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# Backstepping Based State Observer Design Vlasov-Poisson Equation

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## Abstract

In this work, we present some new results on state observers design, in infinite dimension, for the  $1D \times 1D$ -Vlasov-Poisson transport equation. We show that the observers gains may be deduced from the backstepping technique. The obtained kernel equations are solved by the prime integrals of the ordinary differential system and, by the way, lead to assure convergence of the state observer. To show high performances, numerical simulations of the distribution function and its state observer are presented at the last section.

## 1. Introduction

We start by defining the Vlasov-Poisson equation  $1D \times 1D$ :

$$\partial_t f(t, x, v) + v \partial_x f(t, x, v) - \partial_x \phi(t, x) \partial_v f(t, x, v) = 0, \quad (1)$$

where

- $(t, x, v) \in \Omega_T \times \Omega_x \times \Omega_v$
- $f(t, x, v)$  is the electron distribution function in the phase space  $\Omega = \Omega_x \times \Omega_v$  (with the mass normalized to one and the charge of an electron to plus one)
- $\phi(t, x)$  Electrostatic potential of the  $E(t, x)$  field.

In the following, we will take  $x \in \Omega_x = [0, L]$ ,  $t \in \Omega_T = [0, T]$ ,  $v \in \Omega_v = [-V_c, V_c]$  with  $V_c > 0$ .

In order to describe the motion of charged particles in plasmas, we must calculate the force field from the macroscopic density of the particles

$$\rho(t, x) = \int_{\Omega_v} f(t, x, v) dv. \quad (2)$$

We calculate the force field from the Poisson equation,

$$-\partial_{xx} \phi = \rho(t, x) - 1, \quad (t, x) \in \Omega_T \times \Omega_x, \quad (3)$$

with a positive initial data,

$$f(t_0, x, v) = f_0(x, v), \quad (x, v) \in \Omega_x \times \Omega_v. \quad (4)$$

We impose periodic boundary conditions in  $x$ :

$$\phi(t, 0) = \phi(t, L), \quad \partial_x \phi(t, 0) = \partial_x \phi(t, L) \quad (5)$$

and

$$f(t, 0, v) = f(t, L, v) = q(t, v), \quad (t, v) \in \Omega_T \times \Omega_v. \quad (6)$$

We assume that  $q(t, v)$  is  $C^2$ -function and  $T$ -periodic.

A condition of compatibility signifying the "global neutrality" of the plasma is given by

$$\int_{\Omega_x} \int_{\mathbb{R}} f(t, x, v) dv dx = 1. \quad (7)$$

To uniquely determine the electric field  $E(t, x)$ , we add a condition of zero mean we have

$$\int_{\Omega_x} E(t, x) dx = 0, \quad t \in \Omega_T, \quad (8)$$

The output of the system is restricted to  $x = 0$ , this is due to the fact that the sensor is placed at the beginning of the membrane. we have :

$$y(t, v) = f(t, 0, v) = q(t, v), \quad (t, v) \in \Omega_T \times \Omega_v. \quad (9)$$

$T$ -periodic function  $q(t, v) \neq 0$  guarantee the observability of the system (1)-(9).

## 2. Objective

Consider the Luenberger state Observer Vlasov equation:

$$\partial_t \hat{f}(t, x, v) = -v \partial_x \hat{f}(t, x, v) + \partial_x \hat{\phi}(t, x) \partial_v \hat{f}(t, x, v) + l(t, x, v)(y - \hat{y}). \quad (10)$$

Periodic boundary conditions in  $x$

$$\hat{f}(t, 0, v) = q(t, v) + l_0(t, v)(y - \hat{y}), \quad (v, t) \in \Omega_v \times \Omega_T, \quad (11)$$

$$\hat{f}(t, L, v) = q(t, v), \quad (v, t) \in \Omega_T \times \Omega_v, \quad (12)$$

Initial condition

$$\hat{f}(t_0, x, v) = \hat{f}_0(x, v), \quad (x, v) \in \Omega_x \times \Omega_v. \quad (13)$$

While  $l(t, x, v)$  and  $l_0(t, v)$  denote the observer gains to be determined.

Introducing the observer error

$$e(t, x, v) = f(t, x, v) - \hat{f}(t, x, v)$$

. The observer error dynamics satisfies

$$\partial_t e(t, x, v) = -v \partial_x e(t, x, v) + \partial_x \phi(t, x) \partial_v e(t, x, v) - \partial_x \hat{\phi}(t, x) \partial_v \hat{f} - l(t, x, v)(y - \hat{y}). \quad (14)$$

Boundary conditions

$$e(t, 0, v) = -l_0(y - \hat{y}), \quad (15)$$

$$e(t, L, v) = 0 \quad (16)$$

Initial condition

$$e(t_0, x, v) = e_0(x, v), \quad (x, v) \in \Omega_x \times \Omega_v \quad (17)$$

where  $\hat{\phi}_x$  is given by

$$-\partial_{xx} \hat{\phi} = \hat{\rho}(t, x) - 1, \quad (t, x) \in \Omega_T \times \Omega_x, \quad (18)$$

Boundary Conditions Poisson equation

$$\hat{\phi}(t, 0) = \hat{\phi}(t, L), \quad \partial_x \hat{\phi}(t, 0) = \partial_x \hat{\phi}(t, L) \quad (19)$$

The electrostatic field is calculated in an explicit way then one can consider without ambiguity that:

$$\partial_x \phi(t, x) = \partial_x \hat{\phi}(t, x)$$

The dynamic error (14) becomes

$$\partial_t \tilde{e}(t, x, v) = -v \partial_x \tilde{e}(t, x, v) + \partial_x \phi(t, x) \partial_v \tilde{e}(t, x, v) - l(t, x, v)(y - \hat{y}). \quad (20)$$

## 3. Methodology

The idea is to determine  $l_0$  and  $l$  by applying the Backstepping method. For that we define the target system of the dynamic error. The choice of the system must be made in such a way that the boundary conditions for the transformation of the invertible volterra integral are verified. It must also guarantee the existence of the integral. Now we consider the following target linear system :

$$\partial_t \tilde{e}(t, x, v) = -v \partial_x \tilde{e}(t, x, v) + x \partial_v \tilde{e}(t, x, v). \quad (21)$$

Boundary Conditions

$$\tilde{e}(t, 0, v) \neq 0, \quad (22)$$

$$\tilde{e}(t, L, v) = 0. \quad (23)$$

Initial condition

$$\tilde{e}(t_0, x, v) = \tilde{e}_0(x, v), \quad (x, v) \in \Omega_x \times \Omega_v \quad (24)$$

The system is a transport equation whose resolution is simple with the method of characteristics. The general solution of this equation is:

$$\tilde{e} = \Theta(vt - x, x^2 + v^2), \quad (25)$$

where  $\Theta$  is a continuously differentiable function.

With the following lemma We prove that the equations system (21)-(24) is asymptotically stable in the norm  $L^2$ .

**Lemma 1** The transport equation (21)-(24) is exponentially stable in  $L^2$ -norm, if

$$\left( 2V_c - \frac{\gamma L^2}{2} - \frac{\sup_{\Omega} \alpha(t, x, v)}{2\gamma} \right) > 0$$

with  $V_c, \gamma > 0$ .

## 4. Results and discussion

The backstepping approach is utilized to connect the linear observer error system (20) and a suitable target system in the state  $\tilde{e}(t, x, v)$  by the invertible Volterra integral transformation

$$e(t, x, v) = \tilde{e}(t, x, v) - \int_0^x k(t, x, \xi, v) \tilde{e}(t, \xi, v) d\xi \quad (26)$$

where  $k(t, x, \xi, v)$  is the Kernel transformation. By calling Leibniz's differentiation rule and differentiating the transformation (26) we obtain:

$$l(t, x, v) = (\partial_x \phi(t, x) + x - v)k(t, x, 0, v) \quad (27)$$

$l_0$  is given by (15)

$$l_0(t, v) = -1 \quad (28)$$

We also obtain the following Kernel equation:

$$\partial_t k(t, x, \xi, v) + 2v \partial_x k(t, x, \xi, v) - x \partial_v k(t, x, \xi, v) = 0 \quad (29)$$

Choose  $\xi \in (0, L)$  and  $x \in (\xi, L)$  we get the Boundary Conditions

$$(\partial_x \phi(t, x) + x - v)k(t, x, x, v) = 0 \quad (30)$$

Choose  $x = L$  in (26) we have:

$$k(t, x, L, v) = 0 \quad (31)$$

then we have:

$$k(t, L, L, v) = 0 \quad (32)$$

$$k(t, 0, 0, v) = 0 \quad (33)$$

Initial condition

$$k(t_0, x, \xi, v) = k_0(x, \xi, v) \quad (34)$$

In order to analyze the exponential stability of the observer error dynamics it is necessary to consider the inverse

back-stepping transformation from the  $e(t, x, v)$ -system to  $\tilde{e}(t, x, v)$ -system in the terms of

$$\tilde{e}(t, x, v) = e(t, x, v) + \int_0^x g(t, x, \xi, v) e(t, \xi, v) d\xi \quad (35)$$

where  $g(t, x, \xi, v)$  is the transformation Kernel. We proceed in the same way as (29)-(33) to determine the kernel-PDE for  $g(t, x, v)$  and is given by

$$\partial_t g(t, x, \xi, v) + 2v \partial_x g(t, x, \xi, v) + x \partial_v g(t, x, \xi, v) = 0 \quad (36)$$

for  $\xi \in (0, L)$  and  $x \in (\xi, L)$  we get the Boundary Conditions

$$(\partial_x \phi(t, x) - x - v)g(t, x, x, v) = 0 \quad (37)$$

By taking  $x = L$  in (35) we have :

$$g(t, x, L, v) = 0 \quad (38)$$

thus we have:

$$g(t, L, L, v) = 0 \quad (39)$$

$$g(t, 0, 0, v) = 0 \quad (40)$$

Initial condition

$$g(t_0, x, \xi, v) = g_0(x, \xi, v) \quad (41)$$

with the function  $g_0(x, \xi, v)$  being consistent with (38)-(40) at  $t = t_0$ .

Based on the solution properties for the two kernels  $k(t, x, \xi, v)$  and  $g(t, x, \xi, v)$ , the exponential stability of the linearized observer error dynamics with output injection weights  $l(t, x, v)$  and  $l_0(t, v)$  can be verified.

**Theorem 1** Let  $l(t, x, v)$  and  $l_0(t, v)$  be as in (27), (28). Then the equilibrium  $e(t, x, v) \equiv 0$  of (14)-(20) is exponentially stable for all  $t \in \mathbb{R}_+^{t_0}$  in the  $L^2$ -norm.

## 5. Numerical results

In this section we present the numerical results of the vlasov-Poisson system (1)-(6) and observer (10)-(13) by the Time-Splitting method (see [1]). The initial condition is chosen as being a perturbation of a homogeneous equilibrium  $f^0(v)$

$$f_0(x, v) = f^0(v)(1 + \alpha \cos(sx)), \quad (42)$$

$s$  being the wave number.

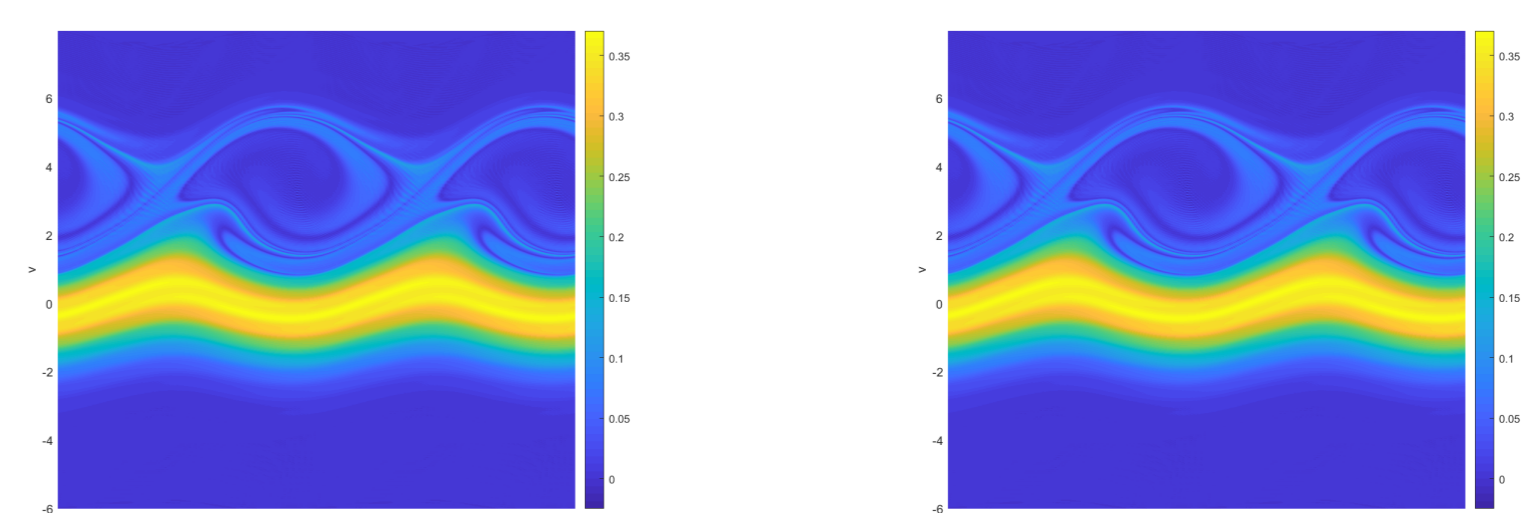
The equilibrium  $f^0(v)$  can be a Maxwellienne (Landau test case)

$$f^0(v) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{v^2}{2}\right). \quad (43)$$

We suppose then that we have  $L = 2\pi/s$ ,  $s = 0.5$  and that the perturbation is equal to  $\alpha = 0.5$  (Landau nonlinear).

The numerical parameters are chosen as follows: the domain  $\Omega_v = [-6, 6]$  and the time step is chosen equal to  $\Delta t = 0.05$ . The phase domain is meshed by  $(N_x, N_v)$  with  $N_v = 2N_x$  where  $N_x, N_v$  are respectively the points of the grids of  $\Omega_x, \Omega_v$ .

We will then present below two results of the systems (1) - (6) and (10) - (13) at times  $t = 40$



**Figure 1:** Solutions  $f(t, x, v)$  and  $\hat{f}(t, x, v)$  in phase domain  $(256, 512)$  at time  $t = 40$  respectively left and right,  $l(t, x, v) = \left(\frac{1}{\sqrt{\pi}} \sin\left(\frac{\pi}{2}x + 2\pi t\right) + x - v\right)x$  and  $l_0(t, v) = -1$ .

## 6. Conclusion

In this work we have established the Luenberger observer of the Vlasov-Poisson system  $1D \times 1D$  of the phase space. We have studied the stabilization of the dynamic error of the target linear system (transport equation). The observer output injection weights are determined by make use of a backstepping-based design.

## References

- [1] B. Afeyan, F. Casas, N. Crouseilles, A. Dodhy, E. Faou, M. Mehrenberger, and E. Sonnerup. Simulations of kinetic electrostatic electron nonlinear (keen) waves with 275 variable velocity resolution grids and high-order time-splitting. The European Physical Journal D, 68 (2014), pp. 1–21.2.