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# Fractional elliptic reaction-diffusion systems with coupled gradient terms and different diffusion

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## Abstract

In this work, we study the existence and nonexistence of nonnegative solutions to a class of nonlocal elliptic Systems set in a bounded open subset of  $\mathbb{R}^N$  with different diffusion, where the gradients of the unknowns act as source terms (see (S) below). This work extends previous results we obtained in the case  $s_1 = s_2$  (see [8]).

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**Keywords.** Reaction-diffusion system, Fractional diffusion, Different diffusion, Nonlinear gradient terms, Nonlinear elliptic system, Fixed-point theorem, A priori estimates.

## 1 Introduction

The main objective of this paper is to investigate the existence and nonexistence of weak solutions to fractional elliptic systems of type

$$(S) \quad \begin{cases} (-\Delta)^{s_1} u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^{s_2} v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where  $\frac{1}{2} < s_1 < s_2 < 1$ ,  $\Omega \subset \mathbb{R}^N$  is a bounded regular domain with  $N > 2s_2$ ,  $p, q \geq 1$ ,  $\lambda, \mu > 0$ ,  $f$  and  $g$  are measurable nonnegative functions.

The operator  $(-\Delta)^s$  denotes the fractional Laplacian defined as :

$$(1.1) \quad (-\Delta)^s u(x) := a_{N,s} \text{ P.V. } \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy, \quad s \in (0, 1).$$

The constant

$$a_{N,s} := \frac{s2^{2s}\Gamma\left(\frac{N+2s}{2}\right)}{\pi^{\frac{N}{2}}\Gamma(1-s)}$$

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is a normalization constant chosen so that the following pair of identities :

$$\lim_{s \rightarrow 0^+} (-\Delta)^s u = u \quad \text{and} \quad \lim_{s \rightarrow 1^-} (-\Delta)^s u = -\Delta u$$

holds. For further details, see, *e.g.*, [32, Proposition 4.4] or [31, Proposition 2.1].

Let us emphasize that Fractional Laplacians are the prototypic nonlocal operators. They occur naturally in diverse scientific disciplines, including Biology, Ecology, Finance, Materials Science, Nonlinear Dynamics and Anomalous Diffusion. In fact, such operators play a fundamental role in modeling complex systems and phenomena with long-range interactions. For more details, we refer the reader, for instance, to [1, 31, 26] and references therein.

Besides, it is worth noting that many types of elliptic systems with gradient terms arise when studying electrochemical models in engineering and some fluid dynamics models. More details can be found in [30, 33] and their references.

As for models incorporating fractional diffusion, we refer to [47, 43, 42] for a rigorous justification for introducing the fractional Laplacian into the continuum equation of the growing surface as another relaxation mechanism.

In the present work, we look for *natural* conditions on  $(f, g)$ ,  $(\lambda, \mu)$  and  $(p, q)$  which allow us to establish existence of (weak) solutions to System  $(S)$ .

Before going further, let us recall some previous results related to our study.

• **Case of single equation.** System  $(S)$  is then reduced to

$$(1.2) \quad \begin{cases} (-\Delta)^s u = |\nabla u|^p + \lambda f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

— *Local case*  $s = 1$ . When  $f = 0$ , Problem (1.2) is no other than the stationary viscous *Hamilton-Jacobi* Problem (also known as the stationary diffusive *Hamilton-Jacobi* Problem), see, *e.g.*, [55] and references cited therein. When  $f \neq 0$ , we find the stationary generalized *Kardar-Parisi-Zhang (KPZ)* Problem (see, for instance, [41] and its references). Problem (1.2) has been extensively investigated, see [7, 15, 16, 17, 18, 19, 14, 38, 39, 4, 35, 36] and their references. We note that the references provided are not exhaustive, but rather a selection of some relevant sources.

— *Nonlocal case*  $\frac{1}{2} < s < 1$ . Although Problem (1.2) has received significant attention when  $s = 1$ , there has been relatively limited research in the case  $\frac{1}{2} < s < 1$ . In [29], the authors established the existence of solutions when  $p < \frac{N}{N - 2s + 1}$ . Additionally, they treated the case where  $f$  is a Radon measure. In the same paper, as well as in [6], the authors have considered natural conditions on  $f$  in order to examine the existence of solutions to Problem (1.2) depending on the value of  $p$ .

• **Case of systems** when  $s_1 = s_2 = s$ .

— *Local case*  $s = 1$ . Unlike the case of single equation, few results are known on systems where the gradients of the unknowns act as source terms. Let us review the known results regarding such systems. In [10], the fourth author and his coworkers have established the existence of nonnegative solutions to System  $(S)$  for any  $(p, q) \in [1, +\infty)^2$  such that  $pq > 1$ , under suitable integrability conditions on  $f$  and  $g$ , and smallness conditions on  $\lambda$  and  $\mu$ . In addition, nonexistence results were also identified for  $\lambda$  (or  $\mu$ ) large, or  $f$  and  $g$  of low integrability. Similar results was obtained in [3] about an elliptic System where the right-hand sides are potential and gradient terms, namely,  $-\Delta u = v^q + \lambda f$ ,  $-\Delta v = |\nabla u|^p + \mu g$ , as well as its parabolic version. Moreover, we refer to [9] where the authors studied the particular case  $0 < p \leq 2$ ,  $q > 0$  with  $pq < 1$ , and  $\lambda = \mu = 1$ . Many other studies, such as [20, 21, 33], have also investigated the existence of solutions to local elliptic systems with gradient terms.

— *Nonlocal case*  $\frac{1}{2} < s < 1$ . We have recently studied System (S) in [8]. Our work appears to be a novel contribution in this case. Let us summarize the main existence and nonexistence results that we obtained.

◊ The existence of weak solutions was obtained for  $(p, q) \in \left[1, \frac{s}{1-s}\right)^2$ , under appropriate integrability assumptions on  $f$  and  $g$  and smallness conditions on  $\lambda$  and  $\mu$ .

◊ Nonexistence results was proven in the following three cases :

- 1)  $\lambda$  and  $\mu$  are large even if  $1 < p, q < \frac{s}{1-s}$ ;
- 2)  $\max\{p, q\} \geq \frac{1}{1-s}$  even if  $f$  and  $g$  are bounded and no matter how small are  $\lambda$  and  $\mu$ ;
- 3)  $f$  and  $g$  have less regularity.

To the best of our knowledge, the case of systems where  $s_1 \neq s_2$  and  $0 < s_1, s_2 < \frac{1}{2}$  has not yet been considered. This is the objective of this work which complements our earlier study in [8] ( $s_1 = s_2$ ). Both studies represent an extension of the results obtained in [10] for the local version of System (S), and those obtained in [6] for the nonlocal single equation (1.2). Needless to say, this extension is much more difficult than it appears at first sight. Indeed, in addition to the difficulties faced in the case  $s_1 = s_2$ , there are other challenges that arise due to the fact that the diffusion operators are different. Hence, the techniques used in the local case are no longer applicable to handle the new situation. In fact, the major difficulties are coming from the loss of regularity caused by the fractional Laplacian and the lack of the homogeneity of this operator.

In order to overcome these difficulties we will use in a convenient way (as we have done in [8]) :

- (i) sharp estimates on the Green's function of the fractional Laplacian obtained in [23];
- (ii) fine weighted estimates on the solution to the corresponding fractional Poisson equation, see Theorems 3.1 and 3.2 below. We mention that the latter estimates improve those obtained in [28, 29, 6] and extend those of [13] regarding the local case.

Relying on these estimates, the existence of weak solution will be obtained for  $1 \leq q < \frac{s_1}{1-s_1}$  and  $1 \leq p < \frac{s_2}{1-s_1}$  under suitable regularity assumptions on  $f$  and  $g$ , and smallness conditions on  $\lambda$  and  $\mu$ , see Theorems 4.1 and 4.2. Nonexistence results will be also obtained in these three cases :

- (i)  $\lambda$  and  $\mu$  are large even if  $1 < q < \frac{s_1}{1-s_1}$  and  $1 < p < \frac{s_2}{1-s_1}$ , see Theorem 4.3;
- (ii)  $\max\{p, q\} \geq \frac{1}{1-s_1}$  even if  $f$  and  $g$  are bounded and no matter how small are  $\lambda$  and  $\mu$ , see Theorem 4.4;
- (iii)  $f$  and  $g$  are less regular, see Theorem 4.5.

We point out that our nonexistence Theorems 4.3, 4.4 and 4.5 show, in some sense, that our assumptions on the data in Theorem 4.1 are indeed necessary.

Regarding the existence, we believe that Theorem 4.1 holds for  $q \in \left[\frac{s_1}{1-s_1}, \frac{1}{1-s_1}\right)$  and  $p \in \left[\frac{s_2}{1-s_1}, \frac{1}{1-s_1}\right)$ . However, we have not been able to prove the existence in this case. We leave it as an open problem.

As a matter of fact, this question is still open even in the case  $s_1 = s_2$  as well as in the case of a single equation.

The rest of this paper is organized as follows. In Section 2, we give some useful tools such as the regularity results for nonlocal Poisson problem with general datum in  $L^1$  or in the space of Radon measures. Furthermore, we recall important results on Green function which will be used to prove our existence theorems. In Section 3, we prove some new regularity and compactness

results in weighted spaces. In Section 4, we prove the main existence results of nonnegative solutions to System (S) in suitable Sobolev spaces. Moreover, we present some nonexistence results that show, in some sense, the optimality of the assumptions considered in our existence theorems. In the last section, we study different variant of System (S) where the question of existence and nonexistence of solutions is also treated.

**Notations.** Let us fix some notations which will be used throughout this paper.

- $\Omega$  is a bounded regular open subset of  $\mathbb{R}^N$  with  $N > 2s_2$  and  $s_2 \in (\frac{1}{2}, 1)$ .
- By definition  $\delta(x) := \text{dist}(x, \partial\Omega)$ .
- For  $\beta > 0$ ,  $L^1(\Omega, \delta^\beta(x)dx) := \{h : \Omega \rightarrow \mathbb{R} \text{ measurable} ; \int_{\Omega} |h(x)|\delta^\beta(x)dx < +\infty\}$  and

$$\|h\|_{L^1(\Omega, \delta^\beta)} = \int_{\Omega} |h(x)|\delta^\beta(x) dx.$$

- If  $\varphi, \psi$  are nonnegative functions, then we write  $\varphi \simeq \psi$  if there exist two positive constants  $C_1$  and  $C_2$  such that

$$C_1\varphi \leq \psi \leq C_2\varphi.$$

□

## 2 Preliminaries and tools

This section is devoted to the basic knowledge on functional settings and some auxiliary results that will play an important role throughout this paper.

### 2.1 Fractional Sobolev Spaces

Let  $s \in (0, 1)$  and  $p \in [1, +\infty)$

- Fractional Sobolev Space  $W^{s,p}(\mathbb{R}^N)$

The space

$$W^{s,p}(\mathbb{R}^N) := \left\{ \phi \in L^p(\mathbb{R}^N) ; \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy < +\infty \right\},$$

endowed with its canonical norm

$$\|\phi\|_{W^{s,p}(\mathbb{R}^N)} := \left( \|\phi\|_{L^p(\mathbb{R}^N)}^p + \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}$$

is a Banach space.

- Fractional Sobolev Space  $\mathbb{W}_0^{s,p}(\Omega)$

The space  $\mathbb{W}_0^{s,p}(\Omega)$  is defined by

$$\mathbb{W}_0^{s,p}(\Omega) := \{ \phi \in W^{s,p}(\mathbb{R}^N) ; \phi = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}.$$

Endowed with the norm induced by  $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$  which we denote  $|\cdot|_{\mathbb{W}_0^{s,p}(\Omega)}$ ,  $\mathbb{W}_0^{s,p}(\Omega)$  is a Banach space.

For  $\phi \in \mathbb{W}_0^{s,p}(\Omega)$ , we set

$$\|\phi\|_{\mathbb{W}_0^{s,p}(\Omega)} := \left( \iint_{D_\Omega} \frac{|\phi(x) - \phi(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{1/p},$$

where

$$D_\Omega := (\mathbb{R}^N \times \mathbb{R}^N) \setminus ((\mathbb{R}^N \setminus \Omega) \times (\mathbb{R}^N \setminus \Omega)).$$

It is clear that  $\|\cdot\|_{\mathbb{W}_0^{s,p}(\Omega)}$  defines a norm on  $\mathbb{W}_0^{s,p}(\Omega)$ . In addition,  $\Omega$  is assumed to be bounded, then the norms  $|\cdot|_{\mathbb{W}_0^{s,p}(\Omega)}$  and  $\|\cdot\|_{\mathbb{W}_0^{s,p}(\Omega)}$  are equivalent.

## 2.2 Green's function

Let us recall the definition of the Green's function of the fractional Laplacian  $(-\Delta)^s$  in  $\Omega$ . For  $x \in \Omega$  fixed, then  $\mathcal{G}_s(x, y)$  solves the problem

$$(2.1) \quad \begin{cases} (-\Delta)_y^s \mathcal{G}_s(x, y) = D_x & \text{if } y \in \Omega, \\ \mathcal{G}_s(x, y) = 0 & \text{if } y \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where  $D_x$  is the Dirac mass at  $x$ .

In the following Lemma, we give some useful properties of this function and its gradient. We refer the interested readers to [6, 27, 25, 23] for a detailed proof (which relies on probabilistic tools).

**Lemma 2.1.** *Assume that  $s \in (0, 1)$ . Then, for a.e.  $x, y \in \Omega$ , we have*

$$(2.2) \quad \mathcal{G}_s(x, y) \simeq \frac{1}{|x - y|^{N-2s}} \left( \frac{\delta^s(x)}{|x - y|^s} \wedge 1 \right) \left( \frac{\delta^s(y)}{|x - y|^s} \wedge 1 \right).$$

In particular, there exists  $C > 0$  such that

$$(2.3) \quad \mathcal{G}_s(x, y) \leq C \min \left\{ \frac{1}{|x - y|^{N-2s}}, \frac{\delta^s(x)}{|x - y|^{N-s}}, \frac{\delta^s(y)}{|x - y|^{N-s}} \right\}.$$

Moreover, for any  $\eta \in (0, 1)$ , we get

$$(2.4) \quad \mathcal{G}_s(x, y) \leq C \frac{\delta^{\eta s}(y) \delta^{(1-\eta)s}(x)}{|x - y|^{N-s}} \text{ and } \mathcal{G}_s(x, y) \leq C \frac{\delta^{\eta s}(y)}{|x - y|^{N-s(2-\eta)}}.$$

If  $s \in (\frac{1}{2}, 1)$ , then there exists  $C > 0$  such that

$$(2.5) \quad |\nabla_x \mathcal{G}_s(x, y)| \leq C \mathcal{G}_s(x, y) \max \left\{ \frac{1}{|x - y|}, \frac{1}{\delta(x)} \right\}.$$

Therefore,

$$(2.6) \quad |\nabla_x \mathcal{G}_s(x, y)| \leq \frac{C}{|x - y|^{N-2s+1} \delta^{1-s}(x)}. \quad \square$$

## 2.3 Fractional Poisson equation: Regularity and compactness

Let us consider the Problem

$$(2.7) \quad \begin{cases} (-\Delta)^s w = h & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where  $0 < s < 1$  and  $h$  is a measurable function such that  $h \in L^m(\Omega)$  with  $m \geq 1$ . But before giving the definition of weak solution to Problem (2.7). Let us first recall that the solution  $w$  to Problem (2.7) is giving by

$$(2.8) \quad w(x) = \int_{\Omega} \mathcal{G}_s(x, y) h(y) dy.$$

**Definition 2.1.** *We say that  $w$  is a weak solution to Problem (2.7) if  $w \in L^1(\Omega)$ , and for any  $\phi \in \mathbb{X}_s$ , we have*

$$\int_{\Omega} w (-\Delta)^s \phi dx = \int_{\Omega} \phi h dx,$$

where

$$\mathbb{X}_s := \left\{ \phi \in \mathcal{C}(\mathbb{R}^N); \text{supp}(\phi) \subset \overline{\Omega}, (-\Delta)^s \phi \text{ exists and } (-\Delta)^s \phi \in L^\infty(\Omega) \right\}.$$

**Theorem 2.1** ([46, Theorem 23]). *Assume that  $h \in L^1(\Omega)$ . Then, Problem (2.7) has a unique weak solution  $w$ , which is the limit of the sequence  $\{w_n\}_{n \geq 1}$ , where  $w_n$  is the unique solution to the approximating Problem*

$$(2.9) \quad \begin{cases} (-\Delta)^s w_n = T_n(h) & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Namely

$$(2.10) \quad w_n \rightarrow w \text{ strongly in } \mathbb{W}_0^{s,\gamma}(\Omega) \quad \forall \gamma < \frac{N}{N-2s+1}.$$

Moreover,

$$(2.11) \quad T_k(w_n) \rightarrow T_k(w) \text{ strongly in } \mathbb{W}_0^{s,2}(\Omega),$$

where  $T_k$  is the truncating function defined by  $T_k(\varsigma) := \max(-k, \min(k, \varsigma))$  for any  $\varsigma \in \mathbb{R}^N$  and  $k > 0$ .

$$(2.12) \quad w \in L^\theta(\Omega), \quad \forall \theta \in \left(1, \frac{N}{N-2s}\right);$$

and

$$(2.13) \quad (-\Delta)^{\frac{s}{2}} w \in L^r(\Omega), \quad \forall r \in \left(1, \frac{N}{N-s}\right).$$

The following result will be used to prove our existence results.

**Proposition 2.1** ([29, Proposition 2.3]). *Let  $s \in (\frac{1}{2}, 1)$ . Assume that  $h \in L^1(\Omega, \delta^\beta)$  with  $0 \leq \beta < 2s - 1$ . Then, Problem (2.7) has a unique weak solution  $w \in \mathbb{W}_0^{2s-\gamma,p}(\Omega)$  for any  $p \in (1, \frac{N}{N+\beta-2s})$  where  $\gamma = \beta + \frac{N}{p'}$  if  $\beta > 0$  and  $\gamma > \frac{N}{p'}$  if  $\beta = 0$ . Moreover, there exists  $C := C(p, s, \Omega) > 0$  independent of  $w$  and  $h$  such that*

$$\|w\|_{\mathbb{W}_0^{2s-\gamma,p}(\Omega)} \leq C \|h\|_{L^1(\Omega, \delta^\beta)}.$$

For a datum  $h\delta^{a(1-s)} \in L^m(\Omega)$  with  $a \geq 0$  and  $m \geq 1$ , we have the following weighted regularity result.

**Theorem 2.2** ([8, Theorem 3.2]). *Let  $s \in (\frac{1}{2}, 1)$  and  $h$  be a measurable function such that  $h\delta^{a(1-s)} \in L^m(\Omega)$  with  $m \geq 1$  and  $0 \leq a < \frac{s}{1-s}$ . Let  $w$  be the unique weak solution to Problem (2.7). Moreover, let us set  $\widehat{m}_{s,a} := \frac{mN}{(N-m(s-a(1-s)))_+}$  and  $\widetilde{m}_{s,a} := \min\{\widehat{m}_{s,a}, \frac{mN}{(N-m(2s-1))_+}\}$ . Then, we have the following points.*

- (i) *If  $m < \max\{\frac{N}{2s-1}, \frac{N}{s-a(1-s)}\}$ , then  $|\nabla w|\delta^{1-s} \in L^\gamma(\Omega)$  for any  $\gamma < \widetilde{m}_{s,a}$ , and there exists  $C > 0$  such that:*

$$(2.14) \quad \|\nabla w|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C \left( \left\| \frac{w}{\delta^s} \right\|_{L^{\widehat{m}_{s,a}}(\Omega)} + \|\mathfrak{S}(h\delta^{a(1-s)})\|_{L^\gamma(\Omega)} \right),$$

where  $\mathfrak{S} : L^m(\Omega) \rightarrow L^\gamma(\Omega)$  is a linear operator defined by

$$(2.15) \quad \mathfrak{S}(g)(x) := \begin{cases} \int_{\Omega} \frac{g(y)}{|x-y|^{N-(s-a(1-s))}} dy & \text{if } a \geq 1, \\ \int_{\Omega} \frac{g(y)}{|x-y|^{N-2s+1}} dy & \text{if } 0 \leq a < 1. \end{cases}$$

Moreover for any  $\gamma < \widetilde{m}_{s,a}$ , there exists a constant  $C > 0$  such that

$$(2.16) \quad \|\nabla w|\delta^{1-s}\|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s)}\|_{L^m(\Omega)}.$$

(ii) If  $m = \max\{\frac{N}{2s-1}, \frac{N}{s-a(1-s)}\}$ , then  $|\nabla w|\delta^{1-s} \in L^\gamma(\Omega)$  for any  $\gamma < +\infty$  and there exists  $C > 0$  such that

$$(2.17) \quad \left\| |\nabla w|\delta^{1-s} \right\|_{L^\gamma(\Omega)} \leq C \left( \left\| \frac{w}{\delta^s} \right\|_{L^\gamma(\Omega)} + \|\Im(h\delta^{a(1-s)})\|_{L^\gamma(\Omega)} \right).$$

(iii) If  $m > \max\{\frac{N}{2s-1}, \frac{N}{s-a(1-s)}\}$ , then  $|\nabla w|\delta^{1-s} \in L^\infty(\Omega)$  and there exists  $C > 0$  such that

$$(2.18) \quad \left\| |\nabla w|\delta^{1-s} \right\|_{L^\infty(\Omega)} \leq C \left( \left\| \frac{w}{\delta^s} \right\|_{L^\infty(\Omega)} + \|\Im(h\delta^{a(1-s)})\|_{L^\infty(\Omega)} \right).$$

Moreover, for  $\gamma < \min\{\frac{N}{N-(s-a(1-s))}, \frac{N}{N-(2s-1)}\}$  fixed, the operator  $\Gamma : L^1(\Omega, \delta^{a(1-s)}) \rightarrow \mathbb{W}_0^{1,\gamma}(\Omega, \delta^{\gamma(1-s)})$  defined by  $\Gamma(h) = w$ , where  $w$  is the unique weak solution to (2.7), is compact.

## 2.4 Some useful tools

Let us begin by the next useful tool that will be used systematically in the rest of the paper.

### 2.4.1 Stein Lemma

**Lemma 2.2** ([54, Chapter V, Section 1.2]). Let  $0 < \lambda < N$ ,  $1 \leq p < \ell < +\infty$  be such that  $\frac{1}{\ell} + 1 = \frac{1}{p} + \frac{\lambda}{N}$ . For  $g \in L^p(\mathbb{R}^N)$ , we define

$$J_\lambda(g)(x) := \int_{\mathbb{R}^N} \frac{g(y)}{|x-y|^\lambda} dy.$$

a)  $J_\lambda$  is well defined in the sense that the integral converges absolutely for almost any  $x \in \mathbb{R}^N$ .

b) If  $p > 1$ , then  $\|J_\lambda(g)\|_\ell \leq c_{p,q} \|g\|_p$ .

c) If  $p = 1$ , then  $|\{x \in \mathbb{R}^N \mid J_\lambda(g)(x) > \sigma\}| \leq \left( \frac{A\|g\|_1}{\sigma} \right)^\ell$ .

### 2.4.2 Interpolation Inequality

**Proposition 2.2** ([12, Theorem 1]). Let  $h \in L^1(\Omega)$  and  $\{h_n\}_n \subset L^r(\Omega)$  with  $1 < r < +\infty$ . Assume that  $h_n \rightarrow h$  strongly in  $L^1(\Omega)$  and  $\{h_n\}_n$  is bounded in  $L^r(\Omega)$ .

Then  $h \in L^r(\Omega)$ , and for any  $a \in [1, r)$  we have the following interpolation inequality

$$(2.19) \quad \|h_n - h\|_{L^a(\Omega)} \leq \|h_n - h\|_{L^1(\Omega)}^\theta \|h_n - h\|_{L^r(\Omega)}^{1-\theta} \quad \text{where } \theta := \frac{r-a}{a(r-1)}.$$

In particular,  $h_n \rightarrow h$  strongly in  $L^a(\Omega)$ .

### 2.4.3 Weighted Hardy inequality

In the following theorem, we give a version of the weighted Hardy inequality that will be used throughout this paper.

**Theorem 2.3** ([50, Theorem 1.6]). Let  $\nu \in (1, +\infty)$  and  $0 \leq \rho < \nu - 1$ . Then, there exists a positive constant  $C = C(\Omega, \rho, \nu)$  such that for any  $\varphi \in C_0^\infty(\Omega)$ , we have

$$(2.20) \quad \int_{\Omega} \delta^{\rho-\nu}(x) |\varphi(x)|^\nu dx \leq C \int_{\Omega} \delta^\rho(x) |\nabla \varphi(x)|^\nu dx.$$

Regarding the proof, we refer also to [34].



### 3 Regularity and compactness results in weighted spaces

The main goal of this section is to prove a weighted regularity result to the linear Problem

$$(3.1) \quad \begin{cases} (-\Delta)^{s_2} w = h & \text{in } \Omega, \\ w = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

The first result reads as:

**Theorem 3.1.** *Let  $h$  be a measurable function such that  $h\delta^{a(1-s_1)} \in L^m(\Omega)$  with  $m \geq 1$  and  $0 \leq a < \frac{s_2}{1-s_1}$  with  $s_2 > s_1$ . Let  $w$  be the unique solution to Problem (3.1), then there exists a positive constant  $C$  such that*

(i) *if  $1 \leq m < \frac{N}{2s_2-s_1-a(1-s_1)}$ , then  $\frac{w}{\delta^{s_1}} \in L^\gamma(\Omega)$  for any  $\gamma \leq \widehat{m}_{s_1, s_2, a} := \frac{mN}{N-m(2s_2-s_1-a(1-s_1))}$ . Moreover,*

$$(3.2) \quad \left\| \frac{w}{\delta^{s_1}} \right\|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)};$$

(ii) *if  $m = \frac{N}{2s_2-s_1-a(1-s_1)}$ , then  $\frac{w}{\delta^{s_1}} \in L^\gamma(\Omega)$  for any  $\gamma < \infty$  and*

$$(3.3) \quad \left\| \frac{w}{\delta^{s_1}} \right\|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)};$$

(iii) *if  $m > \frac{N}{2s_2-s_1-a(1-s_1)}$ , then  $\frac{w}{\delta^{s_1}} \in L^\infty(\Omega)$  and*

$$(3.4) \quad \left\| \frac{w}{\delta^{s_1}} \right\|_{L^\infty(\Omega)} \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)}.$$

*Proof.* We give the proof for the case (i). The demonstration of the other cases is quite similar. We follow closely the arguments used in [8]. To make the paper self-contained as possible, we include here some details. In what follows, we denote by  $C$  a positive constant that may change from line to line.

Let  $h \geq 0$  and let  $w$  be the unique solution to Problem (3.1). Then, from the representation formula, we get

$$w(x) = \int_{\Omega} \mathcal{G}_{s_2}(x, y) h(y) dy.$$

Thus,

$$(3.5) \quad \left\{ \begin{aligned} \left| \frac{w(x)}{\delta^{s_1}(x)} \right| &\leq \int_{\Omega} \frac{\mathcal{G}_{s_2}(x, y)}{\delta^{s_1}(x)} h(y) dy, \\ &\leq \int_{\{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}} \frac{\mathcal{G}_{s_2}(x, y)}{\delta^{s_1}(x)} h(y) dy + \int_{\{z \in \Omega; \frac{1}{2}\delta(x) \leq |x-z| < \delta(x)\}} \frac{\mathcal{G}_{s_2}(x, y)}{\delta^{s_1}(x)} h(y) dy \\ &+ \int_{\{z \in \Omega; \delta(x) \leq |x-z| < \delta(z)\}} \frac{\mathcal{G}_{s_2}(x, y)}{\delta^{s_1}(x)} h(y) dy \\ &+ \int_{\{z \in \Omega; |x-z| \geq \max\{\delta(z), \delta(x)\}\}} \frac{\mathcal{G}_{s_2}(x, y)}{\delta^{s_1}(x)} h(y) dy, \\ &= J_1(x) + J_2(x) + J_3(x) + J_4(x). \end{aligned} \right.$$

• Let us start by estimating  $J_1$ . Notice that, since  $\delta$  is a Lipschitz function. Then, for any  $x \in \Omega$  and  $y \in \{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}$ , we have  $|\delta(x) - \delta(y)| \leq |x-y| \leq \frac{1}{2}\delta(x)$ . Hence,

$$|x-y| \leq \frac{1}{2}\delta(x) \leq \delta(y) \leq \frac{3}{2}\delta(x).$$

On the other hand, from (2.2) we get  $\mathcal{G}_{s_2}(x, y) \leq \frac{C}{|x-y|^{N-2s_2}}$ . Thus,

$$(3.6) \quad \left\{ \begin{aligned} J_1(x) &\leq C \int_{\{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}} \frac{h(y)}{\delta^{s_1}(x)|x-y|^{N-2s_2}} dy, \\ &\leq C \int_{\{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}} \frac{h(y)}{|x-y|^{N-2s_2+s_1}} dy, \\ &\leq C \int_{\{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}} \frac{h(y)\delta^{a(1-s_1)}(y)\delta^{-a(1-s_1)}(y)}{|x-y|^{N-2s_2+s_1}} dy, \\ &\leq C \int_{\{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}} \frac{h(y)\delta^{a(1-s_1)}(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy, \\ &\leq C \int_{\Omega} \frac{h(y)\delta^{a(1-s_1)}(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy. \end{aligned} \right.$$

Since  $h\delta^{a(1-s_1)} \in L^m(\Omega)$ , then by using Lemma 2.2, it follows that  $J_1 \in L^{\widehat{m}_{s_1, s_2, a}}(\Omega)$  and

$$\|J_1\|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)} \quad \text{for any } \gamma \leq \widehat{m}_{s_1, s_2, a}.$$

• Now, we deal with  $J_2$ . Using estimate (2.4) with  $\eta = a\frac{1-s_1}{s_2} < 1$ , it holds that

$$\mathcal{G}_{s_2}(x, y) \leq C \frac{\delta^{a(1-s_1)}(y)\delta^{s_2-(1-s_1)a}(x)}{|x-y|^{N-s_2}}.$$

Hence, it follows that

$$(3.7) \quad \left\{ \begin{aligned} J_2(x) &= \int_{\{z \in \Omega; \frac{1}{2}\delta(x) \leq |x-z| < \delta(x)\}} \frac{\mathcal{G}_{s_2}(x, y)}{\delta^{s_1}(x)} h(y) dy, \\ &\leq C \int_{\{z \in \Omega; \frac{1}{2}\delta(x) \leq |x-z| < \delta(x)\}} \frac{h(y)\delta^{a(1-s_1)}(y)\delta^{s_2-a(1-s_1)}(x)}{|x-y|^{N-s_2+s_1}} dy, \\ &\leq C \int_{\{z \in \Omega; \frac{1}{2}\delta(x) \leq |x-z| < \delta(x)\}} \frac{h(y)\delta^{a(1-s_1)}(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy, \\ &\leq C \int_{\Omega} \frac{h(y)\delta^{a(1-s_1)}(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy. \end{aligned} \right.$$

Thus, we conclude as in the estimating of the term  $J_1$ .

• Now, let us treat  $J_3$ . Since  $s_2 > s_1$ , then by using estimate (2.2) we get

$$(3.8) \quad \left\{ \begin{aligned} J_3(x) &\leq C \frac{1}{\delta^{s_1}(x)} \int_{\{z \in \Omega; \delta(x) \leq |x-z| < \delta(z)\}} \frac{h(y)\delta^{s_2}(x)}{|x-y|^{N-s_2}} dy, \\ &\leq C \int_{\{z \in \Omega; \delta(x) \leq |x-z| < \delta(z)\}} \frac{h(y)}{|x-y|^{N-2s_2+s_1}} dy, \\ &\leq C \int_{\{z \in \Omega; \delta(x) \leq |x-z| < \delta(z)\}} \frac{h(y)\delta^{a(1-s_1)}(y)}{|x-y|^{N-2s_2+s_1}\delta^{a(1-s_1)}(y)} dy, \\ &\leq C \int_{\Omega} \frac{h(y)\delta^{a(1-s_1)}(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy. \end{aligned} \right.$$

Hence, using again Lemma 2.2, it follows that

$$\|J_3\|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)} \quad \text{for any } \gamma \leq \widehat{m}_{s_1, s_2, a}.$$

Now, let us deal with the last term  $J_4$ . From (2.2) and since  $a < \frac{s_2}{1-s_1}$  with  $s_2 > s_1$ , then

$$(3.9) \quad \begin{cases} J_4(x) \leq C \int_{\{z \in \Omega; |x-z| \geq \max\{\delta(z), \delta(x)\}\}} \frac{h(y)\delta^{s_2}(y)\delta^{s_2}(x)}{\delta^{s_1}(x)|x-y|^N} dy, \\ \leq C \int_{\{z \in \Omega; |x-z| \geq \max\{\delta(z), \delta(x)\}\}} \frac{h(y)\delta^{a(1-s_1)}(y)\delta^{s_2-a(1-s_1)}(y)\delta^{s_2-s_1}(x)}{|x-y|^N} dy, \\ \leq C \int_{\{z \in \Omega; |x-z| \geq \max\{\delta(z), \delta(x)\}\}} \frac{h(y)\delta^{a(1-s_1)}(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy. \end{cases}$$

Hence, we conclude as in the previous estimates. Thus, we have proven that

$$\left\| \frac{w}{\delta^{s_1}} \right\|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)} \quad \text{for any } \gamma \leq \widehat{m}_{s_1, s_2, a}.$$

□

In the next result, we give a weighted regularity result for the gradient of the solution to Problem (3.1). This result will play a crucial role in the proof of the main existence result for System (S).

**Theorem 3.2.** *Let  $s_1, s_2 \in (\frac{1}{2}, 1)$  be such that  $s_2 > s_1$ . Assume that all the hypotheses of Theorem 3.1 hold. Let  $w$  be the unique weak solution to Problem (3.1). Furthermore, let us set  $\widetilde{m}_{s_1, s_2, a} := \min\{\widehat{m}_{s_1, s_2, a}, \frac{mN}{(N-m(2s_2-1))_+}\}$  where  $\widehat{m}_{s_1, s_2, a} := \frac{mN}{(N-m(2s_2-s_1-a(1-s_1)))_+}$ . Then, we have the following points.*

(i) *If  $m < \min\{\frac{N}{2s_2-1}, \frac{N}{2s_2-s_1-a(1-s_1)}\}$ , then  $|\nabla w|\delta^{1-s_1} \in L^\gamma(\Omega)$  for any  $\gamma < \widetilde{m}_{s_1, s_2, a}$  and*

$$(3.10) \quad \left\| |\nabla w|\delta^{1-s_1} \right\|_{L^\gamma(\Omega)} \leq C \left( \left\| \frac{w}{\delta^{s_1}} \right\|_{L^{\widehat{m}_{s_1, s_2, a}}(\Omega)} + \|\mathbb{P}(h\delta^{a(1-s_1)})\|_{L^\gamma(\Omega)} \right),$$

where  $\mathbb{P} : L^m(\Omega) \rightarrow L^\gamma(\Omega)$  is a linear operator defined by

$$(3.11) \quad \mathbb{P}(g)(x) := \begin{cases} \int_{\Omega} \frac{g(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy & \text{if } a \geq 1, \\ \int_{\Omega} \frac{g(y)}{|x-y|^{N-2s_2+1}} dy & \text{if } 0 \leq a < 1. \end{cases}$$

Moreover, we have

$$(3.12) \quad \left\| |\nabla w|\delta^{1-s_1} \right\|_{L^\gamma(\Omega)} \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)} \quad \forall \gamma < \widetilde{m}_{s_1, s_2, a}.$$

(ii) *If  $m = \max\{\frac{N}{2s_2-1}, \frac{N}{2s_2-s_1-a(1-s_1)}\}$ , then  $|\nabla w|\delta^{1-s_1} \in L^\gamma(\Omega)$  for any  $\gamma < \infty$*

$$(3.13) \quad \left\| |\nabla w|\delta^{1-s_1} \right\|_{L^\gamma(\Omega)} \leq C \left( \left\| \frac{w}{\delta^{s_1}} \right\|_{L^\gamma(\Omega)} + \|\mathbb{P}(h\delta^{a(1-s_1)})\|_{L^\gamma(\Omega)} \right).$$

(iii) *If  $m > \max\{\frac{N}{2s_2-1}, \frac{N}{2s_2-s_1-a(1-s_1)}\}$ , then  $|\nabla w|\delta^{1-s_1} \in L^\infty(\Omega)$  and*

$$(3.14) \quad \left\| |\nabla w|\delta^{1-s_1} \right\|_{L^\infty(\Omega)} \leq C \left( \left\| \frac{w}{\delta^{s_1}} \right\|_{L^\infty(\Omega)} + \|\mathbb{P}(h\delta^{a(1-s_1)})\|_{L^\infty(\Omega)} \right).$$

Hence, for  $\gamma < \min\{\frac{N}{N-(2s_2-s_1-a(1-s_1))}, \frac{N}{N-(2s_2-1)}\}$  fixed, the operator  $\Gamma : L^1(\Omega, \delta^{a(1-s_1)}) \rightarrow \mathbb{W}_0^{1,\gamma}(\Omega, \delta^{\gamma(1-s_1)})$  defined by  $\Gamma(h) = w$ , where  $w$  is the unique weak solution to (3.1), is compact.

*Proof.* The idea of the proof is the same used in our previous paper [8] ( see also [6]). We will split it into two parts:

- 1) Weighted regularity.
- 2) Compactness.

• **Weighted Regularity:**

Recall that  $w$  is given by

$$w(x) = \int_{\Omega} \mathcal{G}_{s_2}(x, y)h(y) dy.$$

Without loss of generalities, we suppose that  $h \geq 0$ . Then,

$$|\nabla w(x)| \leq \int_{\Omega} |\nabla_x \mathcal{G}_{s_2}(x, y)|h(y) dy \leq \int_{\Omega} \frac{|\nabla_x \mathcal{G}_{s_2}(x, y)|}{\mathcal{G}_{s_2}(x, y)} \mathcal{G}_{s_2}(x, y)h(y)dy.$$

Using the estimate (2.5), we get

$$|\nabla w(x)| \quad \mathcal{D}' \leq C \int_{\{z \in \Omega; |x-z| < \delta(x)\}} \frac{\mathcal{G}_{s_2}(x, y)}{|x-y|} h(y)dy + \frac{C}{\delta(x)} \int_{\{z \in \Omega; |x-z| \geq \delta(x)\}} \mathcal{G}_{s_2}(x, y)h(y)dy.$$

Thus,

$$(3.15) \quad \left\{ \begin{array}{l} |\nabla w(x)|\delta^{1-s_1}(x) \leq C\delta^{1-s_1}(x) \left( \int_{\{z \in \Omega; |x-z| < \delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_{s_2}(x, y)dy \right) \\ \quad \quad \quad \quad \quad + C\delta^{1-s_1}(x) \left( \int_{\{z \in \Omega; |x-z| \geq \delta(x)\}} \frac{h(y)}{\delta(x)} \mathcal{G}_{s_2}(x, y)dy \right), \\ \quad \quad \quad \quad \quad = Q_1(x) + Q_2(x). \end{array} \right.$$

— Let us begin by estimating  $Q_2$ . We have

$$(3.16) \quad Q_2(x) \leq C \frac{w(x)}{\delta^{s_1}(x)}.$$

As soon as  $\gamma < \tilde{m}_{s_1, s_2, a} \leq \hat{m}_{s_1, s_2, a}$ , we obtain

$$(3.17) \quad \int_{\Omega} Q_2^\gamma(x)dx \leq C \int_{\Omega} \frac{w^\gamma(x)}{\delta^{s_1\gamma}(x)} dx \leq C \|h\delta^{a(1-s_1)}\|_{L^m(\Omega)}^\gamma.$$

— Now, we deal with  $Q_1$ . We have

$$(3.18) \quad \left\{ \begin{array}{l} Q_1(x) = \delta^{1-s_1}(x) \left( \int_{\{z \in \Omega; |x-z| < \delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_{s_2}(x, y)dy \right), \\ \leq \delta^{1-s_1}(x) \left( \int_{\{z \in \Omega; \frac{1}{2}\delta(x) \leq |x-z| < \delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_{s_2}(x, y)dy \right) \\ \quad \quad \quad \quad \quad + \delta^{1-s_1}(x) \left( \int_{\{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}} \frac{h(y)}{|x-y|} \mathcal{G}_{s_2}(x, y)dy \right), \\ = Q_{11}(x) + Q_{12}(x). \end{array} \right.$$

Regarding  $Q_{11}$ , we have

$$(3.19) \quad Q_{11}(x) \leq C\delta^{-s_1}(x) \int_{\{z \in \Omega; \frac{1}{2}\delta(x) \leq |x-z| < \delta(x)\}} h(y)\mathcal{G}_{s_2}(x, y)dy \leq C \frac{w(x)}{\delta^{s_1}(x)}.$$

As above, since  $\gamma < \tilde{m}_{s_1, s_2, a} \leq \hat{m}_{s_1, s_2, a}$ , then

$$(3.20) \quad \int_{\Omega} Q_{11}^{\gamma}(x) dx \leq C \int_{\Omega} \frac{w^{\gamma}(x)}{\delta^{s_1 \gamma}(x)} dx \leq C \|h \delta^{a(1-s_1)}\|_{L^m(\Omega)}^{\gamma}.$$

Now, we estimate the term  $Q_{12}$ . Recall that for any  $x \in \Omega$  and  $y \in \{z \in \Omega; |x-z| < \frac{1}{2}\delta(x)\}$ , we have  $\frac{1}{2}\delta(x) \leq \delta(y) \leq \frac{3}{2}\delta(x)$ . Hence, from (2.2), we find

$$(3.21) \quad Q_{12}(x) \leq C \left( \int_{\{z \in \Omega; |x-z| \leq \frac{1}{2}\delta(x)\}} \frac{h(y) \delta^{a(1-s_1)}(y) \delta^{1-s_1-a(1-s_1)}(x)}{|x-y|^{N-2s_2+1}} dy \right).$$

Hence, according to the value of  $a$ , we have

$$(3.22) \quad Q_{12}(x) \leq \begin{cases} C \left( \int_{\Omega} \frac{h(y) \delta^{a(1-s_1)}(y)}{|x-y|^{N-(2s_2-s_1-a(1-s_1))}} dy \right) & \text{if } a \geq 1, \\ C \left( \int_{\Omega} \frac{h(y) \delta^{a(1-s_1)}(y)}{|x-y|^{N-2s_2+1}} dy \right) & \text{if } 0 \leq a < 1. \end{cases}$$

Taking into consideration the definition of  $\mathbb{P}$ , it follows that

$$\int_{\Omega} Q_{12}^{\gamma}(x) dx \leq C \int_{\Omega} (\mathbb{P}(h \delta^{a(1-s_1)})(x))^{\gamma} dx.$$

Combining the estimates (3.16) and (3.22), it holds that

$$(3.23) \quad |\nabla w(x)| \delta^{1-s_1}(x) \leq C \frac{w(x)}{\delta^{s_1}(x)} + \mathbb{P}(h \delta^{a(1-s_1)})(x).$$

Hence (3.10) follows.

Finally, by using Theorem 3.1 and Lemma 2.2, we deduce that from (3.23),

$$\| |\nabla w| \delta^{1-s_1} \|_{L^{\gamma}(\Omega)} \leq C \|h \delta^{a(1-s_1)}\|_{L^m(\Omega)},$$

for any  $\gamma < \tilde{m}_{s_1, s_2, a}$ .

• **Compactness:**

Now, let us prove that  $\Gamma$  is compact. So, let  $\{h_n\}_n \subset L^1(\Omega, \delta^{a(1-s_1)})$  be a bounded sequence and set  $w_n = \Gamma(h_n)$ . Then,  $w_n$  solves

$$(3.24) \quad \begin{cases} (-\Delta)^{s_2} w_n = h_n & \text{in } \Omega, \\ w_n = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

For  $q < \frac{N}{N-(2s_2-s_1-a(1-s_1))}$  fixed, by Theorem 3.2, there exists  $C > 0$  independent of  $n$  such that

$$(3.25) \quad \int_{\Omega} \frac{|w_n(x)|^q}{\delta^{s_1 q}(x)} dx \leq C \|h_n \delta^{a(1-s_1)}\|_{L^1(\Omega)}^q \leq C.$$

Hence, using the estimate (3.10) and choosing  $q$  such that  $\gamma < q < \frac{N}{N-(2s_2-s_1-a(1-s_1))}$ , we conclude that

$$(3.26) \quad \begin{cases} \| |\nabla w_n| \delta^{(1-s_1)} \|_{L^{\gamma}(\Omega)} \leq C \left( \left\| \frac{w_n}{\delta^{s_1}} \right\|_{L^q(\Omega)} + \| \mathbb{P}(h_n \delta^{a(1-s_1)}) \|_{L^{\gamma}(\Omega)} \right), \\ \leq C \|h_n \delta^{a(1-s_1)}\|_{L^1(\Omega)} \leq C. \end{cases}$$

Taking into consideration that  $|\nabla(w_n \delta^{1-s_1})| \leq \delta^{1-s_1} |\nabla w_n| + (1-s_1) \frac{|w_n(x)|}{\delta^{s_1}(x)}$ , we deduce that the sequence  $\{w_n \delta^{1-s_1}\}_n$  is bounded in the space  $\mathbb{W}_0^{1,\gamma}(\Omega)$  for any  $\gamma < \min\left\{\frac{N}{N-(2s_2-s_1-a(1-s_1))}, \frac{N}{N-(2s_2-1)}\right\}$ .

Then, up to a subsequence, we get the existence of a measurable function  $w$  such that  $w\delta^{1-s_1} \in \mathbb{W}_0^{1,\gamma}(\Omega)$ ,  $w_n\delta^{1-s_1} \rightharpoonup w\delta^{1-s_1}$  weakly in  $\mathbb{W}_0^{1,\gamma}(\Omega)$  and  $w_n \rightarrow w$  *a.e.* in  $\Omega$ . Thanks to (3.25),  $\left\{\frac{w_n}{\delta^{s_1}}\right\}_n$  is bounded in  $L^q(\Omega)$  for any  $q < \frac{N}{N-(2s_2-s_1-a(1-s_1))}$ . Therefore, we conclude that  $\frac{w}{\delta^{s_1}} \in L^q(\Omega)$  for any  $q < \frac{N}{N-(2s_2-s_1-a(1-s_1))}$  by Fatou's Lemma. Finally, since  $\Omega$  is a bounded domain, by using Vitali's Lemma we obtain that

$$\frac{w_n}{\delta^{s_1}} \longrightarrow \frac{w}{\delta^{s_1}} \text{ strongly in } L^q(\Omega),$$

for any  $q < \frac{N}{N-(2s_2-s_1-a(1-s_1))}$ . Now, setting  $v_{ij} = w_i - w_j$  and  $h_{ij} = h_i - h_j$ , it holds that  $v_{ij}, \frac{v_{ij}}{\delta^{s_1}} \rightarrow 0$  *a.e.* in  $\Omega$  as  $i, j \rightarrow +\infty$  and  $\frac{v_{ij}}{\delta^{s_1}} \rightarrow 0$  strongly in  $L^q(\Omega)$  for any  $q < \frac{N}{N-(2s_2-s_1-a(1-s_1))}$ . Since  $v_{ij}$  solves Problem

$$(3.27) \quad \begin{cases} (-\Delta)^{s_2} v_{ij} = h_{ij} & \text{in } \Omega, \\ v_{ij} = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

From (3.10), we get

$$\|\nabla v_{ij}\delta^{1-s_1}\|_{L^\gamma(\Omega)} \leq C \left( \left\| \frac{v_{ij}}{\delta^{s_1}} \right\|_{L^q(\Omega)} + \|\mathbb{P}(h_{ij}\delta^{a(1-s_1)})\|_{L^\gamma(\Omega)} \right).$$

Recall that  $\frac{v_{ij}}{\delta^{s_1}} \rightarrow 0$  as  $i, j \rightarrow +\infty$  strongly in  $L^q(\Omega)$  for any  $q < \frac{N}{N-(2s_2-s_1-a(1-s_1))}$ . Now, we deduce from [44, Theorem 8.1] that the operator  $\mathbb{P} : L^1(\Omega) \rightarrow L^\gamma(\Omega)$  is compact for any  $\gamma < \min\left\{\frac{N}{N-(2s_2-s_1-a(1-s_1))}, \frac{N}{N-(2s_2-1)}\right\}$ . Hence, up to an other subsequence, we conclude that  $w_n \rightarrow w$  strongly in  $\mathbb{W}_0^{1,\gamma}(\Omega, \delta^{\gamma(1-s_1)})$ . This ends the proof.  $\square$

## 4 Existence and nonexistence results

As we mentioned in the introduction, our main objective is to discuss the existence and nonexistence of solutions to

$$(S) \quad \begin{cases} (-\Delta)^{s_1} u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^{s_2} v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

In the first subsection, we will establish the existence of a solution to System (S) under natural assumptions on data. Then, we will establish nonexistence results in the second subsection.

### 4.1 Existence results

In this subsection, we first specify what we mean by a weak solution to System (S). Then, we state and prove the existence theorems in the two cases:

- 1)  $1 \leq q < \frac{s_1}{1-s_1}$ ,  $1 \leq p < \frac{s_2}{1-s_1}$  and  $pq > 1$ ;
- 2)  $p = q = 1$ .

**Definition 4.1.** *Let  $(f, g) \in (L^1(\Omega)^+)^2$ . We say that  $(u, v) \in \mathbb{W}_0^{1,p}(\Omega) \times \mathbb{W}_0^{1,q}(\Omega)$  is a weak solution to System (S) if for any  $(\varphi, \psi) \in \mathbb{X}_{s_1} \times \mathbb{X}_{s_2}$  (see Definition 2.1), we have*

$$(4.1) \quad \begin{cases} \int_{\Omega} u(-\Delta)^{s_1} \varphi \, dx = \int_{\Omega} |\nabla v|^q \varphi \, dx + \lambda \int_{\Omega} f \varphi \, dx \\ \int_{\Omega} v(-\Delta)^{s_2} \psi \, dx = \int_{\Omega} |\nabla u|^p \psi \, dx + \mu \int_{\Omega} g \psi \, dx. \end{cases}$$

**4.1.1 Case:**  $1 \leq q < \frac{s_1}{1-s_1}$ ,  $1 \leq p < \frac{s_2}{1-s_1}$  and  $pq > 1$

**Theorem 4.1.** Assume that  $1 \leq q < \frac{s_1}{1-s_1}$ ,  $1 \leq p < \frac{s_2}{1-s_1}$  such that  $pq > 1$  and  $s_2 > s_1$ . Suppose that  $(f, g) \in L^m(\Omega)^+ \times L^\sigma(\Omega)^+$  where  $(m, \sigma) \in [1, +\infty)^2$  satisfies one of the following set of conditions:

$$(4.2) \quad \begin{cases} \frac{q+1}{q+2} < s_1 < 1, \quad \frac{p+1}{p+2} < s_2 < 1, \\ 1 < m < \frac{N}{s_1 - q(1-s_1)}, \quad 1 < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q} := \frac{\hat{m}_{s_1, q}}{1 + \hat{m}_{s_1, q}(s_2 - s_1)} \quad \text{where } \hat{m}_{s_1, q} := \frac{mN}{N - m(s_1 - q(1-s_1))}, \\ qm < \hat{\sigma}_{s_1, s_2, p} := \frac{\sigma N}{N - \sigma(2s_2 - s_1 - p(1-s_1))}, \end{cases}$$

or

$$(4.3) \quad \begin{cases} \frac{1}{2} < s_1 < \frac{q+1}{q+2}, \quad \frac{1}{2} < s_2 < \frac{p+1}{p+2}, \\ \frac{1}{2s_1 - q(1-s_1)} < m < \frac{N}{s_1 - q(1-s_1)}, \quad \frac{1}{2s_2 - p(1-s_2)} < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q}, \quad qm < \hat{\sigma}_{s_1, s_2, p}, \end{cases}$$

or

$$(4.4) \quad \begin{cases} \frac{q+1}{q+2} < s_1 < 1, \quad \frac{1}{2} < s_2 < \frac{p+1}{p+2}, \quad q < p, \\ 1 < m < \frac{N}{s_1 - q(1-s_1)}, \quad \frac{1}{2s_2 - p(1-s_2)} < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q}, \quad qm < \hat{\sigma}_{s_1, s_2, p}, \end{cases}$$

or

$$(4.5) \quad \begin{cases} \frac{1}{2} < s_1 < \frac{q+1}{q+2}, \quad \frac{p+1}{p+2} < s_2 < 1, \\ \frac{1}{2s_1 - q(1-s_1)} < m < \frac{N}{s_1 - q(1-s_1)}, \quad 1 < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q}, \quad qm < \hat{\sigma}_{s_1, s_2, p}, \end{cases}$$

or

$$(4.6) \quad m \geq \frac{N}{s_1 - q(1-s_1)}, \quad \sigma > \frac{qmN}{N + qm(2s_2 - s_1 - p(1-s_1))} \quad \text{and} \quad p\sigma < \frac{1}{s_2 - s_1},$$

or

$$(4.7) \quad \sigma \geq \frac{N}{2s_2 - s_1 - p(1-s_1)} \quad \text{and} \quad p\sigma < \bar{m}_{s_1, s_2, q}.$$

Then, there exists  $A > 0$  such that for any  $(\lambda, \mu) \in \Pi$  where

$$(4.8) \quad \Pi := \left\{ (\lambda, \mu) \in [0, +\infty) \times [0, +\infty) ; \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq A \right\},$$

System (S) has a nonnegative solution  $(u, v)$ . Moreover,  $(u, v) \in \mathbb{W}_0^{1, \theta_1}(\Omega, \delta^{(1-s_1)}) \times \mathbb{W}_0^{1, \theta_2}(\Omega, \delta^{(1-s_2)})$  for any  $\theta_1 < \bar{m}_{s_1, s_2, q}$  and  $\theta_2 < \hat{\sigma}_{s_1, s_2, p}$ .

*Proof.* We follow closely the arguments used in [6, 8, 10]. Also, we will give the proof under the assumptions (4.2) and (4.3). The other cases follow by using the same reasoning. Since we will use Schauder's fixed point Theorem, and in order to make our demonstration clear for the readers, we will proceed by three steps.

**First case.** Set (4.2).

Let us define the following function

$$\Upsilon(\alpha) := \alpha^{\frac{1}{pq}} - \tilde{C}\alpha \quad \forall \alpha > 0,$$

where  $\tilde{C}$  is a positive constant depending only on the data and which we will specify later. Since  $pq > 1$ , then there exists  $\alpha_0 > 0$  such that  $\Upsilon(\alpha_0) = 0$ ,  $\Upsilon(\alpha) > 0$  for any  $\alpha \in (0, \alpha_0)$  and  $\Upsilon(\alpha) < 0$  for any  $\alpha \in (\alpha_0, +\infty)$ . Hence, there exist two positive constants  $\ell > 0$  and  $\Lambda^* > 0$  such that

$$\max_{\alpha > 0} \Upsilon(\alpha) = \Upsilon(\ell) = \Lambda^*.$$

Thus,

$$\ell^{\frac{1}{pq}} = \tilde{C} \left( \ell + \frac{\Lambda^*}{\tilde{C}} \right).$$

Let  $\ell > 0$  be fixed as above. Since  $(\lambda, \mu) \in \Pi$  with the choice  $A = \frac{\Lambda^*}{\tilde{C}}$ , we have

$$(4.9) \quad \tilde{C} \left( \ell + \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \right) \leq \ell^{\frac{1}{pq}}.$$

Since  $qm < \hat{\sigma}_{s_1, s_2, p} := \frac{\sigma N}{N - \sigma(2s_2 - s_1 - p(1 - s_1))}$ , then there exists  $r > 0$  such that  $qm < r < \hat{\sigma}_{s_1, s_2, p}$ . Furthermore, let us consider the set  $H$  such that

$$(4.10) \quad H := \left\{ \varphi \in \mathbb{W}_0^{1,1}(\Omega) ; \varphi \delta^{1-s_1} \in \mathbb{W}_0^{1,r}(\Omega) \text{ and } \left( \int_{\Omega} |\nabla(\varphi \delta^{1-s_1})|^r dx \right)^{\frac{1}{r}} \leq \ell^{\frac{1}{pq}} \right\}.$$

It is clear that  $H$  is a convex and closed subset of  $\mathbb{W}_0^{1,1}(\Omega)$ .

Notice that, if  $\varphi \in H$ , then setting  $\hat{\varphi} = \varphi \delta^{1-s_1}$ , we deduce that  $\hat{\varphi} \in \mathbb{W}_0^{1,r}(\Omega)$ . Then, by using Hardy's inequality (Theorem 2.3) with  $\rho = 0$ . Then, we get

$$\int_{\Omega} \frac{|\hat{\varphi}|^r}{(\delta(x))^r} dx \leq C(\Omega) \int_{\Omega} |\nabla \hat{\varphi}|^r dx.$$

Thus,

$$\int_{\Omega} \frac{|\varphi|^r}{(\delta(x))^{rs_1}} dx \leq C(\Omega) \int_{\Omega} |\nabla \hat{\varphi}|^r dx.$$

Since  $|\nabla \delta(x)| = 1$  a.e. in  $\Omega$ , we deduce that

$$|\nabla \varphi|^r \delta^{r(1-s_1)} = |\nabla(\varphi \delta^{1-s_1}) - \varphi \nabla \delta^{1-s_1}|^r \leq C |\nabla \hat{\varphi}|^r + \frac{|\varphi|^r}{\delta^{rs_1}}.$$

Therefore,  $|\nabla \varphi|^r \delta^{r(1-s_1)} \in L^1(\Omega)$  and

$$\left( \int_{\Omega} |\nabla \varphi|^r \delta^{r(1-s_1)} dx \right)^{\frac{1}{r}} \leq C \ell^{\frac{1}{pq}}.$$

We observe that, if  $\varphi \in H$ , then  $|\nabla \varphi|^a \delta^{a(1-s_1)} \in L^1(\Omega)$  for any  $a \leq r$ . In particular,  $|\nabla \varphi|^q \delta^{q(1-s_1)} \in L^m(\Omega)$ .



**Step 1.** We claim that if  $\varphi \in H$ , then there exists  $\beta < 2s_1 - 1$  such that  $|\nabla\varphi|^q \delta^\beta \in L^1(\Omega)$ . Indeed, let  $\varphi \in H$ , then by using Hölder's inequality, we get

$$(4.11) \quad \begin{cases} \int_{\Omega} |\nabla\varphi|^q \delta^\beta dx &= \int_{\Omega} |\nabla\varphi|^q \delta^{q(1-s_1)} \delta^{\beta-q(1-s_1)} dx, \\ &\leq \left( \int_{\Omega} |\nabla\varphi|^{qm} \delta^{q(1-s_1)m} dx \right)^{\frac{1}{m}} \left( \int_{\Omega} \delta^{(\beta-q(1-s_1))m'} dx \right)^{\frac{1}{m'}}, \\ &\leq C \left( \int_{\Omega} \delta^{(\beta-q(1-s_1))m'} dx \right)^{\frac{1}{m'}}. \end{cases}$$

Since  $\frac{q+1}{q+2} < s_1$  (from the set of conditions (4.2)), then there exists  $\beta > 0$  such that  $q(1-s_1) < \beta < 2s_1 - 1$ . Therefore,  $|\nabla\varphi|^q \delta^\beta + \lambda f \in L^1(\Omega)$ . By Proposition 2.1, we get the existence of a unique weak solution  $u$  to Problem

$$(4.12) \quad \begin{cases} (-\Delta)^{s_1} u &= |\nabla\varphi|^q + \lambda f & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover,  $u \in \mathbb{W}_0^{1,\alpha}(\Omega)$  for any  $\alpha < \frac{N}{N-2s_1+1+\beta}$ .

Now, we claim that  $|\nabla u|^p \delta^{\beta_1} \in L^1(\Omega)$  for any  $\beta_1 < 2s_2 - 1$ . Indeed, by Theorem 2.2 and by taking  $s = s_1$ ,  $a = q$  and  $h = |\nabla\varphi|^q + \lambda f$ , we have

$$\| |\nabla u| \delta^{1-s_1} \|_{L^\gamma(\Omega)} \leq C \left( \| |\nabla\varphi| \delta^{1-s_1} \|_{L^{qm}(\Omega)}^q + \lambda \| f \|_{L^m(\Omega)} \right) \quad \text{for any } \gamma \leq \widehat{m}_{s_1,q}.$$

Since  $qm < r$ , then by Hölder's inequality we get

$$(4.13) \quad \begin{cases} \| |\nabla u| \delta^{1-s_1} \|_{L^\gamma(\Omega)} &\leq C \left( \| |\nabla\varphi| \delta^{1-s_1} \|_{L^r(\Omega)}^q + \lambda \| f \|_{L^m(\Omega)} \right), \\ &\leq C \left( \ell^{\frac{1}{p}} + \lambda \| f \|_{L^m(\Omega)} \right), \end{cases}$$

for any  $\gamma \leq \widehat{m}_{s_1,q}$ . Thus,  $|\nabla u|^p \delta^{p(1-s_1)} \in L^\sigma(\Omega) \subset L^1(\Omega)$ . Since  $p\sigma < \overline{m}_{s_1,s_2,q}$ , then by using Hölder's inequality, we obtain  $|\nabla u|^p \delta^{p(1-s_2)} \in L^\sigma(\Omega)$ .

On the other hand, as  $\frac{p+1}{p+2} < s_2$ , we get by using the same computation as above, the existence of  $\beta_1 < 2s_2 - 1$ , such that  $|\nabla u|^p \delta^{\beta_1} \in L^1(\Omega)$ .

Thus, using again Proposition 2.1, we get the existence of a weak solution  $v$  to the following Problem

$$(4.14) \quad \begin{cases} (-\Delta)^{s_2} v &= |\nabla u|^p + \mu g & \text{in } \Omega, \\ v &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

such that  $v \in \mathbb{W}_0^{1,\alpha_1}(\Omega)$  for any  $\alpha_1 < \frac{N}{N-2s_2+1+\beta_1}$ .

Therefore, the operator

$$\begin{aligned} T : H &\longrightarrow \mathbb{W}_0^{1,1}(\Omega) \\ \varphi &\longmapsto T(\varphi) = v \end{aligned}$$

is well defined. Moreover, if  $v$  is a fixed point of  $T$ , then  $(u, v)$  is a weak solution to System (S).

**Step 2.** In this step, we want to show that  $T(H) \subset H$ .

Indeed, by taking  $h = |\nabla u|^p + \mu g$  with  $a = p$  and by using Theorem 3.2, it holds that

$$(4.15) \quad \left\| |\nabla v| \delta^{1-s_1} \right\|_{L^\nu(\Omega)} \leq C \left( \left\| |\nabla u| \delta^{1-s_1} \right\|_{L^{p\sigma}(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \right) \quad \text{for any } \nu \leq \widehat{\sigma}_{s_1, s_2, p}.$$

Since,  $p\sigma < \overline{m}_{s_1, s_2, q} < \widehat{m}_{s_1, q}$  and thanks to estimate (4.13), it follows that

$$(4.16) \quad \left\| |\nabla v| \delta^{1-s_1} \right\|_{L^\nu(\Omega)} \leq C \left( \left\| |\nabla \varphi| \delta^{1-s_1} \right\|_{L^r(\Omega)}^{pq} + \mu \|g\|_{L^\sigma(\Omega)} + \lambda^p \|f\|_{L^m(\Omega)}^p \right).$$

Since  $r < \widehat{\sigma}_{s_1, s_2, p}$ , then by choosing  $\nu = r$  in the previous inequality, we get

$$\left\| |\nabla v| \delta^{1-s_1} \right\|_{L^r(\Omega)} \leq C \left( \left\| |\nabla \varphi| \delta^{1-s_1} \right\|_{L^r(\Omega)}^{pq} + \mu \|g\|_{L^\sigma(\Omega)} + \lambda^p \|f\|_{L^m(\Omega)}^p \right).$$

Recall that  $\varphi \in H$ , thus

$$\left\| |\nabla v| \delta^{1-s_1} \right\|_{L^r(\Omega)} \leq C \left( \ell + \mu \|g\|_{L^\sigma(\Omega)} + \lambda^p \|f\|_{L^m(\Omega)}^p \right).$$

By choosing  $\widetilde{C} = C$  and taking into consideration the definition of  $\ell$ , we conclude that

$$\left\| |\nabla v| \delta^{1-s_1} \right\|_{L^r(\Omega)} \leq \ell^{\frac{1}{pq}}.$$

Thus,  $v \in H$  and therefore  $T(H) \subset H$ .

**Step 3.** Continuity and Compactness of  $T$ .

• **Continuity**

– Let  $\{\varphi_n\}_n \subset H$  and  $\varphi \in H$  be such that  $\varphi_n \rightarrow \varphi$  strongly in  $\mathbb{W}_0^{1,1}(\Omega)$ . Moreover, let us define  $v_n = T(\varphi_n)$ ,  $v = T(\varphi)$ . Then,  $(u_n, v_n)$  and  $(u, v)$  satisfy

$$(4.17) \quad \begin{cases} (-\Delta)^{s_1} u_n = |\nabla \varphi_n|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^{s_1} u = |\nabla \varphi|^q + \lambda f & \text{in } \Omega, \\ u_n = u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$(4.18) \quad \begin{cases} (-\Delta)^{s_2} v_n = |\nabla u_n|^p + \mu g & \text{in } \Omega, \\ (-\Delta)^{s_2} v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ v_n = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Thus, by using Proposition 2.1, we get

$$\|\nabla u_n - \nabla u\|_{L^\alpha(\Omega)} \leq C \left\| |\nabla \varphi_n|^q - |\nabla \varphi|^q \right\|_{L^1(\Omega, \delta^\beta)},$$

for any  $\alpha < \frac{N}{N-2s_1+1+\beta}$  where  $\beta < 2s_1 - 1$ .

Since  $\varphi_n \rightarrow \varphi$  in  $\mathbb{W}_0^{1,1}(\Omega)$  and  $\{\varphi_n\}_n \subset H$ , then, by Fatou's Lemma, we get

$$\left( \int_\Omega |\nabla \varphi|^r \delta^{r(1-s_1)} dx \right)^{\frac{1}{r}} \leq C \ell^{\frac{1}{pq}}.$$

Hence, for  $a \in (1, r)$  fixed and due to Proposition 2.2, we get

$$(4.19) \quad \left\{ \begin{aligned} \left( \int_{\Omega} |\nabla \varphi_n - \nabla \varphi|^a \delta^{a(1-s_1)} dx \right)^{\frac{1}{a}} &\leq \left( \int_{\Omega} |\nabla \varphi_n - \nabla \varphi| \delta^{1-s_1} dx \right)^{\eta} \\ &\times \left( \int_{\Omega} |\nabla \varphi_n - \nabla \varphi|^r \delta^{r(1-s_1)} dx \right)^{\frac{1-\eta}{r}}, \\ &\leq C \left( \int_{\Omega} |\nabla \varphi_n - \nabla \varphi| \delta^{1-s_1} dx \right)^{\eta} \rightarrow 0 \text{ as } n \rightarrow +\infty, \end{aligned} \right.$$

where  $\eta = \frac{r-a}{a(r-1)}$ .

Since  $qm < r$ , thus by choosing  $a = qm$  we can find as in Step 1,  $\beta < 2s_1 - 1$  such that

$$\int_{\Omega} |\nabla \varphi_n - \nabla \varphi|^q \delta^{\beta} dx \leq C \left( \int_{\Omega} |\nabla \varphi_n - \nabla \varphi|^{qm} \delta^{qm(1-s_1)} dx \right)^{\frac{1}{m}} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Thus, it follows that

$$\|\nabla u_n - \nabla u\|_{L^{\alpha}(\Omega)} \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

for any  $\alpha < \frac{N}{N-2s_1+1+\beta}$ . In particular,  $u_n \rightarrow u$  strongly in  $\mathbb{W}_0^{1,1}(\Omega)$ .

Now going back to (4.13), we deduce that

$$(4.20) \quad \|\nabla u_n \delta^{1-s_1}\|_{L^{\widehat{m}_{s_1,q}}(\Omega)} \leq C \left( \ell^{\frac{1}{p}} + \lambda \|f\|_{L^m(\Omega)} \right).$$

Thus,  $\{|\nabla u_n \delta^{1-s_1}|\}_n$  is bounded in  $L^{\widehat{m}_{s_1,q}}(\Omega)$ .

Now, we claim that

$$(4.21) \quad \{|\nabla u_n \delta^{1-s_2}|\}_n \text{ is bounded in } L^{\gamma}(\Omega) \text{ for any } \gamma < \overline{m}_{s_1,s_2,q}.$$

Indeed, let  $\gamma < \overline{m}_{s_1,s_2,q}$ , then by using the fact that  $\{|\nabla u_n \delta^{1-s_1}|\}_n$  is bounded in  $L^{\widehat{m}_{s_1,q}}(\Omega)$  and Hölder's inequality, we get

$$\int_{\Omega} |\nabla u_n|^{\gamma} \delta^{\gamma(1-s_2)} dx \leq \left( \int_{\Omega} |\nabla u_n|^{\widehat{m}_{s_1,q}} \delta^{\widehat{m}_{s_1,q}(1-s_1)} dx \right)^{\frac{\gamma}{\widehat{m}_{s_1,q}}} \left( \int_{\Omega} \frac{dx}{\delta^{\frac{\widehat{m}_{s_1,q}\gamma(s_2-s_1)}{\widehat{m}_{s_1,q}-\gamma}}} \right)^{1-\frac{\gamma}{\widehat{m}_{s_1,q}}} < +\infty.$$

Here, we have used the fact that  $\gamma < \overline{m}_{s_1,s_2,q}$ . Then, the claim follows. Therefore, by Fatou's Lemma, it follows that

$$\int_{\Omega} |\nabla u|^{\gamma} \delta^{\gamma(1-s_2)} dx < +\infty.$$

Let  $\gamma < \overline{m}_{s_1,s_2,q}$  be fixed such that  $p\sigma < \gamma$ , then by using Proposition 2.2 we get

$$(4.22) \quad \left\{ \begin{aligned} \left( \int_{\Omega} |\nabla u_n - \nabla u|^{p\sigma} \delta^{p\sigma(1-s_2)} dx \right)^{\frac{1}{p\sigma}} &\leq \left( \int_{\Omega} |\nabla u_n - \nabla u| \delta^{1-s_2} dx \right)^{\eta_1} \\ &\times \left( \int_{\Omega} |\nabla u_n - \nabla u|^{\gamma} \delta^{\gamma(1-s_2)} dx \right)^{\frac{1-\eta_1}{\gamma}}, \\ &\leq C \left( \int_{\Omega} |\nabla u_n - \nabla u| \delta^{1-s_2} dx \right)^{\eta_1} \rightarrow 0 \text{ as } n \rightarrow +\infty, \end{aligned} \right.$$

where  $\eta_1 = \frac{\gamma-p\sigma}{p\sigma(\gamma-1)}$ .

Since  $p\sigma < \overline{m}_{s_1,s_2,q}$ , then by repeating the same computations as in the first step, we can find  $\beta_1 < 2s_2 - 1$  such that

$$\int_{\Omega} |\nabla u_n - \nabla u|^p \delta^{\beta_1} dx \leq C \left( \int_{\Omega} |\nabla u_n - \nabla u|^{p\sigma} \delta^{p\sigma(1-s_2)} dx \right)^{\frac{1}{\sigma}} \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Hence, by using Proposition 2.1, we deduce that  $v_n \rightarrow v$  strongly in  $\mathbb{W}_0^{1,1}(\Omega)$  and then the continuity of  $T$  follows.

• **Compactness**

– Let  $\{\varphi_n\}_n \subset H$  be such that  $\|\varphi_n\|_{\mathbb{W}_0^{1,1}(\Omega)} \leq C$  and  $v_n = T(\varphi_n)$ . Since  $\{\varphi_n\}_n \subset H$ , then

$$\left( \int_{\Omega} |\nabla \varphi_n|^r \delta^{r(1-s_1)} dx \right)^{\frac{1}{r}} \leq C \text{ for any } n.$$

Thus, as in the first step, there exists  $\beta < 2s_1 - 1$  such that  $\left\{ |\nabla \varphi_n|^{q\delta^\beta} + \lambda f \right\}_n$  is bounded in  $L^1(\Omega)$ . Then, according to Proposition 2.1, we deduce that, up to a subsequence,  $u_n \rightarrow u$  strongly in  $\mathbb{W}_0^{1,\alpha}(\Omega)$  for any  $\alpha < \frac{N}{N - 2s_1 + 1 + \beta}$ .

Recall that, from (4.22), we have proven that  $|\nabla u_n| \delta^{1-s_2} \rightarrow |\nabla u| \delta^{1-s_2}$  strongly in  $L^{p\sigma}(\Omega)$ . Thus by using the same arguments as in the first and second steps, we get the existence of  $\beta_1 < 2s_2 - 1$  such that  $(|\nabla u_n|^p + \mu g) \delta^{\beta_1} \rightarrow (|\nabla u|^p + \mu g) \delta^{\beta_1}$  strongly  $L^1(\Omega)$ . Thanks to Proposition 2.1, we conclude that  $v_n \rightarrow v$  strongly in  $\mathbb{W}_0^{1,\alpha_1}(\Omega)$  for any  $\alpha_1 < \frac{N}{N - 2s_2 + 1 + \beta}$ . In particular,  $v_n \rightarrow v$  strongly in  $\mathbb{W}_0^{1,1}(\Omega)$ . Hence,  $T$  is compact.

Therefore, by applying Schauder fixed point Theorem, the operator  $T$  admits at least one fixed point  $v \in H$  such that  $T(v) = v$ . Hence, we get the existence of  $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$  which is the solution of System (S) such that  $(u, v) \in \mathbb{W}_0^{1,\theta_1}(\Omega, \delta^{1-s_1}) \times \mathbb{W}_0^{1,\theta_2}(\Omega, \delta^{1-s_2})$  for any  $\theta_1 < \bar{m}_{s_1, s_2, q}$  and  $\theta_2 < \hat{\sigma}_{s_1, s_2, p}$ .

Now, we briefly describe the main changes regarding the set (4.3).

**Second case :** The set (4.3).

The main difference between the two sets is revealed when we show that for every  $\varphi \in H$ , there exists  $\beta < 2s_1 - 1$  and  $\beta_1 < 2s_2 - 1$  such that  $|\nabla \varphi|^q \delta^{\beta_1}, |\nabla u|^p \delta^\beta, |\nabla \varphi|^q \delta^\beta, |\nabla u|^p \delta^{\beta_1} \in L^1(\Omega)$ .

Since  $s_1 < \frac{q+1}{q+2}$  and  $m > \frac{1}{2s_1 - q(1-s_1)}$ , it follows that  $(q+1-s_1(q+2))m' < 1$ . Then, we can

find  $\beta < 2s_1 - 1$  such that  $(q(1-s_1) - \beta)m' < 1$ . Hence,  $\int_{\Omega} \delta^{(\beta - q(1-s_1))m'} dx < \infty$  (respectively

$$\int_{\Omega} \delta^{(\beta_1 - p(1-s_2))\sigma'} dx < +\infty).$$

□

**4.1.2 Case:  $p = q = 1$**

Notice that, the above convexity argument can not be used to get the existence of a solution. Then, we need to find a priori estimate to the solution to the following System

$$(4.23) \quad \begin{cases} (-\Delta)^{s_1} u = \lambda (|\nabla v| + f) & \text{in } \Omega, \\ (-\Delta)^{s_2} v = \mu (|\nabla u| + g) & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

By using a suitable compactness result, we get the following existence result for this case.

**Theorem 4.2.** *Assume that  $s_1, s_2 \in (\frac{1}{2}, 1)$  with  $s_2 > s_1$ . Suppose that  $(f, g) \in L^m(\Omega)^+ \times L^\sigma(\Omega)^+$  where  $(m, \sigma)$  satisfies one of the following conditions:*

$$(4.24) \quad \begin{cases} \frac{2}{3} < s_1 < 1, \frac{2}{3} < s_2 < 1, \\ 1 < m < \min \left\{ \frac{N}{2s_1 - 1}, \hat{\sigma}_{s_2} \right\}, 1 < \sigma < \min \left\{ \frac{N}{2s_2 - 1}, \frac{\hat{m}_{s_1, 1}}{1 + \hat{m}_{s_1, 1}(s_2 - s_1)} \right\}, \\ \text{where } \hat{m}_{s_1, 1} := \frac{mN}{N - m(2s_1 - 1)} \text{ and } \hat{\sigma}_{s_2} := \frac{\sigma N}{N - \sigma(2s_2 - 1)}, \end{cases}$$

or

$$(4.25) \quad \left\{ \begin{array}{l} \frac{1}{2} < s_1 < \frac{2}{3}, \quad \frac{1}{2} < s_2 < \frac{2}{3}, \\ \frac{1}{3s_1 - 1} < m < \min \left\{ \frac{N}{2s_1 - 1}, \widehat{\sigma}_{s_2} \right\}, \quad \frac{1}{3s_2 - 1} < \sigma < \min \left\{ \frac{N}{2s_2 - 1}, \frac{\widehat{m}_{s_1,1}}{1 + \widehat{m}_{s_1,1}(s_2 - s_1)} \right\}, \end{array} \right.$$

or

$$(4.26) \quad \left\{ \begin{array}{l} \frac{1}{2} < s_1 < \frac{2}{3}, \quad \frac{2}{3} < s_2 < 1, \\ \frac{1}{3s_1 - 1} < m < \min \left\{ \frac{N}{2s_1 - 1}, \widehat{\sigma}_{s_2} \right\}, \quad 1 < \sigma < \min \left\{ \frac{N}{2s_2 - 1}, \frac{\widehat{m}_{s_1,1}}{1 + \widehat{m}_{s_1,1}(s_2 - s_1)} \right\}, \end{array} \right.$$

or

$$(4.27) \quad m \geq \frac{N}{2s_1 - 1}, \quad \sigma > \frac{mN}{N + m(2s_2 - 1)} \text{ and } \sigma < \frac{1}{s_2 - s_1},$$

or

$$(4.28) \quad \sigma \geq \frac{N}{2s_2 - 1} \text{ and } \sigma < \frac{\widehat{m}_{s_1,1}}{1 + \widehat{m}_{s_1,1}(s_2 - s_1)}.$$

Then, there exist  $\lambda^* > 0$  and  $\mu^* > 0$  such that if  $\lambda < \lambda^*$  and  $\mu < \mu^*$ , System (4.23) has a solution  $(u, v)$  with  $(u, v) \in \mathbb{W}_0^{1,\theta_1}(\Omega, \delta^{1-s_1}) \times \mathbb{W}_0^{1,\theta_2}(\Omega, \delta^{1-s_2})$  for any  $\theta_1 < \overline{m}_{s_1, s_2}$  and  $\theta_2 < \widehat{\sigma}_{s_2}$ .

*Proof.* We give the proof under the assumption (4.24). The others are similar.

By using Schauder's fixed point Theorem. We get the existence of a solution  $(u_n, v_n)$  to the following System

$$(4.29) \quad \left\{ \begin{array}{l} (-\Delta)^{s_1} u_n = \lambda \left( \frac{|\nabla v_n|}{1 + \frac{1}{n}|\nabla v_n|} + \frac{f}{1 + \frac{1}{n}f} \right) \quad \text{in } \Omega, \\ (-\Delta)^{s_2} v_n = \mu \left( \frac{|\nabla u_n|}{1 + \frac{1}{n}|\nabla u_n|} + \frac{g}{1 + \frac{1}{n}g} \right) \quad \text{in } \Omega, \\ u_n = v_n = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{array} \right.$$

Let us

$$H_{1n}(x) = \frac{|\nabla v_n|}{1 + \frac{1}{n}|\nabla v_n|} + \frac{f}{1 + \frac{1}{n}f} \quad \text{and} \quad H_{2n}(x) = \frac{|\nabla u_n|}{1 + \frac{1}{n}|\nabla u_n|} + \frac{g}{1 + \frac{1}{n}g}.$$

By using the regularity result in Theorem 2.2 with  $a = 1$  and  $s = s_1$ , it follows that

$$(4.30) \quad \left\| |\nabla u_n| \delta^{1-s_1} \right\|_{L^{\theta_1}(\Omega)} \leq C \lambda \left( \left\| |\nabla v_n| \delta^{1-s_1} \right\|_{L^m(\Omega)} + \|f\|_{L^m(\Omega)} \right) \quad \forall \theta_1 \leq \frac{mN}{N - m(2s_1 - 1)}.$$

Since  $\sigma < \overline{m}_{s_1, s_2}$ , then as in the proof of Theorem 4.1, it holds that

$$(4.31) \quad \left\| |\nabla v_n| \delta^{1-s_2} \right\|_{L^{\theta_2}(\Omega)} \leq C \mu \left( \left\| |\nabla u_n| \delta^{1-s_1} \right\|_{L^\sigma(\Omega)} + \|g\|_{L^\sigma(\Omega)} \right) \quad \forall \theta_2 \leq \frac{\sigma N}{N - \sigma(2s_2 - 1)}.$$

Thus,

$$(4.32) \quad \left\{ \begin{array}{l} \left\| |\nabla u_n| \delta^{1-s_1} \right\|_{L^{\theta_1}(\Omega)} + \left\| |\nabla v_n| \delta^{1-s_2} \right\|_{L^{\theta_2}(\Omega)} \leq \\ C \left( \mu \left\| |\nabla u_n| \delta^{1-s_1} \right\|_{L^\sigma(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)} + \lambda \left\| |\nabla v_n| \delta^{1-s_1} \right\|_{L^m(\Omega)} + \lambda \|f\|_{L^m(\Omega)} \right). \end{array} \right.$$

Since  $\sigma < \overline{m}_{s_1, s_2}$ , we can choose  $\theta_1$  close to  $\overline{m}_{s_1, s_2}$  such that  $\sigma < \theta_1 < \overline{m}_{s_1, s_2}$ . Thus using Hölder's inequality, we deduce that

$$\left\| |\nabla u_n| \delta^{1-s_1} \right\|_{L^\sigma(\Omega)} \leq C \left\| |\nabla u_n| \delta^{1-s_1} \right\|_{L^{\theta_1}(\Omega)}.$$

In the same way, by using the fact that  $s_2 > s_1$ , we obtain

$$\| |\nabla v_n| \delta^{1-s_1} \|_{L^m(\Omega)} \leq C \| |\nabla v_n| \delta^{1-s_2} \|_{L^m(\Omega)} \leq C \| |\nabla v_n| \delta^{1-s_2} \|_{L^{\theta_2}(\Omega)}.$$

Hence, by choosing  $\max\{\lambda, \mu\} < C_0$  such that  $C_0$  is small enough, it follows that

$$\| |\nabla u_n| \delta^{1-s_1} \|_{L^{\theta_1}(\Omega)} + \| |\nabla v_n| \delta^{1-s_2} \|_{L^{\theta_2}(\Omega)} \leq C(\Omega, C_0) \left( \lambda \|f\|_{L^m(\Omega)} + \mu \|g\|_{L^\sigma(\Omega)} \right).$$

As a consequence,  $\{(u_n, v_n)\}_n$  is bounded in  $\mathbb{W}_0^{1, \theta_1}(\Omega, \delta^{\theta_1(1-s_1)}) \times \mathbb{W}_0^{1, \theta_2}(\Omega, \delta^{\theta_2(1-s_2)})$  for any for any  $\theta_1 < \bar{m}_{s_1, s_2}$  and  $\theta_2 < \hat{\sigma}_{s_2, p}$ .

Thus, the rest of the proof follows by using the same compactness computations as in the proof of Theorem 4.1.  $\square$

**Remark 4.1.**

1. The same existence result holds without any positivity assumptions on the data.
2. Under the same hypothesis of Theorem 4.1, we can prove that the following System

$$(4.33) \quad \begin{cases} (-\Delta)^{s_1} u + |\nabla v|^q = \lambda f & \text{in } \Omega, \\ (-\Delta)^{s_2} v + |\nabla u|^p = \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits a nontrivial weak solution  $(u, v)$  for  $s_2 > s_1$ .

**4.2 Nonexistence results**

In this subsection, we prove the optimality of the assumptions imposed on the data in Theorem 4.1. We start by the next nonexistence result to prove that the smallness condition on  $(\lambda, \mu)$  is necessary to obtain the existence of a solution to System (S).

**4.2.1 Nonexistence: Optimality of smalness condition on  $\lambda$  and  $\mu$**

**Theorem 4.3.** *Let  $s_1, s_2 \in (\frac{1}{2}, 1)$  with  $s_2 > s_1$ ,  $1 < q < \frac{s_1}{1-s_1}$  and  $1 < p < \frac{s_2}{1-s_1}$ . Assume that  $(f, g) \in (L^\infty(\Omega)^+)^2$ . Then, there exist two positive constants  $\Gamma(f)$  and  $\Gamma(g)$  such that if  $\lambda > \Gamma(f)$  or  $\mu > \Gamma(g)$ , System (S) does not have any nonnegative solution.*

*Proof.* Let  $\lambda, \mu > 0$  be fixed such that System (S) has a nonnegative solution  $(u, v)$ . Given  $\theta \in C_0^\infty(\Omega)$  with  $\theta \geq 0$ . Let  $\phi_\theta \in \mathbb{W}_0^{s_1, 2}(\Omega) \cap L^\infty(\Omega)$  and  $\varphi_\theta \in \mathbb{W}_0^{s_2, 2}(\Omega) \cap L^\infty(\Omega)$  be the unique positive solutions to the following Problems respectively

$$(4.34) \quad \begin{cases} (-\Delta)^{s_1} \phi_\theta = \theta & \text{in } \Omega, \\ \phi_\theta = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and

$$(4.35) \quad \begin{cases} (-\Delta)^{s_2} \varphi_\theta = \theta & \text{in } \Omega, \\ \varphi_\theta = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Since  $\theta$  is bounded, and according to [53], we know that  $\phi_\theta \asymp \delta^{s_1}$  and  $\varphi_\theta \asymp \delta^{s_2}$ . Let  $\psi_\theta$  be the unique solution to the following quasi-linear Problem

$$(4.36) \quad \begin{cases} -\operatorname{div}(\varphi_\theta |\nabla \psi_\theta|^{\alpha-2} \nabla \psi_\theta) = \theta & \text{in } \Omega, \\ \psi_\theta = 0 & \text{on } \partial\Omega, \end{cases}$$

where  $1 < \alpha \leq \min\{p, q\}$ . Notice that the existence of  $\psi_\theta$  follows by using an approximating argument. It is clear that  $(\alpha - 1)p' \leq \alpha$  and  $(\alpha - 1)q' \leq \alpha$ .

Now, we use  $\phi_\theta$  as a test function in the first equation of System (S), we get

$$(4.37) \quad \begin{cases} \int_{\Omega} (|\nabla v|^q + \lambda f) \phi_\theta dx &= \int_{\Omega} \theta u dx, \\ &\leq \int_{\Omega} \varphi_\theta |\nabla \psi_\theta|^{\alpha-1} |\nabla u| dx. \end{cases}$$

Therefore, by using Hölder's and Young's inequalities, it follows that

$$(4.38) \quad \begin{cases} \int_{\Omega} (|\nabla v|^q + \lambda f) \phi_\theta dx &\leq \left( \int_{\Omega} |\nabla u|^p \varphi_\theta dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \varphi_\theta dx \right)^{\frac{1}{p'}}, \\ &\leq \varepsilon \int_{\Omega} |\nabla u|^p \varphi_\theta dx + C'(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \varphi_\theta dx. \end{cases}$$

Analogously, using  $\varphi_\theta$  as a test function in the second equation of System (S), we get

$$(4.39) \quad \int_{\Omega} (|\nabla u|^p + \mu g) \varphi_\theta dx \leq \varepsilon \int_{\Omega} |\nabla v|^q \varphi_\theta dx + C'_2(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{q'(\alpha-1)} \varphi_\theta dx.$$

Combining (4.38) and (4.39), it holds that

$$(4.40) \quad \begin{cases} \int_{\Omega} (|\nabla v|^q + \lambda f) \phi_\theta dx &\leq \varepsilon^2 \int_{\Omega} |\nabla v|^q \varphi_\theta dx + C_1(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{q'(\alpha-1)} \varphi_\theta dx \\ &+ C'(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \varphi_\theta dx. \end{cases}$$

By using the fact that,  $\varphi_\theta \simeq \delta^{s_2}$  and  $\phi_\theta \simeq \delta^{s_1}$  in the previous inequality, we obtain

$$(4.41) \quad \begin{cases} \int_{\Omega} (|\nabla v|^q + \lambda f) \delta^{s_1}(x) dx &\leq \varepsilon^2 C \int_{\Omega} |\nabla v|^q \delta^{s_2}(x) dx + \overline{C}_1(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{q'(\alpha-1)} \delta^{s_2}(x) dx \\ &+ \overline{C}_2(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \delta^{s_2}(x) dx. \end{cases}$$

Since  $\Omega$  is bounded and  $s_2 > s_1$ , then  $\delta^{s_2}(x) \leq C(\Omega) \delta^{s_1}(x)$  for any  $x \in \Omega$ .

Thus, from the inequality (4.41), it follows that

$$(4.42) \quad \begin{cases} \int_{\Omega} (|\nabla v|^q + \lambda f) \delta^{s_1}(x) dx &\leq \varepsilon^2 C \int_{\Omega} |\nabla v|^q \delta^{s_1}(x) dx + \overline{C}_1(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{q'(\alpha-1)} \delta^{s_2}(x) dx \\ &+ \overline{C}_2(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \delta^{s_2}(x) dx. \end{cases}$$

Hence, by choosing  $0 < \varepsilon \ll \sqrt{\frac{1}{C}}$ , we get

$$(4.43) \quad \lambda \int_{\Omega} f \delta^{s_1} dx \leq \overline{C}_1(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \delta^{s_2}(x) dx + \overline{C}_2(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{q'(\alpha-1)} \delta^{s_2}(x) dx.$$

Otherwise, we obtain

$$(4.44) \quad \begin{cases} \lambda \int_{\Omega} f \phi_\theta dx &\leq C_3(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \varphi_\theta dx + C_4(\varepsilon) \int_{\Omega} |\nabla \psi_\theta|^{q'(\alpha-1)} \varphi_\theta dx \\ &\leq C(\varepsilon) \left( \int_{\Omega} |\nabla \psi_\theta|^{p'(\alpha-1)} \varphi_\theta dx + \int_{\Omega} |\nabla \psi_\theta|^{q'(\alpha-1)} \varphi_\theta dx \right), \end{cases}$$

where  $C(\varepsilon) = \max\{C_3(\varepsilon), C_4(\varepsilon)\}$ .

Let us set

$$(4.45) \quad F(\theta) := \int_{\Omega} |\nabla \psi_{\theta}|^{p'(\alpha-1)} \varphi_{\theta} dx + \int_{\Omega} |\nabla \psi_{\theta}|^{q'(\alpha-1)} \varphi_{\theta} dx.$$

Then,  $\lambda \leq \Gamma(f) = C(\varepsilon)\lambda^*(f)$  where

$$\lambda^*(f) := \inf_{\{0 \leq \theta \in C_0^{\infty}(\Omega)\}} \left\{ F(\theta) ; \int_{\Omega} f \phi_{\theta} dx = 1 \right\}.$$

Conversely, if  $\lambda > \Gamma(f)$ , System (S) does not have any nonnegative solution. By a similar way, we get the result for  $\mu$ . □

#### 4.2.2 Nonexistence: Optimality of smallness conditions on the exponents $p$ and $q$

Now, we give the optimality result of the assumption on the exponents  $q$  and  $p$ . More precisely, we have the following nonexistence result.

**Theorem 4.4.**

1) Assume that  $\max\{p, q\} \geq \frac{1+s_2}{1-s_1}$  with  $s_2 > s_1$ . Then, System (S) does not have any nonnegative solution  $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$  such that

$$\int_{\Omega} |\nabla u|^p \delta^{s_2}(x) dx < +\infty \quad \text{and} \quad \int_{\Omega} |\nabla v|^q \delta^{s_1}(x) dx < +\infty.$$

2) Assume that  $\max\{p, q\} \geq \frac{1}{1-s_1}$ . Then, System (S) does not have any nonnegative solution  $(u, v) \in \mathbb{W}_0^{1,p}(\Omega) \times \mathbb{W}_0^{1,q}(\Omega)$ .

*Proof.*

**First case :** Let us start by proving the first case. Without loss of generality, we can suppose that  $p \geq q$  and  $p \geq \frac{1+s_2}{1-s_1}$ . Assume by contradiction that System (S) has a solution  $(u, v)$  such that  $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$  and

$$\int_{\Omega} |\nabla u|^p \delta^{s_2}(x) dx < +\infty \quad \text{and} \quad \int_{\Omega} |\nabla v|^q \delta^{s_1}(x) dx < +\infty.$$

Since  $s_2 < p-1$ , by using the weighted Hardy's inequality (Theorem 2.3) with  $\rho = \delta^{s_2}$ , we obtain

$$(4.46) \quad \int_{\Omega} \frac{u^p}{\delta^p} \delta^{s_2}(x) dx \leq \int_{\Omega} |\nabla u|^p \delta^{s_2}(x) dx < +\infty.$$

From the maximum principle proved in [11], we have  $u \geq C\delta^{s_1}$ . Hence,

$$\int_{\Omega} \frac{1}{\delta^{p(1-s_1)-s_2}} dx < +\infty.$$

Thus,  $p < \frac{1+s_2}{1-s_1}$  which is contradiction.

**Second case :** Now, we prove the second case. Suppose that System (S) has a nonnegative solution  $(u, v)$  with  $(u, v) \in \mathbb{W}_0^{1,p}(\Omega) \times \mathbb{W}_0^{1,q}(\Omega)$ . Then, by the same reasoning as in first case, by using Hardy's inequality (Theorem 2.3 with  $\rho = 0$ ), it follows that

$$\int_{\Omega} \frac{u^p}{\delta^p} dx \leq \int_{\Omega} |\nabla u|^p dx < +\infty.$$

Furthermore, by using the fact that  $u \geq C\delta^{s_1}$  it holds that

$$\int_{\Omega} \frac{1}{\delta^{p(1-s_1)}} dx < +\infty.$$

Hence,  $p < \frac{1}{1-s_1}$  which is a contradiction. This completes the proof. □



### 4.2.3 Nonexistence: Optimality of the integrability conditions on $f$ and $g$

Before ending this subsection, we prove an optimality result related to the assumption on  $f$  and  $g$  in some particular cases. More precisely, we have the following result.

**Theorem 4.5.** *Let  $s_1, s_2 \in (\frac{1}{2}, 1)$  with  $s_2 > s_1$ . Assume that the following condition*

$$(4.47) \quad p\sigma > \frac{mN}{N - m(2s_1 - 1)} \text{ and } qm > \frac{\sigma N}{N - \sigma(2s_2 - 1)}$$

*holds. Then, there exist  $(m, \sigma)$  and  $f \in L^m(\Omega), g \in L^\sigma(\Omega)$  with  $f, g \geq 0$  such that System (S) does not have any positive solution for any  $\lambda, \mu > 0$ .*

*Proof.* Let us construct  $f, g$  such that, under the hypotheses of Theorem 4.3, we get  $\Gamma(f) = 0$  or  $\Gamma(g) = 0$ .

Without loss of generalities, we suppose that  $\Omega = B(0, 1)$  and  $m, \sigma < N$ . Let  $\varepsilon > 0$  (it will be chosen later), and define  $f(x) = \frac{1}{|x|^{\frac{N-\varepsilon}{m}}}$ . Then,  $f \in L^m(\Omega)$ . Assume by contradiction that System (S) has a solution  $(u, v) \in \mathbb{W}_0^{1,1}(\Omega) \times \mathbb{W}_0^{1,1}(\Omega)$  such that  $(u, v) \in \mathbb{W}_{loc}^{1,p}(\Omega) \times \mathbb{W}_{loc}^{1,q}(\Omega)$ . Then, by following the calculations done to prove Theorem 4.3, we deduce that

$$(4.48) \quad \lambda \int_{\Omega} f \phi_{\theta} dx \leq C \int_{\Omega} |\nabla \psi_{\theta}|^{p'(\alpha-1)} \phi_{\theta} dx + C \int_{\Omega} |\nabla \psi_{\theta}|^{q'(\alpha-1)} \phi_{\theta} dx.$$

Moreover, let us consider the function  $\theta$  such that

$$\theta(x) = \begin{cases} \frac{1}{|x|^a} & \text{if } |x| \leq \frac{1}{4} \\ (1 - |x|)^{\gamma} & \text{if } \frac{1}{2} \leq |x| \leq 1, \end{cases}$$

where  $\gamma > 1$  and  $a > 2s_1$  satisfying

$$\begin{cases} N + 2s_1 - \frac{N}{m} < a < N + 2s_1 - p'(2s_2 - 1), \\ N + 2s_1 - \frac{N}{m} < a < N + 2s_1 - q'(2s_2 - 1). \end{cases}$$

Notice that the existence of  $a$  follows using the hypothesis (4.47).

By using the definition of  $\phi_{\theta}$  as the solution to Problem (4.34), we obtain that  $\phi_{\theta} \simeq |x|^{2s_1-a}$  in a small  $B(0, r_0)$ .

Regarding the function  $\psi_{\theta}$  which is the solution to Problem (4.36), choosing  $1 < \alpha < 2s_2$ , we can prove that in a small  $B(0, r_1)$ , we have

$$(4.49) \quad \psi_{\theta}(x) \simeq |x|^{-\frac{2s_2-\alpha}{\alpha-1}} \text{ and } |\nabla \psi_{\theta}| \leq C|x|^{-\frac{2s_2-1}{\alpha-1}}.$$

Thus, taking into consideration estimates (4.48) and (4.49) and choosing  $\varepsilon$  small enough such that  $\frac{N-\varepsilon}{m} + a - 2s_1 \geq N$ , we deduce that

$$\int_{\Omega} f \phi_{\theta} dx \geq C \int_{B(0, \frac{1}{5})} \frac{1}{|x|^{\frac{N-\varepsilon}{m} + a - 2s_1}} dx = +\infty.$$

Moreover, we have

$$\begin{cases} \int_{\Omega} |\nabla \psi_{\theta}|^{p'(\alpha-1)} dx + \int_{\Omega} |\nabla \psi_{\theta}|^{q'(\alpha-1)} \phi_{\theta} dx \\ \leq C \int_{\Omega} \left( \frac{1}{|x|^{(2s_2-1)p'+a-2s_2}} + \frac{1}{|x|^{(2s_2-1)q'+a-2s_2}} \right) dx < +\infty. \end{cases}$$

Thus, using a suitable approximation argument, we conclude that  $\Gamma(f) = 0$ . Hence, the nonexistence result follows.  $\square$

## 5 Extensions and further results

In this section, we deal with some extensions of the existence result to other nonlinear systems with different structures. By using the same approach as in the previous section, we can establish existence results for different cases.

### 5.1 Systems with drift

Let us consider the following System

$$(SD) \quad \begin{cases} (-\Delta)^{s_1} u + B_1(x) \cdot \nabla u = |\nabla v|^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^{s_2} v + B_2(x) \cdot \nabla v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where  $s_2 > s_1$  and  $B_i$  is a vector fields satisfies  $|B_i| \in L^{\sigma_i}(\Omega)$  with  $\sigma_i > \frac{N}{2s_i-1}$ . In our previous paper [8], we have explained how for Systems of type (SD), we can establish the existence of solutions.

### 5.2 Systems with potential-gradient term

Now, let us consider the following System with potential-gradient in the right-hand side ;

$$(SPG) \quad \begin{cases} (-\Delta)^{s_1} u = v^q + \lambda f & \text{in } \Omega, \\ (-\Delta)^{s_2} v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then, we have the following existence result.

**Theorem 5.1.** *Let  $s_1, s_2 \in (\frac{1}{2}, 1)$  with  $s_2 > s_1$ ,  $1 \leq q < \frac{s_1}{1-s_1}$  and  $1 \leq p < \frac{s_2}{1-s_1}$  such that  $pq > 1$ . Assume that  $(f, g) \in L^m(\Omega)^+ \times L^\sigma(\Omega)^+$  where  $(m, \sigma) \in [1, +\infty)^2$  satisfies one of the following set of conditions:*

$$(5.1) \quad \begin{cases} \frac{q+1}{q+2} < s_1 < 1, \frac{p+1}{p+2} < s_2 < 1, \\ 1 < m < \frac{N}{s_1 - q(1-s_1)}, \quad 1 < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q} := \frac{\hat{m}_{s_1, q}}{1 + \hat{m}_{s_1, q}(s_2 - s_1)} \text{ where } \hat{m}_{s_1, q} := \frac{mN}{N - m(s_1 - q(1-s_1))}, \\ \frac{qmN}{N + qm} < \hat{\sigma}_{s_1, s_2, p} := \frac{\sigma N}{N - \sigma(2s_2 - s_1 - p(1-s_1))}, \end{cases}$$

or

$$(5.2) \quad \begin{cases} \frac{1}{2} < s_1 < \frac{q+1}{q+2}, \quad \frac{1}{2} < s_2 < \frac{p+1}{p+2}, \\ \frac{1}{2s_1 - q(1-s_1)} < m < \frac{N}{s_1 - q(1-s_1)}, \quad \frac{1}{2s_2 - p(1-s_2)} < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q}, \quad \frac{qmN}{N + qm} < \hat{\sigma}_{s_1, s_2, p}, \end{cases}$$

or

$$(5.3) \quad \begin{cases} \frac{q+1}{q+2} < s_1 < 1, \quad \frac{1}{2} < s_2 < \frac{p+1}{p+2}, \quad q < p \\ 1 < m < \frac{N}{s_1 - q(1-s_1)}, \quad \frac{1}{2s_2 - p(1-s_2)} < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q}, \quad \frac{qmN}{N + qm} < \hat{\sigma}_{s_1, s_2, p}, \end{cases}$$

or

$$(5.4) \quad \begin{cases} \frac{1}{2} < s_1 < \frac{q+1}{q+2}, \frac{p+1}{p+2} < s_2 < 1, \\ \frac{1}{2s_1 - q(1-s_1)} < m < \frac{N}{s_1 - q(1-s_1)}, \quad 1 < \sigma < \frac{N}{2s_2 - s_1 - p(1-s_1)}, \\ p\sigma < \bar{m}_{s_1, s_2, q}, \quad \frac{qmN}{N+qm} < \hat{\sigma}_{s_1, s_2, p}, \end{cases}$$

(5.5)

$$\text{or} \quad m \geq \frac{N}{s_1 - q(1-s_1)}, \quad \sigma > \frac{qmN}{N + qm(2s_2 - s_1 + 1 - p(1-s_1))} \quad \text{and} \quad p\sigma < \frac{1}{s_2 - s_1}$$

(5.6)

$$\text{or} \quad \sigma \geq \frac{N}{2s_2 - s_1 - p(1-s_1)} \quad \text{and} \quad p\sigma < \bar{m}_{s_1, s_2, q}.$$

Then, there exists  $A > 0$  such that for any  $(\lambda, \mu) \in \Pi$  where

$$(5.7) \quad \Pi := \left\{ (\lambda, \mu) \in [0, +\infty) \times [0, +\infty) ; \lambda^p \|f\|_{L^m(\Omega)}^p + \mu \|g\|_{L^\sigma(\Omega)} \leq A \right\},$$

System (SPG) has a nonnegative solution  $(u, v)$ . Moreover,  $(u, v) \in \mathbb{W}_0^{1, \theta_1}(\Omega, \delta^{1-s_1}) \times \mathbb{W}_0^{1, \theta_2}(\Omega, \delta^{1-s_2})$  for any  $\theta_1 < \bar{m}_{s_1, s_2, q}$  and  $\theta_2 < \hat{\sigma}_{s_1, s_2, p}$ .

*Proof.* The proof follows closely the arguments used in the proof of Theorem 4.1 by taking into consideration the potential term in the first equation. For the reader's convenience, we give some details under the condition (5.1). The others are quite similar.

Since  $\frac{qmN}{N+qm} < \hat{\sigma}_{s_1, s_2, p}$ , then we can find  $r > 1$  such that  $\frac{qmN}{N+qm} < r < \hat{\sigma}_{s_1, s_2, p}$ . Therefore, if  $r < N$ , thus we have

$$(5.8) \quad qm < r^* := \frac{rN}{N-r},$$

and if  $r \geq N$ , then it is clear that  $r^* = \infty$ .

Let  $\ell$  and  $r$  be fixed where  $\ell$  satisfies (4.9). Furthermore, let us consider the set  $H$  defined as follows

$$(5.9) \quad H := \left\{ \varphi \in \mathbb{W}_0^{1,1}(\Omega) ; \varphi \delta^{1-s_1} \in \mathbb{W}_0^{1,r}(\Omega) \text{ and } \left( \int_{\Omega} |\nabla(\varphi \delta^{1-s_1})|^r dx \right)^{\frac{1}{r}} \leq \ell^{\frac{1}{pq}} \right\}.$$

Notice that, if  $\varphi \in H$ , then by Sobolev's inequality, it holds that

$$\left( \int_{\Omega} |\varphi \delta^{1-s_1}|^{r^*} dx \right)^{\frac{1}{r^*}} \leq C \left( \int_{\Omega} |\nabla(\varphi \delta^{1-s_1})|^r dx \right)^{\frac{1}{r}} \leq C \ell^{\frac{1}{pq}}.$$

We observe that if  $\varphi \in H$ , then  $\varphi^a \delta^{a(1-s_1)} \in L^1(\Omega)$  for any  $a \leq r^*$ . In particular,  $\varphi^q \delta^{q(1-s_1)} \in L^m(\Omega)$ . Hence, by the same computations as in the first step of the proof of Theorem 4.1, we can prove the existence of  $\beta < 2s_1 - 1$  such that  $\varphi^q \delta^\beta \in L^1(\Omega)$  for every  $\varphi \in H$ .

Thus, from Proposition 2.1, there exists a unique weak solution  $u$  to Problem

$$(5.10) \quad \begin{cases} (-\Delta)^{s_1} u = \varphi_+^q + \lambda f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Moreover,  $|\nabla u| \in L^\alpha(\Omega)$  for any  $\alpha < \frac{N}{N-2s_1+1+\beta}$ .

Now by applying Theorem 2.2, with  $s = s_1$  and  $a = q$ , we have

$$(5.11) \quad \|\nabla u|\delta^{1-s_1}\|_{L^\gamma\Omega} \leq C \left( \|\varphi\delta^{1-s_1}\|_{L^{qm}(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)} \right) \quad \text{for any } \gamma \leq \widehat{m}_{s_1,q},$$

where  $C$  is positive constant. Since  $qm \leq r^*$ , it follows that

$$(5.12) \quad \|\nabla u|\delta^{1-s_1}\|_{L^\gamma(\Omega)} \leq C \left( \|\varphi\delta^{1-s_1}\|_{L^{r^*}(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)} \right)$$

for every  $\gamma \leq \widehat{m}_{s_1,q}$ . Thus, by Sobolev's inequality, we get

$$(5.13) \quad \|\nabla u|\delta^{1-s_1}\|_{L^\gamma(\Omega)} \leq C \left( \|\nabla(\varphi\delta^{1-s_1})\|_{L^r(\Omega)}^q + \lambda\|f\|_{L^m(\Omega)} \right) \quad \text{for any } \gamma \leq \widehat{m}_{s_1,q}.$$

Since  $p\sigma < \overline{m}_{s_1,s_2,q} < \widehat{m}_{s_1,q}$ , then by (5.13), it holds that  $|\nabla u|^p\delta^{p(1-s_2)} \in L^\sigma(\Omega)$ .

Moreover,  $p\sigma < \overline{m}_{s_1,s_2,q}$  and  $\frac{p+1}{p+2} < s_2$ . Thus, by using Hölder's inequality, we get the existence of  $\beta_1 < 2s_2 - 1$  such that  $|\nabla u|^p\delta^{\beta_1} \in L^1(\Omega)$ .

Therefore by Proposition 4.1, we get the existence of  $v$  as a unique weak solution to Problem

$$(5.14) \quad \begin{cases} (-\Delta)^{s_2}v = |\nabla u|^p + \mu g & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

such that  $v \in \mathbb{W}_0^{1,\alpha_1}(\Omega)$  for any  $\alpha_1 < \frac{N}{N-2s_2+1+\beta_1}$  with  $\beta_1 < 2s_2 - 1$ .

Hence, the operator

$$\begin{aligned} T : H &\longrightarrow \mathbb{W}_0^{1,1}(\Omega) \\ \varphi &\longmapsto T(\varphi) = v \end{aligned}$$

is well defined. Moreover, if  $v$  is a fixed point of  $T$ , then  $(u, v)$  is a weak solution to System (SPG).

Now, by using Theorem 3.2 with  $h = |\nabla u|^p + \mu g$ , we get

$$(5.15) \quad \|\nabla v|\delta^{1-s_1}\|_{L^\rho(\Omega)} \leq C \left( \|\nabla u|\delta^{1-s_1}\|_{L^{p\sigma}(\Omega)}^p + \mu\|g\|_{L^\sigma(\Omega)} \right) \quad \text{for any } \rho \leq \widehat{\sigma}_{s_1,s_2,p}.$$

Since  $p\sigma < \overline{m}_{s_1,s_2,q}$  and  $qm < r^*$ , then by using (5.13) and (5.15), we get

$$(5.16) \quad \|\nabla v|\delta^{1-s_1}\|_{L^\rho(\Omega)} \leq C \left( \|\nabla\varphi|\delta^{1-s_1}\|_{L^r(\Omega)}^{pq} + \mu\|g\|_{L^\sigma(\Omega)} + \lambda^p\|f\|_{L^m(\Omega)}^p \right).$$

Since  $r < \widehat{\sigma}_{s_1,s_2,p}$ , by choosing  $\rho = r$  in the previous inequality it holds that

$$(5.17) \quad \|\nabla v|\delta^{1-s_1}\|_{L^r(\Omega)} \leq C \left( \|\nabla\varphi|\delta^{1-s_1}\|_{L^r(\Omega)}^{pq} + \mu\|g\|_{L^\sigma(\Omega)} + \lambda^p\|f\|_{L^m(\Omega)}^p \right).$$

Recall that  $\varphi \in H$ . Thus,

$$\|\nabla v|\delta^{1-s_1}\|_{L^r(\Omega)} \leq C \left( \ell + \mu\|g\|_{L^\sigma(\Omega)} + \lambda^p\|f\|_{L^m(\Omega)}^p \right).$$

By choosing  $\tilde{C} = C$  and taking into consideration the definition of  $\ell$ , we conclude that

$$\|\nabla v|\delta^{1-s_1}\|_{L^r(\Omega)} \leq C\ell^{\frac{1}{pq}}.$$

Therefore,  $v \in H$  and then  $T(H) \subset H$ .

Thus, we conclude that  $T$  has a fixed point in  $H$ . So, System (SPG) has a solution  $(u, v)$  such that  $(u, v) \in \mathbb{W}_0^{1,\theta_1}(\Omega, \delta^{1-s_1}) \times \mathbb{W}_0^{1,\theta_2}(\Omega, \delta^{1-s_2})$  for any  $\theta_1 < \overline{m}_{s_1,s_2,q}$  and  $\theta_2 < \widehat{\sigma}_{s_1,s_2,p}$ .  $\square$

To end this section, we prove that the smallness condition on the parameters  $\lambda, \mu$  is necessary to guarantee the existence of a nonnegative solution to System (SPG). To be more precise, we have the following nonexistence result.

**Theorem 5.2.** *Assume that  $s_1, s_2 \in (\frac{1}{2}, 1)$  with  $s_2 > s_1$ ,  $p > 1$  and  $q \geq 1$ . For  $\theta \in C_0^\infty(\Omega)$  with  $\theta \not\equiv 0$ , let us consider  $\phi_\theta \in \mathbb{W}_0^{s_1, 2}(\Omega) \cap L^\infty(\Omega)$ ,  $\varphi_\theta \in \mathbb{W}_0^{s_2, 2}(\Omega) \cap L^\infty(\Omega)$  and  $\psi_\theta \in \mathbb{W}_0^{1, \alpha}(\Omega, \delta^{s_2}(x))$  to be the solutions to Problems (4.34)-(4.36) respectively.*

Moreover, let us set

$$H(\theta) := \left( \int_{\Omega} |\nabla \psi_\theta|^\alpha \varphi_\theta dx \right)^{\frac{p'(\alpha-1)}{\alpha}} + \int_{\Omega} \theta^{q'} \phi_\theta^{1-q'} dx,$$

and define

$$\Lambda^*(f) := \inf_{\{0 \leq \theta \in C_0^\infty(\Omega)\}} \left\{ H(\theta) ; \int_{\Omega} f \phi_\theta dx = 1 \right\},$$

and

$$\Gamma^*(g) := \inf_{\{0 \leq \theta \in C_0^\infty(\Omega)\}} \left\{ H(\theta) ; \int_{\Omega} g \varphi_\theta dx = 1 \right\}.$$

Then, there exists  $C(\varepsilon)$  such that if System (SPG) has a solution  $(u, v)$ , then  $\lambda \leq C(\varepsilon)\Lambda^*(f)$  and  $\mu \leq C(\varepsilon)\Gamma^*(g)$ .

*Proof.* Without loss of generality, we assume that  $f, g \in L^\infty(\Omega)$ . By using  $\phi_\theta$  as a test function in the first equation of System (SPG), we get

$$(5.18) \quad \begin{cases} \int_{\Omega} (v^q + \lambda f) \phi_\theta dx = \left| \int_{\Omega} \theta u dx \right|, \\ \leq \int_{\Omega} \varphi_\theta |\nabla \psi_\theta|^{\alpha-1} |\nabla u| dx. \end{cases}$$

Since  $p'(\alpha - 1) < \alpha$ , then by Hölder's and Young's inequalities we have that

$$(5.19) \quad \int_{\Omega} (v^q + \lambda f) \phi_\theta dx \leq \varepsilon \int_{\Omega} |\nabla u|^p \varphi_\theta dx + C_1(\varepsilon) \left( \int_{\Omega} |\nabla \psi_\theta|^\alpha \varphi_\theta dx \right)^{\frac{p'(\alpha-1)}{\alpha}}.$$

Now, by using  $\varphi_\theta$  as a test function in the second equation of System (SPG) and by applying Young's inequality, we deduce that

$$(5.20) \quad \begin{aligned} \int_{\Omega} (|\nabla u|^p + \mu g) \varphi_\theta dx &\leq \int_{\Omega} v \theta dx \leq \int_{\Omega} v \phi_\theta \frac{\theta}{\phi_\theta} dx \\ &\leq \varepsilon \int_{\Omega} v^q \phi_\theta dx + C_2(\varepsilon) \int_{\Omega} \theta^{q'} \phi_\theta^{1-q'} dx. \end{aligned}$$

The last integral is finite using the fact that  $0 \leq \theta \in C_c^\infty(\Omega)$  and  $\phi_\theta \simeq \delta^{s_2}(x)$ .

Combining (5.19) and (5.20), we get

$$(5.21) \quad \begin{aligned} \lambda \int_{\Omega} f \phi_\theta dx &\leq C_1(\varepsilon) \left( \int_{\Omega} |\nabla \psi_\theta|^\alpha \varphi_\theta dx \right)^{\frac{p'(\alpha-1)}{\alpha}} + C_3(\varepsilon) \int_{\Omega} \theta^{q'} \phi_\theta^{1-q'} dx, \\ &\leq C(\varepsilon) H(\theta) \end{aligned}$$

for some  $0 < \varepsilon \ll 1$ .

Hence,  $\lambda \leq C(\varepsilon)\Lambda^*(f)$  with

$$\Lambda^*(f) := \inf_{\{0 \leq \theta \in C_0^\infty(\Omega)\}} \left\{ H(\theta) ; \int_{\Omega} f \phi_\theta dx = 1 \right\}.$$

In the same way, we get the estimate on  $\mu$ . □

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