

On the termination of the X-ray constrained wavefunction procedure: reformulation of the method for a univocal determination of λ

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Synopsis The long-standing problem of determining the correct point at which stopping the calculations in the X-ray constrained wavefunction (XCW) approach is formally solved on solid mathematical grounds and a new and more physically meaningful variant of the XCW method is proposed.

Abstract The X-ray constrained/restrained wavefunction (XCW/XRW) approach of quantum crystallography is revisited by introducing the stationary condition of the Jayatilaka functional with respect to the Lagrange multiplier λ . The theoretical derivation has unequivocally shown that the right value of λ is a maximum stationary point of the functional to optimise, thus enabling to solve the long-standing problem of establishing the point at which halting the XCW/XRW procedure. Based on the new finding, a reformulation of the X-ray constrained wavefunction algorithm is proposed and its implementation is envisaged. In addition to relying on more solid mathematical grounds, the new variant of the method will be intrinsically more physically meaningful, allowing a straightforward evaluation of the highest level of confidence with which the experimental X-ray diffraction data can be possibly reproduced.

Keywords: X-ray constrained/restrained wavefunction; Lagrange multipliers; constraints; restraints; quantum crystallography

1. Introduction and theoretical background

The X-ray constrained wavefunction (XCW) approach (Jayatilaka, 1998; Jayatilaka & Grimwood, 2001; Grimwood & Jayatilaka, 2001; Bytheway, Grimwood & Jayatilaka, 2002; Bytheway, Grimwood, Figgis *et al.*, 2002; Grimwood *et al.*, 2003) is undoubtedly one of the cornerstone methods of modern quantum crystallography (Genoni *et al.*, 2018; Grabowsky, Genoni & Bürgi, 2017; Massa & Matta, 2017; Genoni & Macchi, 2020; Grabowsky, Genoni, Thomas & Jayatilaka, 2020; Macchi, 2020). It was originally proposed by Jayatilaka in 1998 and it can be considered as a computational technique having the goal of determining electronic wavefunctions compatible with sets of experimental X-ray diffraction data. To accomplish this task, the XCW strategy aims at looking for wavefunctions that minimise the energy of the investigated system and that, at the same time, reproduce sets of experimental structure factor amplitudes within a desired level of agreement. This is equivalent to minimising the energy of the system under the constraint that the squared goodness-of-fit (GoF²) is equal to a desired value Δ .¹ Namely, the imposed constraint is

$$\text{GoF}^2 = \Delta \quad (1)$$

with GoF² given by the following expression:

$$\text{GoF}^2 = \frac{1}{N_r - N_p} \sum_{\mathbf{h}} \frac{(\eta |F_{\mathbf{h}}^{\text{calc}}| - |F_{\mathbf{h}}^{\text{exp}}|)^2}{\sigma_{\mathbf{h}}^2} \quad (2)$$

where N_r is the number of considered experimental structure factor amplitudes, N_p the number of adjustable parameters, \mathbf{h} the triad of Miller indices labelling the reflection, $|F_{\mathbf{h}}^{\text{calc}}|$ and $|F_{\mathbf{h}}^{\text{exp}}|$ the calculated and experimental structure factor amplitudes, respectively, η an external \mathbf{h} -independent scale factor that is used to put the calculated structure factor amplitudes on the same scale of the experimental ones, and $\sigma_{\mathbf{h}}$ the experimental uncertainty associated with $|F_{\mathbf{h}}^{\text{exp}}|$ (provided that the statistics obeys the normal (Gaussian) law and as long as the estimated error bars are close enough to the true standard deviation of the law).

According to the original formulation of the technique, it is suggested to set Δ in equation (1) equal to 1.0 (Jayatilaka, 1998; Jayatilaka & Grimwood, 2001). This means that the goal of the technique is to obtain “constrained wavefunctions” that provide structure factor amplitudes that are, on average, within one standard deviation of the experimental values. Therefore, to minimise the total energy of the system in presence of constraint (1), the following functional was introduced:

$$J = E_{QM} + \lambda \{\text{GoF}^2 - \Delta\} \quad (3)$$

¹ Note that throughout the text it was decided to adopt the new terminology recently introduced by Davidson and coworkers (Davidson, Grabowsky & Jayatilaka, 2022a), who correctly recommended to use GoF² instead of χ^2 .

with E_{QM} as the quantum mechanical energy of the investigated system and λ as a Lagrange multiplier. By inspecting equation (3), it is also worth noting that λ is not dimensionless, but it has the physical dimensions of an energy.

Although multi-determinant variants of the XCW method have been also proposed in recent years (Genoni, 2017; Casati *et al.*, 2017; Genoni, Franchini *et al.*, 2018; Genoni, Macetti, Franchini *et al.*, 2019), the original and most used versions of the Jayatilaka approach assume that the wavefunction to be determined is a single Slater determinant (Jayatilaka, 1998; Jayatilaka & Grimwood, 2001; Grimwood *et al.*, 2003; Hudák *et al.*, 2010; Jayatilaka, 2012; Genoni, 2013a; Genoni, 2013b; Dos Santos *et al.*, 2014; Genoni & Meyer, 2016; Bučinský *et al.*, 2016). This will also be the choice that will be considered throughout this work. Under this hypothesis, obtaining a wavefunction that minimises the energy of the system under exam and that simultaneously reproduces experimental X-ray diffraction data within a desired limit is equivalent to making functional J (see equation (3) above) stationary with respect to the molecular orbitals of the single Slater determinant and with respect to the Lagrange multiplier λ .

In the case of a $2N$ -electron closed-shell system, making J stationary with respect to the molecular orbitals (under the additional condition that the molecular orbitals are also orthonormal) leads to the following modified Hartree-Fock equations:

$$\left[-\frac{1}{2}\nabla^2 + v(\mathbf{r}) + v_H(\mathbf{r}) + \widehat{K} + \lambda v_{XCW}(\mathbf{r}) \right] \phi_i(\mathbf{r}) = \epsilon_i \phi_i(\mathbf{r}) \quad (4)$$

with $v(\mathbf{r}) = \sum_A \frac{Z_A}{|\mathbf{r}-\mathbf{R}_A|}$ as the external potential, $v_H(\mathbf{r}) = \int d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$ as the Hartree potential ($\rho(\mathbf{r})$ being the electron density of the examined system), and with \widehat{K} as the exchange integral operator defined as follows:

$$\widehat{K} \phi_i(\mathbf{r}) = -\frac{1}{2} \int d\mathbf{r}' \frac{\gamma(\mathbf{r}, \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \phi_i(\mathbf{r}') \quad (5)$$

where

$$\gamma(\mathbf{r}, \mathbf{r}') = 2 \sum_{j=1}^N \phi_j^*(\mathbf{r}') \phi_j(\mathbf{r}) \quad (6)$$

Furthermore, again in equation (4), $v_{XCW}(\mathbf{r})$ can be expressed as

$$v_{XCW}(\mathbf{r}) = \sum_{\mathbf{h}} K_{\mathbf{h}} \left[\text{Re}\{F_{\mathbf{h}}^{calc}\} \cos[2\pi (\mathbf{Q}_k \mathbf{r} + \mathbf{q}_k) \cdot (\mathbf{B}\mathbf{h})] + \text{Im}\{F_{\mathbf{h}}^{calc}\} \sin[2\pi (\mathbf{Q}_k \mathbf{r} + \mathbf{q}_k) \cdot (\mathbf{B}\mathbf{h})] \right] \quad (7)$$

which derives from the introduction of the so-called one-electron scattering operator:

$$\hat{I}_{\mathbf{h}} = \hat{I}_{\mathbf{h},R} + i \hat{I}_{\mathbf{h},C} = \sum_{k=1}^{N_m} \exp[i2\pi (\mathbf{Q}_k \mathbf{r} + \mathbf{q}_k) \cdot (\mathbf{B}\mathbf{h})] \quad (8)$$

with $\hat{I}_{\mathbf{h},R}$ and $\hat{I}_{\mathbf{h},C}$ as its real and imaginary parts (which are hermitian, while $\hat{I}_{\mathbf{h}}$ is not), N_m as the number of symmetry-unique positions in the unit-cell, $\{\mathbf{Q}_k, \mathbf{q}_k\}$ as the roto-translations associated with the unit-cell symmetry operations (being \mathbf{Q}_k and \mathbf{q}_k a rotation matrix and a translation vector, respectively), and \mathbf{B} as the reciprocal lattice matrix. Finally, $K_{\mathbf{h}}$ in equation (7) is given by the following expression:

$$K_{\mathbf{h}} = \frac{2\eta}{N_r - N_p} \frac{\eta |F_{\mathbf{h}}^{calc}| - |F_{\mathbf{h}}^{exp}|}{\sigma_{\mathbf{h}}^2 |F_{\mathbf{h}}^{calc}|} \quad (9).$$

For the sake of completeness, it is also worthwhile to point out that, while λ has the physical dimensions of an energy (see above), the potential $v_{XCW}(\mathbf{r})$ is dimensionless.

In the current implementation of the XCW method, equation (4) is solved self-consistently for different values of λ in a sort of “trial & error” procedure. In other words, the molecular orbitals resulting from the calculation with $\lambda^{(i)}$ at the i -th step are used as guess for the computation at the following step with $\lambda^{(i+1)} = \lambda^{(i)} + \Delta\lambda$. The procedure is usually iterated, but without a clear and definitive criterion to stop. Different suggestions have been proposed over the years: from the simple and arbitrary idea of halting at the λ value for which the SCF process stops converging to a very recent and sophisticated approach introduced by Davidson and coworkers (Davidson, Grabowsky & Jayatilaka, 2022b) who adapted an asymptotic extrapolation technique originally devised by Tozer, Ingamells and Handy (Tozer, Ingamells & Handy, 1996) in methods aiming at obtaining Kohn-Sham-like wavefunctions from high-level *ab initio* electron densities given as sets of grid points in real space (Zhao & Parr, 1993; Zhao, Morrison & Parr, 1994; Tozer, Ingamells & Handy, 1996). For a more comprehensive overview on the different attempts of establishing a halting criterion, the reader can refer to a recent review on the XCW strategy by Davidson, Grabowsky and Jayatilaka (Davidson, Grabowsky & Jayatilaka, 2022a).

Nevertheless, despite all the remarkable efforts, none of the proposed approaches seems the definitive solution to the halting problem of the XCW method. In fact, although in principle the procedure should be stopped when $\text{GoF}^2 = 1$, this goal is seldom achieved (i.e., $\text{GoF}^2 > 1$) or it may also occur that the computations continue beyond the desired limit, thus leading to final values of GoF^2 that are lower than 1.0. Therefore, in the current XCW implementations, the starting constraint given by equation (1) is never really considered and it does not play any role in the working equations. Due to this reason and since the calculated structure factor amplitudes would never exactly match the experimental ones even in the case in which we fully accounted for constraint (1), all the variants of the Jayatilaka approach developed so far should be more correctly renamed as X-ray restrained wavefunction (XRW) methods (Jayatilaka, 2012; Grabowsky, Genoni & Bürgi, 2017; Ernst, Genoni & Macchi, 2020; Macetti, Macchi

& Genoni, 2021) or X-ray regularisation procedures (Davidson, Grabowsky & Jayatilaka, 2022a; Davidson, Grabowsky & Jayatilaka, 2022b).

Therefore, based on the previous considerations, it is as the following functional was optimised in the current implementation of the Jayatilaka technique (not the functional given in equation (3)):

$$J = E_{QM} + \lambda \text{GoF}^2 \quad (10)$$

In other words, in the current XCW calculations, one tries to determine the wavefunction that minimises the energy of the system and that (on average, not exactly) reproduces as much as possible the given experimental X-ray structure factor amplitudes. Consequently, λ is not a Lagrange multiplier, but only an external parameter that is manually adjusted during the XCW procedure to modulate the weight of the experimental data in the computations.

The above-described situation stems from the fact that, in the current formulation of the XCW approach, the stationary condition of the original functional J (i.e., equation (3)) with respect to λ is completely neglected. In the next section, we will show that if we account for this stationary condition, λ regains its original meaning of Lagrange multiplier and a clear and univocal way to determine its value simply arises from the derived equations. This will also lead to a slight reformulation of the XCW algorithm.

2. Determination of λ and reformulation of the XCW approach

We start from equations (4) with the idea of solving the constrained problem set by relation (3). First of all, by inspecting equations (4), we can observe that they are like usual Hartree-Fock equations except for the additional term $\lambda v_{XCW}(\mathbf{r})$, which can be considered as a “constraint potential”. Furthermore, $v_{XCW}(\mathbf{r})$ is known explicitly (see equation (7)) and, for a given set of molecular orbitals, it is predefined. On the contrary, λ is known only implicitly and its correct value is the one that should make the wavefunction (and, in our particular case, the molecular orbitals) satisfy the condition expressed by means of equation (1). It is also worth noting that each value of λ is uniquely associated with a set of molecular orbitals. When this set is used to calculate the wavefunction and the corresponding electron density, then functional J becomes a function of λ only. Now, following a reasoning analogous to the one adopted in constrained Density Functional Theory (Wu & Van Voorhis, 2005; Wu & Van Voorhis, 2006; Kaduk, Kowalczyk & Van Voorhis, 2012), it is possible to show that $J(\lambda)$ is a strictly concave function of λ , which has an important repercussion on the way in which λ can be determined.

Let us consider the first derivative of J with respect to λ :²

$$\frac{dJ}{d\lambda} = 2 \sum_{i=1}^N \left(\frac{\delta J}{\delta \phi_i^*} \frac{\partial \phi_i^*}{\partial \lambda} + \text{c.c.} \right) + \frac{\partial J}{\partial \lambda} \quad (11)$$

However, for a given set of molecular orbitals that are solution of the modified Hartree-Fock equations (4), $\delta J / \delta \phi_i^* = 0$ and, consequently, equation (11) simply becomes:

$$\frac{dJ}{d\lambda} = \frac{\partial J}{\partial \lambda} = \text{GoF}^2 - \Delta \quad (12)$$

The meaning of equation (12) is that the stationary point of $J(\lambda)$ (for which $dJ/d\lambda = 0$) provides the constraint expressed by equation (1). To determine the nature (minimum or maximum) of the stationary point, one needs to evaluate the second derivative of $J(\lambda)$:

$$\frac{d^2J}{d\lambda^2} = \frac{\partial}{\partial \lambda} (\text{GoF}^2 - \Delta) = \frac{\partial \text{GoF}^2}{\partial \lambda} = \sum_{\mathbf{h}} \frac{2\eta}{N_r - N_p} \frac{\eta |F_{\mathbf{h}}^{\text{calc}}| - |F_{\mathbf{h}}^{\text{exp}}|}{\sigma_{\mathbf{h}}^2} \frac{\partial |F_{\mathbf{h}}^{\text{calc}}|}{\partial \lambda} \quad (13)$$

Considering that $|F_{\mathbf{h}}^{\text{calc}}| = [F_{\mathbf{h}}^{\text{calc}} (F_{\mathbf{h}}^{\text{calc}})^*]^{1/2}$, and exploiting the definitions of one-electron scattering operator (see equation (8)), we obtain:

$$\frac{d^2J}{d\lambda^2} = \sum_{\mathbf{h}} K_{\mathbf{h}} \text{Re} \left\{ F_{\mathbf{h}}^{\text{calc}} \frac{\partial (F_{\mathbf{h}}^{\text{calc}})^*}{\partial \lambda} \right\} \quad (14),$$

with $K_{\mathbf{h}}$ defined by equation (9). Afterwards, exploiting again the definition of one-electron scattering operator and the definition of potential $v_{\text{XCW}}(\mathbf{r})$ (see equation (7)), it is possible to show that

² Hereinafter, we will extensively use the concept of functional derivative (e.g., $\delta J / \delta \phi_i^*$ in equation (11)). The functional derivative can be easily seen as a mathematical quantity that connects the variation in a functional (which is a function having another function as variable) to the variation in the function on which the functional depends.

More mathematically, given a functional $F[f]$ (namely, a function F having another function $f(x)$ as variable), its differential δF is that part of the difference $F[f + \delta f] - F[f]$ that depends linearly on the variation δf of function $f(x)$; since, in principle, each $\delta f(x)$ may contribute to the previous difference, when δf is small the differential δF can be expressed as follows:

$$\delta F = \int \frac{\delta F}{\delta f(x)} \delta f(x) dx$$

where $\delta F / \delta f(x)$ is the so-called functional derivative of $F[f]$ with respect to function $f(x)$ at point x . For further details, the reader may refer to the appendix dedicated to the concepts of functional and functional derivative in the monography on density functional theory written by Parr and Yang (Parr & Yang, 1989).

$$\frac{d^2J}{d\lambda^2} = 4 \sum_{i=1}^N \int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \frac{\partial \phi_i(\mathbf{r})}{\partial \lambda} \quad (15)$$

For interested readers, the full details of the derivation of equation (15) from (13) are given in Appendix A. However, at this point, it is more important to observe that $\phi_i(\mathbf{r})$ depends on the actual constraint potential $\lambda v_{XCW}(\mathbf{r})$. Therefore, we can write:

$$\frac{\partial \phi_i(\mathbf{r})}{\partial \lambda} = \int d\mathbf{r}' \frac{\delta \phi_i(\mathbf{r})}{\delta[\lambda v_{XCW}(\mathbf{r}')] } \frac{\partial(\lambda v_{XCW}(\mathbf{r}'))}{\partial \lambda} = \int d\mathbf{r}' \frac{\delta \phi_i(\mathbf{r})}{\delta[\lambda v_{XCW}(\mathbf{r}')] } v_{XCW}(\mathbf{r}') \quad (16)$$

In the previous equation,³ $\delta \phi_i(\mathbf{r})/\delta[\lambda v_{XCW}(\mathbf{r}')]$ is the functional derivative of the i -th occupied molecular orbital with respect to the constraint potential $\lambda v_{XCW}(\mathbf{r})$ and it can be essentially seen as the first-order response of the orbital ($\delta \phi_i(\mathbf{r})$) caused by a small change in the constraint potential ($\delta[\lambda v_{XCW}(\mathbf{r}')]$). Therefore, first-order perturbation theory can be exploited to evaluate this functional derivative, with $\delta \phi_i(\mathbf{r})$ expanded in terms of the eigenstates of the modified Fock operator of the XCW method:

$$\delta \phi_i(\mathbf{r}) = \sum_{a \neq i} \phi_a(\mathbf{r}) \frac{\int d\mathbf{r}'' \phi_a^*(\mathbf{r}'') \delta[\lambda v_{XCW}(\mathbf{r}'')] \phi_i(\mathbf{r}'')}{\epsilon_i - \epsilon_a} \quad (17),$$

where ϵ_i and ϵ_a are the orbital energies associated with molecular orbitals $\phi_i(\mathbf{r})$ and $\phi_a(\mathbf{r})$, respectively, obtained through the resolution of equations (4).

Therefore, using relation (17), we have

$$\frac{\delta \phi_i(\mathbf{r})}{\delta[\lambda v_{XCW}(\mathbf{r}')] } = \sum_{a \neq i} \phi_a(\mathbf{r}) \frac{\phi_a^*(\mathbf{r}') \phi_i(\mathbf{r}')}{\epsilon_i - \epsilon_a} \quad (18),$$

and substituting into (16) we get:

$$\frac{\partial \phi_i(\mathbf{r})}{\partial \lambda} = \sum_{a \neq i} \phi_a(\mathbf{r}) \int d\mathbf{r}' \frac{\phi_a^*(\mathbf{r}') \phi_i(\mathbf{r}')}{\epsilon_i - \epsilon_a} v_{XCW}(\mathbf{r}') \quad (19)$$

Consequently, exploiting equation (19) in (15), the expression of the second derivative $d^2J/d\lambda^2$ becomes:

³ It is worth noting that equation (16) corresponds to the special case in which the variable of functional F (i.e., the function f) depends on a parameter λ , namely $F=F[f(x,\lambda)]$. In this situation, one can write:

$$\frac{\partial F}{\partial \lambda} = \int \frac{\delta F}{\delta f(x,\lambda)} \frac{\partial f(x,\lambda)}{\partial \lambda} dx$$

More details can be found again in the monography about density functional theory by Parr and Yang (Parr & Yang, 1989).

$$\frac{d^2J}{d\lambda^2} = 4 \sum_{i=1}^N \sum_{a \neq i} \frac{|\int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \phi_a(\mathbf{r})|^2}{\epsilon_i - \epsilon_a} \quad (20),$$

The sum over a can be divided into two contributions (Szabo & Ostlund, 1996): i) one involving a summation over all the virtual orbitals, and ii) another one with a summation over all the occupied orbitals (except $\phi_i(\mathbf{r})$):

$$\frac{d^2J}{d\lambda^2} = 4 \sum_{i=1}^N \sum_a^{virt} \frac{|\int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \phi_a(\mathbf{r})|^2}{\epsilon_i - \epsilon_a} + 4 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{|\int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \phi_j(\mathbf{r})|^2}{\epsilon_i - \epsilon_j} \quad (21)$$

If we consider only the second term and we interchange the indices i and j , we get (Szabo & Ostlund, 1996):

$$Y = 4 \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{|\int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \phi_j(\mathbf{r})|^2}{\epsilon_i - \epsilon_j} = 4 \sum_{j=1}^N \sum_{\substack{i=1 \\ i \neq j}}^N \frac{|\int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \phi_j(\mathbf{r})|^2}{\epsilon_j - \epsilon_i} = -Y \quad (22),$$

namely the second term on the right-hand side of equation (21) is equal to zero. Therefore, the second derivatives $d^2J/d\lambda^2$ can be simply written as follows:

$$\frac{d^2J}{d\lambda^2} = 4 \sum_{i=1}^N \sum_a^{virt} \frac{|\int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \phi_a(\mathbf{r})|^2}{\epsilon_i - \epsilon_a} \quad (23)$$

where it is worth stressing again that the sum over i runs over the occupied molecular orbitals, while the sum over a runs over the virtual molecular orbitals. If we assume that, as usual, the occupied molecular orbitals are the lowest eigenfunctions (i.e., $\epsilon_i < \epsilon_a \forall i, a$), the second derivative $d^2J/d\lambda^2$ given by equation (23) is always negative. This means that, with respect to λ , we only have a maximum as stationary point.

By optimising J with respect to λ , it is thus possible to find the correct value of λ that provides the electronic wavefunction of the investigated system in presence of the experimental constraint introduced through equation (1). This can be done through rapidly converging numerical techniques (e.g., Newton's method) that make use of the easy-to-calculate first and second derivatives considered above. Therefore, following what is currently done in constrained DFT (Wu & Van Voorhis, 2005; Wu & Van Voorhis, 2006; Kaduk, Kowalczyk & Van Voorhis, 2012), one could envisage to re-implement the XCW method as an optimisation procedure consisting in an algorithm with two nested loops: i) an *outer loop* that will be very similar to the current XCW calculations, with the SCF iterations that are carried out to optimise the molecular orbitals (see equation (4)); ii) an *inner loop* of microiterations (that will be nested within each iteration of the outer loop) with the goal of determining the value of the Lagrange multiplier λ that makes the wavefunction/electron density satisfy the constraint given by

equation (1). When convergence is achieved, the process should provide both the desired X-ray constrained wavefunction and the potential that is necessary to fulfil the constraint.

3. Conclusions and perspectives

In this short communication, we have shown that, by introducing the stationary condition of functional J with respect to λ , the long-standing problem of determining the point at which halting the XCW procedure can be elegantly solved. In fact, it has been shown analytically that the right λ -value is a maximum stationary point of J with respect to the auxiliary variable λ . In principle, this value could be determined by transforming the current XCW procedure into an algorithm essentially made of two nested loops, an outer one for the optimisation of the molecular orbitals and an inner one for the estimation of λ . Operating in this way, at convergence the desired constraint would be necessarily fulfilled and, unlike the current implementations of the Jayatilaka technique, λ would regain its original meaning of Lagrange multiplier and we would have a computational strategy more similar to a real X-ray constrained wavefunction approach.

Concerning the last point, it is again important to observe that, by introducing a constraint in the form of equation (1), even the new version of the method would not allow an exact reproduction of all the considered experimental structure factor amplitudes and, for this reason, it should be also not defined as “X-ray constrained wavefunction technique”. However, to distinguish the new variant of the strategy from the old one (which should be better called as “X-ray restrained wavefunction approach”; see above), we propose to introduce the term “weakly X-ray constrained wavefunction (wXCW) method”. In fact, through the new version, at least the final GoF^2 would be indeed constrained to its desired value Δ . If one wanted to reproduce the values of all the structure amplitudes, it would be necessary to introduce one Lagrange multiplier for each experimental observation. This would not modify the mathematical derivation outlined above (actually, as a generalisation, the values of the different Lagrange multipliers would still define a maximum stationary point of functional J) and would give rise to a fully (or strongly) “X-ray constrained wavefunction technique”. It is also very interesting to note that a strongly XCW method would be practically identical to a strategy originally proposed in the 1960s by Weiss, who envisaged a technique based on perturbation theory to obtain a wavefunction corrected by experimental X-ray diffraction intensities (Weiss, 1966). In this context, an interesting re-evaluation of Weiss’ approach and a thorough discussion of its differences as compared to the Jayatilaka method can be found in a pamphlet on quantum crystallography recently published by Macchi (Macchi, 2022).

Finally, we want to point out that, in the new variant of the XCW technique, the uncertainty shifts from λ to Δ (i.e., the desired level of agreement), but with a significant advantage in terms of physical meaning for the method. In fact, instead of varying λ , in the new version of the strategy one could

imagine considering different Δ values (with lower bound equal to 1.0 in case of experimental data) and performing a wXCW calculation for each of them. The lowest value of Δ for which convergence was achieved would then represent the highest degree of confidence with which the experimental X-ray diffraction data could be reproduced on average, given the wavefunction *ansatz* and the basis set chosen for the calculations.

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Appendix A. Further details of the mathematical derivation

As mentioned in the main text, here we report the full derivation of equation (15) from equation (13). Therefore, the starting point is the initial expression of the second derivative $d^2J/d\lambda^2$:

$$\frac{d^2J}{d\lambda^2} = \frac{\partial}{\partial\lambda}(\text{GoF}^2 - \Delta) = \frac{\partial\text{GoF}^2}{\partial\lambda} = \sum_{\mathbf{h}} \frac{2\eta}{N_r - N_p} \frac{\eta |F_{\mathbf{h}}^{\text{calc}}| - |F_{\mathbf{h}}^{\text{exp}}|}{\sigma_{\mathbf{h}}^2} \frac{\partial |F_{\mathbf{h}}^{\text{calc}}|}{\partial\lambda} \quad (\text{A1})$$

Considering that $|F_{\mathbf{h}}^{\text{calc}}| = [F_{\mathbf{h}}^{\text{calc}} (F_{\mathbf{h}}^{\text{calc}})^*]^{1/2}$, we have:

$$\frac{\partial |F_{\mathbf{h}}^{\text{calc}}|}{\partial\lambda} = \frac{1}{2 |F_{\mathbf{h}}^{\text{calc}}|} \left\{ \frac{\partial F_{\mathbf{h}}^{\text{calc}}}{\partial\lambda} (F_{\mathbf{h}}^{\text{calc}})^* + F_{\mathbf{h}}^{\text{calc}} \frac{\partial (F_{\mathbf{h}}^{\text{calc}})^*}{\partial\lambda} \right\} \quad (\text{A2})$$

Plugging equation (S2) into equation (S1), we obtain:

$$\frac{d^2J}{d\lambda^2} = \sum_{\mathbf{h}} \frac{\eta}{N_r - N_p} \frac{\eta |F_{\mathbf{h}}^{\text{calc}}| - |F_{\mathbf{h}}^{\text{exp}}|}{\sigma_{\mathbf{h}}^2 |F_{\mathbf{h}}^{\text{calc}}|} \left\{ \frac{\partial F_{\mathbf{h}}^{\text{calc}}}{\partial\lambda} (F_{\mathbf{h}}^{\text{calc}})^* + F_{\mathbf{h}}^{\text{calc}} \frac{\partial (F_{\mathbf{h}}^{\text{calc}})^*}{\partial\lambda} \right\} \quad (\text{A3})$$

Now, let us consider $F_{\mathbf{h}}^{\text{calc}}$ and $(F_{\mathbf{h}}^{\text{calc}})^*$. By exploiting the definition of one-electron scattering operator (see equation (8) in the main text), we have:

$$F_{\mathbf{h}}^{\text{calc}} = 2 \sum_{i=1}^N \langle \phi_i | \hat{I}_{\mathbf{h}} | \phi_i \rangle = 2 \sum_{i=1}^N \langle \phi_i | \hat{I}_{\mathbf{h},R} | \phi_i \rangle + i \langle \phi_i | \hat{I}_{\mathbf{h},C} | \phi_i \rangle \quad (\text{A4})$$

$$\begin{aligned}
(F_{\mathbf{h}}^{calc})^* &= 2 \sum_{i=1}^N \langle \phi_i | \hat{I}_{\mathbf{h}} | \phi_i \rangle^* = 2 \sum_{i=1}^N \langle \hat{I}_{\mathbf{h}} \phi_i | \phi_i \rangle \\
&= 2 \sum_{i=1}^N \langle \phi_i | \hat{I}_{\mathbf{h}}^\dagger | \phi_i \rangle = 2 \sum_{i=1}^N \langle \phi_i | \hat{I}_{\mathbf{h},R} | \phi_i \rangle - i \langle \phi_i | \hat{I}_{\mathbf{h},C} | \phi_i \rangle \quad (A5)
\end{aligned}$$

Starting from (A4) and (A5) we can determine $\frac{\partial F_{\mathbf{h}}^{calc}}{\partial \lambda}$ and $\frac{\partial (F_{\mathbf{h}}^{calc})^*}{\partial \lambda}$:

$$\begin{aligned}
\frac{\partial F_{\mathbf{h}}^{calc}}{\partial \lambda} &= 2 \sum_{i=1}^N \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h}} \middle| \phi_i \right\rangle + \langle \phi_i | \hat{I}_{\mathbf{h}} \middle| \frac{\partial \phi_i}{\partial \lambda} \rangle = 2 \sum_{i=1}^N \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h}} \middle| \phi_i \right\rangle + \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h}}^\dagger \middle| \phi_i \right\rangle^* \\
&= 2 \sum_{i=1}^N \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h},R} \middle| \phi_i \right\rangle + i \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h},C} \middle| \phi_i \right\rangle + \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h},R} \middle| \phi_i \right\rangle + i \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h},C} \middle| \phi_i \right\rangle \\
&= 4 \sum_{i=1}^N \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h}} \middle| \phi_i \right\rangle \quad (A6)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial (F_{\mathbf{h}}^{calc})^*}{\partial \lambda} &= 2 \sum_{i=1}^N \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h}}^\dagger \middle| \phi_i \right\rangle + \langle \phi_i | \hat{I}_{\mathbf{h}}^\dagger \middle| \frac{\partial \phi_i}{\partial \lambda} \rangle \\
&= 2 \sum_{i=1}^N \left\langle \hat{I}_{\mathbf{h}}^\dagger \phi_i \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle^* + \langle \phi_i | \hat{I}_{\mathbf{h}}^\dagger \middle| \frac{\partial \phi_i}{\partial \lambda} \rangle = 2 \sum_{i=1}^N \left\langle \phi_i | \hat{I}_{\mathbf{h}} \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle^* + \langle \phi_i | \hat{I}_{\mathbf{h}}^\dagger \middle| \frac{\partial \phi_i}{\partial \lambda} \rangle \\
&= 2 \sum_{i=1}^N \left\langle \phi_i | \hat{I}_{\mathbf{h},R} \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle - i \left\langle \phi_i | \hat{I}_{\mathbf{h},C} \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle + \left\langle \phi_i | \hat{I}_{\mathbf{h},R} \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle - i \left\langle \phi_i | \hat{I}_{\mathbf{h},C} \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle \\
&= 4 \sum_{i=1}^N \left\langle \phi_i | \hat{I}_{\mathbf{h}}^\dagger \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle \quad (A7)
\end{aligned}$$

And it is easy to show that:

$$\frac{\partial (F_{\mathbf{h}}^{calc})^*}{\partial \lambda} = 4 \sum_{i=1}^N \left\langle \phi_i | \hat{I}_{\mathbf{h}}^\dagger \middle| \frac{\partial \phi_i}{\partial \lambda} \right\rangle = 4 \sum_{i=1}^N \left\langle \hat{I}_{\mathbf{h}}^\dagger \frac{\partial \phi_i}{\partial \lambda} \middle| \phi_i \right\rangle^* = 4 \sum_{i=1}^N \left\langle \frac{\partial \phi_i}{\partial \lambda} \middle| \hat{I}_{\mathbf{h}} \middle| \phi_i \right\rangle^* = \left(\frac{\partial F_{\mathbf{h}}^{calc}}{\partial \lambda} \right)^* \quad (A8)$$

Therefore, exploiting equations (A4-A8), equation (A3) becomes

$$\frac{d^2 J}{d\lambda^2} = \sum_{\mathbf{h}} K_{\mathbf{h}} \operatorname{Re} \left\{ F_{\mathbf{h}}^{calc} \frac{\partial (F_{\mathbf{h}}^{calc})^*}{\partial \lambda} \right\} \quad (A9),$$

which is equation (14) in the main text and where $K_{\mathbf{h}}$ is defined by equation (9).

Now, let us analyse the term $\operatorname{Re} \left\{ F_{\mathbf{h}}^{calc} \frac{\partial (F_{\mathbf{h}}^{calc})^*}{\partial \lambda} \right\}$ in equation (A9):

$$\begin{aligned}
\operatorname{Re} \left\{ F_h^{calc} \frac{\partial (F_h^{calc})^*}{\partial \lambda} \right\} &= 4 \operatorname{Re} \left\{ F_h^{calc} \sum_{i=1}^N \left\langle \phi_i \left| \hat{I}_h^\dagger \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle \right\} \\
&= 4 \operatorname{Re} \left\{ \left[\operatorname{Re}\{F_h^{calc}\} + i \operatorname{Im}\{F_h^{calc}\} \right] \left[\sum_{i=1}^N \left\langle \phi_i \left| \hat{I}_{h,R} \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle - i \sum_{i=1}^N \left\langle \phi_i \left| \hat{I}_{h,C} \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle \right] \right\} \\
&= 4 \operatorname{Re}\{F_h^{calc}\} \sum_{i=1}^N \left\langle \phi_i \left| \hat{I}_{h,R} \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle + 4 \operatorname{Im}\{F_h^{calc}\} \sum_{i=1}^N \left\langle \phi_i \left| \hat{I}_{h,C} \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle \\
&= 4 \sum_{i=1}^N \left\langle \phi_i \left| \operatorname{Re}\{F_h^{calc}\} \hat{I}_{h,R} + \operatorname{Im}\{F_h^{calc}\} \hat{I}_{h,C} \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle \tag{A10}
\end{aligned}$$

By substituting (A10) into (A9), we obtain:

$$\frac{d^2 J}{d\lambda^2} = 4 \sum_{i=1}^N \left\langle \phi_i \left| \sum_h K_h \left[\operatorname{Re}\{F_h^{calc}\} \hat{I}_{h,R} + \operatorname{Im}\{F_h^{calc}\} \hat{I}_{h,C} \right] \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle \tag{A11}$$

However, due to the definition of $v_{XCW}(\mathbf{r})$ (see equations (7) and (8) in the main text), we have:

$$\hat{v}_{XCW} = \sum_h K_h \left[\operatorname{Re}\{F_h^{calc}\} \hat{I}_{h,R} + \operatorname{Im}\{F_h^{calc}\} \hat{I}_{h,C} \right] \tag{A12}$$

and, therefore, relation (A11) becomes:

$$\frac{d^2 J}{d\lambda^2} = 4 \sum_{i=1}^N \left\langle \phi_i \left| \hat{v}_{XCW} \left| \frac{\partial \phi_i}{\partial \lambda} \right. \right. \right\rangle = 4 \sum_{i=1}^N \int d\mathbf{r} \phi_i^*(\mathbf{r}) v_{XCW}(\mathbf{r}) \frac{\partial \phi_i(\mathbf{r})}{\partial \lambda} \tag{A13}$$

which is exactly equation (15) in the main text.

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