

MULTIPLICITY RESULTS FOR NONLOCAL CRITICAL PROBLEMS INVOLVING HARDY POTENTIAL IN THE WHOLE SPACE

B. ABDELLAOUI^{1,2}, A. ATTAR², Y.O. BOUKARABILA², E.-H. LAAMRI*

ABSTRACT. The aim of this paper is to study a non-local elliptic problem in \mathbb{R}^N (denoted as (P_λ) below) involving the fractional Laplacian, a linear Hardy potential term and a critical non-linear term. According to suitable assumptions on the set of extremal points of the functional coefficients, we prove that (P_λ) has multiple positive solutions, and we determine their precise behavior near the extremal points. This work extends a previous article by the first author et al. on the local case.

1. INTRODUCTION

The main objective of this work is to study a class of nonlocal nonlinear Schrödinger equations with singular potential. More precisely, we consider the problem

$$(P_\lambda) \begin{cases} (-\Delta)^s u = \frac{(\lambda + h(x))}{|x|^{2s}} u + k(x) u^{2_s^*-1} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \end{cases}$$

where $s \in (0, 1)$, $N > 2s$, $\lambda > 0$, $2_s^* = \frac{2N}{N-2s}$, and the functions h, k are nonnegative, continuous and bounded with additional assumptions that we will specify later. The fractional Laplacian of order s is given by

$$(-\Delta)^s u(x) := c_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad s \in (0, 1), \quad (1.1)$$

where

$$c_{N,s} := \frac{s4^s \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

is the normalization constant such that the following identity

$$(-\Delta)^s u(x) = \mathcal{F}^{-1}[|\xi|^{2s} \mathcal{F}u(\xi)](x), \quad x \in \mathbb{R}^N$$

holds in $\mathcal{S}(\mathbb{R}^N)$, the class of Schwartz functions. For an overview of the fractional laplacians, the interested reader is referred to [12, 27] and the references therein.

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* Corresponding author : El-Haj Laamri.

Nonlinear Schrödinger equations with singular potentials arise in different areas of quantum field theory where the singular potential is used to describe a particle which is free to move between two or more regions of the space separated by some barrier. See for instance [28, 29, 21, 20, 6] and the references therein for more general discussion and other models and applications. They are also used in some models of condensed matter physics, see for instance [36].

Related equations with Hardy potential appear also when analyzing the properties of Aharonov–Bohm magnetic potentials, we refer to [14, 7, 26] for more details about this relation.

As we will see later on, Problem (P_λ) is related to the classical Hardy–Sobolev inequality. In order to be precise and for the ease of the reader, let us recall this classical inequality: for all $u \in \mathcal{C}_0^\infty(\mathbb{R}^N)$

$$\Lambda_{N,s} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} \leq c_{N,s} \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad (1.2)$$

where

$$\Lambda_{N,s} = 2^{2s} \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}. \quad (1.3)$$

One should note that $\Lambda_{N,s}$ is optimal but not achieved ; we refer to [16] for more details about (1.2) and its improved version.

To put our work in context, let us review what has been done concerning Problem (P_λ) .

—*Local case* $s = 1$. When $h = 0$ and $k = 1$, the problem was treated in [37]. In that paper, the author proved that if $\lambda \geq \Lambda_{N,1}$, Problem (P_λ) does not have any positive solution, while if $\lambda \in (0, \Lambda_{N,1})$, (P_λ) has a one-dimensional manifold of positive solutions given by

$$z_\mu(x) = \mu^{-\frac{N-2}{2}} z\left(\frac{x}{\mu}\right),$$

where $\mu > 0$ and

$$z(x) = \frac{(N(N-2)\eta^2)^{\frac{N-2}{2}}}{(|x|^{1-\eta}(1+|x|^{2\eta}))^{\frac{N-2}{2}}} \text{ with } \eta = \left(1 - \frac{\lambda}{\Lambda_{N,1}}\right)^{\frac{1}{2}}.$$

The case where $h, k \neq 0$ was studied by the first author et al. in [2]. Using concentration-compactness arguments and a suitable version of the Palais–Smale condition, the authors showed the existence of a solution.

In addition, denoting $A = \{x_i \in \mathbb{R}^N \setminus \{0\} \text{ where } x_i \text{ is a local maximum of } k\}$, then under additional assumptions on the behavior of k near the points of A , they also got a multiplicity of positive solutions.

—*Nonlocal case* $0 < s < 1$. The case $h = 0$ and $k = 1$ was analyzed in depth in [13]. Using the moving plane method, the authors proved that the problem, in this case, has also a one-dimensional manifold of positive solutions given by $z_\mu(x) = \mu^{-\frac{N-2s}{2}} z(\frac{x}{\mu})$ where z is a normalized radial solution to the same problem. The asymptotic behavior of this solution as $x \rightarrow 0$ or $x \rightarrow +\infty$ was also obtained. However, an explicit formula as in the local case is yet missing.

We refer also to [33], [11], [15] where some partial existence results are obtained using the Caffarelli–Silvestre extension. In those works, the main problem is transformed to another local one defined in $\mathbb{R}^N \times (0, +\infty)$. The analysis of Palais–Smale condition is obtained using suitable trace Theorem in $\mathbb{R}^N \times (0, +\infty)$. It is worth to notice that the authors in [15] consider also the case of multi-pole

singular potential.

It is also interesting to note that in the case where Problem (P_λ) is defined in a bounded star-shaped domain Ω with Dirichlet boundary condition and $h = k = 1$, It is proved in [32] that the unique solution is $u = 0$ by using the Pohozaev-type identity, see [32, Proposition 1.6].

One should not end this short review without mentioning that Problems with Hardy potential and critical nonlinearity have already been studied in the literature, see for instance [3, 15, 17, 33, 31] and the references therein.

Our goal in this work is to extend the results obtained in [2] to the nonlocal case, especially the multiplicity results. To this end, we shall construct a separation functional that allows us to get the existence of multiple positive solutions. Contrary to the local case, some fine estimates are needed in order to show this multiplicity.

Let us summarize the main results and difficulties of the paper.

- We will begin by analyzing the perturbed problem in the Hardy potential, namely we assume that $h \neq 0$ and $k = 1$. We prove that under suitable assumptions on the size and the shape of h , the associated energy function satisfies a local Palais–Smale condition. The proof is based on a nonlocal version of the concentration-compactness argument obtained in [30, Theorem 5]. However, some very technical difficulties are to be solved due to the nonlocal character of the operator and to the fact that the fractional laplacian of a product is not easy to estimate.
- In the second part of the paper, we deal with the question of the multiplicity of positive solutions. In general this phenomenon occurs under additional assumptions on the set of critical points of k . Hence, in order to simplify the presentation, we will assume that $h = 0$ and $k \neq 1$. We begin by proving a local Palais–Smale condition. Then to achieve the desired multiplicity result we assume that k has a finite set of local maximum points with some precise behavior. Our objective is to prove the existence of a solution with some prescribed behavior in the neighborhood of each maximum. The main difficulty will be to distinguish between the different solutions. This will be achieved by defining a suitable separable functional. Let us recall in the local case, the form of the separable functional follows by using a cut-off function that concentrates near a fixed maximum point. However, in the nonlocal case, we have to modify this form since we have to take into consideration the nonlocal character of the problem.

The paper is organized as follows. In Section 2, we gather some useful tools: we introduce the right Sobolev space associated to Problem (P_λ) , and we recall some classical functional inequalities, including Hardy–Sobolev inequalities and a Picone’s inequality, which will be useful to prove the nonexistence results.

In Section 3, we deal with the case $k = 1$ and $h \neq 0$. For the sake of clarity, we have split this section into two subsections. The first one establishes a nonexistence result using a suitable version of the Pohozaev identity. Then, in Subsection 3.2, we show a local Palais–Smale condition and, as a consequence, the existence of non-trivial solutions will be proved under suitable assumptions on the behavior of h near to 0 or near to ∞ . The case $h = 0$ and $k \geq 0$ is considered in Section 4, where we first prove another Palais–Smale condition, and then that the number of positive solutions depends on the set of points where k reaches its maximum. ■

2. PRELIMINARY AND USEFUL TOOLS

In this section, we will recall some useful tools that we will be used in this paper. First, we introduce a natural functional framework for our Problem (P_λ) . Then, we will recall some classic inequalities that we will use to prove our results. Also, we specify the sense in which solution are considered.

• **Functional framework** The appropriate space for our study is the classical fractional Sobolev space

$$\mathcal{D}^{s,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*_s}(\mathbb{R}^N); \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy < \infty \right\}, \quad (2.1)$$

endowed with the norm

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 = c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \quad (2.2)$$

which makes it a Hilbert space.

The fact that the above formula actually defines a norm on $\mathcal{D}^{s,2}(\mathbb{R}^N)$ is a consequence of the following point.

• **Classical Sobolev inequality** Let $s \in (0, 1)$ and $N > 2s$, there exists a positive constant $S = S(N, s)$ such that, for any function $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, we have

$$S \|u\|_{L^{2^*_s}(\mathbb{R}^N)}^2 \leq c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy, \quad (2.3)$$

where $2^*_s = \frac{2N}{N-2s}$.

Now, let us introduce the following functional defined for $A \in (0, \Lambda_{N,s})$ by

$$Q_A(u) = c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - A \int_{\mathbb{R}^N} \frac{u^2(x)}{|x|^{2s}} dx, \quad (2.4)$$

with $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$.

Thanks to Hardy–Sobolev inequality (1.2), we obtain

$$\left(1 - \frac{A}{\Lambda_{N,s}}\right) \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 \leq Q_A(u) \leq \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2.$$

Now, we define the quantity $S(A)$ by

$$S(A) := \inf_{\{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}\}} \frac{Q_A(u)}{\left(\int_{\mathbb{R}^N} |u|^{2^*_s} dx\right)^{\frac{2}{2^*_s}}}. \quad (2.5)$$

It is clear that $S(A) > 0$ and $S(0) = S$ (the Sobolev constant defined in (2.3)).

From [13], we know that $S(A)$ is achieved in one-dimensional manifold of $w_{A,\mu}$ given by $w_{A,\mu}(x) = \mu^{-\frac{N-2s}{2}} w\left(\frac{x}{\mu}\right)$ where w is a normalized radial minimizer of $S(A)$. The behavior of w near 0 and $+\infty$ is also known and it is given by :

$$\frac{C_1}{(|x|^{1-\eta_{s,A}}(1 + |x|^{2\eta_{s,A}}))^{\frac{N-2s}{2}}} \leq w(x) \leq \frac{C_2}{(|x|^{1-\eta_{s,A}}(1 + |x|^{2\eta_{s,A}}))^{\frac{N-2s}{2}}}$$

where $\eta_{s,A} \in (0, \frac{N-2s}{2})$ is the unique solution to the following equation

$$\Psi(\eta_{s,A}) = A \text{ where } \Psi(\sigma) = 2^{2s} \left(\frac{\Gamma(\frac{\sigma+2s}{2})\Gamma(\frac{N-\sigma}{2})}{\Gamma(\frac{N-\sigma-2s}{2})\Gamma(\frac{\sigma}{2})} - \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})} \right). \quad (2.6)$$

We refer to [13] for more details about the existence and the behavior of w .

• **Notion of solution** Let us now make precise what we mean by a solution to Problem (P_λ) .

Definition 2.1. Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and $u \geq 0$. We say that u is a solution to Problem (P_λ) if for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$c_{N,s} P.V \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))}{|x - y|^{N+2s}} (\varphi(x) - \varphi(y)) dy dx = \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u}{|x|^{2s}} \varphi dx + \int_{\mathbb{R}^N} k(x) u^{2_s^* - 1} \varphi dx. \quad (2.7)$$

It is clear that, if u is a solution to Problem (P_λ) , then u is a critical point of the functional

$$E_\lambda(u) = \frac{c_{N,s}}{2} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \frac{1}{2} \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u_+^2}{|x|^{2s}} dx - \frac{1}{2_s^*} \int_{\mathbb{R}^N} k(x) u_+^{2_s^*} dx. \quad (2.8)$$

• **Fractional version of Picone's inequality**

The next Picone type inequality will be used in order to show the nonexistence results. For a proof of this inequality, see for instance [22, Proposition 18] or [1, Lemma 2.3].

Proposition 2.2. Assume that Ω is a strict subset of \mathbb{R}^N or $\Omega = \mathbb{R}^N$. Let $w \in H^s(\Omega)$ be such that $w > 0$ in Ω and $w \geq 0$ in \mathbb{R}^N . Assume that $(-\Delta)^s w = \vartheta$ with $\vartheta \in L_{loc}^1(\mathbb{R}^N)$. Then, for all $\varphi \in C_0^\infty(\Omega)$, we have

$$c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+2s}} dx dy \geq \int_{\Omega} \frac{(-\Delta)^s w}{w} \varphi^2 dx. \quad (2.9)$$

• **An useful Lemma**

The following lemma will be useful to estimate some singular integrals and will be used systematically in the paper. see [35, Chapter V].

Lemma 2.3. Let $0 < \alpha < N$ and $1 \leq p < q < \infty$ be such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{N}$. For $g \in L^p(\mathbb{R}^N)$, we define

$$h_\alpha(g)(x) = \int_{\mathbb{R}^N} \frac{g(y)}{|x - y|^{N-\alpha}} dy.$$

Then, it follows that:

- h_α is well defined in the sense that the integral converges absolutely for almost all $x \in \mathbb{R}^N$.
- If $p > 1$, then $\|h_\alpha(g)\|_{L^q(\mathbb{R}^N)} \leq c_{p,q} \|g\|_{L^p(\mathbb{R}^N)}$.
- If $p = 1$, then $|\{x \in \mathbb{R}^N ; h_\alpha(g)(x) > \sigma\}| \leq c_{1,q} \left(\frac{\|g\|_{L^1(\mathbb{R}^N)}}{\sigma} \right)^q$.

■

3. EXISTENCE AND NONEXISTENCE RESULTS: PERTURBATION IN THE POTENTIAL TERM.

In this section, we take $k \equiv 1$ and we consider a perturbation on the Hardy potential. More precisely, we study the existence and the nonexistence of solutions to the following Problem

$$\begin{cases} (-\Delta)^s u = \frac{(\lambda + h(x))u}{|x|^{2s}} + u^{2_s^*-1} & \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{s,2}(\mathbb{R}^N), \\ u \geq 0 & \text{in } \mathbb{R}^N \end{cases} \quad (3.1)$$

3.1. Nonexistence Results.

In this subsection, we will prove some nonexistence results according to some hypotheses on the datum h . Namely, we shall show that the size and the shape of the perturbation are important for existence of a solution. More precisely, we have the following nonexistence result.

Theorem 3.1. *Assume that one of the following conditions hold:*

(1)

$$I := \inf_{u \in D} \left\{ c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u^2}{|x|^{2s}} dx \right\} < 0,$$

where

$$D := \{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) ; \int_{\mathbb{R}^N} |u|^{2_s^*} dx = 1\},$$

or

(2) h is a differential function where $(x \cdot \nabla h)$ has a fixed sign.

Then, Problem (3.1) does not have any positive solution.

Proof.

• Let us begin by proving the first point. We proceed by contradiction. Assume that Problem (3.1) has a positive solution $u_0 \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ and let $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. Using the Picone inequality of Proposition 2.2, it follows that

$$c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{N+2s}} dx dy \geq \int_{\mathbb{R}^N} \frac{(-\Delta)^s u_0}{u_0} \phi^2 dx \geq \int_{\mathbb{R}^N} \frac{(\lambda + h(x))\phi^2}{|x|^{2s}} dx + \int_{\mathbb{R}^N} u_0^{2_s^*-2} \phi^2 dx.$$

Thus, for all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))\phi^2}{|x|^{2s}} dx \geq \int_{\mathbb{R}^N} u_0^{2_s^*-2} \phi^2 dx.$$

Hence, by a density argument,

$$I := \inf_{\phi \in D} c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(\phi(x) - \phi(y))^2}{|x - y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))\phi^2}{|x|^{2s}} dx \geq 0,$$

which is a contradiction.

•• Now, we deal with the second case. The main idea is to use a Pohozaev-type identity. Let h be a bounded differentiable function and suppose that u is a nonnegative solution to (3.1). Using u as a test function in (3.1), it holds that

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u^2}{|x|^{2s}} dx + \int_{\mathbb{R}^N} u^{2_s^*} dx.$$

Now, using $(x \cdot \nabla u)$ as a test function in (3.1) yields

$$\int_{\mathbb{R}^N} (x \cdot \nabla u)(-\Delta)^s u \, dx = \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u}{|x|^{2s}} (x \cdot \nabla u) \, dx + \int_{\mathbb{R}^N} u^{2^* - 1} (x \cdot \nabla u) \, dx.$$

Or

$$\int_{\mathbb{R}^N} u^{2^* - 1} (x \cdot \nabla u) \, dx = \frac{1}{2^*} \int_{\mathbb{R}^N} (x \cdot \nabla u^{2^*}) \, dx, \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u}{|x|^{2s}} (x \cdot \nabla u) \, dx = \frac{1}{2} \int_{\mathbb{R}^N} \frac{(\lambda + h(x))}{|x|^{2s}} x \cdot \nabla u^2 \, dx.$$

Thus, we obtain

$$\int_{\mathbb{R}^N} \frac{(\lambda + h(x))u}{|x|^{2s}} (x \cdot \nabla u) \, dx = -\frac{1}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} (x \cdot \nabla h) \, dx - \frac{(N - 2s)}{2} \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u^2}{|x|^{2s}} \, dx$$

and

$$\int_{\mathbb{R}^N} (x \cdot \nabla u^{2^*}) \, dx = -N \int_{\mathbb{R}^N} u^{2^*} \, dx.$$

On the other hand, from [32, Proposition 1.6], we get

$$\int_{\mathbb{R}^N} (x \cdot \nabla u)(-\Delta)^s u \, dx = \frac{2s - N}{2} \int_{\mathbb{R}^N} u(-\Delta)^s u \, dx = \frac{2s - N}{2} \|u\|_{D^{s,2}(\mathbb{R}^N)}^2.$$

Putting together the above estimates, we conclude that

$$\frac{1}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} (x \cdot \nabla h) \, dx = 0,$$

which is a contradiction if $(x \cdot \nabla h)$ has a fixed sign. Thus, Problem (3.1) does not have any positive solution. \square

Applying again the Picone inequality, it is easy to prove the following result

Corollary 3.2. *If there exists $r > 0$ such that $\lambda + h(x) > \Lambda_{N,s}$ in $\overline{B_r(0)}$. Then, Problem (3.1) does not have any positive solution. \blacksquare*

3.2. Existence Results.

We recall that the solutions to Problem (3.1) are the critical points of the functional E_λ . a continuous bounded function in \mathbb{R}^N such that the following conditions hold: As already in Section 2, solutions to Problem (3.1) are the critical points of the functional E_λ . So, we shall use a variational approach in order to prove the existence results. On the other hand, in what follows we will assume that h is a continuous bounded function in \mathbb{R}^N such that the following conditions hold:

(H₁) $\lambda + h(0) > 0$;

(H₂) there exists $c_0 > 0$ such that $\lambda + \|h\|_\infty \leq \Lambda_{N,s} - c_0$ where $\Lambda_{N,s}$ is given by Formula (1.3).

Our first main result is the next nonlocal Palais-Smale conditions.

Lemma 3.3. *Suppose that the conditions (H₁) and (H₂) hold, and denote $h(\infty) = \limsup_{|x| \rightarrow \infty} h(x)$.*

Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a Palais-Smale sequence for E_λ , namely,

$$E_\lambda(u_n) \rightarrow c \text{ and } E'_\lambda(u_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.2)$$

If

$$c < c^* = \frac{s}{N} \min \left\{ S(\lambda + h(\infty))^{\frac{N}{2s}}, S(\lambda + h(0))^{\frac{N}{2s}} \right\}, \quad (3.3)$$

then, $\{u_n\}_{n \in \mathbb{N}}$ has a converging subsequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

Proof Let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a Palais-Smale sequence. Using (H_2) and applying Hardy–Sobolev inequality, it holds

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u_n^2}{|x|^{2s}} dx &\leq (\lambda + \|h\|_\infty) \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^{2s}} dx \leq (\Lambda_{N,s} - c_0) \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^{2s}} dx \\ &\leq \left(1 - \frac{c_0}{\Lambda_{N,s}}\right) \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2. \end{aligned}$$

In addition, we have

$$E_\lambda(u_n) - \frac{1}{2_s^*} \langle E'_\lambda(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \left(\frac{1}{2} - \frac{1}{2_s^*}\right) \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u_n^2}{|x|^{2s}} dx = c + o(1).$$

Hence

$$\left(\frac{1}{2} - \frac{1}{2_s^*}\right) \frac{c_0}{\Lambda_N} \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 \leq c + o(1).$$

Thus, $\{u_n\}_n$ is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Consequently, up to a subsequence, $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, $u_n \rightarrow u$ strongly in $L_{loc}^\alpha(\mathbb{R}^N)$ for all $\alpha < 2_s^*$ and $u_n \rightarrow u$ a.e. in \mathbb{R}^N .

Moreover, $|(-\Delta)^{\frac{s}{2}} u_n|^2$ is bounded in $L^1(\mathbb{R}^N)$. Therefore, up to a subsequence, $|(-\Delta)^{\frac{s}{2}} u_n|^2 dx$ converges in the weak* topology to some bounded Radon measure $d\varpi$. Similarly, $|u_n|^{2_s^*} dx \rightharpoonup d\omega$ where $\omega \in \mathcal{M}_b(\mathbb{R}^N)$ (the set of bounded Radon measures).

Using the generalization of the concentration-compactness (see [30, Theorem 5]), there exists a set of distinct points $\{x_j\}_{j \in J} \subset \mathbb{R}^N$ and nonnegative numbers $\{\mu_j, \nu_j\}_{j \in J}$ such that

- (1) $|(-\Delta)^{\frac{s}{2}} u_n|^2 dx \rightharpoonup d\varpi \geq |(-\Delta)^{\frac{s}{2}} u|^2 dx + \sum_{j \in J} \mu_j \delta_{x_j}$;
- (2) $|u_n|^{2_s^*} dx \rightharpoonup d\omega = |u|^{2_s^*} dx + \sum_{j \in J} \nu_j \delta_{x_j}$;
- (3) $S \nu_j^{\frac{2}{2_s^*}} \leq \mu_j$;
- (4) $\frac{u_n^2}{|x|^{2s}} \rightharpoonup d\varsigma = \frac{u^2}{|x|^{2s}} dx + \gamma_0 \delta_0$;
- (5) $\Lambda_{N,s} \gamma_0 \leq \mu_0$.

In order to analyze the concentration at infinity, we proceed in the same way as before. To this end, let us introduce the following quantities

$$\begin{aligned} \nu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{2_s^*} dx, & \mu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy \\ \gamma_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} \frac{u_n^2}{|x|^{2s}} dx \quad \text{where } \gamma_\infty, \mu_\infty \text{ and } \nu_\infty \text{ are concentrations at } \infty. \end{aligned}$$

From now on, we will present the remainder of the proof in two steps.

First step: J is finite.

— Let $j \in J$ be fixed. For $\varepsilon > 0$, let us consider a regular function ϕ_ε such that $0 \leq \phi_\varepsilon \leq 1$ and

$$\phi_\varepsilon(x) = \begin{cases} 1 & \text{if } |x - x_j| \leq \frac{\varepsilon}{2}, \\ 0 & \text{if } |x - x_j| \geq \varepsilon, \end{cases} \quad (3.4)$$

and $|\nabla\phi_\varepsilon| \leq \frac{8}{\varepsilon}$.

So,

$$\begin{aligned} o(1) = \langle E'_\lambda(u_n), \phi_\varepsilon u_n \rangle &= \int_{\mathbb{R}^N} (-\Delta)^s u_n (u_n \phi_\varepsilon) dx \\ &\quad - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))}{|x|^{2s}} u_n^2 \phi_\varepsilon dx - \int_{\mathbb{R}^N} |u_n|^{2^*_s} \phi_\varepsilon dx. \end{aligned}$$

For $x_j \neq 0$, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{(\lambda + h(x))}{|x|^{2s}} u_n^2 \phi_\varepsilon dx = \int_{\mathbb{R}^N} \frac{(\lambda + h(x)) u^2 \phi_\varepsilon}{|x|^{2s}} dx,$$

then

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{(\lambda + h(x))}{|x|^{2s}} u_n^2 \phi_\varepsilon dx = \lim_{\varepsilon \rightarrow 0} \int_{B_{x_j}(\varepsilon)} \frac{(\lambda + h(x)) u^2 \phi_\varepsilon}{|x|^{2s}} dx = 0.$$

On the other hand, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_s} \phi_\varepsilon dx = \int_{\mathbb{R}^N} \phi_\varepsilon d\omega = \int_{\mathbb{R}^N} |u|^{2^*_s} \phi_\varepsilon dx + \sum_{i \in J} \nu_i \phi_\varepsilon(x_i).$$

Thus

$$\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^{2^*_s} \phi_\varepsilon dx = \nu_j. \quad (3.5)$$

— Now, we deal with the term $\int_{\mathbb{R}^N} (-\Delta)^s u_n (u_n \phi_\varepsilon) dx$. As we have

$$\int_{\mathbb{R}^N} (-\Delta)^s u_n (u_n \phi_\varepsilon) dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}}(u_n) (-\Delta)^{\frac{s}{2}}(u_n \phi_\varepsilon) dx,$$

we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^s u_n (u_n \phi_\varepsilon) dx &= \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n(x) c_{N,s} \int_{\mathbb{R}^N} \frac{(u_n(x) \phi_\varepsilon(x) - u_n(y) \phi_\varepsilon(y))}{|x-y|^{N+s}} dy dx \\ &= c_{N,s} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n(x) \left(\int_{\mathbb{R}^N} \frac{\phi_\varepsilon(x) [u_n(x) - u_n(y)]}{|x-y|^{N+s}} dy \right) dx \\ &\quad + c_{N,s} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n(x) \left(\int_{\mathbb{R}^N} u_n(y) \frac{[\phi_\varepsilon(x) - \phi_\varepsilon(y)]}{|x-y|^{N+s}} dy \right) dx \end{aligned}$$

Consequently,

$$\begin{aligned} \int_{\mathbb{R}^N} (-\Delta)^s u_n (u_n \phi_\varepsilon) dx &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(u_n)|^2 \phi_\varepsilon(x) dx \\ &\quad + c_{N,s} \int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n(x) \left(\int_{\mathbb{R}^N} u_n(y) \frac{[\phi_\varepsilon(x) - \phi_\varepsilon(y)]}{|x-y|^{N+s}} dy \right) dx. \end{aligned} \quad (3.6)$$

Moreover

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(u_n)|^2 \phi_\varepsilon(x) dx = \int_{\mathbb{R}^N} \phi_\varepsilon(x) d\mu(x) + o(1), \quad \text{as } n \rightarrow \infty,$$

hence

$$\limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(u_n)|^2 \phi_\varepsilon(x) dx = \mu_j. \quad (3.7)$$

— Now we analyze the term

$$\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n(x) \left(\int_{\mathbb{R}^N} u_n(y) \frac{[\phi_\varepsilon(x) - \phi_\varepsilon(y)]}{|x-y|^{N+s}} dy \right) dx.$$

Given that $(-\Delta)^{\frac{s}{2}} u_n$ is bounded in $L^2(\mathbb{R}^N)$, we will prove that

$$\limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} u_n(y) \frac{[\phi_\varepsilon(x) - \phi_\varepsilon(y)]}{|x-y|^{N+s}} dy \right) = 0 \text{ in } L^2(\mathbb{R}^N).$$

For this, let us define

$$\ell_{\varepsilon,n}(x) := \int_{\mathbb{R}^N} u_n(y) \frac{\phi_\varepsilon(x) - \phi_\varepsilon(y)}{|x-y|^{N+s}} dy.$$

So we have,

$$\begin{aligned} \int_{\mathbb{R}^N} \ell_{\varepsilon,n}^2(x) dx &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} u_n(y) \frac{\phi_\varepsilon(x) - \phi_\varepsilon(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &= \int_{B_{x_j}(\varepsilon)} \left(\int_{\mathbb{R}^N} u_n(y) \frac{\phi_\varepsilon(x) - \phi_\varepsilon(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &\quad + \int_{\mathbb{R}^N \setminus B_{x_j}(\varepsilon)} \left(\int_{B_{x_j}(\varepsilon)} u_n(y) \frac{\phi_\varepsilon(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &= I_1(\varepsilon, n) + I_2(\varepsilon, n). \end{aligned}$$

Note that,

$$\begin{aligned} I_2(\varepsilon, n) &\leq C \int_{\mathbb{R}^N \setminus B_{x_j}(1)} \frac{dx}{|x|^{N+s}} \left(\int_{B_{x_j}(\varepsilon)} u_n(y) \phi_\varepsilon(y) dy \right)^2 \\ &\quad + \int_{B_{x_j}(1)} \left(\int_{B_{x_j}(\varepsilon)} u_n(y) \frac{\phi_\varepsilon(x) - \phi_\varepsilon(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &\leq C \left(\int_{B_{x_j}(\varepsilon)} u_n(y) \phi_\varepsilon(y) dy \right)^2 + J_2(\varepsilon, n) \\ &\leq \|u_n\|_{L^2(\mathbb{R}^N)}^2 \|\phi_\varepsilon\|_{L^2(B_{x_j}(\varepsilon))}^2 + J_2(\varepsilon, n) \\ &\leq C\varepsilon^N + J_2(\varepsilon, n). \end{aligned}$$

Let us estimate $J_2(\varepsilon, n)$. We have

$$\begin{aligned} J_2(\varepsilon, n) &\leq C\varepsilon^{-2} \int_{B_{x_j}(1)} \left(\int_{B_{x_j}(\varepsilon)} \frac{u_n(y)}{|x-y|^{N+s-1}} dy \right)^2 dx \\ &\leq C\varepsilon^{-2} \int_{B_{x_j}(1)} \left(\int_{\mathbb{R}^N} \frac{v_n(y)}{|x-y|^{N+s-1}} dy \right)^2 dx, \end{aligned}$$

where $v_n(y) := u_n(y) \chi_{B_{x_j}(\varepsilon)}(y)$. Since $\{v_n\}_n$ is bounded in $L^q(\mathbb{R}^N)$ for all $1 \leq q \leq 2_s^*$, let introduce

$$h_{n,\varepsilon}(x) := \int_{\mathbb{R}^N} \frac{v_n(y)}{|x-y|^{N+s-1}} dy.$$

Thanks to Lemma 2.3, $\{h_n\}_n$ is bounded in $L^\beta(\mathbb{R}^N)$ for all $\beta \in (\frac{N}{N-(1-s)}, 2^*]$ and

$$\|h_{n,\varepsilon}\|_{L^\beta(\mathbb{R}^N)} \leq C \|u_n\|_{L^q(B_{x_j}(\varepsilon))}$$

where $\frac{1}{\beta} = \frac{1}{q} - \frac{1-s}{N}$.

Fix $1 < q < \frac{2N}{N+2-2s}$. Then, we have

$$J_2(n, \varepsilon) \leq C \varepsilon^d \|u_n\|_{L^{2^*}(B_\varepsilon(x_j))}^2 \leq C \varepsilon^d$$

where $d = N(\frac{2}{q} - \frac{2}{2^*}) - 2 > 0$. Hence we conclude that

$$\limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} I_2(\varepsilon, n) = 0. \quad (3.8)$$

— Next, we estimate the term $I_1(\varepsilon, n)$.

$$\begin{aligned} I_1(\varepsilon, n) &= \int_{B_{x_j}(\varepsilon)} \left(\int_{\mathbb{R}^N} u_n(y) \frac{\phi_\varepsilon(x) - \phi_\varepsilon(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &= \int_{B_{x_j}(\varepsilon)} \phi_\varepsilon(x)^2 \left(\int_{\mathbb{R}^N \setminus B_{x_j}(\varepsilon)} \frac{u_n(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &\quad + \int_{B_{x_j}(\varepsilon)} \left(\int_{B_{x_j}(\varepsilon)} u_n(y) \frac{\phi_\varepsilon(x) - \phi_\varepsilon(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &= D_1(\varepsilon, n) + D_2(\varepsilon, n). \end{aligned}$$

With the same decomposition made for the term $J_2(\varepsilon, n)$, we get

$$\limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} D_2(\varepsilon, n) = 0. \quad (3.9)$$

For $D_1(\varepsilon, n)$, we have

$$\begin{aligned} D_1(\varepsilon, n) &= \int_{B_{x_j}(\varepsilon)} \phi_\varepsilon^2(x) \left(\int_{\mathbb{R}^N \setminus B_{x_j}(\varepsilon)} \frac{u_n(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &\leq \int_{B_{x_j}(\varepsilon)} \phi_\varepsilon^2(x) \left(\int_{\mathbb{R}^N \setminus B_{x_j}(1)} \frac{u_n(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &\quad + C\varepsilon^{-2} \int_{B_{x_j}(\varepsilon)} \left(\int_{B_{x_j}(1) \setminus B_{x_j}(\varepsilon)} \frac{u_n(y)}{|x-y|^{N+s-1}} dy \right)^2 dx. \end{aligned}$$

By Cauchy-Schwarz inequality and the fact that $\{u_n\}_n$ is bounded in $L^2(\mathbb{R}^N)$, we get

$$\begin{aligned} &\int_{B_{x_j}(\varepsilon)} \phi_\varepsilon^2(x) \left(\int_{\mathbb{R}^N \setminus B_{x_j}(1)} \frac{u_n(y)}{|x-y|^{N+s}} dy \right)^2 dx \\ &\leq \int_{B_{x_j}(\varepsilon)} \phi_\varepsilon^2(x) \left(\int_{\mathbb{R}^N \setminus B_{x_j}(1)} u_n^2(y) dy \right) \left(\int_{\mathbb{R}^N \setminus B_{x_j}(1)} \frac{1}{|x-y|^{2(N+s)}} dy \right) \end{aligned}$$

Since $x \in B_{x_j}(\varepsilon)$ and $y \in \mathbb{R}^N \setminus B_{x_j}(1)$, then $|x - y| \geq 1 - \varepsilon \geq \frac{1}{2}$. Hence

$$\sup_{x \in B_{x_j}(\varepsilon)} \int_{\mathbb{R}^N \setminus B_{x_j}(1)} \frac{1}{|x - y|^{2(N+s)}} dy \leq C.$$

Thus

$$\int_{B_{x_j}(\varepsilon)} \phi_\varepsilon^2(x) \left(\int_{\mathbb{R}^N \setminus B_{x_j}(1)} \frac{u_n(y)}{|x - y|^{N+s}} dy \right)^2 dx \leq C \int_{B_{x_j}(\varepsilon)} \phi_\varepsilon^2(x) dx \leq C\varepsilon^N.$$

We note

$$h_{n,\varepsilon}(x) := \int_{B_{x_j}(1) \setminus B_{x_j}(\varepsilon)} \frac{|u_n(y)|}{|x - y|^{N+s-1}} dy, \quad \text{and } h(x) := \int_{B_{x_j}(1) \setminus B_{x_j}(\varepsilon)} \frac{|u(y)|}{|x - y|^{N+s-1}} dy.$$

Thanks to Lemma 2.3, we obtain

$$\|h_n\|_{L^{2^*_s}(B_{x_j}(1) \setminus B_{x_j}(\varepsilon))} \leq C \|u_n\|_{L^{2^*_s}(\mathbb{R}^N)} \leq C.$$

Moreover, for fixed $\varepsilon > 0$, we have for all $m < 2^*_s$,

$$\|h_n - h\|_{L^q(B_{x_j}(1) \setminus B_{x_j}(\varepsilon))} \leq C \|u_n - u\|_{L^m(B_{x_j}(\varepsilon))},$$

where $q = \frac{sm}{s - m(1 - s)} < 2^*$. Thus, h_n converges strongly to h in $L^q(B_{x_j}(1) \setminus B_{x_j}(\varepsilon))$, in particular for $q = 2$. Therefore

$$\lim_{n \rightarrow \infty} \varepsilon^{-2} \int_{B_{x_j}(\varepsilon)} \left(\int_{B_{x_j}(1) \setminus B_{x_j}(\varepsilon)} \frac{u_n(y)}{|x - y|^{N+s-1}} dy \right)^2 dx = \varepsilon^{-2} \int_{B_{x_j}(\varepsilon)} \left(\int_{B_{x_j}(1) \setminus B_{x_j}(\varepsilon)} \frac{u(y)}{|x - y|^{N+s-1}} dy \right)^2 dx.$$

Moreover

$$\begin{aligned} & \varepsilon^{-2} \int_{B_{x_j}(\varepsilon)} \left(\int_{B_{x_j}(1) \setminus B_{x_j}(\varepsilon)} \frac{u(y)}{|x - y|^{N+s-1}} dy \right)^2 dx \\ & \leq C \left(\int_{B_{x_j}(\varepsilon)} h^{2^*}(x) dx \right)^{\frac{2}{2^*}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

So, we obtain

$$\limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \varepsilon^{-2} \int_{B_{x_j}(\varepsilon)} \left(\int_{B_{x_j}(1) \setminus B_{x_j}(\varepsilon)} \frac{u_n(y)}{|x - y|^{N+s-1}} dy \right)^2 dx = 0.$$

Consequently

$$\limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} D_1(\varepsilon, n) = 0. \quad (3.10)$$

So, by combining (3.6) and (3.7), we conclude that

$$\limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (-\Delta)^s u_n (u_n \phi_\varepsilon) dx = \limsup_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}(u_n)|^2 \phi_\varepsilon(x) dx = \mu_j. \quad (3.11)$$

Finally, combing (3.5) and (3.11) yields $\mu_j - \nu_j \leq 0$. Since $S\nu_j^{\frac{2}{2^*_s}} \leq \mu_j$, then we get that either $\nu_j = 0$ or $S^{\frac{N}{2s}} \leq \nu_j$ which implies that J is finite.

Hence we conclude the proof of the first step.

Second step: Now, we analyze the concentration at $x_j = 0$ and at ∞ .

For $R > 1$, we consider a regular bounded function ψ such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \geq 2R \\ 0 & \text{if } |x| < R \end{cases} \quad (3.12)$$

and $|\nabla\psi| \leq \frac{4}{R}$. In addition, let us introduce

$$Q_{\lambda+h(\infty)}(u) := c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - (\lambda + h(\infty)) \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx,$$

and consider

$$S(\lambda + h(\infty)) := \inf_{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}} \frac{Q_{\lambda+h(\infty)}(u)}{\left(\int_{\mathbb{R}^N} |u|^{2^*_s} dx \right)^{\frac{2}{2^*_s}}}.$$

We deduce from [13, Theorem 1.5] that $S(\lambda + h(\infty)) > 0$. Setting $w = u_n\psi$, it holds

$$\begin{aligned} c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|(u_n\psi)(x) - (u_n\psi)(y)|^2}{|x - y|^{N+2s}} dx dy &\geq \\ (\lambda + h(\infty)) \int_{\mathbb{R}^N} \frac{u_n^2 \psi^2}{|x|^{2s}} dx + S(\lambda + h(\infty)) \left(\int_{\mathbb{R}^N} |u_n\psi|^{2^*_s} dx \right)^{\frac{2}{2^*_s}} \end{aligned} \quad (3.13)$$

We will estimate each term of (3.13). For this purpose, we will write $|(u_n\psi)(x) - (u_n\psi)(y)|^2$ under the following form

$$|u_n(x) - u_n(y)|^2 \psi^2(x) + |\psi(x) - \psi(y)|^2 u_n^2(y) + 2\psi(x)u_n(y)(\psi(x) - \psi(y))(u_n(x) - u_n(y)). \quad (3.14)$$

and we set

$$\begin{aligned} I_1(n, R) &:= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy, \\ I_2(n, R) &:= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\psi(x)u_n(y)(\psi(x) - \psi(y))(u_n(x) - u_n(y))}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

We claim that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_1(n, R) = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_2(n, R) = 0. \quad (3.15)$$

• **Case of $I_1(n, R)$**

Thanks to (3.14), we have

$$\begin{aligned} I_1(n, R) &= \int_{B_{2R}(0)} \int_{B_{2R}(0)} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy + \int_{B_{2R}^c(0)} \int_{B_{2R}(0)} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy \\ &+ \int_{B_{2R}(0)} \int_{B_{2R}^c(0)} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy + \int_{B_{2R}^c(0)} \int_{B_{2R}^c(0)} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy \\ &= I_1^1(n, R) + I_1^2(n, R) + I_1^3(n, R) + I_1^4(n, R). \end{aligned}$$

First, it is clear that $I_1^4(n, R) = 0$.

Second, taking into consideration that for all $(x, y) \in B_{2R}(0) \times B_{2R}(0)$, $|\psi(x) - \psi(y)| \leq C \frac{|x-y|}{R}$,

we reach that

$$\begin{aligned} I_1^1(n, R) &= \int_{|y| < 2R} \int_{|x| < 2R} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy \\ &\leq C \int_{|y| < 2R} u_n^2(y) \int_{|\xi| > 2R} \frac{d\xi}{|\xi|^{N+2s}} dy + \frac{C}{R^2} \int_{|y| < 2R} u_n^2(y) \int_{|\xi| < 2R} \frac{d\xi}{|\xi|^{N+2s}} dy. \end{aligned}$$

So,

$$\limsup_{n \rightarrow \infty} I_1^1(n, R) \leq \frac{C}{R^{2s}} \int_{|y| < 2R} u^2(y) dy,$$

which implies

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_1^1(n, R) = 0. \quad (3.16)$$

Third, we deal now with $I_1^2(n, R)$.

$$\begin{aligned} I_1^2(n, R) &= \int_{B_{2R}^c(0)} \int_{B_{2R}(0)} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy = \int_{|y| > 2R} \int_{|x| < 2R} \frac{|\psi(x) - 1|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy \\ &= \int_{|y| > 2R} u_n^2(y) \int_{|x| < \frac{R}{2}} \frac{|\psi(x) - \psi(y)|^2 dx}{|x - y|^{N+2s}} dy + \int_{|y| > 2R} u_n^2(y) \int_{\frac{R}{2} < |x| < 2R} \frac{|\psi(x) - 1|^2 dx}{|x - y|^{N+2s}} dy \\ &= I_{11}^2(n, R) + I_{12}^2(n, R). \end{aligned}$$

Notice that, if $|y| > 2R$ and $|x| < \frac{R}{2}$, then $|x - y| \geq \frac{|x|}{2}$ and we have

$$I_{11}^2(n, R) \leq \frac{C_2}{R^2} \int_{|y| > 2R} u_n^2(y) \int_{|x| < \frac{R}{2}} \frac{dx}{|x|^{N+2s-2}} dy \leq \frac{C}{R^{2s}}.$$

Now, we have

$$\begin{aligned} I_{12}^2(n, R) &\leq \int_{|y| > 2R} u_n^2(y) \int_{\frac{R}{2} < |x| < 2R, |x-y| < 2R} \frac{|\psi(x) - \psi(y)|^2}{|x - y|^{N+2s}} dx dy \\ &\quad + \int_{|y| > 2R} u_n^2(y) \int_{\frac{R}{2} < |x| < 2R, |x-y| > 2R} \frac{1}{|x - y|^{N+2s}} dx dy \\ &\leq \frac{c}{R^2} \int_{|y| > 2R} u_n^2(y) \int_{|\xi| < 2R} \frac{d\xi}{|\xi|^{N+2s-2}} dy + \int_{|y| > 2R} u_n^2(y) \int_{|\xi| > 2R} \frac{d\xi}{|\xi|^{N+2s}} dy \\ &\leq \frac{c}{R^{2s}} \int_{|y| > 2R} u_n^2(y) dy \leq \frac{C}{R^{2s}}. \end{aligned}$$

Thus

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_1^2(n, R) = 0.$$

Finally we deal with $I_1^3(n, R)$. First, we have

$$\begin{aligned} I_1^3(n, R) &= \int_{B_{2R}(0)} \int_{B_{2R}^c(0)} \frac{|\psi(x) - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy = \int_{|y| < 2R} \int_{|x| > 2R} \frac{|1 - \psi(y)|^2 u_n^2(y)}{|x - y|^{N+2s}} dx dy \\ &= \int_{|y| < R} u_n^2(y) \int_{|x| > 2R} \frac{dx}{|x - y|^{N+2s}} dy + \int_{R < |y| < 2R} |1 - \psi(y)|^2 u_n^2(y) \int_{|x| > 2R} \frac{dx}{|x - y|^{N+2s}} dy. \end{aligned}$$

By letting $n \rightarrow \infty$, we obtain

$$\limsup_{n \rightarrow \infty} I(n, R) = \int_{|y| < R} u^2(y) \int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} dy + \int_{R < |y| < 2R} |1-\psi(y)|^2 u^2(y) \int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} dy.$$

Moreover, we have

$$\int_{|y| < R} u^2(y) \int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} dy \leq \frac{C}{R^{2s}} \int_{|y| < R} u^2(y) dy \leq \frac{C}{R^{2s}}.$$

Now, let us estimate the term

$$\int_{R < |y| < 2R} |1-\psi(y)|^2 u^2(y) \int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} dy.$$

Putting $\rho = |x|$, $r = |y|$, then $y = ry'$, $x = \rho x'$ with $|x'| = |y'| = 1$. So,

$$\int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} = \int_{2R}^{\infty} \rho^{N-1} \int_{S^{N-1}} \frac{dx'}{|\rho x' - ry'|^{N+2s}} d\rho = \frac{1}{|y|^{N+2s}} \int_{2R}^{\infty} \rho^{N-1} \int_{S^{N-1}} \frac{dx'}{|y' - x' \frac{\rho}{r}|^{N+2s}} d\rho.$$

By denoting $\sigma = \frac{\rho}{r}$, we obtain

$$\int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} = \frac{1}{|y|^{2s}} \int_{\frac{2R}{r}}^{\infty} \sigma^{N-1} K(\sigma) d\sigma,$$

where

$$K(\sigma) := \int_{S^{N-1}} \frac{dx'}{|y' - x' \frac{\rho}{r}|^{N+2s}} = \int_{|x'|=1} \frac{dH^{n-1}(x')}{|y' - \sigma x'|^{N+2s}}.$$

Since $\rho > 2R$, then $r\sigma > 2R$. So, as $\sigma \rightarrow \infty$, we get

$$K(\sigma)\sigma^{N-1} \simeq \sigma^{-1-2s} \in L^1((2, \infty)).$$

On the other hand, as $\sigma \rightarrow 1$, we obtain

$$K(\sigma) \simeq (\sigma - 1)^{-2s} \in L^1((1, 2)).$$

Whenever A is big and ϵ is small enough, we get

$$\begin{aligned} \int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} &= \frac{1}{|y|^{2s}} \int_{\frac{2R}{r}}^{\infty} \sigma^{N-1} K(\sigma) d\sigma \\ &= \frac{1}{|y|^{2s}} \int_A^{\infty} \sigma^{N-1} K(\sigma) d\sigma + \frac{1}{|y|^{2s}} \int_{1-\epsilon}^{1+\epsilon} \sigma^{N-1} K(\sigma) d\sigma + C, \\ &\simeq CA^{-2s} + C + C \frac{1}{|y|^{2s}} \leq C + C \frac{1}{|y|^{2s}} \\ &\leq C + C \frac{1}{R^{2s}}. \end{aligned}$$

Hence, for large $R > 0$,

$$\int_{R < |y| < 2R} |1-\psi(y)|^2 u^2(y) \int_{|x| > 2R} \frac{dx}{|x-y|^{N+2s}} dy \leq C \int_{R < |y| < 2R} u^2(y) dy = o(1) \quad \text{as } R \rightarrow \infty.$$

Consequently, the result follows.

As a conclusion we obtain that

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_1^3(n, R) = 0. \quad (3.17)$$

• **Case of $I_2(n, R)$**

We treat now the term $I_2(n, R)$. Using Cauchy-Schwarz inequality, it holds

$$\begin{aligned} I_2(n, R) &\leq \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u_n^2(y)(\psi(x) - \psi(y))^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_n(x) - u_n(y))^2 \psi(x)^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \\ &\leq I_1^{\frac{1}{2}}(n, R) \left(\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_n(x) - u_n(y))^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}} \leq c I_1^{\frac{1}{2}}(n, R). \end{aligned}$$

Hence

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_2(n, R) = 0. \quad (3.18)$$

Therefore

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_1(n, R) = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} I_2(n, R) = 0. \quad (3.19)$$

Hence the claim is proved.

As a conclusion, we get

$$c_{N,s} \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|(u_n \psi)(x) - (u_n \psi)(y)|^2}{|x - y|^{N+2s}} dx dy = \mu_\infty.$$

Thus, passing to the limit in (3.13), we obtain

$$\mu_\infty \geq (\lambda + h(\infty))\gamma_\infty + S(\lambda + h(\infty))\nu_\infty^{\frac{2}{2^*}}. \quad (3.20)$$

Or

$$\begin{aligned} o(1) = \langle E'_\lambda(u_n), \psi u_n \rangle &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_n(x) - u_n(y))}{|x - y|^{N+2s}} (\psi(x)u_n(x) - \psi(y)u_n(y)) dx dy \\ &\quad - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))}{|x|^{2s}} u_n^2 \psi dx - \int_{\mathbb{R}^N} |u_n|^{2^*} \psi dx. \end{aligned}$$

Then, by using the same decomposition as in (3.14) and following closely the same kind of computations as in the previous integrals, we obtain

$$0 = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \langle E'_\lambda(u_n), \psi u_n \rangle \geq \mu_\infty - (\lambda + h(\infty))\gamma_\infty - \nu_\infty. \quad (3.21)$$

Hence $\nu_\infty \geq \mu_\infty - (\lambda + h(\infty))\gamma_\infty$. From (3.20) and (3.21), it holds that

$$\nu_\infty \geq S(\lambda + h(\infty))\nu_\infty^{\frac{2}{2^*}}.$$

Then either $\nu_\infty = 0$ nor $\nu_\infty^{1 - \frac{2}{2^*}} \geq S(\lambda + h(\infty))$. Then either $\nu_\infty = 0$ nor $\nu_\infty \geq S(\lambda + h(\infty))^{\frac{N}{2^*}}$.

In a similar way we can treat the concentration where $x_j = 0$, to reach that either $\nu_0 = 0$ or $\nu_0 \geq S(\lambda + h(0))^{\frac{N}{2s}}$. Therefore, we obtain

$$\begin{aligned} c &= E_\lambda(u_n) - \frac{1}{2} \langle E'_\lambda(u_n), u_n \rangle + o(1) = \frac{s}{N} \int_{\mathbb{R}^N} |u_n|^{2^*_s} dx + o(1) \\ &= \frac{s}{N} \left\{ \int_{\mathbb{R}^N} |u|^{2^*_s} dx + \nu_0 + \nu_\infty + \sum_{j \in J} \nu_j \right\}. \end{aligned}$$

If there exists $j \in J \cup \{0, \infty\}$ such that $\nu_j \neq 0$, we obtain $\nu_j \frac{s}{N} \leq c$. Then if $j = 0$, it holds that $\frac{s}{N} S(\lambda + h(0))^{\frac{N}{2s}} \leq \nu_0 \frac{s}{N} \leq c$. Thus

$$c \geq \frac{s}{N} S(\lambda + h(0))^{\frac{N}{2s}} \geq c^*. \quad (3.22)$$

If $j = \infty$, then we conclude that

$$c \geq \frac{s}{N} \nu_\infty \geq \frac{s}{N} S(\lambda + h(\infty))^{\frac{N}{2s}} \geq c^*. \quad (3.23)$$

Finally, if $j \neq 0$ and $j \neq \infty$, it holds that

$$c \geq \frac{s}{N} \nu_j \geq \frac{s}{N} S^{\frac{N}{2s}} \geq c^*. \quad (3.24)$$

As a conclusion, we get that $c \geq c^*$. Or, this is in contradiction with the main assumption on c in (3.3). Hence, for all $j \in J \cup \{0, \infty\}$, $\nu_j = 0$ and consequently

$$\int_{\mathbb{R}^N} |u_n|^{2^*_s} = \int_{\mathbb{R}^N} |u|^{2^*_s} + o(1).$$

By Brezis–Lieb Lemma (see [8]), we deduce that

$$u_n \rightarrow u \quad \text{strongly in } L^{2^*_s}(\mathbb{R}^N),$$

which implies that

$$u_n \rightarrow u \quad \text{strongly in } \mathcal{D}^{s,2}(\mathbb{R}^N).$$

And the result follows. \square

Now let us set $H := \max\{h(0), h(\infty)\}$ and consider $\{w_\mu\}$, the parameter family of minimizer of $S(A)$, where A is replaced by $\lambda + H$. Then, we have the following result :

Theorem 3.4. *Suppose that (H_1) and (H_2) hold. Assume the existence of $\mu_0 > 0$ such that*

$$\int_{\mathbb{R}^N} \frac{\lambda + h(x)}{|x|^{2s}} w_{\mu_0}^2 dx > H \int_{\mathbb{R}^N} \frac{w_{\mu_0}^2}{|x|^{2s}} dx. \quad (3.25)$$

Then, Problem (3.1) has at least one positive solution.

Proof.

Let μ_0 satisfying (3.25). Then,

$$\begin{aligned} & E_\lambda(tw_{\mu_0}) \\ &= \frac{t^2}{2} \left(c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_{\mu_0}(x) - w_{\mu_0}(y)|^2}{|x - y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))w_{\mu_0}^2}{|x|^{2s}} dx \right) - \frac{t^{2^*_s}}{2^*_s} \int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*_s} dx. \end{aligned}$$

We can easily see that E_λ achieves its maximum at some $t_0 > 0$. Since $\langle E'_\lambda(t_0 w_{\mu_0}), t_0 w_{\mu_0} \rangle = 0$, then

$$t_0^{2^*_s-2} = \frac{c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_{\mu_0}(x) - w_{\mu_0}(y)|^2}{|x-y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))w_{\mu_0}^2}{|x|^{2s}} dx}{\int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*_s} dx}.$$

Thus

$$\begin{aligned} E_\lambda(t_0 w_{\mu_0}) &= \max_{t \geq 0} E_\lambda(t w_{\mu_0}) \\ &= \frac{s}{N} \left[\frac{c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_{\mu_0}(x) - w_{\mu_0}(y)|^2}{|x-y|^{N+2s}} dx dy - \int_{\mathbb{R}^N} \frac{(\lambda + h(x))w_{\mu_0}^2}{|x|^{2s}} dx}{\int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*_s} dx} \right]^{\frac{2^*_s}{2s}}. \end{aligned} \quad (3.26)$$

Using (3.25), we obtain that

$$E_\lambda(t_0 w_{\mu_0}) < \frac{s}{N} \left[\frac{c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_{\mu_0}(x) - w_{\mu_0}(y)|^2}{|x-y|^{N+2s}} dx dy - (\lambda + H) \int_{\mathbb{R}^N} \frac{w_{\mu_0}^2}{|x|^{2s}} dx}{\left(\int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*_s} dx \right)^{\frac{2^*_s}{2s}}} \right]^{\frac{2^*_s}{2s}} < c^*. \quad (3.27)$$

In the same way, we can prove the existence of $\rho > 0$ such that if $\|t w_{\mu_0}\| > \rho$, we have $E_\lambda(t w_{\mu_0}) < 0$. On the other hand, let us introduce

$$\Gamma := \{\gamma \in C([0, 1], \mathcal{D}^{s,2}(\mathbb{R}^N)) ; \gamma(0) = \gamma(1) = 0 \text{ and } E_\lambda(\gamma(1)) < 0\},$$

and set $c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_\lambda(\gamma(t))$.

Since we have the geometry of Mountain pass, then by [4, Theorem 2.1], we get the existence of a Mountain pass sequence $\{u_n\}_n \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $E_\lambda(u_n) \rightarrow c$ and $E'_\lambda(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Thanks to Lemma 3.3, we reach that, up to a subsequence, $u_n \rightarrow u_0$ where u_0 is a critical point of E_λ . So, in order to complete the proof, we have just to prove that we can choose $u_0 \geq 0$. Notice that, since $c > 0$, then $u_0 \neq 0$. Consider the Nehari manifold

$$\begin{aligned} M &= \{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) ; u \neq 0 \text{ and } \langle E'_\lambda(u), u \rangle = 0\} \\ &= \left\{ u \in \mathcal{D}^{s,2}(\mathbb{R}^N) ; u \neq 0 \text{ and } \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} \frac{(\lambda + h(x))u^2}{|x|^{2s}} dx + \int_{\mathbb{R}^N} |u|^{2^*_s} dx \right\}. \end{aligned}$$

It is clear that $u_0 \in M$. Since u_0 is a solution to Problem (3.1), we can prove easily that

$$c = E_\lambda(u_0) = \min_{u \in M} E_\lambda(u).$$

Moreover, as $E_\lambda(|u_0|) = \min_{u \in M} E_\lambda(u)$, then $|u_0|$ is also a critical point of E_λ . \square

As a consequence and under suitable behavior of h near 0 or ∞ , we have the next existence result.

Corollary 3.5. *Assume that $k = 1$ and $h \geq 0$ such that $\lambda + \|h\|_\infty < \Lambda_{N,s}$. Moreover, assume that one of the following conditions holds*

- *If $h(0) > h(\infty)$, there exists $r > 0$ such that $h(x) > h(0) + c|x|^{(N-2s)\eta_{s,\lambda+H}}$ for all $x \in B_r(0)$;*

or

- if $h(\infty) > h(0)$, there exists $R \gg 1$ such that $h(x) > h(\infty) + c|x|^{-(N-2s)\eta_{s,\lambda+H}}$ for all $|x| > R$, where $\eta_{s,\lambda+H}$ is defined in (2.6) by substituting A by $\lambda + H$. Then, Problem (3.1) has a positive solution.

Proof.

For simplicity of notation, we note $\eta := \eta_{s,\lambda+H}$. Consider a small $\varepsilon > 0$ such that if $|x| < \varepsilon$ one has $h(x) > h(0) + c|x|^{(N-2s)\eta}$. Taking in consideration the behavior of $w(\cdot)$, let define

$$I_{\varepsilon,\mu} = \int_{|x|<\varepsilon} (h(x) - H) \frac{dx}{|x|^{(1-\eta)N+2s\eta} (\mu^{2\eta} + |x|^{2\eta})^{N-2s}}.$$

We have

$$I_{\varepsilon,\mu} \geq C \int_{|x|<\varepsilon} \frac{|x|^{\eta(N-2s)} dx}{|x|^{(1-\eta)N+2s\eta} (\mu^{2\eta} + |x|^{2\eta})^{N-2s}}.$$

By Fatou's lemma and since $(1-\eta)N + 2s\eta + 2\eta(N-2s) - \eta(N-2s) = N$, we get that $I_{\varepsilon,\mu} \rightarrow +\infty$ as $\mu \rightarrow 0$. Moreover,

$$\int_{|x|\geq\varepsilon} |h(x) - H| \frac{dx}{|x|^{(1-\eta)N+2s\eta} (\mu^{2\eta} + |x|^{2\eta})^{N-2s}} \leq C \int_{|x|\geq\varepsilon} \frac{dx}{|x|^{N+\eta(N-2s)}} \leq C(\varepsilon).$$

Consequently, there exists $\mu_0 > 0$ big enough such that

$$\int_{\mathbb{R}^N} \frac{(h(x) - H)}{|x|^{2s}} w_{\mu_0}^2 dx \geq \int_{|x|<\varepsilon} \frac{(h(x) - H)}{|x|^{2s}} w_{\mu_0}^2 dx - \int_{|x|>\varepsilon} \frac{|h(x) - H|}{|x|^{2s}} w_{\mu_0}^2 dx > 0.$$

Therefore the condition (3.25) is fulfilled, which implies the existence of a solution.

Concerning the second case, we use closely the same argument as above.

Hence the result follows. ■

4. PERTURBATION IN THE CRITICAL TERM

In this section, we deal with the following Problem

$$\begin{cases} (-\Delta)^s u = \lambda \frac{u}{|x|^{2s}} + k(x) u^{2^*_s-1} & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{and } u \in \mathcal{D}^{s,2}(\mathbb{R}^N). \end{cases} \quad (4.1)$$

As mentioned before, solutions to Problem (4.1) are critical points to the functional

$$E_\lambda(u) = \frac{c_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx - \frac{1}{2^*_s} \int_{\mathbb{R}^N} k(x) |u(x)|^{2^*_s} dx.$$

In what follows, we suppose the function k is continuous, bounded and such that

$$(K_0) \quad \|k\|_\infty > \max\{k(0), k(\infty)\} \quad \text{where } k(\infty) = \limsup_{|x| \rightarrow \infty} k(x).$$

4.1. A compactness result and application.

Let us begin by proving the following Palais–Smale condition.

Lemma 4.1. *(P.S condition) Suppose that (K_0) holds and let $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}^{s,2}(\mathbb{R}^N)$ be a Palais–Smale sequence for E_λ , namely*

$$E_\lambda(u_n) \rightarrow c < \infty \text{ and } E'_\lambda(u_n) \rightarrow 0.$$

If

$$c < c^* = \frac{s}{N} \|k\|_\infty^{1-\frac{N}{2s}} S(\lambda)^{\frac{N}{2s}},$$

then $\{u_n\}_n$ has a converging subsequence in $\mathcal{D}^{s,2}(\mathbb{R}^N)$.

Proof.

We argue as in the proof of Lemma 3.3. It is not difficult to show that $\{u_n\}_n$ is bounded. Hence, up to a subsequence,

$$u_n \rightharpoonup u \text{ weakly in } \mathcal{D}^{s,2}(\mathbb{R}^N) \text{ and } u_n \rightarrow u \text{ strongly in } L_{loc}^\alpha(\mathbb{R}^N) \quad \forall \alpha \text{ with } 1 < \alpha < 2_s^*.$$

Using the concentration compactness Theorem [30, Theorem 5], we deduce that

- (1) $|(-\Delta)^{\frac{s}{2}} u_n|^2 dx \rightharpoonup d\varpi \geq |(-\Delta)^{\frac{s}{2}} u|^2 dx + \sum_{j \in J} \mu_j \delta_{x_j}$;
- (2) $|u_n|^{2_s^*} dx \rightharpoonup d\omega = |u|^{2_s^*} + \sum_{j \in J} \nu_j \delta_{x_j}$;
- (3) $S \nu_j^{\frac{2}{2_s^*}} \leq \mu_j$;
- (4) $\frac{u_n^2}{|x|^{2s}} \rightharpoonup d\varsigma = \frac{u^2}{|x|^{2s}} + \gamma_0 \delta_0$;
- (5) $\Lambda_N \gamma_0 \leq \mu_0$.

In the same way, we define the concentration at infinity by

$$\begin{aligned} \nu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |u_n|^{2_s^*} dx, & \mu_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} \int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^2}{|x - y|^{N+2s}} dx dy \\ \gamma_\infty &= \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} \frac{u_n^2}{|x|^{2s}} dx \quad \text{where } \gamma_\infty, \mu_\infty \text{ and } \nu_\infty \text{ are concentration at } \infty. \end{aligned}$$

Following the estimates as in the proof of Lemma 3.3, it follows that E_λ is finite and for all $x_j \neq 0$, we have either $\nu_j = 0$ or $S^{\frac{N}{2s}} k(x_j)^{-\frac{N}{2s}} \leq \nu_j$.

Respect to ν_0 and ν_∞ , we reach that

$$\text{either } \nu_0 = 0 \text{ nor } \nu_0 \geq k(0)^{-1} \|k\|_\infty^{1-\frac{N}{2s}} S(\lambda)^{\frac{N}{2s}},$$

and

$$\text{either } \nu_\infty = 0 \text{ or } \nu_\infty \geq k(\infty)^{-1} \|k\|_\infty^{1-\frac{N}{2s}} S(\lambda)^{\frac{N}{2s}}.$$

Therefore, it holds that

$$\begin{aligned} c &= E_\lambda(u_n) - \frac{1}{2} \langle E'_\lambda(u_n), u_n \rangle + o(1) = \left(\frac{1}{2} - \frac{1}{2_s^*} \right) \int_{\mathbb{R}^N} k(x) |u_n|^{2_s^*} dx + o(1) \\ &= \frac{s}{N} \left\{ \int_{\mathbb{R}^N} k(x) |u|^{2_s^*} dx + k(0) \nu_0 + k(\infty) \nu_\infty + \sum_{j \in J} k(x_j) \nu_j \right\}. \end{aligned}$$

If there exists $j \in J$ such that $\nu_j \neq 0$, then $\frac{s}{N}k(x_j)\nu_j \leq c$. So, if $j \neq 0$ and $j \neq \infty$,

$$\frac{s}{N} S^{\frac{N}{2s}} \|k\|_{\infty}^{1-\frac{N}{2s}} \leq c.$$

If $j = 0$ or $j = \infty$ such that $\nu_j \neq 0$, then

$$\frac{s}{N} S(\lambda)^{\frac{N}{2s}} \|k\|_{\infty}^{1-\frac{N}{2s}} \leq c.$$

As a conclusion, we reach that $c \geq c^*$ which contradicts the main assumption on c . Thus $\nu_j = 0$ for all j . Hence

$$\int_{\mathbb{R}^N} k(x)|u_n|^{2_s^*} = \int_{\mathbb{R}^N} k(x)|u|^{2_s^*} + o(1).$$

On the other hand

$$\int_{\mathbb{R}^N} k(x)|u_n|^{2_s^*} = \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^{2s}} dx + o(1),$$

we conclude that

$$\|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} \frac{u_n^2}{|x|^{2s}} dx \rightarrow \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx + o(1), \quad \text{as } n \rightarrow \infty.$$

Moreover $\lambda < \Lambda_{N,s}$, then by using the above estimate, we deduce that $u_n \rightarrow u$ strongly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. \square

As a direct application of the previous compactness result, we get the next existence result that can be proved in a similar way as Theorem 3.4.

Theorem 4.2. *Suppose that (K_0) holds. Assume the existence of $\mu_0 > 0$ such that*

$$\int_{\mathbb{R}^N} k(x)w_{\mu_0}^{2_s^*} dx > \bar{K} \int_{\mathbb{R}^N} w_{\mu_0}^{2_s^*} dx, \quad (4.2)$$

where $\bar{K} = \max\{k(0), k(\infty)\}$. Then, Problem (4.1) has at least one positive solution.

Some sufficient conditions such that estimate (4.2) holds are given in the next corollary.

Corollary 4.3. *Assume that (K_0) holds and one of the following conditions is satisfied*

• *if $k(0) > k(\infty)$, there exists $r > 0$ such that $k(x) > k(0) + c|x|^{\eta_{s,\lambda}}$ for all $x \in B_r(0)$,*

or

• *$k(x) > k(\infty) + c|x|^{-\eta_{s,\lambda}}$ if $|x| > R$ for some $R \gg 1$ if $k(\infty) > k(0)$.*

Then, Problem (4.1) has a positive solution.

\square

4.2. Multiplicity of positive solutions.

The goal of this subsection is to analyze the existence of multiple positive solutions to Problem (4.1) according to the existence of critical points of the function k . More precisely, we will assume that k satisfies the following assumptions :

(K_1) there exists a finite set of points $\mathcal{C}(k)$ such that for all $a \in \mathcal{C}(k) : k(a) = \max_{x \in \mathbb{R}^N} k(x)$;

(K_2) there exists $\theta \in (2, N)$ such that if $a_j \in \mathcal{C}(k)$, then $k(a_j) - k(x) = o(|x - a_j|^\theta)$ as $x \rightarrow a_j$.

Let us fix $0 < r_0 \ll 1$ such that $B_{r_0}(a_i) \cap B_{r_0}(a_j) = \emptyset$ for $i \neq j$, with $1 \leq i, j \leq \text{Card}(\mathcal{C}(k))$. Let $\delta = \frac{r_0^2}{5}$ and define the next separation functionals

$$T_{ij}(u) = \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \psi_i(x) \psi_j(y) dx dy}{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy} \quad (4.3)$$

where $\psi_j(x) = \min\{1, |x - a_j|\}$. Note that $T_{ij} = T_{ji}$, for $i = j$ we simply denote $T_i = T_{ii}$.

We begin by proving the next technical Lemma.

Lemma 4.4. *Assume that $u \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that $T_{ij}(u) \leq \delta$. Then*

$$5 \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} \int_{\mathbb{R}^N \setminus B_{r_0}(a_i)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy. \quad (4.4)$$

Proof.

Let $u \neq 0$ such that $T_{ij}(u) \leq \delta$, then

$$\begin{aligned} & r_0^2 \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} \int_{\mathbb{R}^N \setminus B_{r_0}(a_i)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \\ & \leq \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} \int_{\mathbb{R}^N \setminus B_{r_0}(a_i)} \psi_j(x) \psi_i(y) \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \\ & \leq \iint_{\mathbb{R}^N \times \mathbb{R}^N} \psi_j(x) \psi_i(y) \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \\ & \leq \delta \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \\ & \leq \frac{r_0^2}{5} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

■

As a consequence, we get the first separation result that will be useful to show the multiplicity of positive solutions.

Corollary 4.5. *Let $u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}$ such that $\max\{T_i(u), T_j(u)\} \leq \delta$. If $T_{ij}(u) \leq \delta$, then $i = j$.*

Proof.

Let suppose $i \neq j$, then from the previous Lemmas, we get

$$\begin{aligned} & 5 \left[2 \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} \int_{\mathbb{R}^N \setminus B_{r_0}(a_i)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy + \int_{\mathbb{R}^N \setminus B_{r_0}(a_i)} \int_{\mathbb{R}^N \setminus B_{r_0}(a_i)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right. \\ & \left. + \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} \int_{\mathbb{R}^N \setminus B_{r_0}(a_j)} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right] \leq 4 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy. \end{aligned}$$

Therefore,

$$5 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \leq 4 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy$$

which is a contradiction. Hence we conclude. \square

Now let us consider the Nehari manifold associated to Problem (4.1),

$$M(\lambda) = \{u \in \mathcal{D}^{s,2}(\mathbb{R}^N) ; u \neq 0 \text{ and } \langle J'_\lambda(u), u \rangle = 0\}. \quad (4.5)$$

It is clear that if $u \in \mathcal{D}^{s,2}(\mathbb{R}^N) \setminus \{0\}$, then $t \cdot u \in M(\lambda)$ where

$$t := \left(\frac{\|u\|_{\mathcal{D}^{s,2}}^2 - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx}{\int_{\mathbb{R}^N} k(x) \cdot |u(x)|^{2^*_s} dx} \right)^{\frac{N-2s}{4s}}. \quad (4.6)$$

Using Sobolev and Hardy–Sobolev inequalities, we deduce that if $u \in M(\lambda)$

$$\|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} > \left[\left(1 - \Lambda_{N,s}^{-1} \lambda\right) (2^*_s - 1)^{-1} \|k\|_{\infty^{-1}} S^{\frac{2^*_s}{2}} \right]^{\frac{N-2s}{4s}} = c_1 > 0. \quad (4.7)$$

For $1 \leq j \leq \text{Card}(\mathcal{C}(k))$, we introduce the following family of subsets of $M(\lambda)$,

$$M_j(\lambda) := \{u \in M(\lambda) \text{ such that } \forall i \quad T_{ij}(u) < \delta\},$$

and its boundary

$$\Gamma_j(\lambda) := \{u \in M(\lambda) \text{ such that } \forall i \quad T_{ij}(u) = \delta\}.$$

Let us set

$$m_j(\lambda) := \inf\{E_\lambda(u) ; u \in M_j(\lambda)\} \quad \text{and} \quad \eta_j(\lambda) := \inf\{E_\lambda(u) ; u \in \Gamma_j(\lambda)\}.$$

We have the next lemma.

Lemma 4.6. *Suppose that (K_0) , (K_1) and (K_2) hold. Then, $M_j(\lambda) \neq \emptyset$ and there exists $\epsilon_1 > 0$ such that*

$$m_j(\lambda) < \tilde{c} \quad \text{for all } 0 < \lambda < \epsilon_1 \text{ and } 1 \leq j \leq \text{Card}(\mathcal{C}(k)). \quad (4.8)$$

Proof.

Let us set

$$v_{\mu,j}(x) = \frac{1}{(\mu^2 + |x - a_j|^2)^{\frac{N-2s}{2}}} \quad \text{and} \quad u_{\mu,j} = \frac{v_{\mu,j}}{\|v_{\mu,j}\|_{2^*_s}}. \quad (4.9)$$

Then $\|u_{\mu,j}\|_{2^*_s} = 1$ and $\|u_{\mu,j}\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = S$ where S was given by (2.3). Notice that $t_{\mu,j}(\lambda) \cdot u_{\mu,j} \in M(\lambda)$ where

$$t_{\mu,j}(\lambda) = \left(\frac{\|u_{\mu,j}\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} u_{\mu,j}^2 |x|^{-2s} dx}{\int_{\mathbb{R}^N} k(x) \cdot |u_{\mu,j}(x)|^{2^*_s} dx} \right)^{\frac{N-2s}{4s}}.$$

For the clarity of the exposition, we will use the following notations. More precisely, we set $x - a_j = \mu \cdot x_1$, then $u_{\mu,j}(a_j + \mu x_1) = \mu^{\frac{2s-N}{2}} u_0(x_1)$ where $u_0 = u_{\mu,j}$ with $\mu = 1$ and $x_j = 0$.

For j fixed, going back to the definition of T_{ij} given in (4.3) and setting $y - a_j = \mu \cdot y_1$, it holds that

$$T_{ij}(t_{\mu,j}(\lambda) u_{\mu,j}) = \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \psi_i(a_j + \mu y_1) \psi_j(a_j + \mu x_1) \frac{|u_0(x_1) - u_0(y_1)|^2}{|x_1 - y_1|^{N+2s}} dx_1 dy_1}{\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x_1) - u_0(y_1)|^2}{|x_1 - y_1|^{N+2s}} dx_1 dy_1}.$$

Thanks to Lebesgue's dominated convergence Theorem, we obtain

$$\forall i \quad \lim_{\mu \rightarrow 0} T_{ij}(t_{\mu,j}(\lambda) \cdot u_{\mu,j}) = \psi_j(a_j) \cdot \psi_i(a_j) = 0,$$

uniformly in λ . Hence, we get the existence of $\mu_0 > 0$ independent of λ such that if $\mu < \mu_0$, we have $t_{\mu,j} u_{\mu,j} \in M_j(\lambda)$. Thus $M_j(\lambda) \neq \emptyset$.

Now, we will prove (4.8).

First, by using Hardy–Sobolev inequality, we get

$$t_{\mu,j}(\lambda) \geq \|k\|_{\infty}^{\frac{2s-N}{4s}} \left(\left(1 - \frac{\lambda}{\Lambda_{N,s}}\right) S \right)^{\frac{N-2s}{4s}} \quad (4.10)$$

for all μ, j .

As in [2], to prove (4.8) it suffices to show the existence of $\mu < \mu_0$ such that if $0 < \lambda < \epsilon_1$, then

$$\max_{t \geq t_1(\lambda)} J_\lambda(t(\lambda) \cdot u_{\mu,j}) < \tilde{c}.$$

Moreover, we have

$$E_\lambda(t \cdot u_{\mu,j}) = \frac{t^2}{2} \|u_{\mu,j}\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \frac{t^{2^*_s}}{2^*_s} \int_{\mathbb{R}^N} k(x) |u_{\mu,j}|^{2^*_s} dx - \frac{\lambda}{2} t^2 \int_{\mathbb{R}^N} \frac{|u_{\mu,j}|^2}{|x|^{2s}} dx.$$

Then

$$\max_{t \geq t_1(\lambda)} E_\lambda(t \cdot u_{\mu,j}) \leq \max_{t > 0} \left\{ \frac{t^2}{2} \|u_{\mu,j}\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \frac{t^{2^*_s}}{2^*_s} \int_{\mathbb{R}^N} k(x) |u_{\mu,j}|^{2^*_s} dx \right\} - \frac{\lambda}{2} t_1(\lambda)^2 \cdot \int_{\mathbb{R}^N} \frac{|u_{\mu,j}|^2}{|x|^{2s}} dx.$$

So, we obtain

$$\max_{t > 0} \left\{ \frac{t^2}{2} \|u_{\mu,j}\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \frac{t^{2^*_s}}{2^*_s} \int_{\mathbb{R}^N} k(x) |u_{\mu,j}|^{2^*_s} dx \right\} = \frac{s}{N} \left(\frac{S}{\int_{\mathbb{R}^N} k(x) |u_{\mu,j}(x)|^{2^*_s} dx} \right)^{\frac{N}{2s}}. \quad (4.11)$$

As a consequence of the assumption (K_2) , there exists two positive constants c_1 and c_2 such that

$$\begin{aligned} \int_{\mathbb{R}^N} k(x) \cdot |u_{\mu,j}|^{2^*_s} dx &= \|k\|_\infty - \int_{\mathbb{R}^N} (k(a_j) - k(x)) |u_{\mu,j}|^{2^*_s} dx \\ &= \|k\|_\infty - c_1 \mu^N \int_{\mathbb{R}^N} \frac{k(a_j) - k(x)}{(\mu^2 + |x - a_j|^2)^N} dx \\ &\geq \|k\|_\infty - c_1 \mu^N \left\{ \int_{B_\delta(a_j)} \frac{c_2 |x - a_j|^\theta dx}{(\mu^2 + |x - a_j|^2)^N} + 2\|k\|_\infty \int_{\mathbb{R}^N \setminus B_\delta(a_j)} \frac{dx}{(\mu^2 + |x - a_j|^2)^N} \right\} \\ &\geq \|k\|_\infty - c_1 \mu^N \left\{ \mu^{\theta-N} c_2 \int_{\mathbb{R}^N} \frac{|y|^\theta}{(1 + |y|^2)^N} dy + 2\|k\|_\infty \int_{|y| \geq \delta} \frac{dy}{|y|^{2N}} \right\} \end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} k(x) \cdot |u_{\mu,j}|^{2^*_s} dx \geq \|k\|_\infty + O(\mu^\theta).$$

Consequently,

$$\max_{t \geq t_1(\lambda)} E_\lambda(t \cdot u_{\mu,j}) \leq \frac{s}{N} \frac{S^{\frac{N}{2s}}}{\|k\|_\infty^{\frac{N}{2s}}} + O(\mu^\theta) - \frac{1}{2} \lambda t_1(\lambda)^2 \int_{\mathbb{R}^N} \frac{u_{\mu,j}^2}{|x|^{2s}} dx \quad (4.12)$$

Following closely the same argument as in [34], we obtain

$$\int_{\mathbb{R}^N} \frac{u_{\mu,j}^2}{|x|^{2s}} dx \geq c \mu^{2s}. \quad (4.13)$$

Therefore,

$$\begin{aligned} \max_{t \geq t_1(\lambda)} E_\lambda(t \cdot u_{\mu,j}) &\leq \frac{s}{N} \frac{S^{\frac{N}{2s}}}{\|k\|_\infty^{\frac{N}{2s}}} + O(\mu^\theta) - \frac{1}{2} \lambda t_1(\lambda)^2 \mu^{2s} \\ &\leq \tilde{c} + c_3 \mu^\theta - \frac{1}{2} c \lambda t_1(\lambda)^2 \mu^{2s}. \end{aligned}$$

Hence, there exists ϵ_1 and μ_0 such that if $\mu < \mu_0$ and $0 < \lambda < \epsilon_1$, we get

$$\max_{t \geq t_1(\lambda)} E_\lambda(t u_{\mu,j}) < \tilde{c},$$

which is the desired result. \square

Lemma 4.7. *Suppose that (K_0) , (K_1) and (K_2) are satisfied. Then, there exists ϵ_2 such that for all $0 < \lambda < \epsilon_2$, we have*

$$\tilde{c} < \eta_j(\lambda). \quad (4.14)$$

Proof.

We proceed by contradiction. Assume there exists a sequence $\{(\lambda_n, u_n)\}_n \subset \mathbb{R}^+ \times \Gamma_j(\lambda_n)$ such that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$E_{\lambda_n}(u_n) \rightarrow c \leq \tilde{c} = \frac{s}{N} S^{\frac{N}{2s}} \|k\|_\infty^{1-\frac{N}{2s}}.$$

It is not difficult to show that $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $\mathcal{D}^{s,2}(\mathbb{R}^N)$. Thus, up to subsequence, we get the existence of $\ell > 0$ such that

$$\lim_{n \rightarrow \infty} \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} k(x) \cdot |u_n(x)|^{2_s^*} dx = \ell.$$

Or

$$\int_{\mathbb{R}^N} k(x) |u_n|^{2_s^*} dx \leq \|k\|_{\infty} S^{-\frac{2_s^*}{2}} \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^{2_s^*},$$

then

$$\ell \geq \|k\|_{\infty}^{1-\frac{N}{2s}} \cdot S^{\frac{N}{2s}}.$$

On the other hand,

$$\tilde{c} \geq E_{\lambda_n}(u_n) = \frac{s}{N} \ell + o(1).$$

Therefore $\ell = \|k\|_{\infty}^{1-\frac{N}{2s}} \cdot S^{\frac{N}{2s}}$ and then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (\|k\|_{\infty} - k(x)) \cdot |u_n|^{2_s^*} = 0. \quad (4.15)$$

Let $w_n = \frac{u_n}{\|u_n\|_{2_s^*}}$, then $\|w_n\|_{2_s^*} = 1$ and $\lim_{n \rightarrow \infty} \|w_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 = S$.

Thus, up to a subsequence, $w_n \rightharpoonup \bar{w}$ weakly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$, and again by the concentration-compactness principle [30, Theorem 5], we conclude that only one of the following alternatives holds:

- 1) $\bar{w} \neq 0$ and $w_n \rightarrow \bar{w}$ strongly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$
- 2) $\bar{w} = 0$ and either
 - i) $|(-\Delta)^{\frac{s}{2}} w_n|^2 \rightharpoonup d\varpi = S \delta_{x_0}$ and $|w_n|^{2_s^*} \rightarrow d\omega = \delta_{x_0}$
 - or
 - ii) $|(-\Delta)^{\frac{s}{2}} w_n|^2 \rightharpoonup d\varpi_{\infty} = S \delta_{\infty}$ and $|w_n|^{2_s^*} \rightarrow d\omega_{\infty} = \delta_{\infty}$.

where

$$\begin{cases} \nu_{\infty} = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} |w_n|^{2_s^*} dx = 1, \\ \mu_{\infty} = \lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \int_{|x| > R} \int_{\mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dy dx = S. \end{cases}$$

If the first alternative holds, then

$$\int_{\mathbb{R}^N} (\|k\|_{\infty} - k(x)) \cdot |\bar{w}|^{2_s^*} dx = 0$$

which is impossible since k is not a constant function.

Now assume that $\bar{w} = 0$ and 2.i) holds. Since for all $i \in \{1, \dots, \text{Card}(\mathcal{C}(k))\}$, we have $T_{ij}(w_n) = T_{ij}(u_n) = \delta$. Then, for all $i \in \{1, \dots, \text{Card}(\mathcal{C}(k))\}$,

$$\begin{aligned} \delta &= \lim_{n \rightarrow \infty} T_{ij}(w_n) \\ &= \frac{\lim_{n \rightarrow \infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} \psi_i(x) \psi_j(y) dx dy}{\lim_{n \rightarrow \infty} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy} \\ &= \psi_j(x_0). \end{aligned}$$

Thus, using the main assumption on k , it holds that $x_0 \neq a_j$ for all j . Hence, we deduce from (4.15) that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (\|k\|_\infty - k(x)) \cdot |w_n|^{2_s^*} = 0.$$

Consequently

$$0 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (\|k\|_\infty - k(x)) |w_n|^{2_s^*} = \|k\|_\infty - k(x_0).$$

Hence we reach a contradiction.

Finally, let us suppose $\bar{w} = 0$ such that 2.ii) holds. Consider a regular function ξ satisfying

$$\xi(x) = \begin{cases} 1 & \text{if } |x| > R + 1 \\ 0 & \text{if } |x| < R \end{cases}$$

where R is chosen in such a way that $|a_j| < R - 1$ for all j . So we have

$$\begin{aligned} \delta &= \lim_{n \rightarrow \infty} T_{ij}(w_n) \\ &= \lim_{n \rightarrow \infty} \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} \psi_i(x) \psi_j(y) dx dy}{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy} \\ &= \lim_{n \rightarrow \infty} \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} \xi(y) \psi_i(x) \psi_j(y) dx dy}{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy} \\ &\quad + \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} (1 - \xi(y)) \psi_i(x) \psi_j(y) dx dy}{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy}. \end{aligned}$$

Since $\text{Supp}(1 - \xi)$ is a compact subset of \mathbb{R}^N , then for R fixed, we have

$$\limsup_{n \rightarrow \infty} \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} (1 - \xi(y)) \psi_i(x) \psi_j(y) dx dy}{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy} = 0.$$

Thus,

$$\delta = \lim_{R \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} \xi(y) \psi_i(x) \psi_j(y) dx dy}{\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|w_n(x) - w_n(y)|^2}{|x - y|^{N+2s}} dx dy}. \quad (4.16)$$

By a duality argument, the right-hand side of (4.16) is actually equal to 1. So, $\delta = 1$ which is a contradiction. \square

Let $u \in M_j(\lambda)$ be fixed and consider the functional $G : \mathbb{R} \times \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined by

$$G(t, \vartheta) = t \left(\|u - \vartheta\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} \frac{(u - \vartheta)^2}{|x|^{2s}} dx \right) - |t|^{2_s^* - 1} \int_{\mathbb{R}^N} k(x) |u - \vartheta|^{2_s^*} dx. \quad (4.17)$$

It is clear that $G(1, 0) = 0$ and

$$G_t(1, 0) = \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx - (2_s^* - 1) \int_{\mathbb{R}^N} k(x) |u|^{2_s^*} dx \neq 0 \text{ since } u \in M_j(\lambda).$$

Thanks to the implicit functions Theorem, there exists $\rho_u > 0$ and a differentiable function $f^u : B(0, \rho_u) \subset \mathcal{D}^{s,2}(\mathbb{R}^N) \rightarrow \mathbb{R}$ such that $f^u(0) = 1$ and for all $\vartheta \in B_{\rho_u}(0)$, we have $f^u(\vartheta) \cdot (u - \vartheta) \in M_j(\lambda)$. Moreover for all $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$,

$$\begin{aligned} \langle Df^u(0), v \rangle &= - \frac{\langle G_{\vartheta(1,0)}, v \rangle}{G_t(1,0)} \\ &= \frac{2c_{n,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x - y|^{N+2s}} dx dy - 2\lambda \int_{\mathbb{R}^N} \frac{uv}{|x|^{2s}} dx - 2_s^* \int_{\mathbb{R}^N} k(x) \cdot |u|^{2_s^* - 2} u \cdot v dx}{c_{N,s} \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^{2s}} dx - (2_s^* - 1) \int_{\mathbb{R}^N} k(x) \cdot |u|^{2_s^*} dx}. \end{aligned}$$

Remark 4.8. Let $C > 0$. Using the implicit functions theorem, one can get a quantitative estimate on ρ which in general is very small. However, in this case we can show that in the set

$$A(C) := \{u \in M_\lambda(\lambda) ; \|u\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \geq C\} \text{ with } \lambda \in [0, \Lambda_{N,s} - a],$$

we get the existence of $\rho > 0$ depending only on C and $\Lambda_{N,s} - a$ such that f^u is well defined in $B_\rho(0)$ for all $u \in A(C)$ and for all $\lambda < \Lambda_{N,s} - a$.

This is due to the fact that

$$G_t(1, \vartheta) = \left(\|u - \vartheta\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} \frac{(u - \vartheta)^2}{|x|^{2s}} dx \right) - (2_s^* - 1) \int_{\mathbb{R}^N} k(x) |u - \vartheta|^{2_s^*} dx.$$

Indeed, let $u \in A(C)$ and $\lambda < \Lambda_{N,s} - a$. By using Sobolev and Hardy-Sobolev inequalities, we can show that if $\|\vartheta\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} < \rho$ for $\rho > 0$ depending only on C and $\Lambda_{N,s} - a$, there exists $C_2 > 0$ such that $|G_t(1, \vartheta)| \geq C_2 > 0$. Hence, we can conclude.

It is worth to point out that for $\vartheta \in \mathcal{D}^{s,2}(\mathbb{R}^N)$, $f^u(\vartheta)(\vartheta - u) \in M_j(\lambda)$ if and only if

$$f^u(\vartheta) = \left(\frac{\|u - \vartheta\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 - \lambda \int_{\mathbb{R}^N} \frac{(u - \vartheta)^2}{|x|^{2s}} dx}{\int_{\mathbb{R}^N} k(x) |u - \vartheta|^{2^*_s} dx} \right)^{\frac{N-2s}{4}}. \quad \square$$

Now, we are in position to state the main multiplicity result of this section.

Theorem 4.9. *Assume that the conditions (K_0) , (K_1) and (K_2) hold. Then, there exists $\epsilon_3 > 0$ such that, for all $0 < \lambda < \epsilon_3$, Problem (4.1) has $\text{Card}(\mathcal{C}(k))$ distinct positive solutions $u_{j,\lambda}$. Moreover, as $\lambda \rightarrow 0$, we have*

$$|(-\Delta)^{\frac{s}{2}} u_{j,\lambda}|^2 \rightarrow S^{\frac{N}{2s}} \|k\|_{\infty}^{1-\frac{N}{2s}} \cdot \delta_{a_j} \quad \text{and} \quad |u_{j,\lambda}|^{2^*_s} \rightarrow S^{\frac{N}{2s}} \|k\|_{\infty}^{-\frac{N}{2s}} \cdot \delta_{a_j}. \quad (4.18)$$

Proof.

Assume that $0 < \lambda < \epsilon_3 = \min\{\epsilon_1, \epsilon_2\}$ where ϵ_1 and ϵ_2 are given by Lemmas 4.6 and 4.7. For j fixed, we consider $\{u_n\}_{n \in \mathbb{N}}$, a minimizing sequence to $E_\lambda(\cdot)$ in $M_j(\lambda)$, that is

$$\begin{cases} u_n \in M_j(\lambda) \\ E_\lambda(u_n) \rightarrow m_j(\lambda) \quad \text{as } n \rightarrow \infty. \end{cases}$$

Since $E_\lambda(|u_n|) \leq E_\lambda(u_n)$ we can assume from the beginning that $u_n \geq 0$. Using the definition of $M_j(\lambda)$, we obtain that

$$c_1 \leq \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \leq c_2 \quad \text{uniformly in } n.$$

By the Ekeland variational principle, there exists a subsequence, still denoted $\{u_n\}_n$, such that

$$\begin{cases} E_\lambda(u_n) \leq m_j(\lambda) + \frac{1}{n} \\ E_\lambda(w) \geq E_\lambda(u_n) - \frac{1}{n} \|w - u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \quad \forall w \in M_j(\lambda). \end{cases}$$

We claim that $E'_\lambda(u_n) \rightarrow 0$, which means that $\{u_n\}_n$ is Palais-Smale sequence for E_λ .

In fact, as $u_n \in M_j(\lambda)$, then $\|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} \geq c_1$ and $\lambda < \Lambda_{N,s} - a < \Lambda_{N,s}$. So according to Remark 4.8, there exists $0 < \rho < \rho_n \equiv \rho_{u_n}$ and $f_n := f^{u_n}$, defined above, such that :

for $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ with $\|v\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = 1$, we define $v_\rho = \rho \cdot v$.

Consequently, $v_\rho \in B_{\rho_n}(0)$ and $w_\rho := f_n(v_\rho)(u_n - v_\rho) \in M_\lambda(j)$. Thus

$$\begin{aligned} \frac{1}{n} \|w_\rho - u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} &\geq E_\lambda(u_n) - E_\lambda(w_\rho) \\ &\geq \langle E'_\lambda(u_n), u_n - w_\rho \rangle + o(\|u_n - w_\rho\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}) \\ &\geq \rho f_n(\rho v) < E'_\lambda(u_n), v \rangle + o(\|u_n - w_\rho\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}). \end{aligned}$$

Therefore,

$$\langle E'_\lambda(u_n), v \rangle \leq \frac{1}{n} \cdot \frac{\|w_\rho - u_n\|}{\rho \cdot f_n(\rho v)} (1 + o(1)) \quad \text{as } \rho \rightarrow 0.$$

On the other hand, $|f_n(\rho v)| \rightarrow |f_n(0)| \geq c$ as $\rho \rightarrow 0$ and

$$\begin{aligned} \frac{\|w_\rho - u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}{\rho} &= \frac{\|f_n(0)u_n - f_n(\rho v)(u_n - \rho v)\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}}{\rho} \\ &\leq \frac{\|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} |f_n(0) - f_n(\rho v)| + \rho \cdot |f_n(\rho v)|}{\rho} \\ &\leq C \cdot |f'_n(0)| \cdot \|v\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} + c_3 \leq c. \end{aligned}$$

Hence, for all $v \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ with $\|v\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)} = 1$, we have

$$\langle E'_\lambda(u_n), v \rangle \leq \frac{C}{n}.$$

Thus $\|E'_\lambda(u_n)\|_{(\mathcal{D}^{s,2}(\mathbb{R}^N))'} \rightarrow 0$ as $n \rightarrow \infty$ and the claim follows.

So, $\{u_n\}_{n \in \mathbb{N}}$ is a Palais–Smale sequence for E_λ . Since $m_j(\alpha) < \tilde{c} = \tilde{c}(\lambda)$, thanks to Lemma 4.1, we reach that, up to a subsequence, $u_n \rightarrow u_j$ strongly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ where $u_j \in M_j(\lambda)$, $E_\lambda(u_j) = m_j(\lambda)$ and u_j is a critical point of E_λ . Thus, u_j is a positive solution to Problem (4.1).

Now, let us prove (4.18). Consider $\lambda_n \rightarrow 0$ and let $u_n \equiv u_{j_0, \lambda_n} \in M_{j_0}(\lambda_n)$ be the solution obtained above. Then, up to a subsequence, there exists $\ell_1 > 0$ such that

$$\lim_{n \rightarrow \infty} \|u_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} k(x) \cdot |u_n|^{2^*} dx = \ell_1.$$

Using the same approach as in Lemma 4.7, we obtain $\ell_1 = S^{\frac{N}{2s}} \cdot \|k\|_\infty^{1 - \frac{N}{2s}}$ and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (\|k\|_\infty - k(x)) |u_n|^{2^*} dx = 0.$$

As before, let us set $w_n = \frac{u_n}{\|u_n\|_{2^*}}$, then $\|w_n\|_{2^*} = 1$ and $\lim_{n \rightarrow \infty} \|w_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 = S$. Hence, we get the existence of $\bar{w} \in \mathcal{D}^{s,2}(\mathbb{R}^N)$ such that either $\bar{w} \not\equiv 0$ and $w_n \rightarrow \bar{w}$ strongly in $\mathcal{D}^{s,2}(\mathbb{R}^N)$ or $\bar{w} \equiv 0$ and

- 1) $|(-\Delta)^{\frac{s}{2}} w_n|^2 \rightharpoonup d\varpi = S \delta_{x_0}$ and $|w_n|^{2^*} \rightarrow d\omega = \delta_{x_0}$,
- or
- 2) $|(-\Delta)^{\frac{s}{2}} w_n|^2 \rightharpoonup d\varpi_\infty = S \delta_\infty$ and $|w_n|^{2^*} \rightarrow d\omega_\infty = \delta_\infty$.

As in the proof of Lemma 4.7, the cases $\bar{w} \not\equiv 0$ and $\bar{w} \equiv 0$ with the alternative 2) can be ruled out using the same computations. Hence the only possible case is that $\bar{w} \equiv 0$ and

$$\bar{w} \equiv 0; \quad |(-\Delta)^{\frac{s}{2}} w_n|^2 \rightharpoonup d\varpi = S \delta_{x_0} \quad \text{and} \quad |w_n|^{2^*} \rightarrow d\omega = \delta_{x_0},$$

for some point $x_0 \in \mathbb{R}^N$. Or,

$$\begin{aligned} \|w_n\|_{\mathcal{D}^{s,2}(\mathbb{R}^N)}^2 &= S + o(1) = S \int_{\mathbb{R}^N} |w_n|^{2^*} dx + o(1) \\ &= \frac{S}{\|k\|_\infty} \int_{\mathbb{R}^N} k(x) |w_n|^{2^*} dx + o(1) \\ &= \frac{S}{\|k\|_\infty} k(x_0) + o(1). \end{aligned}$$

then $x_0 = a_j \in \mathcal{C}(k)$. Corollary 4.5 allows us to conclude that $x_0 = a_{j_0}$ and the result follows. \blacksquare

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B. ABDELLAOUI, A. ATTAR, Y.O. BOUKARABILA, LABORATOIRE D'ANALYSE NONLINÉAIRE ET MATHÉMATIQUES APPLIQUÉES.
 DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ ABOU BAKR BELKAÏD, TLEMCEM,
 TLEMCEM 13000, ALGERIA.

E.-H. LAAMRI, INSTITUT ELIE CARTAN DE LORRAINE, UNIVERSITÉ DE LORRAINE, B.P. 239, 54506 VANDŒUVRE-LÉS- NANCY, FRANCE.

E-mail addresses:

boumediene.abdellaoui@inv.uam.es,
 ahm.attar@yahoo.fr,
 boukarabila.youcef.oussama@gmail.com,
 el-haj.laamri@univ-lorraine.fr.