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On the number of crossings and bouncings of a diffusion at a sticky threshold

Alexis Anagnostakis^{*1} Sara Mazzonetto^{†2}

¹Université Grenoble-Alpes, CNRS, LJK, F-38000, Grenoble, France

²Université de Lorraine, CNRS, IECL, Inria, F-54000 Nancy, France

Abstract

We distinguish between three types of crossing of a sticky threshold by a one-dimensional diffusion. We examine the limit behavior of the corresponding number of crossings statistics and prove that each of these statistics has a different asymptotic regime. We define the notions of bouncing as the symmetric counterparts of crossings. We prove similar results for the number of bouncings at reflecting and non-reflecting sticky thresholds. Using these, we develop consistent stickiness parameter estimators for sticky diffusions and sticky-reflecting Brownian motion.

1. Introduction

The subject of threshold crossing by a process has been extensively studied in probabilistic literature. For a one-dimensional diffusion that solves a classical stochastic differential equation (SDE) with sufficiently regular coefficients, the number of crossings of a threshold $\zeta \in \mathbb{R}$ is known to provide consistent estimations of the local time at ζ (e.g. [6, 20, 21]). For a definition of local time see Section 2.1. Similar results were devised for a diffusion with a skew threshold or with irregular coefficients. These allow to estimate localized features of a diffusion, like the localized diffusion coefficient (e.g., [14]), the skew parameter [25, 26] and the oscillation jump of the diffusion coefficient [26]. Despite these advances, at a sticky threshold, the limit behavior is not well understood.

In this paper, we study the limit behavior of the number of crossings of the sticky threshold 0 by the sticky Brownian motion. The sticky Brownian motion is a diffusion process that behaves like the standard Brownian motion away from 0, and whose zeroes form a totally disconnected random set of positive measure (see a definition in Section 2.2). This contrasts with the standard Brownian motion, whose zeroes form an uncountable, totally disconnected set of Lebesgue measure 0 (see, e.g., [29, Proposition III.3.12]).

*alexis.anagnostakis@univ-grenoble-alpes.fr

†sara.mazzonetto@univ-lorraine.fr

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To the best of our knowledge, the only established result about number of crossings for a sticky Brownian motion $(X_t)_{t \geq 0}$ is the following: as n goes to infinity, the statistic

$$\sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0}$$

converges in law to some non-trivial discrete random variable (see [28, §8], stated in the next Theorem 2.8). This contrasts with the standard Brownian motion, where this statistic, renormalized by the sequence $\sqrt{\pi/2n}$, converges in probability to its local time at 0.

In the case of sticky Brownian motion, we distinguish between three different kinds of crossings and show that, for each type of crossing, establish the limit behavior of the respective re-normalized number of crossings statistic. Unlike the case of the non-sticky Brownian motion, we show that the normalizing sequence depends on the type of crossing considered. Understanding the limit behavior of these statistics is crucial for obtaining the convergence rates to the local time at a sticky threshold. See [20, Theorem 1.2] and its proof for a classical diffusion and [11, Theorem 1] and its proof for a skew threshold.

We introduce the symmetric counterpart of crossings of the process, called bouncings. As with crossings, we distinguish three kinds of bouncing and prove that for each kind of bouncing, the respective number of bouncings has the same asymptotic properties as the number of crossings of the same kind. We combine the results on bouncings and crossings to establish results on the Brownian motion with sticky reflection at 0 (see Section 5.1 for a definition), where only bouncing behavior is possible, and also on some sticky Itô diffusions. We use these results to set up a consistent stickiness parameter estimator and numerically show that it converges at least as fast as the one devised from other statistics in [3, 5].

The paper is organized as follows: In Section 2, we introduce the sticky Brownian motion along with several notions and results, which are useful for this paper. In Section 3, we state the main results of this paper, namely, the limit behavior of the considered number of crossings and bouncings statistics (Theorems 3.1, 3.2) and the consistency of a stickiness parameter estimator based on these (Proposition 3.4). In Section 4, we prove these results. In Section 5, we extend the results to the sticky-reflected Brownian motion and to smooth Itô diffusions with a sticky point. Section 6 is dedicated to numerical experiments on the stickiness parameter estimator devised in this paper.

In the Appendix we prove several useful results for our derivations. More precisely, in Appendix A, we prove asymptotic results on the sticky Brownian motion transition kernel. In Appendix B, we prove a reflection principle at zero for the sticky Brownian motion.

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2. Preliminary notions and results

In this section we state preliminary notions and results useful for the rest of the paper. We begin by defining the sticky Brownian motion, recall its probability transition kernel

and its space-time scaling property. We then recall two notions of convergence. The first is the convergence uniform in time, in probability, in which the local time approximation results are expressed and which retains the functional character of the approximation. The second is the conditional convergence in probability, in which the estimation results are expressed. The estimators are indeed consistent only on the event that the threshold of interest (0 in our case) is hit by the process. Last, we state convergence results, in particular a local time approximation result previously established in [5], to which the proofs of our main results are reduced to.

2.1. Local time

We now define the local time which is an object that plays a fundamental role in our analysis. Indeed, it will be in many cases the limiting process of our statistics of interest.

Let $\mathcal{P} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space. The local time of a semimartingale X , defined on this space, is the random field $(L_t^a(X); a \in \mathbb{R}, t \geq 0)$, defined for all $t \geq 0, a \in \mathbb{R}$ as the term $L_t^a(X)$ so that the following equation holds (see [29, Theorem 1.2])

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + L_t^a(X).$$

In the above equation we used the convention $\operatorname{sgn}(x) = \mathbb{1}_{x>0} - \mathbb{1}_{x \leq 0}$.

For all $a \in \mathbb{R}, t \mapsto L_t^a(X)$ is increasing and is fully supported on $\{t \geq 0 : X_t = a\}$. This means that

$$\forall t \geq 0 : \int_0^t \mathbb{1}_{|X_s - a| \neq 0} dL_s^a(X) = 0. \quad (1)$$

The local time field can alternatively be defined as an almost sure limit (see [29, Corollary VI.1.9]):

$$\forall t \geq 0, a \in \mathbb{R}, \quad \mathbb{P}\text{-a.s.} : L_t^a(X) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{X_s \in [a, a+\epsilon)} d\langle X \rangle_s.$$

2.2. The sticky Brownian motion

The sticky Brownian motion of stickiness parameter $\rho > 0$ is the diffusion process on \mathbb{R} , on natural scale, with speed measure $m(dx) = dx + \rho \delta_0(dx)$ (see, e.g., [8, p.123-124]). The parameter $\rho > 0$, called stickiness parameter, expresses the propensity of the process to stick at 0. The higher it is, the more time it spends on average at 0. The asymptotic case $\rho = 0$ corresponds to the standard Brownian motion and $\rho = \infty$ to the Brownian motion with an absorbing boundary at 0. In this paper we deal only with the case $\rho \in (0, \infty)$. The standard Brownian motion case is known and the absorbing case is trivial.

We now provide some useful equivalent characterizations of sticky Brownian motion. Let $\mathcal{P}_x = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ be a filtered probability space. The following hold (see e.g. [19, Sections 5.1, 5.2], [30, Theorem 47.1 and Remark (ii), p.277]):

- t1** Let X be a sticky Brownian motion of stickiness parameter ρ , defined on \mathcal{P}_x such that \mathbb{P}_x -a.s., $X_0 = x$. (In particular, X is $(\mathcal{F}_t)_{t \geq 0}$ -adapted.) There exists a Brownian

motion Z , defined on an extension of \mathcal{P}_x , such that $X = (Z_{\gamma(t)})_{t \geq 0}$, where the time-change γ is the right-inverse of

$$A(t) := t + \rho L_t^0(Z), \quad t \geq 0,$$

given by $\gamma(t) := \inf\{s > 0: A(s) > t\}$ and where $L_t^0(Z)$ is the right local time at 0 of the process Z .

t2 Let Z be a standard Brownian motion, defined on the probability space \mathcal{P}_x , such that \mathbb{P}_x -a.s.: $Z_0 = x$. Then, the process $X := (Z_{\gamma(t)})_{t \geq 0}$ with γ defined in **t1**, is a sticky Brownian motion of stickiness parameter ρ , and, \mathbb{P}_x -a.s.: $X_0 = x$.

Also, the following SDE characterization holds.

p1 Let X be a sticky Brownian motion of stickiness parameter ρ , defined on the probability space \mathcal{P}_x , such that \mathbb{P}_x -a.s., $X_0 = x$. There exists a Brownian motion W defined on an extension of \mathcal{P}_x such that (X, W) solves

$$\begin{cases} dX_t = \mathbb{1}_{X_t \neq 0} dW_t, & X_0 = x, \\ \mathbb{1}_{X_t = 0} dt = \rho dL_t^0(X). \end{cases} \quad (2)$$

p2 The system (2) has a jointly unique weak solution [12, Theorem 1] and it is a sticky Brownian motion of stickiness parameter ρ .

A proof of the last two statements for some sticky diffusions can be found respectively in [3, Proposition 4.2] for **p1** and in [3, Theorem 4.1] for **p2**. For sticky Brownian motion these proofs are contained in the proof of [12, Theorem 1].

For an historical overview of the sticky Brownian motion, see [27]. For results on this process, see, e.g., [2, 3, 5, 10, 18, 31]. For applications, see [9] and references therein.

Remark 2.1 (Occupation time and local time). The second line of the system (2) states that the occupation time of the process at 0 is proportional to its local time at 0. This shows that the process spends a positive amount of time at the threshold 0.

2.3. Notions of convergence

The local time approximation results are expressed in the following type of convergence.

Definition 2.2. Let $(A^n)_{n \geq 0}$ be a sequence of processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We say that $(A^n)_{n \geq 1}$ converges locally uniformly in time, in probability to A^0 if

$$\forall t \geq 0 : \sup_{s \leq t} |A_s^n - A_s^0| \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

We denote this convergence with

$$A^n \xrightarrow[n \rightarrow \infty]{\mathbb{P}\text{-ucp}} A^0.$$

The following result gives a sufficient condition for ucp convergence to occur.

Lemma 2.3 (cf. [22], §2.2.3). *Assume that A^n and A have increasing paths and A is continuous. If there exists D dense in $[0, \infty)$ such that $A_t^n \xrightarrow{P} A_t$, $\forall t \in D$, then $A^n \xrightarrow[n \rightarrow \infty]{P\text{-ucp}} A$.*

For the estimation result, we consider the notion of *conditional convergence in probability*. Indeed, the estimators we are going to consider are consistent conditionally on the event that the threshold of interest (that we consider to be 0) is reached. For an event \mathcal{H} , let $P^{\mathcal{H}}(\cdot) := P(\cdot | \mathcal{H})$ be the conditional probability on \mathcal{H} .

Definition 2.4. Let $(A^n)_{n \geq 0}$ be a sequence of processes defined on the probability space (Ω, \mathcal{F}, P) . We say that $(A^n)_{n \geq 1}$ converges to A^0 in probability, conditionally on \mathcal{H} , if $A^n \rightarrow A^0$ in $P^{\mathcal{H}}$ -probability. We denote this convergence with

$$A^n \xrightarrow[n \rightarrow \infty]{P^{\mathcal{H}}} A^0.$$

2.4. Existing useful asymptotic results

For establishing the limit behavior of the number of crossings statistics we need the following existing results on the sticky Brownian motion: an approximation of the local time, an approximation of the occupation time at 0, and a result on discrete martingales. We conclude the section by recalling a previously established limit in law of the number of crossings of type 0, that we do not use in the paper because we do not have access to the proof.

Proposition 2.5 ([5] Proposition 6.2). *Let X be the sticky Brownian motion of stickiness parameter $\rho > 0$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P_x)$ such that P_x -a.s., $X_0 = x$. Let $m_{\sqrt{n}\rho}$ be the measure defined for all $x \in \mathbb{R}$ by*

$$m_{\sqrt{n}\rho}(dx) = dx + \sqrt{n}\rho \delta_0(dx) \quad (3)$$

and $(g_n)_n$ a sequence of measurable real functions such that for all $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \left(\frac{g_n^2(\sqrt{n}x)}{n} + \frac{m_{\sqrt{n}\rho}(g_n^2)}{\sqrt{n}} + \frac{\left(\int_{-\infty}^{+\infty} |x| g_n(x) dx \right) (1 + \log(n)) g_n(\sqrt{n}x)}{n} + \frac{(1 + \log(n)) m_{\sqrt{n}\rho}(|g_n|)}{\sqrt{n}} \right) = 0 \quad (4)$$

and $\lim_{n \rightarrow \infty} m_{\sqrt{n}\rho}(g_n) = M$. Then, for all $t \geq 0$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} g_n(\sqrt{n}X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{P_x} ML_t^0(X).$$

Also, if (4) holds and $\sup_n (m_{\sqrt{n}\rho}(|g_n|)) < \infty$ then the above convergence is locally uniform in time, in P_x -probability.

In the above proposition, as well as in the entire paper, we use the notation

$$m_{\sqrt{n}\rho}(g) = \int_{\mathbb{R}} g(x)m_{\sqrt{n}\rho}(dx) = \int_{-\infty}^{+\infty} g(x) dx + \sqrt{n}\rho g(0) \quad (5)$$

for any integrable function g .

Lemma 2.6 ([5], Lemma 2.9). *Let X be a sticky Brownian motion of stickiness parameter $\rho > 0$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that \mathbb{P}_x -a.s., $X_0 = x$. Then, it holds that, for all intervals U (including singletons),*

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \in U} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} \int_0^t \mathbb{1}_{X_s \in U} ds.$$

In particular, taking $U = \{0\}$, it yields that

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} = 0} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \rho L_t^0(X).$$

See also [1], for the approximation of smooth occupation time functionals.

Lemma 2.7 ([16], Lemma 9). *Let $\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $((\mathcal{F}_i^n)_i)_n$ a family of discrete filtrations on \mathcal{P} , $(\chi_i^n)_{i,n}$ a family of random variables on \mathcal{P} such that for all (i, n) , χ_i^n being \mathcal{F}_i^n -measurable and U a random variables on \mathcal{P} . The following two conditions imply $\sum_{i=1}^n \chi_i^n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} U$:*

$$(i) \sum_{i=1}^n \mathbb{E}(\chi_i^n | \mathcal{F}_{i-1}^n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} U,$$

$$(ii) \sum_{i=1}^n \mathbb{E}((\chi_i^n)^2 | \mathcal{F}_{i-1}^n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Theorem 2.8 (Efimenko-Portenko (1989), see [28]). *We consider the setting of Lemma 2.6. For all $t > 0$, $\sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0}$ converges in law to some discrete random variable Z_t with values in \mathbb{N}_0 such that for all $k \in \mathbb{N}_0$*

$$\mathbb{P}_0(Z_t = k) = b_k(t),$$

where $(b_k(t))_{k,t}$ are defined via their Laplace transform

$$\int_0^\infty \lambda e^{-\lambda t} b_k(t) dt = \frac{A_\lambda}{1/\rho^2 + A_\lambda} \left(\frac{1/\rho^2}{1/\rho^2 + A_\lambda} \right)^k,$$

with $A_\lambda = \sqrt{\lambda} (\sqrt{\lambda} + \sqrt{2}/\rho)$.

3. Main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider three types behaviors on the time interval $[t, t + s]$ of the process X at the threshold 0. For simplicity, we consider the threshold to be located at 0.

1. The *crossing of type 0*: the process crosses from one open half-plane to the other. For example if $X_t < 0$ and $X_{t+s} > 0$.
2. The *crossing of type 1*: the process crosses from one open or closed half-plane to its complementary. For example if $X_t \leq 0$ and $X_{t+s} > 0$ or if $X_t < 0$ and $X_{t+s} \geq 0$.
3. The *crossing of type 2*: the process crosses from one closed half-plane to the other. For example $X_t \leq 0$ and $X_{t+s} \geq 0$.
4. The *bouncing of type 0*: the process bounces from $(0, \infty)$ back into $(0, \infty)$ after hitting 0. For example if $X_t > 0$, $X_{t+s} > 0$, and for some $h \in [0, s]$, $X_{t+h} = 0$.
5. The *bouncing of type 1*: the process bounces from the closed half-plane to the open one or vice-versa. For example if $X_t > 0$ and $X_{t+s} \geq 0$ or if $X_t > 0$ and $X_{t+s} \geq 0$, and for some $h \in [0, s]$, $X_{t+h} = 0$.
6. The *bouncing of type 2*: the process bounces from $[0, \infty)$ to $[0, \infty)$. For example $X_t \geq 0$, $X_{t+s} \geq 0$, and for some $h \in [0, s]$, $X_{t+h} = 0$.

We consider the following crossings statistics. Let us assume we are given n observations of the process per unit of time on $[0, t]$, equally spaced in time, i.e. we are given $(X_{i/n})_{i \leq [nt]}$. The sampling frequency is therefore n : we have n observations on a unit time interval. For $j \in \{0, 1, 2\}$, the *number of crossings statistic of type j* is the number of times the high-frequency discretization $(X_{i/n})_{i \leq [nt]}$ of the process executes a crossing of type j at 0. In particular, for all $t \in (0, \infty)$, $n \in \mathbb{N}$,

$$C_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0}, \quad C_{n,t}^{(1)}(X) = \sum_{i=1}^{[nt]} \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} \leq 0} - \mathbb{1}_{X_{\frac{i-1}{n}} = X_{\frac{i}{n}} = 0} \right),$$

$$C_{n,t}^{(2)}(X) = \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} \leq 0}.$$

Similarly, for all $t \in (0, \infty)$, $n \in \mathbb{N}$, we define the number of bouncings statistics as

$$B_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \mathbb{1}_{U_i^n(X)} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} > 0}, \quad B_{n,t}^{(1)}(X) = \sum_{i=1}^{[nt]} \mathbb{1}_{U_i^n(X)} \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} \geq 0} - \mathbb{1}_{X_{\frac{i-1}{n}} = X_{\frac{i}{n}} = 0} \right),$$

$$B_{n,t}^{(2)}(X) = \sum_{i=1}^{[nt]} \mathbb{1}_{U_i^n(X)} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} \geq 0},$$

where, for all n, i , $U_i^n(X)$ is the event defined as

$$U_i^n(X) = \left\{ \exists s \in \left[\frac{i-1}{n}, \frac{i}{n} \right] : X_s = 0 \right\}.$$

We note that these statistics are different when the process spends a positive amount of time at 0 with positive probability. Indeed, if we define the *difference statistics*:

$$\begin{aligned}\mathbf{Z}_{n,t}^{(1)}(X) &:= C_{n,t}^{(1)}(X) - C_{n,t}^{(0)}(X) = B_{n,t}^{(1)}(X) - B_{n,t}^{(0)}(X) \\ &= \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0; X_{\frac{i}{n}} \neq 0} + \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \neq 0; X_{\frac{i}{n}}=0}, \\ \mathbf{Z}_{n,t}^{(2)}(X) &:= C_{n,t}^{(2)}(X) - C_{n,t}^{(1)}(X) = B_{n,t}^{(2)}(X) - B_{n,t}^{(1)}(X) \\ &= \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=X_{\frac{i}{n}}=0},\end{aligned}$$

then, for all t, n and $i = 1, 2$, it holds that $\mathbb{P}(\mathbf{Z}_{n,t}^{(i)}(X) > 0) > 0$. If X is a Brownian motion, then $\mathbb{P}(\mathbf{Z}_{n,t}^{(i)}(X) = 0) = 1$.

One may say that $B^{(j)}$ are not statistics, since they depend on the events $(U_{n,i})_{n,i}$ which are not in the σ -algebra generated by high-frequency samples of the process. However, they are related to the *difference statistics* $\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}$.

3.1. Limit behavior of the considered statistics

The main result of this paper is:

Theorem 3.1. *Let X be the sticky Brownian motion of stickiness parameter $\rho > 0$ defined on the filtered probability space $\mathcal{P}_x = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that \mathbb{P}_x -a.s., $X_0 = x$ and that X is $(\mathcal{F}_t)_{t \geq 0}$ -adapted, and let $L^0(X)$ be the right local time at 0 of X . Then, the following convergences holds,*

- (i) *for every diverging sequence u_n , it holds that: $C_{n,\cdot}^{(0)}(X)/u_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} 0$,*
- (ii) *$\mathbf{Z}_{n,\cdot}^{(1)}(X)/\sqrt{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} 4\sqrt{\frac{2}{\pi}}L^0(X)$ and $C_{n,\cdot}^{(1)}(X)/\sqrt{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} 4\sqrt{\frac{2}{\pi}}L^0(X)$,*
- (iii) *$\mathbf{Z}_{n,\cdot}^{(2)}(X)/n \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} \rho L^0(X) = \int_0^\cdot \mathbb{1}_{X_s=0} ds$ and $C_{n,\cdot}^{(2)}(X)/n \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} \rho L^0(X)$.*

We note that in the case of the standard Brownian motion W , all three statistics have the same non-trivial limit for the same normalizing sequence \sqrt{n} . Indeed, from, e.g., [20, Theorem 1.1],

$$\mathbb{P} - \lim_{n \rightarrow \infty} \left(\frac{C_{n,t}^{(0)}(W)}{\sqrt{n}} \right) = \mathbb{P} - \lim_{n \rightarrow \infty} \left(\frac{C_{n,t}^{(1)}(W)}{\sqrt{n}} \right) = \mathbb{P} - \lim_{n \rightarrow \infty} \left(\frac{C_{n,t}^{(2)}(W)}{\sqrt{n}} \right) = \sqrt{\frac{2}{\pi}}L_t^0(W).$$

Also, the rates of convergence are known. In [20, Theorem 1.2], a central limit theorem is proven for this convergence, with a rate of $n^{1/4}$.

Unlike the standard Brownian motion, we observe that there is a factor 4 in the limits of $C_{n,\cdot}^{(1)}/\sqrt{n}, \mathbf{Z}_{n,\cdot}^{(1)}/\sqrt{n}$ for the sticky Brownian motion (see Theorem 3.1(ii)). The factor 4

comes from the fact that $\mathbf{Z}_{n,t}^{(1)}/\sqrt{n}$ rewrites as sum of 4 different terms, all converging to $\sqrt{2/\pi}L_t^0(X)$:

$$\frac{1}{\sqrt{n}}\mathbf{Z}_{n,t}^{(1)}(X) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \left(\mathbb{1}_{X_{\frac{i-1}{n}}=0; X_{\frac{i}{n}}>0} + \mathbb{1}_{X_{\frac{i-1}{n}}=0; X_{\frac{i}{n}}<0} + \mathbb{1}_{X_{\frac{i-1}{n}}>0; X_{\frac{i}{n}}=0} + \mathbb{1}_{X_{\frac{i-1}{n}}<0; X_{\frac{i}{n}}=0} \right).$$

Each term converges to 0 if the process is a standard Brownian motion W . Instead, for the sticky Brownian motion, the limit is non trivial and it corresponds to the case of the renormalized number of crossings statistic for Brownian motion: $C_{n,\cdot}^{(0)}(W)/\sqrt{n}$.

Regarding the bouncing statistics, we prove that:

Theorem 3.2. *Consider the setting of Theorem 3.1. Then*

- (i) *for every diverging sequence u_n , $B_{n,\cdot}^{(0)}/u_n$ has the same limit as $C_{n,\cdot}^{(0)}(X)/u_n$,*
- (ii) *$B_{n,\cdot}^{(j)}/n^{j/2}$ has the same limit as $C_{n,\cdot}^{(j)}(X)/n^{j/2}$, for $j = 1, 2$.*

The difference statistics $\mathbf{Z}^{(\cdot)}$ are the leading parts of number of crossings and bouncings statistics of type 1 and 2.

For better dealing with the result in Theorem 3.1, we define the conditional versions of the statistics $C^{(0)}, C^{(1)}, C^{(2)}$. Let $\tilde{C}^{(0)}, \tilde{C}^{(1)}, \tilde{C}^{(2)}$ be the statistics defined, for all n, t , by

$$\begin{aligned} \tilde{C}_{n,t}^{(0)}(X) &= \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right), & \tilde{C}_{n,t}^{(1)}(X) &= \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} \leq 0} - \mathbb{1}_{X_{\frac{i-1}{n}} = X_{\frac{i}{n}} = 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right), \\ \tilde{C}_{n,t}^{(2)}(X) &= \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} \leq 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right). \end{aligned}$$

For these, the following result holds.

Proposition 3.3. *Consider the setting of Theorem 3.1. Then*

- (i) $\tilde{C}_{n,\cdot}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{P_x\text{-ucp}} (1/\rho) L^0(X)$,
- (ii) $\tilde{C}_{n,\cdot}^{(j)}(X)/n^{j/2}$ has the same limit as $C_{n,\cdot}^{(j)}(X)$ for $j = 1, 2$.

The proof of Theorem 3.1(i) is done by bounding the distance between $\tilde{C}^{(0)}$ and $C^{(0)}$ and showing that, if renormalized by $1/u_n$, both the distance and the limit of $\tilde{C}^{(0)}(X)/u_n$ vanish in probability, as $n \rightarrow \infty$.

Let us note that the only case, for which the limits of the expected and non-expected versions of the number of crossings statistics are not necessarily the same, is for the 0-type of crossings (strict crossings). In fact, they cannot be the same since $C_{n,t}^{(0)}(X)$ and its limit take values in $\mathbb{N} \cup (+\infty)$, while the law of $(1/\rho)L_t^0(X)$, the limit of $\tilde{C}_{n,t}^{(0)}(X)$, is continuous. The discrete limit law of $C_{n,t}^{(0)}(X)$ is described in [28, §8], which is stated here in Theorem 2.8. Let us also mention that Theorem 3.1(i) can also be inferred directly from Theorem 2.8.

3.2. Estimation

In [3, 5] an estimator was proposed based on a test function g and a normalizing sequence u_n . More precisely, let g be a bounded integrable function that satisfies $g(0) = 0$ and $(u_n)_n$ a sequence that as $n \rightarrow \infty$: $u_n \rightarrow \infty$, $u_n/n \rightarrow 0$, \mathcal{H}_t be the event $\{\tau_0^X < t\} = \{\exists s < t : X_s = 0\}$ and $\varrho_n^{(0)}(X)$ the statistic

$$\varrho_n^{(0)}(X) := \left(\int_{\mathbb{R}} g \, dx \right) \frac{1}{u_n} \frac{\sum_{i=1}^{\lfloor nt \rfloor} \mathbb{1}_{X_{\frac{i-1}{n}}=0}}{\sum_{i=1}^{\lfloor nt \rfloor} g(u_n X_{\frac{i-1}{n}})}. \quad (6)$$

Then, $\varrho_n^{(0)}(X)$ is a consistent estimator of ρ , conditionally on the event \mathcal{H}_t , i.e. $\varrho_n^{(0)}(X) \rightarrow \rho$, in $\mathbb{P}_x^{\mathcal{H}_t}$ -probability, as $n \rightarrow \infty$.

We now devise a consistent stickiness parameter estimator based on number of crossings statistics. This provides an alternative estimator to (6).

Like (6), this new estimator also converges conditionally on the event that 0 has been reached before time t . If the threshold is not reached, the number of crossings statistics are all trivially null.

Then, the following conditional convergences in probability hold:

Proposition 3.4. *Consider the setting of Theorem 3.1 and let $t > 0$. The statistic*

$$\varrho_n(X) = \left(4\sqrt{\frac{2}{\pi}} \right) \frac{1}{\sqrt{n}} \frac{C_{n,t}^{(2)}(X)}{C_{n,t}^{(1)}(X)} \quad (7)$$

is a consistent estimator of ρ , conditionally on the event \mathcal{H}_t , i.e. $\varrho_n(X) \rightarrow \rho$, in $\mathbb{P}_x^{\mathcal{H}_t}$ -probability, as $n \rightarrow \infty$.

Remark 3.5. The result holds also if in (7) one replaces $(C_{n,t}^{(1)}(X), C_{n,t}^{(2)}(X))$ with $(B_{n,t}^{(1)}(X), B_{n,t}^{(2)}(X))$ or $(\mathbf{Z}_{n,t}^{(1)}(X), \mathbf{Z}_{n,t}^{(2)}(X))$.

4. Proof of the main results

In this section we prove Theorem 3.1 separately for each type of crossing. By the way, we prove Proposition 3.3 as well. Next we prove Theorem 3.2. Finally, in Section 4.5, we prove Proposition 3.4.

For these, we first establish the following result on the asymptotic behavior of the kernel, whose proof is the subject of Appendix A. In the statement we use the notation (3)-(5).

Lemma 4.1. *Consider the setting of Theorem 3.1. Let $(f_n, k_n, g_n, h_n; n \in \mathbb{N})$ be the functions defined for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ by $f_n(x) := \mathbb{P}_x(X_{1/n} = 0)$, $k_n(x) := \mathbb{1}_{x \neq 0} f_n(x/\sqrt{n})$, $g_n(x) := \mathbb{P}_x(xX_{1/n} < 0)$, $h_n(x) := \sqrt{n}g_n(x/\sqrt{n})$. It holds that*

- (i) $\lim_{n \rightarrow \infty} f_n(0) = 1$ and $\lim_{n \rightarrow \infty} \sqrt{n}(1 - f_n(0)) = 2\sqrt{2}/\rho\sqrt{\pi}$,
- (ii) the sequence $(k_n)_n$ satisfies (4) and $\lim_{n \rightarrow \infty} m_{\sqrt{n}\rho}(k_n) = 2\sqrt{2/\pi}$,

(iii) the sequence $(h_n)_n$ satisfies (4) and $\lim_{n \rightarrow \infty} m_{\sqrt{n}\rho}(h_n) = 1/\rho$.

For convenience, we define the conditional version of the $\tilde{\mathbf{Z}}^{(\cdot)}$ statistics as

$$\begin{aligned}\tilde{\mathbf{Z}}_{n,t}^{(1)}(X) &:= \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}}=0, X_{\frac{i}{n}} \neq 0} + \mathbb{1}_{X_{\frac{i-1}{n}} \neq 0, X_{\frac{i}{n}}=0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right) \\ \tilde{\mathbf{Z}}_{n,t}^{(2)}(X) &:= \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}}=X_{\frac{i}{n}}=0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right).\end{aligned}$$

4.1. Crossings of type 0

Proof of Proposition 3.3(i). Let $(g_n)_n, (h_n)_n$ be defined as in Lemma 4.1. Then, it holds that

$$\tilde{C}_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right) = \sum_{i=1}^{[nt]} g_n(X_{\frac{i-1}{n}}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} h_n(\sqrt{n} X_{\frac{i-1}{n}}).$$

From Proposition 2.5 and Lemma 4.1(iii), we have

$$\tilde{C}_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \frac{1}{\rho} L_t^0(X) = \frac{1}{\rho^2} \mathcal{O}_t^0(X).$$

This finishes the proof. \square

Proof of Theorem 3.1(i). We consider the families of σ -algebras $(\mathcal{F}_i^n)_{i,n}$ and random variables $(\chi_i^n)_{i,n}$, defined for all i, n by

$$\mathcal{F}_i^n = \mathcal{F}_{i/n}, \quad \text{and} \quad \chi_i^n = \frac{1}{u_n} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0}$$

so that

$$\frac{1}{u_n} C_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \chi_i^n, \quad \text{and} \quad \frac{1}{u_n} \tilde{C}_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \mathbb{E}_x (\chi_i^n | \mathcal{F}_{i-1}^n).$$

From Proposition 3.3(i),

$$\sum_{i=1}^{[nt]} \mathbb{E}_x (\chi_i^n | \mathcal{F}_{i-1}^n) = \frac{1}{u_n} \tilde{C}_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 0$$

and, since $(\chi_i^n)^2 = \chi_i^n / u_n$,

$$\sum_{i=1}^{[nt]} \mathbb{E}_x ((\chi_i^n)^2 | \mathcal{F}_{i-1}^n) = \frac{1}{u_n} \sum_{i=1}^{[nt]} \mathbb{E}_x (\chi_i^n | \mathcal{F}_{i-1}^n) = \frac{1}{u_n^2} \tilde{C}_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 0.$$

Thus, from Lemma 2.7,

$$\frac{1}{u_n} C_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 0.$$

From Lemma 2.3, the convergence is also uniform in time, in probability (ucp). This finishes the proof. \square

4.2. Crossings of type 1

Lemma 4.2. *The following convergences hold:*

$$\frac{1}{\sqrt{n}} \tilde{\mathbf{Z}}_{n,t}^{(1)}(X) \xrightarrow[n \rightarrow \infty]{P_x} 4\sqrt{\frac{2}{\pi}} L_t^0(X).$$

Proof. Regarding the first term of $\tilde{\mathbf{Z}}^{(1)}$, from Lemmas 2.6 and 4.1(i),

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \mathbb{E}_x \left[\mathbb{1}_{X_{\frac{i-1}{n}}=0; X_{\frac{i}{n}} \neq 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right] &= \frac{1}{\sqrt{n}} \mathbb{P}_0 \left(X_{\frac{1}{n}} \neq 0 \right) \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} \\ &= \sqrt{n} (1 - f_n(0)) \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} \xrightarrow[n \rightarrow \infty]{P_x} \frac{2\sqrt{2}}{\rho\sqrt{\pi}} \mathcal{O}_t^0(X) = 2\sqrt{\frac{2}{\pi}} L_t^0(X). \end{aligned} \quad (8)$$

Regarding the second term of $\tilde{\mathbf{Z}}^{(1)}$,

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \mathbb{E}_x \left[\mathbb{1}_{X_{\frac{i-1}{n}} \neq 0; X_{\frac{i}{n}}=0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right] &= \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}} \neq 0} \mathbb{P}_{X_{\frac{i-1}{n}}} \left(X_{\frac{i}{n}} = 0 \right) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} k_n(\sqrt{n} X_{\frac{i-1}{n}}), \end{aligned}$$

where $(k_n)_n$ is the sequence of functions defined in Lemma 4.1. From Proposition 2.5 and Lemma 4.1(ii),

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} k_n(\sqrt{n} X_{\frac{i-1}{n}}) \xrightarrow[n \rightarrow \infty]{P_x} \left(\lim_{n \rightarrow \infty} m_{\sqrt{n}\rho}(k_n) \right) L_t^0(X) = 2\sqrt{\frac{2}{\pi}} L_t^0(X). \quad (9)$$

Combining the relations (8) and (9) yields the desired result. \square

Proof of Proposition 3.3(ii) for $j = 1$. Let us observe that

$$\frac{1}{\sqrt{n}} \tilde{\mathcal{C}}_{n,t}^{(1)}(X) = \frac{1}{\sqrt{n}} \tilde{\mathbf{Z}}_{n,t}^{(1)}(X) + \frac{1}{\sqrt{n}} \tilde{\mathcal{C}}_{n,t}^{(0)}(X).$$

From Lemma 4.2 and Proposition 3.3(i),

$$\frac{1}{\sqrt{n}} \tilde{\mathcal{C}}_{n,t}^{(1)}(X) \xrightarrow[n \rightarrow \infty]{P_x} 4\sqrt{\frac{2}{\pi}} L_t^0(X).$$

From Lemma 2.3, the convergence is also uniform in time, in probability (**ucp**). \square

Proof of Theorem 3.1(ii). We consider the families of σ -algebras $(\mathcal{F}_i^n)_{i,n}$ and random variables $(\chi_i^n)_{i,n}$, defined for all $i \in \mathbb{N}, n > 0$ by

$$\mathcal{F}_i^n = \mathcal{F}_{i/n}, \quad \text{and} \quad \chi_i^n = \frac{1}{\sqrt{n}} \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}}=0} - \mathbb{1}_{X_{\frac{i-1}{n}}=X_{\frac{i}{n}}=0} \right)$$

so that

$$\frac{1}{\sqrt{n}}\mathbf{Z}_{n,t}^{(1)}(X) = \sum_{i=1}^{[nt]} \chi_i^n, \quad \frac{1}{\sqrt{n}}\tilde{\mathbf{Z}}_{n,t}^{(1)}(X) = \sum_{i=1}^{[nt]} \mathbb{E}_x(\chi_i^n | \mathcal{F}_{i-1}^n).$$

From Lemma 4.2,

$$\sum_{i=1}^{[nt]} \mathbb{E}_x(\chi_i^n | \mathcal{F}_{i-1}^n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 4\sqrt{\frac{2}{\pi}}L_t^0(X)$$

and, since $(\chi_i^n)^2 = \chi_i^n / \sqrt{n}$,

$$\sum_{i=1}^{[nt]} \mathbb{E}_x((\chi_i^n)^2 | \mathcal{F}_{i-1}^n) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 0.$$

Thus, from Lemma 2.7,

$$\frac{1}{\sqrt{n}}\mathbf{Z}_{n,t}^{(1)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 4\sqrt{\frac{2}{\pi}}L_t^0(X).$$

Combining this with Theorem 3.1(i) ensures that $\frac{1}{\sqrt{n}}C_{n,t}^{(1)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 4\sqrt{\frac{2}{\pi}}L_t^0(X)$. From Lemma 2.3, the convergence is also uniform in time, in probability (ucp). This finishes the proof. \square

4.3. Crossings of type 2

Proof of Theorem 3.1(iii). We recall that $C_{n,t}^{(2)}(X) = C_{n,t}^{(1)}(X) + \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0, X_{\frac{i}{n}}=0}$. We note that

$$\mathbb{1}_{X_{\frac{i-1}{n}}=0, X_{\frac{i}{n}}=0} = \mathbb{1}_{X_{\frac{i-1}{n}}=0} - \mathbb{1}_{X_{\frac{i-1}{n}}=0, X_{\frac{i}{n}} \neq 0}.$$

Since $\{X_{\frac{i-1}{n}} = 0, X_{\frac{i}{n}} \neq 0\}$ corresponds to a specific case of type 1 crossing, we have $0 \leq C_{n,t}^{(1)}(X) - \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0, X_{\frac{i}{n}} \neq 0} \leq C_{n,t}^{(1)}(X)$. Hence,

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} \leq \frac{1}{n} C_{n,t}^{(2)}(X) \leq \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} + \frac{1}{n} C_{n,t}^{(1)}(X)$$

From Item (ii) and Lemma 2.6, we have that

$$\forall t \geq 0 : \quad \frac{1}{n} C_{n,t}^{(2)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \int_0^t \mathbb{1}_{X_s=0} ds = \rho L_t^0(X).$$

Lemma 2.3 ensures ucp convergence. The proof is thus completed. \square

Proof of Proposition 3.3(ii) for $j = 2$. We observe that

$$\frac{1}{n} \tilde{C}_{n,t}^{(2)}(X) = \frac{1}{n} \tilde{\mathbf{Z}}_{n,t}^{(2)}(X) + \frac{1}{n} \tilde{C}_{n,t}^{(1)}(X).$$

With the same argument as in the proof of Theorem 3.1(iii),

$$\frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} \leq \frac{1}{n} \tilde{C}_{n,t}^{(2)}(X) \leq \frac{1}{n} \sum_{i=1}^{[nt]} \mathbb{1}_{X_{\frac{i-1}{n}}=0} + \frac{1}{n} \tilde{C}_{n,t}^{(1)}(X)$$

From Item (ii) and Lemma 2.6, we have that

$$\forall t \geq 0 : \quad \frac{1}{n} \tilde{C}_{n,t}^{(2)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \int_0^t \mathbb{1}_{X_s=0} ds = \rho L_t^0(X).$$

From Lemma 2.3, the convergence is ucp. This finishes the proof. \square

4.4. Proof of Theorem 3.2

For the proof we use the following result which is a reflection principle at 0 for the sticky Brownian motion. The proof is the subject of Appendix B.

In the statement and the proof, for a process Y , we set τ_0^Y to be the hitting time of 0 by Y , defined by

$$\tau_0^Y = \inf\{t > 0 : Y_t = 0\}. \quad (10)$$

Lemma 4.3. *Let X be the sticky Brownian motion of stickiness parameter $\rho > 0$ defined on the filtered probability space $\mathcal{P}_x = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that \mathbb{P}_x -a.s., $X_0 = x$. Let X' be the process defined for all $t \geq 0$ by*

$$X'_t = \begin{cases} X_t & t < \tau_0^X, \\ -X_t & t \geq \tau_0^X. \end{cases}$$

The process X' defined this way is a sticky Brownian motion of stickiness parameter ρ .

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2(i). In this proof, let $\tilde{B}^{(0)}$ be the conditional version of $B^{(0)}$, defined for all $n, t > 0$ by

$$\tilde{B}_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \mathbb{E}_x \left(\mathbb{1}_{U_i^n(X)} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} > 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right).$$

We also introduce the notation of Markovian families used in [15, Section 2.1]. Let X' be a diffusion on J , an interval of \mathbb{R} , defined on the probability space \mathcal{P}_x such that \mathbb{P}_x -a.s., $X'_0 = x$. We denote by $(Q_y^{X'}, y \in J)$ the canonical diffusion on the path-space $C([0, \infty), \mathbb{R})$ and Y be the coordinate process. Then, for every almost surely finite stopping time τ , we have

$$Q_{X'_\tau}^{X'}(h(Y_t)) = \mathbb{E}_x(h(X'_{\tau+t}) | \mathcal{F}_\tau).$$

From the strong-Markov property, using the above notation and (10), we have

$$\begin{aligned} \mathbb{E}_x \left(\mathbb{1}_{U_i^n(X)} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} > 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right) &= Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{\tau_0^Y < \frac{1}{n}} \mathbb{1}_{Y_0 Y_{\frac{1}{n}} > 0} \right) \\ &= \mathbb{1}_{X_{\frac{i-1}{n}} > 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{\tau_0^Y < \frac{1}{n}} \mathbb{1}_{Y_{\frac{1}{n}} > 0} \right) + \mathbb{1}_{X_{\frac{i-1}{n}} < 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{\tau_0^Y < \frac{1}{n}} \mathbb{1}_{Y_{\frac{1}{n}} < 0} \right), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right) &= Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{Y_0 Y_{\frac{1}{n}} < 0} \right) \\ &= \mathbb{1}_{X_{\frac{i-1}{n}} < 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{Y_{\frac{1}{n}} > 0} \right) + \mathbb{1}_{X_{\frac{i-1}{n}} > 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{Y_{\frac{1}{n}} < 0} \right). \end{aligned}$$

From Lemma 4.3, we have

$$\mathbb{1}_{X_{\frac{i-1}{n}} > 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{\tau_0^Y < \frac{1}{n}} \mathbb{1}_{Y_{\frac{1}{n}} > 0} \right) = \mathbb{1}_{X_{\frac{i-1}{n}} > 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{\tau_0^Y < \frac{1}{n}} \mathbb{1}_{Y_{\frac{1}{n}} < 0} \right) = \mathbb{1}_{X_{\frac{i-1}{n}} > 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{Y_{\frac{1}{n}} < 0} \right)$$

and

$$\mathbb{1}_{X_{\frac{i-1}{n}} < 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{\tau_0^Y < \frac{1}{n}} \mathbb{1}_{Y_{\frac{1}{n}} < 0} \right) = \mathbb{1}_{X_{\frac{i-1}{n}} < 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{\tau_0^Y < \frac{1}{n}} \mathbb{1}_{Y_{\frac{1}{n}} > 0} \right) = \mathbb{1}_{X_{\frac{i-1}{n}} < 0} Q_{X_{\frac{i-1}{n}}}^X \left(\mathbb{1}_{Y_{\frac{1}{n}} > 0} \right).$$

Therefore, we obtain

$$\mathbb{E}_x \left(\mathbb{1}_{U_i^n(X)} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} > 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right) = \mathbb{E}_x \left(\mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} < 0} \middle| \mathcal{F}_{\frac{i-1}{n}} \right)$$

and thus

$$\tilde{B}_{n,t}^{(0)} = \tilde{C}_{n,t}^{(0)}. \quad (11)$$

Let us consider the families of σ -algebras $(\mathcal{F}_i^n)_{i,n}$ and random variables $(\chi_i^n)_{i,n}$, defined for all i, n by

$$\mathcal{F}_i^n = \mathcal{F}_{i/n} \quad \text{and} \quad \chi_i^n = \frac{1}{u_n} \mathbb{1}_{U_i^n(X)} \mathbb{1}_{X_{\frac{i-1}{n}} X_{\frac{i}{n}} > 0}.$$

Then,

$$\frac{1}{u_n} B_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \chi_i^n \quad \text{and} \quad \frac{1}{u_n} \tilde{B}_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \mathbb{E}_x(\chi_i^n | \mathcal{F}_{i-1}^n).$$

From Proposition 3.3(i) and (11),

$$\sum_{i=1}^{[nt]} \mathbb{E}_x(\chi_i^n | \mathcal{F}_{i-1}^n) = \frac{1}{u_n} \tilde{B}_{n,t}^{(0)}(X) = \frac{1}{u_n} \tilde{C}_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{P_x} 0$$

and, since $(\chi_i^n)^2 = \chi_i^n / u_n$,

$$\sum_{i=1}^{[nt]} \mathbb{E}_x((\chi_i^n)^2 | \mathcal{F}_{i-1}^n) = \frac{1}{u_n} \sum_{i=1}^{[nt]} \mathbb{E}_x(\chi_i^n | \mathcal{F}_{i-1}^n) = \frac{1}{u_n^2} \tilde{B}_{n,t}^{(0)}(X) = \frac{1}{u_n^2} \tilde{C}_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{P_x} 0.$$

Thus, from Lemma 2.7,

$$\frac{1}{u_n} B_{n,t}^{(0)}(X) = \sum_{i=1}^{[nt]} \chi_i^n \xrightarrow[n \rightarrow \infty]{P_x} 0.$$

From Lemma 2.3, the convergence is also uniform in time, in probability (ucp). This finishes the proof of Item (i). \square

Proof of Theorem 3.2(ii). We observe that, for $j = 1, 2$, $\mathbf{Z}_{n,\cdot}^{(j)}(X) = B_{n,\cdot}^{(j)}(X) - B_{n,\cdot}^{(j-1)}(X)$. Thus, from Theorem 3.1, and Item (i), it holds that

$$\frac{1}{\sqrt{n}}B_{n,\cdot}^{(1)}(X) \xrightarrow[n \rightarrow \infty]{P_x} 4\sqrt{\frac{2}{\pi}}L^0(X), \quad \frac{1}{n}B_{n,\cdot}^{(2)}(X) \xrightarrow[n \rightarrow \infty]{P_x} \rho L_t^0(X).$$

From Lemma 2.3, the convergence is uniform in time, in P_x -probability (P_x -ucp). This completes the proof. \square

4.5. Proof of Proposition 3.4

Proof. From Theorem 3.1, $C_{n,t}^{(2)}(X) \xrightarrow[n \rightarrow \infty]{P_x} \rho L_t^0(X)$ and $C_{n,t}^{(1)}(X) \xrightarrow[n \rightarrow \infty]{P_x} 4\sqrt{\frac{2}{\pi}}L_t^0(X)$. From [29, Theorem VI.2.3], (1), and the Markov property, if B is the standard Brownian motion, P_x -a.s.: $\{\tau_0^B < t\} = \{L_t^0(B) > 0\}$. Hence, from **t1** and [29, Exercise VI.1.27], we have that P_x -a.s.:

$$\mathcal{H}_t := \{\tau_0^X < t\} = \{L_t^0(X) > 0\} = \left\{ \int_0^t \mathbb{1}_{X_s=0} ds > 0 \right\}.$$

Thus,

$$\mathbb{1}_{\mathcal{H}_t} \frac{1}{\sqrt{n}} \frac{C_{n,t}^{(2)}(X)}{C_{n,t}^{(1)}(X)} \xrightarrow[n \rightarrow \infty]{P_x} \frac{\rho}{4\sqrt{\frac{2}{\pi}}}$$

and so,

$$\mathbb{1}_{\mathcal{H}_t} \varrho_n(X) = \mathbb{1}_{\mathcal{H}_t} \left(4\sqrt{\frac{2}{\pi}} \right) \frac{1}{\sqrt{n}} \frac{C_{n,t}^{(2)}(X)}{C_{n,t}^{(1)}(X)} \xrightarrow[n \rightarrow \infty]{P_x} \rho.$$

Since the convergence is in P_x -probability and $P_x(\mathcal{H}_t) > 0$, from Bayes rule, we prove the desired convergence. \square

5. Extensions

In this section we extend the results on the sticky Brownian motion to the sticky-reflected Brownian motion and to sticky Itô diffusions, i.e. processes that solve a homogeneous SDE away from a sticky point.

5.1. Bouncings of the sticky-reflected Brownian motion

The sticky-reflected Brownian motion, also known as slowly reflected Brownian motion, was first discovered by Feller in his attempt to describe all possible ways to define a Brownian motion on the positive semi-axis $[0, \infty)$, see [13]. In particular, he discovered that the operator $L = D_x^2$ with domain

$$\text{dom}(L) = \{f \in C_0([0, \infty)) \cap C^2((0, \infty)) : f'(0+) = (\rho/2)f''(0+)\},$$

with $\rho > 0$, is the infinitesimal generator of a diffusion that spends a positive amount of time at 0 and that behaves like a Brownian motion away from 0. Replacing the boundary condition $f'(0+) = (\rho/2)f''(0+)$ by the lateral condition $f(0+) - f(0-) = (\rho/2)f''(0+)$ defines the sticky Brownian motion, or two-sided sticky Brownian motion, introduced in Section 2.2. A nice historical overview can be found in [27].

For simplicity, we define the sticky–reflecting Brownian motion of stickiness parameter $\rho > 0$ as the unique (in law) weak solution of the system

$$\begin{cases} dX_t = \mathbb{1}_{X_t > 0} dB_t + \frac{1}{2} dL_t^0(X), \\ \mathbb{1}_{X_t = 0} dt = \frac{\rho}{2} dL_t^0(X), \end{cases} \quad (12)$$

where B is a standard Brownian motion (see [12, Theorem 5]).

Let us note that the absolute value of a sticky Brownian motion is a sticky–reflected Brownian motion with the same stickiness parameter. More precisely, if Y is a sticky Brownian motion of stickiness parameter ρ , from the Tanaka formula [29, theorem VI.1.2] and **p1**, there exists a Brownian motion B on an extension of the probability space such that

$$\begin{aligned} d|Y_t| &= \operatorname{sgn}(Y_t) dY_t + dL_t^0(Y) \\ &= \operatorname{sgn}(Y_t) \mathbb{1}_{|Y_t| \neq 0} dB_t + dL_t^0(Y). \end{aligned} \quad (13)$$

This, along with **p1**, yield that $d\langle Y \rangle_t = d\langle |Y| \rangle_t = \mathbb{1}_{Y_t \neq 0} dt$ and that, since Y is a martingale, from [29, Corollary VI.1.9],

$$\begin{aligned} L_t^0(|Y|) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{0 < |Y_s| < \epsilon} ds \\ &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{0 < Y_s < \epsilon} ds + \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_0^t \mathbb{1}_{0 < -Y_s < \epsilon} ds \\ &= 2L_t^0(Y) = 2\rho \int_0^t \mathbb{1}_{Y_s = 0} ds. \end{aligned} \quad (14)$$

From (13) and (14), if $B' = \int_0^\cdot \operatorname{sgn}(Y_s) dB_s$, the pair $(|Y|, B')$ solves (12).

We are now ready to state our main results regarding the number of bouncings of this process.

Theorem 5.1. *Let X be the sticky–reflected Brownian motion of stickiness parameter $\rho > 0$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that \mathbb{P}_x -a.s.: $X_0 = x \geq 0$. Let also $L^0(X)$ be the right local time at 0 of X and $B^{(0)}, B^{(1)}, B^{(2)}, \mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}$ be the quantities defined in Section 1 for the process X . Then, the following convergences holds,*

- (i) *for every diverging sequence u_n , it holds that: $B_{n,\cdot}^{(0)}(X)/u_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} 0$,*
- (ii) *$B_{n,\cdot}^{(1)}(X)/\sqrt{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} 2\sqrt{\frac{2}{\pi}}L^0(X)$,*
- (iii) *$B_{n,\cdot}^{(2)}(X)/n \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x - \text{ucp}} \frac{\rho}{2}L^0(X)$.*

(iv) $\mathbf{Z}_{n,\cdot}^{(j)}(X)/n^{j/2}$ has the same limit as $B_{n,\cdot}^{(j)}(X)/n^{j/2}$ for $j = 1, 2$.

Proof of Theorem 5.1. Let Y be a sticky Brownian motion of stickiness parameter ρ defined on the probability space $\mathcal{P}'_x = (\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t \geq 0}, \mathbb{P}'_x)$ such that \mathbb{P}'_x -a.s., $Y_0 = x$ and Y is $(\mathcal{F}'_t)_{t \geq 0}$ -adapted. We note that X and $|Y|$ are equal in law (see discussion before the statement) and that

$$\begin{aligned} B_{n,\cdot}^{(0)}(|Y|) &= B_{n,\cdot}^{(0)}(Y) + C_{n,\cdot}^{(0)}(Y), \\ B_{n,\cdot}^{(1)}(|Y|) &= \mathbf{Z}_{n,\cdot}^{(1)}(Y) + B_{n,\cdot}^{(0)}(|Y|), \\ B_{n,\cdot}^{(2)}(|Y|) &= \mathbf{Z}_{n,\cdot}^{(2)}(Y) + B_{n,\cdot}^{(0)}(|Y|). \end{aligned} \tag{15}$$

From Theorems 3.1 and 3.2, for every diverging sequence $(u_n)_n$,

$$B_{n,\cdot}^{(0)}(|Y|) \xrightarrow[n \rightarrow \infty]{\mathbb{P}'_x} 0, \quad \frac{1}{\sqrt{n}} B_{n,\cdot}^{(1)}(|Y|) \xrightarrow[n \rightarrow \infty]{\mathbb{P}'_x} 4\sqrt{\frac{2}{\pi}} L_t^0(Y), \quad \frac{1}{n} B_{n,\cdot}^{(2)}(|Y|) \xrightarrow[n \rightarrow \infty]{\mathbb{P}'_x} \rho L^0(Y).$$

Consider the notation (10) for τ_0^X and τ_0^Y . We recall that $L_t^0(|Y|) > 0$ on $\{\tau_0^Y < t\}$ (see proof of Proposition 3.4). Observing $L_t^0(|Y|) = 2L_t^0(Y)$ and $\{\tau_0^Y < t\} = \{\tau_0^X < t\}$, we have

$$B_{n,t}^{(0)}(|Y|) \xrightarrow[n \rightarrow \infty]{\mathbb{P}'_x} 0, \quad \frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(1)}(|Y|)}{\sqrt{n} L_t^0(|Y|)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}'_x} 2\sqrt{\frac{2}{\pi}}, \quad \frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(2)}(|Y|)}{n L_t^0(|Y|)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}'_x} \frac{\rho}{2}.$$

It is known that in the case of constant limits convergence in probability is equivalent to convergence in law (see [7, Theorem 25.2] and the discussion thereafter). Hence,

$$B_{n,t}^{(0)}(|Y|) \xrightarrow[n \rightarrow \infty]{\text{law}} 0, \quad \frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(1)}(|Y|)}{\sqrt{n} L_t^0(|Y|)} \xrightarrow[n \rightarrow \infty]{\text{law}} 2\sqrt{\frac{2}{\pi}}, \quad \frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(2)}(|Y|)}{n L_t^0(|Y|)} \xrightarrow[n \rightarrow \infty]{\text{law}} \frac{\rho}{2}.$$

and since $|Y| = X$ in law (which entails that $(|Y|, L^0(|Y|)) = (X, L^0(X))$ in law), we have the same statement with X instead of $|Y|$. Since convergence in law towards a constant is equivalent to convergence in probability, we get

$$B_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 0, \quad \frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(1)}(X)}{\sqrt{n} L_t^0(X)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 2\sqrt{\frac{2}{\pi}}, \quad \frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(2)}(X)}{n L_t^0(X)} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \frac{\rho}{2}.$$

Hence, it holds

$$\frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(1)}(X)}{\sqrt{n}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 2\sqrt{\frac{2}{\pi}} L_t^0(X) \mathbb{1}_{\tau_0^X < t}, \quad \frac{\mathbb{1}_{\tau_0^X < t} B_{n,t}^{(2)}(X)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \frac{\rho}{2} L_t^0(X) \mathbb{1}_{\tau_0^X < t}.$$

We observe that on $\tau_0^X \geq t$, $L_t^0(X) = 0$, $B_{n,t}^{(0)}(X) = 0$, $\mathbf{Z}_{n,t}^{(2)}(X) = 0$, $B_{n,t}^{(1)}(X) = B_{n,t}^{(2)}(X) = \mathbf{Z}_{n,t}^{(1)}(X) \in \{0, 1\}$.

Therefore, we have

$$B_{n,t}^{(0)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 0, \quad \frac{1}{\sqrt{n}} B_{n,t}^{(1)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} 2\sqrt{\frac{2}{\pi}} L_t^0(X), \quad \frac{1}{n} B_{n,t}^{(2)}(X) \xrightarrow[n \rightarrow \infty]{\mathbb{P}_x} \frac{\rho}{2} L_t^0(X). \tag{16}$$

From Lemma 2.3, the convergences are uniform in time, in \mathbb{P}_x -probability (\mathbb{P}_x -ucp). This proves (i), (ii), (iii). From (15) and (16), we infer (iv). This completes the proof. \square

Corollary 5.2. *We consider the setting of Theorem 5.1 and \mathcal{H}_t be the event $\{\tau_0^X < t\} = \{\exists s \in [0, t): X_s = 0\}$. The statistic*

$$\varrho_n(X) = \left(4\sqrt{\frac{2}{\pi}}\right) \frac{1}{\sqrt{n}} \frac{\mathbf{Z}_{n,\cdot}^{(2)}(X)}{\mathbf{Z}_{n,\cdot}^{(1)}(X)}$$

is a consistent estimators of ρ , conditionally on the event \mathcal{H}_t , i.e. $\varrho_n(X) \rightarrow \rho$, in $\mathbb{P}_x^{\mathcal{H}_t}$ -probability, as $n \rightarrow \infty$.

Proof. Similar to the proof of Proposition 3.4, using $\mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}$ instead of $C^{(1)}, C^{(2)}$ and Theorem 5.1(iv) instead of Theorem 3.1. \square

5.2. Bouncings and crossings of sticky Itô diffusions

In this section we generalize Theorem 3.1 and Theorem 3.2 to sticky Itô diffusions. The generalization procedure is analogue to the ones in, e.g., [3, 5, 6, 20, 26].

We consider the system

$$\begin{cases} dX_t = \mathbb{1}_{X_t \neq 0} (\mu_t dt + \sigma(X_t) dW_t), \\ \mathbb{1}_{X_t=0} dt = \rho dL_t^0(X), \end{cases} \quad (17)$$

with W a standard Brownian motion, defined on the probability space $\mathcal{P}_x = (\Omega, \mathcal{F}, \mathbb{P}_x)$ such that \mathbb{P}_x -a.s., $X_0 = x$. We suppose (μ, σ) satisfy the following conditions.

c1 The diffusion coefficient σ is strictly positive and C^1 .

c2 Weak existence and uniqueness in law holds for the system

$$\begin{cases} dX'_t = \mathbb{1}_{X'_t \neq 0} \sigma(X'_t) dW_t, \\ \mathbb{1}_{X'_t=0} dt = \rho dL_t^0(X'), \end{cases} \quad (18)$$

on \mathcal{P}_x , such that \mathbb{P}_x -a.s., $X'_0 = x$.

c3 The laws of the solutions X, X' of (17),(18) are locally equivalent in the sense of [23, Definition III.3.2]. Basically, this means

$$\forall t \geq 0, \quad Law_{\mathbb{P}_x}(X|_{[0,t]}) \sim Law_{\mathbb{P}_x}(X'|_{[0,t]})$$

c4 The state-space of X' that solves (19) is an open interval J of \mathbb{R} , i.e. X' has only inaccessible boundaries (see, e.g., [24, Section 33]).

Remark 5.3. Condition **c2** is equivalent to the weak existence and uniqueness of classical SDE

$$dX''_t = \sigma(X''_t) dW_t. \quad (19)$$

This is a consequence of [3, Theorem 4.1] and [3, Proposition 4.2].

Under conditions **c1-c4**, the following results hold.

Theorem 5.4. Let $C^{(0)}, C^{(1)}, C^{(2)}, B^{(0)}, B^{(1)}, B^{(2)}, \mathbf{Z}^{(1)}, \mathbf{Z}^{(2)}$ be the quantities defined in Section 1 for the process X . Then,

(i) for every diverging sequence $(u_n)_n$,

$$(B_{n,\cdot}^{(0)}(X)/u_n), (C_{n,\cdot}^{(0)}(X)/u_n) \xrightarrow[n \rightarrow \infty]{P_x - ucp} 0,$$

$$(ii) (B_{n,\cdot}^{(1)}(X)/\sqrt{n}), (C_{n,\cdot}^{(1)}(X)/\sqrt{n}), (\mathbf{Z}_{n,\cdot}^{(1)}(X)/\sqrt{n}) \xrightarrow[n \rightarrow \infty]{P_x - ucp} \frac{4}{\sigma(0)} \sqrt{\frac{2}{\pi}} L^0(X),$$

$$(iii) (B_{n,\cdot}^{(2)}(X)/n), (C_{n,\cdot}^{(2)}(X)/n), (\mathbf{Z}_{n,\cdot}^{(2)}(X)/n) \xrightarrow[n \rightarrow \infty]{P_x - ucp} \rho L^0(X).$$

Proof. Let $T > 0$ be fixed. Consider all processes restricted to the time interval $[0, T]$.

Let us first assume that the diffusion coefficient and its derivative satisfy the following conditions: there exists a real constant $\delta > 0$ such that

$$\forall x \in \mathbb{R} : \delta \leq \sigma(x) \leq 1/\delta, \quad |\sigma'(x)| \leq 1/\delta.$$

We note that under this condition, the state-space of the diffusion J is \mathbb{R} . We consider X' to be the process, solution to (18), defined on \mathcal{P}_x that solves (18). We let $(\mathcal{F}_t)_{t \geq 0}$ be the natural filtration generated by X' . Let $\mathcal{E}(\theta)$, θ be the processes defined by

$$\mathcal{E}_t(\theta) = \exp \left(\int_0^t \theta_s dW_s - \frac{1}{2} \int_0^t \theta_s^2 ds \right), \quad \theta_t = \frac{1}{2} \sigma'(X_t), \quad t \geq 0.$$

Since σ' is bounded, from the Girsanov theorem [3, Lemma 4.4], there exists a measure \mathbb{Q}_x such that the process (X', W^θ) jointly solves

$$\begin{cases} dX'_t = \frac{1}{2} \sigma(X'_t) \sigma'(X'_t) \mathbb{1}_{\{X'_t \neq 0\}} dt + \sigma(X'_t) \mathbb{1}_{\{X'_t \neq 0\}} dW_t^\theta, \\ \mathbb{1}_{X'_t=0} dt = \rho dL_t^0(X'), \end{cases} \quad (20)$$

where W^θ is the \mathbb{Q}_x -Brownian motion defined by $W^\theta = W - \int_0^\cdot \theta_s ds$. Moreover the Radon-Nikodym derivative $d\mathbb{Q}_x / d\mathbb{P}_x |_{\mathcal{F}_T} = \mathcal{E}_T(\theta)$.

Let S be the function defined by

$$S(y) = \int_0^y \frac{1}{\sigma(u)} du, \quad y \in \mathbb{R}.$$

The function S is strictly increasing, $S(0) = 0$, $S \in C^2(\mathbb{R})$, and

$$S'(y) = (\sigma(y))^{-1}, \quad S''(y) = -\sigma'(y)(\sigma(y))^{-2}, \quad y \in \mathbb{R}.$$

Let $Y' := S(X')$, then from [3, Lemma 4.5], (20) and the fact that $S(0) = 0$, it holds that (Y', W^θ) solves

$$\begin{cases} dY'_t = \frac{1}{\sigma(X'_t)} dX'_t - \frac{1}{2} \frac{\sigma'(X'_t)}{\sigma^2(X'_t)} d\langle X' \rangle_t = \mathbb{1}_{\{Y'_t \neq 0\}} dW_t^\theta, \\ \mathbb{1}_{Y'_t=0} dt = \rho dL_t^0(X') = (\rho\sigma(0)) dL_t^0(Y'). \end{cases}$$

From **p2**, the process Y' is a sticky Brownian motion under \mathbb{Q}_x of stickiness parameter $\rho\sigma(0)$ and Theorems 3.1, 3.2 do apply to Y' for the probability \mathbb{Q}_x . In particular, it holds that

1. $(B_{n,\cdot}^{(0)}(Y')/u_n), (C_{n,\cdot}^{(0)}(Y')/u_n) \xrightarrow[n \rightarrow \infty]{\mathbb{Q}_x\text{-ucp}} 0,$
2. $(B_{n,\cdot}^{(1)}(Y')/\sqrt{n}), (C_{n,\cdot}^{(1)}(Y')/\sqrt{n}), (\mathbf{Z}_{n,\cdot}^{(1)}(Y')/\sqrt{n}) \xrightarrow[n \rightarrow \infty]{\mathbb{Q}_x\text{-ucp}} 4\sqrt{\frac{2}{\pi}}L^0(Y'),$
3. $(B_{n,\cdot}^{(2)}(Y')/n), (C_{n,\cdot}^{(2)}(Y')/n), (\mathbf{Z}_{n,\cdot}^{(2)}(Y')/n) \xrightarrow[n \rightarrow \infty]{\mathbb{Q}_x\text{-ucp}} \rho\sigma(0)L^0(Y').$

Since S is strictly increasing and $S(0) = 0$, then,

1. for all n, t and $j \in \{0, 1, 2\}$, $C_{n,t}^{(j)}(Y') = C_{n,t}^{(j)}(X')$, $B_{n,t}^{(j)}(Y') = B_{n,t}^{(j)}(X')$,
2. for all n, t and $j \in \{1, 2\}$, $\mathbf{Z}_{n,t}^{(j)}(Y') = \mathbf{Z}_{n,t}^{(j)}(X')$,
3. for all i, n , $U_i^n(Y') = U_i^n(X')$

and \mathbb{Q}_x -a.s., $L_t^0(X') = \sigma(0)L_t^0(Y')$. From equivalence of measure $\mathbb{P}_x \sim \mathbb{Q}_x$ when restricted to \mathcal{F}_T , and the fact that $\mathcal{E}(\theta)_T$ is an exponential martingale (even square integrable), these yield that the convergences hold also under \mathbb{P}_x .

Let us consider a standard localization argument to go from bounded to unbounded $\sigma, 1/\sigma, \sigma'$, by considering $X'^{\tau'_m} = (X'_{t \wedge \tau'_m})_{t \geq 0}$. More precisely, let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by X' , let $(K_m)_m$ be an increasing sequence of compacts of J such that $\bigcup_m K_m = J$, and let $(\tau'_m)_m$ be the sequence of stopping times defined by $\tau'_m = \inf\{t \geq 0 : X'_t \notin K_m\} \wedge T$ and $X'^{\tau'_m}$ be the stopped process, defined by $X'^{\tau'_m} = (X'_{t \wedge \tau'_m})_{t \geq 0}$. Condition **c4** ensures that

$$\lim_{m \rightarrow \infty} \mathbb{P}_x(t > \tau'_m) = 0. \quad (21)$$

For an process Z and all $t \geq 0$, let $(f_n(Z, t))_n$ be a sequence of statistics such that $f_n(Z, t)$ is \mathcal{F}_t -measurable. These quantities represent the difference between the statistics of interest and their limits. We also observe that

$$\begin{aligned} \mathbb{P}_x(|f_n(X', t)| > \varepsilon) &\leq \mathbb{P}_x(|f_n(X', t)| > \varepsilon; t \leq \tau'_m) + \mathbb{P}_x(t > \tau'_m) \\ &= \mathbb{P}_x(|f_n(X', t \wedge \tau'_m)| > \varepsilon; t \leq \tau'_m) + \mathbb{P}_x(t > \tau'_m) \end{aligned}$$

We use the previous part to pass to the limit as $n \rightarrow \infty$. Next, we let $m \rightarrow \infty$. From (21), this proves the result for X' .

From **c3**, these convergences also hold in \mathbb{P}_x -probability for the solution to (17).

From Lemma 2.3, the convergence is also uniform in time, in \mathbb{P}_x -probability ($\mathbb{P}_x\text{-ucp}$). This completes the proof. \square

Corollary 5.5. *Let $\mathcal{P}_x = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ be a filtered probability space and let X be an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process solution to (17) on \mathcal{P}_x . The statistic defined by*

$$\varrho_n(X) = \left(\frac{4}{\sigma(0)} \sqrt{\frac{2}{\pi}} \right) \frac{1}{\sqrt{n}} \frac{C_{n,t}^{(2)}(X)}{C_{n,t}^{(1)}(X)}, \quad n \in \mathbb{N},$$

is a consistent estimators of ρ , conditionally on the event $\mathcal{H}_t = \{\tau_0^X < t\} = \{\exists s \in [0, t) : X_s = 0\}$. This means that (see Definition 2.4), $\varrho_n(X) \rightarrow \rho$, in $\mathbb{P}_x^{\mathcal{H}_t}$ -probability, as $n \rightarrow \infty$.

Proof. The proof is the same as the proof of Proposition 3.4, using Theorem 5.4 instead of Theorem 3.1 and the fact that $\mathcal{H}_t = \{L_t^0(X) > 0\}$.

For the latter it suffices to remark that, as seen in the proof of Proposition 3.4, this is the case for the sticky Brownian motion, so using the notations of the proof of Theorem 5.4, for Y under \mathbb{Q}_x . From [29, Exercise VI.1.23], this is the case for $X = S^{-1}(Y)$ under \mathbb{Q}_x . From the equivalence of probability measures, this is also the case for X under \mathbb{P}_x . This completes the proof. \square

Remark 5.6. Corollary 5.5 holds by replacing $(C_{n,t}^{(1)}(X), C_{n,t}^{(2)}(X))$ with $(\mathbf{Z}_{n,t}^{(1)}(X), \mathbf{Z}_{n,t}^{(2)}(X))$.

6. Numerical experiments

In this section we aim to numerically establish the properties of the stickiness parameter estimator ϱ_n devised in Proposition 3.4. We compare it with the stickiness parameter estimator $\varrho_n^{(0)}$ of [5] (recalled in (6)) in term of variance and convergence rate.

We simulate sample paths of the sticky Brownian motion of stickiness parameter $\rho = 1$. For this, we use the STMCA approximation scheme with the tuned grid defined in [4, Section 2.3] of size criteria $h = 0.005$ (the nomenclature *tuned grid* and *size criteria* is the one used in [4], it is a grid adapted to the diffusion we aim to approximate). This ensures good convergence properties of the scheme [4, Corollary 2.5]. We consider initial value $X_0 = 0$ and time horizon $T = 1$.

For each generated path $\tilde{X}^{(i)}$ we consider the Monte Carlo estimator of ρ

$$\mu_{\text{MC}} := \frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} \varrho_n(\tilde{X}^{(i)}), \quad \mu_{\text{MC}}^{(0)} := \frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} \varrho_n^{(0)}(\tilde{X}^{(i)}),$$

where N_{MC} is the Monte Carlo simulation size and the estimator $\varrho_n^{(0)}$, defined in (6), is considered for: test function $g : [x \rightarrow \mathbb{1}_{0 < |x| < 5}/10]$ and $u_n = n^\alpha$ for different values of α close to 0.5. For Brownian motion $\alpha = 0.5$ yields the higher convergence speed [20].

estimator	n	Monte Carlo estimation	Monte Carlo std.
$\varrho_n(X)$	200000	1.030	0.0872
$\varrho_n^{(0)}(X)$ with $u_n = n^{0.4}$	200000	1.062	0.3710
$\varrho_n^{(0)}(X)$ with $u_n = n^{0.45}$	200000	1.039	0.2975
$\varrho_n^{(0)}(X)$ with $u_n = n^{0.5}$	200000	1.019	0.3710
$\varrho_n^{(0)}(X)$ with $u_n = n^{0.55}$	200000	1.014	0.1793

Table 1: Stickiness parameter estimation for the sticky Brownian motion: ϱ vs $\varrho^{(0)}$. Simulation size: $N_{\text{MC}} = 2000$. High values of α, n induce some bias to the approximation due to the grid nature of the STMCA approximation scheme (see discussion in [4]).

We also consider the associated Monte Carlo variance estimators:

$$\sigma_{\text{MC}} := \left(\frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} (\varrho_n(\tilde{X}^{(i)}) - \mu_{\text{MC}})^2 \right)^{\frac{1}{2}}, \quad \sigma_{\text{MC}}^{(0)} := \left(\frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} (\varrho_n^{(0)}(\tilde{X}^{(i)}) - \mu_{\text{MC}}^{(0)})^2 \right)^{\frac{1}{2}}.$$

estimator	n	Monte Carlo estimation	Monte Carlo std.
$\varrho_n(X)$	25000	1.018	0.149
$\varrho_n(X)$	50000	1.012	0.133
$\varrho_n(X)$	100000	1.012	0.121
$\varrho_n(X)$	200000	1.011	0.077
$\varrho_n(X)$	300000	1.012	0.079

Table 2: Stickiness parameter estimation of the sticky Brownian motion using the estimator ϱ : variance as function of n . Simulation size: $N_{MC} = 5000$.

Numerical simulations indicate that the choice of the normalizing sequence affects the overall convergence rate of $\varrho_n^{(0)}(X)$. Comparing to $\varrho_n^{(0)}(X)$, the estimator $\varrho_n(X)$ has the advantage that it does not depend upon a test function and a normalizing sequence. Moreover, numerical simulations indicate the following:

1. The estimator ϱ seems superior to $\varrho^{(0)}$ for any normalizing sequence as it has less variance (see Table 1, also compare Table 2 with [3, Table 2]).
2. The Monte Carlo standard deviation decreases as a function of the sampling frequency n (see Table 2).

A theoretical analysis of the convergence speed is object of further research.

A. Proof of Lemma 4.1

For reader's convenience, we recall the statement of Lemma 4.1. Let us also remind that we consider notation (3)-(5).

Lemma. *Let X be the sticky Brownian motion of stickiness parameter $\rho > 0$, defined on the probability space $(\Omega, \mathcal{F}, P_x)$ such that $X_0 = x$, P_x -a.s.. Let $(f_n, k_n, g_n, h_n; n \in \mathbb{N})$ be the functions defined for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ by $f_n(x) := P_x(X_{1/n} = 0)$, $k_n(x) := \mathbb{1}_{x \neq 0} f_n(x/\sqrt{n})$, $g_n(x) := P_x(xX_{1/n} < 0)$, $h_n(x) := \sqrt{n}g_n(x/\sqrt{n})$. It holds that*

- (i) $\lim_{n \rightarrow \infty} f_n(0) = 1$ and $\lim_{n \rightarrow \infty} \sqrt{n}(1 - f_n(0)) = 2\sqrt{2}/\rho\sqrt{\pi}$,
- (ii) the sequence $(k_n)_n$ satisfies (4) and $\lim_{n \rightarrow \infty} m_{\sqrt{n}\rho}(k_n) = 2\sqrt{2/\pi}$,
- (iii) the sequence $(h_n)_n$ satisfies (4) and $\lim_{n \rightarrow \infty} m_{\sqrt{n}\rho}(h_n) = 1/\rho$.

Let us also recall some useful results.

Lemma A.1 (e.g., [8], p.124). *The probability transition kernel of the sticky Brownian motion of stickiness parameter ρ with respect to its speed measure $m_\rho(dy) = dy + \rho\delta_0(dy)$ is the function defined for all $t > 0$, $x, y \in \mathbb{R}$ by*

$$p_\rho(t, x, y) = \frac{e^{-(x-y)^2/2t} - e^{-(|x|+|y|)^2/2t}}{\sqrt{2\pi t}} + \frac{1}{\rho} e^{2(|x|+|y|)/\rho+2t/\rho^2} \operatorname{erfc}\left(\frac{|x|+|y|}{\sqrt{2t}} + \frac{\sqrt{2t}}{\rho}\right).$$

Lemma A.2 (e.g., [3], Corollary 2.7). *The probability transition kernel of the sticky Brownian motion with respect to its speed measure $m_\rho(dx) = dx + \rho\delta_0(dx)$ satisfies for all $c, \rho, t > 0$ and $x, y \in \mathbb{R}$*

$$p_\rho(ct, x, y)m_\rho(dy) = p_{\rho/\sqrt{c}}(t, x/\sqrt{c}, y)m_{\rho/\sqrt{c}}(dy).$$

We are now ready to provide the proof of Lemma 4.1. From Lemma A.2, for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have that:

$$\begin{aligned} f_n(x) &= (\sqrt{n}\rho) p_{\sqrt{n}\rho}(1, \sqrt{n}x, 0), & g_n(x) &= \int_{\mathbb{R}} \mathbb{1}_{xy < 0} p_{\sqrt{n}\rho}(1, \sqrt{n}x, y) dy, \\ k_n(x) &= \mathbb{1}_{x \neq 0} (\sqrt{n}\rho) p_{\sqrt{n}\rho}(1, x, 0), & h_n(x) &= \sqrt{n} \int_{\mathbb{R}} \mathbb{1}_{xy < 0} p_{\sqrt{n}\rho}(1, x, y) dy. \end{aligned} \quad (22)$$

Proof of Item (i). Regarding the first relation, from (22) and Lemma A.1,

$$f_n(0) = (\sqrt{n}\rho) p_{\sqrt{n}\rho}(1, 0, 0) = e^{2/n\rho^2} \operatorname{erfc}\left(\frac{\sqrt{2}}{\sqrt{n}\rho}\right).$$

Taking the limit as $n \rightarrow \infty$ yields the result. Indeed, it holds that

$$\lim_{n \rightarrow \infty} f_n(0) = \lim_{y \rightarrow 0} e^{y^2} \operatorname{erfc}(y) = 1.$$

Regarding the second relation, from the l'Hôpital's rule,

$$\lim_{y \rightarrow 0} \frac{1}{y} \left(1 - e^{y^2} \operatorname{erfc}(y)\right) = -\lim_{y \rightarrow 0} e^{y^2} \operatorname{erfc}'(y) = \frac{2}{\sqrt{\pi}}.$$

Taking the limit as $n \rightarrow \infty$ yields

$$\lim_{n \rightarrow \infty} \sqrt{n}(1 - f_n(0)) = \lim_{n \rightarrow \infty} \frac{\sqrt{2}}{\rho} \left(\frac{\sqrt{2}}{\sqrt{n}\rho}\right)^{-1} (1 - f_n(0)) = \frac{\sqrt{2}}{\rho} \frac{2}{\sqrt{\pi}} = \frac{2}{\rho} \sqrt{\frac{2}{\pi}},$$

which is the desired result. This completes the proof. \square

Proof of Item (ii). Since $k_n(0) = 0$, from (22) and Lemma A.1, we have that

$$\begin{aligned} m_{\sqrt{n}\rho}(k_n) &= \int_{\mathbb{R}} k_n(x) dx = (\sqrt{n}\rho) \int_{\mathbb{R}} p_{\sqrt{n}\rho}(1, x, 0) dx \\ &= \int_{\mathbb{R}} e^{2|x|/\sqrt{n}\rho + 2/n\rho^2} \operatorname{erfc}\left(\frac{|x|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho}\right) dx. \end{aligned}$$

Regarding the function we integrate, from the Mills' ratio on the Gaussian random variable (see [17, p.98]), for some constant $K_{\text{Mills}} > 0$, we have that

$$\begin{aligned} \lim_{n \rightarrow \infty} e^{2|x|/\sqrt{n}\rho + 2/n\rho^2} \operatorname{erfc}\left(\frac{|x|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho}\right) &= \operatorname{erfc}\left(\frac{|x|}{\sqrt{2}}\right), \\ \mathbb{1}_{|x| \geq 1} e^{2|x|/\sqrt{n}\rho + 2/n\rho^2} \operatorname{erfc}\left(\frac{|x|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho}\right) &\leq \mathbb{1}_{|x| \geq 1} K_{\text{Mills}} \frac{\sqrt{2n}\rho}{\sqrt{n}\rho|x| + 2} \operatorname{erfc}\left(-\frac{x^2}{2}\right) \\ &\leq \sqrt{2} K_{\text{Mills}} \operatorname{erfc}\left(-\frac{x^2}{2}\right), \\ \mathbb{1}_{|x| < 1} e^{2|x|/\sqrt{n}\rho + 2/n\rho^2} \operatorname{erfc}\left(\frac{|x|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho}\right) &\leq \mathbb{1}_{|x| < 1} e^{2/\rho + 2/\rho^2} \operatorname{erfc}\left(\frac{|x|}{\sqrt{2}}\right), \end{aligned}$$

where both upper bounds are $L^1(\mathbb{R})$. Thus, from the dominated convergence theorem,

$$\lim_{n \rightarrow \infty} m_{\sqrt{n}\rho}(k_n) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \operatorname{erfc} \left(\frac{|x|}{\sqrt{2}} \right) dx = 2\sqrt{\frac{2}{\pi}}.$$

This proves the second assertion.

Regarding the first assertion, we observe that

$$\begin{aligned} & |x|e^{2|x|/\sqrt{n}\rho+2/n\rho^2} \operatorname{erfc} \left(\frac{|x|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho} \right) \\ & \leq \mathbb{1}_{|x| \geq 1} K_{\text{Mills}} \frac{\sqrt{2n}\rho|x|}{\sqrt{n}\rho|x| + 2} \operatorname{erfc} \left(-\frac{x^2}{2} \right) + \mathbb{1}_{|x| < 1} e^{2/\rho+2/\rho^2} \exp \left(\frac{|x|}{\sqrt{2}} \right) \\ & \leq \mathbb{1}_{|x| \geq 1} K_{\text{Mills}} \sqrt{2} \exp \left(-\frac{x^2}{2} \right) + \mathbb{1}_{|x| < 1} e^{2/\rho+2/\rho^2} \operatorname{erfc} \left(\frac{|x|}{\sqrt{2}} \right), \end{aligned}$$

where the upper bound defined a function in $L^1(\mathbb{R})$. Hence, from the dominated convergence theorem,

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} |x|k_n(x) dx &= \int_{\mathbb{R}} \lim_{n \rightarrow \infty} \left(|x|e^{2|x|/\sqrt{n}\rho+2/n\rho^2} \operatorname{erfc} \left(\frac{|x|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho} \right) \right) dx \\ &= \int_{\mathbb{R}} |x| \operatorname{erfc} \left(\frac{|x|}{\sqrt{2}} \right) dx = 1. \end{aligned}$$

With similar arguments, one can prove that $m(k_n^2)$ converges to some constant $K > 0$. Also,

$$\lim_{n \rightarrow \infty} k_n^2(\sqrt{n}x) = \lim_{n \rightarrow \infty} \left(e^{2|x|/\rho+2/n\rho^2} \operatorname{erfc} \left(\frac{\sqrt{n}|x|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho} \right) \right)^2 = 0.$$

From all the above, $(k_n)_n$ satisfies (4). This completes the proof. \square

Proof of Item (iii). The first assertion is proved with similar arguments as in the proof of Item (ii). Regarding the second assertion, from (22) and Lemma A.1, we have that

$$\begin{aligned} & \sqrt{n} \mathbb{1}_{xy < 0} p_{\sqrt{n}\rho}(1, x, y) \\ &= \frac{1}{\rho} \mathbb{1}_{xy < 0} \left(\mathbb{1}_{|x|+|y| \geq 1} + \mathbb{1}_{|x|+|y| < 1} \right) e^{2(|x|+|y|)/\sqrt{n}\rho+2/n\rho^2} \operatorname{erfc} \left(\frac{|x|+|y|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho} \right) \\ & \leq 2 \left(\mathbb{1}_{|x|+|y| \geq 1} K_{\text{Mills}} \frac{\sqrt{2n}\rho(|x|+|y|)}{\sqrt{n}\rho(|x|+|y|) + 2} e^{-\frac{(|x|+|y|)^2}{2}} + \mathbb{1}_{|x|+|y| < 1} e^{2/\rho+2/\rho^2} \operatorname{erfc} \left(\frac{|x|+|y|}{\sqrt{2}} \right) \right) \\ & \leq 2 \left(\mathbb{1}_{|x|+|y| \geq 1} K_{\text{Mills}} \sqrt{2} e^{-\frac{(|x|+|y|)^2}{2}} + \mathbb{1}_{|x|+|y| < 1} e^{2/\rho+2/\rho^2} \operatorname{erfc} \left(\frac{|x|+|y|}{\sqrt{2}} \right) \right), \end{aligned}$$

where the upper bound defined a function in $L^1(\mathbb{R})$. Hence, from the dominated conver-

gence theorem,

$$\begin{aligned}
m_{\sqrt{n}\rho}(h_n) &= \sqrt{n} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{xy < 0} p_{\sqrt{n}\rho}(1, x, y) \, dy \, dx \\
&= \frac{1}{\rho} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_{xy < 0} e^{2(|x|+|y|)/\sqrt{n}\rho+2/n\rho^2} \operatorname{erfc} \left(\frac{|x|+|y|}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho} \right) \, dy \, dx \\
&= \frac{2}{\rho} \int_0^\infty \int_0^\infty e^{2(x+y)/\sqrt{n}\rho+2/n\rho^2} \operatorname{erfc} \left(\frac{x+y}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{n}\rho} \right) \, dy \, dx \\
&\xrightarrow{n \rightarrow \infty} \frac{2}{\rho} \int_0^\infty \int_0^\infty \operatorname{erfc} \left(\frac{x+y}{\sqrt{2}} \right) \, dy \, dx = \frac{1}{\rho}.
\end{aligned}$$

This completes the proof. \square

B. Proof of Lemma 4.3

For reader's convenience, we recall the statement of Lemma 4.3 using notation (10) for the first hitting time.

Lemma. *Let X be the sticky Brownian motion of stickiness parameter $\rho > 0$ defined on the filtered probability space $\mathcal{P}_x = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}_x)$ such that \mathbb{P}_x -a.s., $X_0 = x$. We consider the process X' , defined for all $t \geq 0$ by*

$$X'_t = \begin{cases} X_t & t < \tau_0^X, \\ -X_t & t \geq \tau_0^X. \end{cases}$$

It holds that X' is also a sticky Brownian motion of stickiness parameter ρ .

Proof. Let Z be the Brownian motion defined in **t1**, on an extension of \mathcal{P}_x , so that

$$\forall t \geq 0: X_t = Z_{\gamma(t)}$$

with γ the time-change defined in **t1**. We observe that $\tau_0^X = \gamma(\tau_0^Z)$. Let Z' be the process defined for all $t \geq 0$ by

$$Z'_t = \begin{cases} Z_t & t < \tau_0^Z, \\ -Z_t & t \geq \tau_0^Z. \end{cases}$$

From the reflection principle of the Brownian motion Z (see [29, Exercise III.3.14]), we have that Z' is also a Brownian motion. Moreover, $L^0(Z) = L^0(Z')$, $\tau_0^Z = \tau_0^{Z'}$. Therefore, $A(t) = t + \rho L_t^0(Z')$ and $\tau_0^X = \gamma(\tau_0^Z) = \gamma(\tau_0^{Z'})$ which, by the definition of X' , is equal also to $\tau_0^{X'}$. From all the above, we show that $X' = (Z'_{\gamma(t)})_{t \geq 0}$. Indeed, on the event $\{t \geq \tau_0^X\}$ it holds that

$$X'_t = -X_t = -Z_{\gamma(t)} = (-Z)_{\gamma(t)} = Z'_{\gamma(t)},$$

and, since $X_t = Z_{\gamma(t)}$,

$$X'_t = Z_{\gamma(t)} \mathbb{1}_{t < \tau_0^X} + Z'_{\gamma(t)} \mathbb{1}_{t \geq \tau_0^X} = Z'_{\gamma(t)}.$$

To finish the proof, it suffices to observe that **t2** ensures that X' is a sticky Brownian motion of stickiness parameter ρ . \square

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