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► **To cite this version:**

| Robin Riblet. Sidon set in a union of intervals. 2022. hal-03519109v1

**HAL Id: hal-03519109**

**<https://hal.univ-lorraine.fr/hal-03519109v1>**

Preprint submitted on 10 Jan 2022 (v1), last revised 2 Feb 2022 (v2)

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# Sidon set in a union of intervals

RIBLET Robin

## Abstract

We study the maximum size of Sidon sets in unions of integers intervals. If  $A \subset \mathbb{N}$  is the union of two intervals and if  $|A| = n$  (where  $|A|$  denotes the cardinality of  $A$ ), we prove that  $A$  contains a Sidon set of size at least  $0,8444\sqrt{n}$ . On the other hand, by using the small differences technique, we establish a bound of the maximum size of Sidon sets in the union of  $k$  intervals.

## 1 Introduction

A Sidon set of integers is a subset of  $\mathbb{N}$  with the property that all sums of two elements are distinct. Working on Fourier series, Simon Sidon [9] was the first to take an interest in these sets. He sought to bound the size of the largest Sidon set in  $\llbracket 1, n \rrbracket$ . The question has been intensively studied and today it is well known (see [5]) that the maximum size of a Sidon set in an interval of size  $n$  is asymptotically equivalent to  $\sqrt{n}$ . We denote by  $F(\llbracket 1, n \rrbracket)$  this maximum size. The lower bound was obtained independently by Chowla [3] and Erdős [4] who established

$$\liminf_{n \rightarrow +\infty} \frac{F(\llbracket 1, n \rrbracket)}{\sqrt{n}} \geq 1.$$

For the upper bound, Erdős and Turán [4] proved that  $F(\llbracket 1, n \rrbracket) < \sqrt{n} + O(n^{1/4})$ . This was sharpened by Lindström [8] who proved that  $F(\llbracket 1, n \rrbracket) < n^{1/2} + n^{1/4} + 1$ . Finally, very recently, Balogh, Füredi and Roy [2] obtained

$$F(\llbracket 1, n \rrbracket) < \sqrt{n} + 0,998n^{1/4}.$$

In this paper we are interested in the size of the largest Sidon set contained in the union of two intervals. For  $A \subseteq \mathbb{N}$  we denote by  $F(A)$  the maximal cardinality of a Sidon set in  $A$ . Erdős conjectured that  $F(A) \geq \sqrt{n}$  for all sets  $A$  of size  $n$ . For more readability, if  $f$  and  $g$  are two functions such that  $f(n) \geq (1 - o(1))g(n)$ , we will write  $f(n) \gtrsim g(n)$ . In the same way if  $f(n) \leq (1 + o(1))g(n)$ , we will write  $f(n) \lesssim g(n)$ . Abbott [1] proved that  $F(A) \gtrsim 0,0805\sqrt{n}$  and so, if  $I_1$  and  $I_2$  are two intervals of respective cardinalities  $n_1$  and  $n_2$

$$F(I_1 \cup I_2) \gtrsim 0,0805\sqrt{n_1 + n_2}.$$

We will prove (Theorem 2.1) that

$$F(I_1 \cup I_2) \gtrsim 0,8444\sqrt{n_1 + n_2}.$$

Conversely, we will show that  $F(I_1 \cup I_2) \lesssim \sqrt{n_1 + n_2}$  and more generally we will give a bound for the maximum cardinality of a Sidon set in a union of  $k$  intervals (Theorem 3.1). In our result the number  $k$  of intervals can grow up with the size of  $A$ .

## 2 Lower bound for the size of the largest Sidon set contained in the union of two intervals

Previous works by Singer [10], Chowla [3], Erdős and Turán [4], lead to

$$F([1, n]) \sim \sqrt{n}. \quad (2.1)$$

Since Sidon's property is stable by translation, if  $A$  is an interval of size  $n$ , (2.1) proves that  $F(A) \sim \sqrt{n}$ . We shall study the case where  $A$  is the union of two intervals. We could simply choose a Sidon set in the largest of the two intervals. Therefore if  $A = I_1 \cup I_2$  where  $I_1$  and  $I_2$  are disjoint intervals of size  $n_1$  and  $n_2$  such that  $n_1 \geq n_2$ , by (2.1) we get a Sidon set of size  $\sqrt{n_1}$  in  $I_1$ , which yields

$$F(A) \geq \sqrt{n_1} = \frac{1}{\sqrt{2}} \sqrt{2n_1} \geq \frac{1}{\sqrt{2}} \sqrt{n_1 + n_2} > 0,707 \sqrt{n_1 + n_2}.$$

We shall get a more precise result in the following statement.

**Theorem 2.1.** *Let  $I_1$  and  $I_2$  be two disjoint intervals of respective cardinalities  $n_1$  and  $n_2$ . We have*

$$F(I_1 \cup I_2) \gtrsim 0,8444 \sqrt{n_1 + n_2}.$$

*Proof.* Let  $A$  be the union of two disjoint intervals of respective cardinalities  $n_1$  and  $n_2$ . Since Sidon's property is stable by translation and symmetry, even if it means translating and considering  $A' = \max A - A + 1$ , we can assume that  $A = I_1 \sqcup I_2$  where  $I_1 = [1, n_1]$ , and  $I_2$  is an interval of cardinality  $n_2 \leq n_1$ .



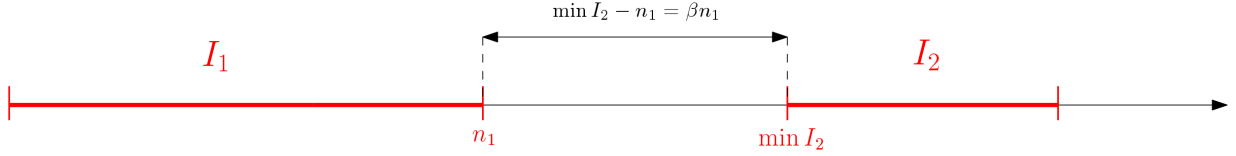
Strategy : We are going to discuss according to two parameters : the size of  $I_2$  compared to  $I_1$ , and the distance between  $I_1$  and  $I_2$ . For that we will consider

$$\alpha = \frac{n_2}{n_1} \quad \text{and} \quad \beta = \frac{\min I_2 - n_1}{n_1}.$$

We will distinguish several cases. First of all, if  $\alpha$  is less than a certain level  $\alpha_0$  (which we will have to optimize at the end of the proof) then we will only have to choose a large Sidon set in  $I_1$ . Indeed, if  $\alpha$  is small, then  $I_2$  is small in front of  $I_1$ . We will therefore not need its contribution to choose our Sidon set. If, on the other hand,  $\alpha$  is greater than  $\alpha_0$ , then we will distinguish two more cases depending on the size of  $\beta$ . If  $\beta$  is less than a certain level  $\beta_0$  (which we will also have to optimize at the end of the proof), then  $I_2$  is sufficiently close to  $I_1$ . To get a big Sidon set in  $I_1 \cup I_2$ , we will remove the middle

elements : those included in  $\llbracket n_1 + 1, \min I_2 - 1 \rrbracket$  to a big Sidon set in  $\llbracket 1, \max I_2 \rrbracket$ . We will use Singer's famous theorem [10] (see also [5] chapter II) to find a large Sidon set in  $\llbracket 1, \max I_2 \rrbracket$  with few elements in  $\llbracket n_1 + 1, \min I_2 - 1 \rrbracket$ . Finally if  $\beta$  is greater than  $\beta_0$ , we will transform a Sidon set in  $\llbracket 1, n_1 + n_2 \rrbracket$  to obtain a large Sidon set in  $I_1 \cup I_2$ .

Let  $\alpha_0 \in ]0, 1]$ ,  $\beta_0 \in \mathbb{R}_+$ ,  $\alpha = \frac{n_2}{n_1}$  et  $\beta = \frac{\min I_2 - n_1}{n_1}$ .



i) If  $\alpha \leq \alpha_0$ .

Write  $n_2 = \alpha n_1$ . It suffices then to choose a Sidon set  $S$  in  $I_1$  of size  $\sqrt{n_1}$ . In this way, we have

$$\begin{aligned} F(A) &\gtrsim \sqrt{n_1} \\ &\gtrsim \sqrt{\frac{1+\alpha}{1+\alpha} n_1} \\ &\gtrsim \frac{1}{\sqrt{1+\alpha}} \sqrt{n_1 + n_2} \\ &\gtrsim \frac{1}{\sqrt{1+\alpha_0}} \sqrt{n_1 + n_2}. \end{aligned}$$

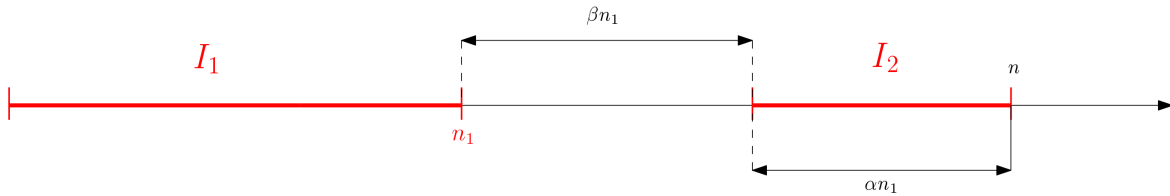
Finally since  $|A| = n_1 + n_2$ , in this case we get

$$F(A) \gtrsim \frac{1}{\sqrt{1+\alpha_0}} \sqrt{|A|}. \quad (2.2)$$

ii) If  $\alpha \geq \alpha_0$  and  $\beta \leq \beta_0$ .

We write  $n_2 = \alpha n_1$  again and we recall that  $\beta = \frac{\min I_2 - n_1}{n_1}$ . In this way, if  $n = \max A$ , we have

$$n = (1 + \alpha + \beta)n_1. \quad (2.3)$$



As explained before, we want to use Singer's theorem.

**Theorem** (Singer, [10]). *Let  $p$  be a prime. Then there exist  $p + 1$  Sidon sets  $S_1, \dots, S_p$  each of size  $p + 1$  such that*

$$\bigcup_{i=1}^p S_i = \{1, \dots, p^2 + p + 1\}.$$

We want to use it in  $\llbracket 1, n \rrbracket$ , so we need to approach  $n$  by  $p^2 + p + 1$  where  $p$  is a prime number. Let  $p$  and  $p'$  be the two consecutive prime numbers such that

$$p^2 + p + 1 \leq n < p'^2 + p' + 1.$$

Since  $p$  and  $p'$  are consecutive, it is well known for instance that  $p' - p = O(p^{5/8})$  (see [7]). (Actually, better results exist on the distance between two consecutive primes (see [6]) but this bound is enough for us). We have  $p^2 + p + 1 \leq n$ . According to Singer's theorem (theorem 2), there exist  $p + 1$  Sidon set  $S_i$  ( $i = 1, \dots, p + 1$ ) each of size  $p + 1$ , whose union is  $\llbracket 1, p^2 + p + 1 \rrbracket$ . Since  $n = p^2 + O(p'^2 - p^2 + p' - p) = p^2 + O(p^{13/8})$ , we have

$$\min I_2 = (1 + \beta)n_1 = \frac{1 + \beta}{1 + \alpha + \beta}n = \frac{1 + \beta}{1 + \alpha + \beta}p^2 + O(p^{13/8}) \leq p^2 + p + 1,$$

for sufficiently large  $n$ . Therefore  $\llbracket n_1, \min I_2 \rrbracket \subset \llbracket 1, p^2 + p + 1 \rrbracket$ , thus

$$\bigcup_{i=1}^{p+1} (S_i \cap \llbracket n_1, \min I_2 \rrbracket) = \llbracket n_1, \min I_2 \rrbracket,$$

and

$$\sum_{i=1}^{p+1} |S_i \cap \llbracket n_1, \min I_2 \rrbracket| = \beta n_1 + o(n_1).$$

So there exists  $i \in \llbracket 1, p + 1 \rrbracket$  such that  $S = S_i$  satisfies

$$|S \cap \llbracket n_1, \min I_2 \rrbracket| \leq \frac{\beta}{p + 1}n_1 + o\left(\frac{n_1}{p}\right).$$

Finally, with  $S' = S \setminus \llbracket n_1, \min I_2 \rrbracket$ , we have  $S' \subset A$  and

$$|S'| \geq p + 1 - \frac{\beta}{p + 1}n_1 + o\left(\frac{n_1}{p}\right).$$

Now  $p \sim \sqrt{n}$ , and so by (2.3) we get

$$\begin{aligned} F(A) &\geq |S'| \\ &\geq \frac{1 + \alpha}{1 + \alpha + \beta}\sqrt{n} + o(\sqrt{n}) \\ &\geq \sqrt{\frac{1 + \alpha}{1 + \alpha + \beta}}\sqrt{n_1 + n_2} + o(\sqrt{n_1 + n_2}) \\ &\geq \sqrt{\frac{1 + \alpha}{1 + \alpha + \beta_0}}\sqrt{n_1 + n_2} + o(\sqrt{n_1 + n_2}). \end{aligned}$$

Moreover the function which associates  $\sqrt{\frac{1 + x}{1 + x + \beta_0}}$  to  $x$  is increasing and we are in case  $\alpha \geq \alpha_0$ , so finally

$$F(A) \gtrsim \sqrt{\frac{1 + \alpha_0}{1 + \alpha_0 + \beta_0}}\sqrt{|A|}. \quad (2.4)$$

iii) If  $\alpha \geq \alpha_0$  and  $\beta \geq \beta_0$ .

Here, we will distinguish between the cases  $\beta_0 \geq 1$  and  $\beta_0 < 1$ .

iii.a. **If  $\beta_0 \geq 1$ .**

Then  $\beta \geq 1$  and therefore  $I_1$  and  $I_2$  are sufficiently far apart.



We then choose a set of Sidon  $S$  in  $\llbracket 1, n_1 + n_2 \rrbracket$  and we define the new set  $S'$  by

$$S' = S_1 \sqcup S_2,$$

where  $S_1 = S \cap \llbracket 1, n_1 \rrbracket$  and  $S_2 = S \cap \llbracket n_1, n_1 + n_2 \rrbracket + \min I_2 - n_1$ . So  $S' \subseteq A$  and we will see that  $S'$  is a Sidon set. First note that  $S_1$  and  $S_2$  are Sidon sets,  $\max S_1 \leq n_1$  and  $\min S_2 > \min I_2 \geq 2n_1$ . Therefore for  $a, b \in S'$ ,  $a \neq b$ , we have

$$a, b \in S_1 \Leftrightarrow a + b < 2n_1, \quad (2.5)$$

$$a, b \in S_2 \Leftrightarrow a + b > 4n_1. \quad (2.6)$$

Let  $a, b, c, d \in S'$  be such that  $a + b = c + d$ . We will distinguish between the following three cases :  $a$  and  $b$  both belong to  $S_1$ , both to  $S_2$ , or one belongs to  $S_1$  and the other to  $S_2$ .

- If  $a, b \in S_1$ , then by (2.5)  $c, d \in S_1$  and since  $S_1$  is a Sidon set,  $\{a, b\} = \{c, d\}$ .
- If  $a, b \in S_2$ , then by (2.6)  $c, d \in S_2$  and  $S_2$  is a Sidon set, so  $\{a, b\} = \{c, d\}$ .
- If  $a \in S_1$  and  $b \in S_2$ , then as seen in previous arguments, necessarily  $c$  and  $d$  cannot belong both to  $S_1$  nor both to  $S_2$ . Suppose therefore without loss of generality that  $c \in S_1$  and  $d \in S_2$ . So we have

$$a + b = c + d \Leftrightarrow a + (b - \min I_2 + n_1) = c + (d - \min I_2 + n_1),$$

and  $a, (b - \min I_2 + n_1), c, (d - \min I_2 + n_1) \in S$  by construction of  $S_1$  and  $S_2$ . So  $\{a, (b - \min I_2 + n_1)\} = \{c, (d - \min I_2 + n_1)\}$  because  $S$  is a Sidon set. Moreover, since  $a, c \in S_1$  and  $b, d \in S_2$ , we have  $a, c \in \llbracket 1, n_1 \rrbracket$  and  $(b - \min I_2 + n_1), (d - \min I_2 + n_1) \in \llbracket n_1, n_1 + n_2 \rrbracket$ . Hence  $a = c$  and  $(b - \min I_2 + n_1) = (d - \min I_2 + n_1)$  so finally  $a = c$  and  $b = d$ .

In conclusion, in any case, we get  $\{a, b\} = \{c, d\}$ , which means that  $S'$  is a Sidon set. It suffices then to notice that  $|S'| = |S|$  and to recall that in  $\llbracket 1, n_1 + n_2 \rrbracket$ , we have Sidon sets of size  $\sqrt{n_1 + n_2}$ , to be able to conclude that when  $\min I_2 - n_1 \geq n_1$ , we have

$$F(A) \gtrsim \sqrt{|A|}. \quad (2.7)$$

**iii.b.** If  $\beta_0 < 1$ .

We will follow the same strategy but as in the case **ii**), we will rely on Singer's theorem in order to remove the annoying elements. This time we work in  $\llbracket 1, n_1 + n_2 \rrbracket$ , thus we approach  $n_1 + n_2$  by  $p^2 + p + 1$  where  $p$  is a prime number. Let  $p$  and  $p'$  be the two consecutive prime numbers such that

$$p^2 + p + 1 \leq n_1 + n_2 < p'^2 + p' + 1.$$

Again, since  $p$  and  $p'$  are consecutive, we know that  $p' - p = O(p^{5/8})$  (see [7]). Thus  $n_1 + n_2 = p^2 + O(p^{13/8})$ . So  $n_1 \leq p^2 + p + 1$  and so we have

$$\llbracket \frac{1+\beta}{2}n_1, n_1 \rrbracket \subset \llbracket 1, p^2 + p + 1 \rrbracket,$$

and as in the previous case, by Singer's theorem there exists a Sidon set  $S$  in  $\llbracket 1, p^2 + p + 1 \rrbracket$  of size  $p + 1$  and such that

$$\left| S \cap \llbracket \frac{1+\beta}{2}n_1, n_1 \rrbracket \right| \leq \frac{1-\beta}{2(p+1)}n_1 + o\left(\frac{n_1}{p}\right).$$

Setting  $S' = S \setminus \llbracket \frac{1+\beta}{2}n_1, n_1 \rrbracket$ , we have

$$\begin{aligned} |S'| &\geq p + 1 - \frac{1-\beta}{2(p+1)}n_1 + o\left(\frac{n_1}{p}\right) \\ &\geq \sqrt{n_1 + n_2} - \frac{1-\beta}{2(1+\alpha)}\sqrt{n_1 + n_2} + o\left(\sqrt{n_1 + n_2}\right) \\ &\geq \frac{1+2\alpha+\beta}{2(1+\alpha)}\sqrt{n_1 + n_2} + o\left(\sqrt{n_1 + n_2}\right) \\ &\geq \frac{1+2\alpha+\beta_0}{2(1+\alpha)}\sqrt{n_1 + n_2} + o\left(\sqrt{n_1 + n_2}\right). \end{aligned}$$

Now the function which associates  $\frac{1+2x+\beta_0}{2(1+x)}$  to  $x$  is increasing, hence

$$|S'| \gtrsim \frac{1+2\alpha_0+\beta_0}{2(1+\alpha_0)}\sqrt{n_1 + n_2}. \quad (2.8)$$

Finally define the new set  $S''$  by

$$S'' = S_1 \sqcup S_2,$$

where  $S_1 = S' \cap \llbracket 1, \frac{1+\beta}{2}n_1 \rrbracket$  and  $S_2 = \left(S' \cap \llbracket \frac{1+\beta}{2}n_1, n_1 + n_2 \rrbracket\right) + \min I_2 - n_1$ . So  $S'' \subseteq A$  and we will see that  $S''$  is a Sidon set. First note that  $S_1$  and  $S_2$  are Sidon sets. Since  $\min S_2 \geq \min I_2 + 1 = (1+\beta)n_1 + 1$  and  $\max S_1 \leq \frac{1+\beta}{2}n_1$ , we get as in **iii.a**, for  $a, b \in S$

$$a, b \in S_1 \Leftrightarrow a + b \leq (1+\beta)n_1,$$

$$a, b \in S_2 \Leftrightarrow a + b > (2+2\beta)n_1,$$

Therefore, as the previous case, we prove that  $S''$  is a Sidon set. So  $F(A) \geq |S''| = |S'|$  and by (2.8), we have

$$F(A) \gtrsim \frac{1 + 2\alpha_0 + \beta_0}{2(1 + \alpha_0)} \sqrt{|A|}. \quad (2.9)$$

**iv) Conclusion.**

Whatever the case we are in, by (2.2), (2.4), (2.9) and (2.7), we have

$$F(A) \gtrsim \min \left( \frac{1}{\sqrt{1 + \alpha_0}}, \sqrt{\frac{1 + \alpha_0}{1 + \alpha_0 + \beta_0}}, \frac{1 + 2\alpha_0 + \beta_0}{2(1 + \alpha_0)} \right) \sqrt{|A|}.$$

It is therefore a question of optimizing the choices of  $\alpha_0$  and  $\beta_0$  so that this minimum is as large as possible. For that we start by asking the first two terms of the minimum to be equal (to find the region where the first term is smaller than the second). Which leads us to the equation  $\alpha_0 = \frac{\sqrt{1+4\beta_0}-1}{2}$ . Then we ask the first and third terms of the minimum to be equal (to find the region where the first term is smaller than the third). This gives us the new equation  $\alpha_0 = \frac{\sqrt{3-2\beta_0}-\beta_0}{2}$ . Finally, in the region between these two curves, the minimum is the first term. It is therefore a question of optimizing this term in this region, which leads us to the intersection of the two curves of equation  $\alpha_0 = \frac{\sqrt{1+4\beta_0}-1}{2}$  and  $\alpha_0 = \frac{\sqrt{3-2\beta_0}-\beta_0}{2}$ , and to the final equation

$$\sqrt{1 + 4\beta_0} - 1 = \sqrt{3 - 2\beta_0} - \beta_0.$$

In order to conclude, we just need to find an approximate value of the solution of this equation then also find an approximate value of  $\alpha_0$  by taking one of our two previous equations. With  $\alpha_0 = 0.402$  and  $\beta_0 = 0.391$ , we get

$$F(A) \gtrsim 0,8444\sqrt{|A|}.$$

□

### 3 Upper bound for the maximum size of a Sidon set in a union of intervals

In the previous section we gave a lower bound for the maximum size of a Sidon set in a union of two intervals. Conversely, we seek in this section an upper bound for the maximum size of a Sidon set in a union of intervals. If we consider two intervals of size  $n/2$  for example, in each of these two intervals, we can only choose at most (asymptotically)  $\sqrt{n/2}$  elements because otherwise it would contradict (2.1). A trivial asymptotic bound would therefore be  $2\sqrt{n/2} = \sqrt{2}\sqrt{n}$ . Using the Erdős-Turán small difference technique [5], we can go down to  $\sqrt{n}$ . Actually, we can prove the result for a fixed number of intervals and even for an increasing number of intervals if it remains  $o(\sqrt{n})$ . This is the content of the following theorem.



**Theorem 3.1.** *If  $E$  is a set of cardinality  $n \in \mathbb{N}^*$  and  $E$  is a union of  $k$  intervals, then any Sidon included in  $E$  has size at most*

i)  $(\alpha + \sqrt{2 + \alpha^2})\sqrt{n} + o(\sqrt{n})$  if  $\limsup_{n \rightarrow +\infty} \frac{k}{\sqrt{n}} = \alpha > 0$

ii)  $\sqrt{n} + o(\sqrt{n})$  si  $k = o(\sqrt{n})$

iii)  $\sqrt{n} + \sqrt{kn}^{1/4} + o(n^{1/4})$  if  $k = o(n^{1/4})$ .

*Proof.* Let  $n, k \in \mathbb{N}^*$  be such that  $k \leq n$ , and

$$E = \bigsqcup_{i=1}^k \llbracket n_i^-, n_i^+ - 1 \rrbracket,$$

where  $n_1^- < n_1^+ < n_2^- < n_2^+ < \dots < n_k^- < n_k^+$  and  $\sum_{i=1}^k (n_i^+ - n_i^-) = n$ . Let  $S \subseteq E$  be a Sidon set. For  $u$  an integer such that  $u < n$ , we define the set  $\mathcal{M}$  by

$$\mathcal{M} = E + \llbracket 1, u \rrbracket = \left( \bigsqcup_{i=1}^k \llbracket n_i^-, n_i^+ - 1 \rrbracket \right) + \llbracket 1, u \rrbracket.$$

We have  $|\mathcal{M}| \leq \sum_{i=1}^k (u + n_i^+ - n_i^-) = n + ku$ . For  $m \in \mathcal{M}$ , we consider the intervals  $I_m$  defined by

$$I_m = \llbracket m - u, m - 1 \rrbracket.$$

Let  $r = |S|$ . Since each element of  $S$  occurs in exactly  $u$  intervals of type  $I_m$ , we have

$$\sum_{m \in \mathcal{M}} |I_m \cap S| = ru. \quad (3.1)$$

Thus by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} (ru)^2 &\leq \left( \sum_{m \in \mathcal{M}} 1 \right) \left( \sum_{m \in \mathcal{M}} |I_m \cap S|^2 \right) \\ &\leq (n + ku) \left( \sum_{m \in \mathcal{M}} |I_m \cap S|^2 \right), \end{aligned}$$

and so

$$\sum_{m \in \mathcal{M}} |I_m \cap S|^2 \geq \frac{(ru)^2}{n + ku}. \quad (3.2)$$

For  $u < n$  and  $m \in \mathcal{M}$ , we define

$$T_u(m) = \left| \left\{ (s_1, s_2) \mid s_1, s_2 \in (S \cap I_m), s_1 < s_2 \right\} \right|,$$

and

$$T_u = \sum_{m \in \mathcal{M}} T_u(m).$$

On the one hand, by (3.1) and (3.2), we have

$$T_u = \sum_{m \in \mathcal{M}} T_u(m) = \sum_{m \in \mathcal{M}} \binom{|I_m \cap S|}{2} \geq \frac{1}{2} \left( \frac{(ru)^2}{n + ku} - ru \right),$$

which yields

$$T_u \geq \frac{ru}{2} \left( \frac{ru}{n+ku} - 1 \right). \quad (3.3)$$

On the other hand, for any couple  $(s_1, s_2)$  counted in  $T_u$ ,  $s_2 - s_1$  is an integer  $d$  satisfying  $0 < d < u$ . Moreover, since  $S$  is a Sidon set, for each  $d$ , there is at most one matching  $(s_1, s_2)$ . Finally, a pair  $(s_1, s_2)$  corresponding to a certain  $d$  appears in exactly  $u - d$  intervals  $I_m$ . Thereby

$$T_u \leq \sum_{d=1}^{u-1} (u-d) = \frac{u(u-1)}{2}.$$

Using (3.3), we get

$$\frac{ru}{2} \left( \frac{ru}{n+ku} - 1 \right) \leq \frac{u(u-1)}{2},$$

which leads to

$$r^2u - (n+ku)r \leq (u-1)(n+ku),$$

and finally

$$r \leq \sqrt{\frac{u-1}{u}(n+ku) + \frac{(n+ku)^2}{4u^2}} + \frac{n+ku}{2u}. \quad (3.4)$$

We just have to choose different values for  $u$  according to the relative size of  $k$  compared to  $n$  in order to conclude.

- If  $\limsup_{k \rightarrow +\infty} \frac{k}{\sqrt{n}} = \alpha \neq 0$ , then we choose  $u = \lceil \sqrt{n}/\alpha \rceil$  and (3.4) gives

$$\begin{aligned} r &\leq \frac{\alpha\sqrt{n}}{2} + \frac{k}{2} + \sqrt{n + \frac{k\sqrt{n}}{\alpha} + \left( \frac{\alpha\sqrt{n}}{2} + \frac{k}{2} \right)^2} + o(n) \\ &\leq \sqrt{n} \left( \alpha + \sqrt{2 + \alpha^2} \right) + o(\sqrt{n}). \end{aligned}$$

**Remark 3.2.** As we want  $r$  to be an  $O(\sqrt{n})$ , in (3.4), on the one hand the  $k(u-1)$  in the root forces us to choose  $u = O(\sqrt{n})$ , and on the other hand the  $\frac{n}{2u}$  outside the root leads us to choose  $\sqrt{n} = O(u)$ . So necessarily, our choice will be of the form  $u = \gamma\sqrt{n}$ . The choice  $u = \lceil \sqrt{n}/\alpha \rceil$  is the simplest giving a good bound for all  $\alpha$ , but at this stage, if we know precisely  $\alpha$ , it is possible to do a little better. For exemple if  $\alpha = 1/\sqrt{2}$ , then choosing  $u = \lceil \beta\sqrt{n} \rceil$  and injecting in (3.4), we obtain a function to be minimized in  $\beta$ . For  $\beta = 1,79$ , we get

$$r \leq 2,266\sqrt{n},$$

whereas our general choice  $u = \lceil \sqrt{n}/\alpha \rceil$ , only yields

$$r \leq 2,29\sqrt{n}.$$

Similarly if  $\alpha = 1$ , we are led to choose  $\beta = \frac{1+\sqrt{5}}{2}$  which gives

$$r \leq \frac{3+\sqrt{5}}{2}\sqrt{n} \leq 2,62\sqrt{n}.$$

- If  $k = o(\sqrt{n})$ , then we choose  $u = \left\lceil \frac{n^{3/4}}{\sqrt{k}} \right\rceil$  and (3.4) gives

$$\begin{aligned}
r &\leq \sqrt{k} \frac{n^{1/4}}{2} + \frac{k}{2} + \sqrt{n + \sqrt{k} n^{3/4} + \left( \sqrt{k} \frac{n^{1/4}}{2} + \frac{k}{2} \right)^2} \\
&\leq \sqrt{n} \sqrt{1 + \frac{\sqrt{k}}{n^{1/4}} + o(1)} + o(\sqrt{n}) \\
&\leq \sqrt{n} + o(\sqrt{n}).
\end{aligned} \tag{3.5}$$

If  $k = o(n^{1/4})$ , we can make the error term more precise.

- If  $k = o(n^{1/4})$ , (3.5) gives

$$\begin{aligned}
r &\leq \sqrt{k} \frac{n^{1/4}}{2} + \frac{k}{2} + \sqrt{n + \sqrt{k} n^{3/4} + \left( \sqrt{k} \frac{n^{1/4}}{2} + \frac{k}{2} \right)^2} \\
&\leq \sqrt{k} \frac{n^{1/4}}{2} + \frac{k}{2} + \sqrt{n} \sqrt{1 + \frac{\sqrt{k}}{n^{1/4}} + \frac{k}{4\sqrt{n}} + \frac{k^{3/2}}{2n^{3/4}} + \frac{k^2}{4n}} \\
&\leq \sqrt{n} + \sqrt{k} n^{1/4} + o(n^{1/4}),
\end{aligned}$$

where the last line comes from the Taylor expansion  $\sqrt{1+x} = 1 + \frac{x}{2} + o(x)$ .

□

## 4 Conclusion and Remarks

Theorems 2.1 and 3.1 prove that if  $A$  is the union of two intervals of respective size  $n_1$  and  $n_2$ , the maximum cardinality of a Sidon set in  $A$  is (asymptotically) between  $0,8444\sqrt{n_1 + n_2}$  and  $\sqrt{n_1 + n_2}$ . Erdős' conjecture claims that it should be equivalent to  $\sqrt{n_1 + n_2}$ . Therefore, it should be very interesting to improve Theorem 2.1 in order to try to bring the constant 0,8444 closer to 1.

It is also surely possible to improve the first point of Theorem 3.1 but we will never be able to reach  $\sqrt{n}$ . Indeed, it is easy to build Sidon sets with a cardinality larger than  $\sqrt{n}$  under the hypothesis of Theorem 3.1.

**Proposition 4.1.** *Let  $n \in \mathbb{N}^*$ . There exists a Sidon set of size  $2n$  in a union of  $n$  intervals each of size  $n$ .*

*Proof.* Let  $n \in \mathbb{N}^*$ ,  $S_1 = \{2^{k+1} \mid k = 1, \dots, n\}$ ,  $S_2 = \{2^{k+1} + k \mid k = 1, \dots, n-1\}$  and  $S = S_1 \sqcup S_2$ . Since  $|S| = 2n - 1$  and

$$S \subseteq \bigsqcup_{k=1}^n \llbracket 2^{k+1}, 2^{k+1} + n - 1 \rrbracket,$$

we just have to check that  $S$  is a Sidon set. Let  $a, b, c, d \in S$  be such that  $a + b = c + d$ .

Since  $2^{k+2} - 2^{k+1} > 2k$ ,  $a + b = c + d$  implies that there exists  $k_1$  and  $k_2$  in  $\{1, \dots, n\}$  such that  $\{a, b\}$  and  $\{c, d\}$  are in  $\llbracket 2^{k_1+1}, 2^{k_1+1} + n \rrbracket \cup \llbracket 2^{k_2+1}, 2^{k_2+1} + n \rrbracket$ . We can assume without loss of generality that  $a, c \in \llbracket 2^{k_1+1}, 2^{k_1+1} + n \rrbracket$  and  $b, d \in \llbracket 2^{k_2+1}, 2^{k_2+1} + n \rrbracket$ . In this way, we have

$$\begin{aligned} a + b = c + d &\Rightarrow a - c = d - b \\ &\Rightarrow a - c \in \{0, k_2\}, \end{aligned}$$

But since  $a, c \in \llbracket 2^{k_1+1}, 2^{k_1+1} + n \rrbracket$ ,  $a - c \in \{0, k_1\}$ . Therefore either  $a = c$  and so  $\{a, b\} = \{c, d\}$ , or  $k_1 = k_2$  and so  $\{a, b\} = \{c, d\}$ , which ends the proof.  $\square$

This proposition proves that the condition  $k = o(\sqrt{n})$  of the point *ii*) of Theorem 3.1 is optimal in a way. To improve this theorem, it would be necessary to reduce the constant  $(\alpha + \sqrt{2 + \alpha^2})$  in front of  $\sqrt{n}$  in the bound of point *i*). However, Proposition 4.1 implies that we cannot go below  $\sqrt{2}$  for  $\alpha = 1$ .

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