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► **To cite this version:**

Somia Atmani, Kheireddine Biroud, Maha Daoud, El Haj Laamri. On Some Nonlocal Elliptic Systems with Gradient Source Terms. 2023. hal-03954615

HAL Id: hal-03954615

<https://hal.univ-lorraine.fr/hal-03954615>

Preprint submitted on 24 Jan 2023

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**ON SOME NONLOCAL ELLIPTIC SYSTEMS WITH GRADIENT SOURCE
TERMS
VERSION DU 15 JUIN 2022**

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ABSTRACT. In this work, we investigate the existence and nonexistence of nonnegative solutions to a class of nonlocal elliptic systems set in a bounded open subset of \mathbb{R}^N , where the gradients of the unknowns act as source terms (see (S) below). Our approach can be also used to treat other nonlinear systems with different structures.

This work extends previous results obtained in the local case by the fourth author and his coworkers, and points to significant differences between the local and the nonlocal cases.

1. INTRODUCTION.

The aim of this paper is to discuss the existence and nonexistence of a weak solution to nonlocal elliptic systems of the form

$$(S) \quad \begin{cases} (-\Delta)^s u &= |\nabla v|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^s v &= |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain, $N > 2s$ with $\frac{1}{2} < s < 1$, $p, q > 1$, f and g are measurable nonnegative functions. Here $(-\Delta)^s$ means the classical fractional Laplacian operator of order s introduced in [56] and defined by

$$(1.1) \quad (-\Delta)^s u(x) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad s \in (0, 1),$$

where

$$a_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}.$$

is a normalization constant chosen so that the following two identities

$$\lim_{s \rightarrow 0^+} (-\Delta)^s u = u \quad \text{and} \quad \lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u$$

hold. We refer readers not familiar with fractional laplacians to [33, 52, 31] and the references therein.

Let us highlight that fractional Laplacians (more generally nonlocal operators) naturally arise in many applications across various scientific fields, including Biology, Ecology, Finance, Materials Science, Nonlinear Dynamics, Anomalous Diffusion. We refer, for instance, to [52, 31, 26] and the references therein. This is why nonlocal operators have drawn the attention of many mathematicians in the recent years.

Before going further, it is worthy mentioning that a class of elliptic systems with gradient term appears when considering electrochemical models in engineering and some other models in fluid dynamics. We refer to [30] and [34] for more details.

Related to the model with fractional diffusion, we refer to [50, 45, 44] for a rigorous justification for the introduction of a fractional Laplacian into the continuum equation of the growing surface as an another relaxation mechanism.

Key words and phrases. fractional diffusion, nonlinear gradient terms, nonlinear elliptic systems, fixed-point theorem, a priori estimates.

Mathematics Subject Classification:MSC 2020: 47G20, 35R11,35J62 ,35G50, 47H10, 35B45.

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In this paper, we look for *natural* conditions on (p, q) , (f, g) and (λ, μ) which allow us to establish existence of solutions to System (S) as is done in [10] where the fourth author and his coworkers studied the local case (*i.e.* $s = 1$). However, there are significative differences between the local and the nonlocal cases as we see later.

To put our work in mathematical context, let us recall some previous results related to our System (S) .

- **Case of single equation.** In this case, System (S) is reduced to

$$(1.2) \quad \begin{cases} (-\Delta)^s u = |\nabla u|^p + \lambda f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

- *Local case* $s = 1$. In this case, Problem (1.2) is known as *stationary Hamilton-Jacobi equation*, or stationary Kardar–Parisi–Zhang equation (see, e.g., [43] for more details in this case).

This Problem has been extensively investigated ; see for instance [7, 14, 16, 17, 20, 40, 41, 3, 37] and the references therein, with no attempt to provide an exhaustive list. It is interesting to note that regularity assumptions on f and smallness condition on λ are necessary in order to show that (1.2) has a solution. We refer to [42, 58] for some physical motivations and existence results for the corresponding evolutionary problem.

- *Nonlocal case* $0 < s < 1$. While considerable effort has been devoted to the study of Problem (1.2) when $s = 1$, relatively little is done in the case $0 < s < 1$. Existence of solution to Problem (1.2) was obtained in [28] under the condition $p < p_* = \frac{N}{N - 2s + 1}$. In the same paper, the case where f is a Radon measure was also studied. Recently, in [28] and [5], the authors have considered natural conditions on f in order to determine the existence of a positive solution to Problem (1.2) depending on the value of p . It is worth mentioning that the existence of solutions to the following fractional elliptic problem

$$(-\Delta)^s w = |\nabla w|^p + h \text{ in } \mathbb{R}^N, \quad w > 0 \text{ in } \mathbb{R}^N, \quad \lim_{|x| \rightarrow +\infty} w(x) = 0.$$

was recently studied in [32] in the case where h is a nonnegative Radon measure and $p > p_* = \frac{N}{N - 2s + 1}$.

- **Case of Systems.**

- *Local case* $s = 1$. Contrary to single equations, only few results are known in the case of systems where the gradients of the unknowns act as source terms, even in the local case. Let us review the known results about such systems.
 - *First*, in [10] the fourth author and his coworkers investigated the local version ($s = 1$) of System (S) . The existence of nonnegative solutions is proved for any $(p, q) \in [1, \infty)^2$ such that $pq > 1$, under suitable integrability conditions of f and g , and smallness conditions on λ and μ . There are also nonexistence results for λ (or μ) large, or f and g of low integrability.
 - *Second*, the fourth author and other coworkers have obtained similar results in [2] about an elliptic System where the right-hand sides are potential and gradient terms, namely, $-\Delta u = v^q + \lambda f$, $-\Delta v = |\nabla u|^p + \mu g$. The same authors also studied [2] the parabolic version *i.e.* $u_t - \Delta u = v^q + \lambda f$, $v_t - \Delta v = |\nabla u|^p + \mu g$.
 - *Third*, regarding other existence results for local elliptic systems with gradient term, let us mention [21, 15, 34].
- *Nonlocal case* $0 < s < 1$. Even less is known in this case: as far as we know, the System (S) has not yet been investigated.

The main goal of the present work is to study the existence and nonexistence of (weak) solutions to System (S) . It is an extension of the results obtained in [10] for the local elliptic System (S) that is when $s = 1$, and those obtained in [5] pertaining to single nonlocal Hamilton–Jacobi equation (1.2). It is needless to say that this extension is far less trivial than it looks at first sight. Indeed, System (S) presents many interesting aspects: it is not variational, even if $s = 1$, with apparently no comparison or monotony properties, and no symmetry properties. Moreover it is nonlocal, which

is a source of new difficulties. Indeed, the arguments used in the local case cannot be adapted to deal with the new situation. The major obstruction to the construction of weak solutions is the loss of regularity caused by the nonlocal operator $(-\Delta)^s$. To circumvent that difficulty, we will use in a convenient way:

- (i) sharp estimates on the Green's function of the fractional Laplacian obtained in [25];
- (ii) fine weighted estimates on the solution to the corresponding fractional Poisson equation, see Proposition 3.1 and Theorem 3.2 below. The latter estimates improve those of [27, 28, 5]. To our knowledge, that theorem is new and interesting in itself. In this respect, one of the two anonymous referees pointed out to us that those results are an extension of the pioneer results of [13] relative to the case $s = 1$.

Then, the existence of weak solutions will be obtained for $(p, q) \in [1, \frac{s}{1-s})^2$ and under smallness conditions on λ and μ , see Theorems 4.2 and 4.4. Nonexistence results are also proved in these two cases :

- (i) if λ and μ are large even if $1 < p, q < \frac{s}{1-s}$, see Theorem 4.5 ;
- (ii) if $\max\{p, q\} \geq \frac{1+s}{1-s}$ even if f and g are bounded and no matter how small are λ and μ , see Theorem 4.6.

Our nonexistence Theorems 4.5 and 4.6 show, in some sense, that our assumptions on the data in Theorem 4.2 are necessary.

Regarding the existence, we believe that Theorem 4.2 holds for $(p, q) \in [\frac{s}{1-s}, \frac{1}{1-s}]^2$. But we have not been able to prove this result in that case. We leave it as an open problem.

As a matter of fact, this question is still open even in the case of a single equation.

As a byproduct, this paper will highlight the significant differences between the local and nonlocal cases. For example, when $s = 1$, the existence of positive solutions to System (S) is proved without any restriction on p, q provided that f and g satisfy some suitable integrability assumptions.

In a forthcoming work [8], we will consider a class of generalized systems with different diffusions, for instance:

$$(1.3) \quad \begin{cases} (-\Delta)^{s_1} u &= |\nabla v|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^{s_2} v &= |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s_1 \neq s_2$. Nonlocal parabolic systems with local gradient terms will also be considered in [9].

The rest of this manuscript is organized as follows. In Section 2, we recall some known results that we will use in our proofs. We begin with useful estimates on the Green's function of $(-\Delta)^s$. Then, we state some regularity results for nonlocal problems with general datum in L^1 . However, to prove our existence results, we need finer and more precise estimates than those stated in subsection 2.2. So we prove some new regularity and compactness results in weighted spaces in Section 3. The section 4 is divided into two subsections. Subsection 4.1 is devoted to the proof of our main existence results. In Subsection 4.2, we prove some nonexistence results in the case where the parameters λ, μ are large or $\max\{p, q\} > \frac{1+s}{1-s}$. In the last section, we explain how to apply the same techniques to study other nonlocal systems with different structures.

Notations. Before ending this section, let us fix some notations.

- Throughout this paper, Ω is a smooth bounded open subset of \mathbb{R}^N with $N > 2s$ and $s \in (\frac{1}{2}, 1)$.
- By definition $\delta(x) := \text{dist}(x, \partial\Omega)$.
- For $\beta > 0$, $L^1(\Omega, \delta^\beta) := \{h : \Omega \rightarrow \mathbb{R} \text{ measurable} ; \int_{\Omega} |h(x)| \delta^\beta(x) dx < +\infty\}$ and

$$\|h\|_{L^1(\Omega, \delta^\beta)} = \int_{\Omega} |h(x)| \delta^\beta(x) dx.$$

- By C , we denote any positive constant which may be different in each inequality.
- For $r \in (1, +\infty)$, we denote by r' its conjugate *i.e.* $r' = \frac{r}{r-1}$.

- Let E be a Borel subset of \mathbb{R}^N , we denote by $|E|$ the Lebesgue measure of E .
- Let φ, ψ be two nonnegative functions. We say that $\varphi \simeq \psi$ if there exist two positive constants C_1 and C_2 such that

$$C_1\varphi \leq \psi \leq C_2\varphi.$$

□

2. USEFUL TOOLS.

In this section, we will recall some (more or less classical) results that we will use in the proofs of our theorems. We refer, for example, to [6, 33] for the definition of the Sobolev spaces $W^{s,p}(\mathbb{R}^N)$ and its canonical norm. We also set

$$\mathbb{W}_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) ; u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

endowed with the norm induced by $\|u\|_{W^{s,p}(\mathbb{R}^N)}$.

2.1. The Green's Function of the fractional Laplacian.

Let us recall the definition of the Green's function of the fractional laplacian $(-\Delta)^s$ in Ω . For $x \in \Omega$ fixed, $\mathcal{G}_s(x, y)$ solves the problem

$$(2.1) \quad \begin{cases} (-\Delta)_y^s \mathcal{G}_s(x, y) = D_x & \text{if } y \in \Omega, \\ \mathcal{G}_s(x, y) = 0 & \text{if } y \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where we denote by D_x the Dirac mass at x (not to be confused with $\delta(x) = \text{dist}(x, \Omega)$).

Unlike in the whole space, there is no explicit formula for this Green's function. However, several useful properties of this function and its gradient are known when $s \in (\frac{1}{2}, 1)$. We just gather them here and refer to [5, 29, 23, 25] for the proof (which relies on probabilistic tools).

Lemma 2.1. *Assume that $s \in (0, 1)$. Then, for a.e. $x, y \in \Omega$, we have*

$$(2.2) \quad \mathcal{G}_s(x, y) \simeq \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^s(x)}{|x-y|^s} \wedge 1 \right) \left(\frac{\delta^s(y)}{|x-y|^s} \wedge 1 \right),$$

in particular, there exists $C_1 > 0$ such that

$$(2.3) \quad \mathcal{G}_s(x, y) \leq C_1 \min \left\{ \frac{1}{|x-y|^{N-2s}}, \frac{\delta^s(x)}{|x-y|^{N-s}}, \frac{\delta^s(y)}{|x-y|^{N-s}} \right\}.$$

Moreover, for all $\eta \in (0, 1)$, we get

$$(2.4) \quad \mathcal{G}_s(x, y) \leq C_1 \frac{\delta^{\eta s}(y) \delta^{(1-\eta)s}(x)}{|x-y|^{N-s}} \text{ and } \mathcal{G}_s(x, y) \leq C_1 \frac{\delta^{\eta s}(y)}{|x-y|^{N-s(2-\eta)}}.$$

Now, if $s \in (\frac{1}{2}, 1)$, then there exists $C_2 > 0$ such that

$$(2.5) \quad |\nabla_x \mathcal{G}_s(x, y)| \leq C_2 \mathcal{G}_s(x, y) \max \left\{ \frac{1}{|x-y|}, \frac{1}{\delta(x)} \right\}.$$

Therefore, there exists $C_3 > 0$ such that

$$(2.6) \quad |\nabla_x \mathcal{G}_s(x, y)| \leq \frac{C_3}{|x-y|^{N-2s+1} \delta^{1-s}(x)}. \square$$

2.2. Fractional Poisson Equation and regularity.

We consider the so-called Fractional Poisson Equation, that is,

$$(2.7) \quad \begin{cases} (-\Delta)^s w = h & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $s \in (0, 1)$ and $h \in L^m(\Omega)$ where $m \geq 1$.

In this subsection, we recall various regularity properties of the solution to (2.7) which will be needed in our proofs.

First of all, let us make precise what we mean by a weak solution to Problem (2.7).

Definition 2.2. We say that w is a weak solution to Problem (2.7) if $w \in L^1(\Omega)$, and for all $\phi \in \mathbb{X}_s$, we have

$$\int_{\Omega} w(-\Delta)^s \phi \, dx = \int_{\Omega} \phi h \, dx,$$

where

$$\mathbb{X}_s := \left\{ \phi \in \mathcal{C}(\mathbb{R}^N); \text{supp}(\phi) \subset \bar{\Omega}, (-\Delta)^s \phi \text{ exists and } |(-\Delta)^s \phi| \in L^\infty(\Omega) \right\}.$$

Before stating the next theorem, let us introduce the truncating function T_k defined for any $t \in \mathbb{R}$ by $T_k(t) := \max(-k, \min(k, t))$ where $k > 0$.

Theorem 2.3. Assume that $h \in L^1(\Omega)$. Then, the Problem (2.7) has a unique weak solution w , which is the limit of the sequence $\{w_n\}_{n \geq 1}$, where w_n is the unique solution to the approximating problem

$$(2.8) \quad \begin{cases} (-\Delta)^s w_n = T_n(h) & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Namely,

$$(2.9) \quad w_n \rightarrow w \text{ strongly in } \mathbb{W}_0^{s, q_1}(\Omega) \text{ for any } q_1 < \frac{N}{N - 2s + 1}.$$

Moreover,

$$(2.10) \quad T_k(w_n) \rightarrow T_k(w) \text{ strongly in } \mathbb{W}_0^{s, 2}(\Omega), \quad \forall k > 0;$$

$$(2.11) \quad w \in L^\theta(\Omega), \quad \forall \theta \in \left(1, \frac{N}{N - 2s}\right);$$

and

$$(2.12) \quad |(-\Delta)^{\frac{s}{2}} w| \in L^r(\Omega), \quad \forall r \in \left(1, \frac{N}{N - s}\right).$$

For the proof, we refer to [48, 27, 1]. \square

Actually, the solution w is more regular in the Sobolev scale than what is stated in the previous theorem. More precisely, one has the next Proposition whose proof can be found in [28, Proposition 2.3].

Proposition 2.4. Let $s \in (\frac{1}{2}, 1)$. Assume that $h \in L^1(\Omega, \delta^\beta)$ with $0 \leq \beta < 2s - 1$. Then, Problem (2.7) has a unique weak solution $w \in \mathbb{W}_0^{2s - \gamma, p}(\Omega)$ for any $p \in (1, \frac{N}{N + \beta - 2s})$ where $\gamma = \beta + \frac{N}{p'}$ if $\beta > 0$ and $\gamma > \frac{N}{p'}$ if $\beta = 0$.

Moreover, there exists $C := C(p, s, \Omega) > 0$ and independent of w and h such that

$$\|w\|_{\mathbb{W}_0^{2s - \gamma, p}(\Omega)} \leq C \|h\|_{L^1(\Omega, \delta^\beta)}. \square$$

For a datum $h \in L^m(\Omega)$ with $m \geq 1$, the following weighted regularity result was proved in [5, Lemma 2.12] using the properties of the Green's function given in Lemma 2.1.

Theorem 2.5. Suppose that $h \in L^m(\Omega)$ with $m \geq 1$ and let w be the unique solution to the Problem (2.7) with $s > \frac{1}{2}$. Then, for any $p < \frac{mN}{(N - m(2s - 1))_+}$, there exists a positive constant $C := C(\Omega, N, s, p)$ such that

$$(2.13) \quad \left\| |\nabla w| \delta^{1-s} \right\|_{L^p(\Omega)} \leq C \|h\|_{L^m(\Omega)}.$$

Moreover,

- (1) if $m = \frac{N}{2s - 1}$, $|\nabla w| \delta^{1-s} \in L^p(\Omega)$ for any $p < +\infty$;
- (2) if $m > \frac{N}{2s - 1}$, $w \in \mathcal{C}^{1, \sigma}(\Omega)$ for some $\sigma \in (0, 1)$. Moreover, there exists $C > 0$ such that

$$\left\| |\nabla w| \delta^{1-s} \right\|_{L^\infty(\Omega)} \leq C \|h\|_{L^m(\Omega)}.$$

Remark 2.6. *It is worth noting that, even if h and Ω are regular, one cannot expect boundedness of the gradient of the solution to Problem (2.7). Actually, consider this explicit example studied in [35] :*

$$(2.14) \quad \begin{cases} (-\Delta)^s w = 1 & \text{in } B_1(0), \\ w = 0 & \text{in } \mathbb{R}^N \setminus B_1(0), \end{cases}$$

where $B_1(0)$ is the ball with center at the origin and radius 1.

It is well known that the unique solution w to (2.14) is given by

$$(2.15) \quad w(x) := \begin{cases} \frac{2^{-2s} \Gamma(\frac{N}{2})}{\Gamma(\frac{N}{2} + s) \Gamma(1 + s)} (1 - |x|^2)^s & \text{in } B_1(0), \\ 0, & \text{in } \mathbb{R}^N \setminus B_1(0). \end{cases}$$

It is clear that $w \in W_0^{1,\eta}(B_1(0))$ if and only if $\eta < \frac{1}{1-s}$. \square

2.3. A useful lemma. We will make use of the following lemma to estimate some singular integrals and prove Theorem 3.2. We refer to [57, Chapter 5] for the proof.

Lemma 2.7. *Let $0 < \lambda < N$, $1 \leq p < \ell < \infty$ be such that $\frac{1}{\ell} + 1 = \frac{1}{p} + \frac{\lambda}{N}$. For $h \in L^p(\mathbb{R}^N)$, we set*

$$J_\lambda(h)(x) := \int_{\mathbb{R}^N} \frac{h(y)}{|x-y|^\lambda} dy.$$

- a) J_λ is well defined in the sense that the integral converges absolutely for almost every $x \in \mathbb{R}^N$.
- b) If $p > 1$, there exists $C_{p,\ell}$ such that $\|J_\lambda(h)\|_{L^\ell(\mathbb{R}^N)} \leq C_{p,\ell} \|h\|_{L^p(\mathbb{R}^N)}$.
- c) If $p = 1$, $|\{x \in \mathbb{R}^N ; J_\lambda(h)(x) > \sigma\}| \leq \left(\frac{C_{1,\ell} \|h\|_{L^1(\mathbb{R}^N)}}{\sigma} \right)^\ell$. \square .

Remark 2.8. *In the case $p = 1$, using the relation between the Marcinkiewicz and Lebesgue spaces, we deduce that if $\Omega \subset \mathbb{R}^N$ is a bounded domain, then*

$$\|J_\lambda(h)\|_{L^\theta(\Omega)} \leq C_{\theta,\ell} \|h\|_{L^1(\mathbb{R}^N)},$$

for any $\theta < \ell$.

3. REGULARITY AND COMPACTNESS RESULTS IN WEIGHTED SPACES

As mentioned in the introduction, the main difficulty is the loss of regularity near the boundary of the domain caused by the nonlocal character of the operator $(-\Delta)^s$. To overcome it, we have to work in the weighted spaces $L^1(\Omega, \delta^\beta)$. In the course of this section, we will improve the regularity result given in Theorem 2.5 as well as those obtained in [27, 12, 1].

Let us begin with...

Proposition 3.1. *Let h be a measurable function such that $h\delta^{a(1-s)} \in L^m(\Omega)$ with $m \geq 1$ and $0 \leq a < \frac{s}{1-s}$. Let w be the unique solution to*

$$(3.1) \quad \begin{cases} (-\Delta)^s w = h & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

We have the following :

- i) if $1 \leq m < \frac{N}{s-a(1-s)}$, then $\frac{w}{\delta^s} \in L^\gamma(\Omega)$ for any $\gamma \leq \frac{mN}{N-m(s-a(1-s))}$, and there exists $C_1 > 0$ such that

$$(3.2) \quad \left\| \frac{w}{\delta^s} \right\|_{L^\gamma(\Omega)} \leq C_1 \|h\delta^{a(1-s)}\|_{L^m(\Omega)} ;$$

- ii) if $m = \frac{N}{s-a(1-s)}$, then $\frac{w}{\delta^s} \in L^\gamma(\Omega)$ for any $\gamma < \infty$ and there exists $C_2 > 0$ such that

$$(3.3) \quad \left\| \frac{w}{\delta^s} \right\|_{L^\gamma(\Omega)} \leq C_2 \|h\delta^{a(1-s)}\|_{L^m(\Omega)} ;$$

iii) if $m > \frac{N}{s-a(1-s)}$, then $\frac{w}{\delta^s} \in L^\infty(\Omega)$ and there exists $C_3 > 0$ such that

$$(3.4) \quad \left\| \frac{w}{\delta^s} \right\|_{L^\infty(\Omega)} \leq C_3 \|h\delta^{\alpha(1-s)}\|_{L^m(\Omega)};$$

Proof. We give the proof in case i). The others are similar.

In what follows we denote by C a positive constant that may only depend on N, s, Ω, a, m , but it is independent of w and h .

The solution w to (3.1) is given in terms of the Green's function \mathcal{G}_s as:

$$w(x) = \int_{\Omega} \mathcal{G}_s(x, y) h(y) dy.$$

Hence

$$(3.5) \quad \left\{ \begin{array}{l} \left| \frac{w(x)}{\delta^s(x)} \right| \leq \int_{\Omega} \frac{\mathcal{G}_s(x, y)}{\delta^s(x)} |h(y)| dy \\ \leq \int_{\{|x-y| < \frac{1}{2}\delta(x)\}} \frac{\mathcal{G}_s(x, y)}{\delta^s(x)} |h(y)| dy + \int_{\{\frac{1}{2}\delta(x) \leq |x-y| < \delta(x)\}} \frac{\mathcal{G}_s(x, y)}{\delta^s(x)} |h(y)| dy \\ + \int_{\{|x-y| \geq \delta(x)\}} \frac{\mathcal{G}_s(x, y)}{\delta^s(x)} |h(y)| dy \\ = J_1(x) + J_2(x) + J_3(x). \end{array} \right.$$

• Let us begin by estimating J_1 . Notice that, since δ is a Lipschitz function, then for all $(x, y) \in \{|x-y| < \frac{1}{2}\delta(x)\}$, we have $|\delta(x) - \delta(y)| \leq |x-y| \leq \frac{1}{2}\delta(x)$. Whence

$$|x-y| \leq \frac{1}{2}\delta(x) \leq \delta(y) \leq \frac{3}{2}\delta(x).$$

On the other hand, $\mathcal{G}_s(x, y) \leq \frac{C\delta^s(x)}{|x-y|^{N-s}}$. Thus,

$$\begin{aligned} J_1(x) &= \int_{\{|x-y| < \frac{1}{2}\delta(x)\}} \frac{\mathcal{G}_s(x, y)}{\delta^s(x)} |h(y)| dy \leq C \int_{\{|x-y| < \frac{1}{2}\delta(x)\}} \frac{|h(y)|}{|x-y|^{N-s}} dy \\ &\leq C \int_{\{|x-y| < \frac{1}{2}\delta(x)\}} \frac{|h(y)| \delta^{\alpha(1-s)}(y) \delta^{-\alpha(1-s)}(y)}{|x-y|^{N-s}} dy \\ &\leq C \int_{\{|x-y| < \frac{1}{2}\delta(x)\}} \frac{|h(y)| \delta^{\alpha(1-s)}(y)}{|x-y|^{N-(s-a(1-s))}} dy. \end{aligned}$$

Now, setting

$$\tilde{h}(x) = \begin{cases} h(x) \delta^{\alpha(1-s)}(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

it follows that $\tilde{h} \in L^m(\mathbb{R}^N)$. Using Lemma 2.7, it holds that $J_1 \in L^{\frac{mN}{N-m(s-a(1-s))}}(\Omega)$ and

$$\|J_1\|_{L^\gamma(\Omega)} \leq C \|\tilde{h}\|_{L^m(\mathbb{R}^N)} = C \|h\delta^{\alpha(1-s)}\|_{L^m(\Omega)} \text{ for any } \gamma \leq \frac{mN}{N-(s-a(1-s))m}.$$

• We now treat J_2 . Using the estimate (2.4) with $\eta = a\frac{1-s}{s} < 1$, we get

$$\mathcal{G}_s(x, y) \leq C \frac{\delta^{\alpha(1-s)}(y) \delta^{s-(1-s)a}(x)}{|x-y|^{N-s}}.$$

Thus:

$$\begin{aligned} J_2(x) &= \int_{\{\frac{1}{2}\delta(x) \leq |x-y| < \delta(x)\}} \frac{\mathcal{G}_s(x, y)}{\delta^s(x)} |h(y)| dy \\ &\leq C \int_{\frac{1}{2}\delta(x) < \{|x-y| < \delta(x)\}} \frac{|h(y)| \delta^{\alpha(1-s)}(y)}{|x-y|^{N-(s-a(1-s))}} dy. \end{aligned}$$

Hence we conclude as in the estimation of J_1 .

• We now deal with J_3 . First, we have

$$(3.6) \quad \left\{ \begin{array}{l} J_3(x) \leq \frac{1}{\delta^s(x)} \int_{\{\delta(x) \leq |x-y| \leq \delta(y)\}} \mathcal{G}_s(x, y) |h(y)| dy \\ \quad + \frac{1}{\delta^s(x)} \int_{\{|x-y| \geq \max\{\delta(y), \delta(x)\}\}} \mathcal{G}_s(x, y) |h(y)| dy \\ \quad = J_{31}(x) + J_{32}(x). \end{array} \right.$$

— The first term J_{31} is bounded as follows:

$$(3.7) \quad \left\{ \begin{array}{l} J_{31}(x) \leq C \frac{1}{\delta^s(x)} \int_{\{\delta(x) \leq |x-y| \leq \delta(y)\}} \frac{|h(y)| \delta^s(x)}{|x-y|^{N-s}} dy \\ \leq C \int_{\{\delta(x) \leq |x-y| \leq \delta(y)\}} \frac{|h(y)|}{|x-y|^{N-s}} dy \\ \leq C \int_{\{\delta(x) \leq |x-y| \leq \delta(y)\}} \frac{|h(y)| \delta^{a(1-s)}(y)}{|x-y|^{N-s} \delta^{a(1-s)}(y)} dy \\ \leq C \int_{\{\delta(x) \leq |x-y| \leq \delta(y)\}} \frac{|h(y)| \delta^{a(1-s)}(y)}{|x-y|^{N-(s-a(1-s))}} dy. \end{array} \right.$$

Hence, using again Lemma 2.7 as in the estimation of J_1 , we get:

$$\|J_{31}\|_{L^\gamma(\Omega)} \leq C \|h \delta^{a(1-s)}\|_{L^m(\Omega)} \text{ for all } \gamma \leq \frac{mN}{N - (s - a(1-s))m}.$$

— Last, we deal with J_{32} . From (2.4), we obtain

$$(3.8) \quad \left\{ \begin{array}{l} J_{32}(x) \leq C \int_{\{|x-y| \geq \max\{\delta(y), \delta(x)\}\}} \frac{|h(y)| \delta^s(y)}{|x-y|^N} dy \\ \leq C \int_{\{|x-y| \geq \max\{\delta(y), \delta(x)\}\}} \frac{|h(y)| \delta^{a(1-s)}(y) \delta^{s-a(1-s)}(y)}{|x-y|^N} dy \\ \leq C \int_{\{|x-y| \geq \max\{\delta(y), \delta(x)\}\}} \frac{|h(y)| \delta^{a(1-s)}(y)}{|x-y|^{N-(s-a(1-s))}} dy. \end{array} \right.$$

We then conclude as in the previous estimates.

Finally, we have proved that

$$\left\| \frac{w}{\delta^s} \right\|_{L^\gamma(\Omega)} \leq C \|h \delta^{a(1-s)}\|_{L^m(\Omega)} \text{ for any } \gamma \leq \frac{mN}{N - m(s - a(1-s))}.$$

□

The next result provides a global weighted regularity for the gradient of the solution to Problem (3.1). This result is interesting by itself, and will play a crucial role in the proof of the main existence result. The compactness result in the space $\mathbb{W}_0^{1,\gamma}(\Omega)$, for some $1 < \gamma < 2s$, without weight, was proved in [28, Proposition 2.3] using [27, Proposition 2.4]. To be more precise, the proof in [27, 28] used the fact that $(-\Delta)^{-s}$ is an isomorphism from $\mathbb{W}^{-\gamma,r}(\Omega)$ into $\mathbb{W}_0^{2s-\gamma,r}(\Omega)$. As far as we know, such a result is not available in weighted spaces. So, we have to devise a new approach that could be generalized to this framework. That is the purpose of the next theorem. It should be noted that this compactness result works in weighted spaces and provides a new proof of the particular case obtained in [28].

Theorem 3.2. *Let $s \in (\frac{1}{2}, 1)$ and suppose that the hypotheses of Proposition 3.1 hold. Let w be the unique weak solution to Problem (3.1). We set $\widehat{m} := \frac{mN}{(N-m(s-a(1-s)))_+}$ and $\widetilde{m} =: \min\{\widehat{m}, \frac{mN}{(N-m(2s-1))_+}\}$.*

We have the following :

- i) *if $\widetilde{m} < \infty$, namely $m < \max\{\frac{N}{2s-1}, \frac{N}{s-a(1-s)}\}$, then $|\nabla w|\delta^{1-s} \in L^\gamma(\Omega)$ for all $\gamma < \widetilde{m}$, and there exists $C_1 > 0$ such that*

$$(3.9) \quad \|\nabla w|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C_1 \left(\left\| \frac{w}{\delta^s} \right\|_{L^{\widehat{m}}(\Omega)} + \|\mathbb{P}(h\delta^{a(1-s)})\|_{L^\gamma(\Omega)} \right),$$

where $\mathbb{P} : L^m(\Omega) \rightarrow L^\gamma(\Omega)$ is a linear operator defined by

$$(3.10) \quad \mathbb{P}(g)(x) = \begin{cases} \int_{\Omega} \frac{g(y)}{|x-y|^{N-(s-a(1-s))}} dy & \text{if } a \geq 1, \\ \int_{\Omega} \frac{g(y)}{|x-y|^{N-2s+1}} dy & \text{if } 0 \leq a < 1. \end{cases}$$

Moreover, for any $\gamma < \widetilde{m}$, there exists $C_2 > 0$ such that

$$(3.11) \quad \|\nabla w|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C_2 \|h\delta^{a(1-s)}\|_{L^m(\Omega)}.$$

- ii) *if $\widetilde{m} = \infty$ and $m > \max\{\frac{N}{2s-1}, \frac{N}{s-a(1-s)}\}$, then $|\nabla w|\delta^{1-s} \in L^\infty(\Omega)$ and there exists $C_3 > 0$ such that*

$$(3.12) \quad \|\nabla w|\delta^{1-s}\|_{L^\infty(\Omega)} \leq C \left(\left\| \frac{w}{\delta^s} \right\|_{L^\infty(\Omega)} + \|\mathbb{P}(h\delta^{a(1-s)})\|_{L^\infty(\Omega)} \right).$$

- iii) *if $\widetilde{m} = \infty$ and $m = \max\{\frac{N}{2s-1}, \frac{N}{s-a(1-s)}\}$, then $|\nabla w|\delta^{1-s} \in L^\gamma(\Omega)$ for any $\gamma < \infty$ and there exists $C_4 > 0$ such that*

$$(3.13) \quad \|\nabla w|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C_4 \left(\left\| \frac{w}{\delta^s} \right\|_{L^\gamma(\Omega)} + \|\mathbb{P}(h\delta^{a(1-s)})\|_{L^\gamma(\Omega)} \right).$$

As a consequence of estimate (3.9), for $\gamma < \min\{\frac{N}{N-(s-a(1-s))}, \frac{N}{N-(2s-1)}\}$ fixed, the operator $T : L^1(\Omega, \delta^{a(1-s)}) \rightarrow \mathbb{W}_0^{1,\gamma}(\Omega, \delta^{\gamma(1-s)})$ defined by $T(h) = w$, where w is the unique weak solution to (3.1), is compact.

Proof. For reasons of concision and clarity of exposition, we give the proof in the case (i). The other cases can be proven in the same way.

Without loss of generality, we can assume that $h \geq 0$. Thus $w \geq 0$. Recall that w is given by

$$w(x) = \int_{\Omega} \mathcal{G}_s(x, y) h(y) dy.$$

Then

$$|\nabla w(x)| \leq \int_{\Omega} |\nabla_x \mathcal{G}_s(x, y)| h(y) dy \leq \int_{\Omega} \frac{|\nabla_x \mathcal{G}_s(x, y)|}{\mathcal{G}_s(x, y)} \mathcal{G}_s(x, y) h(y) dy.$$

Using (2.5), we get,

$$|\nabla w(x)| \leq C \int_{\{|x-y| < \delta(x)\}} \frac{\mathcal{G}_s(x, y)}{|x-y|} h(y) dy + \frac{C}{\delta(x)} \int_{\{|x-y| \geq \delta(x)\}} \mathcal{G}_s(x, y) h(y) dy.$$

Thus

$$\begin{aligned} |\nabla w(x)|\delta^{(1-s)}(x) &\leq C\delta^{(1-s)}(x) \left(\int_{\Omega \cap \{|x-y| < \delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_s(x, y) dy \right) \\ &\quad + C\delta^{(1-s)}(x) \left(\int_{\Omega \cap \{|x-y| \geq \delta(x)\}} \frac{h(y)}{\delta(x)} \mathcal{G}_s(x, y) dy \right) \\ &= Q_1(x) + Q_2(x). \end{aligned}$$

— Let us begin by estimating Q_2 . Notice that

$$(3.14) \quad Q_2(x) \leq C \frac{w(x)}{\delta^s(x)}.$$

Since $\gamma < \tilde{m} \leq \frac{mN}{N - m(s - a(1 - s))}$, then

$$(3.15) \quad \int_{\Omega} Q_2^\gamma(x) dx \leq C \int_{\Omega} \frac{w^\gamma(x)}{\delta^{s\gamma}(x)} dx \leq C \|h\delta^{a(1-s)}\|_{L^m(\Omega)}^\gamma.$$

— We now deal with Q_1 . We have

$$\begin{aligned} Q_1(x) &= \delta^{(1-s)}(x) \left(\int_{\Omega \cap \{|x-y| < \delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_s(x, y) dy \right) \\ &\leq \delta^{(1-s)}(x) \left(\int_{\Omega \cap \{\frac{1}{2}\delta(x) \leq |x-y| < \delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_s(x, y) dy \right) \\ &\quad + \delta^{(1-s)}(x) \left(\int_{\Omega \cap \{|x-y| < \frac{1}{2}\delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_s(x, y) dy \right) \\ &= Q_{11}(x) + Q_{12}(x). \end{aligned}$$

Respect to Q_{11} , we have

$$(3.16) \quad Q_{11}(x) \leq C \delta^{-s}(x) \int_{\Omega \cap \{\frac{1}{2}\delta(x) \leq |x-y| < \delta(x)\}} h(y) \mathcal{G}_s(x, y) dy \leq C \frac{w(x)}{\delta^s(x)}.$$

As above, since $\gamma < \tilde{m} \leq \frac{mN}{N - m(s - a(1 - s))}$, then

$$(3.17) \quad \int_{\Omega} Q_{11}^\gamma(x) dx \leq C \int_{\Omega} \frac{w^\gamma(x)}{\delta^{s\gamma}(x)} dx \leq C \|h\delta^{a(1-s)}\|_{L^m(\Omega)}^\gamma.$$

We now estimate the term Q_{12} . Recall that for $(x, y) \in \{|x-y| < \frac{1}{2}\delta(x)\}$, we have $\frac{1}{2}\delta(x) \leq \delta(y) \leq \frac{3}{2}\delta(x)$. Hence

$$Q_{12}(x) \leq C \left(\int_{\Omega \cap \{|x-y| \leq \frac{1}{2}\delta(x)\}} \frac{h(y) \delta^{a(1-s)}(y) \delta^{1-s-a(1-s)}(x)}{|x-y|^{N-2s+1}} dy \right).$$

Hence, according to the value of a , we have

$$(3.18) \quad Q_{12}(x) \leq \begin{cases} C \left(\int_{\Omega} \frac{h(y) \delta^{a(1-s)}(y)}{|x-y|^{N-(s-a(1-s))}} dy \right) & \text{if } a \geq 1, \\ C \left(\int_{\Omega} \frac{h(y) \delta^{a(1-s)}(y)}{|x-y|^{N-2s+1}} dy \right) & \text{if } 0 \leq a < 1. \end{cases}$$

Taking into consideration the definition of \mathbb{P} , it follows that

$$\int_{\Omega} Q_{12}^\gamma(x) dx \leq C \int_{\Omega} (\mathbb{P}(h\delta^{a(1-s)})(x))^\gamma dx.$$

Combining the estimates (3.14), (3.16) and (3.18), it holds that:

$$(3.19) \quad |\nabla w(x)| \delta^{1-s}(x) \leq C \frac{w(x)}{\delta^s(x)} + \mathbb{P}(h\delta^{a(1-s)})(x).$$

Hence (3.9) follows.

Finally, from (3.19) and using Proposition 3.1, we deduce that

$$\| |\nabla w| \delta^{1-s} \|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s)}\|_{L^m(\Omega)},$$

for any $\gamma < \tilde{m}$.

Now, let us show that T is compact.

Consider a bounded sequence $\{h_n\}_n \subset L^1(\Omega, \delta^{a(1-s)})$ and set $w_n = T(h_n)$. Then w_n solves

$$(3.20) \quad \begin{cases} (-\Delta)^s w_n = h_n & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

For $q < \frac{N}{N-(s-a(1-s))}$ fixed, by Theorem 3.2, there exists $C > 0$ independent of n such that

$$(3.21) \quad \int_{\Omega} \frac{|w_n(x)|^q}{\delta^{qs}(x)} dx \leq C \|h_n \delta^{a(1-s)}\|_{L^1(\Omega)}^q \leq C.$$

Hence, using the estimate (3.9) and choosing q such that $\gamma < q < \frac{N}{N-(s-a(1-s))}$, we infer that

$$\begin{aligned} \|\nabla w_n | \delta^{(1-s)}\|_{L^\gamma(\Omega)} &\leq C \left(\left\| \frac{w_n}{\delta^s} \right\|_{L^q(\Omega)} + \|\mathbb{P}(h_n \delta^{a(1-s)})\|_{L^\gamma(\Omega)} \right) \\ &\leq C \|h_n \delta^{a(1-s)}\|_{L^1(\Omega)} \leq C. \end{aligned}$$

Taking into consideration that $|\nabla(w_n \delta^{1-s})| \leq \delta^{1-s} |\nabla w_n| + (1-s) \frac{|w_n(x)|}{\delta^s(x)}$, we deduce that the sequence $\{w_n \delta^{1-s}\}_n$ is bounded in the space $\mathbb{W}_0^{1,\gamma}(\Omega)$ for all $\gamma < \min\{\frac{N}{N-(s-a(1-s))}, \frac{N}{N-(2s-1)}\}$. Then, up to a subsequence, we get the existence of a measurable function w such that $w \delta^{1-s} \in \mathbb{W}_0^{1,\gamma}(\Omega)$, $w_n \delta^{1-s} \rightharpoonup w \delta^{1-s}$ weakly in $\mathbb{W}_0^{1,\gamma}(\Omega)$ and $w_n \rightarrow w$ a.e. in Ω .

Thanks to (3.21), $\left\{ \frac{w_n}{\delta^s} \right\}_n$ is bounded in $L^q(\Omega)$ for any $q < \frac{N}{N-(s-a(1-s))}$. Therefore, we conclude that $\frac{w}{\delta^s} \in L^q(\Omega)$ for any $q < \frac{N}{N-(s-a(1-s))}$ by Fatou's lemma. Finally, since Ω is a bounded domain, using Vitali's lemma we obtain that

$$\frac{w_n}{\delta^s} \rightarrow \frac{w}{\delta^s} \text{ strongly in } L^q(\Omega),$$

for any $q < \frac{N}{N-(s-a(1-s))}$.

Now, setting $v_{ij} = w_i - w_j$, $h_{ij} = h_i - h_j$, it holds that $v_{ij}, \frac{v_{ij}}{\delta^s} \rightarrow 0$ a.e. in Ω as $i, j \rightarrow \infty$ and $\frac{v_{ij}}{\delta^s} \rightarrow 0$ strongly in $L^r(\Omega)$ for any $r < \frac{N}{N-(s-a(1-s))}$.

Since v_{ij} solves the problem

$$(3.22) \quad \begin{cases} (-\Delta)^s v_{ij} = h_i - h_j & \text{in } \Omega, \\ v_{ij} = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

we deduce from (3.9)

$$\|\nabla v_{ij} | \delta^{1-s}\|_{L^\gamma(\Omega)} \leq C \left(\left\| \frac{v_{ij}}{\delta^s} \right\|_{L^q(\Omega)} + \|\mathbb{P}(h_{ij} \delta^{a(1-s)})\|_{L^\gamma(\Omega)} \right).$$

Recall that $\frac{v_{ij}}{\delta^s} \rightarrow 0$ as $i, j \rightarrow \infty$ strongly in $L^r(\Omega)$ for any $r < \frac{N}{N-(s-a(1-s))}$. Now, we deduce from [46, Theorem 8.1] that the operator $\mathbb{P} : L^1(\Omega) \rightarrow L^\gamma(\Omega)$ is compact for any $\gamma < \min\{\frac{N}{N-(s-a(1-s))}, \frac{N}{N-(2s-1)}\}$.

Hence, up to an other subsequence, we conclude that $w_n \rightarrow w$ strongly in $\mathbb{W}_0^{1,\gamma}(\Omega, \delta^{\gamma(1-s)})$. This ends the proof. \square

4. EXISTENCE AND NONEXISTENCE RESULTS.

Let us come back to our nonlinear System

$$(S) \quad \begin{cases} (-\Delta)^s u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^s v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u, v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded regular domain (C^2 is sufficient), $N > 2s$ with $s > \frac{1}{2}$, $p, q > 1$, f and g are measurable nonnegative functions.

In the first subsection, we will establish the existence of a solution to System (S) under natural assumptions on data. Then we will establish nonexistence results in the second subsection. This shows that our existence assumptions are in some way ‘optimal’.

4.1. Existence results.

In this subsection, we first specify what we mean by a weak solution to System (S). Then, we state and prove the main existence theorem of this paper.

Definition 4.1. Let $(f, g) \in (L^1(\Omega)^+)^2$. We say that $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$ is a weak solution to System (S) if $|\nabla u|^p \delta^s, |\nabla v|^q \delta^s \in L^1(\Omega)$ and for all $\varphi, \psi \in \mathbb{X}_s$, we have

$$(4.1) \quad \begin{cases} \int_{\Omega} u(-\Delta)^s \varphi \, dx = \int_{\Omega} |\nabla v|^q \varphi \, dx + \lambda \int_{\Omega} f \varphi \, dx, \\ \int_{\Omega} v(-\Delta)^s \psi \, dx = \int_{\Omega} |\nabla u|^p \psi \, dx + \mu \int_{\Omega} g \psi \, dx. \square \end{cases}$$

The main existence Theorem of the present work is the following.

Theorem 4.2. Assume $s \in (\frac{1}{2}, 1)$. Let p, q be such that $1 < p, q < \frac{s}{1-s}$. Suppose that $(f, g) \in L^m(\Omega)^+ \times L^\sigma(\Omega)^+$ where (m, σ) satisfies one of the following conditions:

$$(4.2) \quad \begin{cases} \max\{\frac{q+1}{q+2}, \frac{p+1}{p+2}\} < s < 1, \\ 1 < m < \frac{N}{s-q(1-s)}, 1 < \sigma < \frac{N}{s-p(1-s)}, \\ p\sigma < \hat{m} = \frac{mN}{N-m(s-q(1-s))} \text{ and } qm < \hat{\sigma} = \frac{\sigma N}{N-\sigma(s-p(1-s))}, \end{cases}$$

or

$$(4.3) \quad \begin{cases} \frac{1}{2} < s \leq \min\{\frac{q+1}{q+2}, \frac{p+1}{p+2}\}, \\ \frac{1}{2s-q(1-s)} < m < \frac{N}{s-q(1-s)}, \frac{1}{2s-p(1-s)} < \sigma < \frac{N}{s-p(1-s)}, \\ p\sigma < \hat{m}, qm < \hat{\sigma}, \end{cases}$$

or

$$(4.4) \quad m \geq \frac{N}{s-q(1-s)}, \quad \sigma > \frac{qmN}{N+qm(s-p(1-s))},$$

or

$$(4.5) \quad \sigma \geq \frac{N}{s-p(1-s)}, \quad m > \frac{p\sigma N}{N+p\sigma(s-q(1-s))}.$$

Then, there exists $\Lambda^* > 0$ such that for all $(\lambda, \mu) \in \Pi$ where

$$(4.6) \quad \Pi := \{(\lambda, \mu) \in [0, +\infty) \times [0, +\infty) ; \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq \Lambda^*\},$$

System (S) has a nonnegative solution $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$. Moreover $(u\delta^{1-s}, v\delta^{1-s}) \in \mathbb{W}_0^{1,\theta_1}(\Omega) \times \mathbb{W}_0^{1,\theta_2}(\Omega)$ for all $\theta_1 < \frac{mN}{(N-m(s-q(1-s)))_+}$ and $\theta_2 < \frac{\sigma N}{(N-\sigma(s-p(1-s)))_+}$.

Proof. We closely follow the approach developed in [10, 2, 5] while taking into account the difficulties caused by the loss of regularity near the boundary of the domain in the nonlocal case.

As the proof is quite long and for the sake of simplicity, we have chosen to give all the details under the assumptions (4.2), and we briefly explain at the end of proof the main changes under the other assumptions.

The first case. Assume that assumption (4.2) holds. Then, for $\alpha \geq 0$, we define the function

$$\Upsilon(\alpha) := \alpha^{\frac{1}{pq}} - \tilde{C}\alpha,$$

where \tilde{C} is a positive constant depending only on the data, which we will specify later.

Since $pq > 1$, there exists $\alpha_0 > 0$ such that $\Upsilon(\alpha_0) = 0$, $\Upsilon(\alpha) > 0$, $\forall s \in (0, \alpha_0)$, $\Upsilon(\alpha) < 0$, $\forall \alpha \in (\alpha_0, +\infty)$. Hence, there exist two positive constants $\ell > 0$ and $\Lambda^* > 0$ such that

$$\max_{\alpha \geq 0} \Upsilon(\alpha) = \Upsilon(\ell) = \Lambda^*.$$

Thus

$$(4.7) \quad \ell^{\frac{1}{p^q}} = \tilde{C} \left(\ell + \frac{\Lambda^*}{\tilde{C}} \right).$$

Fix $\ell > 0$ satisfying (4.7) and introduce the set

$$\Pi := \left\{ (\lambda, \mu) \in [0, +\infty) \times [0, +\infty) ; \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq \frac{\Lambda^*}{\tilde{C}} \right\}.$$

It is clear that Π is nonempty, bounded and closed in \mathbb{R}^2 . Moreover, for all $(\lambda, \mu) \in \Pi$, we have

$$(4.8) \quad \tilde{C} \left(\ell + \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \right) \leq \ell^{\frac{1}{p^q}}.$$

For a fixed $r \in (qm, \frac{\sigma N}{N - \sigma(s - p(1-s))})$, we define

$$(4.9) \quad H := \left\{ \varphi \in \mathbb{W}_0^{1,1}(\Omega) ; \varphi \delta^{1-s} \in \mathbb{W}_0^{1,r}(\Omega) \text{ and } \left(\int_{\Omega} |\nabla(\varphi \delta^{1-s})|^r dx \right)^{\frac{1}{r}} \leq \ell^{\frac{1}{p^q}} \right\}.$$

It is easy to check that H is a convex and closed subset of $\mathbb{W}_0^{1,1}(\Omega)$. Moreover, if $\varphi \in H$, then setting $\hat{\varphi} = \varphi \delta^{1-s}$, we deduce that $\hat{\varphi} \in \mathbb{W}_0^{1,r}(\Omega)$. Thus, using the Hardy inequality we deduce that

$$\int_{\Omega} \frac{|\hat{\varphi}|^r}{(\delta(x))^r} dx \leq C \int_{\Omega} |\nabla \hat{\varphi}|^r dx.$$

Hence

$$\int_{\Omega} \frac{|\varphi|^r}{(\delta(x))^{rs}} dx \leq C \int_{\Omega} |\nabla \varphi|^r dx.$$

Recall that $|\nabla \delta(x)| = 1$ a.e. in Ω , then

$$|\nabla \varphi|^r \delta^{r(1-s)} = |\nabla(\varphi \delta^{1-s}) - \varphi \nabla \delta^{1-s}|^r \leq C (|\nabla \hat{\varphi}|^r + \frac{|\varphi|^r}{\delta^{rs}}).$$

Therefore $|\nabla \varphi|^r \delta^{r(1-s)} \in L^1(\Omega)$ and

$$\left(\int_{\Omega} |\nabla \varphi|^r \delta^{r(1-s)} dx \right)^{\frac{1}{r}} \leq \hat{C}_0 \ell^{\frac{1}{p^q}}.$$

We observe that, if we take $\varphi \in H$, then using Hölder's inequality we obtain that $|\nabla \varphi|^a \delta^{a(1-s)} \in L^1(\Omega)$ for all $a \leq r$. In particular $|\nabla \varphi|^q \delta^{q(1-s)} \in L^1(\Omega)$.

For the sake of clarity, we articulate the rest of the proof in three steps.

• First step. We claim that if $\varphi \in H$, there exists $\beta < 2s - 1$ such that $|\nabla \varphi|^q \delta^\beta \in L^1(\Omega)$. Indeed, by Hölder's inequality, we get

$$(4.10) \quad \left\{ \begin{aligned} \int_{\Omega} |\nabla \varphi|^q \delta^\beta dx &= \int_{\Omega} |\nabla \varphi|^q \delta^{q(1-s)} \delta^{\beta - q(1-s)} dx \\ &\leq \left(\int_{\Omega} |\nabla \varphi|^{qm} \delta^{q(1-s)m} dx \right)^{\frac{1}{m}} \left(\int_{\Omega} \delta^{(\beta - q(1-s))m'} dx \right)^{\frac{1}{m'}} \\ &\leq C \left(\int_{\Omega} \delta^{(\beta - q(1-s))m'} dx \right)^{\frac{1}{m'}}. \end{aligned} \right.$$

On the other hand, by assumption $\max\{\frac{q+1}{q+2}, \frac{p+1}{p+2}\} < s < 1$. In particular, $s > \frac{q+1}{q+2}$. Thus $q(1-s) < 2s - 1$ and then we can choose $\beta < 2s - 1$, close to $2s - 1$ and such that $q(1-s) < \beta$.

Hence, we conclude that $\int_{\Omega} \delta^{(\beta - q(1-s))m'} dx < +\infty$.

So, we have proved that for $\varphi \in H$, we have $|\nabla \varphi|^q \delta^\beta + \lambda f \in L^1(\Omega)$. Therefore, thanks to Theorem 2.3 and Proposition 2.4, there exists a unique solution u to

$$(4.11) \quad \begin{cases} (-\Delta)^s u = |\nabla \varphi|^q + \lambda f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover, $|\nabla u| \in L^\alpha(\Omega)$ for any $\alpha < \frac{N}{N - 2s + 1 + \beta}$.

Now, we *claim* that

$$(4.12) \quad \|\nabla u|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C(\|\nabla\varphi|\delta^{1-s}\|_{L^{qm}(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)})$$

for any $\gamma \leq \widehat{m} := \frac{mN}{N-m(s-q(1-s))}$ where C is a positive constant depending only on the data and independent of u .

Indeed, setting $h = |\nabla\varphi|^q + \lambda f$, then

$$h\delta^{q(1-s)} = |\nabla\varphi|^q\delta^{q(1-s)} + \lambda f\delta^{q(1-s)} \in L^m(\Omega).$$

Applying estimate (3.11) (in Theorem 3.2), with $a = q$, it follows that

$$\|\nabla u|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C\left(\|\nabla\varphi|\delta^{1-s}\|_{L^{qm}(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)}\right) \text{ for all } \gamma \leq \widehat{m}.$$

Hence the claim follows.

Given that $qm < r$, we deduce by using Hölder's inequality that:

$$(4.13) \quad \|\nabla u|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C\left(\|\nabla\varphi|\delta^{1-s}\|_{L^r(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)}\right),$$

for all $\gamma \leq \widehat{m}$. As $p\sigma < \widehat{m}$, we obtain $|\nabla u|\delta^{(1-s)} \in L^{p\sigma}(\Omega)$.

On the other hand, $s > \frac{p+1}{p+2}$. As above, we get the existence of $\beta_1 < 2s - 1$ such that $|\nabla u|^p\delta^{\beta_1} \in L^1(\Omega)$. Hence, by applying Proposition 2.4, there exists a unique weak solution v to

$$(4.14) \quad \begin{cases} (-\Delta)^s v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $v \in \mathbb{W}_0^{1,a}(\Omega)$ for any $a < \frac{N}{N-2s+1+\beta_1}$.

Therefore, the operator

$$\begin{aligned} T : H &\longrightarrow \mathbb{W}_0^{1,1}(\Omega) \\ \varphi &\longmapsto T(\varphi) = v, \end{aligned}$$

is well defined. Moreover, if v is a fixed point of T , then (u, v) is a weak solution to System (S). Thus, we just have to show that T has a fixed point in H .

• Second step. Let us show that $T(H) \subset H$.

For this purpose, it is sufficient to prove if v solves Problem (4.14), then

$$(4.15) \quad \|\nabla v|\delta^{1-s}\|_{L^\nu(\Omega)} \leq C(\|\nabla u|\delta^{1-s}\|_{L^{p\sigma}(\Omega)}^p + \mu\|g\|_{L^\sigma(\Omega)})$$

where ν will be specified later.

In fact, setting $h = |\nabla u|^p + \mu g$, then $h\delta^{p(1-s)} \in L^\sigma(\Omega)$. Using the estimate (3.11) in Theorem 3.2 with $a = p$, we deduce that, for all $\nu \leq \widehat{\sigma} =: \frac{\sigma N}{N-\sigma(s-p(1-s))}$,

$$\|\nabla v|\delta^{1-s}\|_{L^\nu(\Omega)} \leq C(\|\nabla u|\delta^{1-s}\|_{L^{p\sigma}(\Omega)}^p + \mu\|g\|_{L^\sigma(\Omega)}).$$

Using the fact that $qm < r$ and going back to (4.12), it holds that

$$(4.16) \quad \|\nabla v|\delta^{1-s}\|_{L^\nu(\Omega)} \leq C\left(\|\nabla\varphi|\delta^{1-s}\|_{L^r(\Omega)}^{pq} + \mu\|g\|_{L^\sigma(\Omega)} + \lambda^p\|f\|_{L^m(\Omega)}^p\right).$$

Moreover $r < \widehat{\sigma}$, then by choosing $\nu = r$ in the previous inequality, we get

$$\|\nabla v|\delta^{1-s}\|_{L^r(\Omega)} \leq C\left(\|\nabla\varphi|\delta^{1-s}\|_{L^r(\Omega)}^{pq} + \mu\|g\|_{L^\sigma(\Omega)} + \lambda^p\|f\|_{L^m(\Omega)}^p\right).$$

Recall that $\varphi \in H$, thus

$$\|\nabla v|\delta^{1-s}\|_{L^r(\Omega)} \leq C\left(\ell + \mu\|g\|_{L^\sigma(\Omega)} + \lambda^p\|f\|_{L^m(\Omega)}^p\right).$$

By choosing $\tilde{C} = C$ and using the definition of ℓ , we conclude that $\|\nabla v|\delta^{1-s}\|_{L^r(\Omega)} \leq \ell^{\frac{1}{p\sigma}}$. Thus $v \in H$ and then $T(H) \subset H$.

• Third step. Now, we show that T is a continuous and compact operator on H endowed with the topology of $\mathbb{W}_0^{1,1}(\Omega)$.

Let us begin by proving the continuity of T . Consider $\{\varphi_n\}_n \subset H$ and $\varphi \in H$ be such that $\varphi_n \rightarrow \varphi$ strongly in $\mathbb{W}_0^{1,1}(\Omega)$. Define $v_n = T(\varphi_n)$ and $v = T(\varphi)$. Then (u_n, v_n) and (u, v) satisfy

$$(4.17) \quad \begin{cases} (-\Delta)^s u_n = |\nabla \varphi_n|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^s u = |\nabla \varphi|^q + \lambda f & \text{in } \Omega, \\ u_n = u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$(4.18) \quad \begin{cases} (-\Delta)^s v_n = |\nabla u_n|^p + \mu g & \text{in } \Omega, \\ (-\Delta)^s v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ v_n = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By Proposition 2.4, we get

$$(4.19) \quad \|\nabla u_n - \nabla u\|_{L^\rho(\Omega)} \leq C \left\| |\nabla \varphi_n|^q - |\nabla \varphi|^q \right\|_{L^1(\Omega, \delta^\beta)},$$

for any $\rho < \frac{N}{N-2s+1+\beta}$ with $\beta < 2s-1$.

Since $\{\varphi_n\}_n \subset H$, then $\left(\int_\Omega |\nabla \varphi_n|^r \delta^{r(1-s)} dx \right)^{\frac{1}{r}} \leq \ell^{\frac{1}{p^q}}$ for all n . Hence by Fatou's Lemma, we deduce that $\int_\Omega |\nabla \varphi|^r \delta^{r(1-s)} dx \leq C$.

Now, for $a \in (1, r)$ be fixed, using the interpolation inequality, it holds that

$$(4.20) \quad \begin{cases} \left(\int_\Omega |\nabla \varphi_n - \nabla \varphi|^a \delta^{a(1-s)} dx \right)^{\frac{1}{a}} & \leq \left(\int_\Omega |\nabla \varphi_n - \nabla \varphi| \delta^{(1-s)} dx \right)^\theta \\ & \times \left(\int_\Omega |\nabla \varphi_n - \nabla \varphi|^r \delta^{r(1-s)} dx \right)^{\frac{1-\theta}{r}} \\ & \leq C \left(\int_\Omega |\nabla \varphi_n - \nabla \varphi| \delta^{1-s} dx \right)^\theta \rightarrow 0 \text{ as } n \rightarrow \infty, \end{cases}$$

where $\theta = \frac{r-a}{a(r-1)}$.

Since $qm < r$, choosing $a = qm$. As in the first step, we can establish the existence of $\hat{\beta} < 2s-1$ such that

$$(4.21) \quad \int_\Omega |\nabla \varphi_n - \nabla \varphi|^q \delta^{\hat{\beta}} dx \leq C \left(\int_\Omega |\nabla \varphi_n - \nabla \varphi|^{qm} \delta^{qm(1-s)} dx \right)^{\frac{1}{m}} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

Going back to (4.19), we conclude that

$$\|\nabla u_n - \nabla u\|_{L^\rho(\Omega)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

for any $\rho < \frac{N}{N-2s+1+\hat{\beta}}$. In particular $u_n \rightarrow u$ strongly in $\mathbb{W}_0^{1,1}(\Omega)$.

This time, since $\varphi_n \in H$, using estimate (3.11), we deduce that

$$(4.22) \quad \left\| |\nabla u_n| \delta^{1-s} \right\|_{L^{\hat{m}}(\Omega)} \leq C \left(\ell^{\frac{1}{p}} + \lambda \|f\|_{L^m(\Omega)} \right).$$

Thus $\{|\nabla u_n| \delta^{1-s}\}_n$ is bounded in $L^{\hat{m}}(\Omega)$. Then $\{|\nabla u_n| \delta^{1-s}\}_n$ is bounded in $L^\gamma(\Omega)$ for any $\gamma < \hat{m}$. Let $\gamma < \hat{m}$ be fixed such that $p\sigma < \gamma$. Then, again using the interpolation inequality, we obtain

$$(4.23) \quad \begin{cases} \left(\int_\Omega |\nabla u_n - \nabla u|^{p\sigma} \delta^{p\sigma(1-s)} dx \right)^{\frac{1}{p\sigma}} & \leq \left(\int_\Omega |\nabla u_n - \nabla u| \delta^{(1-s)} dx \right)^\tau \\ & \times \left(\int_\Omega |\nabla u_n - \nabla u|^\gamma \delta^{\gamma(1-s)} dx \right)^{\frac{1-\tau}{\gamma}}, \\ & \leq C \left(\int_\Omega |\nabla u_n - \nabla u| \delta^{(1-s)} dx \right)^\tau \rightarrow 0 \text{ as } n \rightarrow \infty, \end{cases}$$

where $\tau = \frac{\gamma - p\sigma}{p\sigma(\gamma - 1)}$. In addition, $p < p\sigma$. Then by repeating the same computations as in first step, we can find $\beta_1 < 2s - 1$ such that

$$\int_{\Omega} |\nabla u_n - \nabla u|^p \delta^{\beta_1} dx \leq C \left(\int_{\Omega} |\nabla u_n - \nabla u|^{p\sigma} \delta^{p\sigma(1-s)} dx \right)^{\frac{1}{\sigma}} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, thanks to Proposition 2.4, we conclude that $v_n \rightarrow v$ strongly in $\mathbb{W}_0^{1,1}(\Omega)$. Then, the continuity of T follows.

Now we prove that the operator T is compact.

Let $\{\varphi_n\}_n \subset H$ be such that $\|\varphi_n\|_{\mathbb{W}_0^{1,1}(\Omega)} \leq C$ where C is independent of n . Define $v_n = T(\varphi_n)$. Since $\{\varphi_n\}_n \subset H$, we have

$$\left(\int_{\Omega} |\nabla \varphi_n|^r \delta^{r(1-s)} dx \right)^{\frac{1}{r}} \leq C \text{ for any } n.$$

Thus, as in the first step, there exists $\beta < 2s - 1$ such that $\{|\nabla \varphi_n|^q \delta^{\beta} + \lambda f\}_n$ is bounded in $L^1(\Omega)$. According to Proposition 2.4, we deduce that, up to a subsequence, $u_n \rightarrow u$ strongly in $\mathbb{W}_0^{1,\rho}(\Omega)$ for all $\rho < \frac{N}{N - 2s + 1 + \beta}$.

Since $p\sigma < \widehat{m}$, as in above (Continuity of T), we can prove that $|\nabla u_n| \delta^{1-s} \rightarrow |\nabla u| \delta^{1-s}$ strongly in $L^{p\sigma}(\Omega)$. Thus, using the same arguments as in the first and second steps and applying Hölder's inequality, we get the existence of $\beta_2 < 2s - 1$ such that $(|\nabla u_n|^p + \mu g) \delta^{\beta_2} \rightarrow (|\nabla u|^p + \mu g) \delta^{\beta_2}$ strongly $L^1(\Omega)$.

Thanks to Proposition 2.4, we conclude that $v_n \rightarrow v$ strongly in $\mathbb{W}_0^{1,\tau}(\Omega)$ for any $\tau < \frac{N}{N - 2s + 1 + \beta_1}$.

In particular $v_n \rightarrow v$ strongly in $\mathbb{W}_0^{1,1}(\Omega)$. Thus, T is compact.

To conclude the proof, we apply the Schauder fixed-point Theorem¹, see for instance [39, Corollary 11.2]. Hence, we get the existence of $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$, which is solution to System (S) such that $(u \delta^{1-s}, v \delta^{1-s}) \in \mathbb{W}_0^{1,\theta_1}(\Omega) \times \mathbb{W}_0^{1,\theta_2}(\Omega)$ for any $\theta_1 < \widehat{m}$ and $\theta_2 < \widehat{\sigma}$.

This ends up the proof under (4.2).

Now we briefly describe the main changes in the other three cases.

Second case. Under the assumption (4.3), the unique change compared to the first case is to prove that, for any $\varphi \in H$, there exists $\beta_1, \beta_2 < 2s - 1$ such that

$$|\nabla \varphi|^q \delta^{\beta_1}, |\nabla u|^p \delta^{\beta_2} \in L^1(\Omega).$$

Since $\frac{1}{2} < s \leq \min\{\frac{q+1}{q+2}, \frac{p+1}{p+2}\}$, then it follows from (4.3) that $(q + 1 - s(q + 2))m' < 1$. Hence, we get the existence of $\beta_1 < 2s - 1$ such that $(q(1 - s) - \beta_1)m' < 1$. Thus, under (4.3), we get $\int_{\Omega} \delta^{(\beta_1 - q(1-s))m'} dx < \infty$ and we conclude as in the first case.

Third case To prove the existence result under assumption (4.4), we use the fact $m \geq \frac{N}{s - q(1-s)}$. As in the first step of the first case, using Hölder's inequality we can find $\beta < 2s - 1$ such that $|\nabla \varphi|^q \delta^{\beta} \in L^1(\Omega)$ for any $\varphi \in H$. Therefore, using again Proposition 2.4, we get the existence of a weak solution u to Problem (4.11).

On the other hand, by using the same decomposition as in estimate (4.12) and by Theorem 3.2, we obtain

$$(4.24) \quad \|\nabla u| \delta^{1-s}\|_{L^\infty(\Omega)} \leq C(\|\nabla \varphi| \delta^{1-s}\|_{L^{qm}(\Omega)}^q + \lambda \|f\|_{L^m(\Omega)})$$

if $m > \frac{N}{s - q(1-s)}$ and

$$(4.25) \quad \|\nabla u| \delta^{1-s}\|_{L^\rho(\Omega)} \leq C(\|\nabla \varphi| \delta^{1-s}\|_{L^{qm}(\Omega)}^q + \lambda \|f\|_{L^m(\Omega)}), \text{ for any } \rho < \infty,$$

if $m = \frac{N}{s - q(1-s)}$. Therefore, arguing as in the proof of the first case, we can show that the operator T is well defined and has a fixed point in H . Thus we conclude. \square

¹Assume that E is a closed convex set of a Banach space X . Let T be a continuous and compact mapping from E into itself. Then, T has a fixed point in E .

Remark 4.3.

- (1) The same existence result holds without any positivity assumptions on the data.
(2) If $m = \sigma = 1$, the existence results hold for System (S) under the condition

$$(4.26) \quad \begin{cases} \max\{p, q\} < \frac{s}{1-s}, \\ 1 < p < \frac{N}{N - (s - q(1 - s))} \text{ and } q < \frac{N}{N - (s - p(1 - s))}. \end{cases} \square$$

Let us now deal with the case $p = q = 1$. The above convexity argument can not be used to get the existence of a solution. However, if we consider the following System

$$(4.27) \quad \begin{cases} (-\Delta)^s u = \lambda(|\nabla v| + f) & \text{in } \Omega, \\ (-\Delta)^s v = \mu(|\nabla u| + g) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

we have the next existence result.

Theorem 4.4. *Assume that $s \in (\frac{1}{2}, 1)$ and $p = q = 1$. Then, there exist λ^* and μ^* such that if $\lambda < \lambda^*$ and $\mu < \mu^*$, the System (4.27) has a solution (u, v) with $(u\delta^{1-s}, v\delta^{1-s}) \in \mathbb{W}_0^{1,\gamma_1}(\Omega) \times \mathbb{W}_0^{1,\gamma_2}(\Omega)$ for any $\gamma_1 < \frac{mN}{N-m(2s-1)}$ and $\gamma_2 < \frac{\sigma N}{N-\sigma(2s-1)}$, if one of the following conditions holds:*

$$(4.28) \quad \begin{cases} \frac{2}{3} < s < 1, \\ 1 < m < \frac{N}{2s-1}, 1 < \sigma < \frac{N}{2s-1}, \\ \sigma < \frac{mN}{N-m(2s-1)} \text{ and } m < \frac{\sigma N}{N-\sigma(2s-1)}, \end{cases}$$

or

$$(4.29) \quad \begin{cases} \frac{1}{2} < s < \frac{2}{3}, \\ \frac{1}{3s-1} < m < \frac{N}{2s-1}, \frac{1}{3s-1} < \sigma < \frac{N}{2s-1}, \\ \sigma < \frac{mN}{N-m(2s-1)}, m < \frac{\sigma N}{N-\sigma(2s-1)}, \end{cases}$$

or

$$(4.30) \quad m \geq \frac{N}{2s-1}, \quad \sigma > \frac{mN}{N+m(2s-1)},$$

or

$$(4.31) \quad \sigma \geq \frac{N}{2s-1}, \quad m > \frac{\sigma N}{N+\sigma(2s-1)}.$$

Proof. We give the proof under the assumption (4.28). Using the Schauder fixed-point Theorem, we get the existence of a solution (u_n, v_n) to the approximating system

$$(4.32) \quad \begin{cases} (-\Delta)^s u_n = \lambda \left(\frac{|\nabla v_n|}{1 + \frac{1}{n}|\nabla v_n|} + f \right) & \text{in } \Omega, \\ (-\Delta)^s v_n = \mu \left(\frac{|\nabla u_n|}{1 + \frac{1}{n}|\nabla u_n|} + g \right) & \text{in } \Omega, \\ u_n = v_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Setting

$$L_{1n}(x) = \frac{|\nabla v_n|}{1 + \frac{1}{n}|\nabla v_n|} + f, \quad L_{2n}(x) = \frac{|\nabla u_n|}{1 + \frac{1}{n}|\nabla u_n|} + g.$$

Then using the regularity result in Theorem 3.2 with $a = 1$, it holds that

$$(4.33) \quad \begin{cases} \|\nabla u_n \delta^{1-s}\|_{L^{\gamma_1}(\Omega)} \leq C\lambda(\|\nabla v_n \delta^{1-s}\|_{L^m(\Omega)} + \|f\|_{L^m(\Omega)}), \\ \|\nabla v_n \delta^{1-s}\|_{L^{\gamma_2}(\Omega)} \leq C\mu(\|\nabla u_n \delta^{1-s}\|_{L^\sigma(\Omega)} + \|g\|_{L^\sigma(\Omega)}), \end{cases}$$

for any $\gamma_1 \leq \frac{mN}{N-m(2s-1)}$ and $\gamma_2 \leq \frac{\sigma N}{N-\sigma(2s-1)}$.

Thus

$$(4.34) \quad \begin{cases} \|\nabla u_n|\delta^{1-s}\|_{L^{\gamma_1}(\Omega)} + \|\nabla v_n|\delta^{1-s}\|_{L^{\gamma_2}(\Omega)} \leq \\ C \left(\mu \|\nabla u_n|\delta^{1-s}\|_{L^\sigma(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)} + \lambda \|\nabla v_n|\delta^{1-s}\|_{L^m(\Omega)} + \|f\|_{L^m(\Omega)} \right). \end{cases}$$

Since $\sigma < \frac{mN}{N-m(2s-1)}$, we can choose γ_1 close to $\frac{mN}{N-m(2s-1)}$ such that $\sigma < \gamma_1 < \frac{mN}{N-m(2s-1)}$. Thus using Hölder's inequality, we deduce that

$$\|\nabla u_n|\delta^{1-s}\|_{L^\sigma(\Omega)} \leq C \|\nabla u_n|\delta^{1-s}\|_{L^{\gamma_1}(\Omega)}.$$

In the same way, we get

$$\|\nabla v_n|\delta^{1-s}\|_{L^m(\Omega)} \leq C \|\nabla v_n|\delta^{1-s}\|_{L^{\gamma_2}(\Omega)}.$$

Hence for $\max\{\lambda, \mu\} < C_0$ small, we obtain that

$$\|\nabla u_n|\delta^{1-s}\|_{L^{\gamma_1}(\Omega)} + \|\nabla v_n|\delta^{1-s}\|_{L^{\gamma_2}(\Omega)} \leq C(\Omega, C_0)(\lambda \|f\|_{L^m(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)}).$$

As a consequence, $\{(u_n, v_n)\}_n$ is bounded in the space $\mathbb{W}_0^{1,\gamma_1}(\Omega, \delta^{\gamma_1(1-s)}) \times \mathbb{W}_0^{1,\gamma_2}(\Omega, \delta^{\gamma_2(1-s)})$ for any $\gamma_1 < \frac{mN}{N-m(2s-1)}$ and $\gamma_2 < \frac{\sigma N}{N-\sigma(2s-1)}$. The rest of the proof follows using the same compactness computations as in the proof of Theorem 4.2. \square

4.2. Nonexistence results.

In this subsection, we prove that the smallness of λ and μ on the one hand, and the fact that $p, q \in (1, \frac{s}{1-s})$ on the other hand, are necessary in order to get the existence of solutions to (S).

4.2.1. Smallness condition on λ and μ .

It is clear that the set Π defined by (4.6) is bounded in \mathbb{R}^2 . The following result clearly shows that the smallness condition on (λ, μ) is needed for existence of solutions to (S).

Theorem 4.5. *Let $s \in (1/2, 1)$. Suppose that $(f, g) \in L^m(\Omega)^+ \times L^\sigma(\Omega)^+$ where (m, σ) satisfy the conditions of Theorem 4.2 and $1 < p, q < \frac{s}{1-s}$. Then, there exist two positive constants $\Gamma(f)$ and $\Gamma(g)$ such that if $\lambda > \Gamma(f)$ or $\mu > \Gamma(g)$, System (S) does not have any nonnegative solution.*

Proof. Let $\lambda, \mu > 0$ be fixed such that System (S) has a nonnegative solution (u, v) . Given $\theta \in C_0^\infty(\Omega)$ with $\theta \geq 0$. Firstly, introduce $\phi_\theta \in \mathbb{W}_0^{s,2}(\Omega) \cap L^\infty(\Omega)$ to be the unique solution to the following Problem

$$(4.35) \quad \begin{cases} (-\Delta)^s \phi_\theta = \theta & \text{in } \Omega, \\ \phi_\theta = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Let us recall that $\phi_\theta \asymp \delta^s$ according to [11, Lemma 4.2]. Next, consider ψ_θ the unique solution to the quasi-linear Problem :

$$(4.36) \quad \begin{cases} -\operatorname{div}(\phi_\theta |\nabla \psi_\theta|^{\alpha-2} \nabla \psi_\theta) = \theta & \text{in } \Omega \\ \psi_\theta = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < \alpha \leq \min\{p, q\}$. Notice that the existence of ψ_θ can be obtained using an approximating argument. It is clear that $(\alpha-1)p' \leq \alpha$ and $(\alpha-1)q' \leq \alpha$.

Now, using ϕ_θ as a test function in the first equation of System (S), we get:

$$\begin{aligned} \int_{\Omega} (|\nabla v|^q + \lambda f) \phi_\theta dx &= \int_{\Omega} \theta u dx \\ &\leq \int_{\Omega} \phi_\theta |\nabla \psi_\theta|^{\alpha-1} |\nabla u| dx. \end{aligned}$$

Remark 4.7. *To study the question of existence or nonexistence of solutions to System (S) when $p, q \in (\frac{s}{1-s}, \frac{1+s}{1+s})$, it is necessary to know the precise regularity of such solution under minimum conditions given in Definition 4.1. Unfortunately, this question seems to be more difficult and it is not known so far even in the case of a single equation.*

Proof.

1) We begin by the first case. Without loss of generality, we can suppose that $p \geq q$. Assume by contradiction that System (S) has a solution (u, v) such that $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$ and $\int_{\Omega} (|\nabla u|^p + |\nabla v|^q) \delta^s(x) dx < \infty$ and $p \geq \frac{1+s}{1-s}$.

Since $s < p - 1$, then using the weighted Hardy inequality (see [53, Theorem 1.6]), we obtain

$$(4.41) \quad \int_{\Omega} \frac{u^p}{\delta^p} \delta^s(x) dx \leq \int_{\Omega} |\nabla u|^p \delta^s(x) dx < +\infty.$$

Now by the maximum principle (see for instance [11, Lemma 4.2]), we deduce that $u \geq C\delta^s$. Thus

$$\int_{\Omega} \frac{1}{\delta^{p(1-s)-s}} dx < +\infty.$$

Hence $p < \frac{1+s}{1-s}$ and we reach a contradiction.

2) Now, we consider the second case. Suppose that System (S) has a nonnegative solution (u, v) with $(u, v) \in \mathbb{W}_0^{1,p}(\Omega) \times \mathbb{W}_0^{1,q}(\Omega)$, then as above, using Hardy inequality we reach that

$$\int_{\Omega} \frac{u^p}{\delta^p} dx \leq \int_{\Omega} |\nabla u|^p dx < +\infty.$$

Using the fact that $u \geq C\delta^s$, we get

$$\int_{\Omega} \frac{1}{\delta^{p(1-s)}} dx < +\infty.$$

Hence $p < \frac{1}{1-s}$ and we reach a contradiction. \square

Before ending this section, we prove an optimality result related to the assumption on f and g in some particular cases. The next nonexistence result is far from being optimal, but it gives some ideas on the strong relation between the regularity of the data and the exponents of the gradients.

Theorem 4.8. *Assume that one of the following conditions holds*

$$(4.42) \quad p > \frac{N}{N - (2s - 1)m} \text{ and } q > \frac{N}{N - (2s - 1)m},$$

or

$$(4.43) \quad p > \frac{N}{N - (2s - 1)\sigma} \text{ and } q > \frac{N}{N - (2s - 1)\sigma}.$$

Then, there exists (m, σ) and $f \in L^m(\Omega), g \in L^\sigma(\Omega)$ with $f, g \gtrsim 0$ such that the System (S) does not have any positive solution for any $\lambda, \mu > 0$.

Proof. We prove the result under the condition (4.42), the other case follows in a symmetric way.

To do so, we will construct f, g such that, under the hypotheses of Theorem 4.8, we get $\Gamma(f) = 0$ or $\Gamma(g) = 0$.

We suppose that $\Omega = B_1(0)$ and $m, \sigma < N$. Let $\varepsilon > 0$ to be chosen later, and define $f(x) = \frac{1}{|x|^{\frac{N-\varepsilon}{m}}}$, then $f \in L^m(\Omega)$. Assume by contradiction that System (S) has a solution $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$ with $(u, v) \in \mathbb{W}_{loc}^{1,p}(\Omega) \times \mathbb{W}_{loc}^{1,q}(\Omega)$. Then, following the calculations of Theorem 4.5, we deduce that

$$(4.44) \quad \lambda \int_{\Omega} f \phi_{\theta} dx \leq C_2' \int_{\Omega} |\nabla \psi_{\theta}|^{p'(\alpha-1)} \phi_{\theta} dx + C_3' \int_{\Omega} |\nabla \psi_{\theta}|^{q'(\alpha-1)} \phi_{\theta} dx.$$

Consider the function θ given by

$$\theta(x) = \begin{cases} \frac{1}{|x|^a} & \text{if } |x| \leq \frac{1}{4} \\ (1 - |x|)^{\gamma} & \text{if } \frac{1}{2} \leq |x| \leq 1, \end{cases}$$

with $a > 2s$ satisfies

$$\begin{cases} N + 2s - \frac{N}{pq} < a < N + 2s - p'(2s - 1) \text{ and} \\ N + 2s - \frac{N}{m} < a < N + 2s - q'(2s - 1). \end{cases}$$

Notice that the existence of a follows using the hypothesis (4.42).

Now using the definition of ϕ_θ , as the solution to problem (4.35), we obtain that $\phi_\theta \simeq |x|^{2s-a}$ in a small ball $B_{r_0}(0)$.

Respect to the function ψ_θ , the solution of the problem (4.36), choosing $1 < \alpha < 2s$, we can show that in a small ball $B_{r_1}(0)$, we have

$$(4.45) \quad \psi_\theta(x) \simeq |x|^{-\frac{2s-\alpha}{\alpha-1}} \text{ and } |\nabla\psi_\theta| \leq C|x|^{-\frac{2s-1}{\alpha-1}}.$$

Thus, taking into consideration estimates (4.44) and (4.45) and choosing ε small enough such that $\frac{N-\varepsilon}{m} + a - 2s \geq N$, we deduce that

$$\int_{\Omega} f \phi_\theta dx \geq C \int_{B_{\frac{1}{5}}(0)} \frac{1}{|x|^{\frac{N-\varepsilon}{m} + a - 2s}} dx = \infty.$$

Moreover

$$\begin{aligned} & \int_{\Omega} |\nabla\psi_\theta|^{p'(\alpha-1)} \phi_\theta dx + \int_{\Omega} |\nabla\psi_\theta|^{q'(\alpha-1)} \phi_\theta dx \\ & \leq C \int_{\Omega} \left(\frac{1}{|x|^{(2s-1)p' + a - 2s}} + \frac{1}{|x|^{(2s-1)q' + a - 2s}} \right) dx < \infty. \end{aligned}$$

Thus, using a suitable approximation argument, we conclude that $\Gamma(f) = 0$ and the nonexistence result follows. \square

5. EXTENSIONS AND FURTHER RESULTS

In this section, we give some extensions of the existence result to other nonlinear systems with different structures. The approach developed in the previous section can be used, once we have the required regularity result for one equation.

5.1. Systems with absorption gradient terms.

One can also consider a system, where the gradients appear as absorption terms.

$$(5.1) \quad \begin{cases} (-\Delta)^s u + |\nabla v|^q = \lambda f & \text{in } \Omega, \\ (-\Delta)^s v + |\nabla u|^p = \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

At first sight, one may think that this modified system is easier to analyze. This is far from being the case, even when dealing with a single equation. Namely, it is proved in [4] that the following problem

$$\begin{cases} (-\Delta)^s u + |\nabla u|^p = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

has a solution for any $f \in L^1(\Omega)$ provided that $p < 2s$. However, the question of existence is wide open when $p > 2s$, even if f belongs to a suitable Lebesgue space, and even (to our knowledge) in the local case ($s = 1$). The only known result in the latter case is the existence of a solution when $f \in W^{1,\infty}(\Omega)$, see [49].

Nevertheless, by following the same line of proof as in §4.1, one can prove the same existence result under the same conditions as in Theorem 4.2 for the System (5.1). This is a straightforward adaptation, and we leave it to the reader. \square

5.2. Systems with drift.

Here, we consider a version of the System (S) where the fractional Laplacian $(-\Delta)^s$ is replaced by the linear operator L_s defined as $L_s(u) = (-\Delta)^s u + B(x) \cdot \nabla u$, with $|B| \in L^\sigma(\Omega)$ and $\sigma > \frac{N}{2s-1}$. More precisely, we consider the following System

$$(SD) \quad \begin{cases} (-\Delta)^s u + B_1(x) \cdot \nabla u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^s v + B_2(x) \cdot \nabla v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where B_i is a vector fields such that $|B_i| \in L^\sigma(\Omega)$ with $\sigma > \frac{N}{2s-1}$.

As explained previously, in order to prove the existence of a solution to System (SD) , we have to analyze the regularity of the solution to the associate Poisson equation. More precisely, define w to be the unique solution to the problem

$$(5.2) \quad \begin{cases} L_s(w) = h & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $L_s(w) = (-\Delta)^s w + B(x) \cdot \nabla w$ and $|B| \in L^\sigma(\Omega)$ with $\sigma > \frac{N}{2s-1}$.

As defined in [25, page 463], we say that a measurable vector fields $\widehat{B} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ belongs to the Kato's class $\mathcal{K}_N^{\alpha-1}$, with $\alpha \in (1, 2)$, if

$$\lim_{\varepsilon \rightarrow 0} \sup_{x \in \mathbb{R}^N} \int_{|x-y| < \varepsilon} |\widehat{B}(y)| |x-y|^{\alpha-1-N} dy = 0.$$

Since $|B| \in L^\sigma(\Omega)$ with $\sigma > \frac{N}{2s-1}$, then setting

$$\widehat{B}(x) = \begin{cases} B(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

and using Hölder's inequality, it follows that \widehat{B} is in Kato's class \mathcal{K}_N^{2s-1} .

Hence, we can write

$$w(x) = \int_{\Omega} \widetilde{\mathcal{G}}_s(x, y) h(y) dy,$$

where $\widetilde{\mathcal{G}}_s$ is the Green's function associated to the operator L_s . On the other hand, Under the above assumption on B , we know that $\widetilde{\mathcal{G}}_s \simeq \mathcal{G}_s$ thanks to [25, Theorem 1].

Concerning the gradient of $\widetilde{\mathcal{G}}_s$, according to [47, Corollary 1.8] (see also [47, page 284] where the class of operators for which Corollary 1.8 holds is given), we deduce the existence of two positive constants C_1 and C_2 such that for *a.e.* $x, y \in \Omega$

$$|\nabla_x \widetilde{\mathcal{G}}_s(x, y)| \leq C_1 \widetilde{\mathcal{G}}_s(x, y) \max \left\{ \frac{1}{|x-y|}, \frac{1}{\delta(x)} \right\} \leq C_2 \mathcal{G}_s(x, y) \max \left\{ \frac{1}{|x-y|}, \frac{1}{\delta(x)} \right\}.$$

Hence, $\widetilde{\mathcal{G}}_s$ verifies the same estimates as \mathcal{G}_s (the Green's function without drift). Therefore, we get the same regularity result for the solution to Problem (5.2).

Thus, using the approach developed in the proof of Theorem 4.2, we obtain the same existence result for the System (SD) as in the case without drift. Since this extension does not involve any difficulty, we will again leave it to the reader. \square

5.3. Systems with potential-gradient term.

Now, let us consider systems with potential-gradient in the right-side. More precisely, we analyze the following System :

$$(5.3) \quad \begin{cases} (-\Delta)^s u = v^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^s v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

As in the previous section, we assume that $p, q > 1$ and $s > \frac{1}{2}$. Again, we can prove an existence result; this time, we give the proof as this extension is less straightforward than the other two.

Theorem 5.1. *Let $s > \frac{1}{2}$. Assume that $1 < p, q < \frac{s}{1-s}$ and $pq > 1$. In addition, suppose that $(f, g) \in L^m(\Omega)^+ \times L^\sigma(\Omega)^+$ where (m, σ) satisfies one of the following conditions*

$$(5.4) \quad \begin{cases} \max \left\{ \frac{q+1}{q+2}, \frac{p+1}{p+2} \right\} < s < 1, \\ 1 < m < \frac{N}{s-q(1-s)}, \quad 1 < \sigma < \frac{N}{s-p(1-s)}, \\ p\sigma < \widehat{m} = \frac{mN}{N-m(s-q(1-s))} \quad \text{and} \quad \frac{qm}{N+qm} < \frac{\sigma}{N-\sigma(s-p(1-s))}, \end{cases}$$

or

$$(5.5) \quad \begin{cases} \frac{1}{2} < s < \min\{\frac{q+1}{q+2}, \frac{p+1}{p+2}\}, \\ \frac{1}{2s-q(1-s)} < m < \frac{N}{s-q(1-s)}, \quad \frac{1}{2s-p(1-s)} < \sigma < \frac{N}{s-p(1-s)}, \\ p\sigma < \widehat{m} = \frac{mN}{N-m(s-q(1-s))} \quad \text{and} \quad \frac{qm}{N+qm} < \frac{\sigma}{N-\sigma(s-p(1-s))}, \end{cases}$$

or

$$(5.6) \quad m \geq \frac{N}{s-q(1-s)}, \quad \sigma > \frac{qmN}{N+qm\sigma((s+1)-q(1-s))},$$

or

$$(5.7) \quad \sigma \geq \frac{N}{s-p(1-s)}, \quad m > \frac{p\sigma N}{N+p\sigma(s-q(1-s))}.$$

Then, there exists $\widehat{\Lambda} > 0$ such that for any $(\lambda, \mu) \in \Pi$ where

$$(5.8) \quad \Pi := \{(\lambda, \mu) \in [0, +\infty) \times [0, +\infty) ; \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq \widehat{\Lambda}\},$$

System (5.3) has a nonnegative solution $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$. Moreover $(u\delta^{1-s}, v\delta^{1-s}) \in \mathbb{W}_0^{1,\theta_1}(\Omega) \times \mathbb{W}_0^{1,\theta_2}(\Omega)$ for all $\theta_1 < \widehat{m}$ and $\theta_2 < \widehat{\sigma}$.

Proof. The proof closely follows the arguments used in the proof of Theorem 4.2 taking into consideration the potential term in the first equation. For the reader's convenience, we give some details under the condition (5.4).

Since $\frac{qmN}{N+qm} < \widehat{\sigma} = \frac{\sigma N}{N-\sigma(s-p(1-s))}$, then we can fix $r > 1$ such that

$$\frac{qmN}{N+qm} < r < \frac{\sigma N}{N-\sigma(s-p(1-s))} = \widehat{\sigma}.$$

As in the proof of Theorem 4.2, we fix ℓ as the unique solution to algebraic identity

$$\ell^{\frac{1}{pq}} = \widetilde{C} \left(\ell + \frac{\Lambda^*}{\widetilde{C}} \right).$$

Let $\widehat{\Lambda} = \frac{\Lambda^*}{\widetilde{C}}$ and consider the set Π defined in (5.8). In this case Π is given by

$$\Pi := \left\{ (\lambda, \mu) \in [0, +\infty) \times [0, +\infty) ; \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq \frac{\Lambda^*}{\widetilde{C}} \right\}.$$

Obviously, Π is bounded closed set of \mathbb{R}^2 . Moreover, for all $(\lambda, \mu) \in \Pi$, we have

$$(5.9) \quad \widetilde{C} \left(\ell + \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \right) \leq \ell^{\frac{1}{pq}}.$$

Let ℓ and r be fixed and define the set :

$$(5.10) \quad H := \{ \varphi \in \mathbb{W}_0^{1,1}(\Omega) ; \varphi \delta^{1-s} \in \mathbb{W}_0^{1,r}(\Omega) \text{ and } \left(\int_{\Omega} |\nabla(\varphi \delta^{1-s})|^r dx \right)^{\frac{1}{r}} \leq \ell^{\frac{1}{pq}} \}.$$

For $\varphi \in H$, using Sobolev inequality we get

$$(5.11) \quad \left(\int_{\Omega} |\varphi \delta^{1-s}|^{r^*} dx \right)^{\frac{1}{r^*}} \leq C \left(\int_{\Omega} |\nabla(\varphi \delta^{1-s})|^r dx \right)^{\frac{1}{r}} \leq C \ell^{\frac{1}{pq}}.$$

Thus for $\varphi \in H$, we have $\varphi^a \delta^{a(1-s)} \in L^1(\Omega)$ for all $a \leq r^*$; in particular $\varphi^{qm} \delta^{qm(1-s)} \in L^1(\Omega)$. Now, applying Hölder's inequality, we get the existence of $\beta < 2s - 1$ such that $\varphi^q \delta^\beta \in L^1(\Omega)$. Hence according to Proposition 2.4, there exists a unique solution u to

$$(5.12) \quad \begin{cases} (-\Delta)^s u = \varphi_+^q + \lambda f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

In particular, $|\nabla u| \in L^\alpha(\Omega)$ for any $\alpha < \frac{N}{N-2s+1+\beta}$. By Theorem (3.2), with $a = q$, we deduce that

$$(5.13) \quad \|\nabla u| \delta^{1-s}\|_{L^\gamma(\Omega)} \leq C \left(\|\varphi \delta^{1-s}\|_{L^{qm}(\Omega)}^q + \lambda \|f\|_{L^m(\Omega)} \right),$$

for any $\gamma \leq \widehat{m} = \frac{mN}{N-m(s-q(1-s))}$. Recall that $qm < r^*$, then

$$\|\nabla u|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C \left(\|\varphi\delta^{1-s}\|_{L^{r^*}(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)} \right).$$

Thus, by Sobolev's inequality we get

$$(5.14) \quad \|\nabla u|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C \left(\|\nabla(\varphi\delta^{1-s})\|_{L^r(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)} \right) \text{ for any } \gamma \leq \widehat{m}.$$

Given that $p\sigma < \widehat{m}$, then $|\nabla u|^p\delta^{p(1-s)} \in L^\sigma(\Omega)$ by (5.14).

Going back to the condition (5.4) and using Hölder's inequality, then $|\nabla u|^p\delta^{\beta_1} \in L^1(\Omega)$ for some $\beta_1 < 2s-1$. Therefore, by Theorem 2.3 and Remark (2.4), There exists v as a unique weak solution to the Problem

$$(5.15) \quad \begin{cases} (-\Delta)^s v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $v \in \mathbb{W}_0^{1,a}(\Omega)$ for any $a < \frac{N}{N-2s+1+\beta_1}$. Therefore the operator

$$\begin{aligned} T : H &\longrightarrow \mathbb{W}_0^{1,1}(\Omega) \\ \varphi &\longmapsto T(\varphi) = v, \end{aligned}$$

is well defined. Moreover, if v is a fixed point of T , (u, v) is a weak solution to the System (5.3). So, we just have to show that T has a fixed point in H .

The continuity and the compactness of T follow using closely the computations in the proof of Theorem 4.2. Hence we have just to show that $T(H) \subset H$.

Using Theorem 3.2, it holds that for all $\nu \leq \widehat{\sigma}$,

$$(5.16) \quad \|\nabla v|\delta^{1-s}\|_{L^\nu(\Omega)} \leq C \left(\|\nabla u|\delta^{1-s}\|_{L^{p\sigma}(\Omega)}^p + \mu\|g\|_{L^\sigma(\Omega)} \right).$$

Recall that $p\sigma < \widehat{m}$ and $qm < r$. Then, by using (5.14) and (5.16), we get

$$(5.17) \quad \|\nabla v|\delta^{1-s}\|_{L^\nu(\Omega)} \leq C \left(\|\nabla(\varphi\delta^{1-s})\|_{L^r(\Omega)}^{pq} + \mu\|g\|_{L^\sigma(\Omega)} + \lambda\|f\|_{L^m(\Omega)}^p \right).$$

Now, taking into consideration that $r < \widehat{\sigma}$ and choosing $\nu = r$ in the previous inequality,

$$\|\nabla v|\delta^{1-s}\|_{L^r(\Omega)} \leq C \left(\|\nabla(\varphi\delta^{1-s})\|_{L^r(\Omega)}^{pq} + \mu\|g\|_{L^\sigma(\Omega)} + \lambda\|f\|_{L^m(\Omega)}^p \right).$$

Since $\varphi \in H$, then

$$\|\nabla v|\delta^{1-s}\|_{L^r(\Omega)} \leq C \left(\ell + \mu\|g\|_{L^\sigma(\Omega)} + \lambda^p\|f\|_{L^m(\Omega)}^p \right).$$

Taking into consideration the definition of ℓ , we conclude that

$$\|\nabla v|\delta^{1-s}\|_{L^r(\Omega)} \leq \ell^{\frac{1}{pq}}. \text{ Thus } v \in H \text{ and then } T(H) \subset H.$$

Thus, we conclude that T has a fixed point in H . So, System (5.3) has a solution (u, v) with $(u\delta^{1-s}, v\delta^{1-s}) \in \mathbb{W}_0^{1,\theta_1}(\Omega) \times \mathbb{W}_0^{1,\theta_2}(\Omega)$ for any $\theta_1 < \widehat{m}$ and $\theta_2 < \sigma_s^*$. \square

The smallness condition on the parameters λ, μ is needed. Indeed, we can show that for large values of λ or μ , System (5.3) does not have any nonnegative solution. More precisely, we have

Theorem 5.2. *Suppose that $s > \frac{1}{2}$, $p > 1$ and $q \geq 1$. For $\theta \in C_0^\infty(\Omega)$ with $\theta \geq 0$, we consider $\phi_\theta \in \mathbb{W}_0^{s,2}(\Omega) \cap L^\infty(\Omega)$ and $\psi_\theta \in \mathbb{W}_0^{1,\alpha}(\Omega, \delta^s)$ to be the solutions to problems (4.35) and (4.36) respectively.*

We set

$$H(\theta) := \left(\int_\Omega |\nabla \psi_\theta|^\alpha \phi_\theta dx \right)^{\frac{p'(\alpha-1)}{\alpha}} + \int_\Omega \theta^{q'} \phi_\theta^{1-q'},$$

and define

$$(5.18) \quad \Lambda^*(f) := \inf_{\{0 \not\leq \theta \in C_0^\infty(\Omega)\}} \left\{ H(\theta) ; \int_\Omega f \phi_\theta dx = 1 \right\},$$

$$\Gamma^*(g) := \inf_{\{0 \leq \theta \in C_0^\infty(\Omega)\}} \left\{ H(\theta) ; \int_{\Omega} g \phi_{\theta} dx = 1 \right\}.$$

Then there exists $C(p, q)$ such that if the System (5.3) has a solution (u, v) , then $\lambda \leq C(p, q)\Lambda^*(f)$ and $\mu \leq C(p, q)\Gamma(g)$.

Proof. Without loss of generality, we can assume that $f, g \in L^\infty(\Omega)$. Using ϕ_{θ} as a test function in the first equation of System (5.3), we obtain that,

$$\begin{aligned} \int_{\Omega} (v^q + \lambda f) \phi_{\theta} dx &= \left| \int_{\Omega} \theta u dx \right| \\ &\leq \int_{\Omega} \phi_{\theta} |\nabla \psi_{\theta}|^{\alpha-1} |\nabla u| dx. \end{aligned}$$

Since $p'(\alpha - 1) < \alpha$, then, by Hölder and Young's inequalities, we have that,

$$(5.19) \quad \int_{\Omega} (v^q + \lambda f) \phi_{\theta} dx \leq \int_{\Omega} |\nabla u|^p \phi_{\theta} dx + C(p) \left(\int_{\Omega} |\nabla \psi_{\theta}|^{\alpha} \phi_{\theta} dx \right)^{\frac{p'(\alpha-1)}{\alpha}}.$$

Using now ϕ_{θ} as a test function in the second equation of system (5.3) and by Young's inequalities, we deduce

$$(5.20) \quad \begin{aligned} \int_{\Omega} (|\nabla u|^p + \mu g) \phi_{\theta} dx &\leq \int_{\Omega} v \theta dx \leq \int_{\Omega} v \phi_{\theta} \frac{\theta}{\phi_{\theta}} dx \\ &\leq \frac{1}{2} \int_{\Omega} v^q \phi_{\theta} dx + C(q) \int_{\Omega} \theta^{q'} \phi_{\theta}^{1-q'} dx. \end{aligned}$$

The last integral is finite using the fact that $0 \leq \theta \in C_c^\infty(\Omega)$ and $\phi_{\theta} \simeq \delta^s$. Combining (5.19) and (5.20), we get

$$(5.21) \quad \begin{aligned} \lambda \int_{\Omega} f \phi_{\theta} dx &\leq C(p) \left(\int_{\Omega} |\nabla \psi_{\theta}|^{\alpha} \phi_{\theta} dx \right)^{\frac{p'(\alpha-1)}{\alpha}} + C(q) \int_{\Omega} \theta^{q'} \phi_{\theta}^{1-q'} \\ &\leq C(p, q) H(\theta). \end{aligned}$$

Hence $\lambda \leq C(p, q)\Lambda^*(f)$ with

$$\Lambda^*(f) := \inf_{\{0 \leq \theta \in C_0^\infty(\Omega)\}} \left\{ H(\theta) ; \int_{\Omega} f \phi_{\theta} dx = 1 \right\}.$$

In the same way we get the estimate on μ . □

Acknowledgments: The authors would like to thank:

- the first referee for his/her careful reading, for his/her constructive suggestions and for bringing to our attention the references [13] and [32] ;
- the second referee for taking the time to review our manuscript ;
- Prof. Boumediene Abdellaoui for his helpful suggestions and fruitful discussions during the preparation of this work.

Funding: The first and the second authors are supported by a research project from DGRSDT, Algeria. The third author is supported by the research project of CNRS IEA EDPTA.

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