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Directional Pair-Correlation Analysis of Fracture Networks

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Key Points:

- Variability statistics of fracture networks are based on distances between object centers measured in dominant directions.
- An artificial network of a simple structure serves as benchmark.
- Important aspects of spatial correlation and variability become visible and measurable by means of pair correlation functions.

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Abstract

We analyze statistically fractures on surface exposures, which may be considered planar sections through three-dimensional fracture networks (FN). For this planar case, there exist established statistical methods which yield first-order or mean-value characteristics such as fracture density, fracture length distribution or rose of directions.

We extend this situation by presenting a statistical second-order theory, which aims to characterize the inner variability of planar FN. For this purpose we use ideas from the theory of marked point processes or object models, where the ‘points’ are centers of fractures or fracture branches and the ‘marks’ are lengths and strike azimuths. The statistics are based on oriented distances between object centers. These distances are represented by so-called pair correlation and mark correlation functions, which we recommend as new variability characteristics for fracture networks. The form of the corresponding plots gives information on the degree of randomness, on most frequent inter-center distances, and on possible local order, all with respect to fracture directions.

We demonstrate the application of these ideas by analysis of three FN. First, we study a synthetic structure, serving as a benchmark to test the methods under ideal conditions. Then, we analyze two field exposures already studied in the geological literature: a well developed and highly connected FN and a highly irregular FN with many small and isolated fractures. The different spatial arrangements of fractures are characterized by the correlation functions.

Plain Language Summary

The existence of rock fractures has a tremendous impact on rock bodies, on their stability, on their hydraulic properties and others. Therefore, describing and understanding the geometry and organization of fracture networks is an issue for both academic and industrial communities.

One way to tackle this problem is to study fracture network geometries at surface exposures, which are irregular and call for statistical analysis. A classical first step of statistics is the determination of the number and density of fractures. The present paper aims to contribute to the next step, by presenting methods for the description of the variability of fracture networks. Its basic idea is to describe each fracture by a point, its center, which holds two marks, length and orientation of the fracture, and then to use methods for statistical analysis of point patterns. In particular, the distances between the fracture centers are analyzed statistically, measured in the dominant directions of the network. The results are presented in various plots of correlation functions, yielding information on degree of randomness and local order.

The application of this method is demonstrated for a synthetic and two natural fracture networks. The natural network show clear differences in the spatial distribution of fractures, which are well presented by the plots. The synthetic network serves as a benchmark.

1 Introduction

Fractures are mechanical discontinuities in rocks, visible on surface exposures as fracture traces. They form complicated spatial networks called fracture networks (FN), which have a complex geometry and spatial organization. Such networks have great influence on the stability of rock bodies, on ore deposits, on fluid flow and, on reservoir capacity and compartmentalization. Therefore FN are a topic of intensive research, see e.g., Hardebol et al. (2015); Lei et al. (2017); Peacock and Sanderson (2018); Laubach et al. (2019); Sui et al. (2019), and references therein for recent reviews and case studies).

60 A complete and objective description of FN as three-dimensional objects is extremely
 61 difficult as they cannot be observed directly. Methods such as core or log analysis pro-
 62 vide only small-scale and sparse observations. Geophysical surveys provide complemen-
 63 tary data at larger scale and miss the detection of small fractures, i.e. smaller than the
 64 wavelength of the signal propagated. In the end, well data need to be interpreted and
 65 extrapolated using physical and mathematical methods to build a consistent description
 66 of a natural FN.

67 As in many other publications, we study the fracture traces on surface exposures,
 68 which we will further refer to as ‘fractures’. By their nature, they are cross-sections of
 69 the corresponding three-dimensional FN. Such sections provide valuable information in
 70 the context of modeling of three-dimensional FN, see e.g., *Bonneau et al. (2016)*. When
 71 testing the quality of simulated or synthetic FN, planar sections may be compared with
 72 exposure patterns paving the way to realistic three-dimensional FN modeling. For this
 73 purpose, it is important to characterize a planar FN with informative parameters and
 74 statistical summary functions.

75 Following the example of *Bonnet et al. (2001)*; *Darcel et al. (2003b, 2003a)*, we con-
 76 sider FN as samples of ‘object models’ (*Pyrcz & Deutsch, 2014*), i.e. of marked point pro-
 77 cesses where the ‘marks’ are geometrical objects, in our case straight-line segments that
 78 represent fractures or pieces of fractures called branches. The ‘points’ are the centers of
 79 these objects, which are characterized in a natural way by trace length and orientation
 80 marks.

81 For these FN, there exist an established first-order¹ theory, which is presented in
 82 *Zeeb et al. (2013)*. ‘First order’ means that spatial mean-values are determined such as
 83 fracture density, average number of fractures per area unit. While these characteristics
 84 are valuable for a general or global description, they tell nothing about the spatial vari-
 85 ability of FN nor about their inner spatial correlations, and so are not sufficient for rig-
 86 orous model testing. This was already mentioned by *Dershowitz and Einstein (1988)*,
 87 who briefly discussed some early papers in the context of ‘autocorrelation and correla-
 88 tion’ of FN.

89 For the definition of second-order² characteristics for FN, this paper uses ideas of
 90 spatial and point process statistics. The marked-point-process approach offers well-established
 91 functional summary characteristics, which we recommend as descriptors of variability
 92 behavior. Most of them have names that contain the word ‘correlation’ and they are in-
 93 deed related to the well-known correlation functions of geostatistics. In most modern pa-
 94 pers related to statistics of marked point processes, isotropy is assumed. Here, we con-
 95 sider anisotropy as an essential feature and use modified direction-dependent correlation
 96 functions. Even, we perform a separate analysis of subfamilies of objects with similar
 97 orientation.

98 The methods we describe are applicable to fractures as well as to fracture branches.
 99 The analysis of fracture center systems yields information about the spatial distribution
 100 of fractures, while the analysis of branch centers helps to characterize aspects of the in-
 101 teraction of fractures.

102 We apply our methods to the analysis of data from one synthetic and two natu-
 103 ral FN, for comparison. The first natural FN is given by a map from a satellite image
 104 of the Oman mountains and the second is a surface exposure from the Hornelen Basin
 105 in Norway. These data were already studied respectively in *Zeeb et al. (2013)* and *Odling*

¹ The statistical term ‘first-order’ should not be confused with ‘primary’ as used in the term ‘primary FN characteristics’ in *Dershowitz and Einstein (1988)*.

² The statistical term ‘second-order’ should not be confused with ‘secondary’ as used in the term ‘secondary FN characteristics’ in *Dershowitz and Einstein (1988)*.

106 (1997) by means of first-order methods and by fractal methods (Darcel et al., 2003a, 2003b).
 107 Second-order statistics add variability aspects to the description of these patterns.

108 2 Fundamentals of FN Characterization

109 The geometrical-statistical investigation of FN is a well-known challenge. There-
 110 fore, here the usual geological approach combining field observations and modeling is ap-
 111 plied. This section presents a brief overview of useful mathematical models, simplifica-
 112 tions and statistical estimators.

113 2.1 Modeling FN

114 A really accurate way to describe an FN would explicitly represent fractures as in-
 115 dividual objects in the spirit of object models (Pyrzcz & Deutsch, 2014). Each fracture
 116 is then characterized by its spatial location, i.e. a point x_n , and its own irregular geom-
 117 etry, given by a set X_n .

118 Statistics for all the X_n are a rather difficult task, even if the fractures were fully
 119 observable. Therefore and since the observation conditions are usually limited, FN char-
 120 acterization uses models with various simplifying assumptions. The most common math-
 121 ematical idealization of spatial FN is to consider them as systems of two-dimensional flat
 122 surface pieces, i.e., the X_n are assumed to be planar objects such as discs, ellipses or rect-
 123 angles. This means that apertures and deviations from flat shape are ignored.

124 These X_n are randomly scattered in space, where ‘randomly’ does not mean ‘with-
 125 out any rules’ or ‘totally irregular’. In fact, in natural FN there are various spatial cor-
 126 relations between the fractures: younger fractures may be stopped by older ones, in the
 127 close neighborhood of fractures there is often space without fractures resulting from stress
 128 reduction in the shadow zone.

129 A classical model for three-dimensional FN is the Poisson-Baecher disc model, an
 130 object model, where the location points x_n are a homogeneous Poisson process³ and the
 131 objects X_n are discs (Dershowitz & Einstein, 1988). The planar counter-part of a Poisson-
 132 Baecher disc model is a so-called Poisson-Boolean segment process (Chiu et al., 2013),
 133 where the planar points x_n form a planar homogeneous Poisson process⁴ and the objects
 134 X_n are linear segments. Some modern object models for FN are discussed in Stoyan (2021),
 135 focusing on the papers Koike et al. (2015); Ivanova et al. (2014); Cherpeau et al. (2010);
 136 Cherpeau and Caumon (2015); Bonneau et al. (2016).

137 In the present paper, planar FN are considered, which can be understood as result-
 138 ing from planar sections through three-dimensional FN. In this planar setting, FN means
 139 ‘planar FN’, and the fracture traces are approximated by straight-line segments X_n (in
 140 the following: segments). The corresponding location points x_n are the segment centers
 141 in the considered plane. This is a pure geometrical choice, since the segment centers are
 142 hardly related to points where the segments start to grow. The segments are character-
 143 ized by two real marks: their length l_n and azimuth angle a_n . The a_n are angles ($\in [0^{circ}, 180^{circ}]$)
 144 between the north and segment direction, measured in clockwise order. Thus, each frac-
 145 ture is uniquely described by a marked point $[x_n; (l_n, a_n)]$. In this way an FN is mod-
 146 eled by a random collection of marked points.

147 Such collections are called in mathematics marked point processes. These are ran-
 148 dom sequences of location points x_n associated with marks, which are in this case vec-

³ In an homogeneous Poisson process, points are uniformly distributed in space.

⁴ Indeed, it can be shown that the planar section through a Poisson-Baecher disc model yields a collec-
 tion of segments, which centers form a planar homogeneous Poisson process (Chiu et al., 2013).

149 tors (l_n, a_n) . The theory and statistics for marked point processes are described in books
 150 such as Illian et al. (2008) and Baddeley et al. (2016). There are many distributional char-
 151 acteristics of marked point processes, which are explained in this paper when needed.

152 We mention here that there is an alternative approach to statistical modeling and
 153 analysis of FN. Instead of individualize an FN as a set of objects, the whole FN can be
 154 taken as a sample of a segment system. In this way, there is no need to define objects,
 155 but some information about the geometry, about the inner organization and perhaps also
 156 about the genesis of an FN may be lost.

157 This mathematical idea goes under the title ‘segment process’ as in Chiu et al. (2013),
 158 chapter 8, and Stoyan (2021). Some characteristics classically used by geologists can be
 159 seen as belonging to this theory. These are the rose of directions (when including the length
 160 of traces) and the total length of fracture traces per area unit. Also studies of intersec-
 161 tions of FN with scan-lines and statistical investigations of sequences of intersection points
 162 can be considered following segment-process reasoning.

163 2.2 Fractures and Fracture Branches

164 This paper describes FN by patterns of segments that stand for fractures or for parts
 165 of a fracture called ‘branches’.

166 For the construction of segments that represent fractures, we connect their two end-
 167 points by straight lines and ignore deviations from a linear form (Figure 1 (a) and (b)).
 168 This representation is not always unique since it is not always clear where a fracture ends
 169 (Figure 1 (b1) and (b2)).

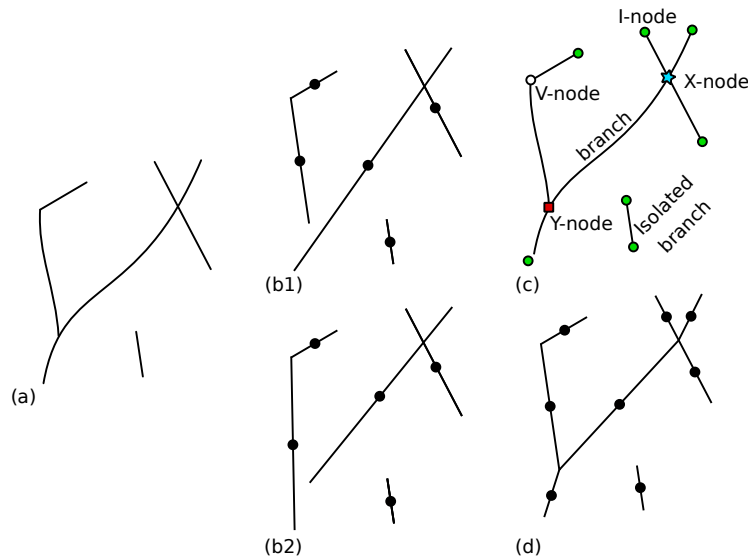


Figure 1. A small piece of a planar FN described as a marked point pattern. (a) A planar FN, (b1) and (b2) two possible representations of the FN in (a) with segments standing for fractures, while the points are fracture centers, (c) topological scheme with branches and nodes, (d) unique representation of the FN in (a) with segments standing for fracture branches; now the points are branch centers.

170 The definition of branches uses the topological scheme introduced by Sanderson
 171 and Nixon (2015); Saevik and Nixon (2017). The original FN is equipped with nodes of
 172 four types that characterize the topology of the FN. The four types are, see Figure 1 (c):

- 173 • I-node: isolated end-point of a fracture,
- 174 • V-node: connects two fractures at their tips,
- 175 • Y-node: connects two fractures, one stops on the another (not at its tip),
- 176 • X-node: crossing point of two fractures.

177 The original fracture branches are curves between nodes. For the statistical anal-
 178 ysis these branches are replaced by straight-line segments that connect the two end-points
 179 of the original branches (Figure 1 (d)). In the following, we simply call these segments
 180 ‘branches’.

181 The approximating segment differs from the geometry of the original fracture and
 182 branch; the deviation is usually smaller for branch than for fracture.

183 Each segment is considered as an object in the sense of object models. Its center
 184 is the corresponding point x_n , and its geometrical information is condensed in two real
 185 numbers (l_n, a_n) , length and azimuth, as defined in Section 2.1. This leads to the clas-
 186 sical case of marked point processes with real-valued marks, and the standard methods
 187 from point process statistics can be applied.

188 In practical work, the determination of the points x_n for fractures and branches
 189 is a non-trivial problem because of edge-effects. Our solution is explained here for frac-
 190 tures: If a fracture is fully in the window, the x_n is simply the center of the correspond-
 191 ing segment. If a fracture is not fully in the window, there are two options according to
 192 the size of the segment inside. Either it is shorter than the median of fracture length dis-
 193 tribution, then no point is constructed, or it is longer and x_n is the center of the segment-
 194 part in the window. Note that not constructing a point for a segment does not mean that
 195 it is excluded from the statistics. The segment is considered in the first-order character-
 196 ization, but excluded from the second-order characterization.

197 The question of which objects one should use remains: fractures or branches? The
 198 decision matters because the second-order statistics are carried out for the object cen-
 199 ters and their marks—and the patterns of the corresponding centers differ greatly, in num-
 200 ber and distribution. It is obvious that fractures are often very long and their centers
 201 are therefore in large distances from big parts of the objects they represent. In contrast,
 202 the branches are shorter and their centers are closer to the ends. The following shows
 203 that both approaches have their benefits, depending on the nature of the FN considered.

204 2.3 Homogeneity

205 In statistical analyses of spatial data the statistician often targets global conclu-
 206 sions for the whole window of observation. In the FN context this may hold for the global
 207 density of fractures. Similarly, the methods discussed in this paper can be used to ob-
 208 tain statements about global variability of FN. Of course, this only makes sense if ev-
 209 erywhere in the window there are the same chances to observe similar fluctuations.

210 The mathematical formulation of this property goes under the name ‘homogene-
 211 ity’. Because of the wide application of this idea, there are equivalent names for the prop-
 212 erty, namely ‘statistical homogeneity’ and ‘stationarity’. The latter is used in the con-
 213 text of point processes. Here, we use the term ‘homogeneity’.

214 A geometrical structure is called *homogeneous* if the statistical distributions of its
 215 elements are invariant with respect to spatial translations (Chiu et al., 2013) For a marked
 216 point process, in particular, this implies that (1) the local point density fluctuates ev-
 217 erywhere around the same value and (2) the marks show everywhere similar variability.
 218 Then, it makes sense to speak about a global value of point density, and a global dis-
 219 tribution of mark values.

220 It is clear that homogeneity is a mathematical concept, which may be true for the-
 221oretical stochastic models of FN. When a real FN is investigated, it is a fundamental de-
 222cision of the geologist together with the statistician to decide if the FN can be consid-
 223ered as a homogeneous structure. This assumption is not justified if there are, for ex-
 224ample, clear spatial trends in fracture density or erratic deformations of the FN. The pa-
 225per Sanderson and Peacock (2019) with its studies of fracture swarms and corridors is
 226a good example of statistics in the case of inhomogeneity.

227 Sometimes, an FN can be partly affected by a local deformation. In that case, the
 228homogeneity hypothesis may call for a restoration to remove translation and/or rotation
 229due to such a deformation (Bergbauer & Pollard, 2004; Macé et al., 2004). Geologists
 230may keep that in mind and use their knowledge to isolate sub-regions where the homo-
 231geneity assumption is acceptable. Bistacchi et al. (2020) give an excellent example of such
 232a regionalization. On the other hand, the size of the window has some influence, a small
 233part of a large homogeneous structure may look like an inhomogeneous pattern.

234 There exist another geometrical invariance property called *isotropy*, which means
 235invariance of distributions with respect to rotations. For FN, this property appears to
 236be totally unrealistic, perhaps with the exception of micro-structures. However, we use
 237statistical summary characteristics that belong to the isotropic case and consider them
 238as directional averages.

239 2.4 Fracture Density Measures

240 Fracture statistics uses a classical system of numerical measures to quantify the amount
 241of fracturing in rock masses (Dershowitz & Herda, 1992; Dershowitz et al., 2020). It is
 242(implicitly) based on the assumption of homogeneity, and therefore care is necessary in
 243its application. The characteristics are given in a notation system which contains sym-
 244bols of the form P_{ij} , where ‘ P ’ means ‘persistence’ (Dershowitz & Einstein, 1988), i de-
 245notes the dimension of the measurement region and j the fracture attribute. Here, we
 246only discuss the planar characteristics, namely *fracture density* P_{20} and *fracture inten-*
 247*sity* P_{21} .

248 The first denotes the mean number of fractures per area unit. In object process think-
 249ing, where each fracture is represented by a marked point, this is a fundamental point-
 250process characteristic. This characteristic is called ‘intensity’ in the point process the-
 251ory, and corresponds to the mean number of points per area unit. Here, we use geolog-
 252ical terms and notation to discuss the number of objects per area unit. Different values
 253of P_{20} appear when for the same pattern fractures and branches are considered in par-
 254allel. For the statistical estimation of P_{20} , it is recommended to count the number of lower
 255end-points of segments, and then to divide it by the window area, see e.g. Chiu et al.
 256(2013) p. 250, Zeeb et al. (2013), and Sanderson et al. (2019).

257 The second characteristic, fracture intensity P_{21} , is the mean of total fracture trace
 258length per unit area. Various techniques exist for its statistical determination, see Chiu
 259et al. (2013); Zeeb et al. (2013). Starting with a map of the surface exposure, methods
 260of image analysis enable direct determination of the total length of all fractures even if
 261they are curved. If the data are given as sets of linear segments, it is even simpler. The
 262values of P_{21} for fractures and branches may differ a little.

263 2.5 Lengths and Orientations

264 The lengths l_n of fractures or branches are important quantities of FN. Their val-
 265ues are characterized by a distribution function, the *fracture (or branch) length distri-*
 266*bution function*. In the theory of marked point processes, such a distribution is called
 267a mark distribution.

268 The best and simplest method to estimate the mean length of fractures (or branches)
 269 is to use the total length of all segments in the window and divide it by the total num-
 270 ber of lower end-points. In this way, edge-effects are eliminated. In contrast, the precise
 271 estimation of length distributions is a difficult task, because of edge-effects, which play
 272 an important role for small windows. We recommend and use in this paper the Miles-
 273 Lantuejoul estimator with minus-sampling, which is explained in Appendix A.

274 Various types of theoretical distributions have been used for the lengths, perhaps
 275 the power-law is still the most common choice, but log-normal, exponential and gamma
 276 distributions are also used, see Bonnet et al. (2001).

277 Since segments are fully characterized by position, length and orientation, orien-
 278 tations are also indispensable characteristics in the study of FN. Statistically it makes
 279 little sense to study a directional mark distribution of the a_n . This way, short and long
 280 fractures or branches would count equally. Therefore FN statistics collects the directional
 281 data in the well-known *rose of directions* or *rose diagram*, which is simply a length-weighted
 282 distribution of directions.

283 The orientations of fractures are closely related to the genesis of FN. Natural FN
 284 are composed of collections often called ‘fracture sets’, which contain fractures with sim-
 285 ilar orientation, determined by the global stress responsible for the fracturing event. These
 286 fracture sets play a central role in this paper.

287 2.6 Fracture Spacing and Scan-Lines

288 A classical method to study the spatial organization of FN is analysis of fracture
 289 spacing (distances between fractures) along scan-lines. These lines are often oriented per-
 290 pendicularly to the main fracture strike azimuth. The intersection points with fractures
 291 on the lines are statistically analyzed using methods for point processes on the real line.

292 This means to study the distances between successive points, and to consider the
 293 corresponding distribution functions and probability densities, assuming homogeneity
 294 (Bistacchi et al., 2020). Deviations from the homogeneous one-dimensional Poisson pro-
 295 cess, the model of complete randomness of point distribution, are of special interest. In
 296 that case, the distance distribution is an exponential distribution and the distances are
 297 independent. As a measure of comparison the coefficient of variation Cv^5 is used (Gillespie
 298 et al., 1993, 1999). Indeed, $Cv = 1$ indicates exponential distribution, while $Cv < 1$
 299 indicates a tendency to regularity, and $Cv > 1$ an irregular distribution such as clus-
 300 tering.

301 The paper Shakiba et al. (2022) introduced second-order statistics for fracture spac-
 302 ings: the one-dimensional analogue of Ripley’s K -function (Section 3.1). This way clus-
 303 tering and regularity can be described better than by simple comparison with the Pois-
 304 son process.

305 The paper Sanderson and Peacock (2019) is an example of analysis under inhomoge-
 306 neous conditions. There the spacings are analyzed in order to characterize swarms and
 307 corridors of fractures.

308 2.7 Connectivity

309 The connectivity of an FN is an important aspect of their characterization because
 310 of its impact on the physical behavior of fractured rock. A complete review of all con-
 311 nectivity measurement is out of the scope of this paper. The interested reader may refer
 312 to Manzocchi (2002); Renard and Allard (2013); Sanderson and Nixon (2018) and

⁵ Cv is the standard deviation of the distribution of spacing divided by its mean.

the references therein for an overview of interesting connectivity metrics. The number of intersections per fracture (C_f) has been largely used particularly in the context of percolation theory (Balberg & Binenbaum, 1983; Berkowitz, 1995; Manzocchi, 2002).

More generally, most of connectivity metrics can be defined in the frame of graph theory. The idea behind is to quantify and characterize intersections of fractures. The topological scheme presented in Section 2.2 is in the spirit of graph theory, where the nodes of the graph are fracture intersections and end-points (I, V, Y and X nodes) and where fracture branches are graph edges. This representation has been used to classify FN using the relative proportion of I,Y and X nodes (Manzocchi, 2002; Sanderson & Nixon, 2018; Sanderson et al., 2019).

In the present paper, we use both the average number of intersections per fracture C_f and the proportions of I,Y and X nodes.

3 Second-Order Characteristics for FN

Second-order characteristics describe the variability of random structures; *variance* and *correlation* are standard terms in this context. Indeed, the variability of a point pattern could be described by the variance of point number in a test circle randomly placed in the pattern. However, the variance obtained would depend on the circle radius; and when using, for example, test squares, other values of variances would be obtained. Therefore, the statistical theory of point processes uses another approach and employs various correlation functions. These functions can be used to calculate, by integration, the variances of random variables (e.g., the number of points or sum of marks) in a test area defined by its size and shape (Chiu et al., 2013; Illian et al., 2008).

The main application of these functions in spatial statistics is the direct characterization of certain aspects of spatial variability. The corresponding statistical estimators work in the center-to-center manner, using point pairs, their inter-point distances and their marks. These distances are then statistically analyzed.

In the present paper, marked point processes are always considered to be homogeneous but anisotropic. Nevertheless, in the explanation of the statistical estimators, isotropy is assumed for better understanding. Also, the corresponding isotropic characteristics play some role as directional averages.

In the following part, we explain various second-order characteristics for marked point processes. We start with characteristics that describe only the points, ignoring the marks first. These are Ripley's K -function $K(r)$ and the pair correlation function $g(r)$. The marks are integrated in mark-correlation functions and in anisotropic pair correlation functions $g(r, \theta)$. The latter, which employ essentially segment orientations, are the most important tools in the paper.

3.1 Ripley's K -Function

The explanation of Ripley's K -function $K(r)$ is as follows: Consider a circle of radius r centered at a randomly chosen point of the point process and count the number of points within the circle, not counting the circle center. This number is random, its mean is denoted by $P_{20}K(r)$ and depends on the radius r . Given fracture density P_{20} , this defines $K(r)$.

Clearly, $K(r)$ is an increasing function of r . Its form gives valuable information on the nature of the point distribution. However, since its definition is based on circles, it is a characteristic of an isotropic nature.

For many point process models the form of $K(r)$ is theoretically known. Here, we mention only the simplest case, the homogeneous Poisson point process, the model of complete spatial randomness. For this model with its strong independence properties it holds simply

$$K(r) = \pi r^2 \text{ for all } r \geq 0. \quad (1)$$

For cluster processes Ripley's $K(r)$ has values larger than πr^2 , while for regular processes the values are smaller.

The statistical estimation of $K(r)$ follows its definition. Here, we describe a naive estimator without edge-correction (thus with some bias) for $K(r)$. Assuming that there are n points x_i in a window W of area $A(W)$, it is plausible to estimate $P_{20}K(r)$ by

$$\mathbf{K}(r) = \frac{1}{n} \sum_{i=1}^n N_i(r), \quad (2)$$

where $N_i(r)$ is the number of points in the circle of radius r centered at x_i (not counting the circle center x_i). An estimator of $K(r)$ is then obtained dividing by an estimator of P_{20} , namely $n/A(W)$. Better estimators of $K(r)$ are described in the literature on point process statistics (Illian et al., 2008; Baddeley et al., 2016).

Clearly, the sum in $\mathbf{K}(r)$ is the same as $n_p(r)$, the number of point pairs in W whose inter-point distances are smaller than r , which is used in fractal statistics (Hentschel & Procaccia, 1983). Then

$$\mathbf{K}(r) = n_p(r)/n. \quad (3)$$

Thus $K(r)$ is closely related to the 'two-point pair correlation function' $C_2(r)$ (Bonnet et al., 2001) defined by

$$C_2(r) = \frac{2n_p(r)}{n(n-1)}. \quad (4)$$

This $C_2(r)$ is used to determine a fractal dimension for fractal point systems: If it scales with r as r^D then the exponent D is called 'correlation dimension'. For dimension estimation, Agterberg (2014) recommends to use edge-corrected estimators of point process statistics, or the pair correlation function (see the discussion below).

3.2 Pair Correlation Functions

Pair correlation functions ('pcf') are the main statistical tools we apply in the present paper. We start with an explanation of the isotropic pcf, which ignores marks and assumes isotropy of the point pattern.

The isotropic pcf $g(r)$ can be formally defined as the derivative of the K -function:

$$g(r) = \frac{K'(r)}{2\pi r}. \quad (5)$$

Thus between $K(r)$ and $g(r)$ there is a relation similar to that between a cumulative distribution function $F(x)$ and the corresponding probability density function $f(x)$.

A heuristic explanation of $g(r)$, which may help in interpretation, is as follows. Consider two deterministic points x and y at distance r . These points are the centers of the infinitesimally small circles $b(x)$ of area dx and $b(y)$ of area dy . The probability to have a point in $b(x)$ is $P_{20} \cdot dx$ and for $b(y)$ it is $P_{20} \cdot dy$. Consider now the probability that there is a point both in $b(x)$ and $b(y)$. Under isotropy assumption this probability depends only on the distance r between x and y . Denote this probability by $p(r)$ and write it as:

$$p(r) = g(r) \cdot P_{20}dx \cdot P_{20}dy \quad (6)$$

The factor $g(r)$, which contains the dependency on r , is the isotropic pcf.

401 In the case of a homogeneous Poisson process, the multiplication theorem of prob-
 402 ability theory yields $p(r) = P_{20}dx \cdot P_{20}dy$ and $g(r) \equiv 1$ because of the independence
 403 property of the process.

404 For all reasonable isotropic point processes the function $g(r)$ takes for large r the
 405 value 1. In fact, for large r , the events ‘there is a point in $b(x)$ ’ and ‘there is a point in
 406 $b(y)$ ’ become independent. Maxima and minima of $g(r)$ mark characteristic values of r ,
 407 related to typical inter-point distances. Values larger than 1 for small r indicate clus-
 408 tering, while values smaller than 1 appear in the case of regularity.

409 A pcf can have very large values for small r in the case of strong clustering. The
 410 case of $g(0) = \infty$ is even possible for some point process models. In this case one speaks
 411 of a pole at $r = 0$. If it is known that the point pattern can be considered as fractal,
 412 and if $g(r)$ scales for small r with r^{-H} , then its fractal dimension D is $D = 2 - H$. How-
 413 ever, non-fractal point processes may also have a pcf with a pole at $r = 0$. Therefore,
 414 the existence of a pcf pole is not a proof of fractality (Stoyan, 1994).

415 For the estimation of $g(r)$ there exists established software (Baddeley et al., 2016),
 416 whose fundamentals are explained in Illian et al. (2008). The main idea is counting the
 417 number of pairs of points in a given inter-point distance r .

418 The interpretation of empirical pcfs is not simple, but we will provide the required
 419 guidance and interested readers may refer to Illian et al. (2008) for a thorough expla-
 420 nation.

421 In the form presented here $g(r)$ is a characteristic for the isotropic case. When we
 422 use it for anisotropic patterns, we speak about the ‘isotropic pcf’. It is understood as
 423 an average over all directions and serves as a characteristic giving global inter-point dis-
 424 tance information.

425 However, when we want to include orientations, we work with an anisotropic ver-
 426 sion of the pcf. The *anisotropic pair correlation function* $g(r, \theta)$ was introduced in Stoyan
 427 and Stoyan (1994), p. 286, and is explained in Appendix B. It is statistically estimated
 428 by counting the number of pairs of points of a given inter-point distance r and connected
 429 by a line of azimuth approximately equal to θ . We use a private implementation for its
 430 estimation, which is a simple modification of the standard algorithm for $g(r)$.

431 3.3 Mark Correlation Functions and Variograms

432 The mark correlation function $k_{mm}(r)$ characterizes aspects of spatial correlation
 433 for a real-valued mark. (The reader may associate the term ‘mark’ with ‘segment length’.)
 434 The variable ‘ r ’ is as above the distance between points. Here, we explain $k_{mm}(r)$ as we
 435 did for $g(r)$ for the isotropic case.

436 The term $k_{mm}(r)$ is the mean of the product of the marks held by two randomly
 437 chosen points separated by a distance r . The estimator considers every pair of points ap-
 438 proximately separated by a distance r , and multiplies their marks. The mean of these
 439 products is divided by a pair correlation estimator, and by the square of the mean mark
 440 $m(l)$. Therefore, $k_{mm}(r)$ has the nature of a conditional mean. A more precise defini-
 441 tion can be found in Illian et al. (2008).

442 If the marks are independent, $k_{mm}(r)$ has the theoretical value of 1 for all r . For
 443 small r , $k_{mm}(r)$ values smaller than 1 indicate that close segments tend to have shorter
 444 lengths than the mean length of segments in the FN. For large r , as mark values become
 445 independent, $k_{mm}(r)$ tends to the theoretical value of 1.

446 Statistical experience shows that it can be accepted to use mark correlation func-
 447 tions in the analysis of anisotropic patterns. In fact, they are obtained by ratio-estimators,
 448 where unwanted fluctuations of numerator and denominator are canceled out.

449 The *mark variogram* $\gamma_m(r)$ is a function similar nature to $k_{mm}(r)$. Instead of the
 450 product of marks the half squared difference is used, and the function is not normalized.
 451 The mark variogram can also be used in anisotropic cases, since the absolute values of
 452 marks do not play a role but only mark differences.

453 For large r , the theoretical value of a mark variogram is equal to the variance σ_m^2
 454 of the marks. Values of $\gamma_m(r)$ smaller than σ_m^2 for small r show that the marks of close
 455 points tend to be similar.

456 The statistics of mark correlation functions are described in Illian et al. (2008).

457 **4 Analysis of an Artificial FN**

458 In this section, we study the Poisson-Boolean segment process (Section 2.1) as a
 459 very simple stochastic model of an FN, where the segments stand for fractures. This is
 460 an analysis under ideal conditions, in order to serve as a benchmark and to help under-
 461 standing the first- and second-order characteristics of an FN with a high degree of ran-
 462 domness.

463 We consider a special Poisson-Boolean segment process, with intensity $P_{20} = 0.2$,
 464 where all segments have the constant length of 5 m and randomly the three directions
 465 0° , 90° and 130° with equal frequency.

466 The fracture centers were simulated in a 105×105 square meter window, as uni-
 467 form random points of density $P_{20} = 0.2$ points per square meter. While the window
 468 W in which we operate is 100×100 , we started with the larger window in order to avoid
 469 edge-effects. So also segments with center outside of W can contribute.

470 The segments for branches are built from the same realization considering the in-
 471 tersections of fractures.

472 Figure 2 shows a typical simulation of the model. The fractures are highly connected
 473 and produce many branches. We counted only 9.0% isolated fractures. The three sub-
 474 sets of fractures or branches with similar orientations are called N0, N90 and N130.

475 **4.1 First-Order Characteristics**

476 We apply the methods described in Section 2 for the statistical analysis of the sim-
 477 ulated pattern.

478 While the fracture density P_{20} is close to the theoretical value 0.20 (with a small
 479 statistical deviation), it is much larger for the branches, namely 0.75 points per square
 480 meter.

481 Concerning the marks, because the fractures are exact linear segments, the roses
 482 of directions of fractures and branches are identical, and follow the three chosen direc-
 483 tions. In contrast, the distributions of lengths for fractures and branches differ (Figure 3).
 484 While the fractures have the constant length of 5 m, the lengths of branches are vari-
 485 able and show a strongly asymmetric distribution, with a median (1.01 m) significantly
 486 smaller than the mean (1.36 m). Following the recommendations discussed in Section 2.5
 487 and Appendix A, objects not completely in the observation window are not used to es-
 488 timate the length distribution. But they are used for estimating P_{21} .

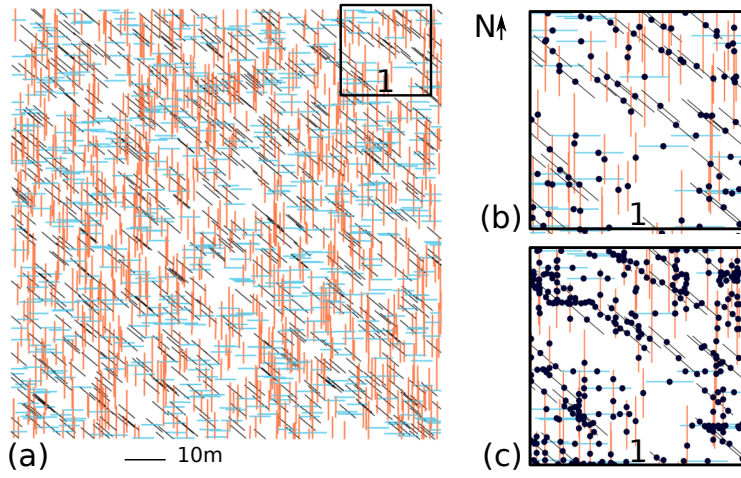


Figure 2. (a) 2D map of the Artificial FN simulated with three fracture sets oriented $0^\circ \pm 5^\circ$ (red), $90^\circ \pm 5^\circ$ (blue) and $130^\circ \pm 5^\circ$ (black). The black square (1) localizes the area shown in (b) and (c). (b) Fracture centers: points of a homogeneous Poisson process. (c) Branch centers constructed by the median rule: a clustered point pattern.

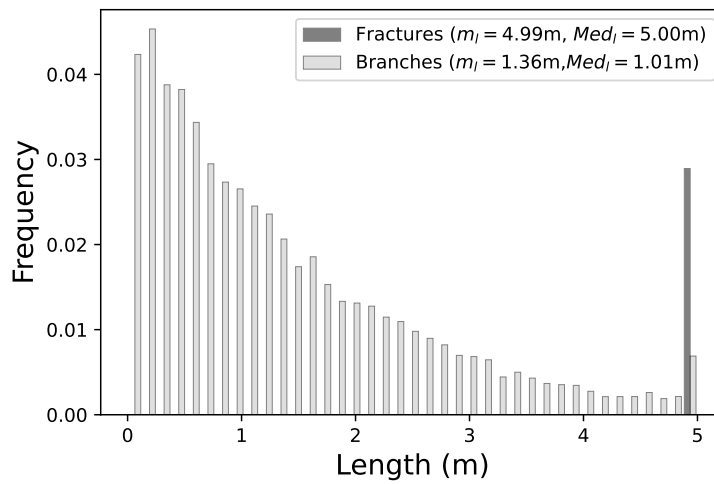


Figure 3. Length distributions of fractures and branches for the Artificial FN (Figure 2).

Table 1. First-order characteristics of the Artificial FN

| | Full set | Fractures | | | Fracture Branches | | | |
|------------------------------------|----------|-----------|--------|--------|-------------------|-------|-------|-------|
| | | N0 | N90 | N130 | Full set | N0 | N90 | N130 |
| P_{20} (nb.m ⁻²) | 0.205 | 0.0689 | 0.0677 | 0.0680 | 0.750 | 0.268 | 0.253 | 0.230 |
| P_{21} (m ⁻¹) | 1.02 | 0.340 | 0.340 | 0.340 | 1.02 | 0.340 | 0.340 | 0.340 |
| m_l mean length (m_l) (m) | 4.99 | 4.94 | 5.02 | 5.01 | 1.36 | 1.27 | 1.35 | 1.48 |
| Med_l median length (m) | 5.0 | 5.0 | 5.0 | 5.0 | 1.01 | 0.96 | 0.98 | 1.12 |
| Connectivity | | | | | | | | |
| C_f (nb.fracture ⁻¹) | 2.52 | 2.76 | 2.61 | 2.21 | | | | |
| $pI-nodes$ | 0.44 | 0.41 | 0.44 | 0.48 | | | | |
| $pY-nodes$ | 0 | 0 | 0 | 0 | | | | |
| $pX-nodes$ | 0.56 | 0.58 | 0.57 | 0.52 | | | | |

489 The empirical first-order characteristics for fractures and branches of the three sets
490 N0, N90 and N130 are presented in Table 1. Consistently with findings in Manzocchi (2002),
491 the probability of occurrence of Y nodes in such a stochastic model is very low: Indeed,
492 no Y nodes are in the simulated Artificial FN.

493 4.2 Second-Order Characteristics

494 This part focuses on the spatial organization of fracture and branch centers. For
495 these centers, here is a special situation, which is never given with natural data: The ex-
496 act fracture centers are known. For validation, we carried out the second-order analy-
497 sis with both the estimated (Section 2.2) and the true fracture centers. We observed no
498 significant differences in the results and report here those from estimated fracture cen-
499 ters. (Note that the median-rule removes all fractures that are not completely inside the
500 window because their length is smaller than the median, which is 5 m.).

501 *Pair Correlation Functions: whole pattern*

502 Figure 4 shows empirical isotropic and anisotropic pcfs for the whole pattern of frac-
503 tures and branches of the Artificial FN. The directions θ considered for the anisotropic
504 pcf $g(r, \theta)$ are those of fracture strike azimuth of the sets N0, N90 and N130.

505 The curves for fracture centers (Figures 4 (a)) have a very simple form since they
506 belong to homogeneous Poisson processes: Constant values with small statistical fluc-
507 tuations around the theoretical value of 1.

508 In contrast, the pcfs for branch centers have a quite different form. They have very
509 large values for small r and tend to 1 for large r . This is the typical form for so-called
510 cluster processes. It even seems to make sense to assume that there are poles of the pcfs
511 at $r = 0$. This behavior can be explained by the spatial arrangement of branches. Branches
512 are the result of fracture intersections, their centers are aligned on fractures, and their
513 lengths are random. Consequently, many branch centers form sub-patterns similar to pieces
514 of linear Poisson processes. Just a model of similar nature is one presented in Stoyan (1994),
515 where the points are randomly scattered on randomly scattered segments. In that model,
516 the pole of the pcf at $r = 0$ is proved mathematically; it does not result from dense clus-
517 ters but from the linear arrangement of points.

518 The pcf values close to 1 for $r > 5$ m are a consequence of the constant fracture
519 lengths: Indeed, for distances larger than 5 m point pairs belong to different fractures
520 and there are no more spatial correlations.

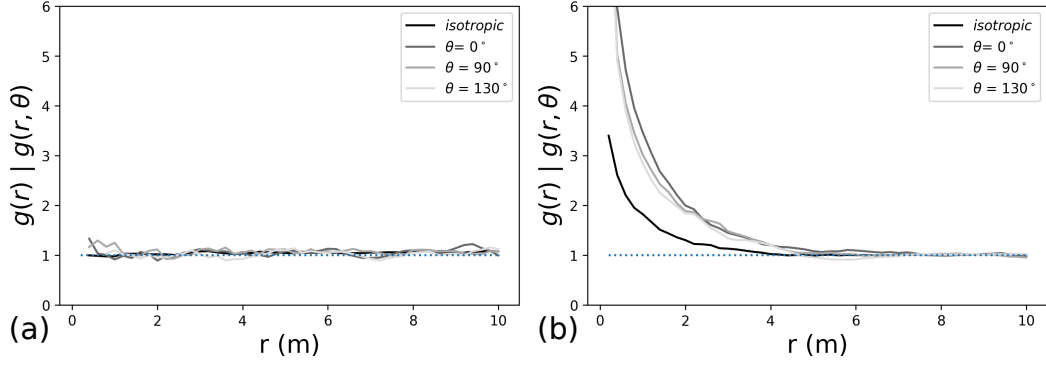


Figure 4. Empirical isotropic pcf $g(r)$ (black curve) and anisotropic pcfs $g(r, \theta)$ (gray curves) for the full set of (a) fractures and (b) branches of the Artificial FN. The blue dot curves indicate the theoretical value of 1.

521 *Pair Correlation Functions: Set N0*

522 Now, we focus on the objects of the set N0, which contains all objects with strike
 523 azimuth 0° shown in red in Figure 2. The empirical isotropic and anisotropic pcfs are
 524 shown in Figure 5. We chose the direction θ corresponding to the azimuth of objects (0°)
 525 and the perpendicular (90°).

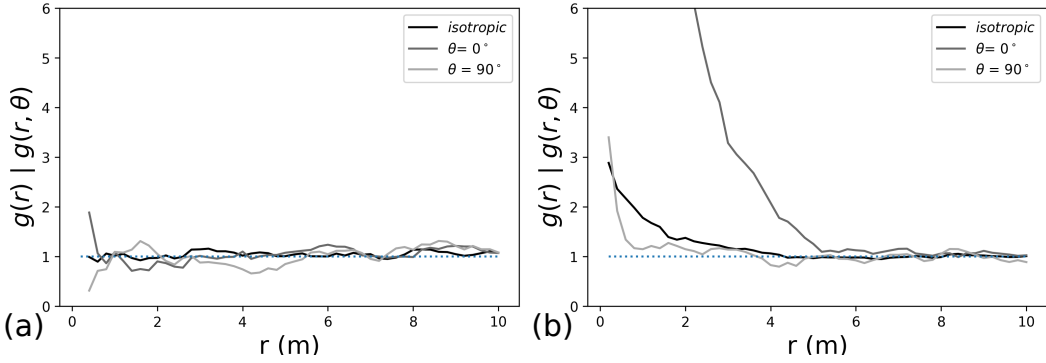


Figure 5. Empirical isotropic and anisotropic pcfs with $\theta = 0^\circ$ and 90° for subset N0 for fractures (a) and branches (b) of the Artificial FN.

526 We observe a similar behavior to the one of the full set, and for the other subsets
 527 N90 and N130 it is similar as well. For fractures, the centers of the N0 subset belong again
 528 to a homogeneous Poisson process and this also holds for the other subsets (N90 and N130).
 529 For branches, the explanation of the large values of $g(r, 0)$ for small r is similar to the
 530 one for the full set. They result from sequences of points aligned on fractures. In the di-
 531 rection 90° , orthogonal to fractures, points are not aligned and $g(r, 90)$ has smaller val-
 532 ues than $g(r, 0)$ for small r .

533 *Mark Correlation Functions and Variograms*

534 The empirical mark correlation functions $k_{mm}(r)$ and mark variograms $\gamma_m(r)$ for
 535 the lengths of fractures and branches of the Artificial FN are shown in 6.

536 For the fractures $k_{mm}(r)$ reflects the simple situation of constant marks: It is con-
 537 stant and equal to 1. In this case, the mark variograms also follow the theory with a con-
 538 stant value equal to 0. Indeed, because the fracture lengths are constant, their differences
 539 vanish and the corresponding variance is zero.

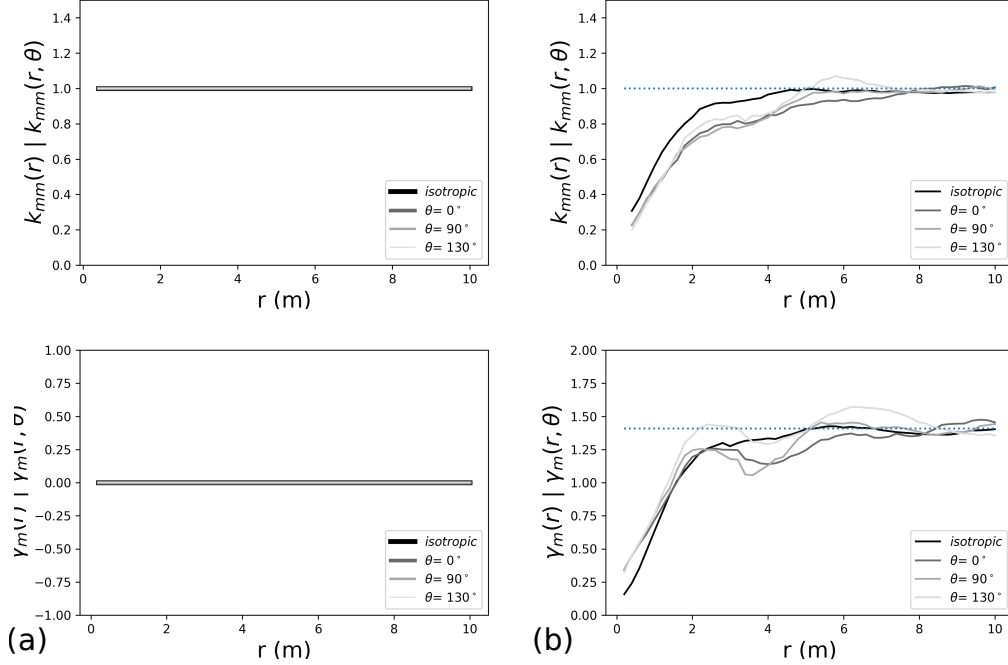


Figure 6. Empirical $0^\circ, 90^\circ, 130^\circ$ anisotropic and isotropic mark correlation function ($k_{mm}(r)$) and mark variogram ($\gamma_m(r)$) for fractures (a) and branches (b) of the Artificial FN. The theoretical values expected for large r are shown as the dotted blue lines.

540 In contrast, for the branches there is non-trivial spatial length-mark correlations,
 541 as shown by Figure 6 (b). The original constant fractures are randomly divided, which
 542 results in random branch lengths, see Figure 3. The form of $k_{mm}(r)$ shows that close branches
 543 tend to have smaller lengths than the average. Indeed, by the construction of branches
 544 frequently short branches are close together, resulting from intersections of the same frac-
 545 ture. Note that the form of the length distribution function shows that almost 62% of
 546 branches are shorter than the mean.

547 The range of correlation is about 5 m, which is explained by the constant length
 548 of fractures. The length variogram $\gamma_m(r)$ has relatively small values for small r , which
 549 indicates similarity of the lengths of close branches, with a range of correlation of 5 m.
 550 The combination of the information given by the two correlation functions yields the state-
 551 ment that close branches tend to be short and similar in length.

552 *General*

553 For the Artificial FN the second-order statistics of fractures well reflects the spa-
 554 tial behavior of a completely random collection of segments. This is shown by the pcf's,
 555 the mark correlation functions $k_{mm}(r)$ and mark variogram $\gamma_m(r)$. In contrast, the re-
 556 sults for branches are more complex. There are spatial correlations, which result only
 557 from the construction of branches and their centers. It simply reveals that branch cen-
 558 ters are randomly aligned on fractures randomly scattered in space.

559 In the following two examples sometimes a similar behavior will be observed, which
 560 would be hardly understood without knowledge of the results for the Artificial FN. Hence
 561 it may serve as a benchmark and an aid for interpretation.

562 5 Analysis of Two Natural FN

563 Two natural fracture patterns are studied in this paper. One presents fractures ob-
 564 served at large scale from a satellite image, which covers 120.000 m^2 of the southern flank
 565 of the Jabal Anhdar dome in the Oman Mountains. This FN has been widely studied
 566 because it can be considered as a typical example of those in large fractured carbonate
 567 reservoirs of the Middle East (Hilgers et al., 2006; Holland, Urai, et al., 2009; Holland,
 568 Saxena, & Urai, 2009; Gomez-Rivas et al., 2014). The map Figure 7 (a) used in this pa-
 569 per is taken from Zeeb et al. (2013) for comparison.

570 The other fracture pattern comes from the Hornelen basin in Norway. The origi-
 571 nal dataset consists of seven maps digitalized at several scales by Odling (1997). We fo-
 572 cus our investigation on the one mapped by hand from an observation height of 4 m by
 573 Odling (1997). The map covers 324 m^2 ($18 \times 18 \text{ m}$, Figure 7 (b)) and describes a small-
 574 scale FN with a high number of small fractures. This is a typical joint pattern, which
 575 may be considered as an example of structures found in a non-stratabound massive rock
 576 (Odling et al., 1999).

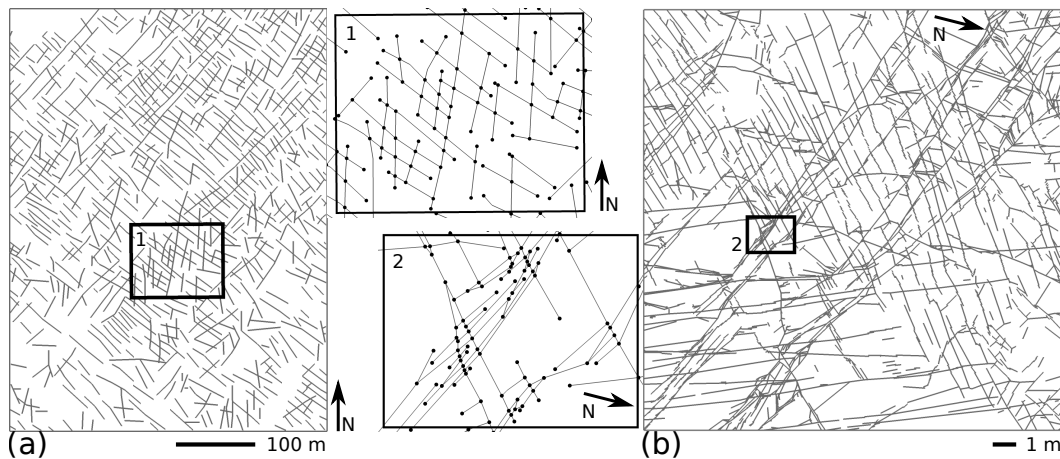


Figure 7. Natural surface exposures from (a) the southern flank of the Jabal Anhdar dome in the Oman Mountains (Zeeb et al., 2013) and (b) the Hornelen basin in Norway (Odling, 1997). The central part of the figure shows zoom of the central part of (1) the Oman dataset and (2) the Hornelen dataset. The I,V,Y, and X nodes are marked as black dots.

577 These datasets were chosen because of their complementary nature in terms of frac-
 578 ture pattern, scale, randomness, and variability. The Oman pattern has a spatially vari-
 579 able fracture density, which is a bit smaller in the lower part. The latter may be con-
 580 sidered as a weak, but acceptable, deviation from homogeneity. For the Hornelen pat-
 581 tern, we decided to consider only sub-samples for the second-order analysis, since devi-
 582 ations from homogeneity turned out to be too strong. Of course, both patterns are clearly
 583 anisotropic.

584 Centers of fractures and branches were constructed by the median rule for segments
 585 not fully in the window of observation.

In the analysis, subsets of objects of similar orientation θ^{circ} are considered, denoted as $N\theta$.

5.1 Statistics for the Oman Fracture Pattern

For the Oman fracture pattern, the map in Figure 7 (a)) was obtained by digitalization using the SKUA-GOCAD software. It was then analyzed by the first- and second-order methods of the present paper.

5.1.1 First-Order Characteristics

The Oman pattern has already been analyzed by Zeeb et al. (2013). Nevertheless, we studied the data again and we confess that we did not exactly obtain the same results. While there were originally 650 fracture traces with lengths ranging between 3 and 179 m, we found 696 fractures with lengths ranging between 1 and 134 m. Furthermore, while Zeeb et al. (2013) reported that approximately 5% of the sampled fractures appear to be cut by the boundary of the observation window, in our digitalization this rate is close to 7%. This illustrates some impact of subjective interpretation; there is some probability that other researchers would produce other maps from the same data.

For an unbiased estimation of the number of fractures, we counted 663 lower endpoints of fractures in the window of 120.000 m^2 . This yields the estimate $5.6 \times 10^{-3} \text{ m}^{-2}$ of fracture density P_{20} . This value is comparable to the result one obtains when using the number of 650 traces reported by Zeeb et al. (2013).

Introducing branches increases the number of objects to 1476, and the lengths range now between 0.5 to 55 m. In parallel, the number of objects hitting the boundary is reduced to approximately 4%. Note that 40% of fractures are isolated (i.e., consisting of one branch bounded by two I-nodes), which means that 40% of the objects are the same in the fracture and branch representation. Segmentation of the other 60% of fractures leads to a density of 1.22×10^{-2} branches per square meter, which is 2.1 times more than for fractures.

The rose of directions (Figure 8 (b)) is computed by weighting object strike azimuths by the corresponding lengths. The result corresponds well to the given chequered pattern, which is strongly anisotropic with two (nearly) orthogonal main fracture sets of orientations 40° and 130° . The fractures associated to the direction 130° are quantitatively predominant. The rose of branch directions is quite similar to the one of the fractures; the small difference is caused by the approximation by segments.

The length distribution of objects was determined for both fractures and branches, see Figure 8 (a). Both distributions are asymmetric with medians clearly smaller than means. Additionally, Table 2 gives some information on the length distributions of the subsets N40 and N130. The mean fracture length of the set N40 is longer than that of N130, and the fractures of N40 are segmented more intensively. This is confirmed by a smaller mean and median of branch length distribution; and larger number of intersections per fracture C_f , which is 2.23 for N40 and 1.10 for N130.

Note that the union of the two sets N40 and N130 does not cover the complete fracture pattern as there exists a minor set with objects aligned to the North. It has been neglected as it contributes only to 10% of total length.

5.1.2 Second-Order Characteristics

Pair Correlation Functions: whole pattern

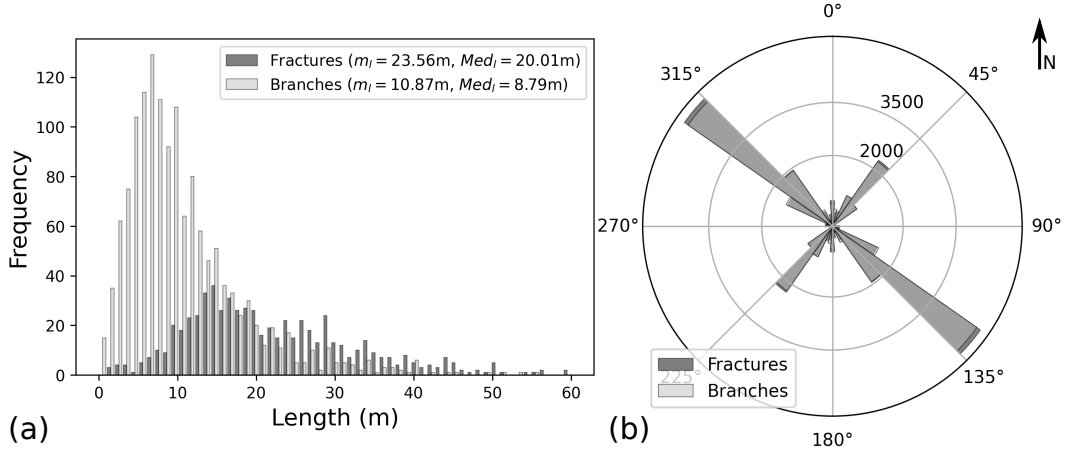


Figure 8. First-order characteristics of the Oman exposure, for fractures (black) and branches (gray). (a) Length distributions (values above 60 m are not plotted), (b) rose of strike azimuths weighted by length (per 10° interval).

Table 2. First-order characteristics of the Oman exposure

| | Fractures | | | Branches | | |
|------------------------------------|-----------|--------|--------|----------|--------|--------|
| | Full set | N40 | N130 | Full set | N40 | N130 |
| P_{20} (nb.m ⁻²) | 0.0056 | 0.0015 | 0.0035 | 0.0122 | 0.0045 | 0.0064 |
| P_{21} (m ⁻¹) | 0.132 | 0.041 | 0.078 | 0.132 | 0.041 | 0.078 |
| mean length (m) | 23.56 | 27.35 | 21.67 | 10.87 | 9.10 | 12.24 |
| median length (m) | 20.01 | 21.66 | 20.00 | 8.79 | 7.05 | 10.29 |
| Connectivity | | | | | | |
| C_f (nb.fracture ⁻¹) | 1.42 | 2.23 | 1.11 | | | |
| $pI-nodes$ | 0.53 | 0.41 | 0.60 | | | |
| $pY-nodes$ | 0.20 | 0.25 | 0.17 | | | |
| $pX-nodes$ | 0.27 | 0.34 | 0.24 | | | |

630 Figure 9 shows the empirical isotropic pcf and two anisotropic pcfs for the whole
 631 pattern of fractures and branches. The directions θ investigated for the anisotropic pcfs
 632 $g(r, \theta)$ are the two main orientations of objects 40° and 130° .

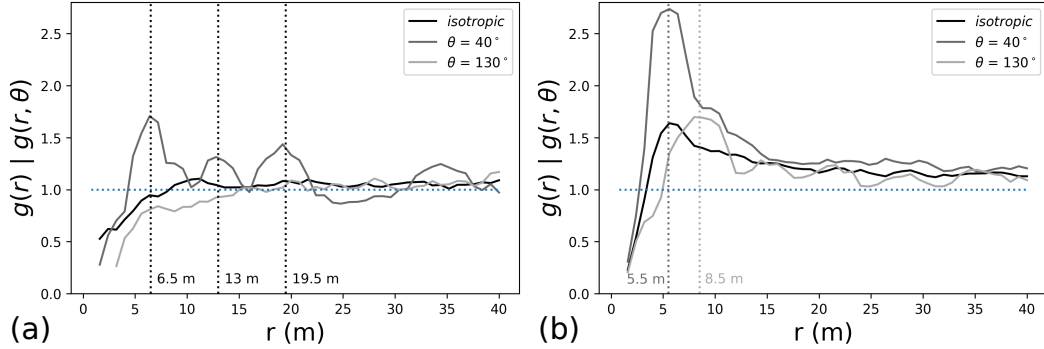


Figure 9. Isotropic pcf (black curve) and anisotropic pcfs (gray curves) for the full set of (a) fractures and (b) branches in the Oman dataset.

633 The pcfs for the fractures in Figure 9 (a) do not deviate largely from the value 1,
 634 which holds for the pcf in the case of complete spatial randomness of segment centers;
 635 the average of both directional pcfs is also close to the constant value of 1. However, the
 636 pcf for the direction 40° deviates a little. The maximum value at $r = 6.5$ m results from
 637 a number of similar inter-center distances close to 6.5 m. This particular shape indicates
 638 some short-range order (See Appendix C for explanation of the terms ‘short-range order’
 639 and ‘long-range order’.) There is even a second and a third maximum at $r = 13$
 640 m and $r = 19.5$ m, which may be seen as a weak trace of long-range order, related to
 641 the visible nearly regular spacing of N130 fractures. Indeed, the pattern in Figure 7 (a)
 642 shows a lot of N130 fracture centers regularly aligned in the direction 40° .

643 The pcfs for the branches in Figure 9 (b) differ clearly from those for the fractures.
 644 They have the typical form of pcfs of point processes with short-range order: All pos-
 645 sess a clear single maximum, which is around $r = 5.5$ m for the isotropic case and for
 646 the direction 40° , but around $r = 8.5$ m for the direction 130° .

647 This behavior can be explained by the chequered nature of the FN. A given frac-
 648 ture is typically intersected by other fractures, which are nearly orthogonal and these
 649 intersection points (which are the ends of branches) form nearly regular sequences. Con-
 650 sequently, the branch centers also form nearly regular sequences, which lead to the form
 651 of the pcfs. Since the intersection points do not form irregular sequences like in the Ar-
 652 tificial FN (Section 4), there is no motive to think about a pole at $r = 0$. The r -value
 653 at the maximum gives information on typical (frequent) inter-point distances and on the
 654 size of the intact rock blocks. The differences in the values of the maxima of the pcfs (2.7
 655 and 1.7) characterize the strength of order in each direction: branch centers are more
 656 regularly spaced in the direction N40 than in N130.

657 In order to obtain more detailed second-order information, we now refine the anal-
 658 ysis by identifying the separate contributions of the sets N40 and N130 to the anisotropic
 659 pcfs.

660 *Pair Correlation Function: Set N40*

661 First we estimated isotropic and anisotropic pcfs for fractures and branches for the
 662 objects belonging to the set N40 (Figure 10). While the isotropic pcfs are close to the

663 typical form of complete spatial randomness, the anisotropic pcfs reflect interesting struc-
 664 ture and reveal real differences in the spatial organization of fractures and branches.

665 In the direction 40° , the pcf $g(r, 40)$ for branches indicates a very high degree of
 666 short-range order by the maximum $g(r, 40) = 10.5$ at $r = 5.5$ m. (Note that the max-
 667 imum value for the isotropic pcf for the whole pattern was clearly lower (2.7) but ap-
 668 peared at the same r value ($r = 5.5$ m).) Perhaps there is even a second maximum at
 669 $r = 11$ m, which may show a trace of long-range order. This behavior reflects statisti-
 670 cally the obvious order of arrangement in direction 40° , which is caused by the nearly
 671 equidistant intersections by the fractures of N130. That means, it reflects the order re-
 672 lated to the nearly regular spacing between N130 fractures.

673 In contrast, $g(r, 40)$ for the fractures indicates repulsion until $r = 25$ m and dis-
 674 order, there is no regular appearance.

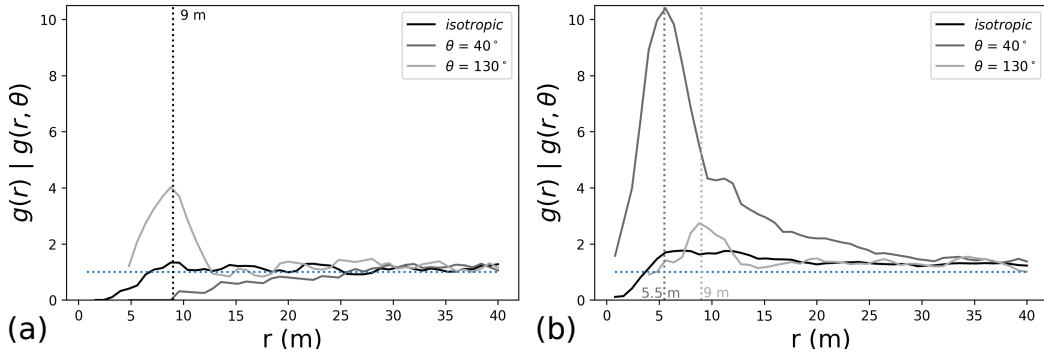


Figure 10. Empirical isotropic and anisotropic pcfs for (a) fractures and (b) branches of the subset N40 in the Oman dataset.

675 We remark that in the orthogonal direction 130° , the pcfs $g(r, 130)$ for fractures
 676 as well as for branches indicate weak short-range order (note the maximum at $r = 9$
 677 m), i.e., they reflect the order related to the nearly regular spacing between N40 frac-
 678 tures.

679 *Pair Correlation Function: Set N130*

680 The pcfs corresponding to the subset N130 (Figure 11) have forms similar to those
 681 for the subset N40, but with lower values for the maxima. (Note the different scales at
 682 the ordinates in Figure 10 and 11.) In particular, the curves of $g(r, 130)$ for branches mark
 683 the difference well: There are maxima at $r = 7.5$ m and $r = 15$ m, but the values at
 684 the maxima are smaller (3.5 at $r = 7.5$ and 1.6 at $r = 15$ m). An interesting fact is
 685 that the curve for branches, also in the orthogonal direction (40°), indicates similar or-
 686 der with maximum values of 2 at $r = 7.5$ m and 1.5 at $r = 15$ m.

687 For fractures the curves of $g(r, 130)$ show repulsion between fracture centers ($g(r) <$
 688 1) until $r = 22$ m and then irregular fluctuations around 1.

689 Finally, in the orthogonal direction 40° , the pcf $g(r, 40)$ for fractures indicates weak
 690 short-range order (indicated by the maxima at around $r = 6.5$ m), related to the nearly
 691 regular spacing between N130 fractures.

692 *Pair Correlation Functions, Lengths and Spacing*

693 For the Oman FN the analysis of branches yields interesting statements about the
 694 spatial distribution of the FN, while for the Artificial FN their role is questionable. In

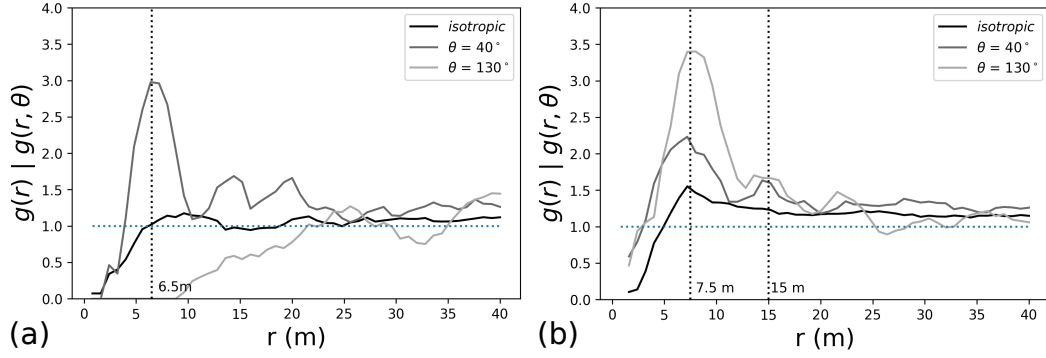


Figure 11. Empirical isotropic and anisotropic pcfs for (a) fractures and (b) branches of the subset N130 in the Oman dataset.

695 fact, the chequered nature of the FN causes that branches may be considered as its nat-
 696 ural constituting elements.

697 The isotropic pcfs reveal little information about the spatial organization of the Oman
 698 FN. However, the anisotropic pcfs describe interesting aspects. In particular, the func-
 699 tions for the branches reflect the regularity caused by regular spacing between fractures
 700 of each sets.

701 A separate analysis of the N40 and N130 fracture set makes the statements even
 702 clearer: The r -values of the maxima of the pcfs for branches indicate very frequent inter-
 703 distance between branch centers, which are slightly different according to the set con-
 704 sidered and the direction investigated. The obtained forms are related to the length of
 705 branches and thus to their spacing. The higher degree of order shown in the direction
 706 40° for branch centers of the N40 fracture set results from the long sequences of nearly
 707 regularly spaced branch centers. In fact, N40 fractures are quite long and also intensively
 708 segmented by N130 fractures, which are also regularly spaced.

709 Such a regularity is visible with fractures but only when considering pcfs in the di-
 710 rection perpendicular to fracture strike azimuth. Then, the spacing between fractures
 711 may be visible. On the contrary, there is some repulsion between fracture centers in the
 712 direction of object strike azimuth. This repulsion extends to the mean fracture length
 713 in each direction. This was expected as fractures of the same set cannot overlap because
 714 of the shadow zones around, see Darcel et al. (2003a).

715 *Mark Correlation Functions and Variograms*

716 Perhaps surprisingly, the empirical mark correlation functions and the mark vari-
 717 ograms for fractures (Figure 12(a)) show a behavior similar to the Artificial FN (Fig-
 718 ure 6(a)). If one ignores statistical fluctuations and deviations from the theoretical val-
 719 ues in the case of independence (which result probably from deviations from homogene-
 720 ity) they look as if they belong to independent length marks (Figure 12 (a)). This dif-
 721 fers from a result of Bour (2002), who wrote “small faults are more clustered than large
 722 ones”⁶. Probably both situations may appear in nature, spatial correlations of fracture
 723 lengths and no-correlations⁷ (Unfortunately, today statistical methods for proving sig-
 724 nificance statements in the context discussed are still missing.)

⁶ In this case the mark correlations would look like the curve for fracture branches (Figure 12(b)).

⁷ See also the mark correlation function for the Hornelen data in Figure 17.

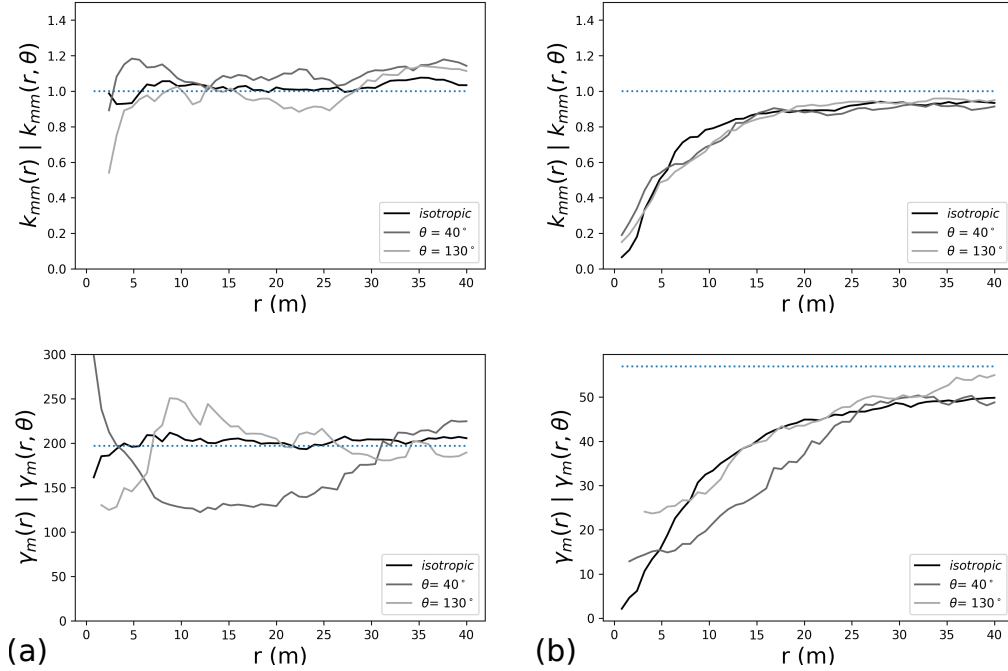


Figure 12. Empirical length-mark correlation function $k_{mm}(r)$ and length-mark variogram $\gamma_m(r)$ for (a) fractures and (b) branches of the Oman exposure. The theoretically expected values for large r or in case of independence are shown as dotted blue lines.

725 The empirical mark correlation functions $k_{mm}(r)$ for branches are shown in Figure
 726 12 (b). They indicate spatial correlations of length marks that can be easily interpreted.
 727 For close branches the lengths tend to be smaller than the average length of branches,
 728 with a range of correlation of 30 m. (This is a clear difference to the case of fractures.)
 729 Also, the empirical length variograms $\gamma_m(r)$ shown in Figure 12 (b) indicate similarity
 730 of lengths for close branches, here again, with a range of correlation of 30 m. The combination
 731 of the information given by mark correlation function and mark variogram leads
 732 to the statement that close branches tend to be short and similar in length. This is plausible
 733 because of the construction of branches and the regularity of fracture spacing already
 734 described.

735 5.2 Statistics for the Hornelen Fracture Pattern

736 5.2.1 First-Order Characteristics

737 The map in Figure 7(b) contains 2109 fractures with lengths ranging between 1 cm
 738 and 11.0 m, 5.4% of fractures end on the boundary. Introducing the branch representation
 739 increases the number of objects to 4350 and reduces the range of object lengths
 740 (between 1.1 mm and 5.6 m) and the proportion of objects cut by the boundary to 2.6%.

741 We counted 2033 fracture lower end-points in the 324 m² window. This edge-corrected
 742 count yields the estimate 6.27 m⁻² of P_{20} . 60% of the fractures are isolated and the segmentation
 743 of the other 40% lead to a density of 13.18 branches per square meter, which
 744 is almost twice the density of fractures.

745 The FN consists of three fracture sets oriented around 50°, 130°, and 170°. We
 746 mention a small difference to Odling (1997), who had for the third fracture set 0° when

747 considering all the seven maps. The rose of branch strike azimuths is, as expected, sim-
 748 ilar to that for the fractures (Figure 13 (b)).

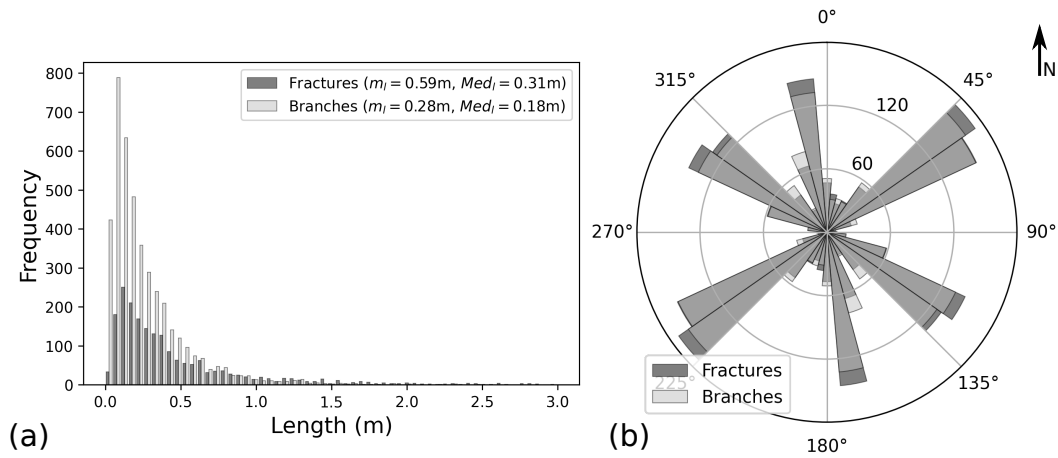


Figure 13. First-order characteristics of the Hornelen exposure, for fractures (black) and branches (gray). (a) Length distributions (values above 3 m are not plotted). (b) Roses of strike azimuths weighted by length (per 10° interval).

749 Here, we used the same statistical analysis as for the Oman data, this time for three
 750 sets of fractures, which we call N50, N130 and N170. As for Oman, a small part of the
 751 FN (9.6%) of the total length is not represented by these three subsets.

752 We learnt that when considering these sub-patterns (Figure 14 (a)) stronger devi-
 753 ations from homogeneity appear. Indeed the objects from the set N130 (green, Fig-
 754 ure 14 (a)) have a higher density in the center of the exposure. Also, those from the set
 755 N170 (red, Figure 14 (a)) have a higher density in the South East of the exposure. Both
 756 sets have rather long fractures. In contrast, the set N50 (blue, Figure 14 (a)) is more dif-
 757 fuse and contains smaller and more isolated fractures. Therefore, we defined the sub-windows
 758 shown in Figure 14 (a) and made the statistical analysis, which assumes homogeneity,
 759 only for these smaller regions.

760 The empirical length distributions are asymmetric (Figure 13 (a)). This is consis-
 761 tent with the findings of Odling (1997), who fitted a lognormal distribution to the frac-
 762 ture lengths of this exposure. The length distributions of branches of the three sub-sets
 763 are presented in Table 3 and Figure 14 (b)).

764 The connectivity measures also reveal that the average number of connections per
 765 fracture of the set N130 is higher than for the sets N50 and N170. The high number of
 766 small fractures in the set N50 may be an explanation for its low connectivity index.

767 **5.2.2 Second-Order Characteristics**

768 *Pair Correlation Functions: whole pattern*

769 The Hornelen FN is characterized by the empirical isotropic and anisotropic pcfs
 770 (Figure 15). The directions are those defining the three fracture sets, 50° , 130° and 170° .

771 The isotropic pcf for fracture centers indicates weak short-range order with a most
 772 frequent inter-center distance of 0.08 m. There is no reason to think of a pole at $r =$
 773 0. For the anisotropic pcfs with $\theta = 50^\circ$ and 130° , we note a weak long-range order re-

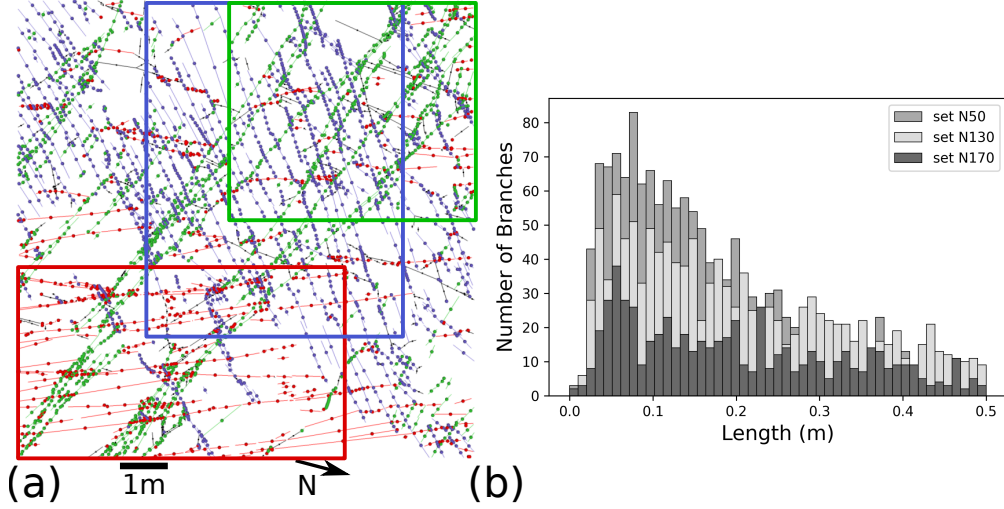


Figure 14. Hornelen surface exposure: (a) Map of the branch centers, N50 (blue), N130 (green), N170 (red), and rest in black. Statistical analysis for these sets is carried out for the three identified sub-regions. (b) Length distributions, N50 (dark gray), N130 (light gray), and N170 (black). The focus is on objects that have a length ranging between 0 and 0.5 m.

Table 3. Hornelen first-order characteristics

| | Full set | Fractures | | | Full set | Branches | | |
|------------------------------------|----------|------------------|-------------------|-------------------|----------|------------------|-------------------|-------------------|
| | | N50 ^a | N130 ^a | N170 ^a | | N50 ^a | N130 ^a | N170 ^a |
| P_{20} (nb.m ⁻²) | 6.27 | 3.23 | 1.76 | 1.63 | 13.20 | 6.32 | 5.65 | 3.89 |
| P_{21} (m ⁻¹) | 3.68 | 1.50 | 1.43 | 1.72 | 3.69 | 1.51 | 1.44 | 1.72 |
| mean length (m) | 0.59 | 0.47 | 0.82 | 1.05 | 0.28 | 0.24 | 0.26 | 0.44 |
| median length (m) | 0.31 | 0.23 | 0.43 | 0.37 | 0.18 | 0.16 | 0.19 | 0.23 |
| Connectivity | | | | | | | | |
| C_f (nb.fracture ⁻¹) | 1.76 | 1.16 | 2.62 | 1.92 | | | | |
| pI -nodes | 0.42 | 0.59 | 0.34 | 0.40 | | | | |
| pY -nodes | 0.41 | 0.26 | 0.46 | 0.37 | | | | |
| pX -nodes | 0.17 | 0.15 | 0.20 | 0.23 | | | | |

^a Statistics for each subset are made on the sub-regions identified in Figure 14 (a).

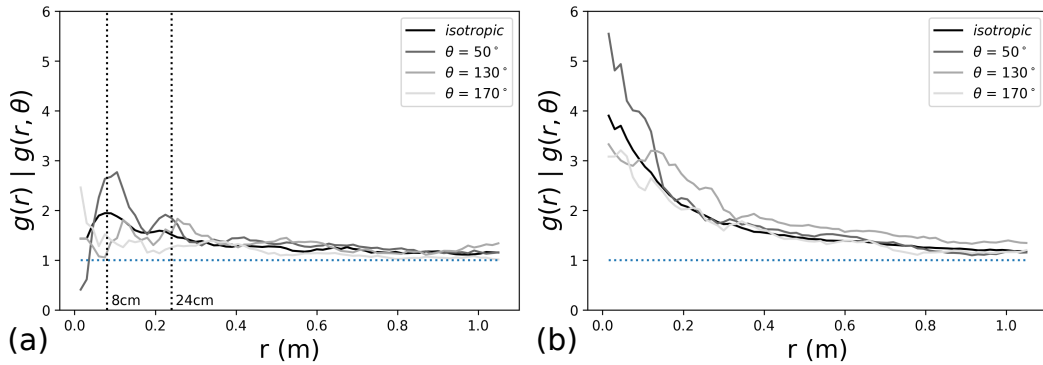


Figure 15. Isotropic pcf (black curve) and anisotropic pcf (gray curves) for the full set of fractures (a) and full set of branches (b) of the Hornelen exposure.

774 spectively with maxima at $r = 0.08$ and 0.24 m for $\theta = 50^\circ$, and at $r = 0.12$ and 0.24
 775 m for $\theta = 130^\circ$. For $\theta = 170^\circ$ the pcf indicates complete spatial randomness.

776 For the branches, the pcfs indicate some form of clustering. The situation here is
 777 similar to the one for the Artificial FN.

778 *Pair Correlation Function: Subsets*

779 We now present empirical isotropic and anisotropic pcfs for the subsets N50, N130,
 780 and N170 separately. As we did for the Oman dataset, we investigate for anisotropic pcfs
 781 the direction defined by the object strike azimuths (dark gray curves Figure 16) and the
 782 perpendicular direction (light gray curves Figure 16).

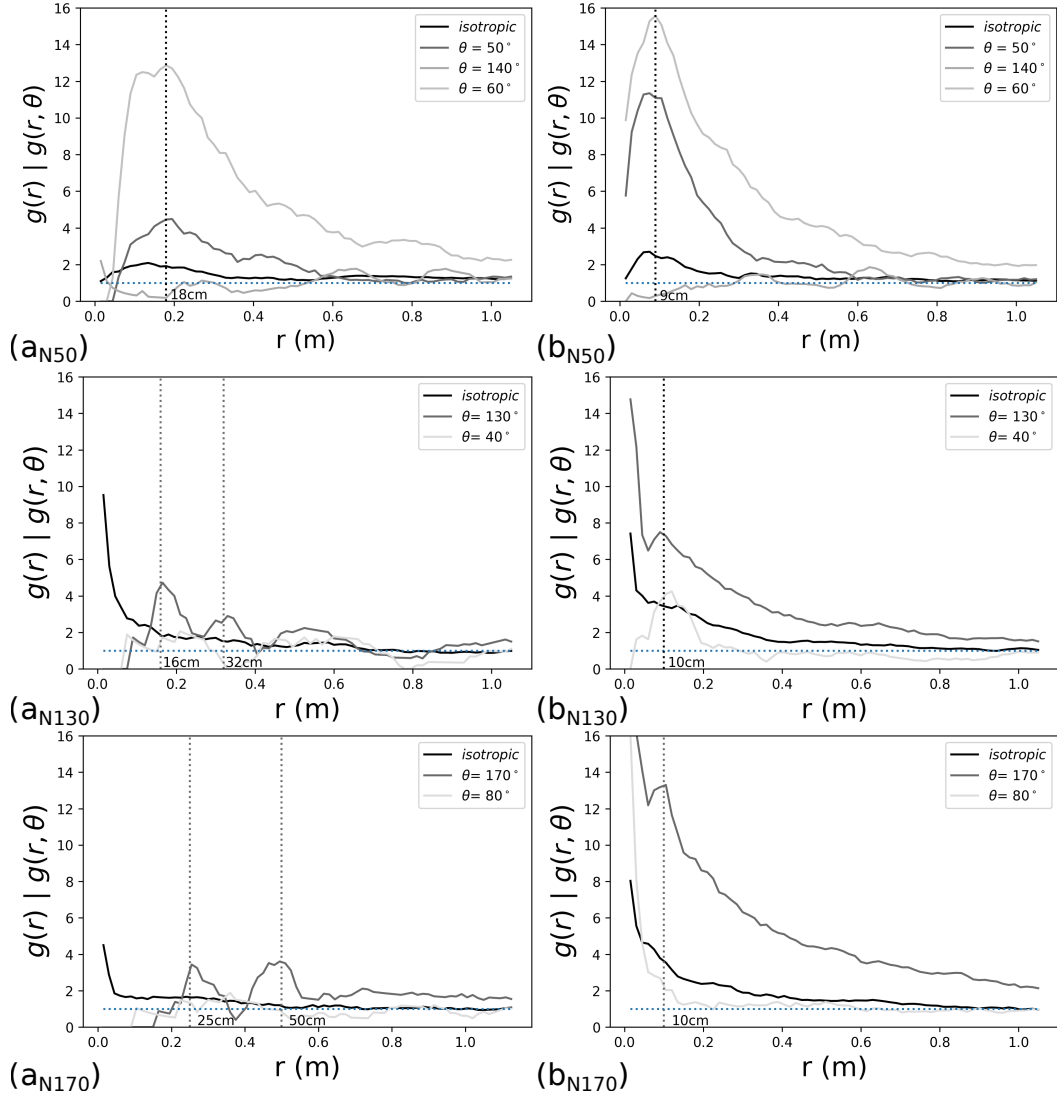


Figure 16. Empirical isotropic and anisotropic pcfs for (a) fractures and (b) branches of the subsets N50, N130 and N170 in the Hornelen dataset.

783 For fractures of the two sets N130 and N170, there are some similarities in their
 784 organization. The anisotropic pcfs in the direction of objects indicate some weak long-
 785 range order with two maxima at $r = 0.16$ and 0.32 m for $\theta = 130^\circ$ (Figure 16 (a_{N130})),

786 and at 0.25 and 0.5 m for $\theta = 170^\circ$ (Figure 16 (a_{N170})). For the set N50, the 50° anisotropic
 787 pcf indicates some short-range order with most frequent inter-center distance $r = 0.18$
 788 m. In all cases, irregularity is shown in the direction perpendicular to objects.

789 For this last set N50, because of en-echelon structures, we observed that many seg-
 790 ment centers are aligned in the direction $\theta = 60^\circ$. Therefore, we investigated addition-
 791 ally this extra direction. The corresponding curve (for $g(r, 60)$, not shown) has a simi-
 792 lar shape as for $\theta = 50^\circ$, but with higher short-range order (Figure 16 (a_{N50})). This
 793 reveals the dominance of en-echelon fractures for this set.

794 For the branches, the pcf curves for both the N130 and N170 subsets indicate clus-
 795 tering of centers. The situation looks like the one observed and explained for the Arti-
 796 ficial FN. The degree of clustering is obviously higher here, because the branch centers
 797 on the fracture traces are even clustered (Figure 7(b)) and not (Poisson-process-like) ran-
 798 dom as with the Artificial FN.

799 The curves for the branches of the N50 fracture set show short-range order with
 800 a very high maximum value of the pcf at $r = 0.09$ m in both directions 50° and 60° .
 801 It reflects a very probable inter-point distance of 0.09 m, which is half of the value mea-
 802 sured on the pcf for N50 fractures. This is in line with the connectivity index of 1.13 in-
 803 tersections per fracture (Table 3) which suggests that N50 fractures are often divided
 804 into two pieces. Also, an inspection of Figure 14(a) reveals that N50 fractures (in blue)
 805 are typically intersected by N130 fractures (in green). And, considering N130 fracture
 806 branches, the 40° pcf (i.e. perpendicular to objects) also shows some short range order
 807 with a maximum at $r = 0.10$ m. Thus, the spacing of the fractures of the set N130 is
 808 related to the segmentation of N50 fractures.

809 *Mark Correlation Functions and Variograms*

810 The empirical isotropic and anisotropic mark correlation functions and variograms
 811 for the branches of the Hornelen dataset in Figure 17 reveal a similar form of spatial cor-
 812 relation as with Oman (Figure 12), of course with quite different length scales and ranges
 813 of correlation.

814 For the fractures, the Oman results are closer to a behavior expected for random-
 815 ness, while with Hornelen there is some inhibition of lengths at short distances (upto 0.6
 816 m), and the mark variogram indicates some tendency of similarity of lengths upto 0.5
 817 m. The combination of information given by the two correlation functions yields the state-
 818 ment that close fractures and branches tend to be short and similar in length.

819 More precisely: Consider a randomly chosen pair of fractures with a distance r be-
 820 tween their centers, denote by l_1 and l_2 the lengths of these fractures. Let the mean of
 821 the product $l_1 \cdot l_2$ be $m(l_1 l_2)$, while the mean fracture length in the FN is m_l . Then

$$k_{mm}(r) = \frac{m(l_1 l_2)}{m_l^2}. \quad (7)$$

822 The curves in Figure 17(a) for $k_{mm}(r)$ can be approximated by

$$k_{mm}(r) = 0.2 + \sqrt{r} \text{ for } r \leq 0.6 \text{ m} \quad (8)$$

823 and $k_{mm}(r) = 1$ for $r > 0.6$ m. This shows that close fractures (closer than 0.6 m)
 824 tend to be shorter than the average in the FN, since the mean of the corresponding lengths
 825 is smaller than the square of total mean length.

826 This result may be compared with findings in Darcel et al. (2003a), where nearest-
 827 neighbor correlations⁸ were studied on the seven maps of the Hornelen dataset. The main

⁸ This statistical approach is an alternative to second-order methods.

828 finding is: The average distance $d_c(l)$ between the center of a fracture of length l to its
 829 nearest-neighbor fracture center depends weakly on l , it is $d_c(l) \sim l^{0.3}$ for small l , but
 830 is independent for larger l , as expected for independent length marks.

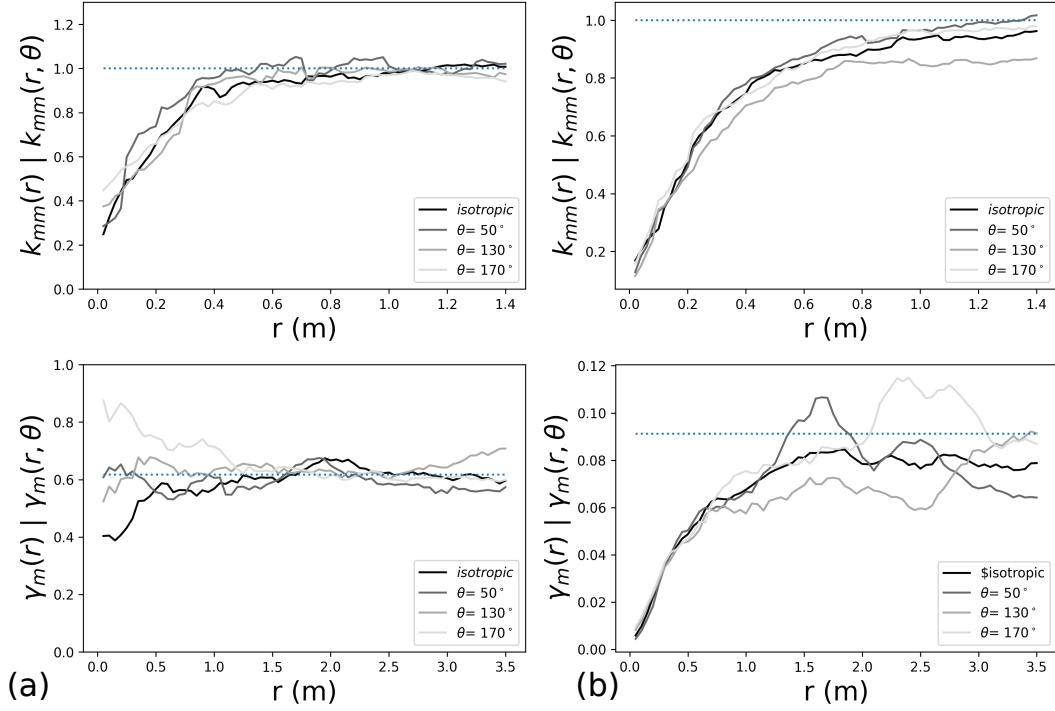


Figure 17. Empirical length-mark correlation function $k_{mm}(r)$ and length-mark variogram $\gamma_m(r)$ for (a) fractures and (b) branches of the Hornelen outcrop. The theoretically expected values for large r are shown as dotted blue lines.

831 5.3 Comparison of the Oman and Hornelen FN

832 The analysis in the Sections 5.1 and 5.2 shows clear differences in various senses
 833 between the two FN. While the Oman FN is clearly anisotropic, with two dominating
 834 directions, almost perpendicular, the Hornelen FN shows directional disorder with per-
 835 haps three main directions. The pattern shows so much variability that we had to con-
 836 sider three sub-samples for an acceptable second-order analysis. There is some similar-
 837 ity to the Artificial FN, however with higher local variability: While for the Artificial
 838 FN the branch centers form pieces of samples of one-dimensional Poisson processes, they
 839 show even clustering for the Hornelen FN.

840 In short, the Oman FN is a well developed chequered pattern with long fractures,
 841 highly connected and regularly spaced. Thus, most of its fractures (60%) are composed
 842 of series of branches. The Hornelen FN is composed of a lot of small fractures (60%)
 843 aligned and almost connected. There is a development and a connection of aligned (en-
 844 echelon) fractures, where probably larger fractures drive the development of smaller ones
 845 at their tips. These observations are refined and quantified by the second-order analysis.

846 For the Oman FN the fractures of the N130 set contribute more to the total frac-
 847 ture length than those of the N40 set. Also, the anisotropic pcfs show that N130 frac-
 848 tures are more regularly spaced than N40 fractures. For both directions (40° and 130°)
 849 there is short-range and partially even long-range order, in a weaker extent for the di-

850 rection 40° . The maxima of the pcfs are at positions (r -values) determined by the modes
 851 of branch lengths, as it is to expect because of the chequered nature of the FN. The anal-
 852 ysis of branches yielded a clearer information than the analysis of fractures.

853 For the Hornelen FN there is disorder in a great extent: We could not identify a
 854 direction that really dominates the FN. The sets N130 and N170 show some degree of
 855 long-range order (in their sub-windows), while N50 seems to reveal only some short-range
 856 order. The N50 fracture centers are not well aligned to fracture azimuths (Figure 5 (a_{N50})),
 857 most likely because of a large proportion of en-echelon fractures. In contrast to the Oman
 858 FN, here the fractures seem to be the more interesting objects in the second-order anal-
 859 ysis.

860 6 Discussion

861 When we started our research for this paper, we assumed that the isotropic pair
 862 correlation function $g(r)$ was a suitable tool for second-order analysis of FN by point-
 863 process methods. Discussions with colleagues and own experience taught us that orien-
 864 tations play a very important role in such a statistical analysis. We are now convinced
 865 that the anisotropic pcf $g(r, \theta)$ is the crucial tool to for quantitative description of the
 866 inner organization of fractures and fracture branches.

867 This paper uses the marked-point-process approach taking both fractures and branches
 868 as marks standing for individual linear segment. This implies that we consider objects
 869 as separate units, as marked points—so we keep the interpretation of fracture and there
 870 is no loss of information.

871 The ‘points’ in this paper, as well as in the literature we know, are geometrically
 872 defined as object centers. However, the ideal choice would be points that determine frac-
 873 tures in a physical sense such as nucleation points. Unfortunately, such points do not
 874 make sense for planar samples, and in three-dimensional samples their detection seems
 875 to be hopelessly complicated. Therefore, for fracture statistics with point-process meth-
 876 ods the use of centers seems to be appropriate.

877 An important question in the statistical characterization of FN is: Which objects
 878 should be used: fractures or branches? On the one hand, fractures are the primary ob-
 879 jects of an FN as they hold the geological interpretation that may reveal some inner or-
 880 ganization. However, fractures may be rather long and their centers may have little to
 881 do with their full appearance. This is particularly true if their lengths span over several
 882 orders of magnitude. On the other hand, branches are only secondary objects as they
 883 subdivide fractures according to their intersections. However, branches have some topo-
 884 logical meaning and capture information about connectivity of FN. Also, the range in
 885 their length distributions is reduced in comparison to fractures. Therefore, the use of branches
 886 may sometimes be a good compromise in variability and correlation analysis.

887 In a concrete application, it probably makes sense to study fractures and branches
 888 in parallel, until some tendency in the data helps to decide. For instance, if the FN is
 889 well developed and highly connected, branches may be considered as its main constitu-
 890 tive elements and yield the more interesting information. But if the FN contains a lot
 891 of isolated fractures (e.g. en-echelon), their analysis may be clearer if branches are not
 892 considered. The Hornelen dataset is a composition of both long interconnected and small
 893 isolated fractures. These are different configurations that may call for separate analy-
 894 sis in order to better understand the transition between ‘poorly-developed’ FN (also known
 895 as ‘unsaturated’ or ‘dilute’) and ‘well-developed’ FN (also known as ‘saturated’ or ‘dense’)
 896 (Wu & D. Pollard, 1995; Josnin et al., 2002; Davy et al., 2010).

897 Relations such as equation (8) and its nearest-neighbor counterpart are probably
 898 valid only for special FN. Only further research for many various FN may lead to gen-
 899 eral results and a deeper understanding.

900 In the definition of fracture branches, nodes are of particular interest. Indeed, the
 901 nodes contain vital information (1) about the relationship between fractures (abutting/cross-
 902 cutting), (2) about potential stress concentration and interaction and (3) on relative tim-
 903 ing of fractures. This even remains true in the planar case. Systematic studies of node-
 904 related statistics may be aims for the future.

905 Several times in this paper, we have mentioned an alternative approach that uses
 906 ideas from the theory of segment processes as in Chiu et al. (2013), chapter 8, and Stoyan
 907 (2021). In this context, the whole FN is considered as a sample of a random collection
 908 of one-dimensional objects, in particular straight-line segments. In contrast to the object-
 909 system approach the individual segments are not considered, only their contribution to
 910 the planar distribution of total segment length. Then, one is free of the need to define
 911 individual object, i.e. ‘points’. The fracture intensity P_{21} and the rose of directions are
 912 characteristics of a long tradition, which can be seen as belonging to the segment-process
 913 approach. The systematic use of segment processes could be an alternative approach in
 914 FN statistics in future. Here we do not recommend it because of difficulties in the cor-
 915 responding second-order theory, which are sketched around page 330 in Chiu et al. (2013):
 916 The segment-process analogue of the pair-correlation function has always a pole at $r =$
 917 0 , which makes fine analyses difficult.

918 Obviously, the Boolean segment process with Poisson-process centers is a model
 919 too simple for realistic stochastic modeling of FN. Nevertheless, our results for the Oman
 920 FN as well as for the Hornelen FN show that better models should be searched in ‘the
 921 neighborhood’ of the Boolean process. The pcfs of fracture centers in the Figures 9(a)
 922 and 15(a) are not very far from the simple form $g(r) \equiv 1$ true for a Poisson process.
 923 Perhaps realistic models may be obtained with other point processes.

924 7 Conclusions

925 Advanced methods of statistics for planar point processes lead to a quantitative
 926 description of variability and inner correlations of FN. The paper presents, in a rigor-
 927 ous and accessible way, the tools for such analyses. They are based on constructed marked
 928 points that stand for fractures or fracture branches, also called ‘objects’.

929 There is no a-priori criterion for the decision to work with fractures or fracture branches.
 930 However, the degree of connectivity of fractures and the proportion of isolated fractures
 931 may help to decide. A safe recommendation is to use both variants in parallel for the
 932 same FN because of the complementary of the information obtained. The study of frac-
 933 tures leads to a quantitative description of their hierarchical organization, while an anal-
 934 ysis based on branches describes fracture spacing and segmentation.

935 A key idea is the use of the anisotropic pair correlation function for analysis of in-
 936 ner correlations of subsets of objects. In the present paper, the subsets consist of objects
 937 of similar orientation, which introduces strong directionality into the statistical analy-
 938 sis, which is indispensable in FN statistics. However, different rules for forming subsets
 939 may be considered and other subsets may be analyzed in the future.

940 The application of second-order analysis to the simple Artificial FN, a Poisson-Boolean
 941 segment process, tests the methods under ideal conditions. It helps to understand the
 942 results for more complicated cases and serves as a benchmark, which describes results
 943 in the case of complete randomness. The analysis of the two natural FN shows clear de-
 944 viations from that ideal case, which, however, explains finer aspects.

945 Application of the presented statistical methods for a large number of surface ex-
 946 posure patterns may be of value for better understanding and classification of FN. FN
 947 are not always so different as the two natural FN considered here, thus also finer differ-
 948 ences may be detected and measured.

949 Finally, information gathered by the presented statistical method opens the path
 950 to better mathematical models for stochastic simulations of FN. At least, it may be used
 951 to test the quality of stochastic models of three-dimensional FN by analyzing some of
 952 their planar sections.

953 Appendix A The Miles-Lantuejoul Estimator

954 The Miles-Lantuejoul sampling method is a general estimation principle in spatial
 955 statistics, created by Miles (1974) and Lantuéjoul (1978). It is applicable to planar and
 956 spatial systems of objects randomly distributed for estimating quantities describing their
 957 size distribution. The structure is assumed to be homogeneous.

958 The basic idea is minus-sampling: The objects can be observed only in a window
 959 W and objects only partly inside are ignored. There is some trick of weighting the ob-
 960 servations so that one is close to unbiased estimation. Of course, the method has no magic
 961 power to use information from very large objects. (By the way, there is also a method
 962 called ‘plus-sampling’, which assumes that all objects in the window W and all objects
 963 hit by W can be measured completely.)

964 Here we explain the Miles-Lantuejoul estimator for the case of segments in the plane.
 965 The aim is to estimate a histogram of segment lengths. We consider the bin $[l_k, l_{k+1}]$.
 966 The window is a rectangle with sides parallel to x - and y -axis of lengths a and b .

967 The total number of lower segment end-points in W is denoted by N . Only seg-
 968 ments fully in W are measured, their number is denoted by n . The segments are described
 969 as follows: The reference points are the lower end-points, the segment lengths are de-
 970 noted by L_i for $i = 1, 2, \dots, n$. The orientation of segment i is described by the rect-
 971 angle with sides parallel to x - and y -axis having the segment as diagonal, with side lengths
 972 x_i and y_i . (If the segment is parallel to x -axis (y -axis), then $x_i = L_i$ and $y_i = 0$ ($y_i =$
 973 L_i and $x_i = 0$).

974 The estimator of the probability p_k that segment length is in the bin $[l_k, l_{k+1}]$ is

$$p_k = \frac{ab}{N} \sum_{i=1}^n \frac{\delta_{ki}}{(a - x_i)(b - y_i)} \quad (\text{A1})$$

975 with

$$\delta_{ki} = 1 \text{ if } l_k \leq L_i < l_{k+1} \text{ and } = 0 \text{ otherwise.} \quad (\text{A2})$$

976 Appendix B Estimating Pair Correlation Functions and the Role of 977 Band-Widths

978 This paper uses extensively statistics for pair correlation functions. This appendix
 979 explains some of the main ideas of the estimators used. Kernel estimators are used in
 980 order to obtain smooth curves for empirical pcfs. The degree of smoothness obtained de-
 981 pends on the choice of a ‘band-width’ h , which determines the region around a distance
 982 r which is used to estimate $g(r)$. While estimators of the isotropic $g(r)$ and of the mark
 983 correlation functions are available in the statistical software R and explained in detail
 984 in Illian et al. (2008) and Baddeley et al. (2016), here the anisotropic pcf $g(r, \theta)$ is in the
 985 focus.

986 The basic idea is to analyze inter-point distances of point pairs. For estimating $g(r, \theta)$
 987 only point-pairs that determine lines of an azimuth direction $\theta \pm \delta$ are considered; they

988 are called “ θ -pairs”. In this paper, we work with $\delta = 5^\circ$. The arc of $\theta \pm \delta$ is denoted
 989 by α ; its numerical value is $\frac{10}{180}\pi$.

990 These distances are collected to obtain groups of point-pairs of an approximate inter-
 991 point distance r . The collection is made with a “kernel function”. The simplest case, which
 992 is also used in the paper, is the “box kernel”: All point-pairs of a distance between $r-h$
 993 and $r+h$ are included with equal weight $1/2h$. The actual distance value between
 994 points is ignored, and h , a positive parameter, is called the “band-width”.

The band-width h has great influence on the results: With a large h the curves ob-
 tained are smooth, fine details are smoothed away. In contrast, with a small h the curves
 become spiky. The optimal choice of h is an art. A simple receipt is to start with

$$h = \frac{0.1}{\sqrt{P_{20}}}$$

995 and then search by experimentation for a good h . The aim is to obtain smooth curves
 996 that show nevertheless interesting details. Values for h used in this paper are either con-
 997 stant or adapted, i.e. piece-wise constant and increasing with r . In the latter case, we
 998 use the notation $(h_{r \leq r_t}, h_{r > r_t}, r_t)$. The Table B1 presents all values of band-width used
 999 in this paper.

1000 There are also other kernels, which give distances closer to r more weight than to
 1001 others.

1002 The estimator of the anisotropic pair correlation function $g(r, \theta)$ is

$$\hat{g}(r, \theta) = \sum_{(x_i, x_j) \in (r, \theta)} \frac{1}{2h \cdot \alpha \cdot r \cdot (A - a_{ij}) \cdot (B - b_{ij})} / \lambda^2, \quad (\text{B1})$$

1003 where x_i and x_j are fracture (or branch) centers and the summation goes over all θ -pairs
 1004 with an inter-point distance between $r-h$ and $r+h$. The window of observation W is
 1005 a rectangle with side-lengths A and B in x - and y -direction, respectively. The absolute
 1006 value of the x -component of $x_i - x_j$ is a_{ij} , and b_{ij} is the same with for the y -component.
 1007 The term $(A - a_{ij})(B - b_{ij})$ replaces the simpler $a = AB$, the window area, in order
 1008 to make an edge-correction.

1009 Finally, λ is the same as P_{20} , fracture density in the case of fracture centers, either
 1010 of all centers or all centers of a subset $N\theta$; and analogously in the case of branches.

1011 Appendix C Short-Range and Long-Range Order

1012 The terms ‘short-range order’ and ‘long-range order’ have their origin in physics
 1013 and chemistry in the context of atomic patterns. In this paper, they are used in a slightly
 1014 modified meaning in the hope to make them useful for spatial statistics in geological ap-
 1015 plications.

1016 The term ‘long-range order’ is in full agreement with the classical wording, if ap-
 1017 plied to a lattice of points. For a quadratic lattice with side length a the pcf has poles
 1018 at $r = a$, $r = \sqrt{2}a$ and $r = 2a$ etc. When the points are slightly randomly moved,
 1019 the pcf would have maxima at these values of r .

1020 In geological contexts, with scales suitable for fractures, such patterns are non-typical.
 1021 Already a sequence of two maxima of a pcf at some r -values a and $\sqrt{2}a$ are somewhat
 1022 remarkable. In such cases, we use the term ‘long-range order’.

1023 If there is at least a clear maximum of a pcf at some r -value, this marks a clear dif-
 1024 ference to patterns of complete randomness belonging to a Poisson process, the pcf of
 1025 which does not have any maximum. In order to show that this is somewhat remarkable,
 1026 we use here the term ‘short-range order’.

Table B1. Band-widths h (in m) for the figures

| Figure | h_{iso} | h_{θ} |
|----------------------|------------------|---|
| 4 (a) | 0.35 | 0°:0.35 ; 90°:0.35 ; 130°:0.35 |
| 4 (b) | 0.15 | 0°:0.4; 90°:0.4 ; 130°:0.4 |
| 5 (a) | 0.3 | 0°:0.7 ; 90°:0.7 |
| 5 (b) | 0.2 | 0°:0.35 ; 90°:0.35 |
| 6 (a) k_{mm} | 0.35 | 0°:0.35 ; 90°:0.35 ; 130°:0.35 |
| 6 (a) γ_{mm} | 0.35 | 0°:0.35 ; 90°:0.35 ; 130°:0.35 |
| 6 (b) k_{mm} | 0.4 | 0°:0.8 ; 90°:0.95 ; 130°:0.8 |
| 6 (b) γ_{mm} | 0.45 | 0°:0.9 ; 90°:0.9 ; 130°:0.9 |
| 9 (a) | (1.8,1.9,16) | 40°:(1.8,2.6,16) ; 130°:(1.9,5,8.8) |
| 9 (b) | (0.8,1.0,8) | 40°:(1.,2.9,12.8) ; 130°:(1.5,1.8,14.4) |
| 10 (a) | 2 | 40°:(4,7,12) ; 130°:(2.,5,14.4) |
| 10 (b) | 1.5 | 40°:(1.9,4,12) ; 130°:(2,2,7,14.4) |
| 11 (a) | 2 | 40°:(1.9,3.6,22.4) ; 130°:(2.6,3.2,14.4) |
| 11 (b) | (0.8,1.4,8) | 40°:(1.9,3.5,16.8) ; 130°:(1.6,2,20) |
| 12 (a) k_{mm} | (2.7,3.6,16) | 40°:(5,6.5,14.4) ; 130°:(5,6.5,5.6) |
| 12 (a) γ_{mm} | (5.5,9.5,6.4) | 40°:(7,14.5,7.2) ; 130°:(5,12,8.0) |
| 12 (b) k_{mm} | (1,3,8.0) | 40°:(4,6,19.2) ; 130°:(5,10,12.8) |
| 12 (b) γ_{mm} | (2,4,9.6) | 40°:(4.5,6,19.2) ; 130°:(6,10,12.8) |
| 15 (a) | 0.03 | 50°:(0.035,0.08,0.33) ; 130°:(0.03,0.08,0.33) ; 170°:(0.07,0.1,0.33) |
| 15 (b) | 0.027 | 50°:(0.027,0.05,0.33) ; 130°:(0.027,0.05,0.33) ; 170°:(0.027,0.05,0.33) |
| 16 (a_{N50}) | 0.06 | 50°:(0.05,0.085,0.09) ; 140°:0.06 ; 60°:(0.035,0.089,0.09) |
| 16 (b_{N50}) | 0.02 | 50°:(0.03,0.045,0.075) ; 140°:0.04 ; 60°:(0.035,0.05,0.3) |
| 16 (a_{N130}) | 0.07 | 40°:(0.07,0.11,0.6) ; 130°:(0.07,0.11,0.6) |
| 16 (b_{N130}) | 0.03 | 40°:(0.05,0.1,0.3) ; 130°:(0.05,0.1,0.3) |
| 16 (a_{N170}) | 0.07 | 80°:(0.07,0.15,0.6) ; 170°:(0.07,0.15,0.6) |
| 16 (b_{N170}) | 0.04 | 80°:(0.04,0.1,0.15) ; 170°:(0.04,0.1,0.15) |
| 17 (a) k_{mm} | (0.09,0.2,0.7) | 50°:(0.35,0.55,0.7) ; 130°:(0.35,0.55,0.7) ; 170°:(0.35,0.55,0.7) |
| 17 (a) γ_{mm} | 0.15 | 50°:0.6 ; 130°:0.6 ; 170°:0.6 |
| 17 (b) k_{mm} | (0.027,0.08,0.4) | 50°:(0.1,0.25,0.4) ; 130°:(0.1,0.25,0.4) ; 170°:(0.1,0.25,0.4) |
| 17 (b) γ_{mm} | 0.075 | 50°:(0.09,0.2,0.7) ; 130°:(0.09,0.2,0.7) ; 170°:(0.09,0.2,0.7) |

1027 When describing short-range and long-range order the r -values at the maxima are
 1028 important because they indicate frequent inter-point distances. Also the maximum val-
 1029 ues are important, since they characterize the relative frequency of these distances.

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