \mathcal{B}_d , $\alpha_{V^{\otimes d}}$ is diagonal with respect to the decomposition (10).

For $\lambda \in \Lambda_G^+$ set $E^\lambda = E^{V^\lambda}$. We conclude that each component

$$\alpha_{V^\lambda \otimes F_d^\lambda} : \mathrm{H}_G^{\leftarrow}(V^\lambda \otimes F_d^\lambda, K) \rightarrow K \boxtimes E^\lambda \otimes F_d^\lambda[1]$$

of $\alpha_{V^{\otimes d}}$ is a \mathcal{B}_d -equivariant isomorphism.

Recall that $\mathrm{D}^{\leftarrow}(S)$ is idempotent complete for an algebraic stack locally of finite type S . By Lemma 4.0.3, the map $\alpha_{V^\lambda \otimes F_d^\lambda}$ writes as an isomorphism denoted

$$\alpha_{V^\lambda, d} : \mathrm{H}_G^{\leftarrow}(V^\lambda, K) \rightarrow K \boxtimes E^\lambda[1]$$

tensored by F_d^λ . Thus, for any $0 \leq r \leq \frac{d}{2}$ and $\lambda \in \text{Par}(d - 2r, n)$ we obtained an isomorphism $\alpha_{V^\lambda, d}$.

Step 2 Consider the isomorphism $\alpha_{V^0, 2} : K \boxtimes \bar{\mathbb{Q}}_\ell \rightarrow K \boxtimes \bar{\mathbb{Q}}_\ell$ on $\text{Bun}_G \times X$, recall that V^0 is the trivial \check{G} -module. Consider the representation $V^{\otimes 2n}$ of \check{H} , it contains the trivial representation $\det V$ of \check{H} . Using H2) of the E -Hecke structure of K with respect to κ for the representations $V^{\otimes 2}, \dots, V^{\otimes 2}$ taken n times, we learn that $\alpha_{V^0, 2} \circ \dots \circ \alpha_{V^0, 2} : K \boxtimes \bar{\mathbb{Q}}_\ell \rightarrow K \boxtimes \bar{\mathbb{Q}}_\ell$ on $\text{Bun}_G \times X$ is the identity, the composition with itself being taken n times.

By Lemma 4.0.3, there is a decomposition $K \xrightarrow{\sim} \bigoplus_{a \in \mu_n} K_a$ in $\text{D}^{\check{r}}(\text{Bun}_G)$ such that $\alpha_{V^0, 2}$ acts on K_a by a . This decomposition is preserved by each isomorphism α_U for $U \in \text{Rep}(\check{H})$. For each $a \in \mu_n$, we twist the E -Hecke property of K_a with respect to κ by a suitable element of the center μ_{2n} of \check{H} as in Remark 4.1.2. So, we may and do assume from now on that $\alpha_{V^0, 2}$ is the identity on each K_a .

Step 3 For $d > 0$ and $\lambda \in \text{Par}(d, n)$ set $\alpha_{V^\lambda} = \alpha_{V^\lambda, d}$. We claim that for any $d \geq 0$, $0 \leq r \leq \frac{d}{2}$, and $\lambda \in \text{Par}(d - 2r, n)$ one has $\alpha_{V^\lambda, d} = \alpha_{V^\lambda}$. Indeed, consider the natural embedding

$$V^\lambda \otimes F_d^\lambda \otimes (F_2^0)^{\otimes r} \hookrightarrow V^{\otimes d-2r} \otimes V^{\otimes 2r} \xrightarrow{\sim} V^{\otimes d}$$

Applying H2) of the E -Hecke property of K with respect to κ for the collection $V^{\otimes d-2r}, V^{\otimes 2}, \dots, V^{\otimes 2} \in \text{Rep}(\check{H})$, one gets

$$(\alpha_{V^\lambda, d-2r}) \circ \alpha_{V^0, 2} \circ \dots \circ \alpha_{V^0, 2} \big|_{\text{Bun}_G \times \Delta(X)} = \alpha_{V^\lambda, d}$$

Step 4 For any $\mathcal{V} \in \text{Rep}(\check{G})$ write $\mathcal{V} = \bigoplus_\lambda (V^\lambda \otimes \text{Hom}(V^\lambda, \mathcal{V}))$. Set

$$\alpha_{\mathcal{V}} = \bigoplus_{\lambda \in \Lambda_G^+} \alpha_{V^\lambda} \otimes \text{id}_{\text{Hom}(V^\lambda, \mathcal{V})}$$

We claim that the isomorphisms $\alpha_{\mathcal{V}}$ provide a $E_{\check{G}}$ -Hecke property of K . It remains to check Properties H1) and H2) of Definition 2.1.1.

For any $\mathcal{V} \in \text{Rep}(\check{G})$ we may pick $\mathcal{W} \in \text{Rep}(\check{H})$ and a decomposition $\text{Res}^\kappa(\mathcal{W}) \xrightarrow{\sim} \mathcal{V} \oplus \mathcal{V}'$ in $\text{Rep}(\check{G})$ such that the isomorphism $\alpha_{\mathcal{W}}$ is diagonal with respect to this decomposition. This formally implies H1) of Definition 2.1.1.

Property H2) of the E -Hecke eigen-sheaf K with respect to κ implies H2) for K as a $E_{\check{G}}$ -Hecke eigensheaf. Namely, given $\lambda, \lambda' \in \Lambda_G^+$ let $d, d' \geq 0$ be such that $\lambda \in \text{Par}(d, n), \lambda' \in \text{Par}(d', n)$. Then the restriction of $\alpha_{V^\lambda \boxtimes V^{\lambda'}} \big|_{\text{Bun}_G \times \Delta(X)}$ is described as the corresponding part of $\alpha_{V^{\otimes d} \boxtimes V^{\otimes d'}} \big|_{\text{Bun}_G \times \Delta(X)}$ by the above.

4) the proof is similar to 3). The inclusion $\kappa : \check{G} \hookrightarrow \check{H}$ is obtained from $\mathbb{G}_m \times \mathbb{S}p_4 \hookrightarrow \mathbb{G}_m \times \mathbb{S}L_4, (x, y) \mapsto (x^{-1}, y)$ by passing to the quotient under the diagonally embedded $(-1, -1)$. Let $\text{Lift}_{E_{\check{H}}}$ be the scheme of liftings of $E_{\check{H}}$ to a \check{G} -local system on X . Then $\text{Lift}_{E_{\check{H}}}$ identifies with μ_2 . Namely, any symplectic form $\wedge^2 E \rightarrow \chi$ compatible with a given isomorphism $\det E \xrightarrow{\sim} \chi^{\otimes 2}$ is $a\omega$ for $a \in \mu_2$. The rest of the proof is as in 3). \square

APPENDIX A. FINITE FIELD CASE

A.1. Assume $k_0 \subset k$ is a finite subfield, X comes from a curve X_0 defined over k_0 . Assume in the situation of Theorem 3.3.4 that $E_{\check{G}}$ comes from a \check{G} -local system $E_{0, \check{G}}$

on X_0 . In this section we check that the function trace of Frobenius of the complex $F_G(K_{E,\mathcal{X},\mathbb{H}})$ is nonzero.

Let \mathbb{A} be the adèle ring of X . Recall that D. Soudry has shown in [34] that irreducible automorphic cuspidal generic representations of $\mathbb{G}(\mathbb{A})$ satisfy the strong multiplicity one property. This is the reason for which we get a particular irreducible automorphic representation of $\mathbb{G}(\mathbb{A})$ attached to $E_{0,\check{\mathbb{G}}}$.

The local Langlands conjecture for \mathbb{G} over a non-archimedean local field of characteristic zero has been established in [17]. It has been extended to the case of local non-archimedean field of characteristic $p > 2$ in [19].

The local theta-correspondence for the dual pair $(\mathrm{GSp}_4, \mathrm{GSO}_6)$ over a local non-archimedean field of characteristic zero and residual characteristic $p > 2$ is completely established in ([16], Theorem 8.3 and Proposition 13.1).

A.1.1. The argument below is due to W. T. Gan. The proof is essentially as in ([17], Theorem 12.1(iii)), where a similar claim is established for number fields instead of the function field of X . Recall that $\mathbb{H} \widetilde{\rightarrow} \mathrm{GL}_4 \times \mathbb{G}_m / \{(z, z^{-2}) \mid z \in \mathbb{G}_m\}$, so an irreducible automorphic representation of $\mathbb{H}(\mathbb{A})$ writes $\Pi \boxtimes \mu$, where Π (resp., μ) is a representation of $\mathrm{GL}_4(\mathbb{A})$ (resp., \mathbb{A}^*) as in *loc.cit.* Let $\Pi \boxtimes \mu$ be the irreducible automorphic cuspidal representation of $\mathbb{H}(\mathbb{A})$ attached to the extension of scalars of $E_{0,\check{\mathbb{G}}}$ via $\kappa : \check{\mathbb{G}} \hookrightarrow \check{\mathbb{H}}$. It suffices to check that the global theta-lift $\Theta(\Pi \boxtimes \mu)$ of $\Pi \boxtimes \mu$ to $\mathbb{G}(\mathbb{A})$ is an irreducible cuspidal globally generic representation attached to $E_{0,\check{\mathbb{G}}}$. By construction, the partial twisted exterior square L -function $L^S(s, \Pi, \wedge \otimes \mu^{-1})$ has a pole at $s = 1$. By a result of Jacquet-Shalika [22], this is equivalent to Π having a nonzero Shalika period with respect to μ . In [34] and ([18], Proposition 3.1), the first Whittaker coefficient of $\Theta(\Pi \boxtimes \mu)$ is expressed in terms of the Shalika period of Π with respect to μ . So, this first Whittaker coefficient is nonzero. The cuspidality of $\Theta(\Pi \boxtimes \mu)$ is proved as in *loc.cit.* Thus, $\Theta(\Pi \boxtimes \mu)$ is a globally generic cuspidal representation of $\mathbb{G}(\mathbb{A})$. We are done.

APPENDIX B. ABELIAN CATEGORIES OVER STACKS

In this section we introduce some notions related to [8] and prove Proposition B.2.4 below.

B.1. Let K be an algebraically closed field of characteristic zero. All the stacks (and morphisms of stacks) we consider are defined over K .

All the stacks we consider in this section are assumed algebraic locally of finite type and such that the diagonal map $\mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$ is affine. For such a stack \mathcal{Y} one has the notion of a sheaf of abelian categories over \mathcal{Y} ([8], Section 9). We use the notions and results of [8] freely. Write Aff/\mathcal{Y} for the category of affine schemes over \mathcal{Y} .

B.1.1. Let \mathcal{C} be an abelian K -linear category, assume \mathcal{C} presentable in the sense of ([26], Definition 5.5.0.1).

Let A be a K -algebra, assume \mathcal{C} is a category over $\mathrm{Spec} A$ and $f : \mathrm{Spec} A \rightarrow \mathrm{Spec} B$ is a morphism of K -schemes. Then \mathcal{C} can also be viewed as a category over $\mathrm{Spec} B$. This is the operation of direct image of \mathcal{C} under f , write $f_*\mathcal{C}$ for this category over $\mathrm{Spec} B$.

Lemma B.1.2. 1) Let M be a B -module. If $X \in \mathcal{C}$ then $M \otimes_B (B \otimes_A X) \xrightarrow{\sim} M \otimes_A X$ canonically.

2) If $B' \leftarrow A' \rightarrow A$ is a diagram of K -algebras, $B = B' \otimes_{A'} A$, and \mathcal{C} is a category over $\text{Spec } A$ then $\mathcal{C} \otimes_{A'} B' \xrightarrow{\sim} \mathcal{C} \otimes_A B$ canonically as B -linear categories.

More generally, if $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ is an affine schematic representable morphism of stacks, and \mathcal{C} is a sheaf of categories over \mathcal{Y} , we define the direct image sheaf $f_*\mathcal{C}$ as a sheaf of categories over \mathcal{Y}' as follows. If $g' : S' \rightarrow \mathcal{Y}'$ is an object of Aff/\mathcal{Y}' and $\bar{f} : S \rightarrow S'$ is obtained from f by the base change under g' then we set $(f_*\mathcal{C})_{S'} = \bar{f}_*\mathcal{C}$. By Lemma B.1.2, we get indeed a sheaf of categories in the sense of [8].

B.1.3. Let \mathcal{C} be a sheaf of categories over \mathcal{Y} , $f : \mathcal{Y} \rightarrow \mathcal{Y}'$ is an affine schematic representable morphism of stacks, $g : \mathcal{Z}' \rightarrow \mathcal{Y}'$ a morphism of stacks. Let $\bar{f} : \mathcal{Z} \rightarrow \mathcal{Z}'$ be obtained from f by the base change under g . Write $\bar{g} : \mathcal{Z} \rightarrow \mathcal{Y}$ be the projection. Then $g^*(f_*\mathcal{C}) \xrightarrow{\sim} \bar{f}_*(\bar{g}^*\mathcal{C})$ canonically.

B.2. From now on the stacks \mathcal{Y} we consider will satisfy the assumptions of ([8], Section 17), so a sheaf of categories over \mathcal{Y} by ([8], Theorem 18) is a datum of a category \mathcal{C} (which we assume K -linear abelian presentable), and an action $* : \text{Vect}_{\mathcal{Y}} \times \mathcal{C} \rightarrow \mathcal{C}$ on \mathcal{C} exact in each variable. Here $\text{Vect}_{\mathcal{Y}}$ is the symmetric monoidal category of vector bundles on \mathcal{Y} .

B.2.1. Let H a connected reductive group over K , $G \subset H$ a closed connected reductive subgroup. Write $\text{Rep}(G)$ for the category of finite-dimensional representations of G , set $\overline{\text{Rep}}(G) = \text{Ind } \text{Rep}(G)$. Let \mathcal{C} be a category over $B(G)$, so $\text{Rep}(G)$ acts on \mathcal{C} .

The category $\text{Hecke}(\mathcal{C}, G)$ of Hecke objects in \mathcal{C} under the action of $\text{Rep}(G)$ is the category of pairs (x, α) , where $x \in \mathcal{C}$, and α is a collection of isomorphisms $\alpha_V : V*x \xrightarrow{\sim} x \otimes V$ for $V \in \text{Rep}(G)$ satisfying the compatibility conditions of ([2], Section 2.2). Recall that $\mathcal{C} \times_{B(G)} \text{Spec } K$ identifies canonically with the category of Hecke objects in \mathcal{C} under the action of $\overline{\text{Rep}}(G)$ by *loc.cit.*

For an algebra \mathcal{A} in $\overline{\text{Rep}}(G)$ write $\mathcal{A} - \text{mod}^r(\mathcal{C})$ for the category of right \mathcal{A} -modules in \mathcal{C} . Consider the space of functions \mathcal{O}_G as an algebra object of $\overline{\text{Rep}}(G)$, where G acts on \mathcal{O}_G by right translations. For $V \in \overline{\text{Rep}}(G)$ write \underline{V} for the underlying vector space. The following is well-known, we give a proof to recall the construction.

Lemma B.2.2. One has canonically $\mathcal{O}_G - \text{mod}^r(\mathcal{C}) \xrightarrow{\sim} \text{Hecke}(\mathcal{C}, G)$.

Proof. Let $(x, \alpha) \in \text{Hecke}(\mathcal{C}, G)$ with $x \in \mathcal{A}$. One gets the action map $a : x * \mathcal{O}_G \rightarrow x$ as the composition

$$x * \mathcal{O}_G \xrightarrow{\alpha_{\mathcal{O}_G}} x \otimes \underline{\mathcal{O}_G} \xrightarrow{\epsilon} x$$

Here $\epsilon : \mathcal{O}_G \rightarrow K$ is the counit, the restriction to $1 \in G$. See also the proof of ([8], Theorem 18), apply it for the map $\text{Spec } K \rightarrow B(G)$.

In the other direction, let $(x, a) \in \mathcal{O}_G - \text{mod}^r(\mathcal{C})$, where $a : x * \mathcal{O}_G \rightarrow x$ is the action map. For $V \in \text{Rep}(G)$ the matrix coefficient gives a map $V \otimes \underline{V}^* \rightarrow \mathcal{O}_G$ in $\overline{\text{Rep}}(G)$. Composing $x * (V \otimes \underline{V}^*) \rightarrow x * \mathcal{O}_G \xrightarrow{a} x$, by adjointness we get $\alpha_V : x * V \rightarrow x \otimes \underline{V}$. \square

B.2.3. Let $f : B(G) \rightarrow B(H)$ be the natural map. Then $f_*\mathcal{C}$ is the same category \mathcal{C} viewed as a category with the action of $\mathrm{Rep}(H)$ via $G \hookrightarrow H$. Note that $G \backslash H \xrightarrow{\sim} B(G) \times_{B(H)} \mathrm{Spec} K$, so $\mathcal{C} \times_{B(H)} \mathrm{Spec} K \xrightarrow{\sim} \mathcal{C} \times_{B(G)} G \backslash H$ is a category over $G \backslash H$.

Assume \mathcal{C}^0 is an abelian K -linear category, in which every object has a finite length, and $\mathcal{C} \xrightarrow{\sim} \mathrm{Ind}(\mathcal{C}^0)$. Since \mathcal{C}^0 admits finite colimits, \mathcal{C} is presentable by ([26], 5.5.1.1). Assume for any $x \in \mathcal{C}^0$, $\dim_K \mathrm{End}_{\mathcal{C}^0}(x) < \infty$. Assume the action of $\mathrm{Rep}(G)$ on \mathcal{C} comes (by the functoriality of Ind) from an action of $\mathrm{Rep}(G)$ on \mathcal{C}^0 .

Write $\mathcal{O}_{G \backslash H}$ for the space of functions on $G \backslash H$, we view it as an algebra object in $\overline{\mathrm{Rep}}(G)$, where G acts by right translations.

Proposition B.2.4. *Let $0 \neq x \in \mathcal{C} \times_{B(G)} G \backslash H$ whose image in \mathcal{C} lies in \mathcal{C}^0 .*

- i) There is a closed point $\mathrm{Spec} K \rightarrow G \backslash H$ such that $x \otimes_{G \backslash H} \mathrm{Spec} K \in \mathcal{C}$ is non zero.*
- ii) x admits a finite filtration $0 = x_0 \subset x_1 \subset \dots \subset x_d = x$ in $\mathcal{C} \times_{B(G)} G \backslash H$ such that for $1 \leq i \leq d$, $\mathcal{O}_{G \backslash H}$ acts on x_i/x_{i-1} via some closed point $\mathcal{O}_{G \backslash H} \rightarrow K$ of $G \backslash H$.*

Remark B.2.5. *View \mathcal{O}_H (resp., \mathcal{O}_G) as an algebra in $\overline{\mathrm{Rep}}(G)$, where G acts by the right translations. We may view x in Proposition B.2.4 as $x \in \mathcal{C}^0$ together with a structure of a right \mathcal{O}_H -module given by the action map $a : x * \mathcal{O}_H \rightarrow x$. Then a closed point $\mathrm{Spec} K \rightarrow G \backslash H$ yields by the base change $H \rightarrow G \backslash H$ a G -equivariant morphism $G \rightarrow H$, hence a morphism of algebras $\mathcal{O}_H \xrightarrow{\tau} \mathcal{O}_G$ in $\overline{\mathrm{Rep}}(G)$. By definition, $x \otimes_{H/G} \mathrm{Spec} K$ is $x \otimes_{\mathcal{O}_H} \mathcal{O}_G \in \mathcal{O}_G - \mathrm{mod}^r(\mathcal{C})$.*

*If $\mathfrak{m} = \mathrm{Ker}(\tau)$ then $x * \mathfrak{m} \rightarrow x \rightarrow x \otimes_{\mathcal{O}_H} \mathcal{O}_G \rightarrow 0$ is exact in \mathcal{C} , so $x \otimes_{H/G} \mathrm{Spec} K$ is a quotient of x in \mathcal{C}^0 .*

B.2.6. *Proof of Proposition B.2.4.* The forgetful functor $\mathcal{C} \times_{B(G)} G \backslash H \rightarrow \mathcal{C}$ is exact and faithful. So, x is of finite length as an object of $\mathcal{C} \times_{B(G)} G \backslash H$. If $y \in \mathcal{C} \times_{B(G)} G \backslash H$ is irreducible such that its image in \mathcal{C} lies in \mathcal{C}^0 then $\dim_K \mathrm{End}_{\mathcal{C} \times_{B(G)} G \backslash H}(y) < \infty$. So, the space of functions $\mathcal{O}_{G \backslash H}$ acts on y via some closed point $\xi : \mathcal{O}_{G \backslash H} \rightarrow K$ of $G \backslash H$. We get $y \otimes_{\mathcal{O}_{G \backslash H}} K \xrightarrow{\sim} y$, where the map $\mathcal{O}_{G \backslash H} \rightarrow K$ is ξ . Since for any $\mu : \mathcal{O}_{G \backslash H} \rightarrow K$ the functor $\mathcal{C} \times_{B(G)} G \backslash H \rightarrow \mathcal{C} \times_{B(G)} \mathrm{Spec} K$, $z \mapsto z \otimes_{\mathcal{O}_{G \backslash H}} K$ of base change by μ is right exact, our claim follows. (See also Proposition B.2.10 below). \square

B.2.7. The rest of Section B.2 is not used in the paper and is added for convenience of the reader. Let A be a K -algebra of finite type, \mathcal{D} is an abelian presentable category over $\mathrm{Spec} A$. Assume \mathcal{D}^0 is an abelian K -linear category, in which every object has a finite length, and $\mathcal{D} \xrightarrow{\sim} \mathrm{Ind}(\mathcal{D}^0)$. Assume for any irreducible object $x \in \mathcal{D}^0$, $\mathrm{End}_{\mathcal{D}}(x) \xrightarrow{\sim} K$.

Lemma B.2.8. *Let $X \in \mathcal{D}^0$ be irreducible. The A -action $A \rightarrow \mathrm{End}_{\mathcal{D}}(X)$ factors through some closed point $A \rightarrow K \xrightarrow{\sim} \mathrm{End}_{\mathcal{D}}(X)$ of $\mathrm{Spec} A$. \square*

The following is an analog of Nakayama's lemma.

Lemma B.2.9. *Let $X \in \mathcal{D}^0$. Assume $X \otimes_A K = 0$ for any K -point $\mathrm{Spec} K \rightarrow \mathrm{Spec} A$. Then $X = 0$.*

Proof. 1) Assume our claim true for any Y irreducible. The functor $\mathcal{C} \rightarrow \mathcal{C}, X \mapsto X \otimes_A K$ is right exact. If $X \rightarrow Y$ is a surjection with Y irreducible then $Y \otimes_A K = 0$

for any closed point of $\text{Spec } A$, so $Y = 0$. Since X is of finite length, $X = 0$. So, it suffices to prove our claim for any X irreducible.

2) Assume X irreducible. By Lemma B.2.8, A acts on X via some closed point $\text{Spec } K \rightarrow \text{Spec } A$. For this point we get $X \otimes_A K \xrightarrow{\sim} X$. So, $X = 0$. \square

The following is an immediate consequence of Lemma B.2.9.

Proposition B.2.10. *If $0 \neq X \in \mathcal{D}^0$ then there is a closed point $\text{Spec } K \rightarrow \text{Spec } A$ such that $X \otimes_A K \neq 0$.*

B.3. Here is a kind of application we have in mind. Use notations of Section 1.2. Let G be a connected reductive group over k , \check{G} its Langlands dual group over $\bar{\mathbb{Q}}_\ell$. Pick $x \in X$. Let $\text{Rep}(\pi_1(X, x))$ be the category of finite-dimensional continuous representations of $\pi_1(X, x)$ over $\bar{\mathbb{Q}}_\ell$. Let \mathcal{C} be an abelian presentable $\bar{\mathbb{Q}}_\ell$ -linear category with commuting actions of $\text{Rep}(\pi_1(X, x))$ and $\text{Rep}(\check{G})$. Both action functors $\mathcal{C} \times \text{Rep}(\check{G}) \rightarrow \mathcal{C}$, $(x, V) \mapsto x * V$ and $\mathcal{C} \times \text{Rep}(\pi_1(X, x)) \rightarrow \mathcal{C}$, $(x, W) \mapsto x * W$ are assumed exact in each variable. Let $\sigma : \pi_1(X, x) \rightarrow \check{G}$ be a continuous homomorphism. For $V \in \text{Rep}(\check{G})$ write V_σ for the composition $\pi_1(X, x) \xrightarrow{\sigma} \check{G} \rightarrow \text{GL}(V)$.

B.3.1. One defines the category $\text{Hecke}(\mathcal{C}, \sigma)$ of σ -Hecke eigen-sheaves in \mathcal{C} as the category of pairs (x, α) , where $x \in \mathcal{C}$, α is a collection of isomorphisms $\alpha_V : x * V \xrightarrow{\sim} x * V_\sigma$ for $V \in \text{Rep}(\check{G})$ satisfying the compatibility conditions as in ([2], Section 2.2). Assume that for any W in $\text{Rep}(\pi_1(X, x))$ or in $\text{Rep}(\check{G})$ the functor $\mathcal{C} \rightarrow \mathcal{C}$, $x \mapsto x * W$ is right adjoint to the functor $\mathcal{C} \rightarrow \mathcal{C}$, $x \mapsto x * W^*$.

Consider $\mathcal{O}_{\check{G}}$ as an algebra object in $\text{Rep}(\check{G} \times \pi_1(X, x))$, where \check{G} (resp., $\pi_1(X, x)$) act on $\mathcal{O}_{\check{G}}$ via left translations (resp., right translations via the homomorphism σ). Then $\mathcal{O}_{\check{G}} - \text{mod}^r(\mathcal{C}) \xrightarrow{\sim} \text{Hecke}(\mathcal{C}, \sigma)$ as in Lemma B.2.2.

Another way to spell this is as follows. The category \mathcal{C} acquires a new action of $\text{Rep}(\check{G})$ as the composition

$$\mathcal{C} \times \text{Rep}(\check{G}) \xrightarrow{\text{id} \times \text{Res}^\sigma} \mathcal{C} \times \text{Rep}(\pi_1(X, x)) \rightarrow \mathcal{C}$$

We refer to it as the *new* action. Let $\text{Rep}(\check{G} \times \check{G})$ act on \mathcal{C} so that the first factor acts through the old action, and the second one through the new one. Then

$$\mathcal{O}_{\check{G}} - \text{mod}^r(\mathcal{C}) \xrightarrow{\sim} \mathcal{C} \times_{B(\check{G} \times \check{G})} B(\check{G}),$$

where the map $\check{G} \rightarrow \check{G} \times \check{G}$ is the diagonal.

B.3.2. Let H be a connected reductive group over k , $\kappa : \check{G} \hookrightarrow \check{H}$ be an inclusion. Let $\text{Hecke}(\mathcal{C}, \kappa\sigma)$ be the category of pairs (x, α) as for $\text{Hecke}(\mathcal{C}, \sigma)$, with the difference that α_V is given for $V \in \text{Rep}(\check{H})$ only. It is understood that $\text{Rep}(\check{H})$ acts on \mathcal{C} via the restriction through κ and the old action of \check{G} .

View \mathcal{C} as a category over $B(\check{G} \times \check{G})$, hence also over $B(\check{H} \times \check{H})$ via the map $\kappa \times \kappa : \check{G} \times \check{G} \rightarrow \check{H} \times \check{H}$. One has naturally

$$\text{Hecke}(\mathcal{C}, \kappa\sigma) \xrightarrow{\sim} \mathcal{C} \times_{B(\check{H} \times \check{H})} B(\check{H}) \xrightarrow{\sim} \mathcal{C} \times_{B(\check{G} \times \check{G})} (\check{G} \times \check{G}) \backslash (\check{H} \times \check{H}) / \check{H}$$

In the latter formula, $(\check{G} \times \check{G}) \backslash (\check{H} \times \check{H}) / \check{H}$ is the stack quotient of $\check{H} \times \check{H}$ by $\check{G} \times \check{G} \times \check{H}$, where $\check{G} \times \check{G}$ (resp., \check{H}) acts by left (resp., right) translations.

Assume \mathcal{C}^0 is an abelian $\bar{\mathbb{Q}}_\ell$ -linear category, in which every object has a finite length, and $\mathcal{C} \xrightarrow{\sim} \mathrm{Ind}(\mathcal{C}^0)$. Assume the action of $\pi_1(X, x) \times \check{G}$ on \mathcal{C} comes by functoriality of Ind from its action on \mathcal{C}^0 .

There is a relation between $\mathrm{Hecke}(\mathcal{C}, \sigma)$ and $\mathrm{Hecke}(\mathcal{C}, \kappa\sigma)$ analogous to Proposition B.2.4, whose precise formulation is left to a reader.

B.3.3. Example. Assume K_1, \dots, K_r are irreducible perverse sheaves on Bun_G such that for any $V \in \mathrm{Rep}(\check{G})$, $H_G^r(V, K_i) \xrightarrow{\sim} \bigoplus_{j=1}^r (E_j^i(V)[1] \boxtimes K_j)$ for some local systems $E_j^i(V)$ on X . Let $\mathrm{Perv}(X \times \mathrm{Bun}_G)$ be the category of perverse sheaves on $X \times \mathrm{Bun}_G$. Let $\mathrm{Rep}(\pi_1(X, x))$ act on $\mathrm{Perv}(X \times \mathrm{Bun}_G)$ so that $W \in \mathrm{Rep}(\pi_1(X, x))$ sends K to $\pi_1^* W \otimes K$ for the projection $\mathrm{pr}_1 : X \times \mathrm{Bun}_G \rightarrow X$. Let $\mathcal{C}^0 \subset \mathrm{Perv}(X \times \mathrm{Bun}_G)$ be the smallest full abelian subcategory containing $\bar{\mathbb{Q}}_\ell \boxtimes K_i[1]$ for all i , stable under extensions and the action of $\mathrm{Rep}(\pi_1(X, x))$. Then it satisfies all the assumptions of Section B.3.

B.3.4. For the derived category $\mathrm{D}(\mathrm{Bun}_G)$ the definition of a Hecke eigen-sheaf as in Section B.3.1 is not satisfactory as in mentioned in ([3], Definition 5.4.2 and Remark after it). This is why we are actually using Definition 2.1.1 taken from ([9], Section 2.8).

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