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Electrical analogy associated with a multi-isotherms medium with internal heat source.

A.Degiovanni[1,2,*], Yves Jannot[1], Christian Moyne[1].

[1] Université de Lorraine, CNRS, LEMTA, F-54500 Vandœuvre-lès-Nancy, France

[2] Université Internationale de Rabat, Collège ingénierie et architecture, LERMA, Rocade Rabat-Salé, 11100, Sala Al Jadida, Morocco

Abstract.

The use of the electrical analogy in heat transfer allows to simply solve many problems in various fields such as building or electrothermics.

Nevertheless, it is limited to systems with two isotherms and a flux tube.

The fin approximation allows to introduce an exchange on the flux tube, but the electrical analogy does not work anymore if there is a heat source inside the fin.

In this paper, we will construct an electrical analogy for a system with:

- On its boundary, several isotherms surfaces, a surface that exchanges with the external medium and a zero flux region,
- In the volume, a heat source with any distribution.

This approach generalizes the notions of thermal resistance and partition coefficient of the internal heat source.

In the applications (here for a cylinder), the only approximation concerns the isotherms surfaces which in reality are not necessarily so (this approximation already exists in the elementary case, a thermal resistance is defined between two isotherms).

Key words.

Conduction, electrical analogy, steady state, internal heat source.

Nomenclature.

A, B, C, D = terms of the transfer matrix (without, KW^{-1}, WK^{-1}, without)

e = length (m)

f = shape factor

h = heat transfer coefficient ($WK^{-1}m^{-2}$)

H = Biot number

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J_0 = Bessel function of the first kind, order 0

J_1 = Bessel function of the first kind, order 1

p = power density (Wm^{-3})

Q = total power (W)

r = radial coordinate (m)

R = thermal resistance (KW^{-1})

or radius (m)

S = section (m^2)

T = temperature (K)

Text = external temperature (K)

V = volume (m^3)

W = Hankel integral transform of p (Wm^{-1})

z = axial coordinate (m)

α = partition coefficient

or eigenvalues (m^{-1})

β = admittance (WK^{-1})

γ = partition coefficient of the resistance

θ = Hankel integral transform of (T-Text) (Km^2)

λ = thermal conductivity ($WK^{-1}m^{-1}$)

ϕ = flux (W)

ω = αR

1. Introduction.

The application of resistance concepts to steady state problems is widely adopted in thermal science. Nonetheless, there are some difficulties when applying the foregoing concepts to problems with non-uniform internal source. Internal sources may be due to the passage of electric current through a solid or nuclear fission or to a chemical reaction taking place within the solid as in the problems of charging and discharging batteries [1]. Two other difficulties make the use of the notion of resistance problematic: the systems with more than two isotherms and the numerous systems exchanging heat with the external medium.

Indeed, the flow tube in heat transfers is in practice non-existent, it is particularly difficult to isolate a surface and there are always convective or radiative exchanges.

For the case of two isotherms with sources, we find a first approach in Bardon [2] for the particular case of sliding contact with friction, the author evaluated the heat transfer rate from both sides of the contact resistance using the notion of 'part of the generated heat that causes a spike in temperature at the interface' or 'heat generation factor'. Various authors used this representation to establish thermal models for sliding contacts with friction [3-7]. Likewise, this representation was applied to various electrothermal applications [8-11]. For our part, we have demonstrated [12], in the general case of non-uniform internal sources, the validity of the partition coefficient and how to calculate it. The coefficient can be applied either to the source or to the resistance. This confirms that the resistance concepts can be used to solve steady-state heat problems with such internal (uniform or non-uniform) sources by introducing a resistance partition coefficient.

The main interest of electrical analogy is to solve complex problems that do not have exact analytical solutions. The electrical analogy allows to obtain an approximate solution very quickly and often very precisely as can be seen in [13] for the calculation of the contact temperature in the case of spot welding.

In this paper we will show that in the general case of a medium whose surface is constituted by several isotherms and whose volume receives a uniform or non-uniform heat source, there is an exact electrical analogy for this type of medium, which generalizes the notion of thermal resistance in a medium with source.

2. The general problem of a multi-isotherms medium with internal heat source in steady state.

The approximate methods of fin type do not allow to obtain a representation by resistances in the presence of internal source. We will show that in a general way there is a representation in resistances and localized sources for a medium with several isotherms and internal heat source. This demonstration generalizes the concepts of resistances and partition coefficient demonstrated for a flow tube with two isotherms [12]. Figure (1) shows the electrical analogy for this case, where two representations are possible (both representations are correct), either resistance partition (1-a) or source partition (1-b). Q is the total power released in the flux tube, R is the total resistance of the flux tube and α is the

partition coefficient which depends only on the power distribution and the geometry of the flux tube.

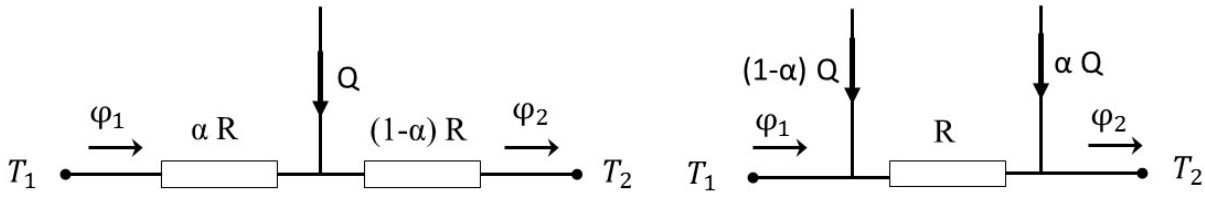

Figure (1-a) Figure (1-b)

Figure (1): Analogical representation of a system with two isotherms:

a) resistance partition, b) source partition.

As an example, for a 1D problem, α is given by:

$$\alpha = \int_0^e \frac{r(s)}{R} \frac{g(s)}{Q} \, S(s) ds$$

Where: $g(s)$ is the power distribution, $S(s)$ is the section of the flux tube, $r(s)$ is the resistance distribution ($r(s) = \int_0^s \frac{1}{\lambda \, S(s)} ds$), where s is a curvilinear coordinate.

For a cylinder of length e:

$$\alpha = \int_0^e \frac{s}{e} \frac{g(s)}{Q} \, S \, ds$$

If the power is evenly distributed, $g(s) = Q/(e\,S)$ and $\alpha = 1/2$

If the power is uniform between a and b, and zero elsewhere, $\alpha = (a+b)/2e$

As in many practical cases, one of the isotherms is the outside temperature, we will introduce, to generalize the problem, a surface exchanging by heat transfer coefficient with the outside.

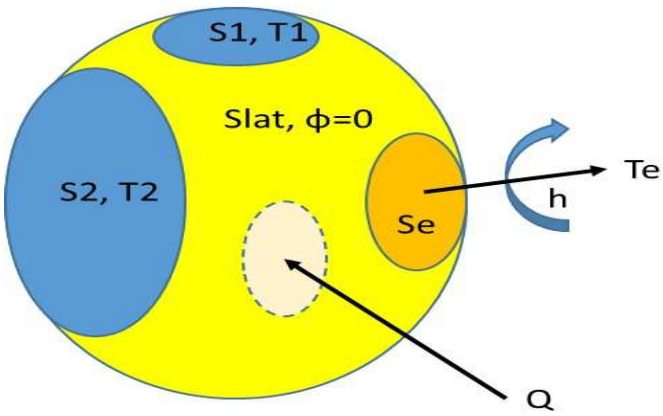

Figure (2): Diagram of a system with several isotherms.

Consider (figure (2)) a medium of volume V with n Si surfaces (1 ≤ n) at an imposed temperature Ti, with a surface Se in exchange with a temperature Te, and an isolated lateral surface Slat. Inside the medium exists a heat source of spatial distribution q* and intensity Q. The steady-state problem is thus written:

In V: $\lambda \nabla^2 T + Q\, q^* = 0$ (1)

On S_i: $T = T_i$

On S_e: $-\lambda \frac{\partial T}{\partial n} = h(T - T_e)$

On S_{lat}: $-\lambda \frac{\partial T}{\partial n} = 0$

The problem being linear we will solve it by superposition (the detailed demonstration is given in appendix).

Consider the following problems:

- n problems :

In V: $\nabla^2 L_i = 0$ (2)

On S_i: $L_i = 1$

On S_j (j ≠ i) $L_i = 0$

On S_e: $-\lambda \frac{\partial L_i}{\partial n} = h\, L_i$

On S_{lat}: $-\frac{\partial L_i}{\partial n} = 0$

- 1 problem :

In V: $\nabla^2 L_e = 0$ (3)

On S_i: $L_e = 0$

On S_e: $-\lambda \frac{\partial L_e}{\partial n} = h(L_e - 1)$

On S_{lat}: $-\frac{\partial L_e}{\partial n} = 0$

- 1 problem :

In V: $\nabla^2 P + q^* = 0$

On S_i: $P = 0$

On S_e: $-\lambda \frac{\partial P}{\partial n} = h\, P$

On S_{lat}: $-\lambda \frac{\partial P}{\partial n} = 0$

It is immediate to verify that:

$$\sum_{i=1}^{n} L_i + L_e = 1 \qquad (4)$$

The solution of problem (1) is therefore given by:

$$T = \sum_{j=1}^{n} T_j\, L_j + T_e L_e + \frac{Q}{\lambda} P \qquad (5)$$

Which can be written taking into account (4):

$$T - T_i = \sum_{j=1}^{n} (T_j - T_i) L_j + (T_e - T_i) L_e + \frac{Q}{\lambda} P \qquad (6)$$

If n is the exterior normal to the volume V, the input fluxes ϕ_i are defined by:

$$\phi_i = \iint_{S_i} \lambda \frac{\partial T}{\partial n} dS$$

From the relation (6), it comes immediately:

$$\phi_i = \sum_{j \neq i} \beta_{ij}\, (T_i - T_j) + \beta_{ie}\, (T_i - T_e) - \alpha_i Q \qquad (7)$$

$$\phi_e = \sum_{j} \beta_{ej}\, (T_e - T_j) - \alpha_e Q \qquad (8)$$

By having posed:

$$\beta_{ij} = \iint_{S_i} -\lambda \frac{\partial L_j}{\partial n} dS \qquad (9)$$

$$\beta_{ie} = \iint_{S_i} -\lambda \frac{\partial L_e}{\partial n} dS \qquad (10)$$

$$\beta_{ei} = \iint_{S_e} -\lambda \frac{\partial L_i}{\partial n} dS \qquad (11)$$

$$\alpha_i = \iint_{S_i} -\frac{\partial P}{\partial n} dS \qquad (12)$$

Relationships (9) to (12) are used to calculate the admittances and partition coefficients. To complete the demonstration, it is necessary to show the symmetry of the admittances, i.e. that:

$\beta_{ij} = \beta_{ji}\, et\, \beta_{ie} = \beta_{ei}$.

- $\beta_{ij} = \beta_{ji}$?

We start from (2) which we multiply by L_j :

$$0 = L_j \nabla^2 L_i = \nabla.\left(L_j \nabla L_i\right) - \nabla L_j.\nabla L_i \qquad (13)$$

The integration of (13) over the volume V of area $S = \sum_k S_k + S_e + S_{lat}$ leads to:

$$\iiint \nabla L_j.\nabla L_i\, dV = \iiint \nabla.\left(L_j \nabla L_i\right) dV = \iint_S L_j \nabla L_i.n\, dS$$

$$= \iint_{S_j} \frac{\partial L_i}{\partial n}\, dS + \iint_{S_e} L_j(-\frac{h}{\lambda} L_i) dS$$

By swapping i and j we get:

$$\iiint \nabla L_i.\nabla L_j\, dV = \iint_{S_i} \frac{\partial L_j}{\partial n}\, dS + \iint_{S_e} L_i(-\frac{h}{\lambda} L_j) dS$$

And:

$$\iint_{S_i} \frac{\partial L_j}{\partial n}\, dS = \iint_{S_j} \frac{\partial L_i}{\partial n}\, dS \qquad et \qquad \beta_{ij} = \beta_{ji}$$

- $\beta_{ie} = \beta_{ei}$?

Similarly, the relation (3) becomes: $0 = L_i \nabla^2 L_e = \nabla.(L_i \nabla L_e) - \nabla L_i.\nabla L_e$ (14)

The integration of (14) over the volume V of area $S = \sum_k S_k + S_e + S_{lat}$ leads to:

$$\iiint \nabla L_i.\nabla L_e\, dV = \iiint \nabla.(L_i \nabla L_e)\, dV = \iint_S L_i \nabla L_e.n\, dS = \iint_{S_i} \frac{\partial L_e}{\partial n}\, dS + \iint_{S_e} L_i(-\frac{h}{\lambda})(L_e - 1) dS \qquad (15)$$

By swapping i and e we get:

$$\iiint \nabla L_e . \nabla L_i \, dV = \iiint \nabla . (L_e \nabla L_i) \, dV = \iint_S L_e \nabla L_i . n \, dS = \iint_{S_e} L_e (-\frac{h}{\lambda}) L_i dS \qquad (16)$$

Combining (15) and (16) we get:

$$\iint_{S_i} \frac{\partial L_e}{\partial n} \, dS + \iint_{S_e} L_i (-\frac{h}{\lambda})(L_e - 1) dS = \iint_{S_e} L_e (-\frac{h}{\lambda}) L_i dS$$

$$\iint_{S_i} \frac{\partial L_e}{\partial n} \, dS = \iint_{S_e} L_i (-\frac{h}{\lambda}) dS = \iint_{S_e} \frac{\partial L_i}{\partial n} dS \qquad and\ thus \quad \beta_{ie} = \beta_{ei}$$

As an example, the analogical representation for a system with 3 isotherms is given in figure (3). This representation is the exact solution of the problem, the only approximation is that the 3 surfaces are isothermal.

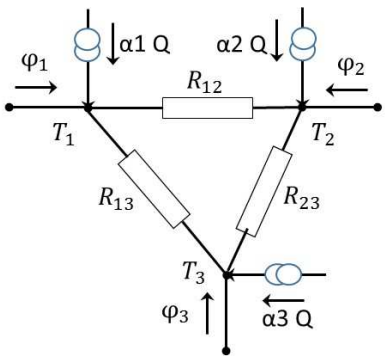

Figure (3): Analogical representation of a system with three isotherms.

3. Application to a cylinder.

To illustrate the method we will treat the problem of the cylinder exchanging through its lateral surface and we will assume a uniform internal heat source.

Let be the system of figure (4), a cylinder of radius R, height e, thermal conductivity λ subjected to uniform boundary conditions (independent of r) in z, presenting exchanges on its lateral surface characterized by an heat transfer coefficient h and a volumic heat generation p_0.

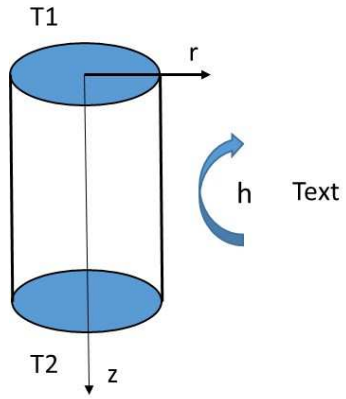

Figure (4): A cylinder exchanging with three temperatures.

3.1. The system of equations:

$$\frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{p_0}{\lambda} = 0 \qquad (17)$$

$$r = 0 \qquad \frac{\partial T}{\partial r} = 0 \qquad (18)$$

$$r = R \qquad -\lambda\frac{\partial T}{\partial r} = h(T - Text) \qquad (19)$$

$$z = 0 \; et \; z = e \qquad Boundary\ conditions\ independent\ of\ r \qquad (20)$$

3.2. The analytical solution:

The solution of the problem is obtained by Hankel integral transformation on r, i.e.:

$$\theta = \int_0^R (T - Text)\, r J_0(\alpha r)\, dr$$

With the eigenvalues given by:

$$\omega_n J_1(\omega_n) = H J_0(\omega_n) \qquad ou \quad \omega_n = \alpha_n R \quad et \quad H = hR/\lambda \qquad (21)$$

And the norm by $\quad N_n = \int_0^R r J_0^2(\alpha_n r)\, dr = \frac{R^2}{2}\left(J_0^2(\alpha_n R) + J_1^2(\alpha_n R)\right) \qquad (22)$

Where $J_1(x)$ is a Bessel function of the first kind, order 1

and $J_0(x)$ is a Bessel function of the first kind, order 0.

The relation (17) becomes after transformation:

$$\frac{\partial^2 \theta}{\partial z^2} - \alpha_n^2 \theta + \frac{W}{\lambda} = 0 \qquad (23)$$

Where $\quad W = p_0 \int_0^R r\, J_0(\alpha r)\, dr = p_0 \frac{R}{\alpha} J_1(\alpha R)$

Whose solution is written in matrix form [14]:

$$\begin{bmatrix}\theta_0\\ \phi_0\end{bmatrix} = \begin{bmatrix}A & B\\ C & D\end{bmatrix}\begin{bmatrix}\theta_e\\ \phi_e\end{bmatrix} - \begin{bmatrix}\frac{A-1}{C}K\\ K\end{bmatrix} \tag{24}$$

With θ_0 Laplace transform of the temperature in $z = 0$

θ_e Laplace transform of the temperature in $z = e$

$\phi_0 = -\lambda\left(\frac{\partial\theta}{\partial z}\right)_0$ Laplace transform of the heat flux density in $z = 0$

$\phi_e = -\lambda\left(\frac{\partial\theta}{\partial z}\right)_e$ Laplace transform of the heat flux density in $z = e$

$A = D = ch(\alpha_n e) \qquad B = \frac{sh(\alpha_n e)}{\lambda\alpha_n} \qquad C = \lambda\alpha_n\, sh(\alpha_n e) \quad et \quad K = \frac{W}{\alpha_n} sh(\alpha_n e)$

Where $\mathrm{ch(x)}$ is a hyperbolic cosine and $\mathrm{sh(x)}$ is a hyperbolic sine.

The return in the real space is done by the following expression:

$$T = \sum_{n=1}^{\infty} \frac{2\,\theta_n\, J_0(\alpha_n r)}{R^2(J_0^2(\alpha_n R) + J_1^2(\alpha_n R))} \tag{25}$$

3.3. The electric analog solution:

Figure (1) is the electrical analogy representation of this problem.

- Let us calculate the resistances, i.e. the inverse of the admittances β_{ij}:

$$\frac{1}{R_{12}} = \beta_{21} = \int_0^R -\lambda\frac{\partial L_1}{\partial z} 2\pi r dr \tag{26}$$

$$\frac{1}{R_{13}} = \frac{1}{R_{23}} = \beta_{31} = \int_0^e -\lambda\frac{\partial L_1}{\partial r} 2\pi R dz \tag{27}$$

With the problem L_1:

$$\frac{\partial^2 L_1}{\partial z^2} + \frac{1}{r}\frac{\partial L_1}{\partial r} + \frac{\partial^2 L_1}{\partial r^2} = 0 \tag{28}$$

$$r = 0 \qquad \frac{\partial L_1}{\partial r} = 0 \tag{29}$$

$$r = R \qquad -\lambda\frac{\partial L_1}{\partial r} = h\, L_1 \tag{30}$$

$$z = 0 \qquad L_1 = 1 \tag{31}$$

$$z = e \qquad L_1 = 0 \tag{32}$$

We get:

$$\frac{1}{R_{12}} = \frac{\lambda \pi R^2}{e} \sum_{n=1}^{\infty} \frac{4\,f}{sh(f\omega_n)} \frac{H^2}{\omega_n(H^2+\omega_n^2)} \tag{33}$$

$$\frac{1}{R_{13}} = \frac{1}{R_{23}} = 4\pi\lambda R \sum_{n=1}^{\infty} \frac{ch(f\omega_n)-1}{sh(f\omega_n)} \frac{H^2}{\omega_n(H^2+\omega_n^2)} \tag{34}$$

Where ω_n solution of:

$$\omega_n\, J_1(\omega_n) = H\, J_0(\omega_n)$$

Where $H = hR/\lambda$ and the shape factor $f = e/R$

We deduce, the partition coefficients of the resistances:

$$\gamma_{12} = \frac{1}{R_{12}} \frac{e}{\lambda \pi R^2} = \sum_{n=1}^{\infty} \frac{4\,f}{sh(f\omega_n)} \frac{H^2}{\omega_n(H^2+\omega_n^2)}$$

$$\gamma_{13} = \gamma_{23} = \frac{1}{R_{13}} \frac{1}{4\pi\lambda R} = \frac{1}{R_{23}} \frac{1}{4\pi\lambda R} = \sum_{n=1}^{\infty} \frac{ch(f\omega_n)-1}{sh(f\omega_n)} \frac{H^2}{\omega_n(H^2+\omega_n^2)}$$

- Similarly for the partition coefficients :

$$\alpha_1 = \alpha_2 = \int_0^R -\frac{\partial P}{\partial z} 2\pi r dr$$

$$\alpha_3 = \int_0^e -\frac{\partial P}{\partial r} 2\pi R dz$$

With the problem P:

$$\frac{\partial^2 P}{\partial z^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial r^2} + \frac{1}{\pi R^2 e} = 0 \tag{35}$$

$$r = 0 \qquad \frac{\partial P}{\partial r} = 0 \tag{36}$$

$$r = R \qquad -\lambda \frac{\partial P}{\partial r} = h\,P \tag{37}$$

$$z = 0 \qquad P = 0 \tag{38}$$

$$z = e \qquad P = 0 \tag{39}$$

We get:

$$\alpha_1 = \alpha_2 = \sum_{n=1}^{\infty} \frac{ch(f\omega_n)-1}{f\omega_n\, sh(f\omega_n)} \frac{4\,H^2}{\omega_n^2(H^2+\omega_n^2)} \tag{40}$$

$$\alpha_3 = \sum_{n=1}^{\infty} \left(\frac{2\big(1-ch(f\omega_n)\big)}{f\omega_n\, sh(f\omega_n)} + 1\right) \frac{4\,H^2}{\omega_n^2(H^2+\omega_n^2)} \tag{41}$$

The partition coefficients of the internal power and the partition coefficients of the resistances depend only on the Biot number (H) and the shape factor (f).

As an example, the variation of γ_{13} , γ_{12} and α_1 as a function of f for three values of H, is given in figures (5 to 7); figure (5) H=0.08, figure (6) H=0.64, figure (7) H=5.12.

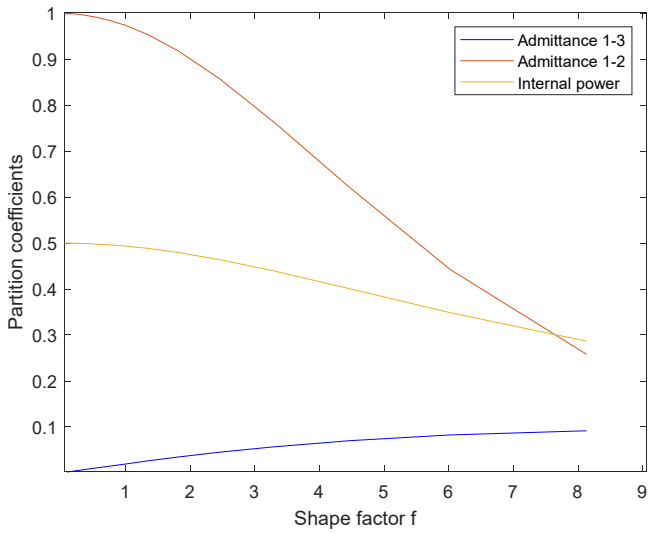

Figure (5): Partition coefficients versus shape factor for H=0.08.

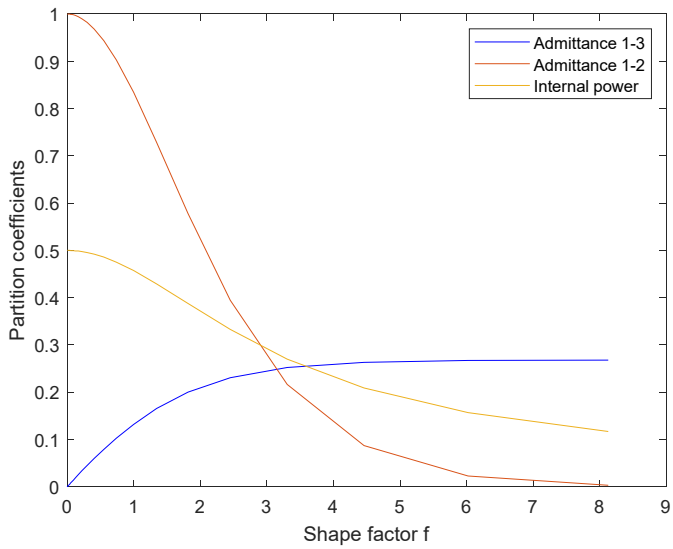

Figure (6): Partition coefficients versus shape factor for H=0.64.

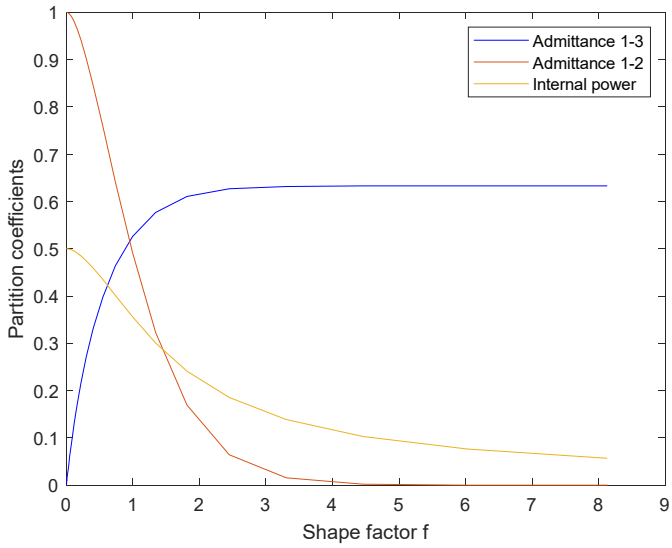

Figure (7): Partition coefficients versus shape factor for H=5.12.

In particular, when the exchange coefficient tends to 0 (H → 0) or the shape factor tends to 0 (f → 0), the heat transfer becomes unidirectional along z, we find the results of [12], the resistances R_{13} and R_{23} tend to infinity (γ_{13} = γ_{23} → 0) and γ_{12} → 1 and $R_{12} = \frac{e}{\lambda\pi R^2}$. For the partition coefficients of the internal power, we find $\alpha_1 = 0.5$, which is normal since the internal power distribution in this problem is uniform.

Conversely, when the shape factor tends to infinity (f → ∞), the heat transfer becomes unidirectional in r (γ_{12} → 0), with a radial resistance that varies with the exchange coefficient H.

3.4. Case of several media:

This solution, allows to solve the problem of figure (8), the analogical diagram is given in figure (9).

This problem has no analytical solution, the electrical analogy allows to obtain an approximate solution. Indeed, for the analogy to be exact it would be necessary that the surface common to both media is isotherm, which is not the case.

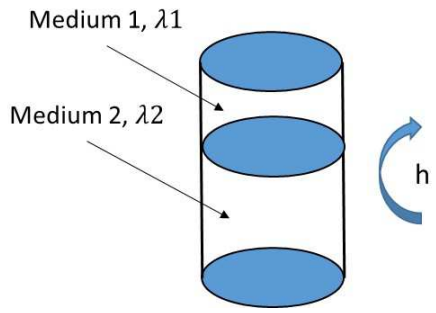

Figure (8): Two media system exchanging with 3 temperatures.

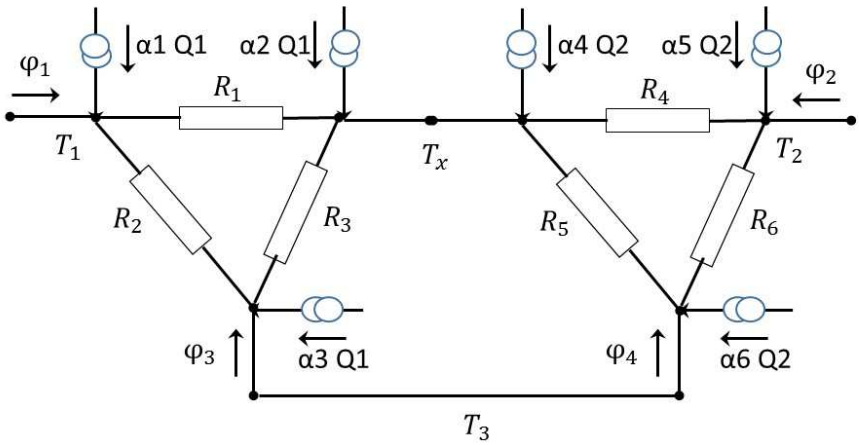

Figure (9): Analogical representation of the two media system.

To this approximation (isotherms surfaces), another difficulty is the calculation of resistances and partition coefficients which requires the resolution of two or three dimensional problems and hence the search for eigenvalues not always easy, so we are going to look at the field of application of the first order approximation.

For the cylinder case, the first eigenvalue α_1 has been tabulated (see figure (10)):

$$\alpha_1 R = \sqrt{\frac{17.76\,H + 5.7831H^2}{8.88 + 5.018H + H^2}} \qquad (42)$$

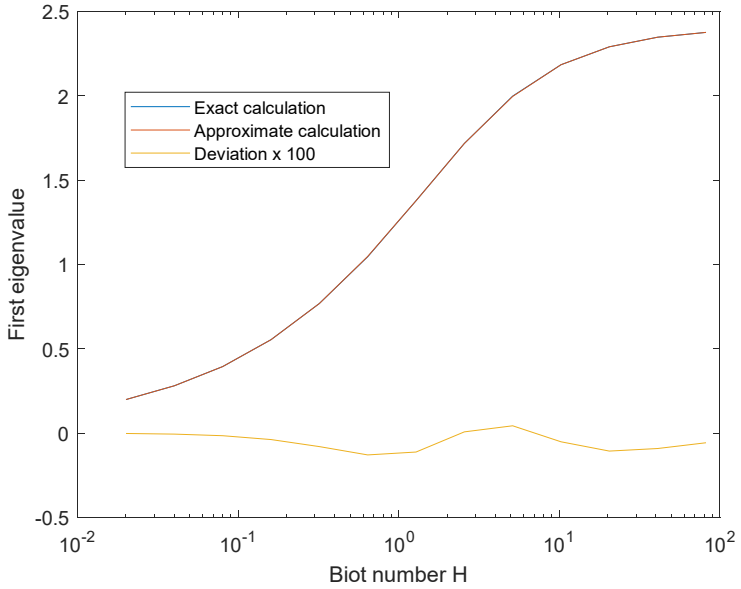

Figure (10): First eigenvalue ($\alpha_1 R$) versus Biot number (H).

4. Examples.

We present three examples:

- For the first one there is an analytical solution, the aim is to show the importance of the error when one of the surfaces is not isotherm (in z=0, the flux density is uniform, not the temperature).
- For the second one, there is also an analytical solution, the aim is to show that if the surfaces are isotherms the analogical solution is perfectly accurate even in the presence of an internal source.
- For the third, there is no analytical solution, the aim is to show that the analogical solution, although approximate, gives very good results.

4.1. Example 1: a single medium without an internal source.

$$\frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} = 0 \qquad (43)$$

$$r = 0 \qquad \frac{\partial T}{\partial r} = 0 \qquad (44)$$

$$r = R \qquad -\lambda \frac{\partial T}{\partial r} = h(T - Text) \qquad (45)$$

$$z = 0 \qquad \lambda \frac{\partial T}{\partial z} = \varphi_0 = 1000 \qquad (46)$$

$$z = e \qquad T = T_0 = 20°C \qquad (47)$$

With thickness e=0.1 m, radius R=0.05 m and conductivity λ=20 $Wm^{-1}K^{-1}$.

We calculate the average temperature at z=0 ($T(0)$) and the flux lost on the lateral surface (ϕ_p) as a function of the Biot number (H), the resistances are given by formulas (33) and (34) :

$$T(0) = \frac{R_{13}}{R_{12} + R_{13}} \left(R_{12}\, \pi R^2 \varphi_0 + T_0\right)$$

$$\phi_p = \frac{R_{12}}{R_{12} + R_{13}} \left(\pi R^2 \varphi_0 + \frac{R_{12} + R_{13} + R_{23}}{R_{23} R_{12}} T_0\right)$$

Figure (11) shows the results for temperature and figure (12) for flux.

Table (1) summarizes the results:

	temperature Error of 0.1 °C	temperature Error of 0.3 °C	flux Error of 1 %	flux Error of 10 %
First order	H < 1.0	H < 3.4	H < 0.25	H < 1.6
complete	H < 0.36	H < 5.4	all H	all H

Table (1): results example 1

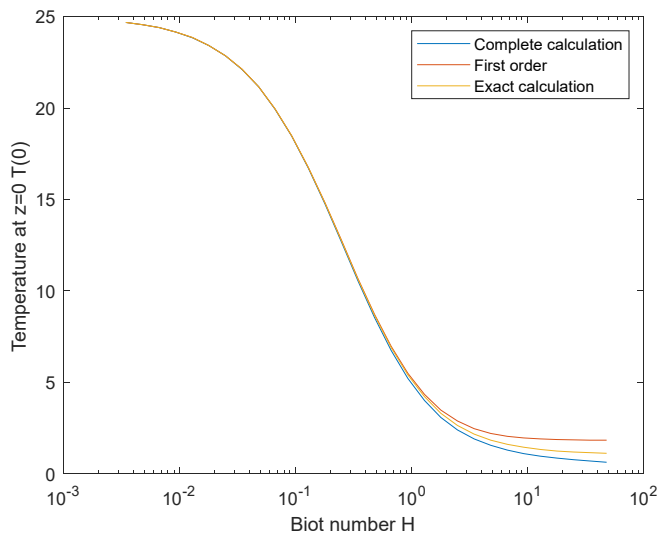

Figure (11): Temperature at z=0 versus Biot number.

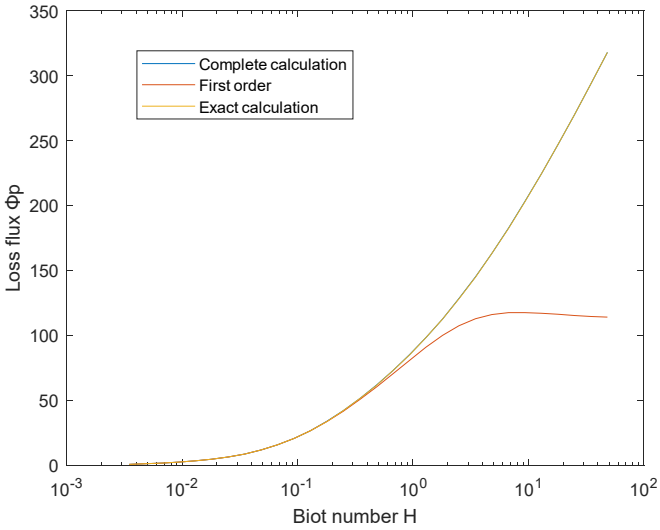

Figure (12): lateral heat flux versus Biot number.

For the full calculation, the error on the temperature comes from the fact that the lateral surface and the surface at z=0, are not isotherms. On the other hand, the flux calculation is correct.

4.2. Example 2: a single medium with an internal source.

Same cylinder as example 1, with $p_0 = 10^5\ Wm^{-3}$:

$$\frac{\partial^2 T}{\partial z^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} + \frac{p_0}{\lambda} = 0 \qquad (48)$$

$$r = 0 \qquad \frac{\partial T}{\partial r} = 0 \qquad (49)$$

$$r = R \qquad -\lambda\frac{\partial T}{\partial r} = h(T - Text) \qquad (50)$$

$$z = 0 \qquad T = T1 = 0\ °C \qquad (51)$$

$$z = e \qquad T = T2 = 0°C \qquad (52)$$

The fluxes at $z = 0$ (ϕ_0), at $z = e$ (ϕ_e) and the lost flux (ϕ_p) on the lateral surface are calculated as a function of the Biot number (H).

The partition coefficients are given by the relations (40) and (41).

In this problem:

$$\phi_0 = -\alpha_1 p_0 \pi R^2 e \qquad \phi_e = -\alpha_2 p_0 \pi R^2 e \qquad \phi_p = -\alpha_3 p_0 \pi R^2 e$$

Table (2) summarizes the results:

	Flux ϕ_0=ϕ_e Error of 1 %	Flux ϕ_0=ϕ_e Error of 10 %		Flux ϕ_p Error of 1 %	Flux ϕ_p Error of 10 %
First order	H < 1.4	H < 20		H < 1.0	all H
complete	all H	all H		all H	all H

Table (2): results example 2.

The results are good, the first order model allows an approximate calculation of the loss flux and the full calculation is accurate for the fluxes as seen on example 1.

4.3. Example 3: multi-cylinder with internal source:

Let's take the case of figure (8), with a source in medium 1, the analog diagram is given in figure (13).

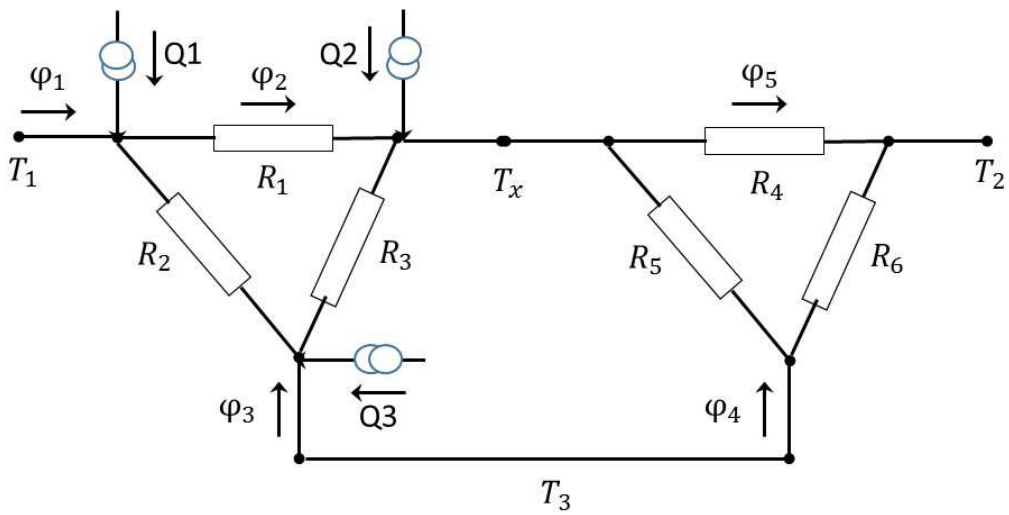

Figure (13): Analogical representation of the example 3.

We used the following values:

- Medium 1: e=0.1 m; R=0.05 m; λ=100 $Wm^{-1}K^{-1}$; p_0=10^5 Wm^{-3}
- Medium 2: e=0.05 m; R=0.05 m; λ=20 $Wm^{-1}K^{-1}$; p_0=0 Wm^{-3}
- Boundary conditions: T1=20 °C; T2=20 °C; T3=0 °C.

The mesh laws allow us to establish the following linear system:

$$T_1 - T_x = R_1\varphi_2$$
$$T_1 - T_3 = R_2(\varphi_1 + Q_1 - \varphi_2)$$
$$T_3 - T_x = R_3(\varphi_3 + Q_3 + \varphi_1 + Q_1 - \varphi_2)$$
$$T_x - T_2 = R_4\varphi_5$$
$$T_x - T_3 = R_5(\varphi_3 + Q_3 + \varphi_1 + Q_1 + Q_2 - \varphi_5)$$
$$T_3 - T_2 = R_6(\varphi_4 + \varphi_3 + Q_3 + \varphi_1 + Q_1 + Q_2 - \varphi_5)$$

Which is solved by formal calculation on Matlab to obtain Tx, the temperature at the interface of the two media. Figure (14) gives Tx as a function of the exchange coefficient h, for the complete calculation, the first order calculation and the results of a numerical calculation on the COMSOL software.

The figure (15) presents the mesh used on COMSOL, it is made of 4800 triangular elements, it was verified that the increase of the number of elements does not modify the value of the average temperature at the interface.

Figure (16) shows the temperature fields for the three values of the exchange coefficient (h=10, 100, 1000).

It is particularly interesting to see that the electrical analogy gives very good values while the temperature fields are very different.

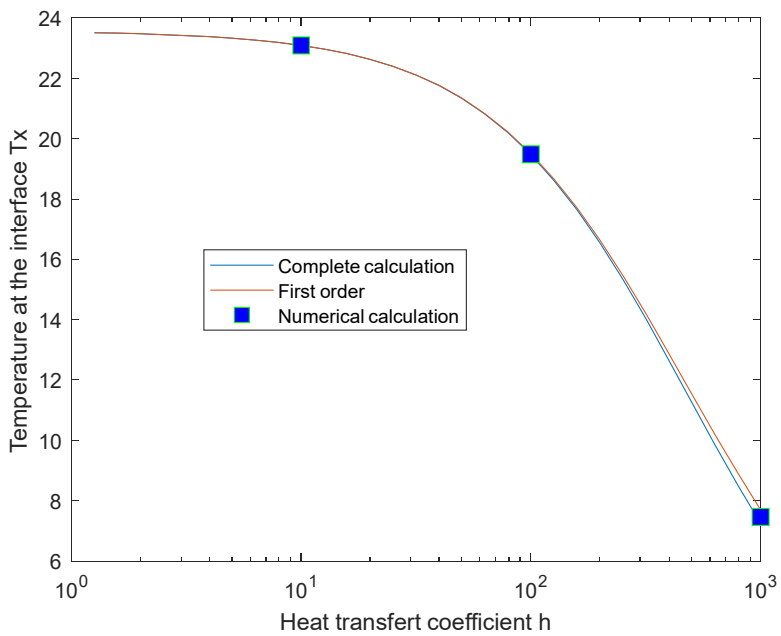

Figure (14): Temperature at the interface versus heat transfer coefficient.

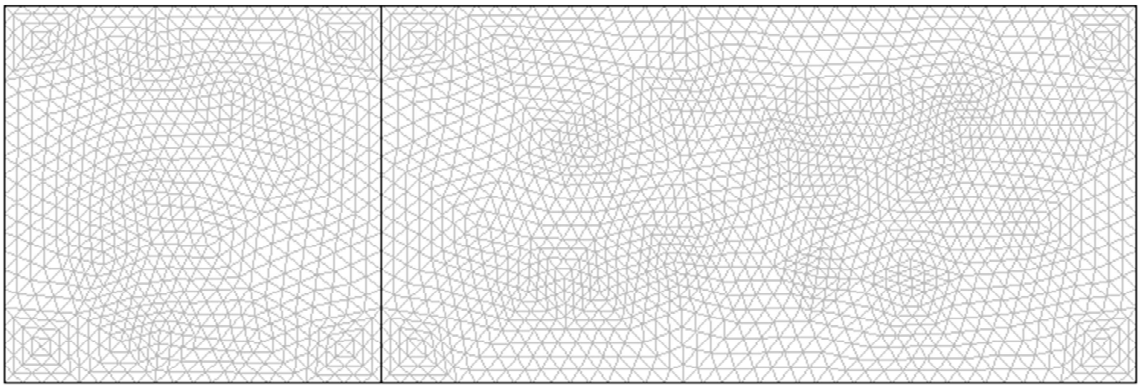

Figure (15): COMSOL mesh.

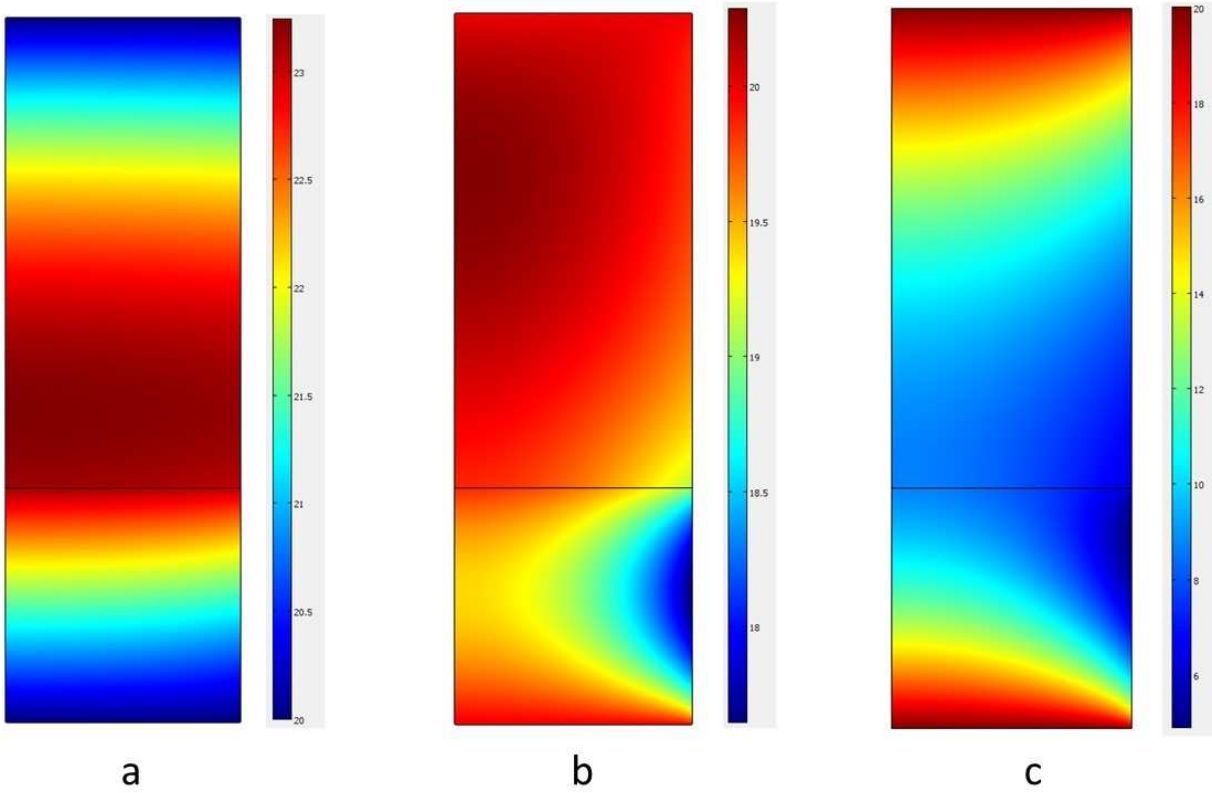

Figure (16): Temperature fields (a : h = 10; b : h = 100; c : h = 1000).

Table (3) summarizes the results:

Tx (°C)	h = 10 $Wm^{-2}K^{-1}$	h = 100 $Wm^{-2}K^{-1}$	h = 1000 $Wm^{-2}K^{-1}$
Numerical calculation	23.09	19.48	7.46
Electrical analogy	23.09	19.47	7.25
Electrical analogy first order	23.09	19.50	7.70

Table (3): results example 3.

The results are very good, for h=1000 $Wm^{-2}K^{-1}$, the difference is only 0.2 °C.

5. Conclusion.

It has been shown that the notion of thermal resistance could be generalized to a flow tube comprising several isotherms surfaces, a surface in exchange with an external temperature and an internal heat source, and that there was an analogical representation of the triangle type with sharing of the flows.

Regarding the first-order calculation:

- In the case without an internal source, an approximation was obtained for the temperatures when a surface is not isotherm (Neumann type boundary condition) not necessarily better than the fin hypothesis. On the other hand, the fluxes are obtained with a very good approximation.
- In the case of a system with an internal source the modeling is very efficient, while the fin approximation is not possible.

The generalization to a cylinder with any base is immediate.

APPENDIX:

The general problem is:

In V: $\lambda \nabla^2 T + Q\, q^* = 0$ (1)

On S_i: $T = T_i$

On S_e: $-\lambda \frac{\partial T}{\partial n} = h(T - T_e)$

On S_{lat}: $-\lambda \frac{\partial T}{\partial n} = 0$

The solution is obtained by superposition of the following problems:

- n problems:

In V: $\nabla^2 L_i^* = 0$ (2)

On S_i: $L_i^* = T_i$

On S_j ($j \neq i$): $L_i^* = 0$

On S_e: $-\lambda \frac{\partial L_i^*}{\partial n} = h\, L_i^*$

On S_{lat}: $-\frac{\partial L_i^*}{\partial n} = 0$

- 1 problem:

In V: $\nabla^2 L_e^* = 0$ (3)

On S_i: $L_e^* = 0$

On S_e: $-\lambda \frac{\partial L_e^*}{\partial n} = h(L_e^* - T_e)$

On S_{lat}: $-\frac{\partial L_e^*}{\partial n} = 0$

- 1 problem:

In V: $\nabla^2 P^* + \frac{Q}{\lambda} q^* = 0$ (4)

On S_i: $P^* = 0$

On S_e: $-\lambda \frac{\partial P^*}{\partial n} = h\, P$

On S_{lat}: $-\lambda \frac{\partial P^*}{\partial n} = 0$

It is immediate to verify that the sum of n problems (2), problem (3) and problem (4) gives problem (1):

$$T = \sum_{i=1}^{n} L_i^* + L_e^* + P^*$$

Moreover, as each problem is a function of only one source, the problems (2) can be written:

In V: $\nabla^2 L_i = 0$ (5)

On S_i: $L_i = 1$

On S_j ($j \neq i$): $L_i = 0$

On S_e: $-\lambda \frac{\partial L_i}{\partial n} = h\, L_i$

On S_{lat}: $-\frac{\partial L_i}{\partial n} = 0$

With: $L_i^* = T_i\, L_i$

The problem (3):

In V: $\nabla^2 L_e = 0$ (6)

On S_i: $L_e = 0$

On S_e: $-\lambda \frac{\partial L_e}{\partial n} = h(L_e - 1)$

On S_{lat}: $-\frac{\partial L_e}{\partial n} = 0$

With: $L_e^* = T_e\, L_e$

The problem (4):

In V: $\nabla^2 P + q^* = 0$ (7)

On S_i: $P = 0$

On S_e: $-\lambda \frac{\partial P}{\partial n} = h\, P$

On S_{lat}: $-\lambda \frac{\partial P}{\partial n} = 0$

With: $P^* = \frac{Q}{\lambda} P$

The solution of problem (1) is therefore given by:

$$T = \sum_{j=1}^{n} T_j\, L_j + T_e L_e + \frac{Q}{\lambda} P \qquad (8)$$

On the other hand the sum of problems (5) and problem (6) gives us:

In V: $\nabla^2 (\sum_{i=1}^{n} L_i + L_e) = 0$ (9)

On S_i: $\sum_{i=1}^{n} L_i + L_e = 1$

On S_e: $-\lambda \frac{\partial (\sum_{i=1}^{n} L_i + L_e)}{\partial n} = h(\sum_{i=1}^{n} L_i + L_e - 1)$

On S_{lat}: $-\frac{\partial (\sum_{i=1}^{n} L_i + L_e)}{\partial n} = 0$

The previous system is a steady state system connected to a single source T=1, the solution is obvious:

$$\sum_{i=1}^{n} L_i + L_e = 1 \qquad (10)$$

The relation (8) can then be written taking into account (10):

$$T - T_i = \sum_{j=1}^{n} (T_j - T_i) L_j + (T_e - T_i) L_e + \frac{Q}{\lambda} P \qquad (11)$$

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