

Nonlinear kinetics: from isolated hole-clump pairs to phase-space turbulence

M. Lesur

Collaborators: A. Guillevic, X. Garbet, P.H. Diamond, G. Lo-Cascio, Y. Kosuga, E. Gravier, D. Mandal, A. Ghizzo, T. Réveillé.

31st International Toki Conference on Plasma and Fusion Research

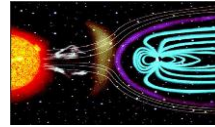
November 8th, 2022



Motivation: essential role of phase-space structures

Vortices in phase-space are observed in experiments

- Space plasmas
- Laboratory linear plasmas
- Magnetic reconnection of toroidal plasmas
- Fusion plasmas
- Laser plasmas



Review: Eliasson & Shukla '06



Saeki '79

Fox '08

Kusama '99 ; Berk, Breizman & Pekker '96

Sarri '10

Deep implications for instabilities, turbulence, transport, heating

- Drive nonlinear instabilities
- Modify the magnitude of saturation, spectrum of turbulence
- Qualitative effect on transport
- Interact with large-scale flows
- Propagation of trapped turbulence

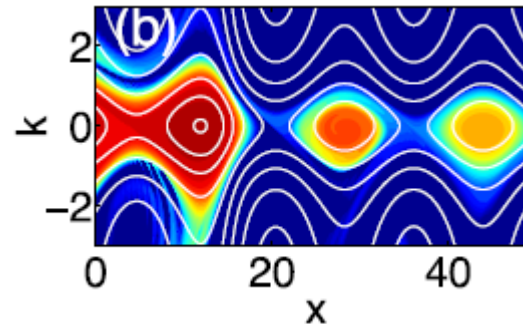
Dupree '82

Terry '90

Biglari '88

Kosuga '11


Sasaki '17



Many open questions remain

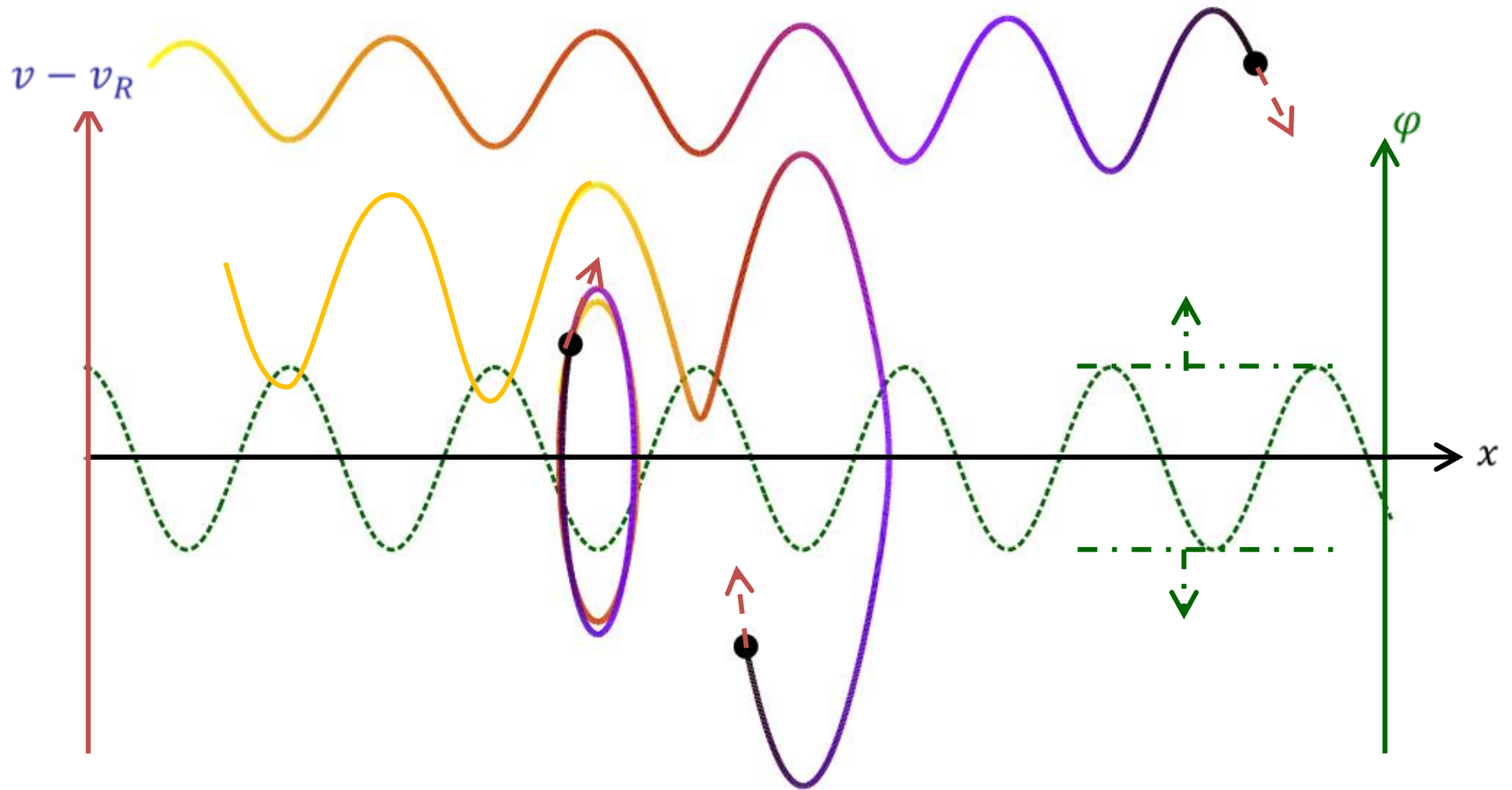
- Need for more detailed analysis in numerical simulations
- Need for more detailed measurements in fusion plasmas

Outline

- 
- I. Phase-space vortices
 - II. Energetic-particle driven modes
 - III. Ion-acoustic turbulence in **homogeneous** plasma
 - IV. Drift-wave turbulence in **inhomogeneous** plasma

Summary

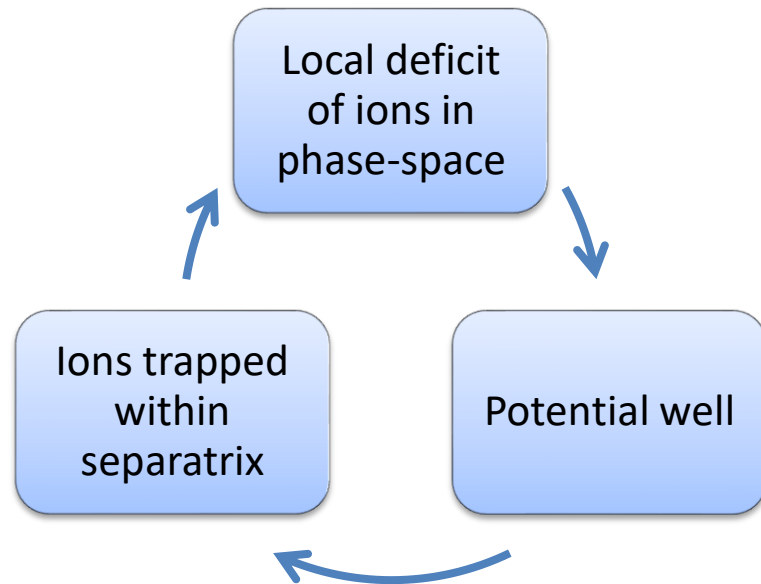
Electrostatic trapping



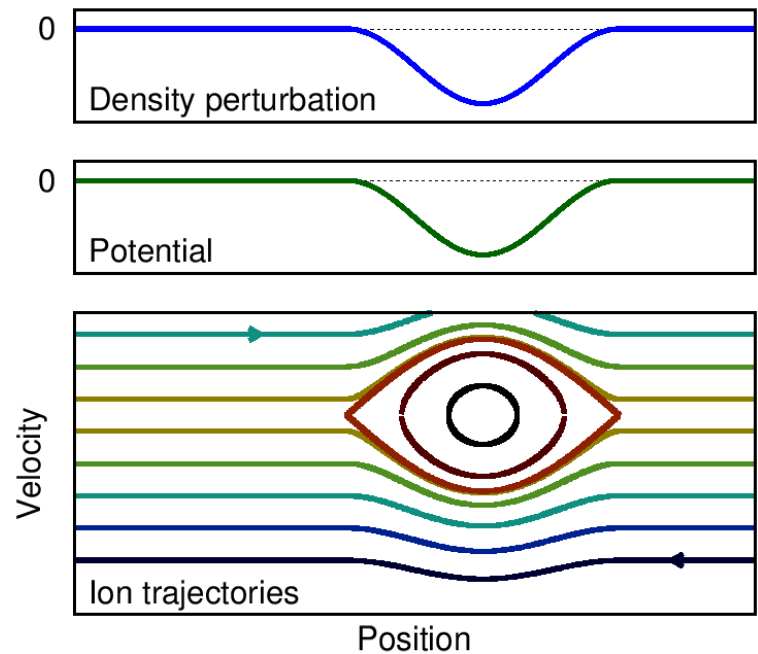
⇒ An electrostatic wave can grow by trapping “barely passing” particles

Electrostatic trapping yields phase-space vortex

Self-sustaining structure



Phase-space vortex formation is a fully nonlinear, kinetic process.



⇒ BGK mode or soliton

Bernstein & Green & Kruskal '57

Dupree '83

Schamel '86

Berk & Breizman '99

⇒ **a PS vortex is not tied to a wave and can evolve independently**

Nonlinear growth of a phase-space vortex

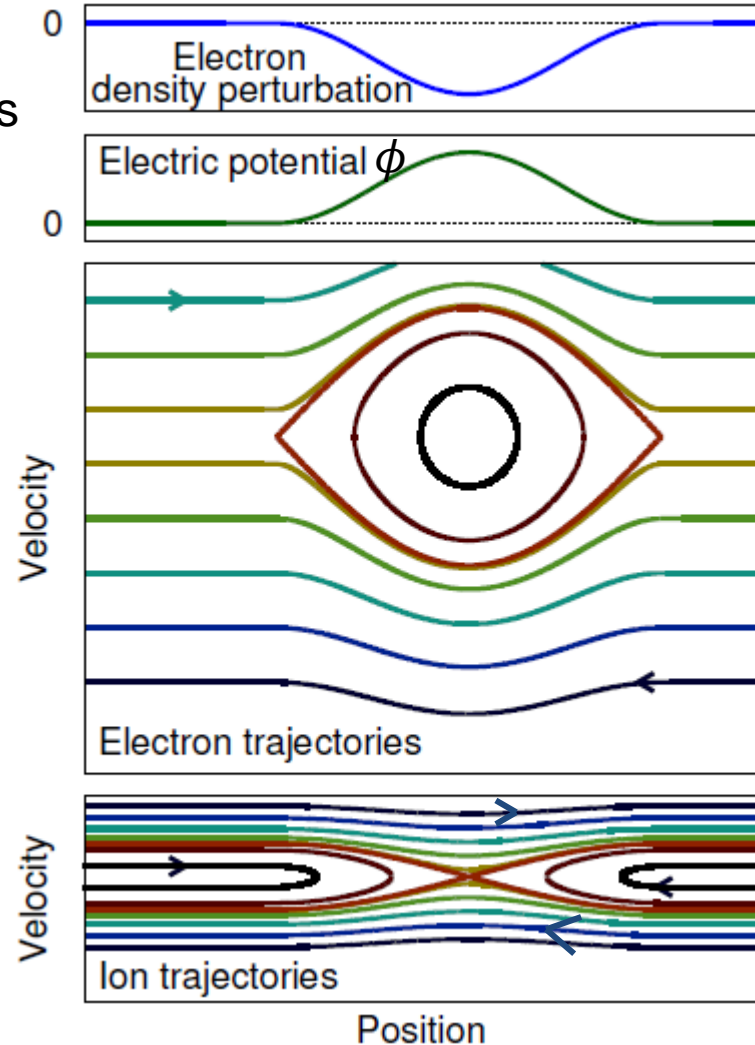
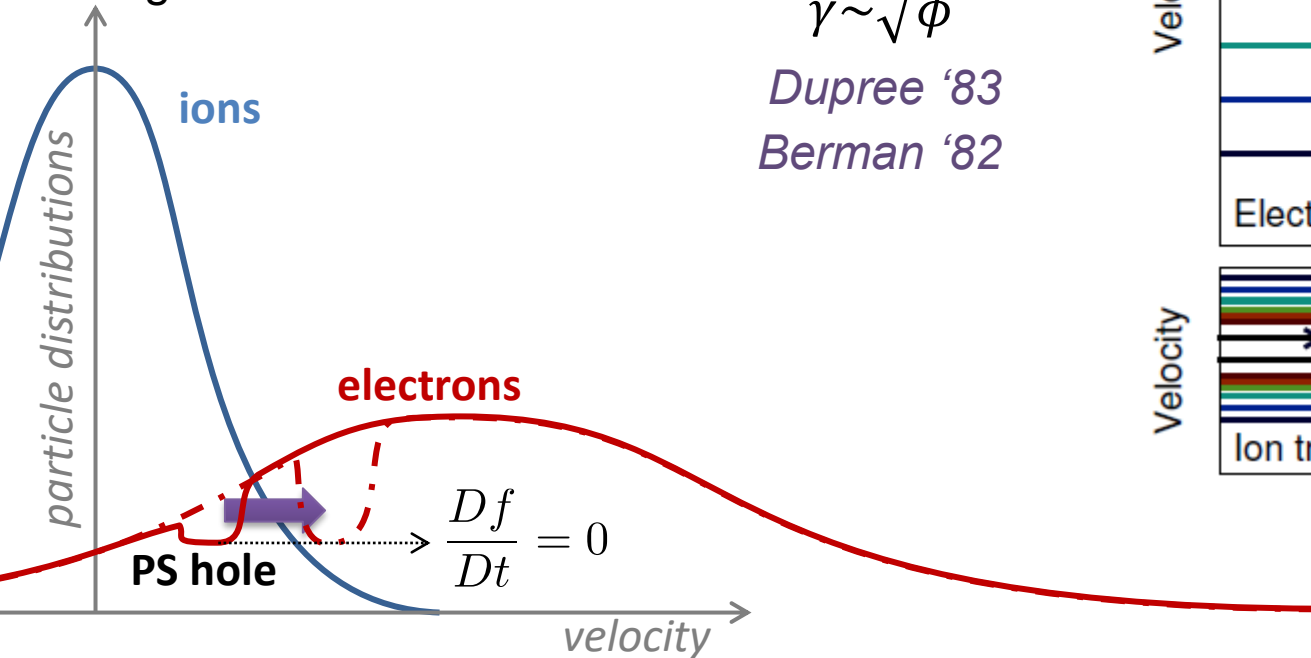
Growth mechanism (ex: electron hole)

- Electron hole = positive charge \Rightarrow scatters ions
- Majority of ions encountering the structure **gain** momentum
- Mostly compensated by **lose of momentum of negative mass electron hole**
 \Rightarrow increase of structure velocity
- Lagrangian conservation of f yields Eulerian growth of δf

$$\gamma \sim \sqrt{\phi}$$

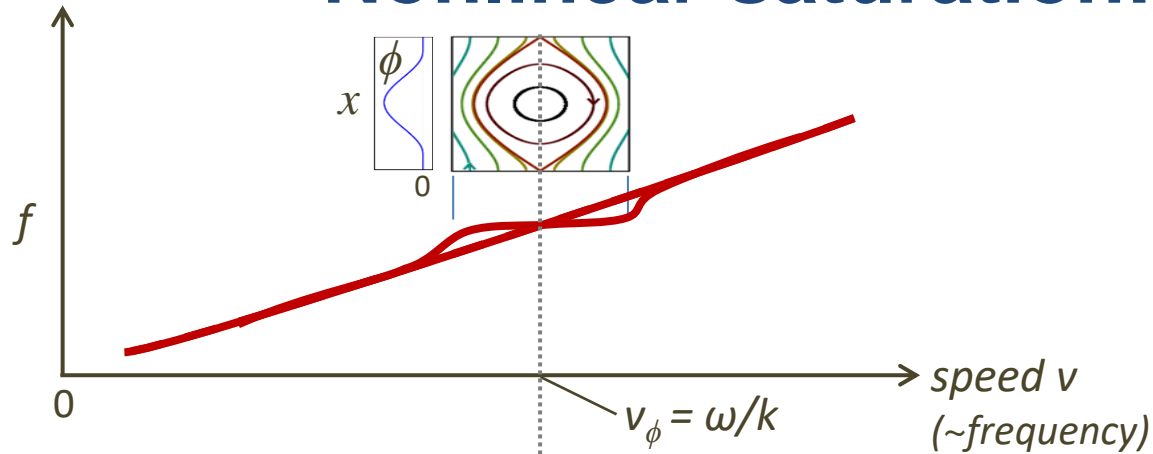
Dupree '83

Berman '82

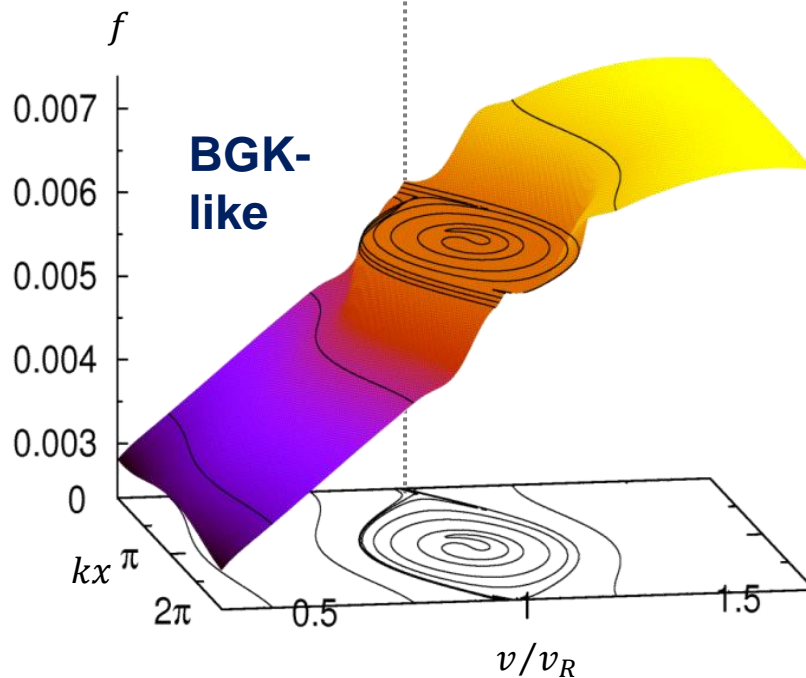


\Rightarrow **Nonlinear growth is due to dissipation (energy lost to ions)**

Nonlinear saturation: BGK limit



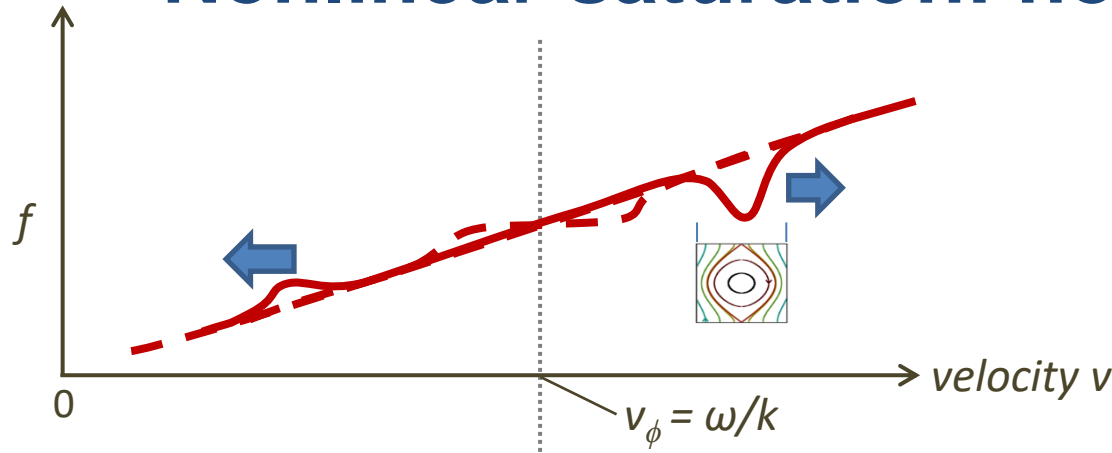
Case of a single mode, in a collisionless, closed system



Analytic theory: O'Neil '65

⇒ Island in phase-space (x,v) , consistent with a finite amplitude potential

Nonlinear saturation: holes and clumps

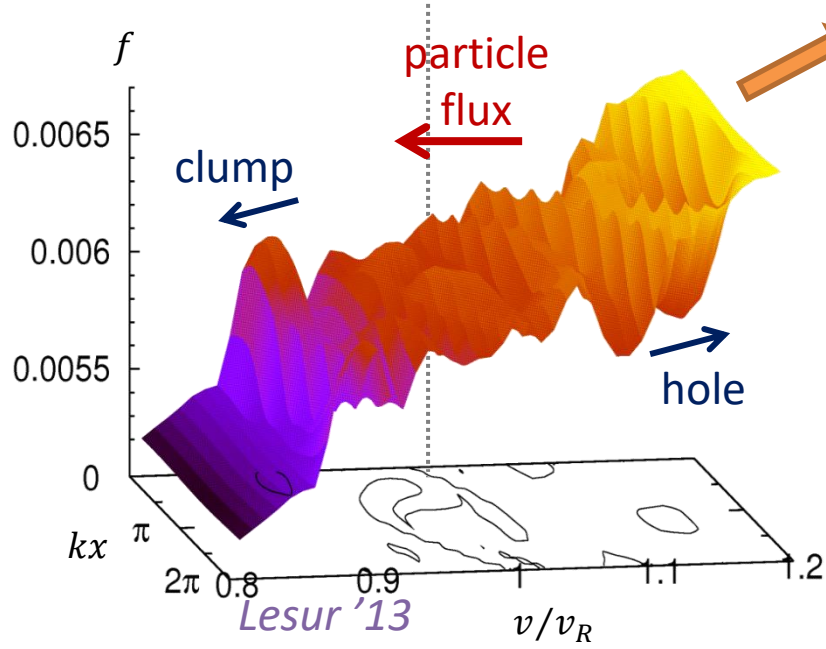


Case of a single mode,
in a dissipative system,
but weakly collisional

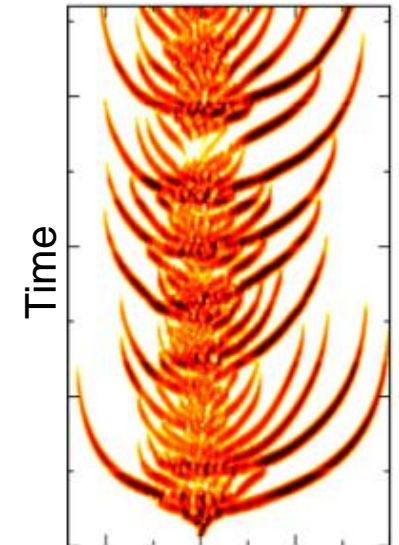
Schamel, '79

Dupree '82, Berman '82

Berk, Breizman, et al., '97



Nonlinear frequency sweeping (chirping)



⇒ Chirping (fast frequency sweeping) = fast vortex dynamics

Phasestrophy and momentum exchange

Phasestrophy is the phase-space density auto-correlation,

$$\Psi \equiv \int \langle \delta f^2 \rangle dv$$

⇒ Measure of phase-space vortices.

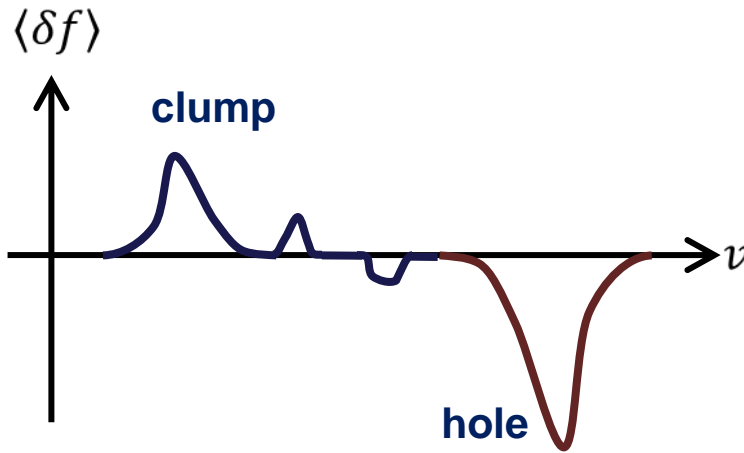
*Diamond, Itoh, Itoh, Modern Plasma Physics
Kosuga & Diamond '11*

Increasing phasestrophy implies:

- Structure growth
- Growth of relative entropy
- Momentum exchange

$$\frac{d\psi}{dt} = -\frac{2}{m} \frac{df_0}{dv} \frac{dp_{struct.}}{dt} \quad \sum_s p_s + p_{wave} = \text{const.}$$

- Particle transport in “velocity space”



Energy/phasestrophy (W- Ψ) theorem *Lesur & Diamond, '13*


Wave energy \longrightarrow

$$\frac{dW}{dt} + 2\gamma_d W = \sum_s \frac{m_s u_s}{d_v f_{0,s}} \left(\gamma_\Psi^{\text{col}} + \frac{d}{dt} \right) \Psi_s$$

Energy dissipation rate \longleftarrow

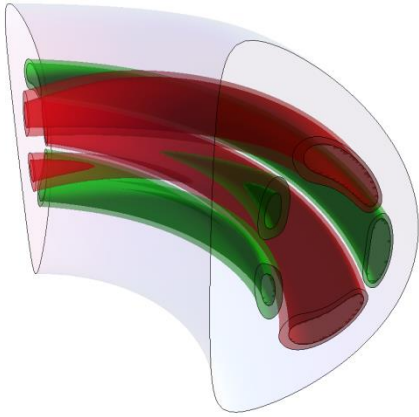
⇒ **4 points of view: momentum, energy, entropy, phasestrophy**

Outline

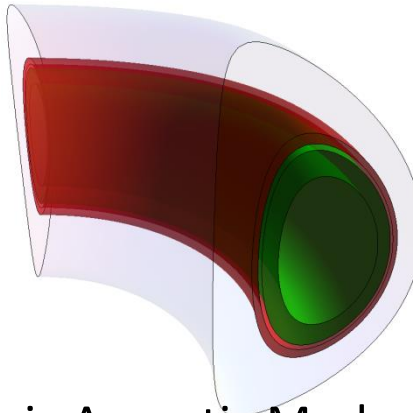
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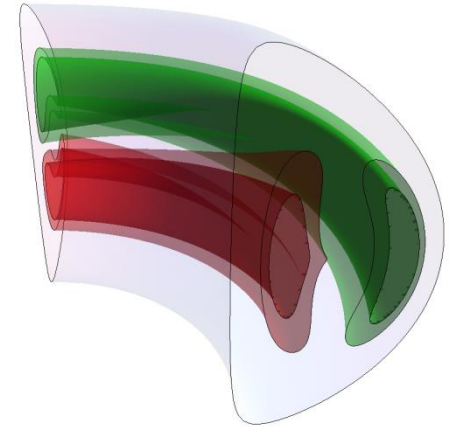
Nonlinear dynamics is essentially 1D



- Alfvén waves



- Geodesic Acoustic Modes (GAMs)



- Fishbones

Complementary approaches

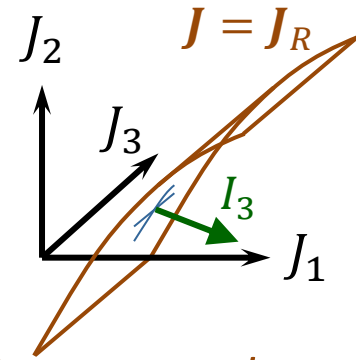
$$H_1(\boldsymbol{\alpha}, \mathbf{J}, t) = V(\mathbf{J}, t) \sin(\mathbf{n} \cdot \boldsymbol{\alpha} - \omega t)$$

$$\theta_3 = \mathbf{n} \cdot \boldsymbol{\alpha} - \omega t$$

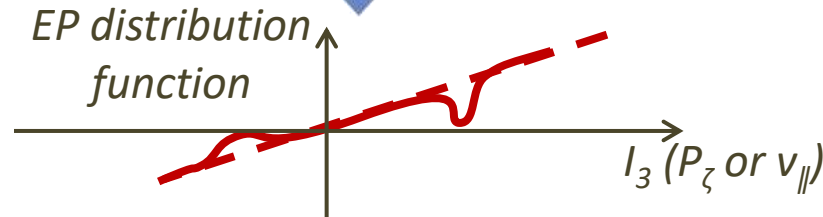
$$I_3 \sim P_\varphi - P_\varphi^{res}$$

Lichtenberg '69

$$H_1(\theta_3, I_3, t) = \frac{1}{2} D I_3^2 + V(t) \sin(\theta_3) + o(I_3, \partial_J V)$$

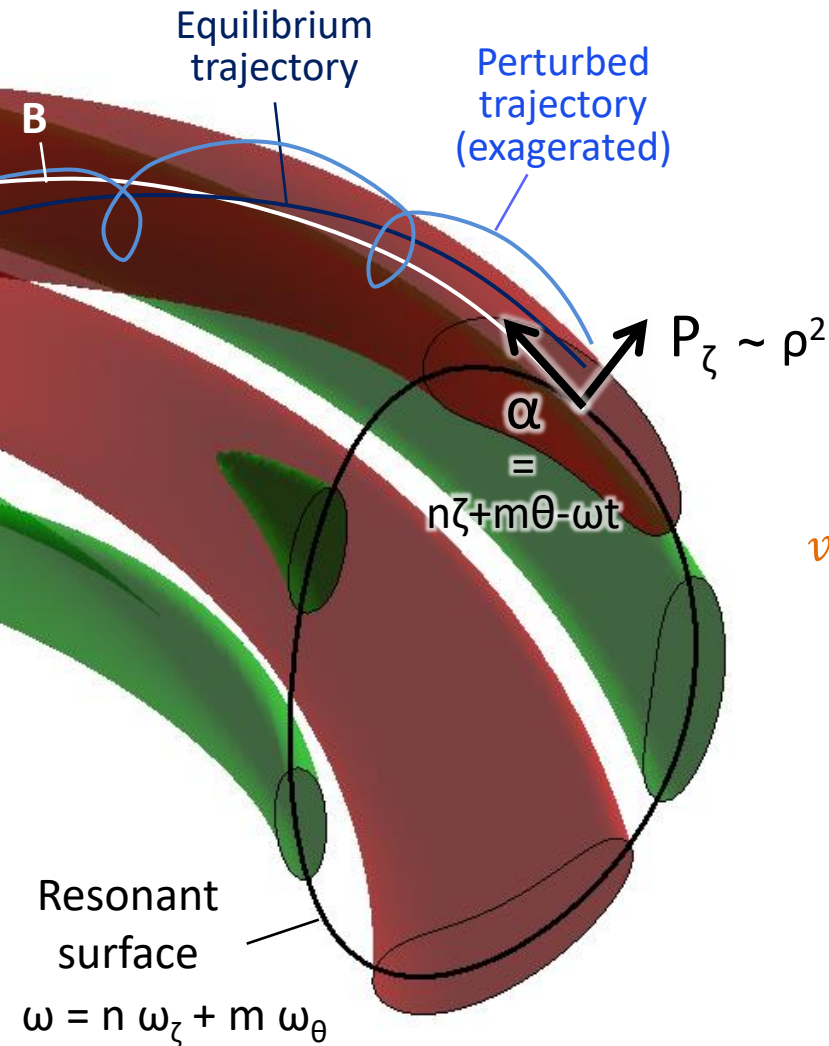


Resonance sheet in phase-space

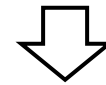


⇒ Perturbative description of nonlinear dynamics for single, isolated mode, with fixed mode structure

Ex: Toroidal Alfvén Eigenmode (TAE)



3D linear theory or experiment



Spatial mode structure, frequency, effective collision frequencies



Perturbation

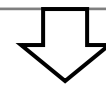
$\alpha \leftrightarrow x$

$v \leftrightarrow P_\zeta$

1D model (2D phase-space)

$V \sim \delta A_\perp \leftrightarrow \phi$

Adding energy dissipation
 \Rightarrow Berk-Breizman model



- **Nonlinear stability**
- Chirping, PS turbulence
- PS structures-driven transport

\Rightarrow **Problem split between slowly-evolving (~ 100 ms) 3D mode structure and fast (~ 1 ms) 1D nonlinear amplitude and phase dynamics**

The Berk-Breizman model

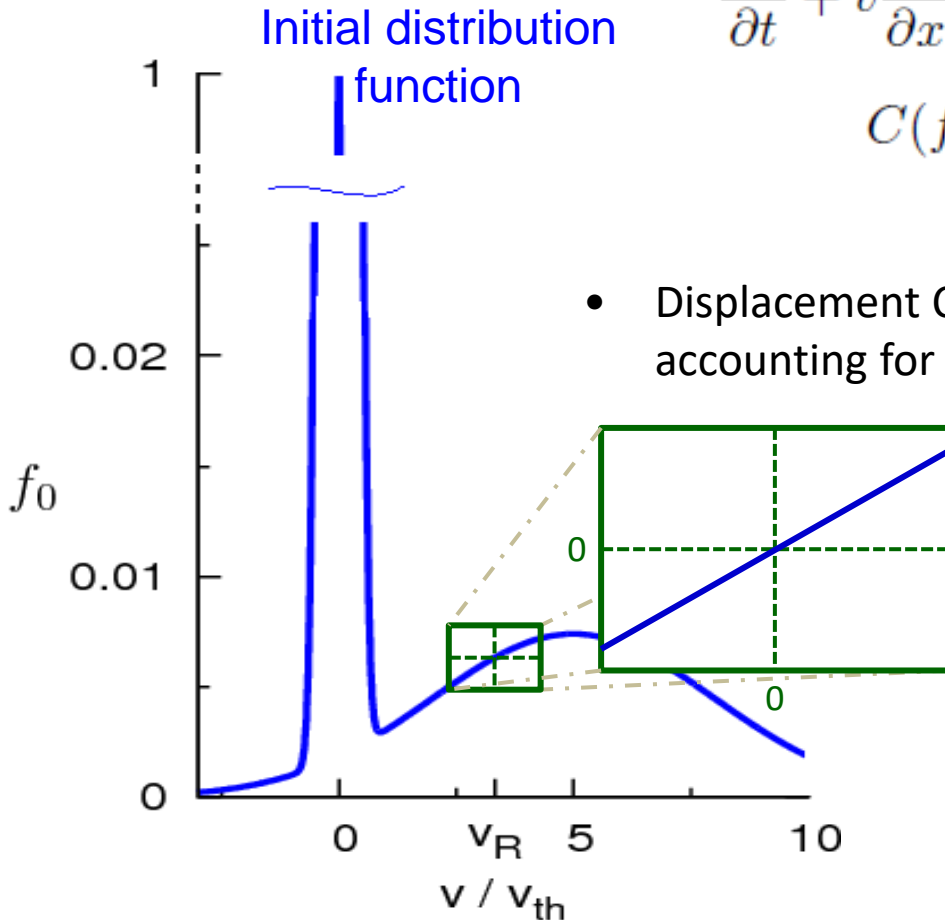
Classic “bump-on-tail” instability, with collisions and external damping.

Berk, Breizman, et al. '90, '92, '93, '95, '96, '97, '98, '99

- 1D kinetic equation with collision operator,

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial v} = C(f - f_0)$$

$$C(f - f_0) = \underbrace{\frac{\nu_f^2}{k} \frac{\partial (f - f_0)}{\partial v}}_{\text{friction (drag)}} + \underbrace{\frac{\nu_d^3}{k^2} \frac{\partial^2 (f - f_0)}{\partial v^2}}_{\text{v-space diffusion}}$$



- Displacement Current Equation with an external wave damping accounting for background dissipative mechanisms,

$$\frac{\partial E}{\partial t} = -4\pi q \int v(f - \bar{f}) dv \quad \boxed{-2\gamma_d E}$$

Linear drive, $\gamma_L \sim \partial f_0 / \partial v$

\neq linear growth rate $\gamma \approx \gamma_L - \gamma_d$

\Rightarrow System with 6 timescales:

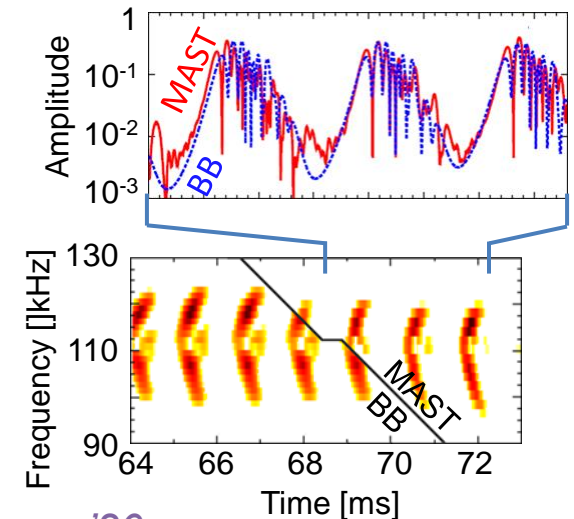
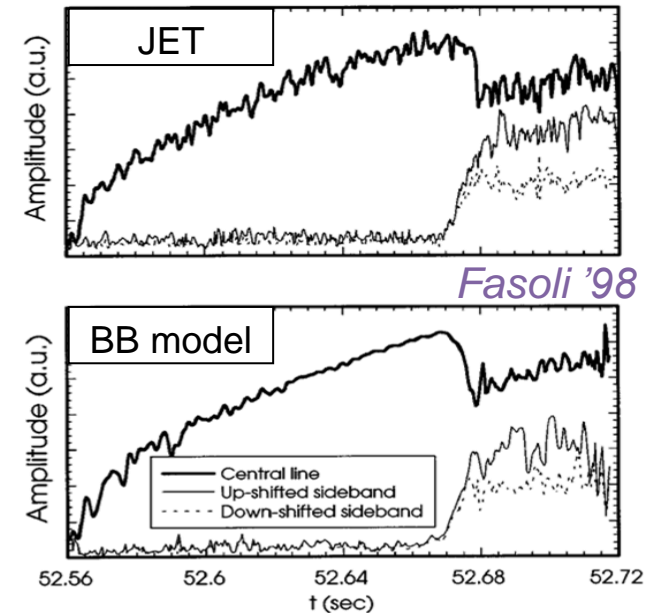
$\omega, \omega_b, \gamma_L, \gamma_d, \nu_f, \nu_d$

The BB model reproduces experiments

Comparing BB model with fusion plasma experiments

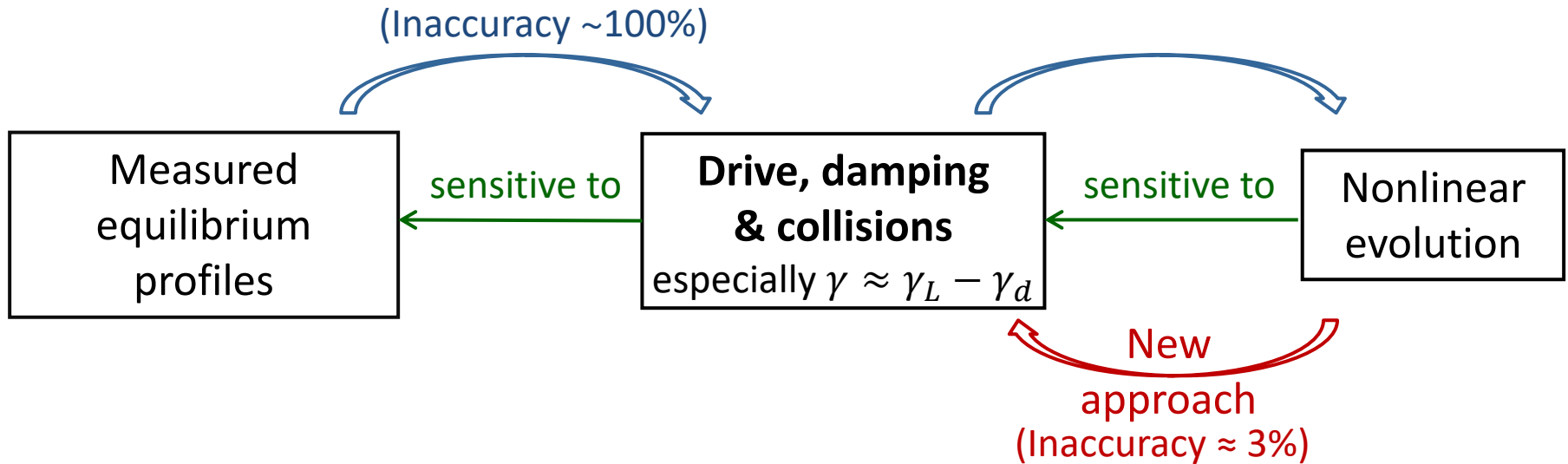
- **Quantitative agreement** (after adjusting free parameters)
 - TAEs on JET, JT-60U, MAST
 - EGAM on the LHD
- **Qualitative agreement** (may be quantitative ?)
 - TAEs on the LHD
 - EGAM on JET
 - e-fishbones on HL-2a
 - many more
- **Predictions**
 - EGAM on LHD: phase relationship and its evolution, amplitude threshold
 - ???

⇒ **Successful reduced modeling, but limited predictive capabilities**



Lesur '20

Application: NL analysis of experimental data



Example with TAEs on JT-60U

Lesur '10

γ_L
8.8%

γ_d
4.7%

ν_f ν_d
0.42% 1.7%

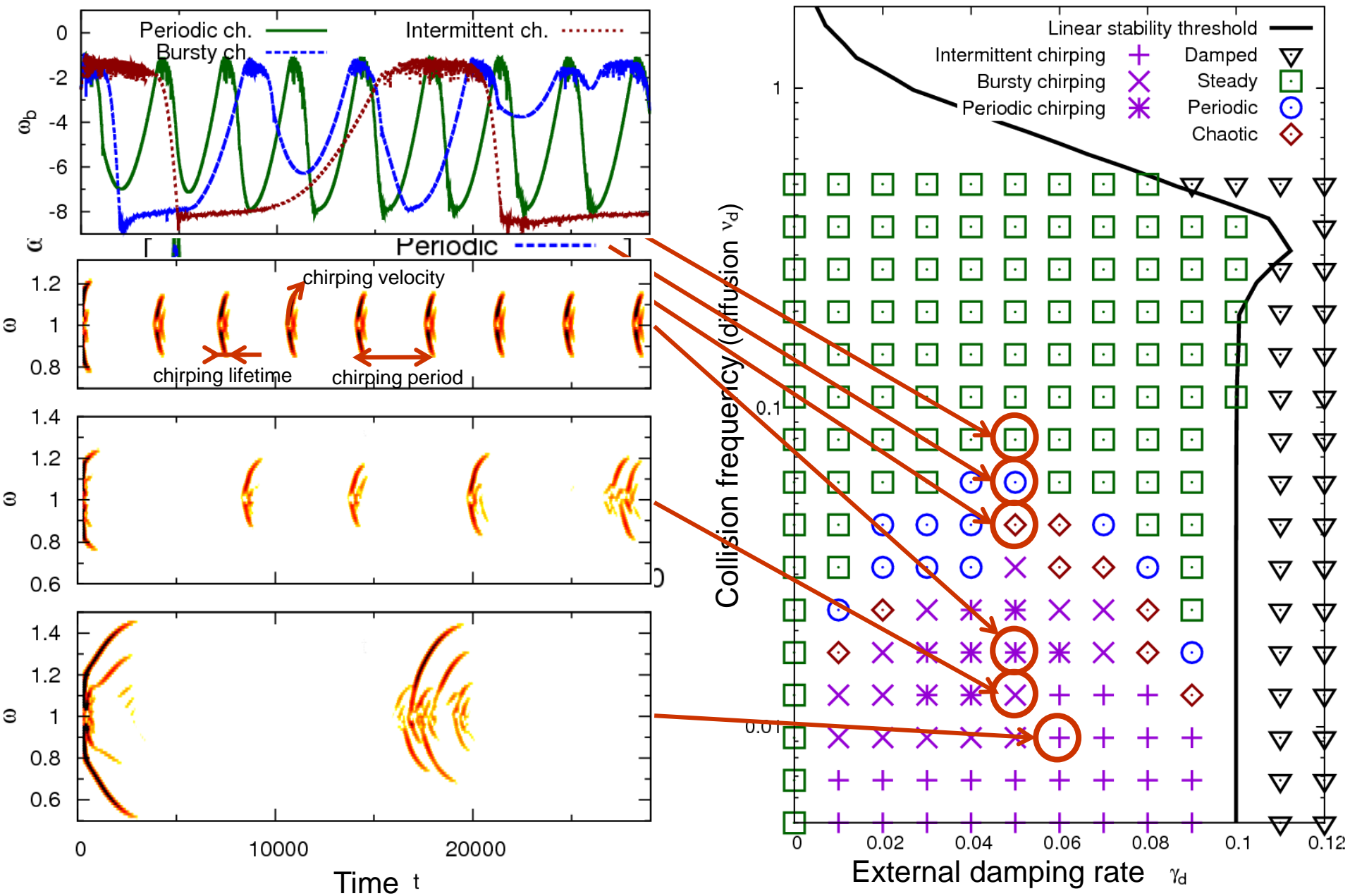
γ
4.6%

(expressed in % of the TAE frequency)

In agreement with first-principle calculation from experimental parameters

⇒ **Non-intrusive diagnostic method**

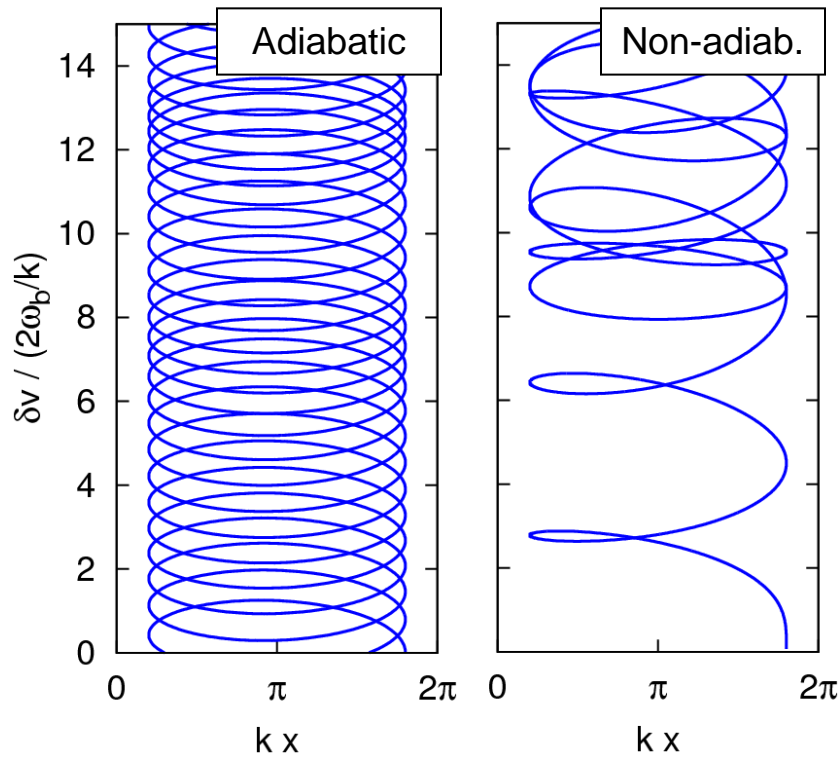
Phenomenology: for small drag ($\nu_d/\nu_f = 5$)



Chirping velocity

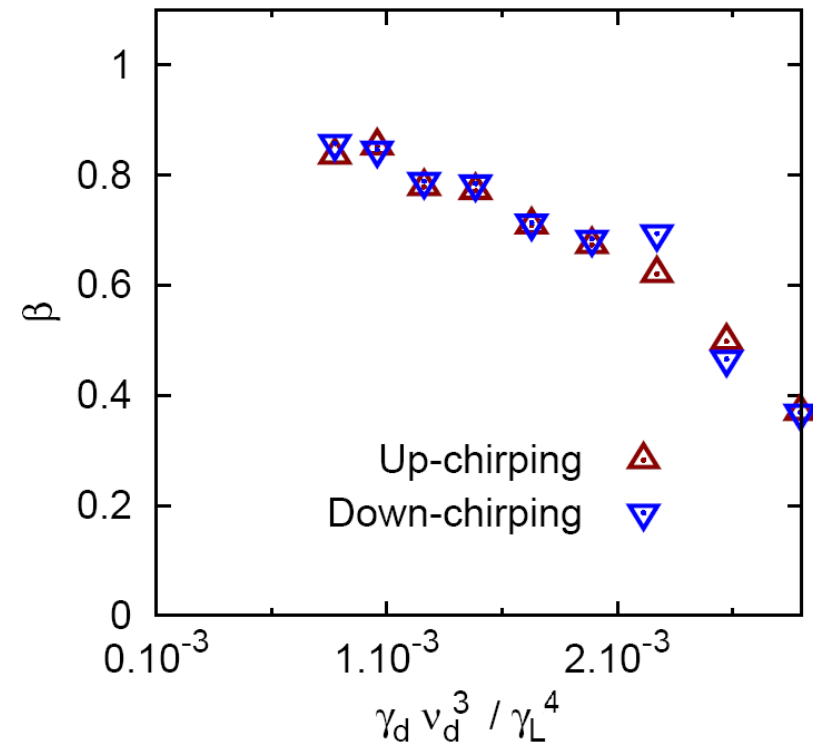
Adiabatic evolution ($\dot{\omega}/\omega_b^2 \ll 1$) $\Rightarrow \delta\omega(t) \approx 0.44 \gamma_L \sqrt{\gamma_d t}$

Berk, Breizman, et al. '99



Strong correction due to collisions:
still analytical, see *Lesur '13*

Lesur '10



When $\dot{\omega}/\omega_b^2 \sim 1$, correction parameter:

$$\beta \equiv \frac{\delta\omega(t)}{0.44 \gamma_L \sqrt{\gamma_d t}}$$

\Rightarrow **Frequency varying as roughly $t^{1/2}$ is a good signature**

Chirping lifetime

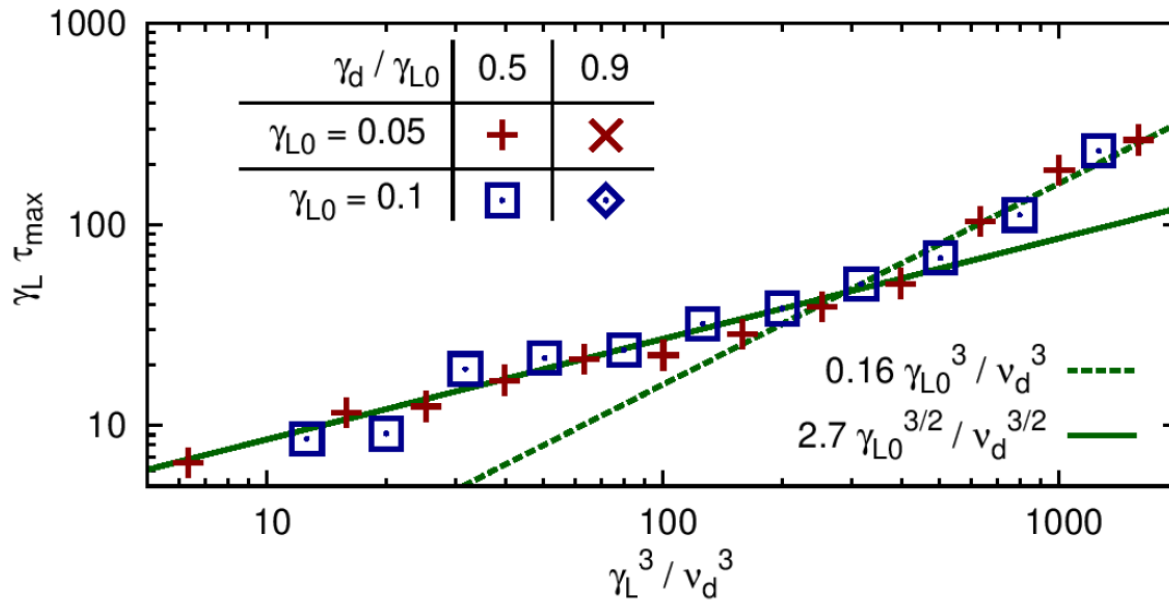
Hole/clump width $\sim \gamma_L$ \Rightarrow diffusion time $\tau_{max} \sim \frac{\gamma_L^2}{v_d^3}$

Berk, Breizman, et al., '97

However, for higher collisionality, diffusion affects hole/clump width

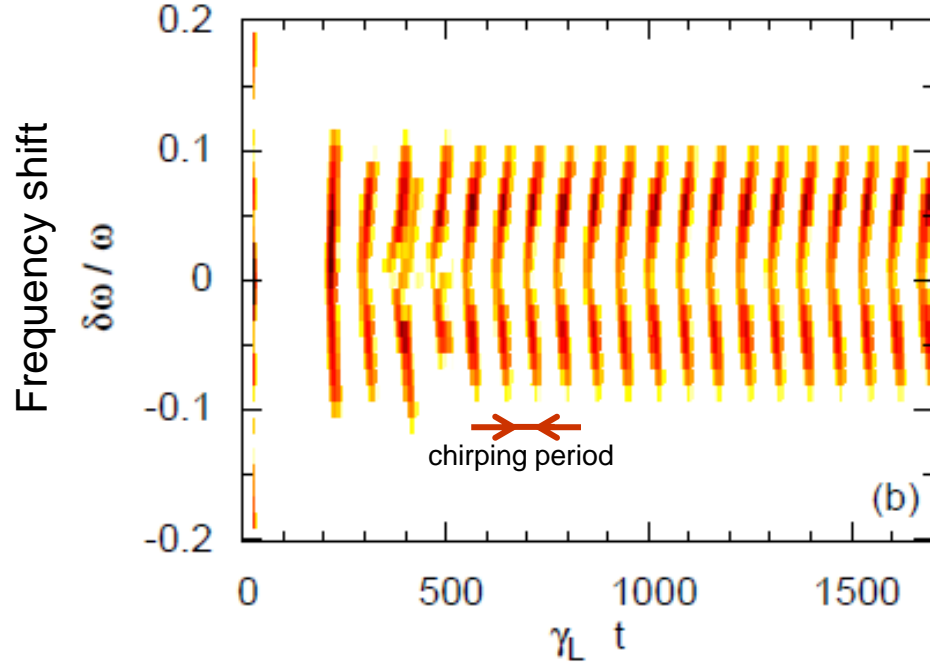
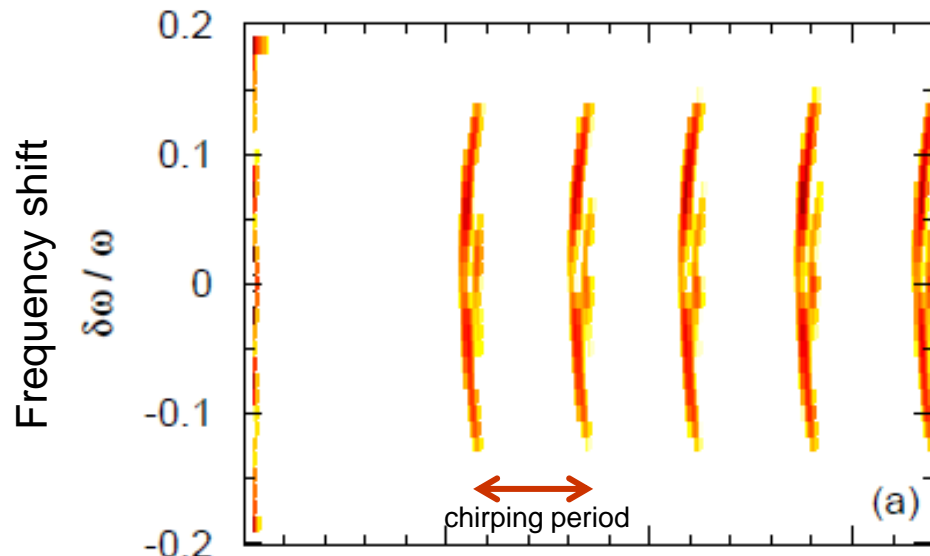
\Rightarrow semi-empirical law $\tau_{max} \sim \frac{\gamma_L^{1/2}}{v_d^{3/2}}$

Lesur, '20



\Rightarrow **Simple expression for chirping lifetime**

Chirping period



In experimentally-relevant conditions, the time between 2 chirping bursts decreases with:

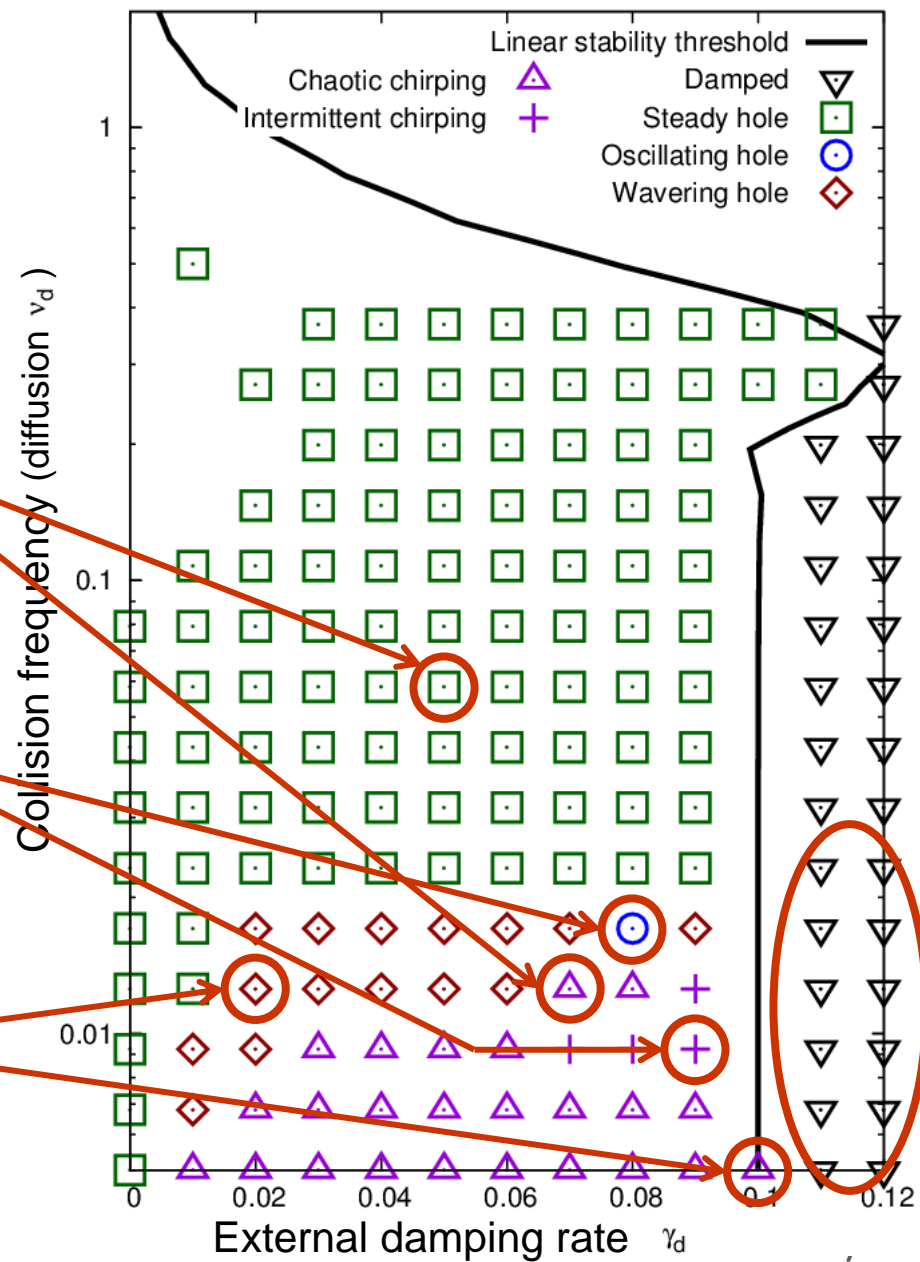
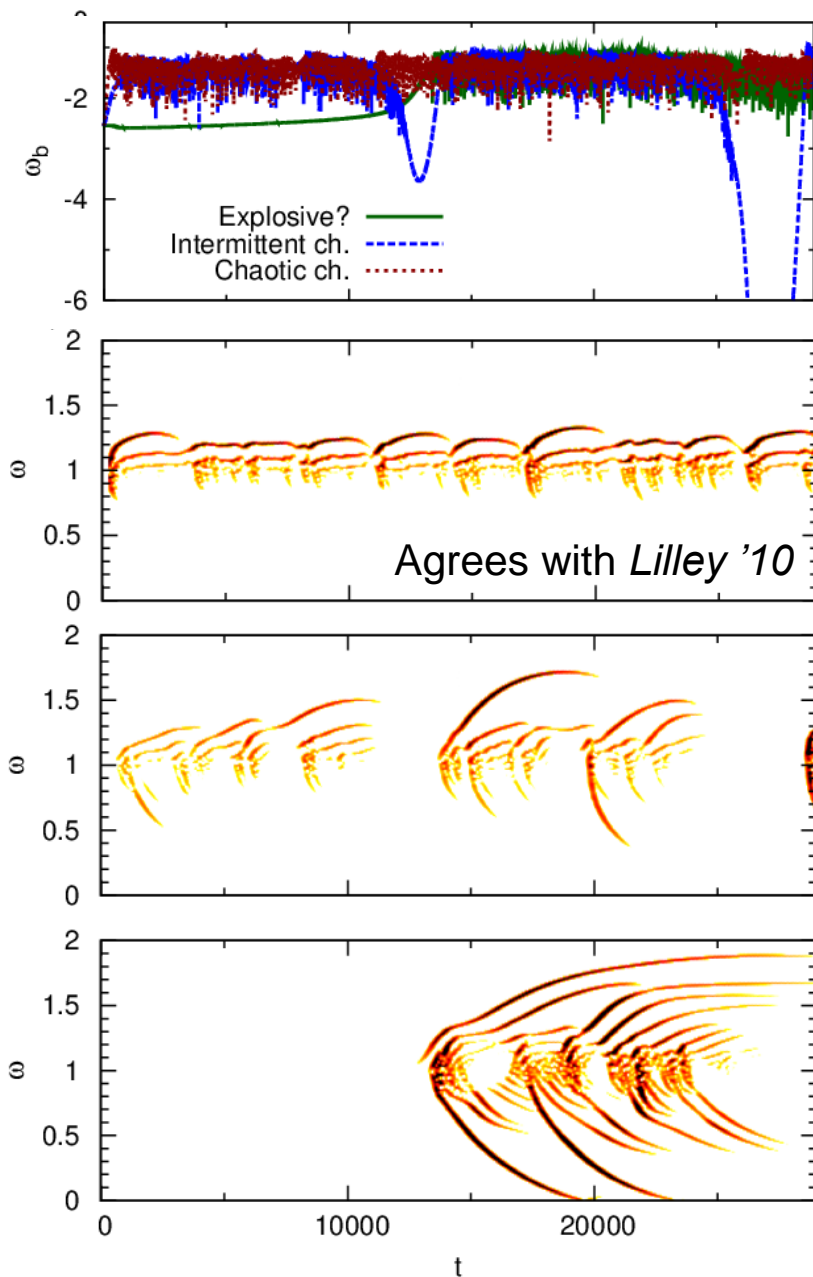
- Increasing ν_f
- Increasing ν_d
- Increasing γ_L

* rough picture
More details in
Lesur '13

ν_f

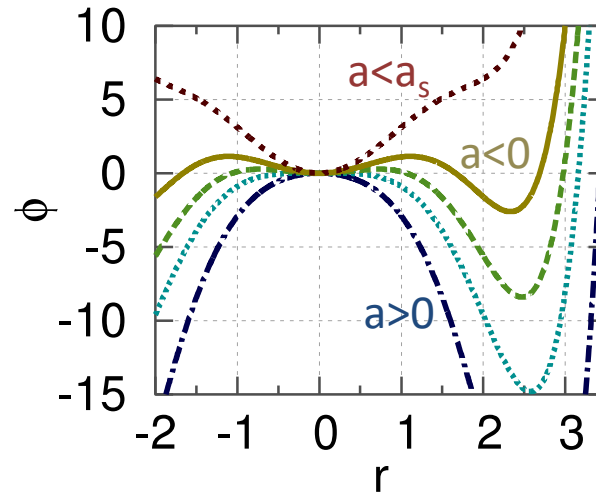
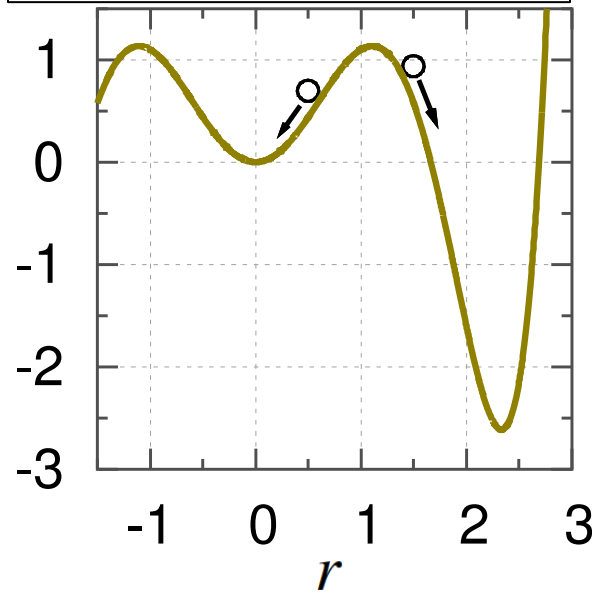
⇒ Reduce velocity diffusion to reduce frequency of bursts

Phenomenology: for large drag ($\nu_d/\nu_f = 1$)



Basic concepts of subcritical instability

$$\phi(r) = -ar^2 - r^4 + cr^6$$

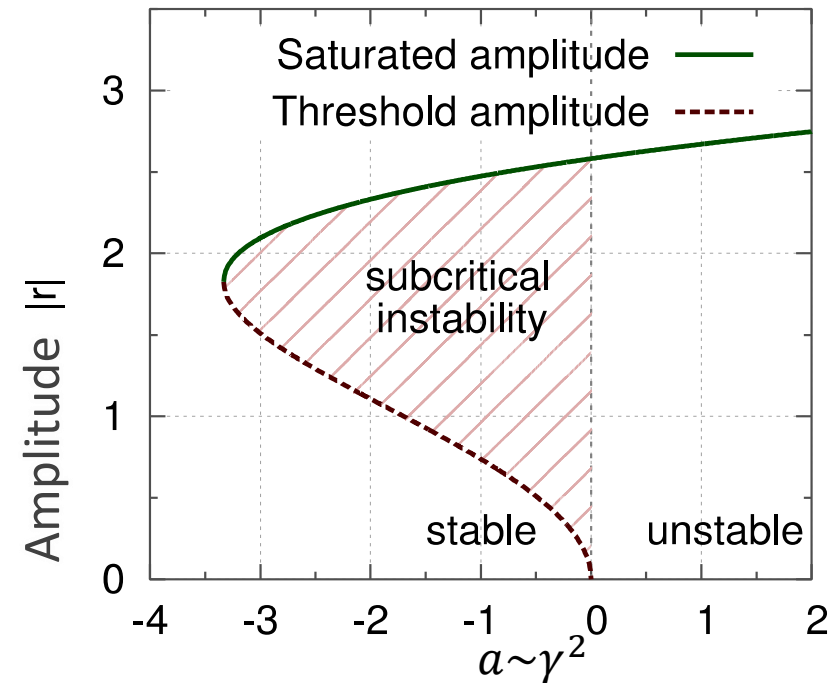


Linearized eqn
of motion

$$\ddot{r} \sim ar$$

- Circumvent linear stability theory
- Threshold in amplitude
- Hysteresis
- Local vs Global

Dauchot & Manneville '97



Tutorial: Lesur '18

⇒ **Subcritical instabilities ubiquitous in plasmas and neutral fluids**

Nonlinear growth rate

Energy/phasesatropy (W-Ψ) theorem

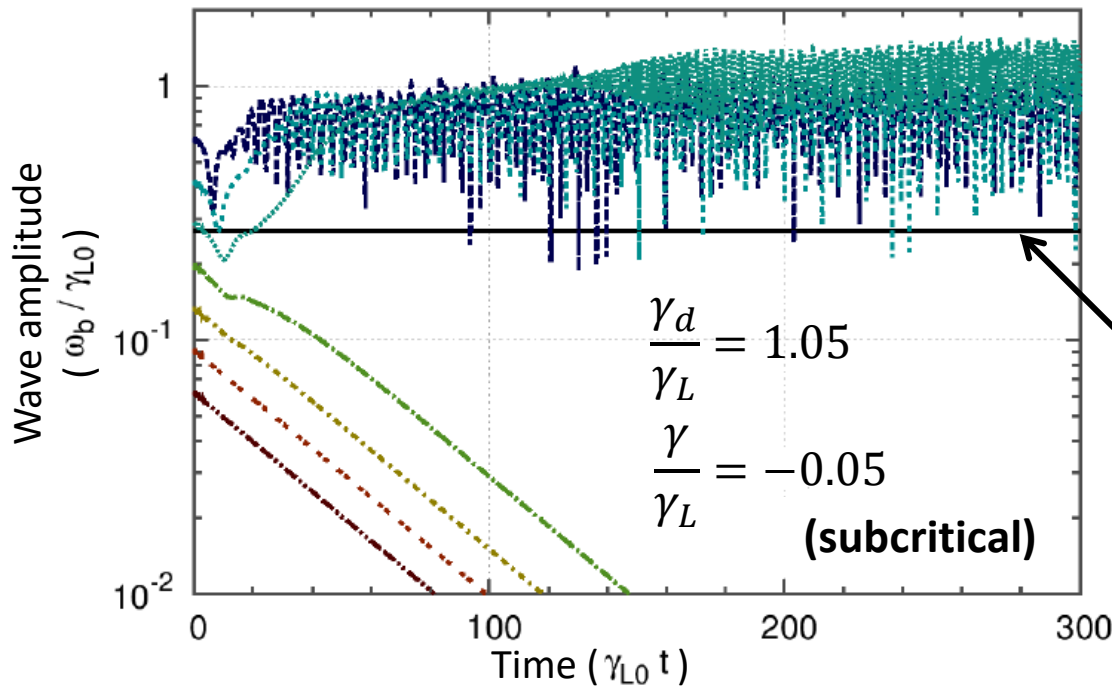
$$\frac{dW}{dt} + 2\gamma_d W = \sum_s \frac{m_s u_s}{d_v f_{0,s}} \left(\gamma_\Psi^{col} + \frac{d}{dt} \right) \Psi_s$$

Model of Gaussian hole

⇒ second relation between W and Ψ (via Poisson)

Lesur, Diamond '13

$$\gamma_\Psi \approx \frac{16}{3\sqrt{\pi}} \frac{\Delta v}{v_R} \frac{\gamma_L}{\omega_p} \gamma_d$$

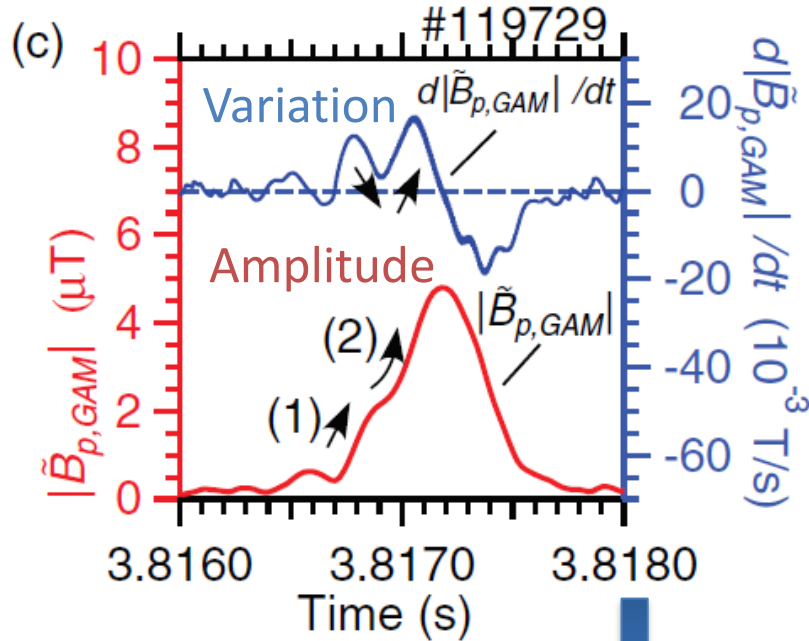


Predicted by balancing γ_Ψ with γ_Ψ^{col}

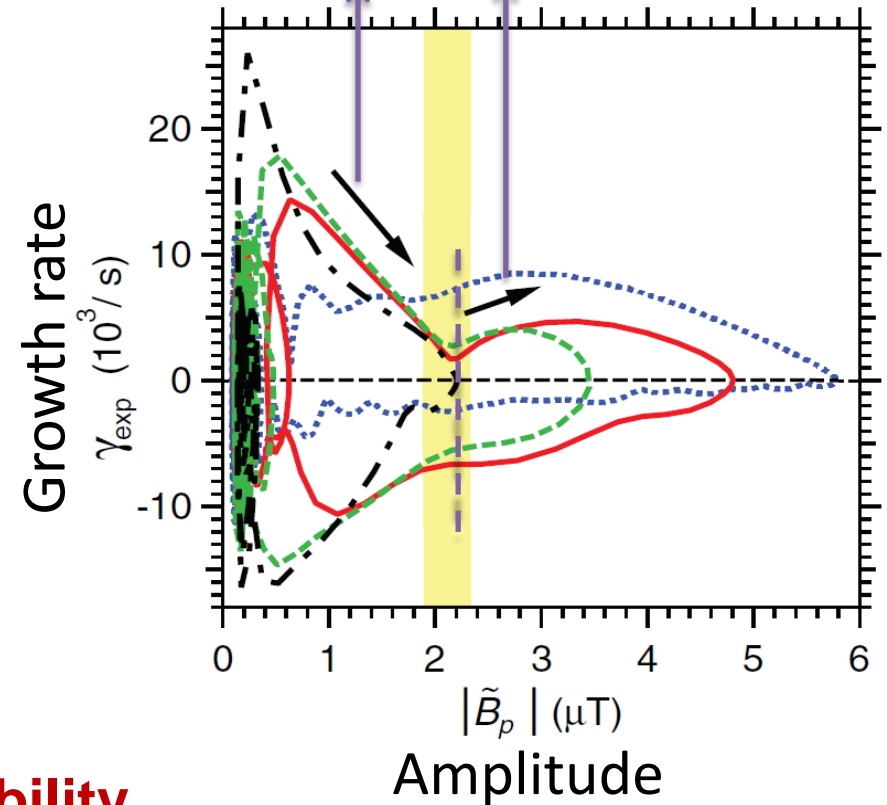
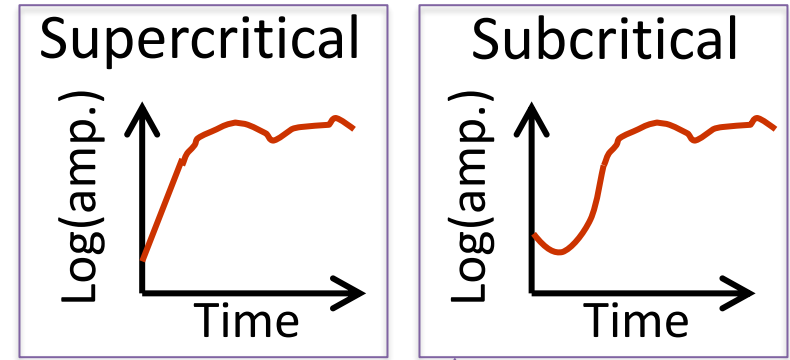
⇒ **Nonlinear growth does not require that free-energy exceeds dissipated energy**

Experimental evidence of subcritical instability

EGAM on the LHD



Ido '16

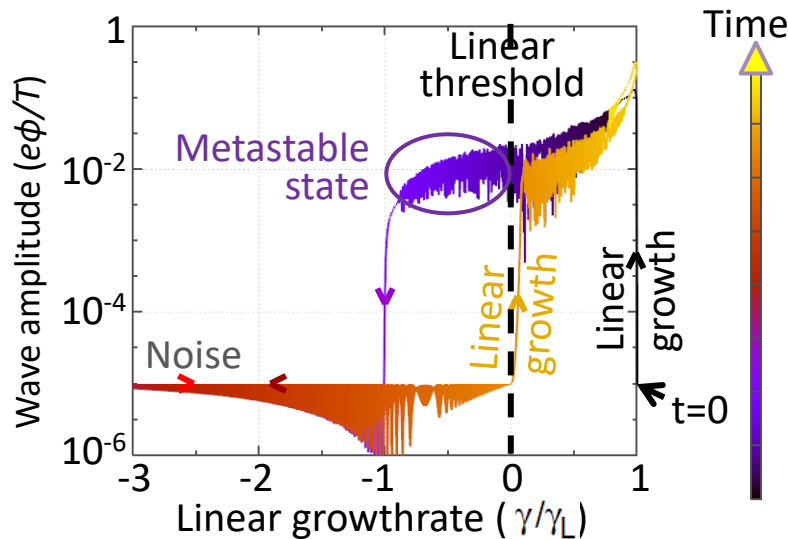


\Rightarrow **Signature of subcritical instability**

Application: control in phase-space

Naive (linear) strategy

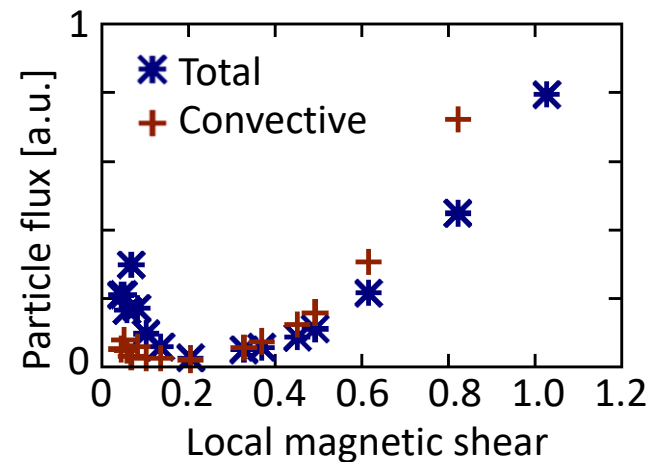
Decrease linear growthrate



The system self-organize to extract more free energy

Efficient (nonlinear) strategy

Diffuse PS structures, e.g. by modifying local magnetic shear




Bifurcation towards a quieter quasilinear regime



⇒ **Technique prometteuse de mitigation**

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Summary

Current-driven ion-acoustic turbulence

1D model for ion-electron plasma with initial velocity drift

- 1D Vlasov for ions and electrons (collisionless) + Poisson equation
- Ion-acoustic modes $\omega \approx kc_s$
- Linear instability threshold $v_d > v_{d,cr}$

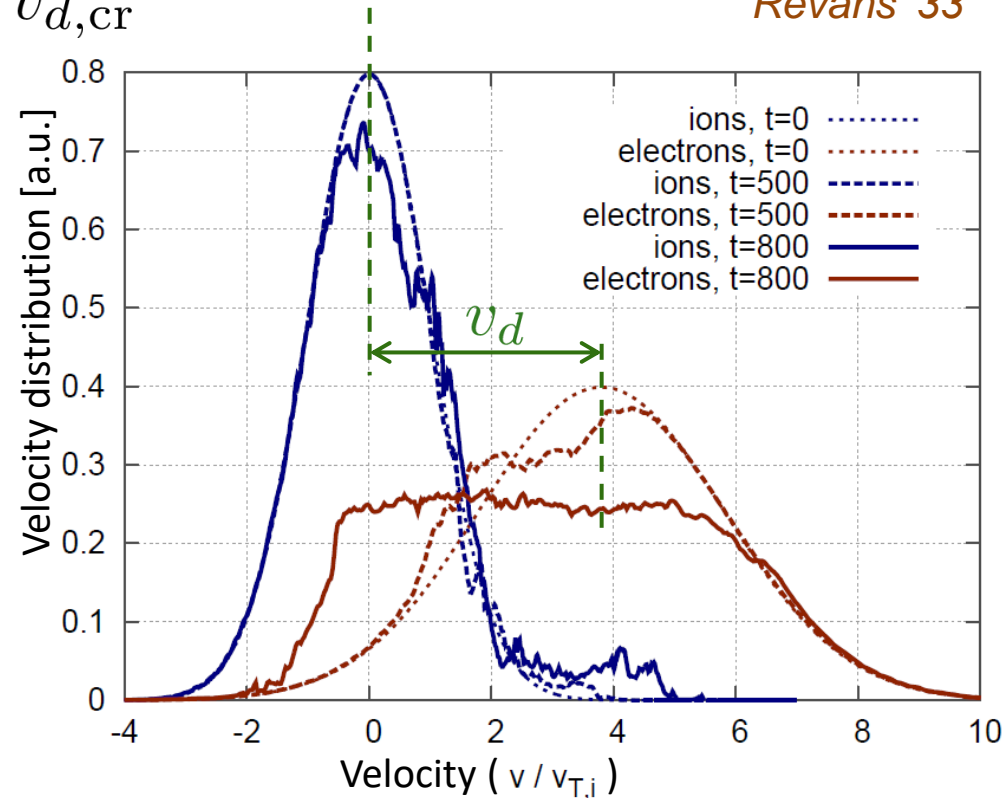
Tonks & Langmuir '29
Revans '33

Velocity redistribution

- Anomalous resistivity
- Turbulent "heating"

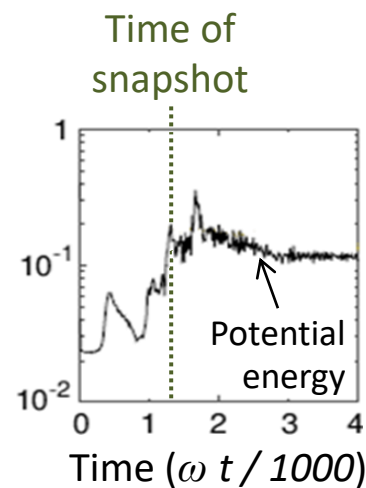
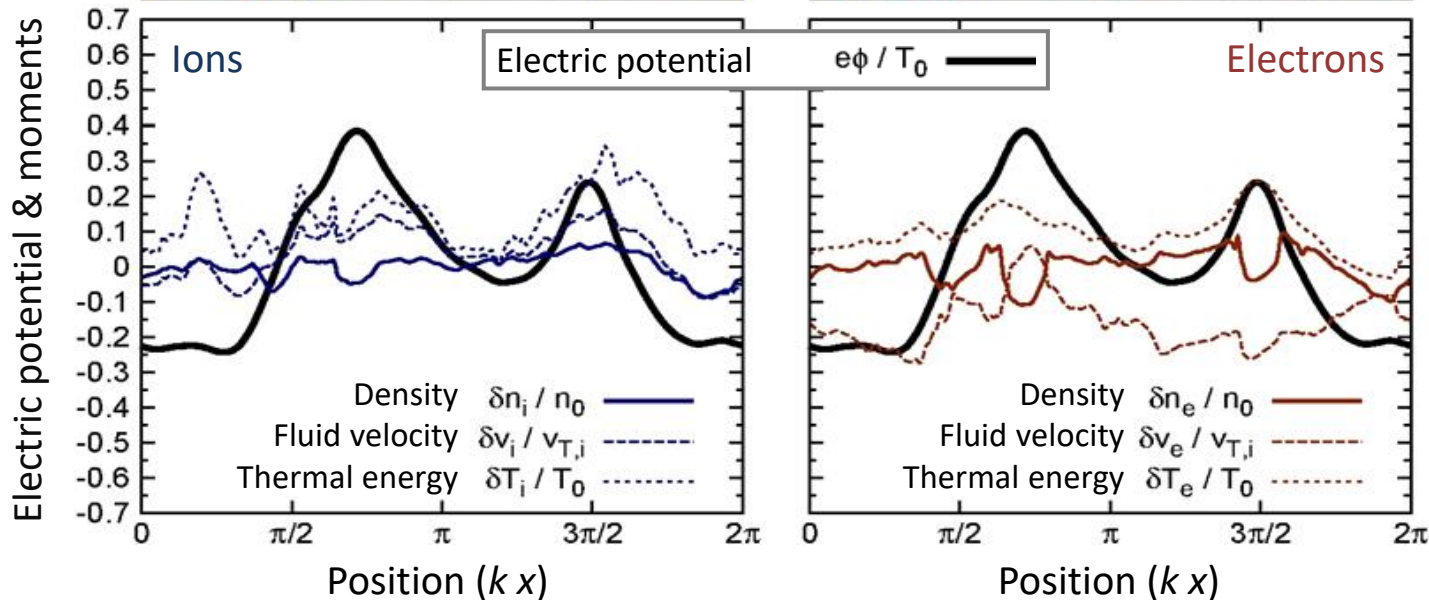
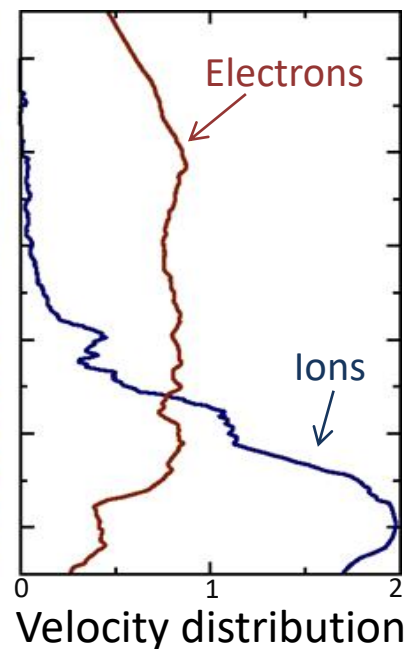
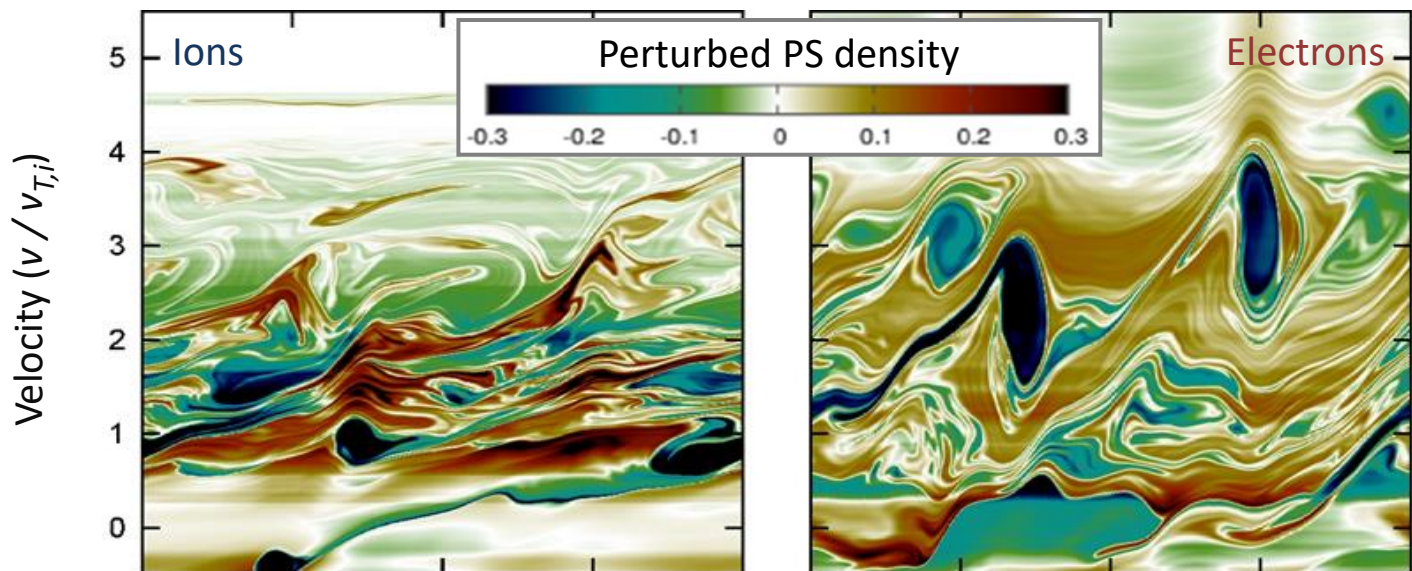
Growth of electron PS hole

- Electron inverse Landau damping plays role of γ_L *Dupree '83*
- Ion Landau damping plays role of γ_d
- Momentum exchange



⇒ **Basic version of many types of fusion plasma turbulence**

Phase-space dynamics



⇒ PS dynamics helps understand real-space dynamics (hidden variable v)

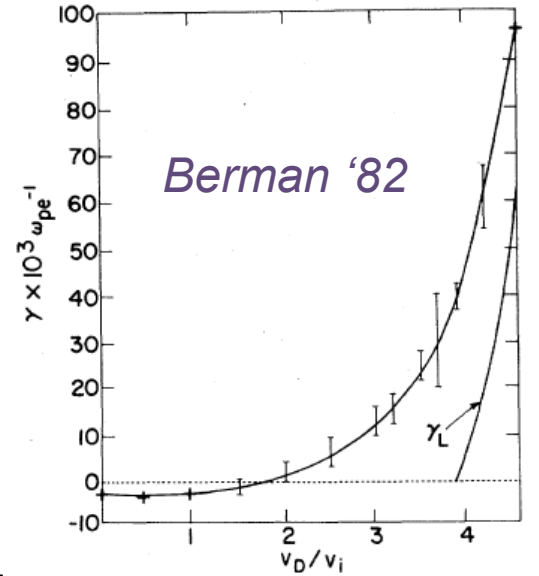
Subcritical ion-acoustic turbulence

As predicted by analytic theory

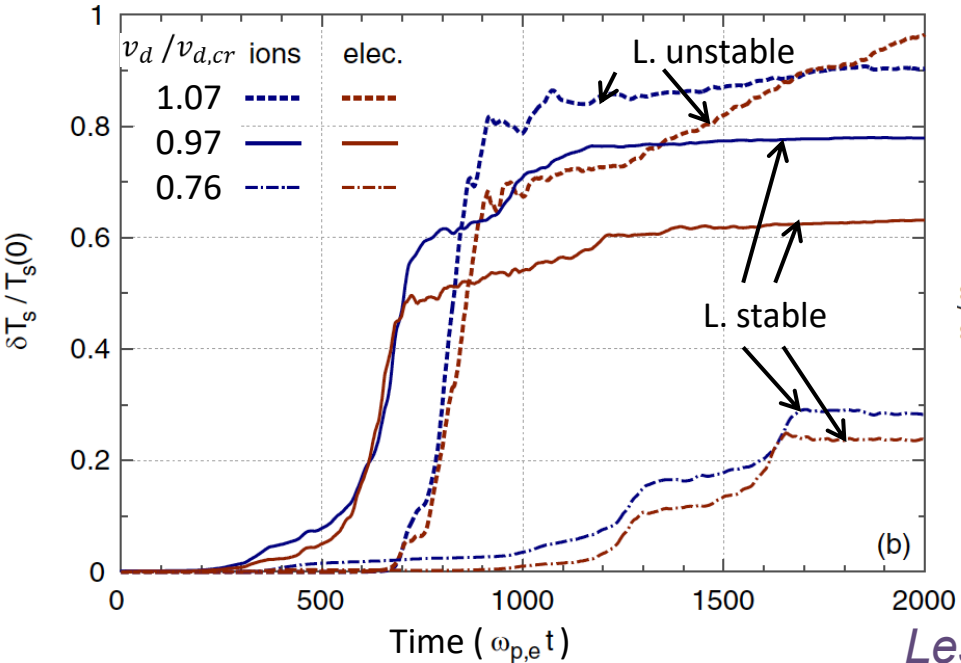
Dupree, Tetreault, Boutros-Ghali

$$\gamma_{NL} \approx -v_{Te}^2 v_{Ti}^2 \frac{df_{e0}}{dv} \frac{df_{i0}}{dv} \omega_b$$

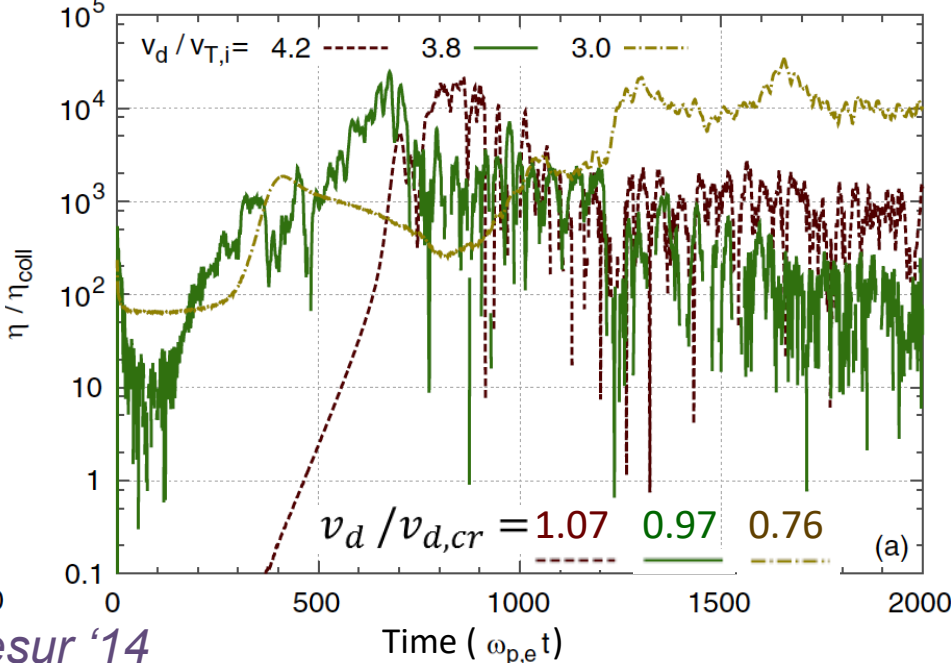
Turbulence dominated by NL growth of PS holes



Turbulent "heating"



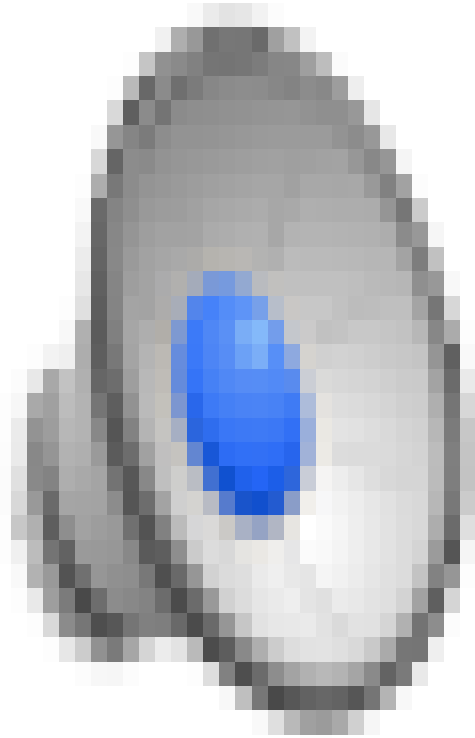
Anomalous resistivity



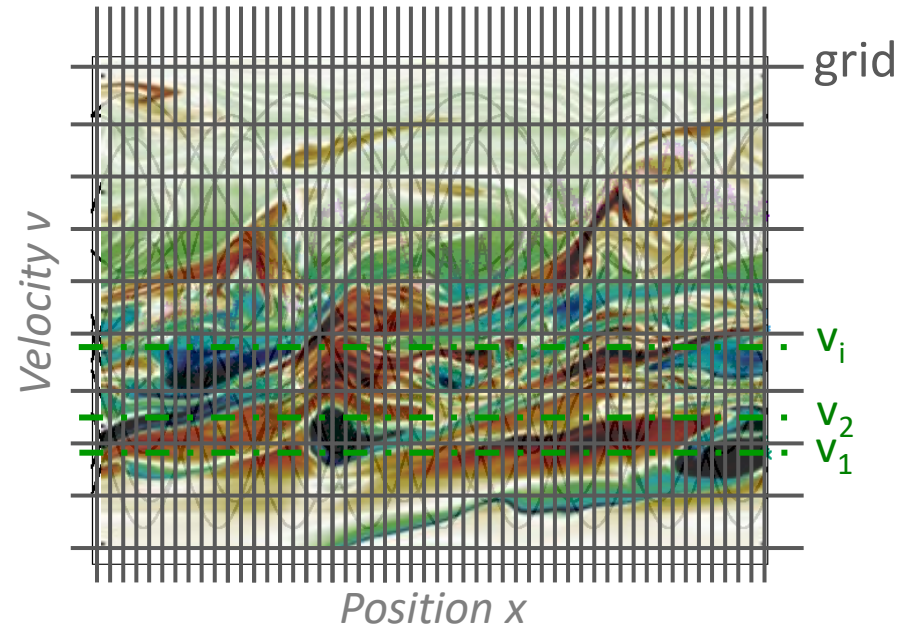
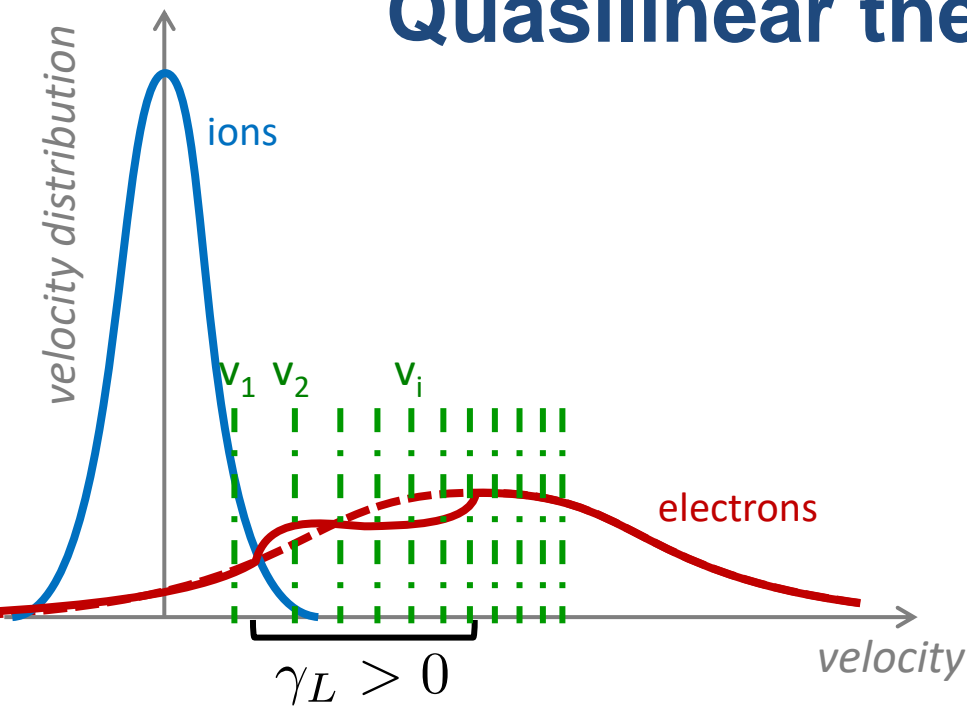
Lesur '14

⇒ NL growthrate, saturated amplitude and other measures of impacts do not "feel" the linear instability threshold

Video

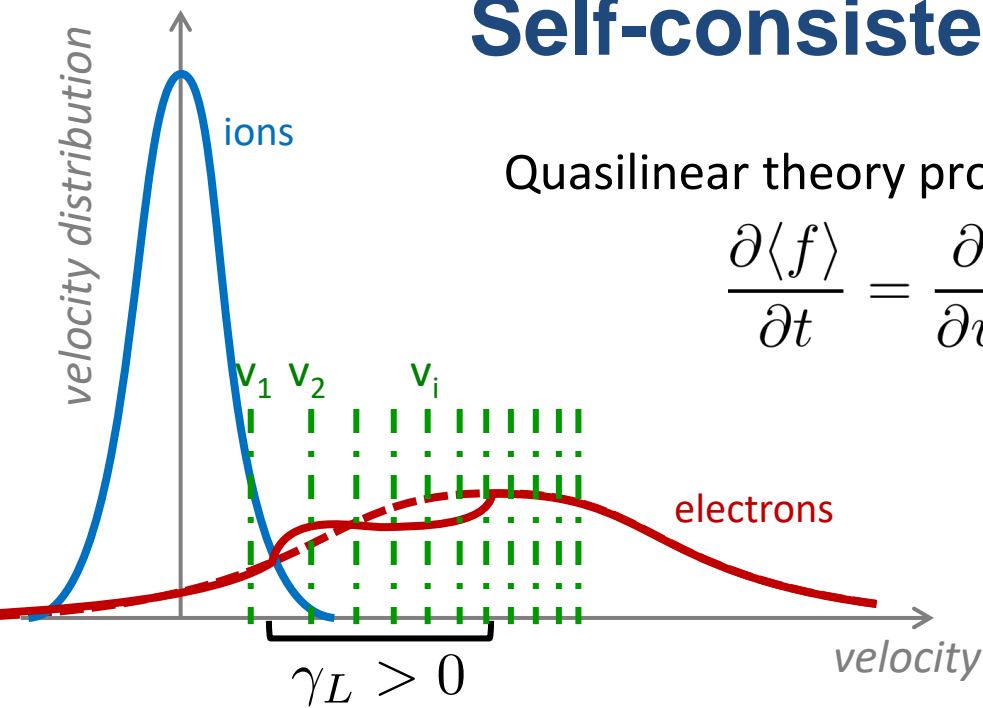


Quasilinear theory: approach



⇒ Random walk based on random phase (or sim.) approximation, incompatible with strong PS structures

Self-consistent evolution



Quasilinear theory provides the evolution of mean distribution

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} \left(D_{\text{QL}}(|\hat{E}_k|, v) \frac{\partial \langle f \rangle}{\partial v} \right)$$

$$\frac{\partial |\hat{E}_k|}{\partial t} = \gamma_{L,k}(\langle f \rangle) |\hat{E}_k|$$

Evolution of turbulent spectrum

Vedenov&Velikov&Sagdeev '61

Drummond&Pines '62

Sagdeev&Galeev '69

Outside the scope of quasilinear theory:

- Phase-space turbulence and granulation

$$\frac{\partial \langle f \rangle}{\partial t} = \frac{\partial}{\partial v} \left(F_{\text{PS}}(\Psi, v) \langle f \rangle + D_{\text{PS}}(\Psi, v) \frac{\partial \langle f \rangle}{\partial v} \right)$$

$$\frac{\partial \Psi}{\partial t} = \gamma_{\text{NL}}(\Psi, \langle f \rangle) \Psi$$

Dupree '72

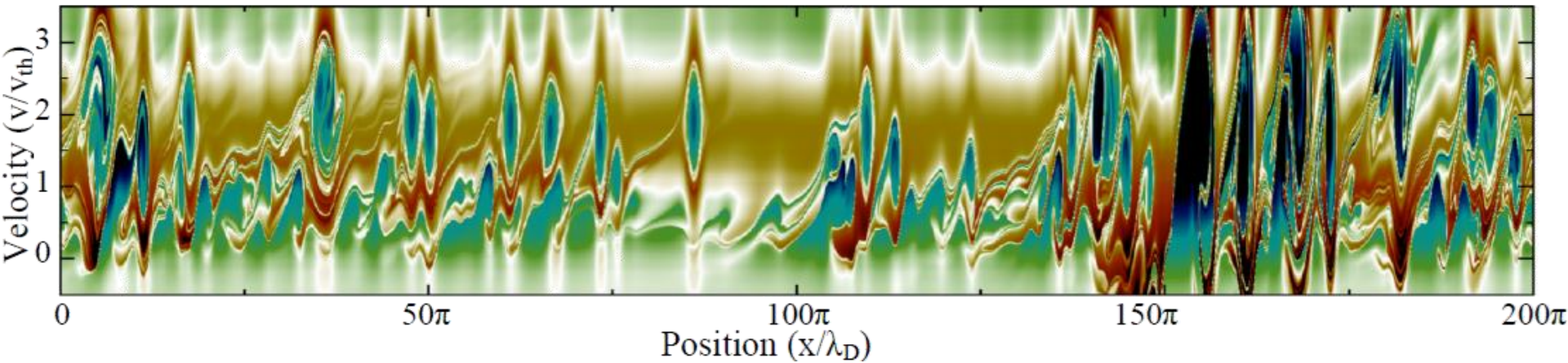
Evolution of phasestrophy

- Subcritical instabilities

Transport even though $\gamma_L < 0$ for all modes

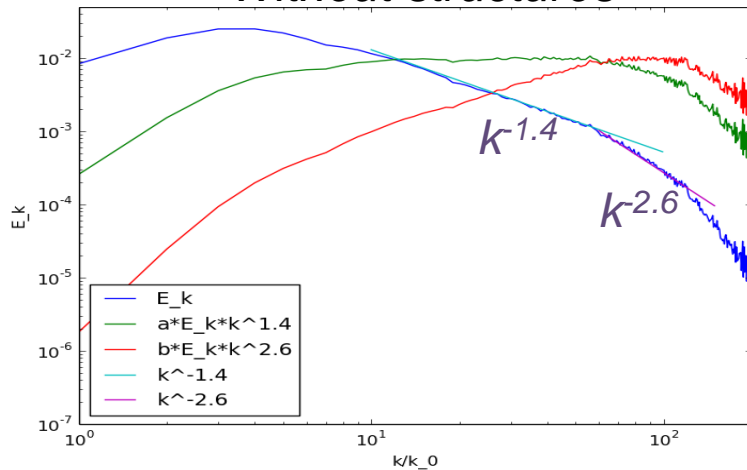
Beyond quasilinear theory

Statistics of interacting structures of various sizes

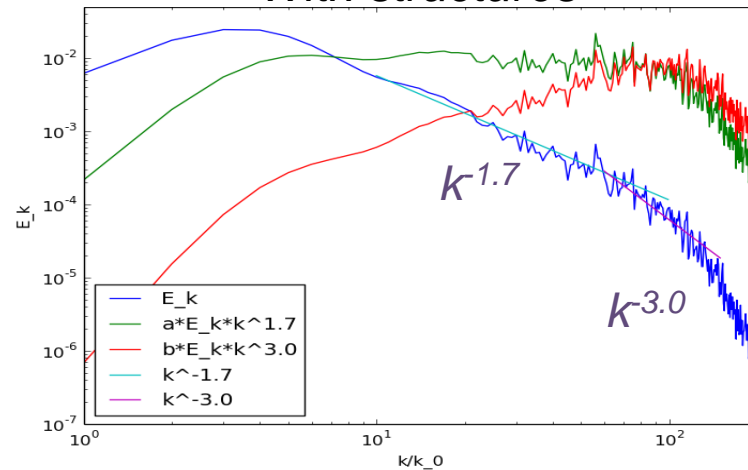


Spectral signature

Without structures



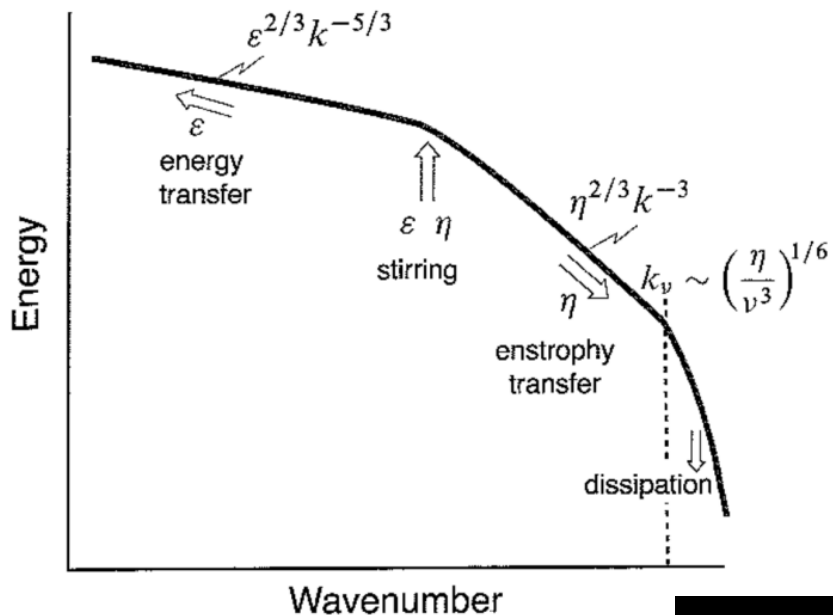
With structures



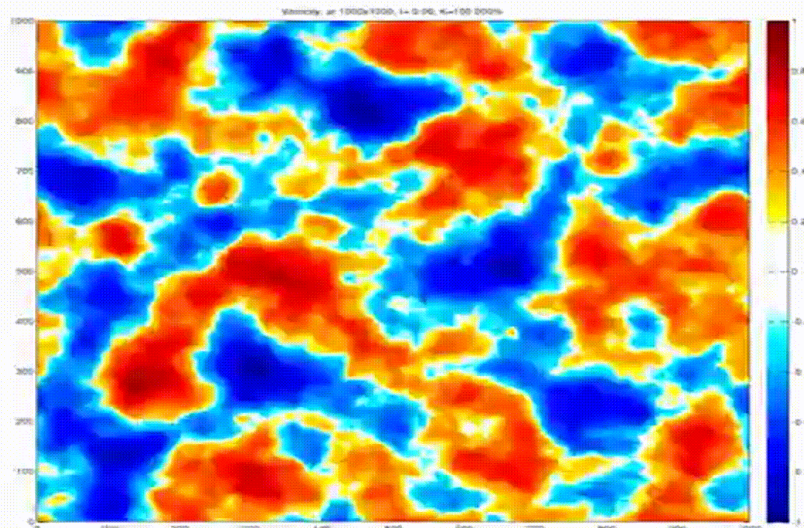
⇒ Many open questions remain

Analogy with descriptions of turbulence in real space

Collection of waves



Collection of vortices

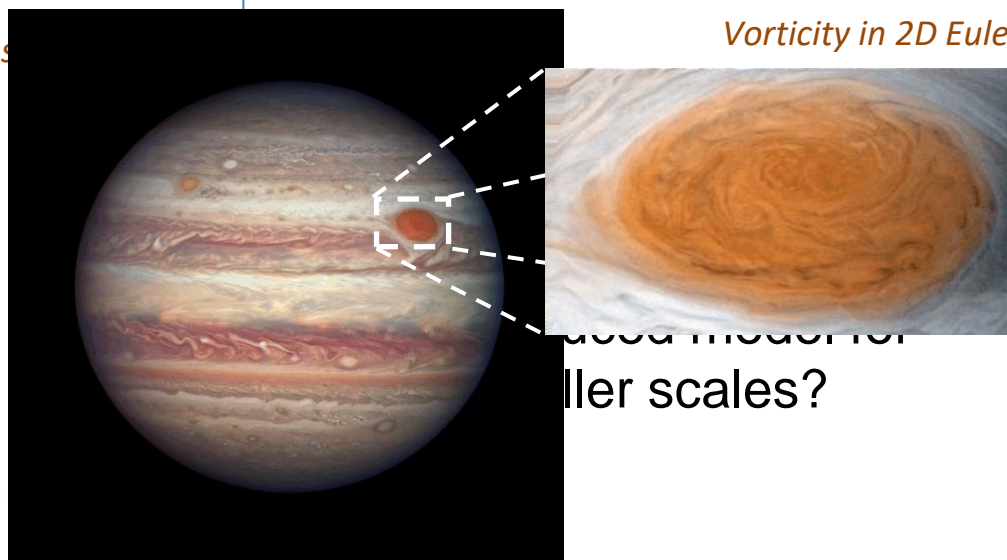


Illus

Vorticity in 2D Euler turbulence

- Energy transfers?
- Phase dynamics?
- Intermittency?

⇒ Limited range of applicability



smaller scales?

Outline

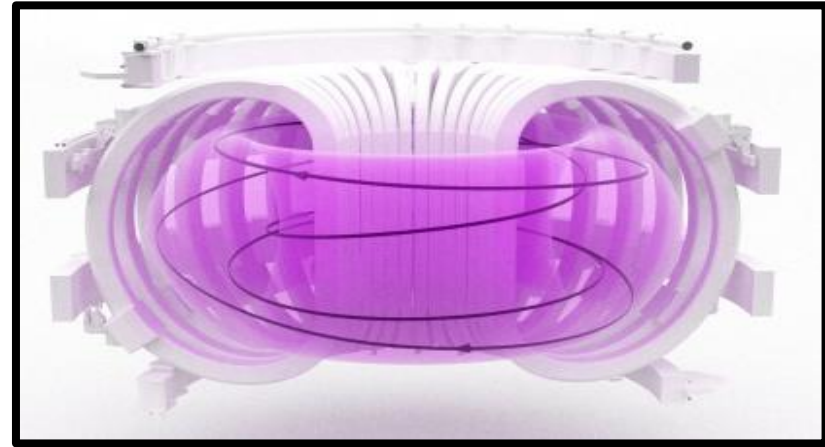
- I. Phase-space vortices
- II. Energetic-particle driven modes
- III. Ion-acoustic turbulence in **homogeneous** plasma
- ➔ IV. Drift-wave turbulence in **inhomogeneous** plasma

Summary

Electrostatic turbulence in tokamak core

Drift waves

- Driven by pressure gradients
- λ (\sim mm) \ll m.f.p. (\sim km)
- Turbulence \Rightarrow diffusion coefficient D
(Cost \sim diameter³ $\sim D^{3/2}$)



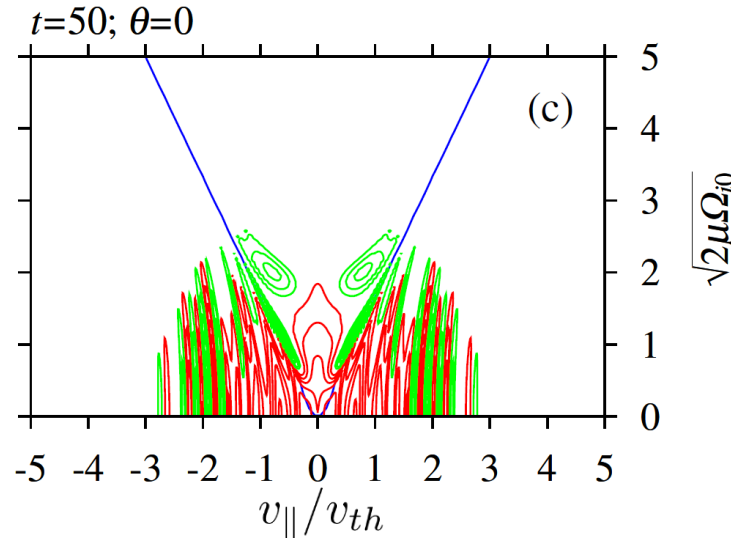
Analytic theory of PS structures

- Either robust structures e.g. “**drift-holes**”, *Terry, Diamond, Hahm '90*
or more subtle “**granulation**”. *Dupree '72* *Biglari, Diamond, Terry '88*
- Essential macroscopic impacts:
 - Stability
 - Turbulence, transport
 - Zonal flows
 - Corrugation, torque *Review: Kosuga '17*
- Challenging for measurements and simulations:
 $\Delta v \ll v_{th}$ $\Delta r \sim \rho_{ci} \sim$ mm $\tau \sim 1 - 10 \mu s$

Small-scale PS structures in gyrokinetics

Fine-scale structures in velocity-space

- Local (flux tube) GK simulation with $N_{v_{\parallel}} \times N_{\mu} = 1024 \times 64$



Watanabe&Sugama '06

Turbulent cascade of entropy in phase-space

- Driven by nonlinear phase-mixing
- Real space and velocity space are intertwined

Schekochihin '08

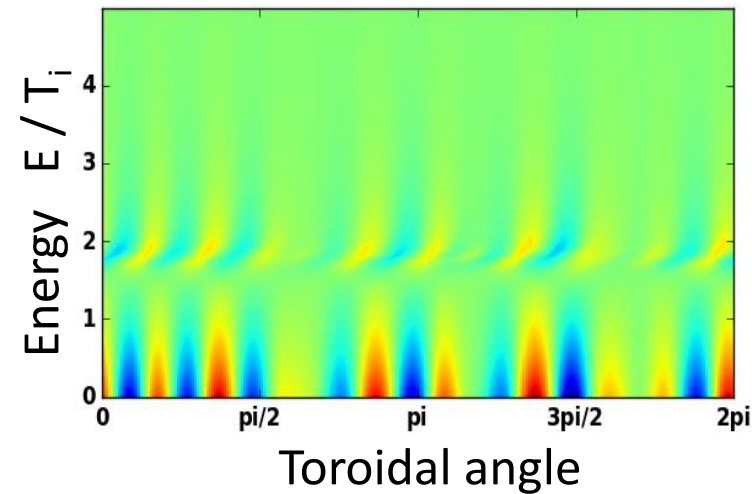
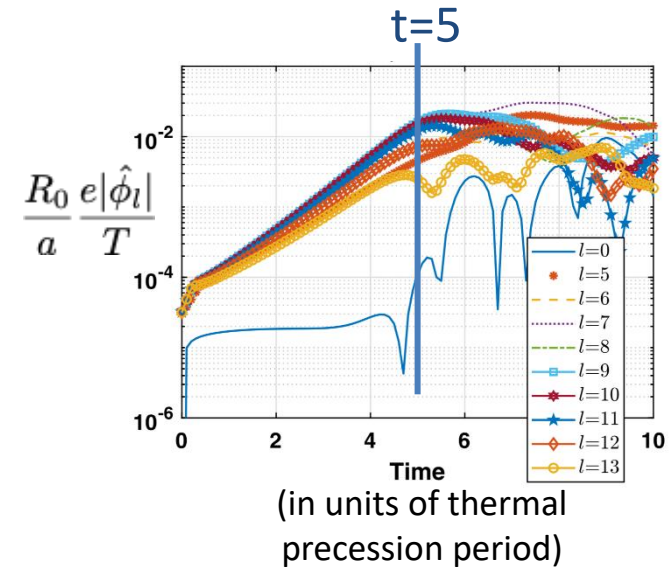
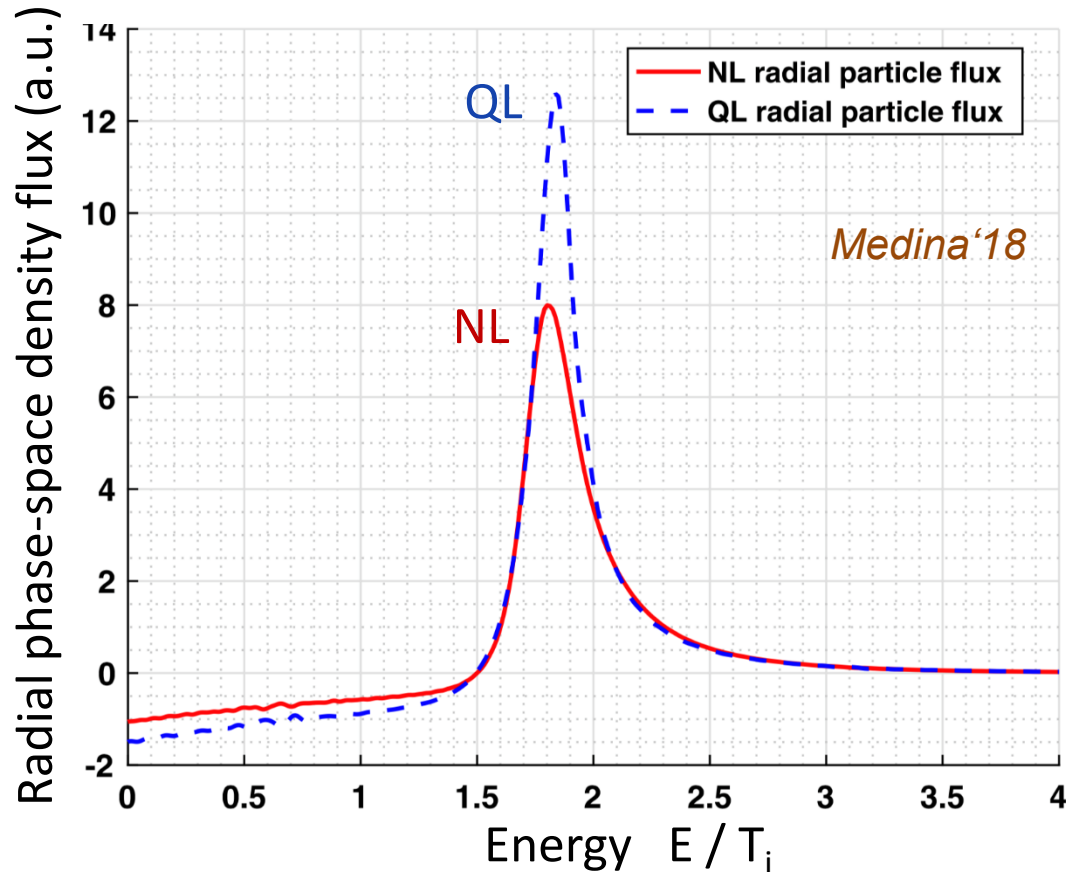
Tatsuno '09

⇒ **GK simulations may miss effects of small-scale PS structures**

Anatomy of TEM-driven transport (1/2)

$$\frac{\partial \langle f \rangle_\alpha}{\partial t} = \frac{\partial}{\partial \psi} \Lambda_\psi^{\text{NL}}(\psi, \underline{E}, t)$$

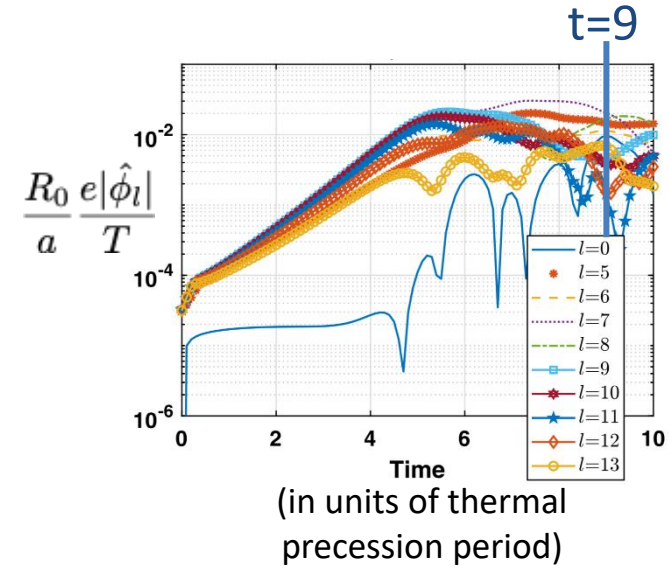
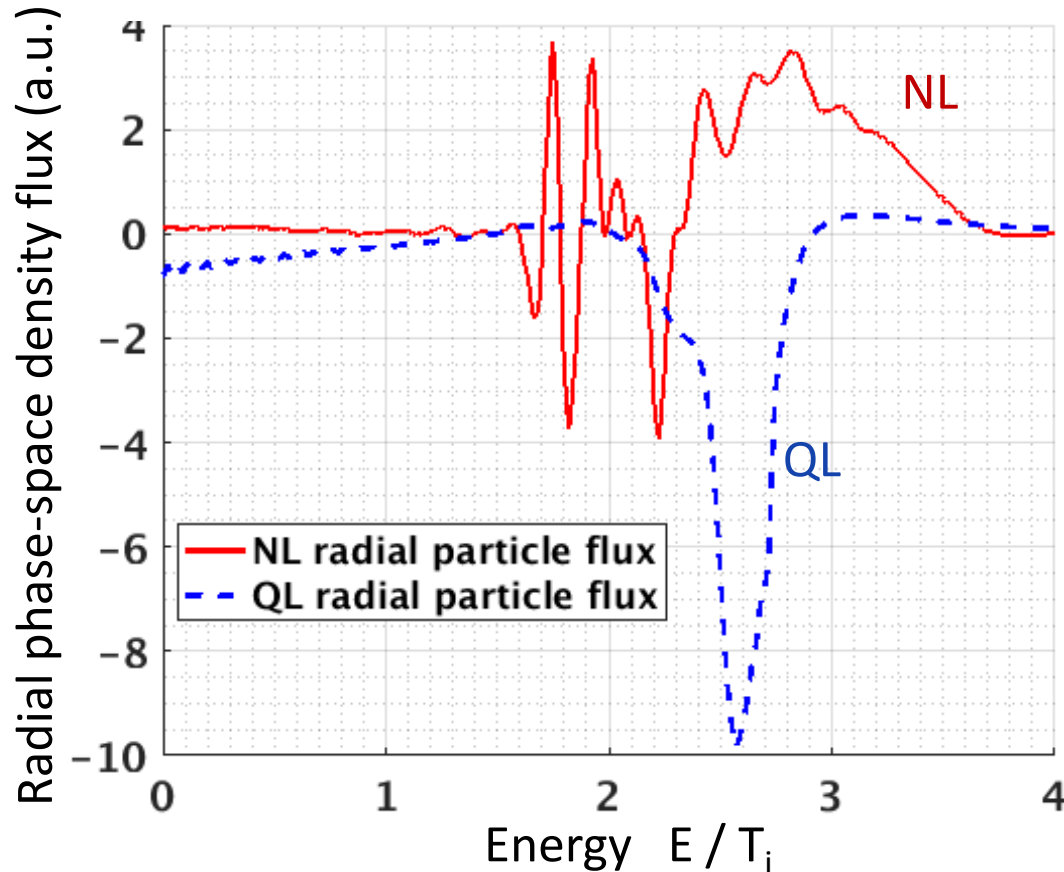
Radial phase-space density flux



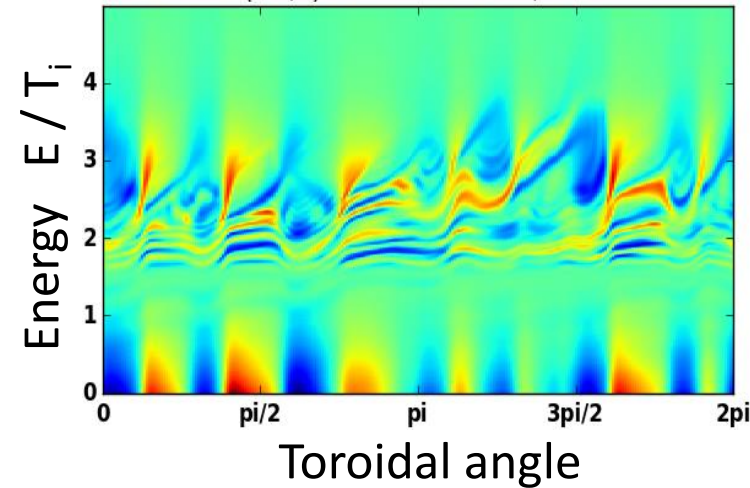
Anatomy of TEM-driven transport (2/2)

$$\frac{\partial \langle f \rangle_\alpha}{\partial t} = \frac{\partial}{\partial \psi} \Lambda_\psi^{\text{NL}}(\psi, \underline{E}, t)$$

Radial phase-space density flux



$$N_\alpha \times N_\psi \times N_E = 1024 \times 513 \times 2048$$

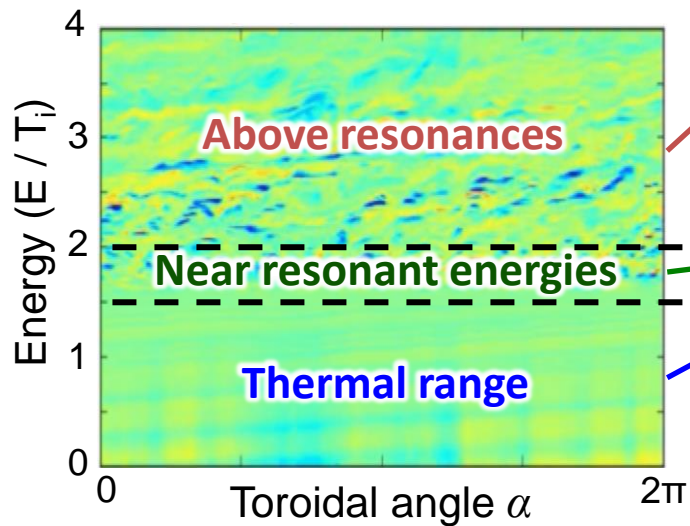


PS turbulence-driven transport

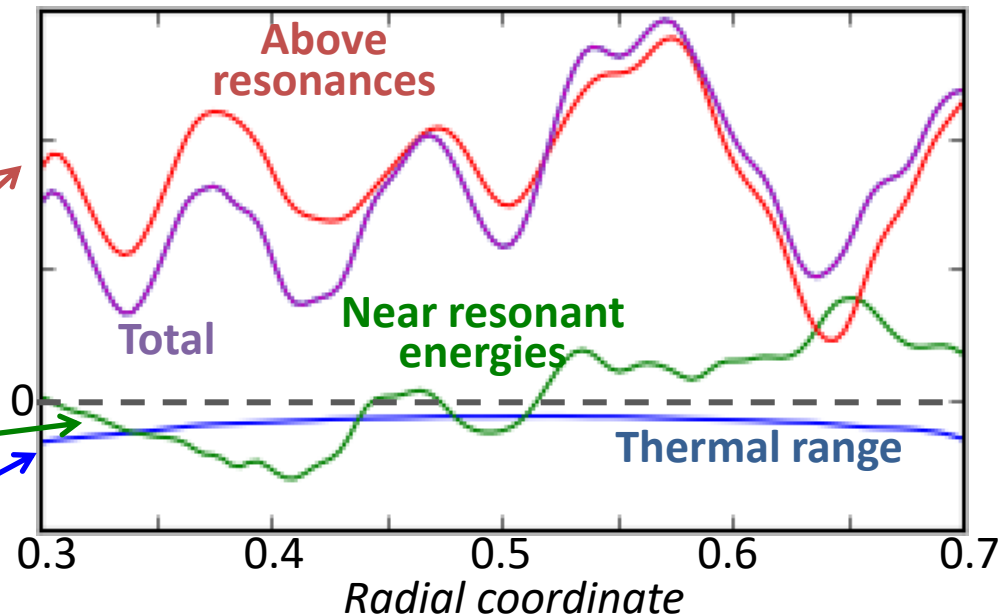
Heat balance $\frac{\partial(nT)}{\partial t} + \frac{\partial q}{\partial \psi} = 0$ $q \equiv \frac{2}{\sqrt{\pi}} \int \langle \Lambda_\psi \rangle E^{3/2} dE.$

Impact on heat transport

- Dominant effect in the flux
- Qualitatively: non-diffusive

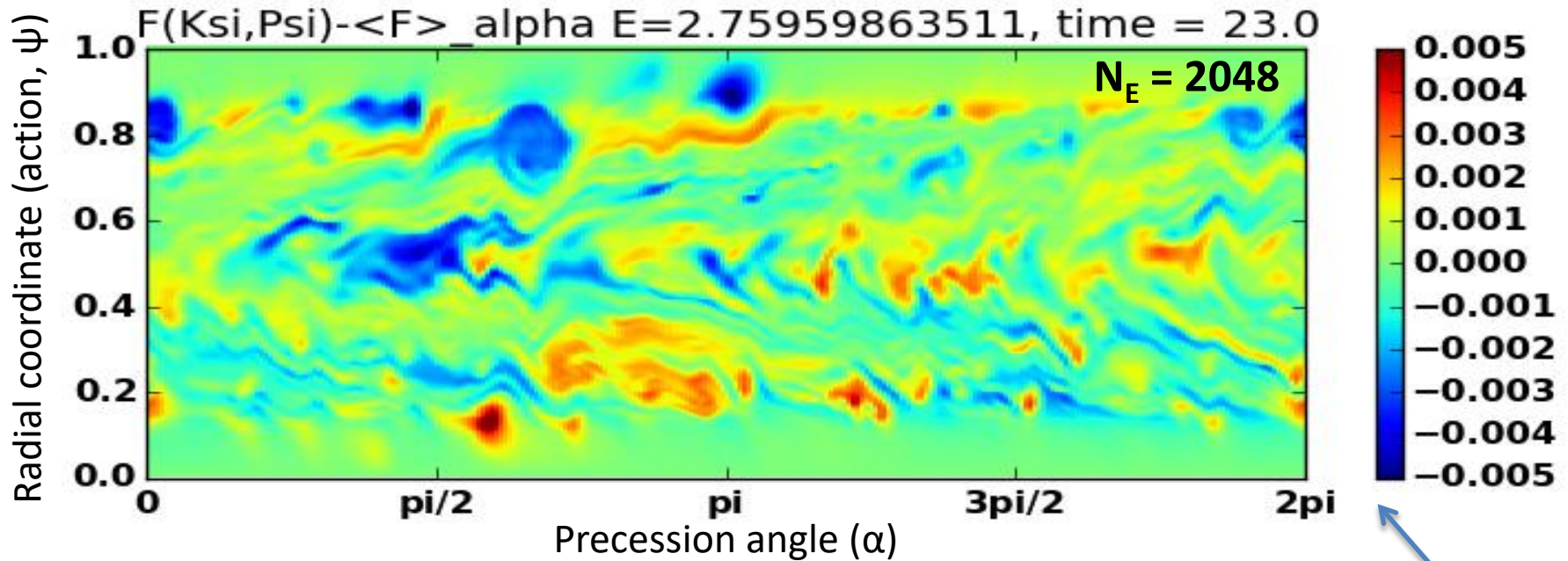


Heat flux q



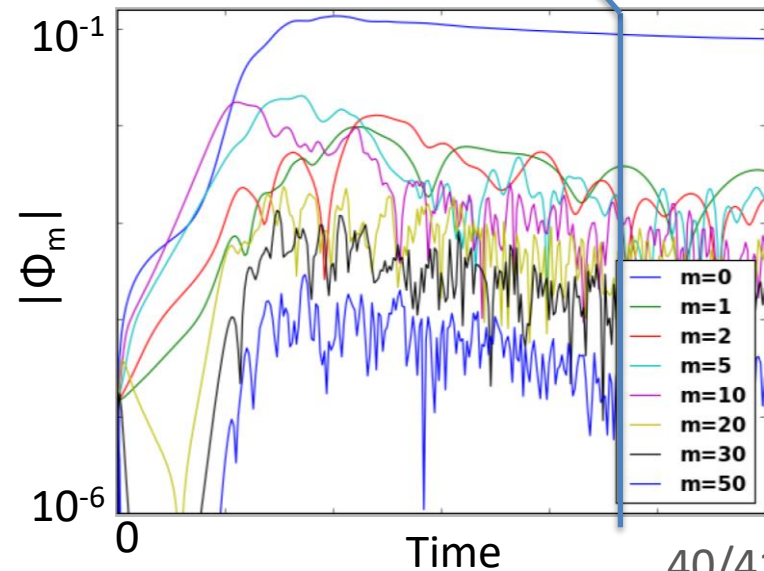
⇒ Small-scale phase-space structures drive particle and heat transport

Properties of drift-holes



Properties

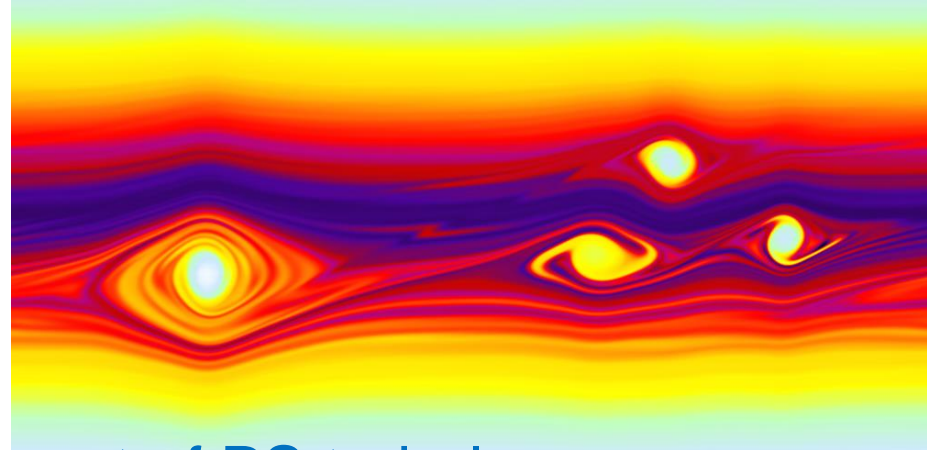
1. Self-organized from seeds born at the resonant energy
2. Evolve toward higher energies
3. Localized in radius
4. Localized in energy, with $\Delta E \ll T$
5. Lifetime \gg drift period
6. Local Kubo ~ 10 (global $K \lesssim 1$)



Summary & perspectives

In collisionless plasmas, PS structures spontaneously emerge, and their dynamics often dominate the nonlinear evolution

- Anomalous resistivity
- Anomalous heating
- Particle transport
- Fine-scales in velocity-space
- Departure from quasilinear formalism



Perspectives: theoretical development of PS turbulence

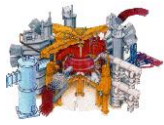
- Dynamics of several PS vortices in interaction
- Statistical mechanics of many PS vortices
- Simulation of steady-state PS turbulence in a stirred (flux-driven) system
- Comparison with experimental measurements

Acknowledgements

- Agence Nationale de la Recherche, project GRANUL (ANR-19-CE30-0005)
- Euratom research and training programmes, Grant Agreement No. 633053, project WP17-ENRCEA-02
- CINECA Marconi (projects GSNTIT and GSNTITS), EXPLOR (project 2017M4XXX0251), IDRIS



Backup slides follow



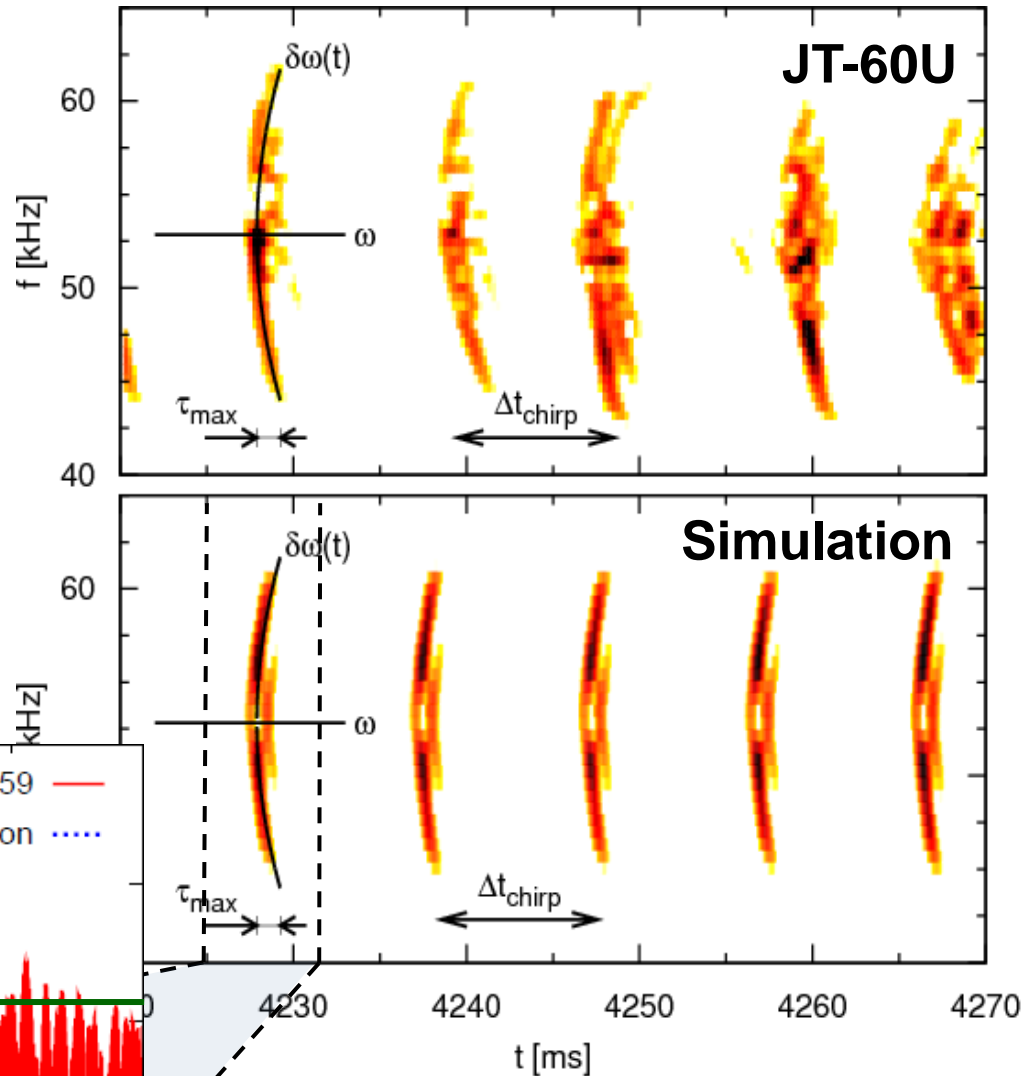
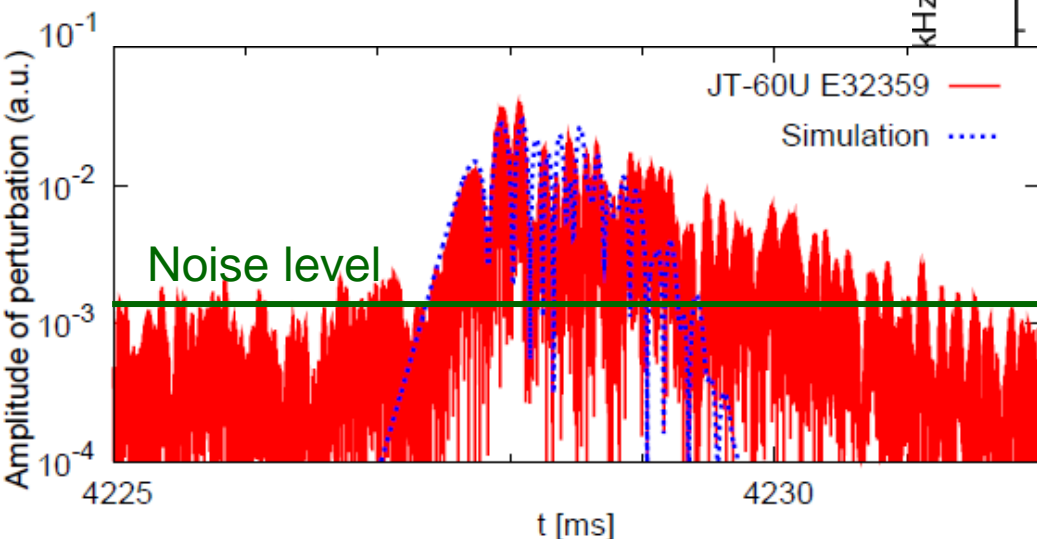
Analysis of JT-60U

Lesur, et al.
PoP(10)

By fitting chirping velocity, lifetime and period, we obtain linear parameters:

γ_L	γ_d	ν_f	ν_d	γ
8.8%	4.7%	0.42%	1.7%	4.6%

In agreement with calculation from experiment parameters



← Quantitative agreement for growth and decay of chirping structures



Analysis of MAST

Lesur, et al.,
submitted to NF

γ_L	γ_d	ν_f	ν_d	γ
9.9%	5.0%	0.55%	2.3%	5.5%
+0.4	+0.5	+0.03	+0.05	+0.1
-0.2	-0.2	-0.05	-0.05	-0.1

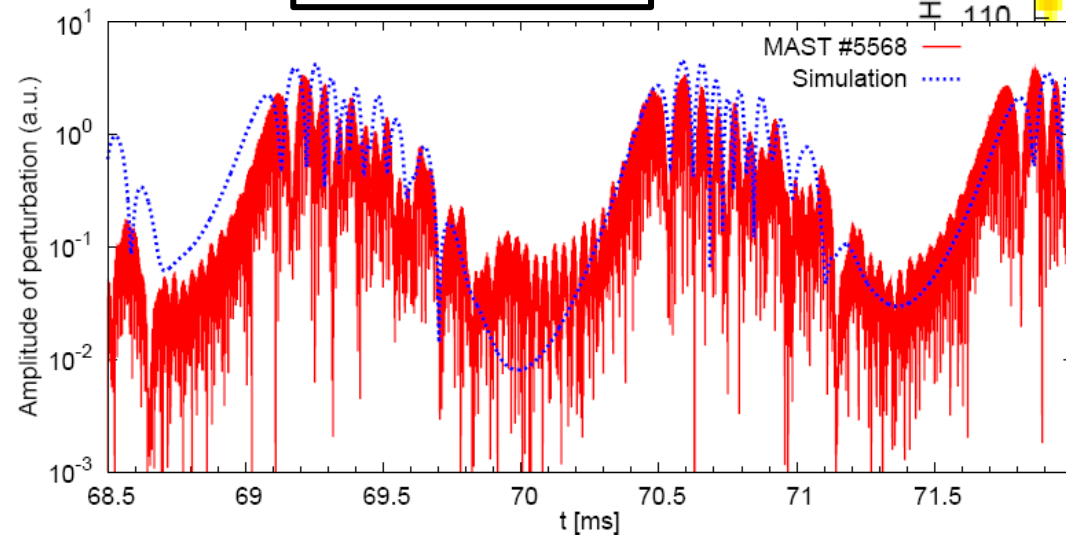
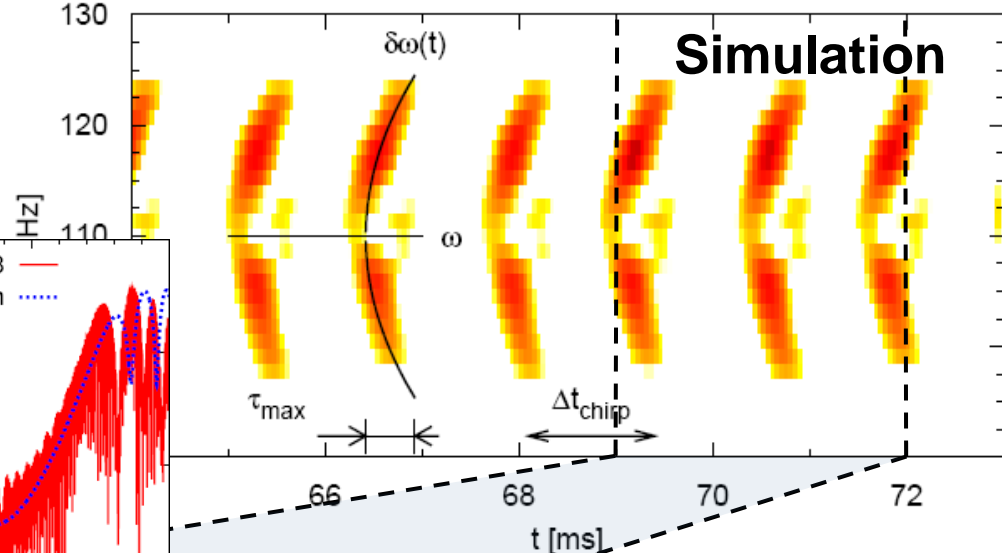
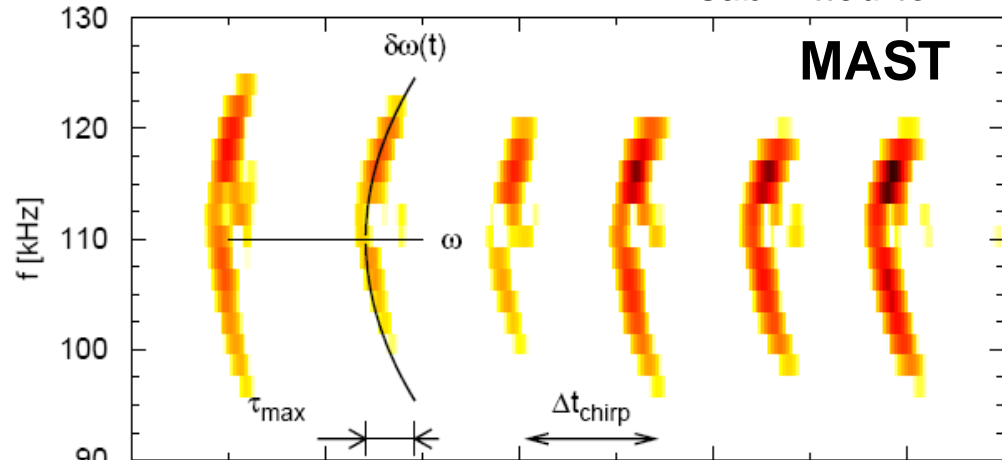
Eq. profiles

very sensitive to

γ_L, γ_d

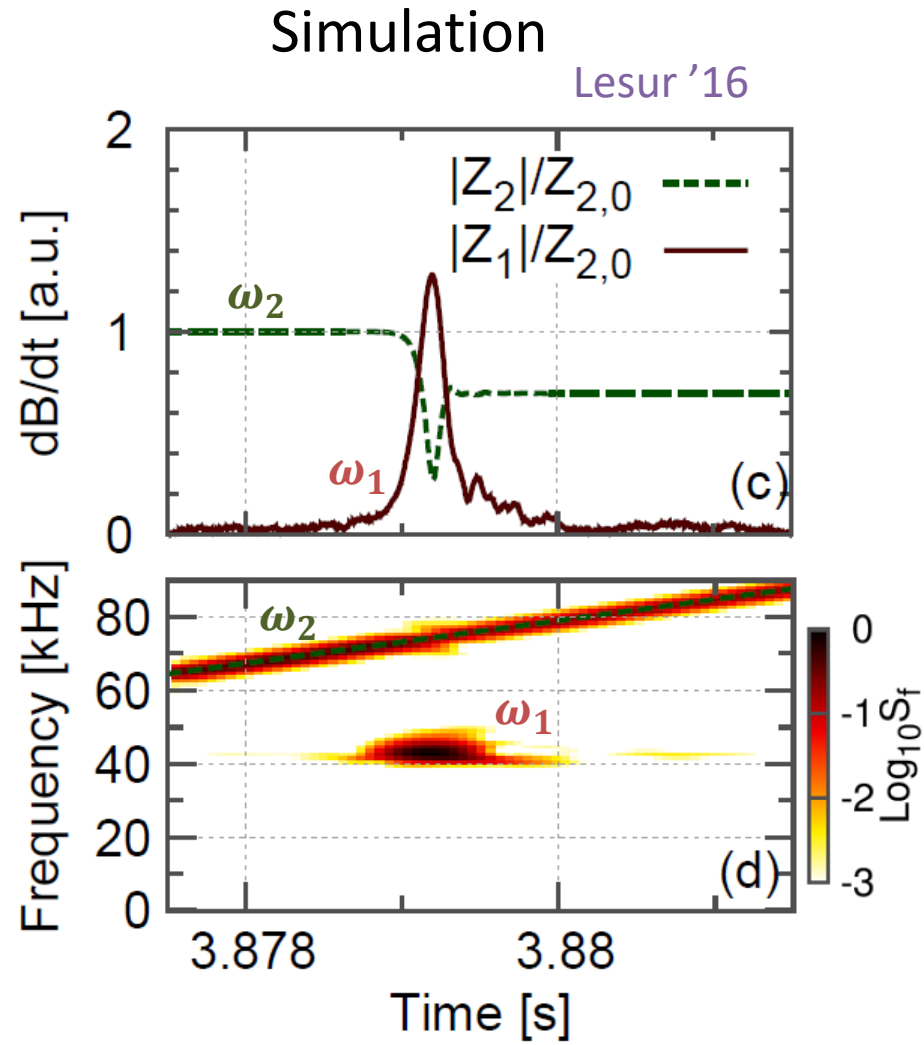
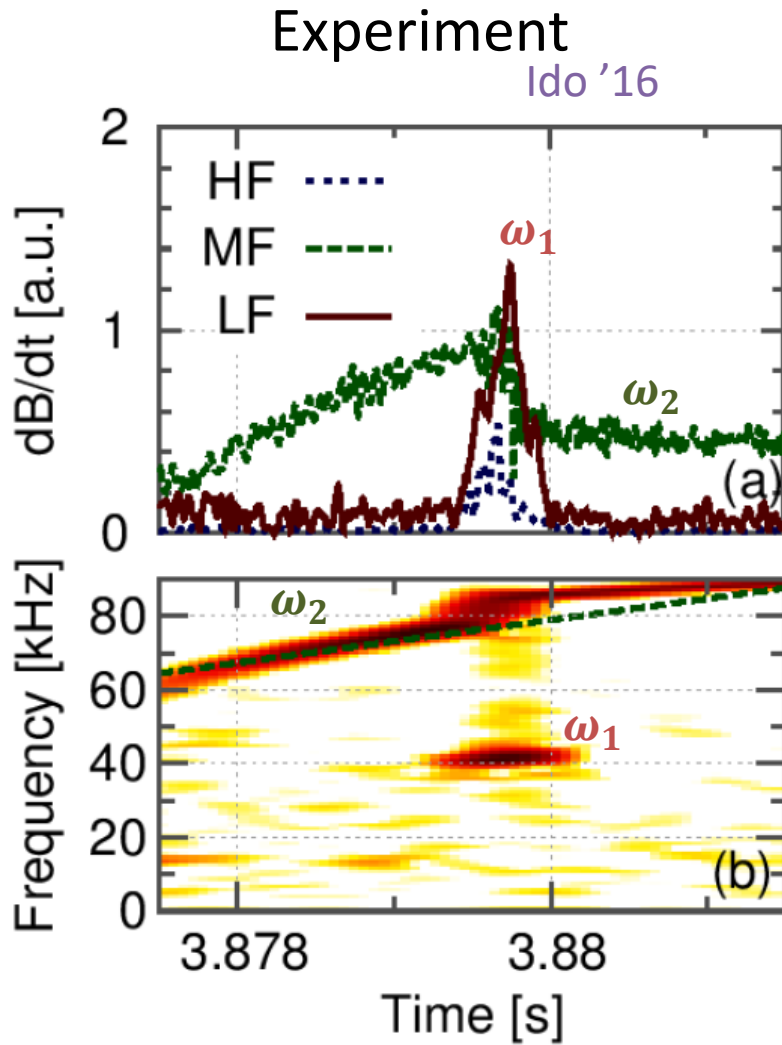
sensitive to

NL features



Quantitative agreement for growth and decay of chirping structures

New model can reproduce the observation



Our new model, which couples 1D kinetic equation with wave coupling equations, reproduces many features of the experiment.

More details on the input parameters of the BB model

- γ_L depends on:
 - the gradients of the distribution function in energy and in P_ζ .
 - the alignment of the orbit with the eigenmode,
 - the strength of the various resonances,

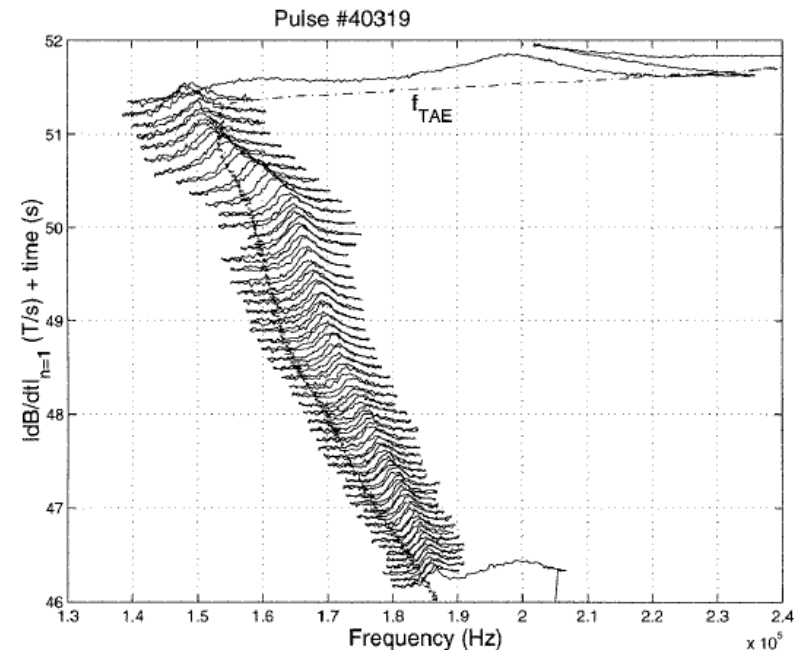
Too complicated to calculate analytically in general.

Numerically, requires kinetic-MHD computations and internal diagnostics.

- γ_d involves still-debated complex mechanisms:
 - Ion Landau damping
Betti, Freidberg, PF(74)
 - Radiative damping
Mett, Mahajan, PF(92)
 - Collisional damping by trapped electrons
Gorelenkov, Sharapov, Phys.Scr.(92)
 - Continuum damping
Rosenbluth, et al., PRL(92)
Zonca, Chen, PRL(92)

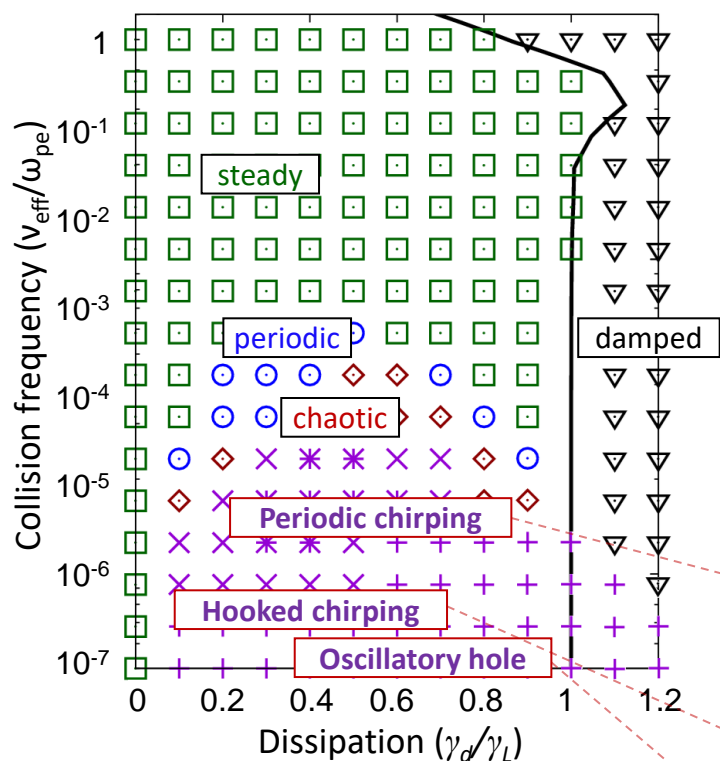
Comprehensive kinetic simulations not available yet.

Measured in dedicated experiments only.



Fasoli, et al., PPCF(97)

Additional examples of comparison with experiments

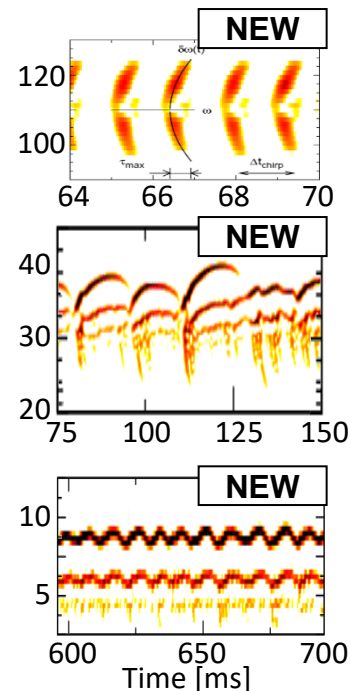
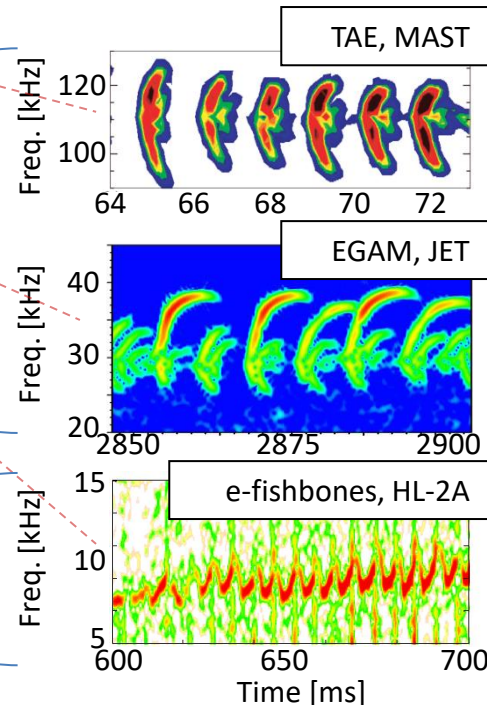


Fort transport

Faible trnspt.

Experiments

BB simulations

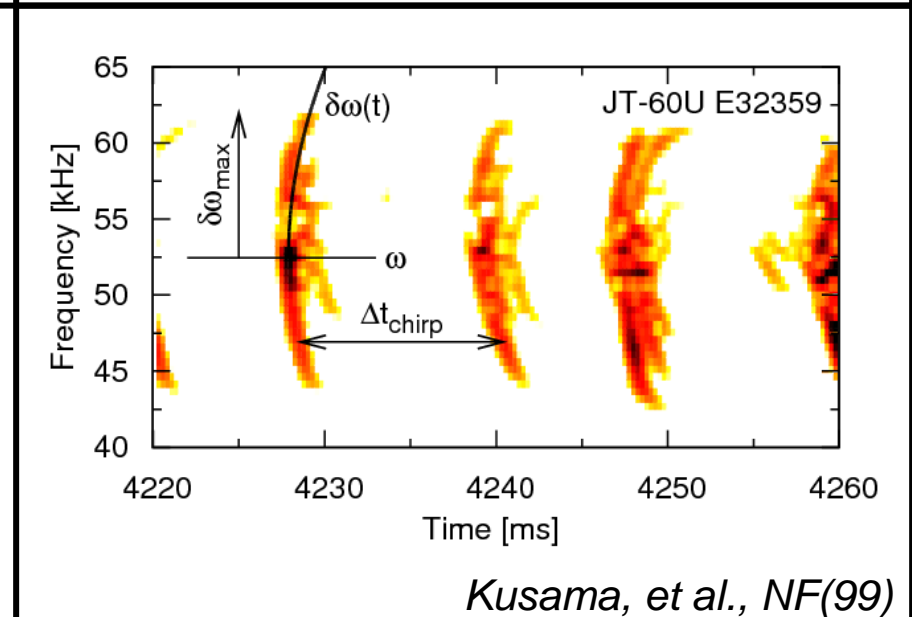
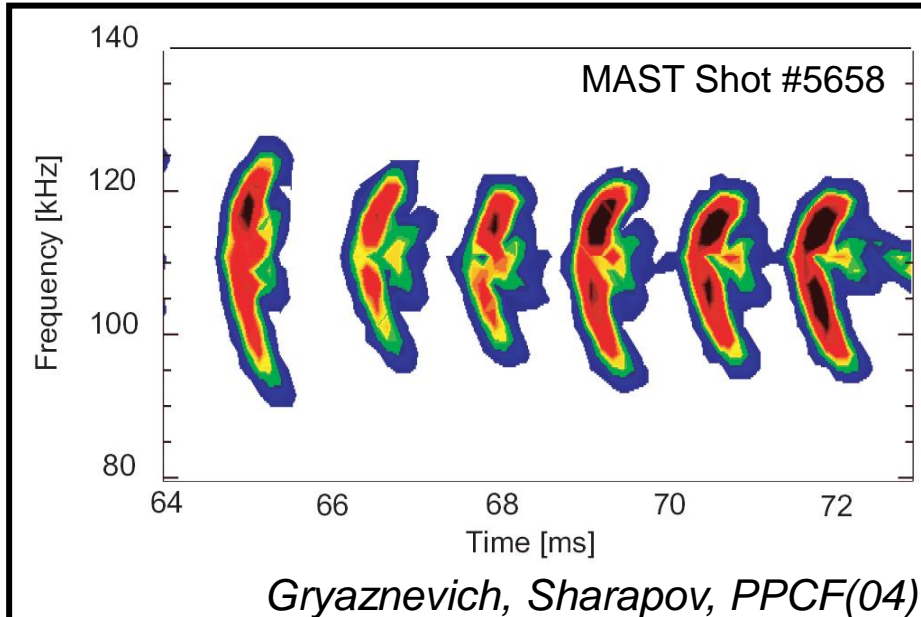
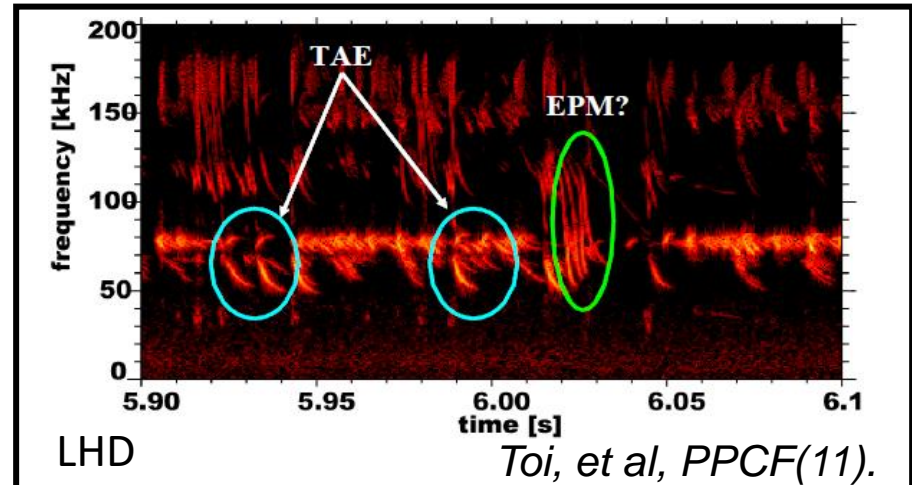




Chirping (frequency sweeping)

NL Chirping

- Chaotic behavior,
- Splitting into two spectral modes, up and down in general,
- Lifetime \ll equilb. evolution time.
- Hole/clumps *Berk, et al., PLA(97)*

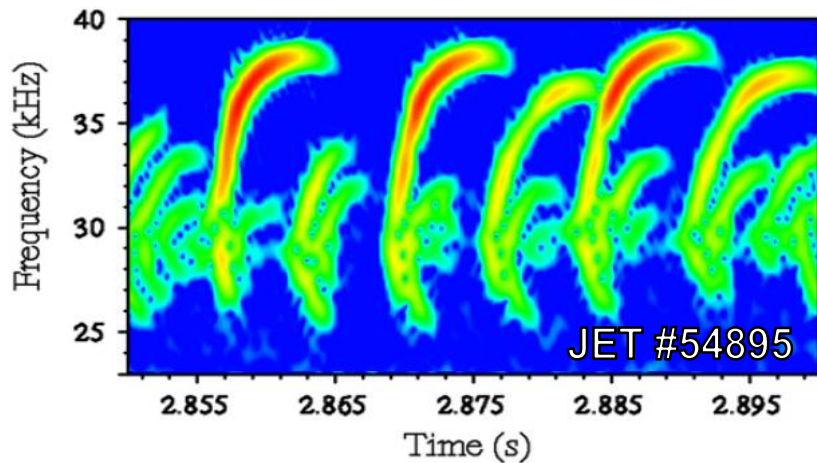


Observations of chirping TAEs

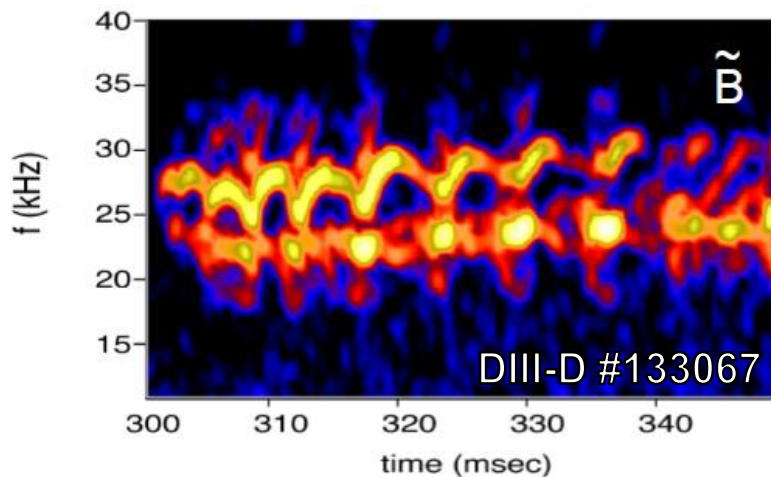


Chirping (2)

Chirping GAMs

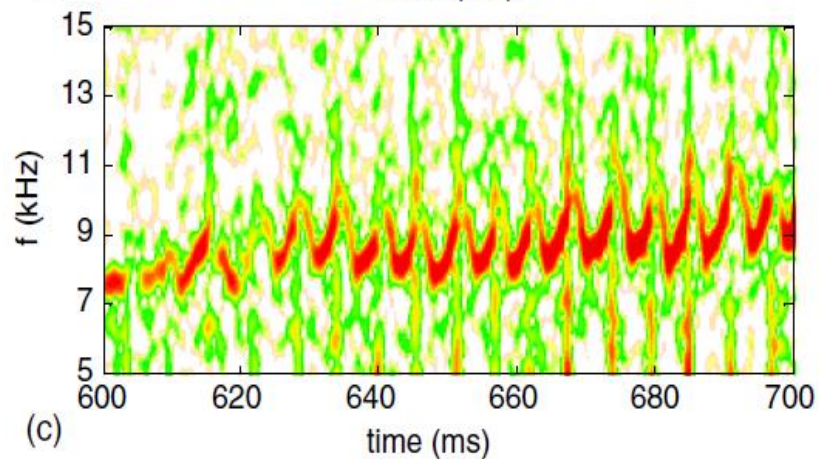
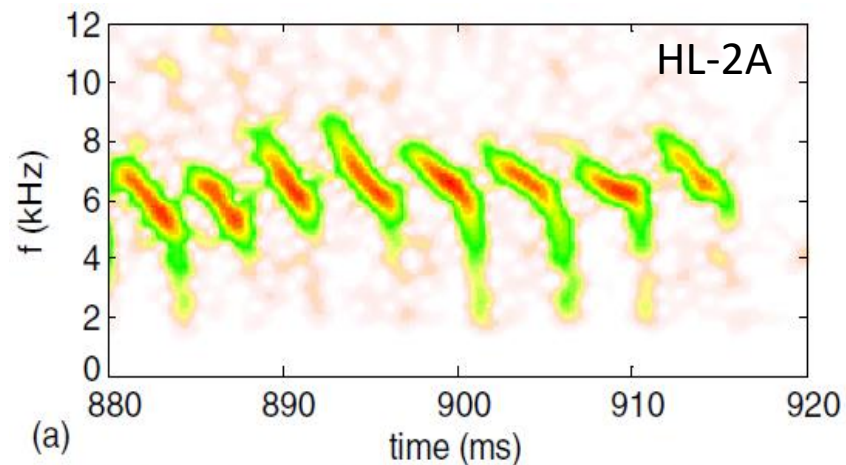


Berk, et al., NF(06)



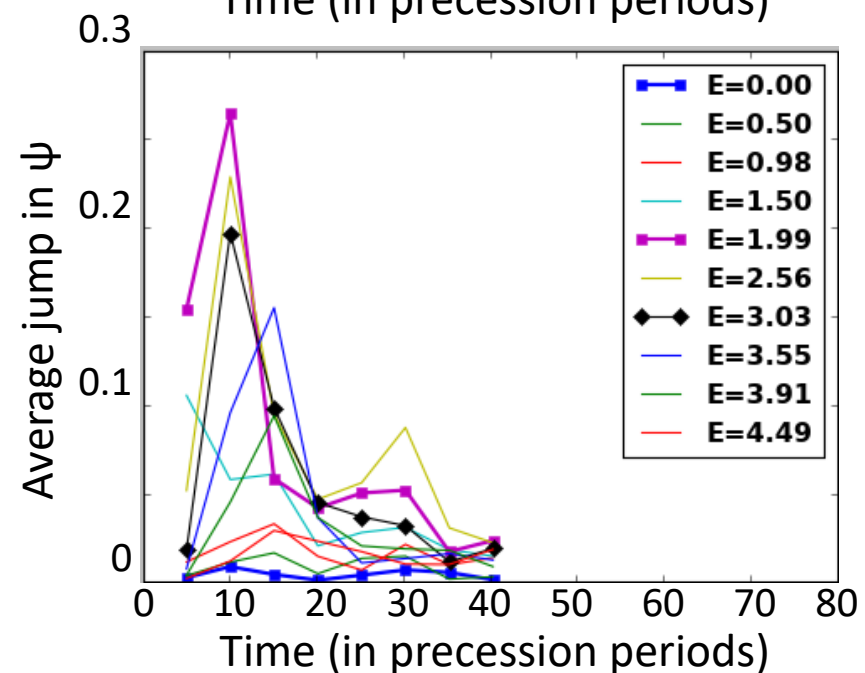
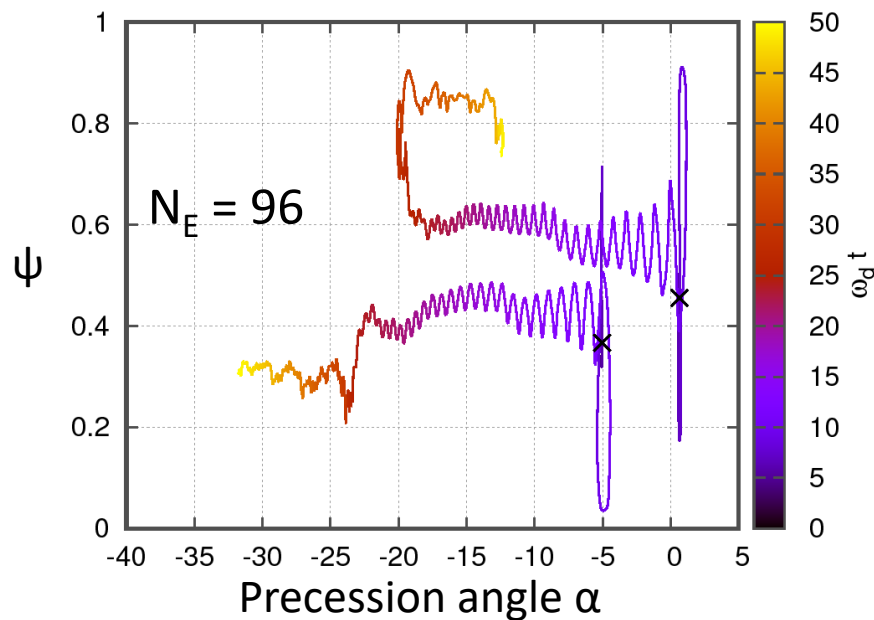
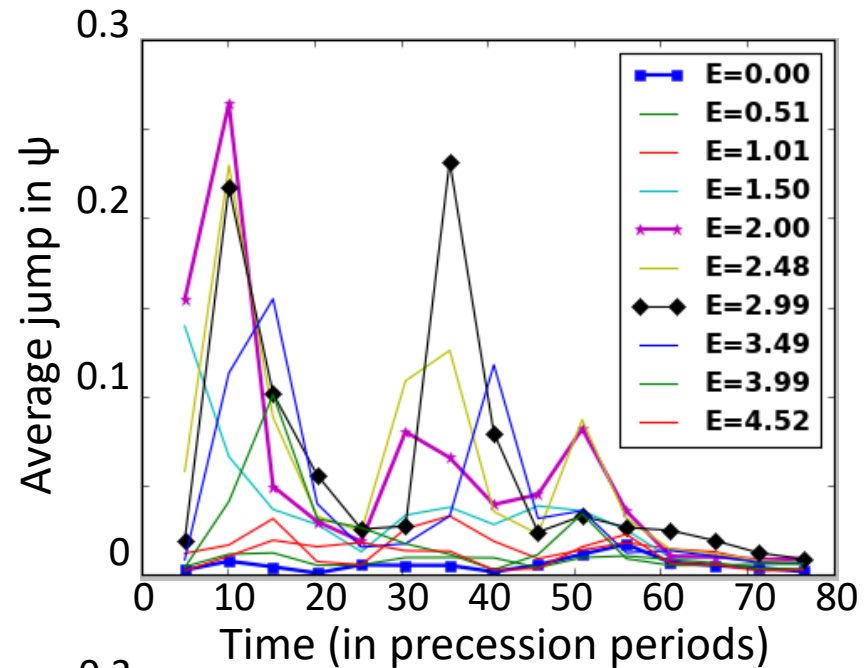
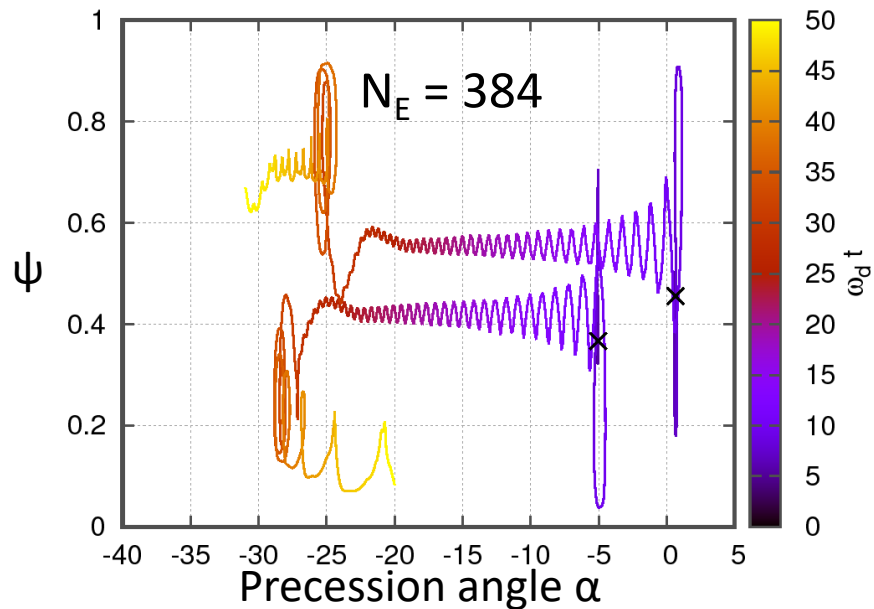
Nazikian, et al., PRL(08)

Chirping e-fishbones



Chen, et al., NF(10)

Particle orbits



... many caveats.

Reduced (toy) model

- Prototype for low-frequency turbulence
 - ⇒ Coupling with higher-frequency turbulence is neglected
 - ⇒ Quasi-periodic orbits with faster time-scale are averaged out
- Heavily simplified geometry (but retain toroidal effects and inhomogeneity of equilibrium plasma)

Decaying turbulence

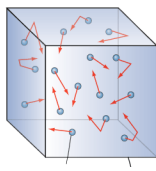
- Initial perturbations, no stirring. Only the transient behavior is analyzed.

Weak turbulence

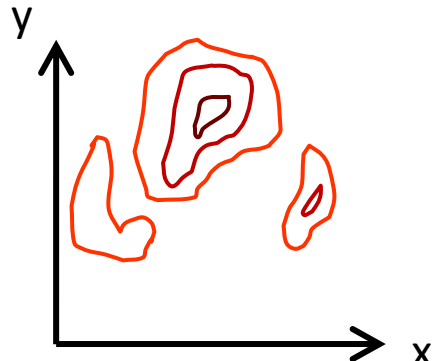
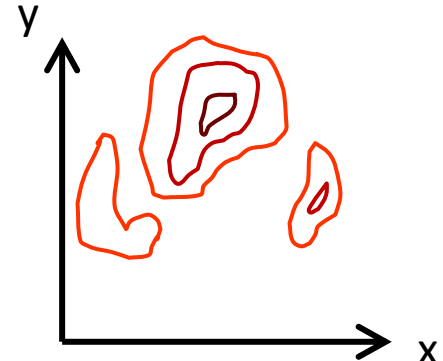
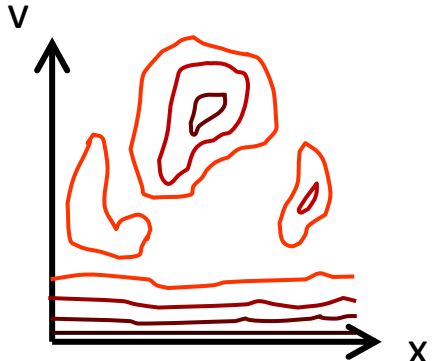
- Small equilibrium gradients
- Small system size

No collision operator

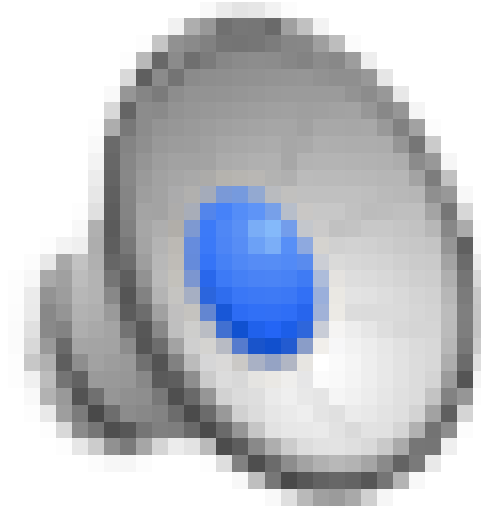
⇒ Future work: remove these caveats one by one



Kinetic theory

System	Many incompress. fluids in 2D	Quasi-geostrophic fluid	1D collisionless plasma
Distribution	$n(x, y, t)$	$\omega(x, y, t)$	$f(x, v, t)$
Description space	2D config. space 	2D config. space 	2D phase space 
Continuity equation	$\frac{\partial n}{\partial t} + u_x \frac{\partial n}{\partial x} + u_y \frac{\partial n}{\partial y} = 0$	$\frac{\partial \omega}{\partial t} + u_x \frac{\partial \omega}{\partial x} + u_y \frac{\partial \omega}{\partial y} = 0$	$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial v} = 0$
Self-consistency	Hierarchy of fluid equations + closure	Stream function $\omega = \nabla^2 \psi$	Poisson $\frac{\partial E}{\partial x} = \sum_s q_s \int f_s dx$

Turbulent case



Single mode, constant precession freq.

