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## ► To cite this version:

Prosper K. Doh, Kondo Hloindo Adjallah, B. Birregah. Thirty-six Full Matrix Forms of the Pascal Triangle: Derivation and Symmetry Relations. Scientific African, 2021, 13, pp.e00932. 10.1016/j.sciaf.2021.e00932 . hal-03321158

**HAL Id: hal-03321158**

**<https://hal.univ-lorraine.fr/hal-03321158>**

Submitted on 16 Oct 2023

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# Thirty-six Full Matrix Forms of the Pascal Triangle: Derivation and Symmetry Relations

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## Abstract

For all  $2 \leq n \in \mathbb{N}$ , the four vertices  $\binom{0}{0}, \binom{n}{0}, \binom{2n}{n}, \binom{n}{n}$  of the Pascal Triangle expanded from level 0 to level  $2n$  define the greatest embedded rhomboid sub-block denoted  $n$ -GRSB in this paper. The  $n$ -GRSB is canonically partitioned into two triangular sub-blocks  $G$  and  $g$ , with respective vertex sets  $\left\{\binom{0}{0}, \binom{n}{0}, \binom{n}{n}\right\}$  and  $\left\{\binom{n+1}{1}, \binom{2n}{n}, \binom{n+1}{n}\right\}$ . The  $G$ -sub-block (resp.  $g$ -sub-block) has twelve distinct triangular matrices arrangements, numbered from 1 to 12 and designated here  $G$ -matrix set (resp.  $g$ -matrix set): three northeast, three northwest, three southwest and three distinct triangular southeast arrangements. So we defined thirty-six matrix forms of the  $n$ -GRSB encompassing the set of the sums of pair-wise complementary  $G$ - and  $g$ -matrices. We refer to the thirty-six as full Pascal matrices (FP-matrices). We then identify and present the invariant groups underlying two significant partitions of the FP-matrix set. The insight gained from a previous study of the twelve  $G$ -matrices led us to derive the 36 full matrix forms presented in this paper. Several papers in the literature have dealt with some matrix forms of the Pascal Triangle. Only two of these are so far encountered in the literature. Our work is the first to focus on the hitherto little-known 36 full matrix forms as mathematical objects in their own right. As novelty, this paper presents, for the first time, the set of the thirty-six Pascal full matrices. This work focuses on a systematic study of matrix forms derived from the Pascal Triangle, on the individual properties of these forms, their applications, and on the groups of transformations that structure their relations.

**Keywords:** Pascal matrix, bipartition, circulant matrix, transformation matrix, symmetry transformation.

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## Introduction

The  $G$ -matrix set, presented in [1], embraces the twelve possible triangular matrix *arrangements* of the Pascal Triangle expanded to level  $n$ , ( $2 \leq n \in \mathbb{N}$ ). This study presents thirty-six full matrix arrangements (referred to as FP-matrices) of the so-called  $n$ -greatest rhomboid sub-block extracted from the Pascal Triangle expanded to level  $2n$ , ( $2 \leq n \in \mathbb{N}$ ). Only two of such matrix forms have hitherto been encountered in the literature [2, 3, 4, 5, 6].

In this paper, every FP-matrix is defined as the sum of a  $G$ -matrix and one of three complementary small  $g$ -matrices. The latter are the twelve possible triangular matrix arrangements of the sub-block of the  $n$ -greatest

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rhomboid comprising the  $n(n+1)/2$  entries from levels  $n+1$  to  $2n$  (see Figure 1 for  $n \in \{3, 4\}$ ). Areas of application of the FP-matrices and the G-matrices include combinatorics, numerical analysis, cryptography, linear algebra, and computational geometry [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

The four so-called circulant  $\Theta$ -compositions ( $\Theta_1 = \alpha\gamma\delta\beta$ ,  $\Theta_2 = \beta\alpha\gamma\delta$ ,  $\Theta_3 = \delta\beta\alpha\gamma$  and  $\Theta_4 = \gamma\delta\beta\alpha$ ) help to structure the derivation of the 36 FP-matrix set and induce a partition of the latter into three twelve-vertex orbits. Every  $\Theta$ -composition is a 3-cycle rotation of matrix entries about the two centers of gravity of the complementary sub-blocks of one of four triangular bipartition of a square matrix. Given an appropriate complementary G- and g-matrix pair and the associated  $\Theta_k$  ( $k = 1, \dots, 4$ ), the procedure is the parallel unfolding of the action of  $(\Theta_k)^3$  on G and g in terms of its twelve constituent circulant factors, applied from right to left. All resulting intermediate matrix pairs are exhibited. The sum of respective pairs in that order yields one of the three induced twelve-vertex orbits of the FP-matrix.

Four six-order invariant groups generated, for example, by paring the  $\Theta$  compositions with square matrix reflections in the main-diagonal ( $T_4$ ) and in the anti-diagonal ( $T_6$ ) account for eight six-vertex partition of the FP-matrix set. Associated group generator pairs are  $\langle T_6, \Theta_1 \rangle$ ,  $\langle T_4, \Theta_2 \rangle$ ,  $\langle T_6, \Theta_3 \rangle$  and  $\langle T_4, \Theta_4 \rangle$ . The concluding section presents their actions along with the associated orbits.

The paper is organized as follows. Section 1 defines the four triangular bipartitions of any square matrix and recalls the definition of the G-matrix set presented in [1]. Section 2 presents the so-called  $n$ -greatest rhomboid sub-block extracted from the Pascal Triangle and defines the g-matrix set. Section 3 is devoted to the construction of the FP-matrix set. The generic definitions of the thirty-six are organized into a grand table divided into four segments comprising nine FP-matrices of the same triangular bipartition. The entries of each segment are derived from the Cartesian product of corresponding rows of the tables of generic definitions of the G- and g-matrix sets. Section 4 relates the vertex orbits of  $\mathcal{G}_n$  and  $\widetilde{\mathcal{G}}_n$  to the three vertex orbits of  $\mathcal{F}_n$  under the action of the  $\Theta$ -compositions. The conclusion of the section considers the special case of  $n = 1$ , corresponding to the cases of the  $2 \times 2$  G- and FP-matrix sets. It turns out that, under  $\mathbb{Z}/2\mathbb{Z}$  algebra,  $\mathcal{G}_1 = \mathcal{F}_1$  and the two sets are simple copies of the four-element Langlet geniton matrix set. Section 5 presents four six-order dihedral groups of transformations that provide an alternate three-orbit structuring of the FP-matrix set.

## 1. Preliminaries

We begin by defining the four triangular bipartitions of any square matrix  $A$  and then recall the generic definition of each of the twelve G-matrices presented in [1]. In the absence of explicit mention, matrix subscripts  $i$  and  $j$  are assumed to run from 0 to  $n$ .

### 1.1. The four triangular bipartitions of a square matrix

In the definitions that follow,  $A$  stands for any  $(n+1) \times (n+1)$  matrix.

**Definition 1.** *The North-East/south-west triangular bipartition of  $A$ .*

$$[A^{NE}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases} \quad \text{and} \quad [A^{sw}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$$

The two matrices  $A^{NE}$  and  $A^{sw}$  together define the North-East/south-west triangular bipartition of  $A$ . For short, we write  $A = A^{NE} + A^{sw} \equiv A^{NE/sw}$ .

**Definition 2.** The **South-West/north-east** triangular bipartition of  $A$ .

$$[A^{SW}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases} \quad \text{and} \quad [A^{ne}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i < j \\ 0 & \text{if } i \geq j \end{cases}$$

The two matrices  $A^{SW}$  and  $A^{ne}$  together define the **South-West/north-east** triangular bipartition of  $A$ . For short, we write  $A = A^{SW} + A^{ne} \equiv A^{SW/ne}$ .

**Definition 3.** The **North-West/south-east** triangular bipartition of  $A$ .

$$[A^{NW}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j \leq n \\ 0 & \text{if } i + j > n \end{cases} \quad \text{and} \quad [A^{se}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j > n \\ 0 & \text{if } i + j \leq n \end{cases}$$

The two matrices  $A^{NW}$  and  $A^{se}$  together define the **North-West/south-east** triangular bipartition of  $A$ . For short, we write  $A = A^{NW} + A^{se} \equiv A^{NW/se}$ .

**Definition 4.** The **South-East/north-west** triangular bipartition of  $A$ .

$$[A^{SE}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j \geq n \\ 0 & \text{if } i + j < n \end{cases} \quad \text{and} \quad [A^{nw}]_{ij} = \begin{cases} [A]_{ij} & \text{if } i + j < n \\ 0 & \text{if } i + j \geq n \end{cases}$$

The two matrices  $A^{SE}$  and  $A^{nw}$  together define the **South-East/north-west** triangular bipartition of  $A$ . For short, we write  $A = A^{SE} + A^{nw} \equiv A^{SE/nw}$ .

Notice that the triangular matrices  $A^{NE}$  and  $A^{SW}$  (resp.  $A^{NW}$  and  $A^{SE}$ ) contain the main-diagonal (resp. anti-diagonal) entry positions. They are referred to as major triangular sub-blocks of  $A$ . The corresponding complementary matrices  $A^{sw}$  and  $A^{ne}$  (resp.  $A^{se}$  and  $A^{nw}$ ) are referred to as the minor triangular sub-blocks of  $A$  as defined in [1] and depicted in Figure SM1 of the Appendix to this paper.

The  $NW/se$  and the  $SE/nw$  bipartitions highlight four types of matrix triangularity rarely encountered in the literature on triangular decomposition of matrices. We refer to the four as non-classical triangular matrix forms and define them as follows.

**Definition 5.** *Non-classical triangular matrix forms.*

$A$  is north-west major (NW) triangular  $\iff [A]_{ij} = 0$  whenever  $i + j > n$ .

$A$  is north-west minor (nw) triangular  $\iff [A]_{ij} = 0$  whenever  $i + j \geq n$ .

$A$  is south-east major (SE) triangular  $\iff [A]_{ij} = 0$  whenever  $i + j < n$ .

$A$  is south-east minor (se) triangular  $\iff [A]_{ij} = 0$  whenever  $i + j \leq n$ .

To be complete, we recall the formal characterization of the four classical triangular matrix forms.

**Definition 6.** *Classic triangular matrix forms.*

$A$  is major north-east (NE) triangular  $\iff [A]_{ij} = 0$  whenever  $i > j$ .

$A$  is minor north-east (ne) triangular  $\iff [A]_{ij} = 0$  whenever  $i \geq j$ .

$A$  is major south-west (SW) triangular  $\iff [A]_{ij} = 0$  whenever  $i < j$ .

$A$  is minor south-west (sw) triangular  $\iff [A]_{ij} = 0$  whenever  $i \leq j$ .

The eight triangular matrix forms highlighted above underlie our procedure for constructing the thirty-six FP-matrices from the so-called  $n$ -greatest rhomboid carved from the Pascal Triangle expanded to level  $2n$ , ( $n \geq 2$ ). This facilitate the understanding of the organization of the 12 matrices as shown in the next section.

### 1.2. The twelve G-matrices

In the sequel, we shall denote the twelve G-matrix set by  $\mathcal{G}_n$  and individual members by  $G_{k,n}$  where  $k \in \mathcal{I} = \{1, \dots, 12\}$ . Table 1 presents the generic definitions of the twelve organized into rows according to the four major triangular forms of Definitions 5 and 6. By construction, the mapping  $\Psi$  of Eq. 1 below

$$\mathcal{I} \xrightarrow{\Psi} \mathcal{G}_n : \Psi(k) = G_{k,n} \quad (1)$$

determines a natural bijection between  $\mathcal{I}$  and  $\mathcal{G}_n$ . So, any partition of  $\mathcal{I}$  in turn induces a corresponding partition on  $\mathcal{G}_n$  and vice-versa.

The row partition of the G-matrix set in Table 1 corresponds to the (mod 4)-classes of  $\mathcal{I}$ . Thus, the four three-order sets  $\mathcal{G}_n^{NE}$ ,  $\mathcal{G}_n^{SE}$ ,  $\mathcal{G}_n^{SW}$  and  $\mathcal{G}_n^{NW}$  constitute a partition of  $\mathcal{G}_n$  just as the four three-order sets  $\mathcal{I}_{NE}$ ,  $\mathcal{I}_{SE}$ ,  $\mathcal{I}_{SW}$  and  $\mathcal{I}_{NW}$  constitute a partition of  $\mathcal{I}$ . It turns out that threading the three columns of Table 1 from left to right yields a twelve-vertex orbit under the composition of the circulant transformations.

North-East G-matrices: $\mathcal{G}_n^{NE} = \{G_{1,n}, G_{5,n}, G_{9,n}\}$ ; $\mathcal{I}_{NE} = \{1, 5, 9\} = \{1 + 4k   k = 0, 1, 2\}$		
$[G_{1,n}]_{ij} = \begin{cases} \binom{j}{i} & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$	$[G_{5,n}]_{ij} = \begin{cases} \binom{n-i}{n-j} & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$	$[G_{9,n}]_{ij} = \begin{cases} \binom{n+i-j}{i} & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases}$
South-East G-matrices: $\mathcal{G}_n^{SE} = \{G_{2,n}, G_{6,n}, G_{10,n}\}$ ; $\mathcal{I}_{SE} = \{2, 6, 10\} = \{2 + 4k   k = 0, 1, 2\}$		
$[G_{2,n}]_{ij} = \begin{cases} \binom{j}{n-i} & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n. \end{cases}$	$[G_{6,n}]_{ij} = \begin{cases} \binom{2n-i-j}{n-j} & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n. \end{cases}$	$[G_{10,n}]_{ij} = \begin{cases} \binom{i}{n-j} & \text{if } i + j \geq n, \\ 0 & \text{if } i + j < n. \end{cases}$
South-West G-matrices: $\mathcal{G}_n^{SW} = \{G_{3,n}, G_{7,n}, G_{11,n}\}$ ; $\mathcal{I}_{SW} = \{3, 7, 11\} = \{3 + 4k   k = 0, 1, 2\}$		
$[G_{3,n}]_{ij} = \begin{cases} \binom{n+j-i}{j} & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$	$[G_{7,n}]_{ij} = \begin{cases} \binom{n-j}{n-i} & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$	$[G_{11,n}]_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$
North-West G-matrices: $\mathcal{G}_n^{NW} = \{G_{4,n}, G_{8,n}, G_{12,n}\}$ ; $\mathcal{I}_{NW} = \{4, 8, 12\} = \{4 + 4k   k = 0, 1, 2\}$		
$[G_{4,n}]_{ij} = \begin{cases} \binom{n-i}{j} & \text{if } i + j \leq n, \\ 0 & \text{if } i + j > n. \end{cases}$	$[G_{8,n}]_{ij} = \begin{cases} \binom{n-j}{i} & \text{if } i + j \leq n, \\ 0 & \text{if } i + j > n. \end{cases}$	$[G_{12,n}]_{ij} = \begin{cases} \binom{i+j}{i} & \text{if } i + j \leq n, \\ 0 & \text{if } i + j > n. \end{cases}$

Table 1: The G-matrix set  $\mathcal{G}_n$ : generic definition of members by triangular type ( $0 \leq i, j \leq n$ ).

## 2. The g-matrix set $\tilde{\mathcal{G}}_n$

We first define the so-called  $n$ -greatest rhomboid sub-block of the Pascal Triangle: the  $n$ -GRSB. In [18], Michael Zaus refers to this as Pascal's square (pp. 40-42, fig. 3.4.3; pp. 93-94) in its binary form. In [17] and [19], Cooper et al. call it Pascal parallelogram (cf. Fig.1).

### Definition 7. (The $n$ -GRSB)

Consider the Pascal Triangle expanded to level  $2n$  ( $n > 1$ ). The  $n$ -greatest rhomboid sub-block is the rhomboid whose vertices are the four binomial entries  $\binom{0}{0}$ ,  $\binom{n}{0}$ ,  $\binom{n}{n}$  and  $\binom{2n}{n}$ . The entries from  $\binom{n}{0}$  to  $\binom{n}{n}$  define the horizontal diagonal of the  $n$ -GRSB and those from  $\binom{0}{0}$  to  $\binom{2n}{n}$  the vertical diagonal.

By way of illustration, Figure 1 presents 3-GRSB and 4-GRSB where the latter is a straightforward extension of the former. As can be seen, for all  $n > 1$ , the associated  $n$ -GRSB in general comprises two triangular sub-blocks:

- an *major upper sub-block*  $U_n^0$  delimited by the level-0 entry  $\binom{0}{0}$  and the two level- $n$  entries  $\binom{n}{0}$  and  $\binom{n}{n}$ ,
- a *minor lower sub-block*  $L_{2n}^{n+1}$  delimited by the two level- $(n+1)$  entries  $\binom{n+1}{n}$  and  $\binom{n+1}{1}$  and the single level- $2n$  entry  $\binom{2n}{n}$ .

It takes little to see that the  $(n+1) \times (n+1)$  G-matrices presented in section 1 are just the twelve possible triangular matrix arrangements of the major upper sub-block  $U_n^0$ . Each of the twelve arrangements preserves the inherent Pascal recursion. Analogous arrangements of the minor lower sub-block  $L_{2n}^{n+1}$  yield the g-matrix set (to be read as *small g-matrix set*)  $\widetilde{\mathcal{G}}_n$  defined hereafter.

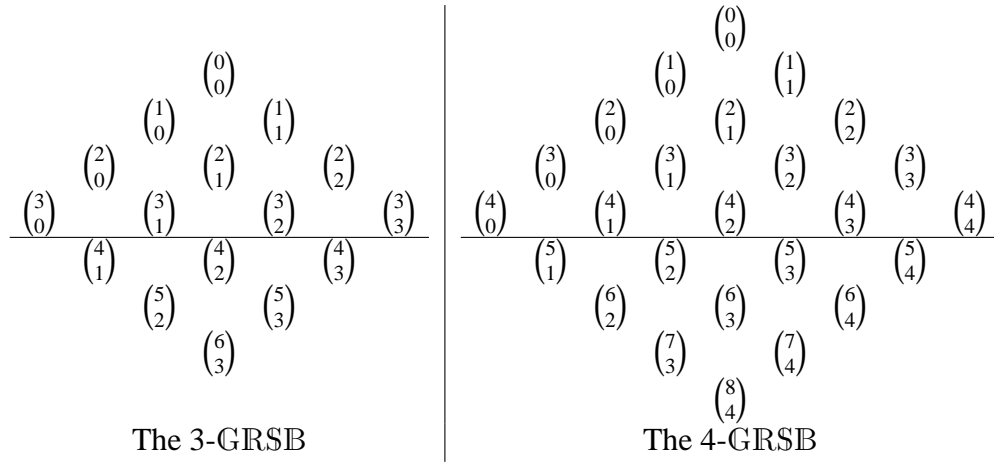


Figure 1: Two consecutive greatest rhomboid sub-blocks ( $n = 3$  and  $4$ ).

As with the G-matrices, generic entries of the twelve  $(n+1) \times (n+1)$  g-matrices in  $\widetilde{\mathcal{G}}_n$  are defined in terms of their row and column subscripts  $i$  and  $j$  where  $0 \leq i, j \leq n \in \mathbb{N}$  and numbered from 1 to 12. Thus,  $\widetilde{\mathcal{G}}_n = \{g_{k,n} | k = 1, \dots, 12\}$ . If we replace  $\mathcal{G}_n$  by  $\widetilde{\mathcal{G}}_n$  in Eq. 1, we transform the mapping  $\Psi$  into a natural bijection between  $\mathcal{I}$  and  $\widetilde{\mathcal{G}}_n$ . Hence any partition of  $\mathcal{I}$  in turn induces a corresponding partition on  $\widetilde{\mathcal{G}}_n$  and vice-versa. Table 2 presents the generic definitions of the twelve g-matrices on four rows. The three matrices of the same row are of the same triangular form.

South-west triangular g-matrices: $\tilde{\mathcal{G}}_n^{sw} = \{g_{1,n}, g_{5,n}, g_{9,n}\}; \mathcal{I}_{sw} = \{1, 5, 9\} = \{1 + 4k   k = 0, 1, 2\}$		
$[g_{1,n}]_{ij} = \begin{cases} 0 & \text{if } i \leq j, \\ \binom{n+1+j}{i} & \text{if } i > j. \end{cases}$	$[g_{5,n}]_{ij} = \begin{cases} 0 & \text{if } i \leq j, \\ \binom{2n+1-i}{n-j} & \text{if } i > j. \end{cases}$	$[g_{9,n}]_{ij} = \begin{cases} 0 & \text{if } i \leq j, \\ \binom{n+i-j}{i} & \text{if } i > j. \end{cases}$
North-west triangular g-matrices: $\tilde{\mathcal{G}}_n^{nw} = \{g_{2,n}, g_{6,n}, g_{10,n}\}; \mathcal{I}_{nw} = \{2, 6, 10\} = \{2 + 4k   k = 0, 1, 2\}$		
$[g_{2,n}]_{ij} = \begin{cases} 0 & \text{if } i + j \geq n, \\ \binom{n+1+j}{n-i} & \text{if } i + j < n. \end{cases}$	$[g_{6,n}]_{ij} = \begin{cases} 0 & \text{if } i + j \geq n, \\ \binom{2n-i-j}{n-j} & \text{if } i + j < n. \end{cases}$	$[g_{10,n}]_{ij} = \begin{cases} 0 & \text{if } i + j \geq n, \\ \binom{n+1+i}{n-j} & \text{if } i + j < n. \end{cases}$
North-east triangular g-matrices: $\tilde{\mathcal{G}}_n^{ne} = \{g_{3,n}, g_{7,n}, g_{11,n}\}; \mathcal{I}_{ne} = \{3, 7, 11\} = \{3 + 4k   k = 0, 1, 2\}$		
$[g_{3,n}]_{ij} = \begin{cases} 0 & \text{if } i \geq j, \\ \binom{n-i+j}{j} & \text{if } i < j. \end{cases}$	$[g_{7,n}]_{ij} = \begin{cases} 0 & \text{if } i \geq j, \\ \binom{2n+1-j}{n-i} & \text{if } i < j. \end{cases}$	$[g_{11,n}]_{ij} = \begin{cases} 0 & \text{if } i \geq j, \\ \binom{n+1+i}{j} & \text{if } i < j. \end{cases}$
South-east triangular g-matrices: $\tilde{\mathcal{G}}_n^{se} = \{g_{4,n}, g_{8,n}, g_{12,n}\}; \mathcal{I}_{se} = \{4, 8, 12\} = \{4 + 4k   k = 0, 1, 2\}$		
$[g_{4,n}]_{ij} = \begin{cases} 0 & \text{if } i + j \leq n, \\ \binom{2n+1-i}{j} & \text{if } i + j > n. \end{cases}$	$[g_{8,n}]_{ij} = \begin{cases} 0 & \text{if } i + j \leq n, \\ \binom{2n+1-j}{i} & \text{if } i + j > n. \end{cases}$	$[g_{12,n}]_{ij} = \begin{cases} 0 & \text{if } i + j \leq n, \\ \binom{i+j}{i} & \text{if } i + j > n. \end{cases}$

Table 2: The g-matrix set  $\tilde{\mathcal{G}}_n$ : generic definition of members by triangular type ( $0 \leq i, j \leq n$ ).

It is clear from Table 2 that  $\tilde{\mathcal{G}}_n^{ne}$ ,  $\tilde{\mathcal{G}}_n^{se}$ ,  $\tilde{\mathcal{G}}_n^{sw}$  and  $\tilde{\mathcal{G}}_n^{nw}$  are disjoint. Moreover,

$$\tilde{\mathcal{G}}_n = \tilde{\mathcal{G}}_n^{ne} \cup \tilde{\mathcal{G}}_n^{se} \cup \tilde{\mathcal{G}}_n^{sw} \cup \tilde{\mathcal{G}}_n^{nw}, \quad |\tilde{\mathcal{G}}_n^{ne}| = |\tilde{\mathcal{G}}_n^{se}| = |\tilde{\mathcal{G}}_n^{sw}| = |\tilde{\mathcal{G}}_n^{nw}| = 3, \quad \mathcal{I} = \mathcal{I}_{ne} \cup \mathcal{I}_{se} \cup \mathcal{I}_{sw} \cup \mathcal{I}_{nw},$$

and  $|\mathcal{I}_k| = 3$  for all  $k \in \{ne, se, sw, nw\}$ .

Table SM1, in Appendix, presents the twelve  $4 \times 4$  g-matrix set ( $n = 3$ ). As can be seen, the twelve non-zero sub-blocks are just matrix arrangements of the minor lower sub-block  $L_6^4$  of 3-GRSB. The stage is now set for constructing the thirty-six  $(n + 1) \times (n + 1)$  FP-matrices ( $2 \leq n \in \mathbb{N}$ ).

### 3. The thirty-six FP-matrices: construction and generic definitions

For the subsequent development, we need the following definition.

**Definition 8.**  $\{G, g\}$ -matrix pairing bijection  $S$

$$S : \mathcal{I} \times \mathcal{I} \longrightarrow \mathcal{G}_n \times \tilde{\mathcal{G}}_n \quad \text{with} \quad (p, q) \in \mathcal{I}^2 \xrightarrow{S} (G_{p,n}, g_{q,n}). \quad (2)$$

In general, the non-zero triangular sub-blocks of  $G_{p,n}$  and  $g_{q,n}$  overlap. So, in general, the couple  $\{G_{p,n}, g_{q,n}\}$  is non-complementary. Inspection of Tables 1 and 2 shows that the condition for complementarity is captured by the

**Theorem 3.1.** (Complementarity condition)

Let  $(p, q) \in \mathcal{I}^2$ .  $G_{p,n}$  and  $g_{q,n}$  are complementary  $\iff p \equiv q \pmod{4}$ .

**Corolary 3.1.** If  $G_{p,n}$  and  $g_{q,n}$  are complementary then so are  $G_{q,n}$  and  $g_{p,n}$ .

Of the 144 ordered couples in  $\mathcal{I}^2$  only exactly 36 verify the complementarity condition, namely those of the four Cartesian products in the set henceforward denoted  $\mathcal{B}$  and defined as follows

**Definition 9.** *Definition of  $\mathcal{B}$*

$$\mathcal{B} = \{NE/sw, SE/nw, SW/ne, NW/se\} \text{ where} \quad (3)$$

$$NE/sw \equiv \mathcal{I}_{NE} \times \mathcal{I}_{sw} = \mathcal{I}_{NE}^2 = \mathcal{I}_{sw}^2 = \{1, 5, 9\}^2, \quad (4)$$

$$SE/nw \equiv \mathcal{I}_{SE} \times \mathcal{I}_{nw} = \mathcal{I}_{SE}^2 = \mathcal{I}_{nw}^2 = \{2, 6, 10\}^2, \quad (5)$$

$$SW/ne \equiv \mathcal{I}_{SW} \times \mathcal{I}_{ne} = \mathcal{I}_{SW}^2 = \mathcal{I}_{ne}^2 = \{3, 6, 11\}^2, \quad (6)$$

$$NW/se \equiv \mathcal{I}_{NW} \times \mathcal{I}_{se} = \mathcal{I}_{NW}^2 = \mathcal{I}_{se}^2 = \{4, 8, 12\}^2. \quad (7)$$

**Definition 10.** *A complementary sub-block selector  $S_{B/b}$  is a restriction of the pairing bijection  $S$  of Eq. 2 to the domain  $B/b \in \mathcal{B}$ .*

Thus for example, the selector  $S_{NE/sw} : NE/sw \rightarrow \mathcal{G}_n^{NE} \times \widetilde{\mathcal{G}}_n^{sw}$  selects the nine complementary pairs in  $\{(G_{p,n}, g_{q,n}) | (p, q) \in \{1, 5, 9\}^2\}$ . *In extensio*, that gives the set of nine ordered couples of complementary G-/g-matrices

$$\left\{ \begin{pmatrix} (G_{1,n}, g_{1,n}) & (G_{1,n}, g_{5,n}) & (G_{1,n}, g_{9,n}) \\ (G_{5,n}, g_{1,n}) & (G_{5,n}, g_{5,n}) & (G_{5,n}, g_{9,n}) \\ (G_{9,n}, g_{1,n}) & (G_{9,n}, g_{5,n}) & (G_{9,n}, g_{9,n}) \end{pmatrix} \right\}. \quad (8)$$

To each of the nine selected pairs  $\{G_{p,n}, g_{q,n}\} \in \mathcal{G}_n^{NE} \times \widetilde{\mathcal{G}}_n^{sw}$ , we associate the FP-matrix defined as follows

$$F_n^{p,q} = G_{p,n} + g_{q,n}, \quad \text{e.g.,} \quad F_n^{1,1} = G_{1,n} + g_{1,n}.$$

We denote their set by

$$\mathcal{F}_n^{NE/sw} = \{F_n^{p,q} = G_{p,n} + g_{q,n} | G_{p,n} \in \mathcal{G}_n^{NE}, g_{q,n} \in \widetilde{\mathcal{G}}_n^{sw}\}.$$

The three remaining cases are summarized in Table 3 presenting the generic definition of the individual 36 FP-matrices. Taking  $n = 3$  for instance, one will find in Table SM2 of the Appendix, the nine FP-matrices of the set  $\mathcal{F}_n^{NE/sw}$  derived from the ordered complementary couples of Eq. 8.

The symmetric FP-matrix  $F_n^{12,12}$  is one of the two most frequently encountered in recent developments in linear and multilinear algebra [3, 4, 9, 14] and in many application fields [10, 11, 20], the remaining FP-matrices are less frequently encountered.

The next section proposes a structured approach to the derivation of the set  $\mathcal{F}_n$  based on the circulant transformations introduced in [1]. The definitions of the latter are briefly recalled. Their structuring actions on the sets  $\mathcal{G}_n$  and  $\widetilde{\mathcal{G}}_n$  are then summarized by corresponding twelve-vertex orbits presented side by side to facilitate visual derivation of the associated three orbits partition of  $\mathcal{F}_n$ .



The set $\mathcal{F}_n^{NE/sw}$ : generic entries of the nine members, $(0 \leq i, j \leq n)$		
$[F_n^{1,1}]_{ij} = \begin{cases} \binom{j}{i} & \text{if } i \leq j, \\ \binom{n+1+j}{i} & \text{if } i > j. \end{cases}$	$[F_n^{1,5}]_{ij} = \begin{cases} \binom{j}{i} & \text{if } i \leq j, \\ \binom{2n+1-j}{n-j} & \text{if } i > j. \end{cases}$	$[F_n^{1,9}]_{ij} = \begin{cases} \binom{j}{i} & \text{if } i \leq j, \\ \binom{n+i-j}{i} & \text{if } i > j. \end{cases}$
$[F_n^{5,1}]_{ij} = \begin{cases} \binom{n-i}{n-j} & \text{if } i \leq j, \\ \binom{n+1+j}{i} & \text{if } i > j. \end{cases}$	$[F_n^{5,5}]_{ij} = \begin{cases} \binom{n-i}{n-j} & \text{if } i \leq j, \\ \binom{2n+1-i}{n-j} & \text{if } i > j. \end{cases}$	$[F_n^{5,9}]_{ij} = \begin{cases} \binom{n-i}{n-j} & \text{if } i \leq j, \\ \binom{n+i-j}{i} & \text{if } i > j. \end{cases}$
$[F_n^{9,1}]_{ij} = \begin{cases} \binom{n+i-j}{i} & \text{if } i \leq j, \\ \binom{n+1+j}{i} & \text{if } i > j. \end{cases}$	$[F_n^{9,5}]_{ij} = \begin{cases} \binom{n+i-j}{i} & \text{if } i \leq j, \\ \binom{2n+1-i}{n-j} & \text{if } i > j. \end{cases}$	$[F_n^{9,9}]_{ij} = \begin{cases} \binom{n+i-j}{i} & \text{if } i \leq j, \\ \binom{n+i-j}{i} & \text{if } i > j. \end{cases}$
The set $\mathcal{F}_n^{SE/nw}$ : generic entries of the nine members, $(0 \leq i, j \leq n)$		
$[F_n^{2,2}]_{ij} = \begin{cases} \binom{j}{n-i} & \text{if } i+j \geq n, \\ \binom{n+1+j}{n-i} & \text{if } i+j < n. \end{cases}$	$[F_n^{2,6}]_{ij} = \begin{cases} \binom{j}{n-i} & \text{if } i+j \geq n, \\ \binom{2n-i-j}{n-j} & \text{if } i+j < n. \end{cases}$	$[F_n^{2,10}]_{ij} = \begin{cases} \binom{j}{n-i} & \text{if } i+j \geq n, \\ \binom{n+1+i}{n-j} & \text{if } i+j < n. \end{cases}$
$[F_n^{6,2}]_{ij} = \begin{cases} \binom{2n-i-j}{n-j} & \text{if } i+j \geq n, \\ \binom{n+1+j}{n-i} & \text{if } i+j < n. \end{cases}$	$[F_n^{6,6}]_{ij} = \begin{cases} \binom{2n-i-j}{n-j} & \text{if } i+j \geq n, \\ \binom{2n-i-j}{n-j} & \text{if } i+j < n. \end{cases}$	$[F_n^{6,10}]_{ij} = \begin{cases} \binom{2n-i-j}{n-j} & \text{if } i+j \geq n, \\ \binom{n+1+i}{n-j} & \text{if } i+j < n. \end{cases}$
$[F_n^{10,2}]_{ij} = \begin{cases} \binom{i}{n-j} & \text{if } i+j \geq n, \\ \binom{n+1+j}{n-i} & \text{if } i+j < n. \end{cases}$	$[F_n^{10,6}]_{ij} = \begin{cases} \binom{i}{n-j} & \text{if } i+j \geq n, \\ \binom{2n-i-j}{n-j} & \text{if } i+j < n. \end{cases}$	$[F_n^{10,10}]_{ij} = \begin{cases} \binom{i}{n-j} & \text{if } i+j \geq n, \\ \binom{n+1+i}{n-j} & \text{if } i+j < n. \end{cases}$
The set $\mathcal{F}_n^{SW/ne}$ : generic entries of the nine members, $(0 \leq i, j \leq n)$		
$[F_n^{3,3}]_{ij} = \begin{cases} \binom{n+j-i}{j} & \text{if } i \geq j, \\ \binom{n+j-i}{j} & \text{if } i < j. \end{cases}$	$[F_n^{3,7}]_{ij} = \begin{cases} \binom{n+j-i}{j} & \text{if } i \geq j, \\ \binom{2n+1-j}{n-i} & \text{if } i < j. \end{cases}$	$[F_n^{3,11}]_{ij} = \begin{cases} \binom{n+j-i}{j} & \text{if } i \geq j, \\ \binom{n+1+i}{j} & \text{if } i < j. \end{cases}$
$[F_n^{7,3}]_{ij} = \begin{cases} \binom{n-j}{n-i} & \text{if } i \geq j, \\ \binom{n+j-i}{j} & \text{if } i < j. \end{cases}$	$[F_n^{7,7}]_{ij} = \begin{cases} \binom{n-j}{n-i} & \text{if } i \geq j, \\ \binom{2n+1-j}{n-i} & \text{if } i < j. \end{cases}$	$[F_n^{7,11}]_{ij} = \begin{cases} \binom{n-j}{n-i} & \text{if } i \geq j, \\ \binom{n+1+i}{j} & \text{if } i < j. \end{cases}$
$[F_n^{11,3}]_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \geq j, \\ \binom{n+j-i}{j} & \text{if } i < j. \end{cases}$	$[F_n^{11,7}]_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \geq j, \\ \binom{2n+1-j}{n-i} & \text{if } i < j. \end{cases}$	$[F_n^{11,11}]_{ij} = \begin{cases} \binom{i}{j} & \text{if } i \geq j, \\ \binom{n+1+i}{j} & \text{if } i < j. \end{cases}$
The set $\mathcal{F}_n^{NW/se}$ : generic entries of the nine members, $(0 \leq i, j \leq n)$		
$[F_n^{4,4}]_{ij} = \begin{cases} \binom{n-i}{j} & \text{if } i+j \leq n, \\ \binom{2n+1-i}{j} & \text{if } i+j > n. \end{cases}$	$[F_n^{4,8}]_{ij} = \begin{cases} \binom{n-i}{j} & \text{if } i+j \leq n, \\ \binom{2n+1-j}{i} & \text{if } i+j > n. \end{cases}$	$[F_n^{4,12}]_{ij} = \begin{cases} \binom{n-i}{j} & \text{if } i+j \leq n, \\ \binom{i+j}{i} & \text{if } i+j > n. \end{cases}$
$[F_n^{8,4}]_{ij} = \begin{cases} \binom{n-j}{i} & \text{if } i+j \leq n, \\ \binom{2n+1-i}{j} & \text{if } i+j > n. \end{cases}$	$[F_n^{8,8}]_{ij} = \begin{cases} \binom{n-j}{i} & \text{if } i+j \leq n, \\ \binom{2n+1-j}{i} & \text{if } i+j > n. \end{cases}$	$[F_n^{8,12}]_{ij} = \begin{cases} \binom{n-j}{i} & \text{if } i+j \leq n, \\ \binom{i+j}{i} & \text{if } i+j > n. \end{cases}$
$[F_n^{12,4}]_{ij} = \begin{cases} \binom{i+j}{i} & \text{if } i+j \leq n, \\ \binom{2n+1-i}{j} & \text{if } i+j > n. \end{cases}$	$[F_n^{12,8}]_{ij} = \begin{cases} \binom{i+j}{i} & \text{if } i+j \leq n, \\ \binom{2n+1-j}{i} & \text{if } i+j > n. \end{cases}$	$[F_n^{12,12}]_{ij} = \begin{cases} \binom{i+j}{i} & \text{if } i+j \leq n, \\ \binom{i+j}{i} & \text{if } i+j > n. \end{cases}$

Table 3: The thirty-six FP-matrix set  $\mathcal{F}_n$ : definition of the generic entries of individual members.

#### 4. The circulant transformations and the twelve-vertex orbits of $\mathcal{G}_n$ , $\tilde{\mathcal{G}}_n$ and $\mathcal{F}_n$

We define the actions of the circulant transformations on an  $(n + 1) \times (n + 1)$  square matrix in terms of transformations of the generic matrix subscript vector  $(i, j) (0 \leq i, j \leq n)$ .

**Definition 11.** *The four circulant transformations*

$$\begin{aligned} \text{The } \alpha\text{-circulant operator: } (i, j) &\xrightarrow{\alpha} (i, i + j) \pmod{n + 1} \\ \text{The } \beta\text{-circulant operator: } (i, j) &\xrightarrow{\beta} (-1 + i - j, j) \pmod{n + 1} \\ \text{The } \gamma\text{-circulant operator: } (i, j) &\xrightarrow{\gamma} (i - j, j) \pmod{n + 1} \\ \text{The } \delta\text{-circulant operator: } (i, j) &\xrightarrow{\delta} (i, 1 + i + j) \pmod{n + 1}. \end{aligned}$$

Cyclic actions of the  $\Theta$ -compositions  $\Theta_1, \Theta_2, \Theta_3$  and  $\Theta_4$  on the appropriate starting G-matrix (resp. g-matrix) vertex, presents the orbits of  $\mathcal{G}_n$  and  $\tilde{\mathcal{G}}_n$ , as displayed on Figure SM2 in the Appendix. Thus for example, taking  $k \in \{1, 5, 9\} = I_{NE} = I_{sw}$ , the  $\mathcal{G}_n$ -orbit (resp.  $\tilde{\mathcal{G}}_n$ -orbit) is obtained by unfolding  $(\Theta_1)^3$  on  $G_{k,n}$  (resp.  $g_{k,n}$ ) and exhibiting every intermediate vertex. This amounts to putting end-to-end the consecutive columns of the defining tables of  $\mathcal{G}_n$  and  $\tilde{\mathcal{G}}_n$  presented earlier on pages 4 and 6 respectively. As Figure 2 on page 10 clearly shows, analogous actions of the composite circulant transformations in  $\{(\alpha\gamma\delta\beta)^3, (\beta\alpha\gamma\delta)^3, (\delta\beta\alpha\gamma)^3, (\gamma\delta\beta\alpha)^3\}$  partition the set  $\mathcal{F}_n$  into the three twelve-vertex orbits:

$$\mathcal{F}_n^{1,1} = \{F_n^{p,p}, 1 \leq p \leq 12\}, \mathcal{F}_n^{1,5} = \{F_n^{p,(p+4)[12]}, 1 \leq p \leq 12\} \text{ and } \mathcal{F}_n^{1,9} = \{F_n^{p,(p+8)[12]}, 1 \leq p \leq 12\},$$

respective orbits of  $F^{1,1}, F^{1,5}$  and  $F^{1,9}$  under the action of  $(\alpha\gamma\delta\beta)^3$ , where  $[12]$  abbreviates  $(\text{mod } 12)$ . Notice that the orbit  $\mathcal{F}_n^{1,1}$  derives from orbits  $\mathcal{G}_n$  and  $\tilde{\mathcal{G}}_n$  simply by adding corresponding orbit vertices. It takes little effort to see how the remaining two orbits are obtained from  $\mathcal{F}_n^{1,1}$ .

The essential relations suggested by the cyclic actions of the four  $\Theta$ -compositions are summarized by the following general theorem.

**Theorem 4.1.** *For all  $(p, q) \in \mathbb{N}^* \times \mathbb{N}^*$  such that  $p \equiv q \pmod{4}$ , the sets*

$$\mathcal{F}_n^{p,q} = \{F_n^{(p+k)[12],(q+k)[12]} \mid 0 \leq k \leq 11\}$$

*verify the following properties:*

- |   |  |
|---|--|
| (i) $\mathcal{F}_n^{1,1} = \mathcal{F}_n^{p[12],p[12]} \quad \forall p \in \mathbb{N}^*$<br>(ii) $\mathcal{F}_n^{1,5} = \mathcal{F}_n^{p[12],(p+4)[12]} \quad \forall p \in \mathbb{N}^*$<br>(iii) $\mathcal{F}_n^{1,9} = \mathcal{F}_n^{p[12],(p+8)[12]} \quad \forall p \in \mathbb{N}^*$ | (iv) $\mathcal{F}_n^{1,p} \cap \mathcal{F}_n^{1,q} = \emptyset \quad \forall p \neq q \in \mathbb{N}^*$<br>(v) $\mathcal{F}_n = \mathcal{F}_n^{1,1} \cup \mathcal{F}_n^{1,5} \cup \mathcal{F}_n^{1,9}$ |
|---|--|

where  $[12]$  abbreviates  $(\text{mod } 12)$  and every multiple of 12 is replaced by 12 instead of 0.

**Proof:**

(i) It suffices to prove the cases of  $p \in \{2, \dots, 12\}$ .

$$F_n^{r,r} \in \mathcal{F}_n^{p,p} \iff \exists k \in \{0, \dots, 11\} \mid r = (p + k)[12]$$

It follows that  $2 \leq r \leq 11$ , and thus  $F_n^{r,r} \in \mathcal{F}_n^{1,1}$

(ii) For a given  $p \in \{1, \dots, 12\}$

$$\begin{aligned}\mathcal{F}_n^{p, p+4[12]} &= \left\{ F_n^{p+k[12], p+4+k[12]}, 0 \leq k \leq 11 \right\} = \left\{ F_n^{p'[12], p'+4[12]}, 1 \leq p' \leq 23 \right\} \\ &= \left\{ F_n^{p, p+4[12]}, 1 \leq p \leq 12 \right\} = \mathcal{F}_n^{1,5}\end{aligned}$$

(iii) Idem.

(iv) The proof follows directly from the fact that  $\mathcal{F}_n^{1,p} \cap \mathcal{F}_n^{1,q} = \emptyset \quad \forall (p, q) \in \{1, 5, 9\} \times \{1, 5, 9\}$  and  $p \neq q$ .

(v) One easily proves that  $\mathcal{F}_n^{1,1} \cup \mathcal{F}_n^{1,5} \cup \mathcal{F}_n^{1,9} = \bigcup_{B/b \in \mathcal{B}} \mathcal{F}_n^{B/b}$  ■

$$\begin{array}{ccccccccccc} F_n^{1,1} & \xrightarrow{\beta} & F_n^{2,2} & \xrightarrow{\delta} & F_n^{3,3} & \xrightarrow{\gamma} & F_n^{4,4} & \xrightarrow{\alpha} & F_n^{5,5} & \xrightarrow{\beta} & F_n^{6,6} \\ \alpha \uparrow & & & & & & & & & & \downarrow \delta \\ F_n^{12,12} & \xleftarrow{\gamma} & F_n^{11,11} & \xleftarrow{\delta} & F_n^{10,10} & \xleftarrow{\beta} & F_n^{9,9} & \xleftarrow{\alpha} & F_n^{8,8} & \xleftarrow{\gamma} & F_n^{7,7} \end{array}$$

(a)  $\mathcal{F}_n^{1,1}$  orbit.

$$\begin{array}{ccccccccccc} F_n^{1,5} & \xrightarrow{\beta} & F_n^{2,6} & \xrightarrow{\delta} & F_n^{3,7} & \xrightarrow{\gamma} & F_n^{4,8} & \xrightarrow{\alpha} & F_n^{5,9} & \xrightarrow{\beta} & F_n^{6,10} \\ \alpha \uparrow & & & & & & & & & & \downarrow \delta \\ F_n^{12,4} & \xleftarrow{\gamma} & F_n^{11,3} & \xleftarrow{\delta} & F_n^{10,2} & \xleftarrow{\beta} & F_n^{9,1} & \xleftarrow{\alpha} & F_n^{8,12} & \xleftarrow{\gamma} & F_n^{7,11} \end{array}$$

(b)  $\mathcal{F}_n^{1,5}$  orbit.

$$\begin{array}{ccccccccccc} F_n^{1,9} & \xrightarrow{\beta} & F_n^{2,10} & \xrightarrow{\delta} & F_n^{3,11} & \xrightarrow{\gamma} & F_n^{4,12} & \xrightarrow{\alpha} & F_n^{5,1} & \xrightarrow{\beta} & F_n^{6,2} \\ \alpha \uparrow & & & & & & & & & & \downarrow \delta \\ F_n^{12,8} & \xleftarrow{\gamma} & F_n^{11,7} & \xleftarrow{\delta} & F_n^{10,6} & \xleftarrow{\beta} & F_n^{9,5} & \xleftarrow{\alpha} & F_n^{8,4} & \xleftarrow{\gamma} & F_n^{7,3} \end{array}$$

(c)  $\mathcal{F}_n^{1,9}$  orbit.

Figure 2: The three orbits partition of  $\mathcal{F}_n$  under the action of the circulant transformations.

*The  $2 \times 2$  Pascal matrix sets  $\mathcal{G}_1$ ,  $\mathcal{F}_1$  and the two-order Langlet geniton matrix set*

Referring to Table 4, it is clear that for the special case of  $n = 1$  and under  $Z/2Z$  algebra, the corresponding singleton sets of the three rows are equal since  $(2 \bmod 2) = 0$ . The four matrix sets of row-one represent the Langlet geniton matrix set [see Michael Zaus, Fig. 5.3, p.107]. So under  $Z/2Z$  algebra,  $\mathcal{G}_1$  and  $\mathcal{F}_1$  coincide with the set of Langlet's geniton matrices. This link with the Langlet system opens up interesting perspectives for solving problems in the fields of numbers and information coding theories.

$G2 = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$	$Gh2 = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$	$Gv2 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$	$Gd2 = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$
$\mathcal{G}_1^{NW} = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$	$\mathcal{G}_1^{SW} = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$	$\mathcal{G}_1^{NE} = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$	$\mathcal{G}_1^{SE} = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$
$\mathcal{F}_1^{NW/se} = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \right\}$	$\mathcal{F}_1^{SW/ne} = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \right\}$	$\mathcal{F}_1^{NE/sw} = \left\{ \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \right\}$	$\mathcal{F}_1^{SE/nw} = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \right\}$

Table 4: The Langlet's geniton, the  $\mathcal{G}_1$  and the  $\mathcal{F}_1$  matrix sets.

## 5. Orbit diagrams of $\mathcal{F}_n$ under four $\mathcal{F}_n$ groups

### 5.1. The two orbits of $\mathcal{F}_n^{NE/sw}$ under $\langle T_6, \Theta_1 \rangle$

By abuse of notation, we write  $\langle T_6, \Theta_1 \rangle$  to denote the group

$$\langle T_6, \Theta_1 \rangle = \{id, \Theta_1, \Theta_1^2, T_6, \rho_{Se}, \kappa_{Nw}\} \quad (9)$$

where  $id$  is the identity transformation,  $\Theta_1 = \alpha\gamma\delta\beta$ , the T-group element  $T_6$  is the reflection of a square matrix in its anti-diagonal denoted by  $\sigma_{Ne}$  in a NE/sw bipartition, by  $\sigma_{Sw}$  in a SW/ne bipartition. Furthermore

$$\rho_{Se} = \Theta_1 T_6 = T_6 \Theta_1^2, \quad (10)$$

$$\kappa_{Nw} = T_6 \Theta_1 = \Theta_1^2 T_6. \quad (11)$$

The  $D_3$  group  $\langle T_6, \Theta_1 \rangle$  bipartitions  $\mathcal{F}_n^{NE/sw}$  into two orbits, a six-vertex orbit and a degenerate orbit with three distinct vertices:  $F_n^{1,1}$ ,  $F_n^{5,5}$  and  $F_n^{9,9}$ . It is readily verified that  $F_n^{5,9}$  and  $F_n^{9,1}$  form a  $T_6$  image pair. Both orbits are respectively depicted in Figure SM3 (a) and (b) in the Appendix to this paper.

### 5.2. The two orbits of $\mathcal{F}_n^{SE/nw}$ under $\langle T_4, \Theta_2 \rangle$

Again by abuse of notation, we write  $\langle T_4, \Theta_2 \rangle$  to denote the group

$$\langle T_4, \Theta_2 \rangle = \{id, \Theta_2, \Theta_2^2, T_4, \rho_{Ne}, \kappa_{Sw}\} \quad (12)$$

where  $\Theta_2 = \beta\alpha\gamma\delta$ , the T-group element  $T_4$  is the classic matrix transposition and

$$\rho_{Ne} = T_4 \Theta_2 = \Theta_2^2 T_4, \quad (13)$$

$$\kappa_{Sw} = \Theta_2 T_4 = T_4 \Theta_2^2. \quad (14)$$

Also, one can verify that the  $D_3$  group  $\langle T_4, \Theta_2 \rangle$  bipartitions  $\mathcal{F}_n^{SE/nw}$  into two orbits, a six-vertex orbit (a) and a degenerate orbit (b) with three duplicated vertices  $F_n^{2,2}$ ,  $F_n^{6,6}$  and  $F_n^{10,10}$ . Both orbits are respectively depicted in Figure SM4 (a) and (b) in the Appendix.

### 5.3. The two orbits of $\mathcal{F}_n^{SW/ne}$ under $\langle T_6, \Theta_3 \rangle$

Again by abuse of notation, we write  $\langle T_6, \Theta_3 \rangle$  to denote the group

$$\langle T_6, \Theta_3 \rangle = \{id, \Theta_3, \Theta_3^2, T_6, \rho_{Nw}, \kappa_{Se}\} \quad (15)$$

where  $id$  is the identity transformation,  $\Theta_3 = \delta\beta\alpha\gamma$ , the T-group element  $T_6$  is the reflection of a square matrix in its anti-diagonal denoted by  $\sigma_{Ne}$  in a NE/sw bipartition, by  $\sigma_{Sw}$  in a SW/ne bipartition. Furthermore

$$\rho_{Nw} = \Theta_3 T_6 = T_6 \Theta_3^2, \quad (16)$$

$$\kappa_{Se} = T_6 \Theta_3 = \Theta_3^2 T_6. \quad (17)$$

Again, one finds the two orbits of  $\mathcal{F}_n^{SW/ne}$  under the action of  $\langle T_6, \Theta_3 \rangle$  in Figure SM5 (a) and (b) of the Appendix. The orbit (a) is a full six-vertex orbit while orbit (b) is a degenerate orbit with three duplicated vertices  $\mathcal{F}_n^{3,3}$ ,  $\mathcal{F}_n^{7,7}$  and  $\mathcal{F}_n^{11,11}$ .

#### 5.4. The two orbits of $\mathcal{F}_n^{NW/se}$ under $\langle T_4, \Theta_4 \rangle$

Again by abuse of notation, we write  $\langle T_4, \Theta_4 \rangle$  to denote the group

$$\langle T_4, \Theta_4 \rangle = \{id, \Theta_4, \Theta_4^2, T_4, \rho_{Sw}, \kappa_{Ne}\} \quad (18)$$

where  $\Theta_4 = \gamma\delta\beta\alpha$ , the T-group element  $T_4$  is the classic matrix transposition and

$$\rho_{Sw} = T_4 \Theta_4 = \Theta_4^2 T_4, \quad (19)$$

$$\kappa_{Ne} = \Theta_4 T_4 = T_4 \Theta_4^2. \quad (20)$$

Here also, the action of the group  $\langle T_4, \Theta_4 \rangle$  highlights, as can be seen in Figure SM6 of the Appendix, two orbits of  $\mathcal{F}_n^{NW/se}$ , a six-vertex orbit (a) and a degenerate orbit (b) with three duplicated vertices  $F_n^{4,4}$ ,  $F_n^{8,8}$  and  $F_n^{12,12}$ . All foregoing partitions offer the potential for applications in several domains.

## Conclusion

This paper presents for the first time the thirty-six full matrix forms of the so-called  $n$ -greatest rhomboid sub-block of the Pascal Triangle expanded to level  $2n$  ( $n \geq 2$ ). To our knowledge, no investigation has so far focused on these matrices as mathematical objects in their own right. The circulant transformations were applied to highlight the partition of the thirty-six full matrix forms into three twelve-vertex orbits, which can also serve to structure the set's derivation.

The new matrix forms of the Pascal triangle presented here open a wide range of applications in fields such as linear algebra, combinatorics, machine-learning, and numerical analysis. In addition, numerical analysis their relations to Fibonacci numbers, fractals and cryptography suggest further exciting directions of investigation. Furthermore, the link established with the Langlet system opens up exciting perspectives for solving problems in the fields of applied science and engineering.

The derived matrix forms, the related matrix transformations, and the software development underway contribute to the knowledgebase building. The appropriation of this knowledgebase by higher education institutions and research centres in Africa offers potential innovative solutions to practical problems in science (e.g. chemistry, fractal-based modelling), engineering (e.g. robotic, distributed systems, urban planning, machine-learning, communication and information), and the fine arts (e.g. textile design, etc.)

To limit the scope of the paper, we left out other inherent FP-matrix group relations, particularly those induced by the actions of the T-group and the median symmetry transformations introduced in [1]. For

example, it can be shown that the action of the T-group partitions the FP-matrix set into four eight-vertex and one four-vertex orbits.

## Acknowledgments

This work got the technical support of the International Organisation for Science and Engineering in Africa (IOSEA). No funding to declare.

## References

- [1] Babiga Birregah, Prosper K. Doh, and Kondo H. Adjallah. A systematic approach to matrix forms of the Pascal triangle: The twelve triangular matrix forms and relations. *European Journal of Combinatorics*, 31:1205–1216, 2010. doi:<https://doi.org/10.1016/j.ejc.2009.10.009>.
- [2] Maurice F. Aburdene, Richard J. Kozick, Ryan S. Magargle, Justin D. Maloney-Hahn, and Christian M. Coviello. Discrete polynomial transform representation using binary matrices and flow diagrams. In *ICASSP '01: Proceedings of the Acoustics, Speech, and Signal Processing, 2001. on IEEE International Conference*, pages 1141–1144, Washington, DC, USA, 2001. IEEE Computer Society. doi:10.1109/ICASSP.2001.941123.
- [3] Lowell Abrams, Donniell E. Fishking, and Silvia Valdes-Leon. Reflecting the Pascal matrix about its main antidiagonal. *Linear and Multilinear Algebra*, 47:129–136, 2000. doi:10.1080/03081080008818638.
- [4] Xiao-Guang Lv, Ting-Zhu Huang, and Zhi-Gang Ren. A new algorithm for linear systems of the Pascal type. *Journal of Computational and Applied Mathematics*, 225(1):309–315, 2009. doi:10.1016/j.cam.2008.07.045.
- [5] Gi-Sang Cheon and Moawwad El-Mikkawy. Extended symmetric Pascal matrices via hypergeometric functions. *Applied Mathematics and Computation*, 158(1):159–168, 2004. doi:10.1016/j.amc.2003.08.095.
- [6] Deveci Ömür and Karaduman Erdal. The cyclic groups via the pascal matrices and the generalized pascal matrices. *Linear Algebra and its Applications*, 437(10):2538–2545, 2012. doi:10.1016/j.laa.2012.06.024.
- [7] Lidia Aceto and Donato Trigianti. The matrices of Pascal and other greats. *American Mathematical Monthly*, 108(3):232–245, March 2001. doi:10.1080/00029890.2001.11919745.
- [8] Geoff Boyd, Charles A. Micchelli, Gilbert Strang, and Ding-Xuan Zhou. Binomial matrices. *Advances in Computational Mathematics*, 14(4):379–391, May 2001. doi:10.1023/A:1012207124894.
- [9] Moawwad E. A. El-Mikkawy. On solving linear systems of the Pascal type. *Applied Mathematics and Computation*, 136:195–202, 2003. doi:10.1016/S0096-3003(02)00034-6.
- [10] Wang Xiang and Zhou Jituan. A fast eigenvalue algorithm for Pascal matrices. *Applied Mathematics and Computation*, 183:711–716, 2006. doi:10.1016/j.amc.2006.05.093.

- [11] Wang Xiang and Lu Linzhang. A fast algorithm for solving linear systems of the Pascal type. *Applied Mathematics and Computation*, 175:441–451, 2006. doi:10.1016/j.amc.2005.07.022.
- [12] Sheng liang Yang and Zhong kui Liu. Explicit inverse of the Pascal matrix plus one. *International Journal of Mathematics and Mathematical Sciences*, 2006:1–7, 2006. doi:10.1155/IJMMS/2006/90901.
- [13] Teijo Arponen. Matrix approach to polynomials 2. *Linear Algebra and its Applications*, 394:257–276, 2005. doi:10.1016/j.laa.2004.07.011.
- [14] Ashkan Ashrafi and Peter M. Gibson. Eigenvalues of Hadamard power of large symmetric Pascal matrices. *Linear Algebra and its Applications*, 405:60–66, 2005. doi:10.1016/j.laa.2005.02.041.
- [15] André Barbé. Symmetric patterns in the cellular automaton that generates Pascal’s triangle modulo 2. *Discrete Applied Mathematics*, 105(1-3):1–38, October 2000. doi:10.1016/S0166-218X(00)00211-0.
- [16] Brawer Robert and Pirovino Magnus. The linear algebra of the pascal matrix. *Linear Algebra and its Applications*, 174:13–23, 1992. doi:10.1016/0024-3795(92)90038-C.
- [17] R. H. Cooper and W. Patterson. A public key encryption system derived from the pascal parallelogram. In *2019 SoutheastCon*, pages 1–5. IEEE, 2019. doi:10.1109/SoutheastCon42311.2019.9020398.
- [18] Michael Zaus. *Crisp and Soft Computing with Hypercubical Calculus: New Approaches to Modeling in Cognitive Science and Technology with Parity Logic, Fuzzy Logic, and Evolutionary Computing*. Studies in Fuzziness and Soft Computing. Physica-Verlag HD, 1999. ISBN 9783790811728. doi:10.1007/978-3-7908-1879-6.
- [19] Rodney H. Cooper, Ron Hunter-Duvar, and Wayne Patterson. A more efficient public-key cryptosystem using the pascal triangle. In *IEEE BOSTONICC’89 Int. Conf. on Communications, World Prosperity Through Communications*, pages 1165–1169. IEEE, 1989. doi:10.1109/ICC.1989.49866.
- [20] Rolan Bacher and Robin Chapman. Symmetric Pascal matrices modulo  $p$ . *European Journal of Combinatorics*, 25:459–473, 2004. doi:10.1016/j.ejc.2003.06.001.