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SOMIA ATMANI, KHEIREDDINE BIROUD, MAHA DAOUD, EL-HAJ LAAMRI*

Abstract. The present paper is concerned with a class of nonlocal parabolic reaction-diffusion systems set in a regular bounded open subset of $\mathbb{R}^N$, where the gradients of the unknowns act as source terms (see (S) below). First, we establish some nonexistence and blow-up in finite time results. Second, we prove some new weighted regularity results. Such results are interesting in themselves and play a crucial role to study local existence of nonnegative solutions to our systems under suitable assumptions on the data.
This work also highlights a substantial difference between the nonlocal case and the local case already studied by the fourth author and his coworkers.

1 Introduction

In this work, we consider a class of nonlinear fractional parabolic reaction-diffusion systems of the following type :

$$(S) \qquad \left\{\begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& |\nabla v|^q + f & \text{in} \quad \Omega \times (0,T), \\ \partial_t v + (-\Delta)^s v &=& |\nabla u|^p + g & \text{in} \quad \Omega \times (0,T), \\ u(x,t) = v(x,t) &=& 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) &=& u_0(x) & \text{in} \quad \Omega, \\ v(x,0) &=& v_0(x) & \text{in} \quad \Omega, \end{array}\right.$$

where Ω is a bounded regular open subset of $\mathbb{R}^N$, $N > 2s$, $\frac{1}{2} < s < 1$, $p, q \geqslant 1$, f, g, u_0 and v_0 are measurable nonnegative functions satisfying some assumptions which will be specified later. The operator $(-\Delta)^s$ is the classical fractional Laplacian defined by

$$(1.1) \qquad (-\Delta)^s u(x) := a_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad s \in (0,1),$$

where $a_{N,s}$ is a normalization constant. The choice

$$a_{N,s} := \frac{s 2^{2s} \Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

makes the above definition coherent with the limiting cases $s = 0$ and $s = 1$, *i.e.*, the following two identities:

$$\lim_{s \to 0^+} (-\Delta)^s u = u \quad \text{and} \quad \lim_{s \to 1^-} (-\Delta)^s u = -\Delta u$$

hold (see, for instance, [31, Proposition 4.4] and [29, Proposition 2.1]). Nevertheless, in the context of System (S), a rescaling of the time variable t allows one to take $a_{N,s} = 1$. This is the choice made in this article.

Let us emphasize that fractional Laplacians play a considerable role in many scientific fields; see, for instance, [29, 31, 58] and their references. In particular, such operators represent the prototypical generators for modeling nonlocal diffusions. For further details, we refer the reader to [23] and the references therein.

Stationary systems with gradient terms appear in some models of electrochemical engineering and fluid mechanics. We refer to [25] and [32] for more details and applications. Related systems also appear when considering Mean Field Games with fractional or nonlocal diffusions: they have many applications in economy, network engineering, biology and many other fields. See for instance [44, 38, 67] and the references therein. In some of those applications, considering evolution version of the model (of generalized reaction-diffusion type) clearly makes sense. This motivates the study which is the object of this paper.

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In order to put our work in mathematical context, we shall recall some previous results related to our System (S).

• **Case of a single equation.** In this case, System (S) reduces to

$$\left\{\begin{array}{rcll} \partial_t w + (-\Delta)^s w &=& |\nabla w|^p + f & \text{in} \quad \Omega \times (0,T), \\ w(x,t) &=& 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ w(x,0) &=& w_0(x) & \text{in} \quad \Omega. \end{array}\right. \tag{1.2}$$

— *Local case* $s = 1$. In this instance, Problem (1.2) is no other than the initial-boundary value problem for the usual Kardar-Parisi-Zhang equation (see [48], [13] and [26] for more details). This Problem and the following Cauchy-Problem

$$\partial_t w - \Delta w = |\nabla w|^p + f \quad \text{in} \quad \mathbb{R}^N \times (0,T)\ ,\ w(x,0) = w_0(x) \quad \text{in} \quad \mathbb{R}^N, \tag{1.3}$$

have been extensively investigated, in both cases $f = 0$ and $f \neq 0$.

◇ Case where $f = 0$. In bounded domains, the question of the existence and uniqueness of solution has been treated in [8, 16, 27, 63] and the references therein. Also, many researchers have devoted their efforts to Cauchy Problem (1.3); see, for example, [9, 17, 15, 14, 43] and the references cited therein.
Needless to say in the two cases, the references mentioned above do not exhaust the rich literature on the subject.

◇ Case where $f \neq 0$. To the best of our knowledge, Problems (1.2) and (1.3) have been treated in [64, 65]. In specific, the case $p = 2$ has received a lot of attention from the point of view of sharp regularity results for positive solutions. Complete classification of the set of nonnegative solutions was obtained in relation with the classical parabolic capacity (see [5]). Concerning the case of unbounded domains strictly included in $\mathbb{R}^N$, we refer to [56] and references cited therein.

— *Nonlocal case* $\frac{1}{2} < s < 1$. As far as we know, Problem (1.2) has only been addressed in [6]. Fortunately, that paper is complete and written with great care, and almost all cases have been studied. To keep short, the authors proved (i) the existence of a unique weak solution when $p < \frac{s}{1-s}$; (ii) the nonexistence when $p > \frac{1}{1-s}$; (iii) blow-up in finite time if $1 + s < p < \frac{s}{1-s}$ and $s \in (\frac{\sqrt{5}-1}{2}, 1)$. Moreover, they established the regularity of the solution according to that of f and u_0.

• **Case of Systems.**

— *Local case* $s = 1$. Contrary to single equations, only few results are known in the case of systems where the gradients of the unknowns act as source terms, even in the local case. Let us review the known results about such systems.

◇ *First*, in [2] the fourth author and his coworkers investigated the local version ($s = 1$) of System (S). They also studied the case where the right-hand sides mix potential and gradient terms, namely, $\partial_t u - \Delta u = v^q + f\ ,\ \partial_t v - \Delta v = |\nabla u|^p + g$. Let us mention that the elliptic version of the latter system, namely, $\Delta u = v^q + \lambda f\ ,\ \Delta v = |\nabla u|^p + \mu g$ was studied by the same authors in [4].

◇ *Second*, other existence results for local parabolic systems with gradient term are found in [2, 39, 33, 46, 50, 61] and the references therein. Let us also mention that the case where the gradient term acts as absorption was considered in [57], see also [18, 33] and the references therein.

— *Nonlocal case* $0 < s < 1$. Even less is known in this case. To our knowledge, System (S) has not been treated yet. On the other hand, we have recently studied in [10] the stationary version of (S), which takes the form

$$\left\{\begin{array}{rcll} (-\Delta)^s u &=& |\nabla v|^q + \lambda f & \text{in} \quad \Omega, \\ (-\Delta)^s v &=& |\nabla u|^p + \mu g & \text{in} \quad \Omega, \\ u = v &=& 0 & \text{in} \quad \mathbb{R}^N \setminus \Omega. \end{array}\right. \tag{1.4}$$

We have also studied in [11] the case where the diffusions are different, namely

$$(-\Delta)^{s_1} u = |\nabla v|^q + \lambda f, \quad (-\Delta)^{s_2} v = |\nabla u|^p + \mu g \quad \text{ where } s_1 \neq s_2.$$

The main goal of the present paper is to assess the issue of existence of nonnegative solutions to System (S) under suitable additional assumptions. By solution, we essentially mean solution in the sense of distributions, see Definition 3.1. This work extends both the results of [2] for the local $(s = 1)$ version of System (S), and those of [6] pertaining to the single nonlocal KPZ equation (1.2).

Those results will either be directly reused or serve as an inspiration in this paper. However, this extension is far less trivial than it looks at first sight. Indeed, System (S) presents many interesting features: it is not variational, even if $s = 1$, it does not apparently have any comparison or monotony properties, and no symmetry properties. Moreover, it is nonlocal, which is a source of new difficulties. Consequently, the arguments used in the local case cannot be adapted to the new situation. The major obstructions to the construction of weak solutions are, firstly, the time-dependent character of the system and, secondly, the loss of regularity caused by the nonlocal operator $(-\Delta)^s$, especially near the boundary. In order to circumvent these difficulties, we will use in a convenient way:
(i) some sharp estimates on the kernel of the fractional heat equation (2.1);
(ii) some fine, and to our knowledge, new weighted estimates on the solution to the corresponding fractional heat equation. See Theorems 4.2 and 4.3. They complement previous results from [55, 20, 6] in many directions.

The other main results of this paper can be summarized in the following points.
1) If $\min\{p,q\} > \frac{1+s}{1-s}$, System (S) does not have any solution even if the data are bounded; see Theorem 3.2.
2) If $\min\{p,q\} > 1+s$, any solution to System (S) blows up in finite time under a suitable assumption on initial data; see Theorem 3.3.
3) If $(p,q) \in \left[1, \frac{2s-1}{1-s}\right)^2$, we are able to prove existence of local solutions to the three following relevant models of System (S), for $\frac{2}{3} < s < 1$:

- *model* 1: we will study System (S) by taking $(f,g) \neq (0,0)$ and $(u_0,v_0) = (0,0)$ for $pq > 1$.
- *model* 2: we will take $(f,g) = (0,0)$ and $(u_0,v_0) \neq (0,0)$ for $pq > 1$.
- *model* 3: we will study System (S) by taking $(f,g) \neq (0,0)$ and $(u_0,v_0) = (0,0)$ for $p = q = 1$.

Unfortunately, existence of solutions to System (S) in the general case *i.e.* $(f,g) \neq (0,0)$ and $(u_0,v_0) \neq (0,0)$ appears much more difficult (indeed, as in [6]) and will be left open for the time being.
As for the unexpected restriction $s > \frac{2}{3}$, it is not clear whether it is only technical, or due to deeper phenomena. Nevertheless, we shall see that it naturally appears in the proof and suggest some possible reasons.
That being said, the above questions will be the subject of our future research. Furthermore, we shall consider variants of the previous systems, called potential-gradient systems, where the term $|\nabla v|^q$ is replaced with v^q in a forthcoming work, see [12]. The following nonlocal Kardar-Parisi-Zhang Problem

$$\partial_t w + (-\Delta)^s w = |(-\Delta)^{s/2} w|^p + f \ , \ w(x,t) = 0 \text{ in } (\mathbb{R}^N \setminus \Omega) \times (0,T) \ , \ w(x,0) = w_0(x) \text{ in } \Omega$$

is also considered in [1].

Although it has been well known for quite a few years, this work also shows significant difference between the local and nonlocal cases. For example, when $s = 1$, the existence of nonnegative solutions to System (S) is proved for a large class of (p,q) provided that f and g satisfy some suitable integrability assumptions; see [2, Theorems 4.1 and 4.2].

The rest of this paper is composed of five sections and an Appendix. In Section 2, we collect some tools that will be used throughout the paper, such as the notion of fractional parabolic-Sobolev spaces and some of their properties. Then, we gather some well-known regularity results of the classical fractional heat equation with Dirichlet conditions, as well as some useful properties of the fractional heat kernel. Section 3 is devoted to nonexistence and blow-up results. In Section 4, we prove two new regularity results for the fractional heat equation by working in weighted fractional Sobolev spaces. Based on the results in Section 4, we state and prove our existence results in Section 5: model 1 in subsection 5.1, model 2 in subsection 5.2 and model 3 in subsection 5.3. As for the Appendix, we introduce the proof of a technical Lemma.

For the sake of completeness and the ease of reading, we have tried to write the paper in an almost self-contained form. Moreover, we will give precise references for all the points that are not detailed in the work.
Notations. Before ending this section, let us fix some notations. Throughout this paper,

— Ω is a bounded regular open subset of $\mathbb{R}^N$ with $N > 2s$ and $s \in (\frac{1}{2}, 1)$.
— $\Omega_T := \Omega \times (0,T)$ where T is an arbitrary positive real.
— For $x \in \Omega$, by definition $\delta(x) := \text{dist}(x, \partial\Omega)$.

— For $\beta > 0$, $L^1(\Omega_T, \delta^\beta(x)dxdt) := \{h : \Omega_T \to \mathbb{R}$ measurable ; $\|h\|_{L^1(\Omega_T,\delta^\beta(x)\,dx\,dt)} < +\infty\}$, where

$$\|h\|_{L^1(\Omega_T,\delta^\beta(x)\,dx\,dt)} := \iint_{\Omega_T} |h(x,t)|\delta^\beta(x)\,dx\,dt.$$

— By C, we denote a positive constant which may be different in each inequality. ■

2 Functional framework and tools

In this section, we will first recall the definitions of the functional spaces relevant to our study. Then, we collect some useful tools that will be used systematically in this paper.

2.1 Some functional spaces.

For the reader's convenience, we will recall in this subsection some (more or less classical) definitions of fractional Sobolev spaces and fractional parabolic spaces (see, e.g., [7], [31], [29], [36], [45], [58], ...). Of course, this subsection can be skipped by readers already familiar with these notions.

2.1.1 Fractional Sobolev Spaces

Let $s \in (0,1)$ and $p \geqslant 1$.
— The fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \Big\{\phi \in L^p(\mathbb{R}^N)\,;\, \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy < +\infty\Big\},$$

and it is a Banach space endowed with its canonical norm

$$\|\phi\|_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\phi(x)|^p dx + \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+ps}}dxdy\right)^{\frac{1}{p}}.$$

— The fractional Sobolev space $\mathbb{W}_0^{s,p}(\Omega)$ is defined by

$$\mathbb{W}_0^{s,p}(\Omega) := \left\{\phi \in W^{s,p}(\mathbb{R}^N)\,;\, \phi = 0 \text{ in } \mathbb{R}^N\setminus\Omega\right\}.$$

Endowed with the norm induced by $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ which we denote $|\cdot|_{\mathbb{W}_0^{s,p}(\Omega)}$, $\mathbb{W}_0^{s,p}(\Omega)$ is a Banach space.

For $\phi \in \mathbb{W}_0^{s,p}(\Omega)$, we set

$$\|\phi\|_{\mathbb{W}_0^{s,p}(\Omega)} := \left(\iint_{D_\Omega} \frac{|\phi(x)-\phi(y)|^p}{|x-y|^{N+sp}}dxdy\right)^{1/p}$$

where

$$D_\Omega := (\mathbb{R}^N\times\mathbb{R}^N)\setminus\big((\mathbb{R}^N\setminus\Omega)\times(\mathbb{R}^N\setminus\Omega)\big).$$

It is clear that $\|\cdot\|_{\mathbb{W}_0^{s,p}(\Omega)}$ defines a norm on $\mathbb{W}_0^{s,p}(\Omega)$. In addition, Ω is assumed to be bounded, then the norms $|\cdot|_{\mathbb{W}_0^{s,p}(\Omega)}$ and $\|\cdot\|_{\mathbb{W}_0^{s,p}(\Omega)}$ are equivalent.

Now, we denote by $\mathcal{C}_0^\infty(\Omega)$ the space

$$\mathcal{C}_0^\infty(\Omega) := \left\{\phi\ :\ \mathbb{R}^N \to \mathbb{R}\,;\ \phi \in \mathcal{C}^\infty(\mathbb{R}^N),\ \operatorname{supp}\phi\ \text{ is compact and }\ \operatorname{supp}\phi \subseteq \Omega\right\},$$

where $\operatorname{supp}\phi := \overline{\{x \in \mathbb{R}^N\ :\ \phi(x) \neq 0\}}$.
It is well-known that

$$\mathbb{W}_0^{s,p}(\Omega) = \overline{\mathcal{C}_0^\infty(\Omega)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}}.$$

See, for instance, [41, Theorem 6] for a proof.
— In the particular case $p = 2$, we denote

$$H^s(\mathbb{R}^N) := W^{s,2}(\mathbb{R}^N) \quad\text{and}\quad \mathbb{H}_0^s(\Omega) := \mathbb{W}_0^{s,2}(\Omega).$$

Furthermore, we denote by $\mathbb{H}^{-s}(\Omega)$ the dual space of $\mathbb{H}_0^s(\Omega)$. □

2.1.2 Weighted Sobolev spaces.

As already said, we shall need to work in weighted spaces, the definition of which we now recall. Let $\omega : \mathbb{R}^N \to [0,+\infty)$ be a measurable function. The weighted Sobolev space $W^{1,p}(\Omega, \omega(x)dx)$ is defined as follows :

$$W^{1,p}(\Omega, \omega(x)dx) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} ; \|u\|_{W^{1,p}(\Omega,\omega(x)dx)} < +\infty\right\},$$

where

$$\|u\|_{W^{1,p}(\Omega,\omega(x)dx)} := \left(\sum_{|\alpha|\leqslant 1} \int_\Omega |D^\alpha u(x)|^p \, \omega(x)dx\right)^{1/p}.$$

In the same way, we have

$$W_0^{1,p}(\Omega, \omega(x)dx) := \left\{\phi \in W^{1,p}(\Omega, \omega(x)dx) \; ; \; \phi = 0 \text{ on } \partial\Omega\right\} \; ;$$

$$\mathbb{W}_0^{1,p}(\Omega, \omega(x)dx) := \left\{\phi \in W^{1,p}(\mathbb{R}^N, \omega(x)dx) \; ; \; \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega\right\}.$$

The case $\omega = 1$ yields the classical Sobolev spaces $W^{1,p}(\Omega)$, $W_0^{1,p}(\Omega)$ and $\mathbb{W}_0^{1,p}(\Omega)$.

Remark 2.1. As Ω is bounded and assumed to be regular, $\mathbb{W}_0^{1,p}(\Omega) \subset \mathbb{W}_0^{s,p}(\Omega)$. See [36, Section 2 and Proposition 3.1]. □

2.1.3 Fractional Parabolic Spaces.

Before ending this subsection, let us recall the definitions of fractional parabolic spaces corresponding to parabolic problems we consider in this work. Let $p, q \geqslant 1$,

— The parabolic Lebesgue space $L^{p,q}(\Omega_T)$ is defined by

$$L^{p,q}(\Omega_T) := \Big\{\phi : \Omega_T \to \mathbb{R} \;\text{ measurable } \; ; \; \|\phi\|_{L^{p,q}(\Omega_T)} < +\infty\Big\},$$

where

$$\|\phi\|_{L^{p,q}(\Omega_T)} := \left(\int_0^T \left(\int_\Omega |\phi(x,t)|^p dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}}.$$

In the case $p = q$, this definition boils down to $L^p(\Omega_T)$.

— The fractional parabolic Sobolev space $L^p(0,T; \mathbb{W}_0^{s,p}(\Omega))$ is the set of functions $\phi \in L^p(\Omega_T)$ such that $\|\phi\|_{L^p(0,T;\mathbb{W}_0^{s,p}(\Omega))} < +\infty$, where

$$\|\phi\|_{L^p(0,T;\mathbb{W}_0^{s,p}(\Omega))} := \Big(\int_0^T \iint_{D_\Omega} \frac{|\phi(x,t) - \phi(y,t)|^p}{|x-y|^{N+ps}} dxdydt\Big)^{\frac{1}{p}}.$$

We can easily check that $L^p(0,T; \mathbb{W}_0^{s,p}(\Omega))$ is a Banach space. □

2.2 The Fractional Heat Equation: existence and regularity.

As said in the introduction, we shall need fine, and (to our knowledge) previously unknown, regularity results for the solution to the fractional heat equation with general data. To begin with, we recall some known results. They will serve as stepping-stones to prove our improved results in Section 5, whose novelty will appear in comparison.

The focus of this Subsection will be on the following problem:

$$\left\{\begin{array}{rcll} \partial_t w + (-\Delta)^s w &=& h & \text{in} \quad \Omega_T, \\ w(x,t) &=& 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ w(x,0) &=& w_0(x) & \text{in} \quad \Omega, \end{array}\right. \tag{2.1}$$

where h and w_0 are functions defined in suitable Lebesgue spaces.

2.2.1 Energy solution, Weak solution and first results of existence and regularity

First, we begin by defining the two notions of solutions of (2.1), that we will use throughout this paper: energy solution and distributional solution.

Definition 2.2 (***Energy solution***)**.** Assume that $(h, w_0) \in L^2(\Omega_T) \times L^2(\Omega)$. We say that w is an energy solution to Problem (2.1) if $w \in L^2(0, T; \mathbb{H}_0^s(\Omega)) \cap \mathcal{C}([0, T]; L^2(\Omega))$, $w_t \in L^2(0, T; \mathbb{H}^{-s}(\Omega))$, and for any $v \in L^2(0, T; \mathbb{H}_0^s(\Omega))$, we have

$$\iint_{\Omega_T} w_t v dx\, dt + \frac{1}{2}\int_0^T \iint_{D\Omega} \frac{(w(x,t) - w(y,t))(v(x,t) - v(y,t))}{|x - y|^{N+2s}} dxdy\; dt = \iint_{\Omega_T} hv dx\, dt.$$

We refer to [55, Section 5] for the existence and properties of energy solutions.

However, Definition 2.2 is meaningless when $(h, w_0) \in L^1(\Omega_T) \times L^1(\Omega)$. To overcome this difficulty, we will use the notion of weak solution; see, e.g., [55, Definition 27]. In order to give precise statement of such definition, let us first introduce the following problem:

$$(P_\varphi) \qquad \begin{cases} -\partial_t \phi + (-\Delta)^s \phi &= \varphi \quad \text{in} \quad \Omega_T, \\ \phi(x,t) &= 0 \quad \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0, T), \\ \phi(x,T) &= 0 \quad \text{in} \quad \Omega. \end{cases}$$

Let $\beta \in (0, 1)$. It is well-known (see [55, 40]) that if $\varphi \in L^\infty(\Omega_T) \cap \mathcal{C}^{0,\beta}(\Omega_T)$, Problem (P_φ) has a regular solution $\phi \in L^\infty(\Omega_T)$ and the equation $-\partial_t\phi + (-\Delta)^s\phi = \varphi$ is satisfied in a pointwise sense.

So, we are ready to define the set of test functions

$$\mathcal{P}(\Omega_T) := \{\phi : \mathbb{R}^N \times [0, T] \to \mathbb{R}\ ;\ \phi \text{ is solution to } (P_\varphi) \text{ where } \varphi \in L^\infty(\Omega_T) \cap \mathcal{C}^{0,\beta}(\Omega_T)\}. \tag{2.2}$$

Then, we have the following definition:

Definition 2.3 (***Weak solution***)**.** Assume that $(h, w_0) \in L^1(\Omega_T) \times L^1(\Omega)$. We say that $w \in \mathcal{C}([0, T]; L^1(\Omega))$ is a weak solution to Problem (2.1) if for any $\phi \in \mathcal{P}(\Omega_T)$, we have

$$\iint_{\Omega_T} w\big(- \phi_t + (-\Delta)^s\phi\big)\, dx\, dt = \iint_{\Omega_T} h\phi\, dxdt + \int_\Omega w_0(x)\phi(x, 0)\, dx. \tag{2.3}$$

The following existence result is proved in [55] (see also [3] and [20] for other approaches).

Theorem 2.4 ([55, Theorem 28])**.** *Assume that $(h, w_0) \in L^1(\Omega_T) \times L^1(\Omega)$. Then, Problem (2.1) has a unique weak solution w such that $w \in \mathcal{C}([0, T]; L^1(\Omega)) \cap L^\alpha(\Omega_T)$ for any $\alpha \in [1, \frac{N+2s}{N})$, $|(-\Delta)^{\frac{s}{2}} w| \in L^r(\Omega_T)$ for any $r \in [1, \frac{N+2s}{N+s})$, and $T_k(w) \in L^2(0, T; \mathbb{H}_0^s(\Omega))$ for any $k > 0$ where $T_k(w) := \max\{-k, \min\{k, w\}\}$.*
Moreover, $w \in L^\gamma(0, T; \mathbb{W}_0^{s,p}(\Omega))$ for any $1 \leqslant \gamma < \frac{N+2s}{N+s}$. In addition, we have

$$\begin{aligned} &\|w\|_{\mathcal{C}([0,T];L^1(\Omega))} + \|w\|_{L^\alpha(\Omega_T)} + \|(-\Delta)^{\frac{s}{2}} w\|_{L^r(\Omega_T)} + \|w\|_{L^\gamma(0,T;\mathbb{W}_0^{s,\gamma}(\Omega))} \\ &\leqslant C(\Omega_T)\bigg(\|h\|_{L^1(\Omega_T)} + \|w_0\|_{L^1(\Omega)} \bigg). \end{aligned} \tag{2.4}$$

Remark 2.5. From now on, we will assume that $s \in (1/2, 1)$. This will allow us to consider the power-gradient terms as a nonlinear perturbation.

The following Theorem deals with the case where data are less regular.

Theorem 2.6 ([6, Theorem 3.22])**.** *Assume that $s \in (\frac{1}{2}, 1)$. Let $w_0 \in L^1(\Omega)$ and h measurable such that $h\delta^\beta \in L^1(\Omega_T)$ for some $\beta < 2s - 1$. Then, Problem (2.1) has a unique weak solution w such that for any $r < \frac{N+2s}{N+\beta+1}$,*

$$\|w\|_{\mathcal{C}([0,T];L^1(\Omega,\delta^\beta dx))} + \|\nabla w\|_{L^r(\Omega_T)} \leqslant C(r, \beta, \Omega_T)\bigg(\|h\delta^\beta\|_{L^1(\Omega_T)} + \|w_0\|_{L^1(\Omega)} \bigg).$$

Moreover, for $r < \frac{N+2s}{N+\beta+1}$ fixed, the operator $\Phi : L^1(\Omega_T, \delta^\beta(x)dxdt) \times L^1(\Omega) \to L^r(0, T; W_0^{1,r}(\Omega))$ defined by $\Phi(h, w_0) = w$ (where w is the unique solution to Problem (2.1)) is compact. □

2.2.2 Some useful properties of the kernel of the Fractional Heat Equation

In order to prove our theorems in the next sections, we will make crucial use of the properties of the kernel of the evolution problem (2.1), which we denote as $P(x,y,t)$. In particular, if w solves (2.1), then it admits the representation formula

$$w(x,t) = \iint_{\Omega_t} P(x,y,t-\tau)h(y,\tau)dyd\tau + \int_{\Omega} P(x,y,t)w_0(y)dy, \tag{2.5}$$

where $\Omega_t = \Omega \times (0,t)$.
Unlike in the whole space, there is no explicit formula for $P(x,y,t)$. However, many properties of this kernel and its gradient are known when $s \in (1/2, 1)$. We gather in the following Lemma those we will use in the next sections. For more details, we refer to [22], [24], [49] and [6].

Lemma 2.7. *Assume that $s \in (\frac{1}{2}, 1)$. Let $(x,y) \in \Omega \times \Omega$ and $0 < t < T$. Then, we have :*

$$P(x,y,t) \backsimeq \Big(1 \wedge \frac{\delta^s(x)}{\sqrt{t}}\Big) \times \Big(1 \wedge \frac{\delta^s(y)}{\sqrt{t}}\Big) \times \Big(t^{-\frac{N}{2s}} \wedge \frac{t}{|x-y|^{N+2s}}\Big) \tag{2.6}$$

and

$$|\nabla_x P(x,y,t)| \leqslant C\Big(\frac{1}{\delta(x) \wedge t^{\frac{1}{2s}}}\Big) P(x,y,t). \tag{2.7}$$

Moreover, there exists $C = C(s,N) > 0$ such that

$$P(x,y,t) \leqslant \frac{2t}{t^{\frac{N+2s}{2s}} + |x-y|^{N+2s}} \leqslant C(s,N)\frac{t}{(t^{\frac{1}{2s}} + |x-y|)^{N+2s}}. \ \square \tag{2.8}$$

2.2.3 Additional regularity results

In this paragraph, we will present some useful results on the existence and regularity of the solution to Problem (2.1), which will be systematically used in our demonstration. All these theorems are taken from [6, Section 3], where the proofs are given in detail.

Theorem 2.8 ([6, Theorem 3.5])**.** *Assume that $s \in (\frac{1}{2}, 1)$ and $(h, w_0) \in L^1(\Omega_T) \times L^1(\Omega)$. Let w be the unique solution to Problem (2.1). Then, for any $\eta < \frac{N+2s}{N+1}$, there exists $C = C(\eta, \Omega_T)$ such that*

$$\|w\|_{\mathcal{C}([0,T];L^1(\Omega))} + \|\nabla w\|_{L^\eta(\Omega_T)} \leqslant C\bigg(\|h\|_{L^1(\Omega_T)} + \|w_0\|_{L^1(\Omega)}\bigg). \tag{2.9}$$

Moreover, for $\eta < \frac{N+2s}{N+1}$ fixed, the operator $\Phi : L^1(\Omega_T) \times L^1(\Omega) \to L^\eta(0,T; \mathbb{W}_0^{1,\eta}(\Omega))$ defined by $\Phi(h, w_0) = w$ is compact.

Theorem 2.9 ([6, Theorems 3.10 and 3.11])**.** *Assume that $w_0 \equiv 0$, $h \in L^m(\Omega_T)$ with $m > 1$ and let w be the unique weak solution to Problem (2.1). Then, $\dfrac{w}{\delta^s} \in L^\gamma(\Omega_T)$ for any $\gamma < \dfrac{m(N+2s)}{N+2s-ms}$ and $|\nabla w|\delta^{1-s} \in L^\nu(\Omega_T)$ for any $\nu < \dfrac{m(N+2s)}{N+2s-m(2s-1)}$. Moreover, there exists a positive constant $C = C(\Omega, T, N, s, m)$ such that*

$$\Big\|\frac{w}{\delta^s}\Big\|_{L^\gamma(\Omega_T)} \leqslant C\|h\|_{L^m(\Omega_T)} \tag{2.10}$$

and

$$\big\||\nabla w|\delta^{1-s}\big\|_{L^\nu(\Omega_T)} \leqslant C\|h\|_{L^m(\Omega_T)}. \tag{2.11}$$

Theorem 2.10 ([6, Proposition 3.20 and Theorem 3.21])**.** *Suppose that $h \equiv 0$ and $w_0 \in L^\sigma(\Omega)$ with $\sigma \geqslant 1$. Let w be the unique weak solution to Problem (2.1). Then, there exists a positive constant $C = C(\Omega, T, N, s, \sigma)$ such that*

$$\Big\|\frac{w}{\delta^s}\Big\|_{L^\alpha(\Omega_T)} \leqslant C\|w_0\|_{L^\sigma(\Omega)}, \tag{2.12}$$

and

$$\big\||\nabla w|\delta^{1-s}\big\|_{L^\beta(\Omega_T)} \leqslant C\|w_0\|_{L^\sigma(\Omega)}, \tag{2.13}$$

for any α, $\beta < \frac{\sigma(N+2s)}{N+\sigma}$.

Setting $\overline{\sigma} = \min\{\sigma, 2s\}$*, then* $|\nabla w| \in L^{\gamma}(\Omega_T)$ *for any* $\gamma < \dfrac{\overline{\sigma}(N+2s)}{N+\overline{\sigma}}$ *and*

$$\| |\nabla w| \|_{L^{\gamma}(\Omega_T)} \leqslant C \|w_0\|_{L^{\sigma}(\Omega)}. \tag{2.14}$$

Remark 2.11. Let $(m,\sigma) \in (1,+\infty)^2$ and $(h, w_0) \in L^m(\Omega_T) \times L^{\sigma}(\Omega)$ with $h \neq 0$ and $w_0 \neq 0$. To our knowledge, one does not have regularity results like those of Theorems 2.9 and 2.10. □

2.3 Some results from integration theory

In this subsection, we will recall two useful results from the theory of integration.

2.3.1 An interpolation result

The following result is rather classical but we tend to forget it quickly. We preferred to recall it to facilitate the reading of the paper. It will be used several times in our proofs.

Proposition 2.12 ([19, Theorem 1])**.** *Let* $h \in L^1(\Omega)$ *and* $\{h_n\}_n \subset L^r(\Omega)$ *with* $1 < r < +\infty$*. Assume that* $h_n \to h$ *strongly in* $L^1(\Omega)$ *and* $\{h_n\}_n$ *is bounded in* $L^r(\Omega)$*.*
Then, $h \in L^r(\Omega)$ *and for any* $a \in [1, r)$*, we have the following interpolation inequality*

$$\|h_n - h\|_{L^a(\Omega)} \leqslant \|h_n - h\|^{\theta}_{L^1(\Omega)} \|h_n - h\|^{1-\theta}_{L^r(\Omega)} \text{ where } \theta := \frac{r-a}{a(r-1)}. \tag{2.15}$$

In particular, $h_n \to h$ *strongly in* $L^a(\Omega)$*.* □

2.3.2 Weighted Hardy Inequality

In order to prove Theorems 3.2, 3.3 and 5.1, we will need the following version of the famous weighted Hardy inequality.

Theorem 2.13 ([59, Theorem 1.6])**.** *Let* $\nu \in (1,+\infty)$ *and* $0 \leqslant \rho < \nu - 1$*. Then, there exists a positive constant* $C = C(\Omega, \rho, \nu)$ *such that for any* $\varphi \in C_0^{\infty}(\Omega)$*, we have*

$$\int_{\Omega} \delta^{\rho-\nu}(x) |\varphi(x)|^{\nu} dx \leqslant C \int_{\Omega} \delta^{\rho}(x) |\nabla \varphi(x)|^{\nu} dx. \tag{2.16}$$

For a recent proof, we refer the interested reader to the nice paper [37] and the references contained therein.

■

3 Nonexistence and blow-up results

First of all, let us specify what we mean by weak solution to System (S).

Definition 3.1. Let $(p,q) \in [1,+\infty)^2$. Assume that $(f,g) \in (L^1(\Omega_T)^+)^2$ and $(u_0, v_0) \in (L^1(\Omega)^+)^2$. We say that $(u,v) \in L^p(0,T; \mathbb{W}_0^{1,p}(\Omega)) \times L^q(0,T; \mathbb{W}_0^{1,q}(\Omega))$ is a weak solution to System (S) if for any $\varphi, \psi \in \mathcal{P}(\Omega_T)$, we have

$$\left\{ \begin{array}{l} \displaystyle\iint_{\Omega_T} u\big(-\varphi_t + (-\Delta)^s \varphi\big)\, dx\, dt = \iint_{\Omega_T} |\nabla v|^q \varphi\, dxdt + \iint_{\Omega_T} f\varphi\, dxdt + \int_{\Omega} u_0(x)\varphi(x,0)\, dx, \\ \displaystyle\iint_{\Omega_T} v\big(-\psi_t + (-\Delta)^s \psi\big)\, dx\, dt = \iint_{\Omega_T} |\nabla u|^p \psi\, dxdt + \iint_{\Omega_T} g\psi\, dxdt + \int_{\Omega} v_0(x)\psi(x,0)\, dx. \ \square \end{array} \right. \tag{3.1}$$

Now, we are ready to state and prove the two main theorems of this section. Such theorems show that p and q must not exceed a certain threshold if one expects the existence of a solution to System (S) even if the data are regular.

3.1 Nonexistence results

Theorem 3.2. *Let* $(f,g) \in (L^{\infty}(\Omega_T)^+)^2$ *and* $(u_0, v_0) \in (L^{\infty}(\Omega)^+)^2$*. Assume that* $\min\{p,q\} \geqslant \dfrac{1+s}{1-s}$ *with* $s \in (\frac{1}{2}, 1)$*. Then, System* (S) *does not have any nonnegative solution* $(u,v) \in L^p(0,\, T;\, \mathbb{W}_0^{1,p}(\Omega)) \times L^q(0,\, T;\, \mathbb{W}_0^{1,q}(\Omega))$ *such that* $\displaystyle\iint_{\Omega_T} |\nabla u(x,t)|^p \delta^s(x) dxdt < +\infty$ *and* $\displaystyle\iint_{\Omega_T} |\nabla v(x,t)|^q \delta^s(x) dxdt < +\infty$*.*

Proof. Without loss of generality, we assume that $q > p \geqslant \frac{1+s}{1-s}$. Assume, by contradiction, that System (S) has a solution (u,v) with $\displaystyle\iint_{\Omega_T} (|\nabla u(x,t)|^p + |\nabla v(x,t)|^q) \delta^s(x) dxdt < +\infty$. Thanks to [6, Proposition 4.2], for any $(t_1, t_2) \subset (0,T)$, there exists a positive constant $C = C(t_1, t_2, \Omega)$ such that

$$u(x,t) \geqslant C\delta^s(x) \ a.e. \text{ for } (x,t) \in \Omega \times (t_1, t_2).$$

Since $s < p-1$, then using the weighted Hardy inequality (2.16), it follows that
$$\int_\Omega \frac{u^p(x,t)}{\delta^{p-s}(x)}dx \leqslant C\int_\Omega |\nabla u(x,t)|^p\delta^s(x)dx.$$
Thus
$$\int_{t_1}^{t_2}\int_\Omega \frac{\delta^{ps}(x,t)}{\delta^{p-s}(x)}dxdt \leqslant \int_{t_1}^{t_2}\int_\Omega \frac{u^p(x,t)}{\delta^{p-s}(x)}dxdt \leqslant C\int_{t_1}^{t_2}\int_\Omega |\nabla u(x,t)|^p\delta^s(x)dxdt < \infty.$$
However
$$\int_{t_1}^{t_2}\int_\Omega \frac{1}{\delta^{p(1-s)-s}(x)}dxdt < +\infty \Leftrightarrow p < \frac{s+1}{1-s}.$$
Contradiction with the fact that $p \geqslant \dfrac{s+1}{1-s}$. □

3.2 Blow-up Result

The second theorem is more precise and shows that we can have explosion in finite time. Namely,

Theorem 3.3. *Suppose that $(f,g)\in (L^1(\Omega_T)^+)^2$ and $\min\{p,q\} > 1+s$ with $s\in(\frac{1}{2},1)$. Let φ_1 be the first eigenfunction of fractional Laplacian. Assume that there exists a constant $C_0 > 0$ such that*
$$\int_\Omega (u_0(x)+v_0(x))\varphi_1(x)dx > C_0.$$
Then, any solution to System (S) blows up in finite time. In other words, there exists $T^* < +\infty$ such that
$$\lim_{t\to T^*}\int_\Omega (u(x,t)+v(x,t))\varphi_1(x)dx = +\infty.$$

Reminding : Before proving Theorem 3.3, let us recall that φ_1 satisfies
$$\begin{cases} (-\Delta)^s\varphi_1 &= \lambda_1\varphi_1 & \text{in} & \Omega,\\ \varphi_1 &= 0 & \text{in} & \mathbb{R}^N\setminus\Omega,\\ \varphi_1 &> 0 & \text{in} & \Omega.\end{cases}$$
Moreover, $\varphi_1\in\mathbb{H}_0^s(\Omega)\cap L^\infty(\Omega)$ and may be chosen such that $||\varphi_1||_{L^2(\Omega)} = 1$. For more details, we refer to [62, Proposition 9]. In addition, $\varphi_1 \simeq \delta^s$ *i.e.* there exists $a,b>0$ such that
$$a\delta^s(x)\leqslant\varphi_1(x)\leqslant b\delta^s(x) \text{ for } a.e.\ x\in\Omega. \tag{3.2}$$

Now we are ready to prove our blow-up result.

Proof.
We follow closely the arguments used in [2] and [6]. Using φ_1 as test function in both equations, we get
$$\frac{d}{dt}\int_\Omega u(x,t)\varphi_1(x)dx + \lambda_1\int_\Omega u(x,t)\varphi_1(x)dx \geqslant \int_\Omega |\nabla v(x,t)|^q\varphi_1(x)dx \tag{3.3}$$
and
$$\frac{d}{dt}\int_\Omega v(x,t)\varphi_1(x)dx + \lambda_1\int_\Omega v(x,t)\varphi_1(x)dx \geqslant \int_\Omega |\nabla u(x,t)|^p\varphi_1(x)dx. \tag{3.4}$$
According to (3.2), we obtain
$$\int_\Omega |\nabla u(x,t)|^p\varphi_1(x)dx \geqslant a\int_\Omega |\nabla u(x,t)|^p\delta^s(x)dx \tag{3.5}$$
and
$$\int_\Omega |\nabla v(x,t)|^q\varphi_1(x)dx \geqslant a\int_\Omega |\nabla v(x,t)|^q\delta^s(x)dx. \tag{3.6}$$
Since $\min\{p,q\} > 1+s$. Then, by using the weighted Hardy inequality (2.16) and Jensen's inequality, we obtain
$$\begin{cases} \displaystyle\int_\Omega |\nabla u(x,t)|^p\delta^s(x)dx & \geqslant & C\displaystyle\int_\Omega u^p(x,t)\delta^{s-p}(x)dx,\\ & \geqslant & C\displaystyle\int_\Omega u^p(x,t)\varphi_1(x)dx,\\ & \geqslant & C\left(\displaystyle\int_\Omega u(x,t)\varphi_1(x)dx\right)^p. \end{cases} \tag{3.7}$$

Applying the same arguments to the equation in v, we get

$$\int_\Omega |\nabla v(x,t)|^q \delta^s(x)dx \geqslant C\bigg(\int_\Omega v(x,t)\varphi_1(x)dx\bigg)^q.$$

Denoting by $Y(t) = \int_\Omega u(x,t)\varphi_1(x)dx$ and $Z(t) = \int_\Omega v(x,t)\varphi_1(x)dx$, it follows that

$$\left\{ \begin{array}{lll} Y'(t) + \lambda_1 Y(t) & \geqslant & CZ^q(t), \\ Z'(t) + \lambda_1 Z(t) & \geqslant & CY^p(t). \end{array} \right. \tag{3.8}$$

Now, setting $\ell = \min\{p,q\} > 1$. Thanks to the convexity, we have

$$(Y+Z)'(t) + \lambda_1(Y+Z)(t) \geqslant (Y^p(t) + Z^q(t)) \geqslant C(Y(t)+Z(t))^\ell - C(Y(t)+Z(t)).$$

Therefore, denoting by $W(t) = (Y+Z)(t)$, we obtain

$$W'(t) + (\lambda_1 + C)W(t) \geqslant CW^\ell(t).$$

Given that $\ell > 1$ and $W(0) > C_0$, then there exists $T^* > 0$ such that

$$\lim_{t\to T^*} W(t) = +\infty.$$

Hence, we get the desired result. ■

4 Weighted regularity and compactness results

As already said, in order to carry out our investigation, we need more regularity results for the solutions to the fractional heat equation, in addition to those recalled in Subsubsection 2.2.3. Thus, we shall introduce some fine weighted estimates in this section. Namely, we will need to know the regularity of $\frac{w}{\delta^s}$ and $|\nabla w|\delta^{1-s}$, where w is the solution to (2.1). To the best of our knowledge, these results are new and interesting in themselves. Moreover, they complement in several respects the previous results obtained in [6, 55, 20].

• One of the fundamental tools that provides the regularity of some types of singular integrals is given in the next Lemma. The proof is postponed to the Appendix.

Lemma 4.1. *Let $s \in (0,1)$, $\alpha, \lambda \in \mathbb{R}$ are such that $\alpha > -1$ and $N + \lambda > 0$. Assume that $h \in L^m(\Omega_T)$ with $m > 1$ and $m' = \frac{m}{m-1}$. Moreover, let us define*

$$H(x,t) := \int_0^t \int_\Omega \frac{h(y,\tau)(t-\tau)^\alpha}{((t-\tau)^{\frac{1}{2s}} + |x-y|)^{N+\lambda}} dyd\tau.$$

Then:

1) *if $\alpha \geqslant \frac{N+\lambda}{2s}$, then $H \in L^\infty(\Omega_T)$ and*

$$\|H\|_{L^\infty(\Omega_T)} \leqslant C(\Omega)T^{\alpha - \frac{N+\lambda}{2s} + \frac{1}{m'}} \|h\|_{L^m(\Omega_T)};$$

2) *if $\frac{\lambda-2s}{2s} < \alpha < \frac{N+\lambda}{2s}$ and $1 < m < \frac{N+2s}{2s(\alpha+1)-\lambda}$, then $H \in L^r(\Omega_T)$ for any $r < \frac{m(N+2s)}{N+2s-m(2s(\alpha+1)-\lambda)}$ and*

$$\|H\|_{L^r(\Omega_T)} \leqslant C(\Omega)T^{\gamma+\frac{1}{a}} \|h\|_{L^m(\Omega_T)};$$

3) *if $\frac{\lambda-2s}{2s} < \alpha < \frac{N+\lambda}{2s}$ and $m \geqslant \frac{N+2s}{2s(\alpha+1)-\lambda}$, then $H \in L^r(\Omega_T)$ for any $r \leqslant +\infty$ and*

$$\|H\|_{L^r(\Omega_T)} \leqslant C(\Omega)T^{\gamma+\frac{1}{a}} \|h\|_{L^m(\Omega_T)},$$

where $a = \frac{m'r}{r+m'}$ and $\gamma = \alpha + \dfrac{N}{2sa} - \dfrac{N+\lambda}{2s}$. □

• Our first regularity theorem reads as follows:

Theorem 4.2. *Assume that $w_0 = 0$ and h is a measurable function such that $h\delta^{a(1-s)} \in L^m(\Omega_T)$ for $0 \leqslant a < \dfrac{s}{1-s}$ and $m \geqslant 1$. Let w be the unique solution to (2.1). Then, there exists a positive constant $C = C(\Omega, T, N, s, m)$ independent of w and h such that:*

(i) *if $1 \leqslant m < \frac{N+2s}{s-a(1-s)}$, $\dfrac{w}{\delta^s} \in L^\gamma(\Omega_T)$ for any $\gamma < \widehat{m} = \frac{m(N+2s)}{N+2s-m(s-a(1-s))}$. Moreover*

$$\left\| \frac{w}{\delta^s} \right\|_{L^\gamma(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}\ ; \tag{4.1}$$

(ii) *if* $m = \frac{N+2s}{s-a(1-s)}$, $\dfrac{w}{\delta^s} \in L^{\gamma}(\Omega_T)$ *for any* $\gamma < +\infty$ *and*

$$\left\|\frac{w}{\delta^s}\right\|_{L^{\gamma}(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}\ ; \tag{4.2}$$

(iii) *if* $m > \frac{N+2s}{s-a(1-s)}$, $\dfrac{w}{\delta^s} \in L^{\infty}(\Omega_T)$ *and*

$$\left\|\frac{w}{\delta^s}\right\|_{L^{\infty}(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}. \tag{4.3}$$

Proof. For reasons of concision, we give the proof only for the case (i). The other cases can be proved in a similar way.
Without loss of generalities, we assume that $h \gneqq 0$. Let w be the unique solution to Problem (2.1). Then

$$w(x,t) = \iint_{\Omega_t} P(x,y,t-\tau)h(y,\tau)dyd\tau.$$

It is clear that $w \gneqq 0$. Moreover, we have

$$\left\{\begin{array}{rcl} \dfrac{w(x,t)}{\delta^s(x)} &=& \dfrac{1}{\delta^s(x)}\displaystyle\iint_{\Omega_t} P(x,y,t-\tau)h(y,\tau)dyd\tau,\\ &=& \dfrac{1}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{(t-\tau)^{\frac{1}{2s}}\leqslant\delta(x)\right\}} P(x,y,t-\tau)h(y,\tau)dyd\tau\\ &+& \dfrac{1}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{(t-\tau)^{\frac{1}{2s}}>\delta(x)\right\}} P(x,y,t-\tau)h(y,\tau)dyd\tau,\\ &=& I_1(x,t)+I_2(x,t). \end{array}\right. \tag{4.4}$$

To estimate I_1, we decompose it as follows

$$\left\{\begin{array}{rcl} I_1(x,t) &=& \dfrac{1}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{\delta(y)<(t-\tau)^{\frac{1}{2s}}\leqslant\delta(x)\right\}} P(x,y,t-\tau)h(y,\tau)dyd\tau\\ &+& \dfrac{1}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{(t-\tau)^{\frac{1}{2s}}\leqslant\min\{\delta(x),\delta(y)\}\right\}} P(x,y,t-\tau)h(y,\tau)dyd\tau,\\ &=& I_{11}(x,t)+I_{12}(x,t). \end{array}\right. \tag{4.5}$$

Regarding I_{11}, using (2.6) and (2.8), we get

$$\left\{\begin{array}{rcl} I_{11}(x,t) &=& \dfrac{C}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{\delta(y)<(t-\tau)^{\frac{1}{2s}}\leqslant\delta(x)\right\}} P(x,y,t-\tau)h(y,\tau)dyd\tau,\\ &\leqslant& \dfrac{1}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{\delta(y)<(t-\tau)^{\frac{1}{2s}}\leqslant\delta(x)\right\}} h(y,\tau)\delta^s(y)\dfrac{\sqrt{t-\tau}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau,\\ &\leqslant& C\displaystyle\iint_{\Omega_t} h(y,\tau)\delta^{a(1-s)}(y)\dfrac{(t-\tau)^{\frac{s-a(1-s)}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau. \end{array}\right. \tag{4.6}$$

Thus by Lemma 4.1, it follows that $I_{11} \in L^{\gamma}(\Omega_T)$ for every $\gamma < \widehat{m}$ and

$$\|I_{11}\|_{L^{\gamma}(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}. \tag{4.7}$$

We treat now I_{12}. From estimates (2.6) and (2.8), we get

$$\left\{\begin{array}{rcl} I_{12}(x,t) &=& \dfrac{1}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{(t-\tau)^{\frac{1}{2s}}\leqslant\min\{\delta(x),\delta(y)\}\right\}} P(x,y,t-\tau)h(y,\tau)dyd\tau,\\ &\leqslant& \dfrac{C}{\delta^s(x)}\displaystyle\iint_{\Omega_t\cap\left\{(t-\tau)^{\frac{1}{2s}}\leqslant\min\{\delta(x),\delta(y)\}\right\}} h(y,\tau)\dfrac{t-\tau}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau,\\ &\leqslant& C\displaystyle\iint_{\Omega_t\cap\left\{(t-\tau)^{\frac{1}{2s}}\leqslant\min\{\delta(x),\delta(y)\}\right\}} \dfrac{h(y,\tau)\delta^{a(1-s)}(y)\delta^{-a(1-s)}(y)\sqrt{t-\tau}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau,\\ &\leqslant& C\displaystyle\iint_{\Omega_t} h(y,\tau)\delta^{a(1-s)}(y)\dfrac{(t-\tau)^{\frac{s-a(1-s)}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau. \end{array}\right. \tag{4.8}$$

Hence, we deduce from Lemma 4.1 that $I_{12} \in L^{\gamma}(\Omega_T)$ for any $\gamma < \widehat{m}$ and

$$\|I_{12}\|_{L^{\gamma}(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}. \tag{4.9}$$

We deal now with I_2. We have

$$\begin{cases} I_2(x,t) = \dfrac{1}{\delta^s(x)} \displaystyle\iint_{\Omega_t \cap \left\{\delta(x)<(t-\tau)^{\frac{1}{2s}}\right\}} h(y,\tau)P(x,y,t-\tau)dyd\tau, \\ = \dfrac{1}{\delta^s(x)} \displaystyle\iint_{\Omega_t \cap \left\{\delta(x)<(t-\tau)^{\frac{1}{2s}} \leqslant \delta(y)\right\}} h(y,\tau)P(x,y,t-\tau)dyd\tau \\ +\dfrac{1}{\delta^s(x)} \displaystyle\iint_{\Omega_t \cap \left\{(t-\tau)^{\frac{1}{2s}} > \max\{\delta(y),\delta(x)\}\right\}} h(y,\tau)P(x,y,t-\tau)dyd\tau, \\ = I_{21}(x,t) + I_{22}(x,t). \end{cases} \tag{4.10}$$

Concerning I_{21}, by using estimate (2.6) and (2.8), it holds that

$$\begin{cases} I_{21}(x,t) &= \dfrac{1}{\delta^s(x)} \displaystyle\iint_{\Omega_t \cap \left\{\delta(x)<(t-\tau)^{\frac{1}{2s}} \leqslant \delta(y)\right\}} P(x,y,t-\tau)h(y,\tau)dyd\tau, \\ &\leqslant C \displaystyle\iint_{\Omega_t \cap \left\{\delta(x)<(t-\tau)^{\frac{1}{2s}} \leqslant \delta(y)\right\}} h(y,\tau)\dfrac{\sqrt{t-\tau}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau, \\ &\leqslant C \displaystyle\iint_{\Omega_t \cap \left\{\delta(x)<(t-\tau)^{\frac{1}{2s}} \leqslant \delta(y)\right\}} \dfrac{h(y,\tau)}{\delta^{a(1-s)}(y)}\dfrac{\delta^{a(1-s)}(y)\sqrt{t-\tau}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau, \\ &\leqslant C \displaystyle\iint_{\Omega_t} h(y,\tau)\delta^{a(1-s)}(y)\dfrac{(t-\sigma)^{\frac{s-a(1-s)}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau. \end{cases} \tag{4.11}$$

Applying again Lemma 4.1, then we have $I_{21} \in L^{\gamma}(\Omega_T)$ for any $\gamma < \widehat{m}$ and

$$\|I_{21}\|_{L^{\gamma}(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}. \tag{4.12}$$

To estimate I_{22}, we follow the same computations as in previous estimate. Indeed,

$$\begin{cases} I_{22}(x,t) = \dfrac{1}{\delta^s(x)} \displaystyle\iint_{\Omega_t \cap \left\{(t-\tau)^{\frac{1}{2s}} > \max\{\delta(y),\delta(x)\}\right\}} h(y,\tau)P(x,y,t-\tau)dyd\tau, \\ \leqslant \dfrac{1}{\delta^s(x)} \displaystyle\iint_{\Omega_t \cap \left\{(t-\tau)^{\frac{1}{2s}} > \max\{\delta(y),\delta(x)\}\right\}} \dfrac{\delta^s(x)}{\sqrt{t-\tau}}\dfrac{\delta^s(y)}{\sqrt{t-\tau}}\dfrac{h(y,\tau)(t-\tau)}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau, \\ \leqslant C \displaystyle\iint_{\Omega_t \cap \left\{(t-\tau)^{\frac{1}{2s}} > \max\{\delta(y),\delta(x)\}\right\}} h(y,\tau)\dfrac{\delta^{a(1-s)}(y)\delta^{s-a(1-s)}(y)}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau, \\ \leqslant C \displaystyle\iint_{\Omega_t} h(y,\tau)\delta^{a(1-s)}(y)\dfrac{(t-\tau)^{\frac{s-a(1-s)}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau. \end{cases} \tag{4.13}$$

Thus, Lemma 4.1 yields

$$\|I_{22}\|_{L^{\gamma}(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)} \qquad \text{for any } \gamma < \widehat{m} = \frac{m(N+2s)}{N+2s-m(s-a(1-s))}. \tag{4.14}$$

Finally, by combining (4.7), (4.9), (4.12) and (4.14), we get the desired result. □

• Now, we are ready to state and prove the second main regularity result of this Section. Surprisingly so, when working in bounded domains, the term δ^{1-s} appears in a natural way describing the gradient's behavior. Of course, this is not the case when working in $\mathbb{R}^N$; we refer to [21] and [47] for more details.

Theorem 4.3. *Assume that $w_0 = 0$ and h is a measurable function such that $h\delta^{a(1-s)} \in L^m(\Omega_T)$ for $0 \leqslant a < \dfrac{2s-1}{1-s}$ and $m \geqslant 1$. Let w be the unique solution to Problem (2.1). Then:*

1) *there exists a positive constant $C = C(\Omega, T, a, m, s)$ independent of h such that:*

(i) *if* $1 \leqslant m < \frac{N+2s}{2s-1-a(1-s)}$, $|\nabla w|\delta^{1-s} \in L^{\gamma}(\Omega_T)$ *for any* $\gamma < \tilde{m} = \frac{m(N+2s)}{N+2s-m(2s-1-a(1-s))}$. *Moreover*

$$\big\||\nabla w|\delta^{1-s}\big\|_{L^{\gamma}(\Omega_T)} \leqslant C\left(\Big\|\frac{w}{\delta^s}\Big\|_{L^{\widehat{m}}(\Omega_T)} + \|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}\right). \tag{4.15}$$

In addition, thanks to Theorem 4.2, we have

$$\big\||\nabla w|\delta^{1-s}\big\|_{L^{\gamma}(\Omega_T)} \leqslant C\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}; \tag{4.16}$$

(ii) *if* $m = \frac{N+2s}{2s-1-a(1-s)}$, $|\nabla w|\delta^{1-s} \in L^{\gamma}(\Omega_T)$ *for any* $\gamma < +\infty$. *Moreover*

$$\big\||\nabla w|\delta^{1-s}\big\|_{L^{\gamma}(\Omega_T)} \leqslant C\left(\Big\|\frac{w}{\delta^s}\Big\|_{L^{\gamma}(\Omega_T)} + \|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}\right); \tag{4.17}$$

(iii) *if* $m > \frac{N+2s}{2s-1-a(1-s)}$, $|\nabla w|\delta^{1-s} \in L^{\infty}(\Omega_T)$ *and*

$$\big\||\nabla w|\delta^{1-s}\big\|_{L^{\infty}(\Omega_T)} \leqslant C\left(\Big\|\frac{w}{\delta^s}\Big\|_{L^{\infty}(\Omega_T)} + \|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}\right). \tag{4.18}$$

2) *For any* $\gamma < \dfrac{N+2s}{N+1+a(1-s)}$ *fixed, the operator*

$$\Phi : L^1(\Omega_T, \delta^{a(1-s)}(x)dxdt)) \to L^{\gamma}(0,T; \mathbb{W}_0^{1,\gamma}(\Omega, \delta^{\gamma(1-s)}dx))$$

defined by $\Phi(h) = w$ (*where* w *is the unique weak solution to* (2.1)) *is compact.*

Proof.
1) This time as well, we will prove case (i). The other cases can be proved in a similar way and are left to the readers.
For the sake of simplicity, we assume that $h \gneqq 0$. Using the representation formula, it holds that

$$w(x,t) = \iint_{\Omega_t} P(x,y,t-\tau)h(y,\tau)dyd\tau.$$

Then

$$|\nabla_x w(x,t)|\delta^{1-s}(x) \leqslant \delta^{1-s}(x)\iint_{\Omega_t} |\nabla_x P(x,y,t-\tau)|h(y,\tau)dyd\tau.$$

From (2.7), we have

$$\left\{\begin{array}{rcl} |\nabla_x w(x,t)|\delta^{1-s}(x) & \leqslant & C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\delta(x)\leqslant(t-\tau)^{\frac{1}{2s}}\right\}} \frac{h(y,\tau)}{\delta(x)}P(x,y,t-\tau)dyd\tau \\ & + & C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\delta(x)>(t-\tau)^{\frac{1}{2s}}\right\}} \frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau, \\ & = & I_1(x,t) + I_2(x,t). \end{array}\right. \tag{4.19}$$

We begin by treating I_1. We have

$$I_1(x,t) \leqslant \frac{C}{\delta^s(x)}\iint_{\Omega_t} h(y,\tau)P(x,y,t-\tau)dyd\tau = C\frac{w(x,t)}{\delta^s(x)}. \tag{4.20}$$

By Theorem 4.2, $I_1 \in L^{\gamma}(\Omega_T)$ for any $\gamma < \widehat{m} = \frac{m(N+2s)}{N+2s-m(s-a(1-s))}$.
Now, we deal with I_2.

$$\left\{\begin{array}{rcl} I_2(x,t) & = & C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\delta(x)>(t-\tau)^{\frac{1}{2s}}\right\}} \frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau, \\ & = & C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\frac{\delta(x)}{2}\leqslant(t-\tau)^{\frac{1}{2s}}<\delta(x)\right\}} \frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau \\ & + & C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\delta(y)\leqslant(t-\tau)^{\frac{1}{2s}}<\frac{\delta(x)}{2}\right\}} \frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau \\ & + & C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\min\{\delta(y),\delta(x)\}>(t-\tau)^{\frac{1}{2s}}\right\}} \frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau, \\ & = & J_1(x,t) + J_2(x,t) + J_3(x,t). \end{array}\right. \tag{4.21}$$

Starting with J_1, we have

$$\tag{4.22}\left\{\begin{array}{lcl} J_1(x,t) &=& \delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\frac{\delta(x)}{2}\leqslant(t-\tau)^{\frac{1}{2s}}<\delta(x)\right\}}\frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau,\\ &=& \displaystyle\frac{1}{\delta^s(x)}\iint_{\Omega_t\cap\left\{\frac{\delta(x)}{2}\leqslant(t-\tau)^{\frac{1}{2s}}<\delta(x)\right\}}h(y,\tau)P(x,y,t-\tau)dyd\tau,\\ &\leqslant& \displaystyle\frac{C}{\delta^s(x)}\iint_{\Omega_t}h(y,\tau)P(x,y,t-\tau)dyd\tau,\\ &\leqslant& C\displaystyle\frac{w(x,t)}{\delta^s(x)}. \end{array}\right.$$

Thus, by Theorem 4.2, we have $J_1\in L^\gamma(\Omega_T)$ for any $\gamma<\widehat{m}$.

Concerning J_2, by using (2.6) and (2.8), we get

$$\tag{4.23}\left\{\begin{array}{lcl} J_2(x,t) &=& \delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\delta(y)\leqslant(t-\tau)^{\frac{1}{2s}}<\frac{\delta(x)}{2}\right\}}\frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau,\\ &\leqslant& C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\delta(y)\leqslant(t-\tau)^{\frac{1}{2s}}<\frac{\delta(x)}{2}\right\}}\frac{h(y,\tau)\delta^s(y)(t-\tau)^{\frac{s-1}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau,\\ &\leqslant& C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\delta(y)\leqslant(t-\tau)^{\frac{1}{2s}}<\frac{\delta(x)}{2}\right\}}\frac{h(y,\tau)\delta^{a(1-s)}(y)\delta^{s-a(1-s)}(y)(t-\tau)^{\frac{s-1}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau,\\ &\leqslant& C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t}h(y,\tau)\delta^{a(1-s)}(y)\frac{(t-\tau)^{\frac{2s-1-a(1-s)}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau. \end{array}\right.$$

Given that $h\delta^{a(1-s)}\in L^m(\Omega_T)$, we deduce from Lemma (4.1) that $J_2\in L^\gamma(\Omega_T)$ for any $\gamma<\widetilde{m}$.

Finally, we deal with the term J_3. Using again (2.6) and (2.8), we have

$$\tag{4.24}\left\{\begin{array}{lcl} J_3(x,t) &=& \delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\min\{\delta(y),\delta(x)\}>(t-\tau)^{\frac{1}{2s}}\right\}}\frac{h(y,\tau)}{(t-\tau)^{\frac{1}{2s}}}P(x,y,t-\tau)dyd\tau,\\ &\leqslant& C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\min\{\delta(y),\delta(x)\}>(t-\tau)^{\frac{1}{2s}}\right\}}h(y,\tau)\frac{(t-\tau)^{\frac{2s-1}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau,\\ &\leqslant& C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t\cap\left\{\min\{\delta(y),\delta(x)\}>(t-\tau)^{\frac{1}{2s}}\right\}}\frac{h(y,\tau)\delta^{a(1-s)}(y)\delta^{-a(1-s)}(y)(t-\tau)^{\frac{2s-1}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau,\\ &\leqslant& C\delta^{1-s}(x)\displaystyle\iint_{\Omega_t}h(y,\tau)\delta^{a(1-s)}(y)\frac{(t-\tau)^{\frac{2s-1-a(1-s)}{2s}}}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+2s}}dyd\tau. \end{array}\right.$$

As above, we conclude that $J_3\in L^\gamma(\Omega_T)$ for any $\gamma<\widetilde{m}$. Then, $I_2\in L^\gamma(\Omega_T)$ for any $\gamma\leqslant\widetilde{m}$ and

$$\tag{4.25}\|I_2\|_{L^\gamma(\Omega_T)}\leqslant C\left(\|h\delta^{a(1-s)}\|_{L^m(\Omega_T)}+\left\|\frac{w}{\delta^s}\right\|_{L^{\widehat{m}}(\Omega_T)}\right).$$

Combining (4.20) et (4.25), we conclude the proof of (i).

2) Now, we prove the compactness of the operator Φ. Let a sequence $\{h_n\}_n\subset L^1(\Omega_T,\delta^{a(1-s)}(x)dxdt)$ be such that $\|h_n\delta^{a(1-s)}\|_{L^1(\Omega_T)}\leqslant C$. Therefore, $w_n=\Phi(h_n)$ solves

$$\tag{4.26}\left\{\begin{array}{lcll} \partial_t w_n+(-\Delta)^s w_n &=& h_n & \text{in } \Omega_T,\\ w_n(x,t) &=& 0 & \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T),\\ w_n(x,0) &=& 0 & \text{in } \Omega. \end{array}\right.$$

Let $\gamma<\frac{N+2s}{N+s+a(1-s)}$. Thanks to Theorem 4.2, there exists $C>0$ independent of n such that

$$\iint_{\Omega_T}\frac{|w_n(x,t)|^\gamma}{\delta^{\gamma s}(x)}dxdt\leqslant C||h_n\delta^{a(1-s)}||^{\gamma_1}_{L^1(\Omega_T)}\leqslant C.$$

Thus, by taking into consideration the estimate (4.15), and by choosing ρ such that $1 \leqslant \rho < \gamma < \frac{N+2s}{N+1+a(1-s)}$, we deduce that

$$\begin{aligned} \big\||\nabla w_n|\delta^{(1-s)}\big\|_{L^\rho(\Omega_T)} &\leqslant C\left(\Big\|\frac{w_n}{\delta^s}\Big\|_{L^{\gamma_1}(\Omega_T)} + \|h_n\delta^{a(1-s)}\|_{L^1(\Omega_T)}\right), \\ &\leqslant C\|h_n\delta^{a(1-s)}\|_{L^1(\Omega_T)} \leqslant C. \end{aligned}$$

Since $|\nabla(w_n\delta^{1-s})| \leqslant \delta^{1-s}|\nabla w_n| + (1-s)\dfrac{|w_n(x)|}{\delta^s(x)}$, then

$$\|w_n\delta^{1-s}\|_{L^\rho(0,T;\,\mathbb{W}_0^{1,\rho}(\Omega))} \leqslant C,$$

for any $\rho < \frac{N+2s}{N+1+a(1-s)}$.

Bearing in mind that $h_n\delta^{a(1-s)} \in L^1(\Omega_T)$ for $0 \leqslant a < \dfrac{2s-1}{1-s}$, we deduce from compactness result in Theorem 2.6 that, up to a subsequence, $w_n \to w$ strongly in $\mathbb{W}_0^{1,\theta}(\Omega)$ for any $\theta < \frac{N+2s}{N+a(1-s)+1}$. Hence, up to an other subsequence, $\delta^{1-s}\nabla w_n \rightharpoonup \delta^{1-s}\nabla w$ a.e. in Ω_T. Using the fact that $\{|\nabla w_n|\delta^{1-s}\}_n$ is bounded in $L^\rho(\Omega_T)$ for any $\rho < \frac{N+2s}{N+1+a(1-s)}$, then using Vitali's Lemma we deduce that $w_n \longrightarrow w$ strongly in $L^\gamma(0,T;\,\mathbb{W}_0^{1,\gamma}(\Omega,\,\delta^{\gamma(1-s)}dx))$. Thus, the result follows. □

Remark 4.4. The above Theorems can be compared with [6, Theorems 3.10 and 3.11] (see Theorem 2.9). Indeed, [6, Theorem 3.10] (resp. [6, Theorem 3.11]) gives the regularity of $\dfrac{w}{\delta^s}$ (resp. $|\nabla w|\delta^{1-s}$) when h is in $L^m(\Omega_T)$, while ours give more precision when h is more regular. ■

5 Existence results

In this section, we will state and prove our main (local) existence results. More precisely, we mainly consider the following three relevant cases of System (S):
(i) $(f,g) \neq (0,0)$, $(u_0,v_0) = (0,0)$ and $pq > 1$;
(ii) $(f,g) = (0,0)$, $(u_0,v_0) \neq (0,0)$ and $pq > 1$;
(iii) $(f,g) \neq (0,0)$, $(u_0,v_0) = (0,0)$ and $p = q = 1$;
under the hypothesis $\frac{2}{3} < s < 1$. As already commented in the introduction, we do not know whether this restriction is necessary or not. We leave it as an open question.

Throughout this section, we assume that $1 \leqslant p,q < \frac{2s-1}{1-s}$, $(f,g) \in L^m(\Omega_T)^+ \times L^\sigma(\Omega_T)^+$, $(u_0,v_0) \in L^{m_1}(\Omega)^+ \times L^{\sigma_1}(\Omega)^+$ where $(m,\sigma,m_1,\sigma_1) \in [1,+\infty)^4$.

Before going further, let us point out that $1 < \frac{2s-1}{1-s}$ implies $\frac{2}{3} < s$.
Hence, the condition $s > \frac{2}{3}$ is needed so that the set of possible pairs (p,q) is not empty.

5.1 Case: $(f,g) \neq (0,0)$ and $(u_0,v_0) = (0,0)$.

In this case, System (S) takes the form

$$\left\{\begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& |\nabla v|^q + f & \text{in} \quad \Omega_T, \\ \partial_t v + (-\Delta)^s v &=& |\nabla u|^p + g & \text{in} \quad \Omega_T, \\ u(x,t) = v(x,t) &=& 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = v(x,0) &=& 0 & \text{in} \quad \Omega. \end{array}\right. \tag{5.1}$$

So, our first local existence result reads as

Theorem 5.1. *Let $pq > 1$. Assume that (m,σ) satisfies one of the following set of conditions:*

$$\left\{\begin{array}{l} 1 \leqslant m < \dfrac{N+2s}{2s-1-q(1-s)},\ 1 \leqslant \sigma < \dfrac{N+2s}{2s-1-p(1-s)}, \\[2ex] p\sigma < \tilde{m} = \dfrac{m(N+2s)}{N+2s-m(2s-1-q(1-s))} \ \textit{and} \ qm < \tilde{\sigma} = \dfrac{\sigma(N+2s)}{N+2s-\sigma(2s-1-p(1-s))}, \end{array}\right. \tag{5.2}$$

$$\textit{or} \qquad m \geqslant \frac{N+2s}{2s-1-q(1-s)} \quad \textit{and} \quad \sigma > \frac{qm(N+2s)}{N+2s+qm(2s-1-p(1-s))}, \tag{5.3}$$

$$\text{(5.4)} \qquad \text{or} \qquad \sigma \geqslant \frac{N+2s}{2s-1-p(1-s)} \quad \text{and} \;\; m > \frac{p\sigma(N+2s)}{N+2s+p\sigma(2s-1-q(1-s))}.$$

Then, there exists $T^* \in (0,T]$ such that System (5.1) has a nonnegative solution (u,v) defined in Ω_{T^*}. Moreover, $(u\delta^{1-s},\, v\delta^{1-s}) \in L^{\theta_1}(0,\, T^*;\, \mathbb{W}_0^{1,\theta_1}(\Omega)) \times L^{\theta_2}(0,\, T^*;\, \mathbb{W}_0^{1,\theta_2}(\Omega))$ for any $\theta_1 < \tilde{m}$ and $\theta_2 < \tilde{\sigma}$. $\square$

Earlier than giving the proof, let us comment on the assumptions of the previous Theorem.
1) It can be checked that, under the given conditions for (p,q,s), all three sets of conditions (5.2), (5.3) and (5.4) define nonempty subsets for (m,σ).
2) Reciprocally, for a given (m,σ,s), one can find p and q satisfying conditions (5.2), (5.3) and (5.4).

To illustrate that, let us shed some light on the conditions (5.2) by giving a simple and significant example. To this end, let us denote $a = N+2s$, $b = 2s-1$ and $c = 1-s$ and suppose that $p = q$ and $m = \sigma$.

- Let p and s be given. It is clear that Condition (5.2) is equivalent to $m \in \left(m^*, \dfrac{a}{b-pc}\right)$ where $m^* := \max\left\{\dfrac{a(p-1)}{p(b-pc)}, 1\right\}$. We can easily check that $\left(m^*, \dfrac{a}{b-pc}\right)$ is nonempty.
- Reciprocally, let m and s be given. Condition (5.2) implies $p \in (1,p^*)$ where $p^* := \min\left\{\dfrac{a}{a-mb}, \dfrac{b}{c}\right\}$. In this case, we can also verify that $p^* > 1$.

Now, let us come back to the proof of Theorem 5.1.

Proof. We give a detailed proof under condition (5.2). The other cases follow in a similar way. The proof will be split into several steps.
First step:
Since $qm < \frac{\sigma(N+2s)}{N+2s-\sigma(2s-1-p(1-s))}$. Then, there exists $r > 1$ such that

$$qm < r < \frac{\sigma(N+2s)}{N+2s-\sigma(2s-1-p(1-s))}. \tag{5.5}$$

We fix $r > 1$ as above and we define the set E as follows:
(5.6)

$$E := \Big\{\varphi \in L^1(0,\, T;\, \mathbb{W}_0^{1,1}(\Omega))\;;\; \varphi\,\delta^{1-s} \in L^r(0,\, T;\, \mathbb{W}_0^{1,r}(\Omega)) \text{ and } \left(\iint_{\Omega_T} |\nabla(\varphi\,\delta^{1-s})|^r dxdt\right)^{\frac{1}{r}} \leqslant \Lambda\},$$

where Λ is a positive constant to be chosen later. It is clear that E is closed and convex subset of $L^1(0,\, T;\, \mathbb{W}_0^{1,1}(\Omega))$.

Moreover, if $\varphi \in E$, then $\hat{\varphi} := \varphi\,\delta^{1-s} \in L^r(0,T;\mathbb{W}_0^{1,r}(\Omega))$. Therefore, by applying Hardy's inequality (2.16) to $\hat{\varphi}$ with $\rho = 0$, we get

$$\iint_{\Omega_T} \frac{|\varphi(x,t)|^r}{\delta^{rs}(x)} dxdt \leqslant C \iint_{\Omega_T} |\nabla(\varphi\delta^{1-s})|^r dxdt < C\Lambda^r.$$

Now, by using the fact that $|\nabla\delta(x)| = 1$ *a.e.* in Ω, we deduce that

$$|\nabla\varphi|^r \delta^{r(1-s)} = |\nabla(\varphi\delta^{1-s}) - \varphi\nabla\delta^{1-s}|^r \leqslant C|\nabla\hat{\varphi}|^r + \frac{|\varphi|^r}{\delta^{rs}}.$$

Thus, $|\nabla\varphi|^r\delta^{r(1-s)} \in L^1(\Omega_T)$ and

$$\left(\iint_{\Omega_T} |\nabla\varphi|^r\, \delta^{r(1-s)} dxdt\right)^{\frac{1}{r}} \leqslant C\Lambda.$$

Using Hölder's inequality, we deduce that $|\nabla\varphi|^\alpha\delta^{\alpha(1-s)} \in L^1(\Omega_T)$ for any $\alpha < r$. In particular, for $\alpha = qm$, we get $|\nabla\varphi|^q\delta^{q(1-s)} \in L^m(\Omega_T) \subset L^1(\Omega_T)$.
Now, we claim that if $\varphi \in E$, then there exists $\beta < 2s-1$ such that $|\nabla\varphi|^q\delta^\beta \in L^1(\Omega_T)$.

Indeed, let $\varphi \in E$, then by using Hölder's inequality, it holds that

$$(5.7)\qquad \begin{cases} \displaystyle\iint_{\Omega_T} |\nabla\varphi|^q\delta^\beta dxdt = \iint_{\Omega_T} |\nabla\varphi|^q\delta^{q(1-s)}\delta^{\beta-q(1-s)}dxdt, \\ \displaystyle\leqslant \Big(\iint_{\Omega_T} |\nabla\varphi|^{qm}\delta^{qm(1-s)}dxdt\Big)^{\frac{1}{m}} \Big(\iint_{\Omega_T} \delta^{(\beta-q(1-s))m'}dxdt\Big)^{\frac{1}{m'}}, \\ \displaystyle\leqslant C\Big(\iint_{\Omega_T} \delta^{(\beta-q(1-s))m'}dxdt\Big)^{\frac{1}{m'}}. \end{cases}$$

On the other hand, as $q(1-s) < 2s-1$ (using the fact that $q < \dfrac{2s-1}{1-s}$). Then, we can choose $\beta < 2s-1$ such that $q(1-s) < \beta < 2s-1$. Hence

$$\iint_{\Omega_T} \delta^{(\beta-q(1-s))m'}dxdt < +\infty,$$

and the claim follows.
Therefore, we get $|\nabla\varphi|^q\delta^\beta + f \in L^1(\Omega_T)$. Now, by Theorem 2.6, we get the existence of a unique solution u to Problem

$$(5.8)\qquad \begin{cases} \partial_t u + (-\Delta)^s u &= |\nabla\varphi|^q + f \quad \text{in } \Omega_T, \\ u(x,t) &= 0 \quad \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T), \\ u(x,0) &= 0 \quad \text{in } \Omega. \end{cases}$$

Moreover, $u \in L^\theta(0,T;\mathbb{W}_0^{1,\theta}(\Omega))$ for any $\theta < \dfrac{N+2s}{N+\beta+1}$.
Setting $h = |\nabla\varphi|^q + f$, it follows that

$$h\delta^{q(1-s)} = |\nabla\varphi|^q\delta^{q(1-s)} + f\delta^{q(1-s)} \in L^m(\Omega_T).$$

Then, by using Theorem 4.3, with $q = a$, we reach to

$$(5.9)\qquad \big\||\nabla u|\delta^{1-s}\big\|_{L^\gamma(\Omega_T)} \leqslant C\left(\Big\||\nabla\varphi|\delta^{(1-s)}\Big\|^q_{L^{qm}(\Omega_T)} + \|f\|_{L^m(\Omega_T)}\right),$$

for any $\gamma < \tilde{m} = \frac{m(N+2s)}{N+2s-m(2s-1-q(1-s))}$.
Since $qm < r$, thanks to Hölder's inequality, we obtain

$$(5.10)\qquad \big\||\nabla u|\delta^{1-s}\big\|_{L^\gamma(\Omega_T)} \leqslant C\left(\Big\||\nabla\varphi|\delta^{(1-s)}\Big\|^q_{L^{r}(\Omega_T)} + \|f\|_{L^m(\Omega_T)}\right),$$

for any $\gamma < \tilde{m}$.
On the other hand, by using the fact that $p\sigma < \tilde{m}$, we get $|\nabla u|^p\delta^{p(1-s)} \in L^\sigma(\Omega_T) \subset L^1(\Omega_T)$.
Moreover $p(1-s) < 2s-1$, so by following the same computations as above, we get the existence of $\beta_1 < 2s-1$ such that $|\nabla u|^p\delta^{\beta_1} \in L^1(\Omega_T)$. Hence, thanks to Theorem 2.6, we get the existence of a unique weak solution v to Problem

$$(5.11)\qquad \begin{cases} \partial_t v + (-\Delta)^s v &= |\nabla u|^p + g \quad \text{in } \Omega_T, \\ v(x,t) &= 0 \quad \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T), \\ v(x,0) &= 0 \quad \text{in } \Omega, \end{cases}$$

with $v \in L^{\tilde\theta}(0,T;\mathbb{W}_0^{1,\tilde\theta}(\Omega))$ for any $\tilde\theta < \dfrac{N+2s}{N+\beta_1+1}$.
Therefore, the operator

$$\begin{aligned} \Phi : E &\longrightarrow L^1(0,T;\mathbb{W}_0^{1,1}(\Omega)) \\ \varphi &\longmapsto \Phi(\varphi) = v, \end{aligned}$$

is well defined. In addition, if v is a fixed point of Φ, then (u,v) is a weak solution to System (5.1). Thence, we will show that Φ has a fixed point in E.

Second step: In this step, we prove the continuity and the compactness of the operator Φ.
• We begin by proving the continuity of Φ with respect to the topology of the space $L^1(0,T;\mathbb{W}_0^{1,1}(\Omega))$.
Let $\{\varphi_n\}_n \subset E$ be such that $\varphi_n \to \varphi$ strongly in $L^1(0,T;\mathbb{W}_0^{1,1}(\Omega))$. Let us define $v_n := \Phi(\varphi_n)$ and $v := \Phi(\varphi)$. Then, (u_n,v_n) and (u,v) satisfy

$$(5.12)\qquad \begin{cases} \partial_t u_n + (-\Delta)^s u_n &= |\nabla\varphi_n|^q + f \quad \text{in } \Omega_T, \\ \partial_t u + (-\Delta)^s u &= |\nabla\varphi|^q + f \quad \text{in } \Omega_T, \\ u(x,t) = u_n(x,t) &= 0 \quad \text{in } (\mathbb{R}^N\setminus\Omega)\times(0,T), \\ u(x,0) = u_n(x,0) &= 0 \quad \text{in } \Omega, \end{cases}$$

and

$$\left\{\begin{array}{lcll} \partial_t v_n + (-\Delta)^s v_n & = & |\nabla u_n|^p + g & \text{in} \quad \Omega_T, \\ \partial_t v + (-\Delta)^s v & = & |\nabla u|^p + g & \text{in} \quad \Omega_T, \\ v_n(x,t) = v(x,t) & = & 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ v_n(x,0) = v(x,0) & = & 0 & \text{in} \quad \Omega. \end{array}\right. \tag{5.13}$$

Now, using Theorem 2.6, it follows that

$$\| |\nabla u_n| - |\nabla u| \|_{L^\theta(\Omega_T)} \leqslant C \, \| |\nabla \varphi_n|^q - |\nabla \varphi|^q \|_{L^1(\Omega_T, \delta^\beta(x)dxdt)}, \tag{5.14}$$

for any $\theta < \frac{N+2s}{\beta+1+N}$ with $\beta < 2s-1$.
On the other hand $\{\varphi_n\}_n \subset E$, then

$$\iint_{\Omega_T} |\nabla \varphi_n|^r \delta^{r(1-s)} dxdt < +\infty.$$

Hence, by taking into consideration that $\varphi_n \to \varphi$ strongly in $L^1(0,T; \mathbb{W}_0^{1,1}(\Omega))$ and by Fatou's lemma, we conclude that

$$\iint_{\Omega_T} |\nabla \varphi|^r \delta^{r(1-s)} dxdt < +\infty.$$

Let $\rho \in (1,r)$ be fixed and $\xi_1 = \frac{r-\rho}{\rho(r-1)}$. Then, by the interpolation inequality (Proposition 2.12), we have

$$\left\{\begin{array}{l} \left(\iint_{\Omega_T} |\nabla \varphi_n - \nabla \varphi_n|^\rho \delta^{\rho(1-s)} dx\right)^{\frac{1}{a}} \leqslant \left(\iint_{\Omega_T} |\nabla \varphi_n - \nabla \varphi| \delta^{(1-s)} dxdt\right)^{\xi_1} \\ \qquad \times \left(\iint_{\Omega_T} |\nabla \varphi_n - \nabla \varphi|^r \delta^{r(1-s)} dxdt\right)^{\frac{1-\xi_1}{r}}, \\ \qquad \leqslant C \left(\iint_{\Omega_T} |\nabla \varphi_n - \nabla \varphi| \delta^{(1-s)} dxdt\right)^{\xi_1}. \end{array}\right. \tag{5.15}$$

Hence

$$\lim_{n\to+\infty} \left(\iint_{\Omega_T} |\nabla \varphi_n - \nabla \varphi_n|^\rho \delta^{\rho(1-s)} dx\right) = 0. \tag{5.16}$$

Now, choosing $\rho = qm$ (let us recall that $qm < r$). Then, as in the first step, we can easily prove the existence of $\beta < 2s-1$ such that

$$\iint_{\Omega_T} |\nabla(\varphi_n - \varphi)|^q \delta^\beta dxdt \quad \leqslant \quad C \left(\iint_{\Omega_T} |\nabla(\varphi_n - \varphi)|^{qm} \delta^{qm(1-s)} dxdt\right)^{\frac{1}{m}}. \tag{5.17}$$

Consequently

$$\lim_{n\to+\infty} \iint_{\Omega_T} |\nabla(\varphi_n - \varphi)|^q \delta^\beta dxdt = 0. \tag{5.18}$$

Going back to (5.14), we conclude that $|\nabla u_n| \to |\nabla u|$ strongly in $L^\theta(\Omega_T)$ for any $\theta < \frac{N+2s}{N+1+\beta}$. In particular, $|\nabla u_n| \to |\nabla u|$ strongly in $L^1(\Omega_T)$. Thus, we deduce from estimate (5.10)

$$\left\| |\nabla u_n| \delta^{1-s} \right\|_{L^\gamma(\Omega_T)} \leqslant C\left(\Lambda^q + \|f\|_{L^m(\Omega_T)}\right) \leqslant C \qquad \text{for any } \gamma < \tilde{m}.$$

Therefore, $\{|\nabla u_n| \delta^{1-s}\}_n$ is bounded in $L^\gamma(\Omega_T)$. Thanks to Fatou's Lemma, we obtain

$$\left\| |\nabla u| \delta^{1-s} \right\|_{L^\gamma(\Omega_T)} \leqslant C \qquad \text{for any } \gamma < \tilde{m}.$$

Let $\gamma < \tilde{m}$ be fixed such that $p\sigma < \gamma$. Using again the interpolation inequality (Proposition 2.12), we get for $\xi_2 = \frac{\gamma - p\sigma}{p\sigma(\gamma-1)}$

$$\left\{\begin{array}{l} \left(\iint_{\Omega_T} |\nabla u_n - \nabla u|^{p\sigma} \delta^{p\sigma(1-s)} dxdt\right)^{\frac{1}{p\sigma}} \leqslant \left(\iint_{\Omega_T} |\nabla u_n - \nabla u| \delta^{(1-s)} dxdt\right)^{\xi_2} \\ \qquad \times \left(\iint_{\Omega_T} |\nabla u_n - \nabla u|^\gamma \delta^{\gamma(1-s)} dxdt\right)^{\frac{1-\xi_2}{\gamma}}, \\ \qquad \leqslant C \left(\iint_{\Omega_T} |\nabla u_n - \nabla u| \delta^{(1-s)} dxdt\right)^{\xi_2}. \end{array}\right. \tag{5.19}$$

So,

$$\lim_{n\to+\infty}\iint_{\Omega_T}|\nabla u_n-\nabla u|^{p\sigma}\delta^{p\sigma(1-s)}dxdt=0. \tag{5.20}$$

In addition, given that $p<p\sigma$, as in (5.17) and (5.18), we can find $\beta_1<2s-1$ such that

$$\iint_{\Omega_T}|\nabla u_n-\nabla u|^{p}\delta^{\beta_1}dxdt \quad\leqslant\quad C\bigg(\iint_{\Omega_T}|\nabla u_n-\nabla u|^{p\sigma}\delta^{p\sigma(1-s)}dxdt\bigg)^{\frac{1}{\sigma}},$$

then

$$\lim_{n\to+\infty}\iint_{\Omega_T}|\nabla u_n-\nabla u|^{p}\delta^{\beta_1}dxdt=0. \tag{5.21}$$

Hence, thanks to Theorem 2.6, $v_n\to v$ strongly in $L^1(0,\,T;\,\mathbb{W}_0^{1,1}(\Omega))$ which implies the continuity of Φ.

• Now, let us prove that Φ is a compact operator. Let $\{\varphi_n\}_n\subset E$ be such that $\|\varphi_n\|_{L^1(0,\,T;\,\mathbb{W}_0^{1,1}(\Omega))}\leqslant C$, where C is independent of n.
Let $v_n=\Phi(\varphi_n)$. Since $\{\varphi_n\}_n\subset E$, then

$$\bigg(\iint_{\Omega_T}|\nabla(\varphi_n\,\delta^{(1-s)}(x))|^r dxdt\bigg)^{\frac{1}{r}}\leqslant C \qquad \text{for any } n.$$

Therefore, as in the first step, we can show the existence of $\beta<2s-1$ such that $\{|\nabla\varphi_n|^q\delta^{\beta}+f\}_n$ is bounded in $L^1(\Omega_T)$. Thus, thanks to Theorem 2.6, we get the existence of a subsequence $\{u_n\}_n$ such that $u_n\to u$ strongly in $L^{\theta}(0,\,T;\,\mathbb{W}_0^{1,\theta}(\Omega))$ for any $\theta<\frac{N+2s}{N+1+\beta}$.
Since $p\sigma<\tilde m$, following the computations as above, we can show that, up to a subsequence, $|\nabla u_n|\delta^{1-s}\to|\nabla u|\delta^{1-s}$ strongly in $L^{p\sigma}(\Omega_T)$.
Thus, as above, we get $\beta_1<2s-1$ such that

$$(|\nabla u_n|^p+g)\;\delta^{\beta_1}\to(|\nabla u|^p+g)\delta^{\beta_1}\quad \text{ strongly in } L^1(\Omega_T).$$

Using Theorem 2.6, we deduce that $v_n\to v$ strongly in $L^{\tilde\theta}(0,\,T;\,\mathbb{W}_0^{1,\tilde\theta}(\Omega))$ for any $\tilde\theta<\frac{N+2s}{N+1+\beta_1}$. In particular, $v_n\to v$ strongly in $L^1(0,\,T;\,\mathbb{W}_0^{1,1}(\Omega))$. Hence, the result follows.

Third step: In this last step, we will prove the existence of $\Lambda>0$ and $T^*>0$ such that $\Phi(E)\subset E$. For $\alpha\geqslant 0$, we consider

$$\Upsilon(\alpha):=\alpha^{\frac{1}{pq}}-\tilde C\alpha, \tag{5.22}$$

where $\tilde C$ is a universal positive constant that will be specified later.
Since $pq>1$, then there exist $\ell,\Lambda>0$ such that

$$\max_{\alpha\geqslant 0}\Upsilon(\alpha)=\Upsilon(\ell)=\Lambda.$$

Thus

$$\ell^{\frac{1}{pq}}=\tilde C\left(\ell+\frac{\Lambda}{\tilde C}\right). \tag{5.23}$$

By fixing $\ell>0$ which satisfies (5.23), we can choose $T^*\in(0,T]$ such that

$$\|f\|^p_{L^m(\Omega_{T^*})}+\|g\|_{L^\sigma(\Omega_{T^*})}\leqslant\frac{\Lambda}{\tilde C}. \tag{5.24}$$

Notice that v solves Problem (5.11), then

$$\big\||\nabla v|\delta^{1-s}\big\|_{L^\nu(\Omega_{T^*})}\leqslant C\bigg(\big\||\nabla u|\delta^{1-s}\big\|^p_{L^{p\sigma}(\Omega_{T^*})}+\|g\|_{L^\sigma(\Omega_{T^*})}\bigg), \tag{5.25}$$

for any $\nu<\tilde\sigma=\frac{\sigma(N+2s)}{N+2s-\sigma(2s-1-p(1-s))}$.
Indeed, setting $h=|\nabla u|^p+g$. Then, $h\delta^{p(1-s)}\in L^\sigma(\Omega_{T^*})$. So, using again Theorem 4.3, with $a=p$, it follows that

$$\left\{\begin{aligned}\big\||\nabla v|\delta^{1-s}\big\|_{L^\nu(\Omega_{T^*})}&\leqslant C\bigg(\big\||\nabla u|^p\delta^{p(1-s)}\big\|_{L^\sigma(\Omega_{T^*})}+\|g\|_{L^\sigma(\Omega_{T^*})}\bigg),\\ &\leqslant C\bigg(\big\||\nabla u|\delta^{(1-s)}\big\|^p_{L^{p\sigma}(\Omega_{T^*})}+\|g\|_{L^\sigma(\Omega_{T^*})}\bigg),\end{aligned}\right. \tag{5.26}$$

for any $\nu < \tilde{\sigma}$.
Thus, using the fact that $p\sigma < \tilde{m}$ and going back to (5.10), it follows that
(5.27)
$$\big\||\nabla v|\delta^{1-s}\big\|_{L^\nu(\Omega_{T^*})} \leqslant C\Big(\big\||\nabla\varphi|\delta^{(1-s)}\big\|^{pq}_{L^r(\Omega_{T^*})} + \|f\|^p_{L^m(\Omega_{T^*})} + \|g\|_{L^\sigma(\Omega_{T^*})}\Big) \qquad \forall\ \nu < \tilde{\sigma}.$$
Recall that $\varphi \in E$ and $r < \tilde{\sigma}$. Hence by choosing $r = \nu$, we get
$$\big\||\nabla v|\delta^{1-s}\big\|_{L^r(\Omega_{T^*})} \leqslant C\Big(\Lambda^{pq} + \|f\|^p_{L^m(\Omega_{T^*})} + \|g\|_{L^\sigma(\Omega_{T^*})}\Big). \tag{5.28}$$
By taking $\Lambda = \ell^{\frac{1}{pq}}$ and combining (5.23) with (5.24), it follows that
$$\tilde{C}\Big(\Lambda^{pq} + \|f\|^p_{L^m(\Omega_{T^*})} + \|g\|_{L^\sigma(\Omega_{T^*})}\Big) \leqslant \ell^{\frac{1}{pq}}. \tag{5.29}$$
Now, by choosing $C = \tilde{C}$ and going back to (5.27), we get
$$\big\||\nabla v|\delta^{1-s}\big\|_{L^r(\Omega_{T^*})} \leqslant \tilde{C}\Big(\Lambda^{pq} + \|f\|^p_{L^m(\Omega_{T^*})} + \|g\|_{L^\sigma(\Omega_{T^*})}\Big) \leqslant \Lambda. \tag{5.30}$$
Hence, $\Phi(E) \subset E$.

As a conclusion, we have proved that the operator Φ satisfies all the conditions of the Schauder Fixed-Point Theorem. Thus, we get the existence of (u, v), a solution to System (5.1) such that $(u\delta^{1-s}, v\delta^{1-s}) \in L^{\theta_1}(0, T^*; \mathbb{W}^{1,\theta_1}_0(\Omega)) \times L^{\theta_2}(0, T^*; \mathbb{W}^{1,\theta_2}_0(\Omega))$ for any $\theta_1 < \tilde{m}$ and $\theta_2 < \tilde{\sigma}$.
$\square$

5.2 Case: $(f, g) = (0, 0)$ and $(u_0, v_0) \neq (0, 0)$.

In this case, System (S) is reduced to the following one:
$$\begin{cases} \partial_t u + (-\Delta)^s u &= |\nabla v|^q & \text{in} & \Omega_T, \\ \partial_t v + (-\Delta)^s v &= |\nabla u|^p & \text{in} & \Omega_T, \\ u(x,t) = v(x,t) &= 0 & \text{in} & (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) &= u_0(x) & \text{in} & \Omega, \\ v(x,0) &= v_0(x) & \text{in} & \Omega. \end{cases} \tag{5.31}$$

Theorem 5.2. *Let $(u_0, v_0) \in L^{m_1}(\Omega)^+ \times L^{\sigma_1}(\Omega)^+$, p and q such that $pq > 1$. Assume that (p, q) satisfies one of the following set of conditions:*
$$q < \min\left\{\frac{\overline{\sigma}_1(N + 2s\overline{m}_1)}{\overline{m}_1(N + (2-s)\overline{\sigma}_1)}, \frac{2s-1}{1-s}\right\} \quad \textit{and} \quad p < \min\left\{\frac{\overline{m}_1(N + 2s\overline{\sigma}_1)}{\overline{\sigma}_1(N + (2-s)\overline{m}_1)}, \frac{2s-1}{1-s}\right\}, \tag{5.32}$$
or
$$q < \frac{2s-1}{1-s} \quad \textit{and} \quad p < \min\left\{\frac{\overline{m}_1(N + 2s\overline{\sigma}_1)}{\overline{\sigma}_1(N + (2-s)\overline{m}_1)}, \frac{2s-1}{1-s}\right\}, \tag{5.33}$$
or
$$q < \min\left\{\frac{\overline{\sigma}_1(N + 2s\overline{m}_1)}{\overline{m}_1(N + (2-s)\overline{\sigma}_1)}, \frac{2s-1}{1-s}\right\} \quad \textit{and} \quad p < \frac{2s-1}{1-s}, \tag{5.34}$$
where $\overline{m}_1 := \min\{m_1, 2s\}$ and $\overline{\sigma}_1 := \min\{\sigma_1, 2s\}$.
Then, there exists $T^* > 0$ such that System (5.31) has a nonnegative solution (u, v) such that $(u, v) \in L^{\theta_1}(0, T^*; \mathbb{W}^{1,\theta_1}_0(\Omega, \delta^{(1-s)}dx)) \times L^{\theta_2}(0, T^*; \mathbb{W}^{1,\theta_2}_0(\Omega, \delta^{(1-s)}dx))$ for any $\theta_1 < \dfrac{\overline{m}_1(N+2s)}{N + \overline{m}_1}$ and $\theta_2 < \dfrac{\overline{\sigma}_1(N+2s)}{N + \overline{\sigma}_1}$. $\square$

Before proceeding to the proof, we shall give a simple and meaningful example to exhibit that the sets defined by conditions (5.32), (5.33) and (5.34) are nonempty. Indeed, under given conditions for (m_1, σ_1, s), all the three sets of conditions (5.32), (5.33) and (5.34) are nonempty for (p, q). For example, let us take $m_1 = \sigma_1$. In this case, we have
$$\min\left\{\frac{N + 2s\overline{m}_1}{N + (2-s)\overline{m}_1}, \frac{2s-1}{1-s}\right\} = \frac{N + 2s\overline{m}_1}{N + (2-s)\overline{m}_1}.$$

Therefore,

$$\begin{cases} \textit{Condition } (5.32) & \Leftrightarrow \quad p, q \in \left[1, \frac{N+2s\overline{m}_1}{N+(2-s)\overline{m}_1}\right) \ ; \\ \textit{Condition } (5.33) & \Leftrightarrow \quad q \in \left[1, \frac{2s-1}{1-s}\right) \ \text{ and } \ p \in \left[1, \frac{N+2s\overline{m}_1}{N+(2-s)\overline{m}_1}\right) \ ; \\ \textit{Condition } (5.34) & \Leftrightarrow \quad p \in \left[1, \frac{2s-1}{1-s}\right) \ \text{ and } \ q \in \left[1, \frac{N+2s\overline{m}_1}{N+(2-s)\overline{m}_1}\right) . \square \end{cases}$$

Now, let us proceed to the proof.

Proof. We follow closely the argument used in [2]. The main idea is to use a suitable change of functions in order to get a relation between System (5.31) and System (5.1). We precise also that the proof will be given under the condition (5.32), the others follow in a similar way.

For $T > 0$, let us introduce φ and ψ as the unique solutions to the following Problems:

$$\begin{cases} \partial_t \varphi + (-\Delta)^s \varphi &= 0 & \text{in } \Omega_T, \\ \varphi(x,t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \varphi(x,0) &= u_0(x) & \text{in } \Omega, \end{cases} \tag{5.35}$$

and

$$\begin{cases} \partial_t \psi + (-\Delta)^s \psi &= 0 & \text{in } \Omega_T, \\ \psi(x,t) &= 0 & \text{in } (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \psi(x,0) &= v_0(x) & \text{in } \Omega. \end{cases} \tag{5.36}$$

Thanks to Theorem 2.10, we have

$$|\nabla \varphi| \in L^{\gamma_1}(\Omega_T) \text{ for any } \gamma_1 < \frac{\overline{m}_1(N+2s)}{N+\overline{m}_1} \qquad \text{where } \overline{m}_1 = \min\{m_1, 2s\}$$

and

$$|\nabla \psi| \in L^{\gamma_2}(\Omega_T) \text{ for any } \gamma_2 < \frac{\overline{\sigma}_1(N+2s)}{N+\overline{\sigma}_1} \qquad \text{where } \overline{\sigma}_1 = \min\{\sigma_1, 2s\}.$$

Observe that $\frac{\overline{m}_1(N+2s)}{N+\overline{m}_1} < \frac{m_1(N+2s)}{N+m_1}$ and $\frac{\overline{\sigma}_1(N+2s)}{N+\overline{\sigma}_1} < \frac{\sigma_1(N+2s)}{N+\sigma_1}$. Again applying Theorem 2.10, we get

$$|\nabla \varphi|\delta^{1-s} \in L^{\gamma_1}(\Omega_T) \text{ for any } \gamma_1 < \frac{\overline{m}_1(N+2s)}{N+\overline{m}_1} \qquad \text{where } \overline{m}_1 = \min\{m_1, 2s\}$$

and

$$|\nabla \psi|\delta^{1-s} \in L^{\gamma_2}(\Omega_T) \text{ for any } \gamma_2 < \frac{\overline{\sigma}_1(N+2s)}{N+\overline{\sigma}_1} \qquad \text{where } \overline{\sigma}_1 = \min\{\sigma_1, 2s\}.$$

Setting $\tilde{u} = u - \varphi$ and $\tilde{v} = v - \psi$, then $(\tilde{u}, \tilde{v})$ solves the following system

$$\begin{cases} \partial_t \tilde{u} + (-\Delta)^s \tilde{u} &= |\nabla \tilde{v} + \nabla \psi|^q & \text{in} & \Omega_T, \\ \partial_t \tilde{v} + (-\Delta)^s \tilde{v} &= |\nabla \tilde{u} + \nabla \varphi|^p & \text{in} & \Omega_T, \\ \tilde{u}(x,t) = \tilde{v}(x,t) &= 0 & \text{in} & (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \tilde{u}(x,0) = \tilde{v}(x,0) &= 0 & \text{in} & \Omega. \end{cases} \tag{5.37}$$

Hence, to prove the main existence result, it suffices to show that System (5.37) possesses a non-negative solution $(\tilde{u}, \tilde{v})$.

Therefore, as in the proof of the previous existence result, fixing $1 < r < \frac{\overline{\sigma}_1(N+2s)}{N+\overline{\sigma}_1}$ where r is close to $\frac{\overline{\sigma}_1(N+2s)}{N+\overline{\sigma}_1}$. To do this, let us first introduce the following set

$$F := \Big\{ \varphi \in L^1(0,T; \mathbb{W}_0^{1,1}(\Omega)) \,;\, \varphi\, \delta^{1-s} \in L^r(0,T;\, \mathbb{W}_0^{1,r}(\Omega)) \text{ and } \left(\iint_{\Omega_T} |\nabla(\varphi\, \delta^{1-s})|^r dxdt \right)^{\frac{1}{r}} \leqslant \Lambda \Big\}.$$

In addition, we the operator

$$\begin{aligned} \mathcal{L} : F &\longrightarrow L^1(0,T;\, \mathbb{W}_0^{1,1}(\Omega)) \\ \varphi &\longmapsto \mathcal{L}(\varphi) = \tilde{v}, \end{aligned}$$

where $\tilde{v}$ is the unique solution to the following Problem:

$$\begin{cases} \partial_t \tilde{v} + (-\Delta)^s \tilde{v} &= |\nabla \tilde{u} + \nabla \varphi|^p & \text{in} & \Omega_T, \\ \tilde{v}(x,t) &= 0 & \text{in} & (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ \tilde{v}(x,0) &= 0 & \text{in} & \Omega, \end{cases} \tag{5.38}$$

and $\tilde{u}$ is the unique solution to Problem

$$
\left\{\begin{array}{rcll}
\partial_t \tilde{u} + (-\Delta)^s \tilde{u} & = & |\nabla \tilde{v} + \nabla\psi|^q & \text{in} \quad \Omega_T,\\
\tilde{u}(x,t) & = & 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T),\\
\tilde{u}(x,0) & = & 0 & \text{in} \quad \Omega.
\end{array}\right. \tag{5.39}
$$

So, if $\tilde{v}$ is a fixed point of $\mathcal{L}$, then $(\tilde{u}, \tilde{v})$ solves System (5.37). Thus, it is sufficient to prove that $\mathcal{L}$ admits a fixed point. To this end, we proceed exactly as in the proof of the previous theorem. We leave the writing to the reader.

However , we will give more details about the regularity. By convexity inequality, we obtain

$$(|\nabla \tilde{v}| + |\nabla\psi|)^q \leqslant C(|\nabla \tilde{v}|^q + |\nabla\psi|^q)$$

and

$$(|\nabla \tilde{u}| + |\nabla\varphi|)^p \leqslant C(|\nabla \tilde{u}|^p + |\nabla\varphi|^p),$$

where C is a universal positive constant.

Now, let us set $f = C|\nabla\psi|^q$ and $g = C|\nabla\varphi|^p$. Since $(u_0, v_0) \in L^{m_1}(\Omega) \times L^{\sigma_1}(\Omega)$, then by Theorem 2.10, it follows that

$$f \in L^m(\Omega_T) \qquad \text{for any } m < \frac{\overline{\sigma}_1(N+2s)}{q(N+\overline{\sigma}_1)},$$

and

$$g \in L^\sigma(\Omega_T) \qquad \text{for any } \sigma < \frac{\overline{m}_1(N+2s)}{p(N+\overline{m}_1)}.$$

Since all the conditions in Theorem 5.2 are satisfied, we can verify that (m,σ) satisfies (5.2). Thus, by Theorem 5.1, we get the existence of a nonnegative solution $(\tilde{u}, \tilde{v})$ such that $(\tilde{u}, \tilde{v}) \in L^{\tilde{\theta}_1}(0,T; \mathbb{W}_0^{1,\tilde{\theta}_1}(\Omega, \delta^{\tilde{\theta}_1(1-s)}dx)) \times L^{\tilde{\theta}_2}(0,T; \mathbb{W}_0^{1,\tilde{\theta}_2}(\Omega, \delta^{\tilde{\theta}_2(1-s)}dx))$ for any $\tilde{\theta}_1 < \frac{m(N+2s)}{N+2s-m(2s-1-q(1-s))}$ and $\tilde{\theta}_2 < \frac{\sigma(N+2s)}{N+2s-\sigma(2s-1-p(1-s))}$.

Since $u = \tilde{u} + \varphi$ and $v = \tilde{v} + \psi$, then $|\nabla u|\delta^{(1-s)} \in L^{\theta_1}(\Omega_T)$ and $|\nabla v|\delta^{(1-s)} \in L^{\theta_2}(\Omega_T)$ where $\theta_1 = \min\{\tilde{\theta}_1, \frac{\overline{m}_1(N+2s)}{N+\overline{m}_1}\}$ and $\theta_2 = \min\{\tilde{\theta}_2, \frac{\overline{\sigma}_1(N+2s)}{N+\overline{\sigma}_1}\}$.

On the other hand, by choosing m very close to $\frac{\overline{\sigma}_1(N+2s)}{q(N+\overline{\sigma}_1)}$, we obtain that

$$\tilde{\theta}_1 \simeq \frac{m(N+2s)}{N+2s-m(2s-1-q(1-s))} \simeq \frac{\overline{\sigma}_1(N+2s)}{q(N+\overline{\sigma}_1(2-s)) - \overline{\sigma}_1(2s-1)},$$

which means that for any $\varepsilon > 0$ small enough, we have

$$\tilde{\theta}_1 < \frac{\overline{\sigma}_1(N+2s)}{q(N+\overline{\sigma}_1(2-s)) - \overline{\sigma}_1(2s-1)} < \tilde{\theta}_1 + \varepsilon.$$

Now using the fact that $q < \dfrac{\overline{\sigma}_1(N+2s\overline{m}_1)}{\overline{m}_1(N+(2-s)\overline{\sigma}_1)}$, it follows that $\tilde{\theta}_1 > \dfrac{\overline{m}_1(N+2s)}{N+\overline{m}_1}$. Hence, we conclude that $\theta_1 \simeq \dfrac{\overline{m}_1(N+2s)}{N+\overline{m}_1}$.

By using the same computations as above, we get $|\nabla v|\delta^{(1-s)} \in L^{\theta_2}(\Omega_T)$ with $\theta_2 \simeq \frac{\overline{\sigma}_1(N+2s)}{N+\overline{\sigma}_1}$.

This concludes the proof of the theorem in the first case. □

We will conclude the two previous subsections with the following remarks.

Remark 5.3. If $\sqrt{3} - 1 < s$, then $\frac{2s-1}{1-s} > 1 + s$. Therefore, any solution to System (S) blows up in finite time according to Theorem 3.3 for any $1 + s < p, q < \frac{2s-1}{1-s}$. However, in the case $2/3 < s < \sqrt{3} - 1$ (and more generally $1/2 < s < \sqrt{3} - 1$), the question of global existence is more difficult. We leave it as an open question. As far as we know, this question is still open even in the case of a single equation. That being said, according to some numerical simulations (see [28] and [30]), we can conjecture that the local solution is actually global. □

Remark 5.4.

(1) The presence of the gradient term, as an absorption term does not generate any additional difficulty and the same existence result holds for the following System

$$\left\{\begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& \pm|\nabla v|^p + f & \text{in} \quad \Omega_T, \\ \partial_t v + (-\Delta)^s v &=& \pm|\nabla u|^q + g & \text{in} \quad \Omega_T, \\ u(x,t) = v(x,t) &=& 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) &=& u_0(x) & \text{in} \quad \Omega, \\ v(x,0) &=& v_0(x) & \text{in} \quad \Omega. \end{array}\right. \tag{5.40}$$

(2) Using the same compactness approach, we can prove the existence of a solution to the next System:

$$\left\{\begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& H_1(|\nabla v|) + f & \text{in} \quad \Omega_T, \\ \partial_t v + (-\Delta)^s v &=& H_2(|\nabla u|) + g & \text{in} \quad \Omega_T, \\ u(x,t) = v(x,t) &=& 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) &=& u_0(x) & \text{in} \quad \Omega, \\ v(x,0) &=& v_0(x) & \text{in} \quad \Omega, \end{array}\right. \tag{5.41}$$

where $H_1, H_2 : \mathbb{R}^+ \to \mathbb{R}$ are continuous and bounded functions. □

5.3 Particular case: $p = q = 1$ and $u_0 = v_0 = 0$

In this case, System (S) becomes

$$\left\{\begin{array}{rcll} \partial_t u + (-\Delta)^s u &=& |\nabla v| + f & \text{in} \quad \Omega_T, \\ \partial_t v + (-\Delta)^s v &=& |\nabla u| + g & \text{in} \quad \Omega_T, \\ u(x,t) = v(x,t) &=& 0 & \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u(x,0) = v(x,0) &=& 0 & \text{in} \quad \Omega. \end{array}\right. \tag{5.42}$$

However, the convexity argument does not work. But, by using the approximating argument and the main regularity result given in Theorem 4.3, we are able to prove the next existence result.

Theorem 5.5. *Assume that (m, σ) satisfies one of the following set of conditions:*

$$1 < m < \min\left\{\frac{N+2s}{3s-2}, \frac{\sigma(N+2s)}{N+2s-\sigma(3s-2)}\right\} \quad \textit{and} \quad 1 < \sigma < \min\left\{\frac{N+2s}{3s-2}, \frac{m(N+2s)}{N+2s-m(3s-2)}\right\}, \tag{5.43}$$

or

$$m \geqslant \frac{N+2s}{3s-2} \quad \textit{and } \sigma > \frac{m(N+2s)}{N+2s+m(3s-2)}, \tag{5.44}$$

or

$$\sigma \geqslant \frac{N+2s}{3s-2} \quad \textit{and } m > \frac{\sigma(N+2s)}{N+2s+\sigma(3s-2)}. \tag{5.45}$$

Then, there exists $T^* \in (0, T]$ such that System (5.42) has a nonnegative solution (u, v). Moreover $(u\delta^{1-s}, v\delta^{1-s}) \in L^{\theta_1}(0, T^*; \mathbb{W}_0^{1,\theta_1}(\Omega)) \times L^{\theta_2}(0, T^*; \mathbb{W}_0^{1,\theta_2}(\Omega))$ for any $\theta_1 < \frac{m(N+2s)}{N+2s-m(3s-2)}$ and $\theta_2 < \frac{\sigma(N+2s)}{N+2s-\sigma(3s-2)}$. □

Before starting the proof of Theorem 5.5, we shall give some light on the conditions (5.43), (5.44) and (5.45).

1) Condition (5.43).

As is done in Subsection 5.1, we denote $a = N + 2s,\ b = 2s - 1$ and $c = 1 - s$, and we take $m = \sigma$. We notice that

$$1 < \frac{a}{a - m(b-c)} \Leftrightarrow m > 1.$$

Thus, condition (5.43) is verified for any $m \in \left(1, \frac{a}{b-c}\right) = \left(1, \frac{N+2s}{3s-2}\right)$.

2) Condition (5.44). We observe that the conditions (5.44) and (5.45) are symmetric. Then, it is sufficient to treat one of the two.

Let $k \in \mathbb{R}^+$ such that $k \geqslant 1$. For $m = k\frac{N+2s}{3s-2}$, condition (5.44) is verified for any $\sigma \in \left(\frac{k}{k+1}\frac{N+2s}{3s-2}, +\infty\right)$.

Now, let us move on to the proof. *Proof.*
For $n \geqslant 1$ fixed, le us consider the following approximate system

$$(5.46)\qquad \begin{cases} \partial_t u_n + (-\Delta)^s u_n &= \dfrac{|\nabla v_n|}{1+\frac{1}{n}|\nabla v_n|} + \dfrac{f}{1+\frac{1}{n}f} = F_n \quad \text{in} \quad \Omega_T, \\ \partial_t v_n + (-\Delta)^s v_n &= \dfrac{|\nabla u_n|}{1+\frac{1}{n}|\nabla u_n|} + \dfrac{g}{1+\frac{1}{n}g} = G_n \quad \text{in} \quad \Omega_T, \\ u_n(x,t) = v_n(x,t) &= 0 \quad \text{in} \quad (\mathbb{R}^N \setminus \Omega) \times (0,T), \\ u_n(x,0) = v_n(x,0) &= 0 \quad \text{in} \quad \Omega. \end{cases}$$

It is clear that $\|F_n\|_{L^\infty(\Omega_T)}$, $\|G_n\|_{L^\infty(\Omega_T)} \leqslant 2n$. So, System (5.46) has at least an energy solution $(u_n, v_n) \in L^2(0,T;\mathbb{H}^s_0(\Omega)) \times L^2(0,T;\mathbb{H}^s_0(\Omega))$.

As we did before, we give the proof under the condition (5.43).
Thanks to Theorem 4.3 with $a = 1$, we get

$$(5.47)\qquad \begin{cases} \big\||\nabla u_n|\delta^{1-s}\big\|_{L^{\theta_1}(\Omega_T)} &\leqslant C(\big\||\nabla v_n|\delta^{1-s}\big\|_{L^m(\Omega_T)} + \|f\|_{L^m(\Omega_T)}), \\ \big\||\nabla v_n|\delta^{1-s}\big\|_{L^{\theta_2}(\Omega_T)} &\leqslant C(\big\||\nabla u_n|\delta^{1-s}\big\|_{L^\sigma(\Omega_T)} + \|g\|_{L^\sigma(\Omega_T)}), \end{cases}$$

for any $\theta_1 < \dfrac{m(N+2s)}{N+2s-m(3s-2)}$ and $\theta_2 < \dfrac{\sigma(N+2s)}{N+2s-\sigma(3s-2)}$.

Thus

$$(5.48)\qquad \begin{cases} \big\||\nabla u_n|\delta^{1-s}\big\|_{L^{\theta_1}(\Omega_T)} + \big\||\nabla v_n|\delta^{1-s}\big\|_{L^{\theta_2}(\Omega_T)} \\ \leqslant C\left(\big\||\nabla u_n|\delta^{1-s}\big\|_{L^\sigma(\Omega_T)} + \big\||\nabla v_n|\delta^{1-s}\big\|_{L^m(\Omega_T)} + \|f\|_{L^m(\Omega_T)} + \|g\|_{L^\sigma(\Omega_T)}\right). \end{cases}$$

Since $\sigma < \frac{m(N+2s)}{N+2s-m(3s-2)}$, we can choose θ_1 close to $\frac{m(N+2s)}{N+2s-m(3s-2)}$ such that $\sigma < \theta_1 < \frac{m(N+2s)}{N+2s-m(3s-2)}$. Hence, using Hölder's inequality, we obtain that

$$(5.49)\qquad \big\||\nabla u_n|\delta^{1-s}\big\|_{L^\sigma(\Omega_T)} \leqslant (|\Omega|T)^{\frac{\theta_1-\sigma}{\sigma\theta_1}} \big\||\nabla u_n|\delta^{1-s}\big\|_{L^{\theta_1}(\Omega_T)}.$$

In the same way, we get

$$(5.50)\qquad \big\||\nabla v_n|\delta^{1-s}\big\|_{L^m(\Omega_T)} \leqslant (|\Omega|T)^{\frac{\theta_2-m}{m\theta_2}} \big\||\nabla v_n|\delta^{1-s}\big\|_{L^{\theta_2}(\Omega_T)}.$$

Hence combining estimates (5.48),(5.49) and (5.50), we reach that

$$\begin{cases} \big\||\nabla u_n|\delta^{1-s}\big\|_{L^{\theta_1}(\Omega_T)} + \big\||\nabla v_n|\delta^{1-s}\big\|_{L^{\theta_2}(\Omega_T)} \\ \leqslant C((|\Omega|T)^{\frac{\theta_1-\sigma}{\sigma\theta_1}} \big\||\nabla u_n|\delta^{1-s}\big\|_{L^{\theta_1}(\Omega_T)} + (|\Omega|T)^{\frac{\theta_2-m}{m\theta_2}} \big\||\nabla v_n|\delta^{1-s}\big\|_{L^{\theta_2}(\Omega_T)} + \|f\|_{L^m(\Omega_T)} + \|g\|_{L^\sigma(\Omega_T)}). \end{cases}$$

Choosing now $T > 0$ such that $C\max\{(|\Omega|T)^{\frac{\theta_1-\sigma}{\sigma\theta_1}}, (|\Omega|T)^{\frac{\theta_2-m}{m\theta_2}}\} = \check{C} < 1$, we obtain that

$$(5.51)\qquad \begin{cases} (1-\check{C})\Big(\big\||\nabla u_n|\delta^{1-s}\big\|_{L^{\theta_1}(\Omega_T)} + \big\||\nabla v_n|\delta^{1-s}\big\|_{L^{\theta_2}(\Omega_T)}\Big) \\ \leqslant C\left(\|f\|_{L^m(\Omega_T)} + \|g\|_{L^\sigma(\Omega_T)}\right). \end{cases}$$

Therefore, $\{(u_n, v_n)\}_n$ is bounded $L^{\theta_1}(0,\,T;\,\mathbb{W}^{1,\theta_1}_0(\Omega,\,\delta^{\theta_1(1-s)}dx)) \times L^{\theta_2}(0,\,T;\,\mathbb{W}^{1,\theta_2}_0(\Omega,\,\delta^{\theta_2(1-s)}dx))$ for any $\theta_1 < \frac{m(N+2s)}{N+2s-m(3s-2)}$ and $\theta_2 < \frac{\sigma(N+2s)}{N+2s-\sigma(3s-2)}$.
The proof of the strong convergence, up to a subsequence, of the sequence $\{(u_n, v_n)_n\}$ to a solution of System (5.1) follows exactly by using the argument in the proof of the compactness part in Theorem 5.1. ∎

6 Appendix

In this paragraph, we give the proof of Lemma 4.1. For the convenience of the reader, we recall its statement.

Lemma 4.1. *Let $s \in (0,1)$, $\alpha, \lambda \in \mathbb{R}$ are such that $\alpha > -1$ and $N + \lambda > 0$. Assume that $h \in L^m(\Omega_T)$ with $m > 1$ and $m' = \frac{m}{m-1}$. Moreover, let us define*

$$H(x,t) := \int_0^t \int_\Omega \frac{h(y,\tau)(t-\tau)^\alpha}{((t-\tau)^{\frac{1}{2s}} + |x-y|)^{N+\lambda}} dy d\tau.$$

Then:

1) *if* $\alpha \geqslant \frac{N+\lambda}{2s}$, $H \in L^\infty(\Omega_T)$ *and*
$$\|H\|_{L^\infty(\Omega_T)} \leqslant C(\Omega)T^{\alpha-\frac{N+\lambda}{2s}+\frac{1}{m'}}\|h\|_{L^m(\Omega_T)};$$
2) *if* $\frac{\lambda-2s}{2s} < \alpha < \frac{N+\lambda}{2s}$ *and* $1 < m < \frac{N+2s}{2s(\alpha+1)-\lambda}$, $H \in L^r(\Omega_T)$ *for any* $r < \frac{m(N+2s)}{N+2s-m(2s(\alpha+1)-\lambda)}$ *and*
$$\|H\|_{L^r(\Omega_T)} \leqslant C(\Omega)T^{\gamma+\frac{1}{a}}\|h\|_{L^m(\Omega_T)};$$
3) *if* $\frac{\lambda-2s}{2s} < \alpha < \frac{N+\lambda}{2s}$ *and* $m \geqslant \frac{N+2s}{2s(\alpha+1)-\lambda}$, $H \in L^r(\Omega_T)$ *for any* $r \leqslant +\infty$ *and*
$$\|H\|_{L^r(\Omega_T)} \leqslant C(\Omega)T^{\gamma+\frac{1}{a}}\|h\|_{L^m(\Omega_T)},$$

where $a = \frac{m'r}{r+m'}$ *and* $\gamma = \alpha + \dfrac{N}{2sa} - \dfrac{N+\lambda}{2s}$.

Proof.

The first case follows easily using the fact that
$$\frac{(t-\tau)^\alpha}{((t-\tau)^{\frac{1}{2s}}+|x-y|)^{N+\lambda}} \leqslant (t-\tau)^{\alpha-\frac{N+\lambda}{2s}}.$$

Now, let us deal with the other cases. In order to prove the regularity of H, we will use a duality argument. Indeed, let $\phi \in \mathcal{C}_0^\infty(\Omega_T)$, then

$$(6.1)\quad \left\{\begin{array}{rcl} \|H\|_{L^r(\Omega_T)} &=& \displaystyle\sup_{\{||\phi||_{L^{r'}(\Omega_T)\leqslant 1}\}} \iint_{\Omega_T} H(x,t)\phi(x,t)dxdt, \\ &\leqslant& \displaystyle\sup_{\{||\phi||_{L^{r'}(\Omega_T)\leqslant 1}\}} \iint_{\Omega_T} |\phi(x,t)| \left(\int_0^t \int_\Omega |h(y,\tau)|K(x-y,t-\tau)dyd\tau\right) dxdt, \end{array}\right.$$

where
$$K(x,t) := \frac{t^\alpha}{(t^{\frac{1}{2s}}+|x|)^{N+\lambda}}.$$

According to generalized Hölder's inequality, we have

(6.2)
$$\|H\|_{L^r(\Omega_T)} \leqslant \sup_{\{\|\phi\|_{L^{r'}(\Omega_T)\leqslant 1}\}} \int_0^T \|\phi(.,t)\|_{L^{r'}(\Omega)} \left(\int_0^t \|h(.,\tau)\|_{L^m(\Omega)}\|K(.,t-\tau)\|_{L^a(\Omega)}d\tau\right) dt$$

where
$$(6.3)\quad \frac{1}{a} + \frac{1}{m} + \frac{1}{r'} = 2.$$

By a direct computation, we get
$$(6.4)\quad \|K(.,t)\|_{L^a(\Omega)} \leqslant Ct^\gamma$$

where
$$(6.5)\quad \gamma := \alpha + \frac{N}{2sa} - \frac{N+\lambda}{2s}.$$

Going back to (6.2), we obtain
$$(6.6)\quad \|H\|_{L^r(\Omega_T)} \leqslant \sup_{\{\|\phi\|_{L^{r'}(\Omega_T)\leqslant 1}\}} \int_0^T \int_0^T ||\phi(.,t)||_{L^{r'}(\Omega)}\|h(.,\tau)||_{L^m(\Omega)}|t-\tau|^\gamma d\tau dt.$$

Using again generalized Hölder's inequality on time, we get
$$(6.7)\quad \|H\|_{L^r(\Omega_T)} \leqslant C \sup_{\{\|\phi\|_{L^{r'}(\Omega_T)\leqslant 1}\}} \|h\|_{L^m(\Omega_T)}\|\phi\|_{L^{r'}(\Omega_T)} \left(\int_0^T |t-\tau|^{\bar{a}\gamma}dt\right)^{\frac{1}{\bar{a}}}.$$

Moreover $|t-\tau|^{\bar{a}\gamma} \leqslant t^{\bar{a}\gamma}$, then
$$(6.8)\quad \|H\|_{L^r(\Omega_T)} \leqslant C \sup_{\{\|\phi\|_{L^{r'}(\Omega_T)\leqslant 1}\}} \|h\|_{L^m(\Omega_T)}\|\phi\|_{L^{r'}(\Omega_T)} \left(\int_0^T t^{\bar{a}\gamma}dt\right)^{\frac{1}{\bar{a}}},$$

where $\dfrac{1}{\bar{a}}+\dfrac{1}{m}+\dfrac{1}{r'}=2$. So $\bar{a}=a=\frac{m'r}{m'+r}$. On the other hand, $\displaystyle\int_0^T t^{a\gamma}dt$ is finite if $a\gamma>-1$. Thus

$$m'r<\left(\frac{N+2s}{N+\lambda-2s\alpha}\right)(r+m'). \tag{6.9}$$

• If $\lambda\geqslant 2s(\alpha+1)$, we deduce from (6.9) that

$$r<\frac{m(N+2s)}{N+2s+m(\lambda-2s(\alpha+1))}.$$

• Now, assume $\lambda<2s(\alpha+1)$.
— If $\lambda<2s(\alpha+1)$ and $m<\frac{N+2s}{2s(\alpha+1)-\lambda}$, then

$$r<\frac{m(N+2s)}{N+2s-m(2s(\alpha+1)-\lambda)}$$

and

$$\|H\|_{L^r(\Omega_T)}\leqslant C(\Omega)T^{\gamma+\frac{1}{a}}\|h\|_{L^m(\Omega_T)}.$$

— If $m\geqslant\frac{N+2s}{2s(\alpha+1)-\lambda}$, then $g\in L^r(\Omega_T)$ for any $r\leqslant\infty$ and

$$\|H\|_{L^r(\Omega_T)}\leqslant C(\Omega)T^{\gamma+\frac{1}{a}}\|h\|_{L^m(\Omega_T)}.$$

This completes the proof. ■

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