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Département de Formation Doctorale en Mathématiques
École Doctorale IAE + M

Contribution à l'étude des processus stochastiques sur les groupes quantiques

S.C.D. - U.H.P. NANCY 1 SCIENCES BIBLIOTHÈQUE

THÈSE

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(Spécialité Mathématiques)

par

Uwe FRANZ

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Résumé

Ce mémoire est rélatif à l'étude des processus stochastiques sur les algèbres de Hopf. Ces algèbres jouent un rôle important en physique mathématique sous le nom groupes quantiques.

Une grande partie de cette thèse est consacrée à l'étude des processus de Lévy, c-à-d des processus à accroissements indépendants et stationnaires, sur ces algèbres. Deux constructions, soit à partir d'un processus de Lévy classique, soit à partir d'une marche aléatoire quantique, sont proprosées. Ces processus sont ensuite étudiés à l'aide des représentations duales et de leurs systèmes d'Appell. En particulier, ceci a permis de démontrer une formule de Feynman-Kac et d'établir un lien étroit entre ces processus et des équations d'évolution sur les groupes quantiques.

Les représentations duales sont également utilisées pour donner des conditions suffisantes pour l'existence de versions classiques des processus de Lévy sur des bigèbres et pour les caractériser. Plusieurs exemples, y compris la martingale d'Azéma, sont traités en détail.

Un autre thème central de ce travail est la caractérisation des lois de Gauss au sens de Bernstein. Il est montré comment les fonctionnelles ainsi que les semi-groupes de convolution sur des algèbres de Hopf qui satisfont l'analogue de la propriété de Bernstein peuvent être calculés. Il est aussi démontré que le plongement d'une fonctionnelle normée infiniment divisible dans un semi-groupe de convolution continu sur un groupe quantique nilpotent ou sur un groupe tressé nilpotent est unique.

Finalement, plusieurs théorèmes limites (loi des grands nombres, théorème de la limite centrale, etc.) sur les groupes quantiques sont présentés.

Mots-clés: Processus de Lévy sur des bigèbres, théorèmes limites

Abstract

In this thesis stochastique processus on Hopf algebras are studied. These algebras, also known under the name quantum group, play in important role in mathematical physics.

A major part of this work is concerned with Lévy processes, i.e. processes with stationary and independent increments. Two construction are proposed, one starts from a classical Lévy process, the other one uses quantom random walks. These processes are then studied with the help of dual representations and Appell systems. This has allowed us to prove an analogue of the Feynman-Kac formula, and to study the relation between the processes and their evolution equations.

Dual representations are also used to give sufficient conditions for the existence of classical versions of Lévy processes on bialgebras, and to calculate the classical generators. Several examples including the Azéma martingale are treated in detail.

Another central theme is the classification of Gaussian laws. It is shown, how the functionals and convolution semigroups that satisfy an analogue of the Bernstein property can be determined. We also prove the the embedding of a normalized functional into a continuous convolution semigroup on a nilpotent quantum group or nilpotent braided group is unique.

Finally, there are also several limit theorems (law of large numbers, central limit theorem, etc.) presented in this work.

Mathematics Subject Classifications (1991): *60B99 Probability theory on general structures, 16W30 Hopf algebras (assoc. rings and algebras), 60F05 Weak limit theorems.

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Avant-propos

Cette thèse se compose de cinq articles en anglais :

- Chapitre 4 : Duality and Multiplicative Stochastic Processes on Quantum Groups
- Chapitre 5 : Diffusions on Braided Spaces
- Chapitre 6 : Gauss Laws in the Sense of Bernstein and Uniqueness of Embedding into Convolution Semigroups on Quantum Groups and Braided Groups
- Chapitre 7 : Evolution Equations and Lévy Processus on Quantum Groups
- Chapitre 8 : Brownian Motion on the Affine Group and Generalized Gegenbauer Polynomials

et de deux chapitres sur des travaux en cours, également en anglais :

- Chapitre 9 : Limit theorems on Quantum Groups
- Chapitre 10 Classical Versions of Quantum Lévy Processes,

qui forment ensemble la deuxième partie.

La première partie, composée de trois chapitres, est une synthèse en français des résultats essentiels. Après l'introduction nous donnons un bref aperçu de la théorie des processus de Lévy sur les bigèbres. Ensuite, dans le Chapitre 3, les résultats propres à ce travail sont énoncés. Une bibliographie se trouve a la fin.

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Première partie

Synthèse

Chapitre 1

Introduction

La manière dont les probabilités quantiques sont obtenues à partir des probabilités classiques ressemble, du moins formellement, à la transition de la mécanique classique à la mécanique quantique. Les observables, appelées variables aléatoires en théorie des probabilités, sont autorisées à vérifier des relations de commutation non triviales (i.e. l'algèbre commutative des fonctions mesurables sur un espace probabilisé est remplacée par une certaine (*-)algèbre non commutative).

Les groupes quantiques dérivent de façon analogue des groupes classiques. L'algèbre de Hopf abélienne des fonctions représentatives est remplaçée par une "déformation" non commutative. De la structure de groupe il reste la structure de cogèbre qui nous permet de définir la plupart des constructions de la théorie des groupes de façon plus générale pour des algèbres de Hopf ou bigèbres.

Le présent travail tente de combiner ces deux notions. Nous étudions des processus quantiques sur des groupes quantiques. Ces processus sont les analogues non-commutatifs naturels des processus stochastiques à valeurs dans les groupes. Nous sommes donc particulièrement interessés par les propriétés qui utilisent la structure de cogèbre, par exemple la théorie des processus à accroissements indépendants et stationnaires (les processus de Lévy), où la notion d'accroissement est maintenant définie via le coproduit au lieu de la loi du groupe. Nous nous sommes aussi intéressés aux théorèmes limites et à la caractérisation de certaines lois de probabilité également basées sur le coproduit.

Nous faisons à présent un bref survol des résultats contenus dans cette thèse, les définitions exactes et les formulations précises sont données aux chapitres 2 et 3.

Les thèmes centraux sont :

- la construction des processus de Lévy et de leurs semi-groupes de convolution sur les bigébres,
- le lien entre les processus de Lévy et les équations d'évolution,
- étude des versions classiques des processus de Lévy sur les bigébres,
- la caractérisation de certaines classes de fonctionnelles á l'aide de leurs propriétés par rapport à la structure de cogèbre,
- des théorèmes limites, avec caractérisation des lois limites.

La première partie est un sommaire. Le chapitre 2 contient toutes les définitions de base et tous les résultats dont nous aurons besoin par la suite.

Les résultats essentiels sont énoncés au chapitre 3.

A la fin de ce chapitre nous avons inclus une liste des publications auxquelles cette thèse a donné lieu (voir section 3.7, page 20).

Chapitre 2

Processus stochastiques sur des bigèbres

Ce chapitre présente une brève introduction à la théorie des processus stochastiques sur des bigèbres, basée essentiellement sur le livre de M. Schürmann[Sch93].

La notion d'indépendance tensorielle à gauche ou à droite est généralisée à l'indépendance tressée.

2.1 Probabilités non-commutatives

Nous résumons les définitions les plus importantes concernant les probabilités quantiques ou non-commutatives, pour une introduction plus détaillée on pourra consulter, par exemple, les livres de Biane [Bia93], de Meyer [Mey93], et de Parthasarathy [Par92].

Un *espace de probabilité non-commutatif* (*quantum probability space*) est un couple $(\mathcal{A}, \Phi)$ où $\mathcal{A}$ est une algèbre involutive sur $\mathbb{C}$ et Φ un *état* (*state*) sur $\mathcal{A}$, càd une fonctionnelle linéaire positive normalisée. Un espace de probabilité classique définit un espace de probabilité non-commutatif si on prend une algèbre convenable de fonctions intégrables à valeurs complexes sur Ω, par exemple $L^\infty(\Omega, \mathcal{F}, P)$, et la fonctionnelle $\Phi : f \mapsto \int_\Omega f dP$.

Une *variable aléatoire non-commutative* (*quantum random variable*) j sur un espace de probabilité non-commutatif $(\mathcal{A}, \Phi)$ est un homomorphisme de *-algèbre $j : \mathcal{B} \to \mathcal{A}$. À partir d'une variable aléatoire classique à valeurs dans un espace mesurable $(E, \mathcal{E})$ on peut définir une variable aléatoire non-commutative en posant $j_X(f) = f \circ X$ pour $f \in \mathcal{B}$ (où l'algèbre $\mathcal{B}$ des fonctions mesurables sur $(E, \mathcal{E})$ est choisie t.q. $f \circ X \in \mathcal{A}$). La fonctionnelle $\varphi_j = \Phi \circ j$ est appelée la *distribution* de j dans l'état Φ.

Un *processus stochastique non-commutatif* (*quantum stochastic process*) n'est rien d'autre qu'une famille de variables aléatoires $\{j_t; t \in I\}$ sur le même espace de probabilité non-commutatif, indexée par un ensemble I, comme dans le cas classique. Ses distributions uni-dimensionnelles (*one-dimensional* ou *marginal distributions*) sont les fonctionnelles $\varphi_t = \Phi \circ j_t$.

Deux processus stochastiques non-commutatifs $\{j_t : \mathcal{B} \to (\mathcal{A}_j, \Phi_j); t \in I\}$ et $\{k_t : \mathcal{B} \to (\mathcal{A}_k, \Phi_k); t \in I\}$, indexés par le même ensemble I, définis sur la même algèbre $\mathcal{B}$, sont

équivalents, si

$$\Phi_j\big(j_{t_1}(b_1)\cdots j_{t_n}(b_n)\big) = \Phi_k\big(k_{t_1}(b_1)\cdots k_{t_n}(b_n)\big),$$

pour tous $n \in \mathbb{N}$, $t_1, \ldots, t_n \in I$, $b_1, \ldots, b_n \in \mathcal{B}$.

Un élément a d'un espace de probabilité non-commutatif définit une variable aléatoire sur $\mathbb{C} < z, z^* >$ (= l'algèbre libre engendrée par z, z^*), ou sur $\mathbb{C}[z]$ (= l'algèbre des polynômes à valeurs complexes sur $\mathbb{R}$), si a est auto-adjoint. On prend simplement l'homomorphisme de *-algèbre défini par $j(z) = a$. De la même façon toute famille $\{a_t; t \in I\}$ d'éléments de $\mathcal{A}$ devient un processus stochastique non-commutatif indexé par I.

Ceci permet d'associer une densité sur la droite réelle à un élément auto-adjoint a d'un espace de probabilité non-commutatif. Une densité de a (dans l'état Φ) est une mesure μ sur $\mathbb{R}$ telle que $\Phi(a^n) = \int_{\mathbb{R}} x^n \mathrm{d}\mu(x)$ pour tout $n \in \mathbb{N}$. Cette mesure n'est pas nécessairement unique. Une variable aléatoire X sur un espace de probabilité $(\Omega, \mathcal{F}, P)$ à valeurs dans $\mathbb{R}$ est une *version classique* de a si sa loi P_X est une densité de a. Pour la version classique $\{\tilde{X}_t; t \in I\}$ d'un processus stochastique non-commutatif on demande seulement que les moments ordonnés coïncident, c-à-d

$$\Phi(a_{t_1}^{k_1}\cdots a_{t_n}^{k_n}) = \mathbb{E}(\tilde{X}_{t_1}^{k_1}\cdots \tilde{X}_{t_n}^{k_n})$$

pour tous $n, k_1, \ldots, k_n \in \mathbb{N}$, $t_1, \ldots, t_n \in I$, $t_1 \leq t_2 \leq \cdots \leq t_n$.

2.2 Bigèbres et algèbres de Hopf

Les textes classiques sur les algèbres de Hopf sont [Swe69, Abe80], mais voir aussi [DHL91, SS93, CP95, Gui95, Kas95, Maj95b].

Une algèbre associative A sur un corps $\mathbb{K}$ est un $\mathbb{K}$-espace vectoriel muni d'une application linéaire $m : A \otimes A \to A$ telle que

$$m \circ (m \otimes \mathrm{id}) = m \circ (\mathrm{id} \otimes m) \qquad \text{(associativity)}.$$

Toutes nos algèbres sont unitaires, ie. il existe un élément $e \in A$ tel que $m(a \otimes e) = m(e \otimes a) = a$ pour tout $a \in A$. Ceci est équivalent à l'existence d'une application linéaire $e : \mathbb{K} \to A$ telle que

$$m \circ (\mathrm{id} \otimes e) = m \circ (e \otimes \mathrm{id}) = \mathrm{id}.$$

Pour voir l'équivalence poser $e(\lambda) = \lambda e$ ou $e = e(1)$.

Le produit tensoriel $A \otimes A$ est une algèbre avec

$$\begin{aligned} e^{\otimes} &= e \otimes e, \\ m^{\otimes} &= (m \otimes m) \circ (\mathrm{id} \otimes \tau \circ \mathrm{id}), \end{aligned}$$

où τ est le 'flip' defini par $\tau(a \otimes b) = b \otimes a$, $\forall a, b \in A$.

Une algèbre est *commutative*, si $m = m \circ \tau$.

On peut considérer la notion de cogèbre comme duale de la notion d'algèbre. Si (A, m, e) est une algèbre (et $\dim A < \infty$), alors les applications $m^* : A^* \to (A \otimes A)^* \cong A^* \otimes A^*$, $e^* : A^* \to \mathbb{K}$, définies sur $A^* = \{\varphi : A \to \mathbb{K};\ \varphi \text{ linear}\}$ par

$$m^*(\varphi)(a \otimes b) = \varphi(m(a \otimes b)), \qquad e^*(\varphi) = \varphi(e),$$

satisfont

$$\begin{aligned}(\mathrm{id} \otimes m^*) \circ m^* &= (m^* \otimes \mathrm{id}) \circ m^*, \\ (e^* \otimes \mathrm{id}) \circ m^* &= (\mathrm{id} \otimes e^*) \circ m^* = \mathrm{id}\end{aligned}$$

On va prendre ces propriétés comme définition d'une cogèbre.

Définition 2.2.1 *Une* cogèbre *(*coalgebra*) sur un corps* $\mathbb{K}$ *est le triplet* (C, Δ, ϵ) *constitué d'un* $\mathbb{K}$*-espace vectoriel* C *et d'applications* $\mathbb{K}$*-linéaires* $\Delta : C \to C \otimes C$, $\epsilon : C \to \mathbb{K}$ *telles que*

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta) \circ \Delta && \textit{(coassociativity)} \\ (\epsilon \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \epsilon) \circ \Delta = \mathrm{id} && \textit{(counit)}\end{aligned}$$

Le produit tensoriel d'une cogèbre est $(C \otimes C, \Delta^{\otimes}, \epsilon^{\otimes})$ où les applications $\Delta^{\otimes}$ et $\epsilon^{\otimes}$ sont définies comme suit :

$$\begin{aligned}\Delta^{\otimes} &: C \otimes C \to (C \otimes C) \otimes (C \otimes C), \\ \Delta^{\otimes} &= (\mathrm{id} \otimes \tau \otimes \mathrm{id}) \circ (\Delta \otimes \Delta), \\ \epsilon^{\otimes} &: C \otimes C \to \mathbb{K}, \\ \epsilon^{\otimes} &= \epsilon \otimes \epsilon\end{aligned}$$

On dit qu'une cogèbre est *co-commutative*, si $\Delta = \tau \circ \Delta$.

Définition 2.2.2 *Une bigèbre (*bialgebra*) est un 5-uplet* $(A, m, e, \Delta, \epsilon)$ *où*

- (A, m, e) *est une* $\mathbb{K}$*-algèbre,*
- (A, Δ, ϵ) *est une* $\mathbb{K}$*-cogèbre,*
- *la structure d'algèbre est la structure de cogèbre sont compatibles dans le sens que :*

  $\Delta : A \to A \otimes A$ *et* $\epsilon : A \to \mathbb{K}$ *sont des homomorphismes d'algèbre*

  ou, d'une façon équivalente,

  $m : A \otimes A \to A$ *et* $e : \mathbb{K} \to A$ *sont des homomorphismes de cogèbre.*

Les conditions de compatibilité s'écrivent aussi sous la forme

$$\begin{aligned}\Delta \circ m &= m^{\otimes} \circ (\Delta \otimes \Delta) = (m \otimes m) \circ \Delta^{\otimes}, \\ \Delta \circ e &= e \otimes e, \\ \epsilon \circ m &= \epsilon \otimes \epsilon, \\ \epsilon \circ e &= \mathrm{id}_{\mathbb{K}}.\end{aligned}$$

Définition 2.2.3 *Soit* $(A, m, e, \Delta, \epsilon)$ *une bigèbre. Une application linéaire* $S : A \to A$ *qui satisfait*

$$m \circ (\mathrm{id} \otimes S) \circ \Delta = m \circ (S \otimes \mathrm{id}) \circ \Delta = e \circ \epsilon$$

est appelée antipode, *et* $(A, m, e, \Delta, \epsilon, S)$ *est appelé* algèbre de Hopf.

Si l'antipode existe, alors il est unique, et est un anti-homomorphisme d'algèbre, ie. $m \circ (S \otimes S) = S \circ m \circ \tau$, ou $S(a)S(b) = S(ba)$ pour tous $a, b \in A$.

Une **-bigèbre* (**-bialgebra*) est une bigèbre (sur un corps involutif, eg. $\mathbb{C}$) muni d'une involution $* : A \to A$ telle que $(A, m, e, *)$ est une *-algèbre (ie. $(e(\lambda))^* = e(\overline{\lambda})$, $* \circ m = m \circ \tau \circ (* \otimes *)$, $* \circ * = \mathrm{id}$), et Δ et ε sont des homomorphismes de *-algèbre (ie. $(* \otimes *) \circ \Delta = \Delta \circ *$ and $\varepsilon(a^*) = \overline{(\varepsilon(a))}$ pour tous $a \in A$). Pour une *-algèbre de Hopf on demande en plus que $S \circ * \circ S \circ * = \mathrm{id}$.

Nous dirons que deux $\mathbb{K}$-bigèbres $(A_1, m_1, e_1, \Delta_1, \epsilon_1)$ et $(A_2, m_2, e_2, \Delta_2, \epsilon_2)$ forment un *couple dual* (*form a dual pair* ou *are dually paired*), s'il existe une application bilinéaire non-dégénérée $< \cdot, \cdot >: A_1 \times A_2 \to \mathbb{K}$, telle que

$$\begin{aligned}
< m_1(a_1 \otimes b_1), c_2 > &= < a_1 \otimes b_1, \Delta_2(c_2) >_{\otimes}, \\
< c_1, m_2(a_2 \otimes b_2) > &= < \Delta_1(c_1), a_2 \otimes b_2 >_{\otimes}, \\
< \epsilon_1, a_2 > &= \epsilon_2(a_2), \\
< a_1, \epsilon_2 > &= \epsilon_1(a_1),
\end{aligned}$$

pour tous $a_1, b_1, c_1 \in A_1$, $a_2, b_2, c_2 \in A_2$. Un couple dual d'algèbres de Hopf $(A_1, m_1, e_1, \Delta_1, \epsilon_1, S_1)$ et $(A_2, m_2, e_2, \Delta_2, \epsilon_2, S_2)$ doit aussi satisfaire à la condition

$$< S_1(a_1), a_2 >=< a_1, S_2(a_2) >$$

pour tous $a_1 \in A_1$, $a_2 \in A_2$.

Si A_1 et A_2 sont des *-algèbres de Hopf, on demande en plus que les involutions soient duales au sens suivant :

$$< a_1^*, a_2 >= \overline{< a_1, (S(a_2))^* >}, \qquad \forall a_1 \in A_1, \quad a_2 \in A_2.$$

Si deux bigèbres forment un couple dual, alors ils agissent l'une sur l'autre par les *représentations duales à gauche et à droite* (right and left *regular* or *dual representation*). Elles sont définies par $\rho_R^*, \rho_L^* : A_1 \to \mathrm{Hom}(A_2, A_2)$, $\rho_R^*(X) = (\mathrm{id} \otimes X) \circ \Delta_2$ et $\rho_L^*(X) = (X \otimes \mathrm{id}) \circ \Delta_2$, respectivement, et satisfont

$$\rho_R^*(XY) = \rho_R^*(X)\rho_R^*(Y), \qquad \rho_L^*(XY) = \rho_L^*(Y)\rho_L^*(X),$$

pour tous $X, Y \in A_1$. Ces représentations satisfont une *propriété de Leibnitz* (*Leibnitz formula*), parce que la multiplication de A_2 et le coproduit de A_1 sont duaux. Soit $\Delta_1(X) = \sum_i X_i^{(1)} \otimes X_i^{(2)}$, alors on a

$$\begin{aligned}
\rho_L^*(X)(ab) &= \textstyle\sum(\rho_L^*(X_i^{(1)})a)(\rho_L^*(X_i^{(2)})b) \\
\rho_R^*(X)(ab) &= \textstyle\sum(\rho_R^* X_i^{(1)})a)(\rho_R^*(X_i^{(2)})b)
\end{aligned} \tag{2.1}$$

pour tous $X \in A_1$, $a, b \in A_2$. Si X est *primitif* (ie. $\Delta X = X \otimes 1 + 1 \otimes X$) on retrouve la formule de Leibnitz classique $\rho_{R,L}^*(X)(ab) = (\rho_{R,L}^*(X)a)b + a(\rho_{R,L}^*(X)b)$.

2.3 Catégories tressées

Les catégories tressées ont été introduites par André Joyal et Ross Street (Macquarie Mathematics Reports 850067 (Dec. 1985) et 86081 (Nov. 1986), voir [JS91a, JS91b, JS93, Kas95, Maj95b]).

Définition 2.3.1 *[Mac71] Une* catégorie momoïdale *ou* tensorielle *(*tensor category *ou* monoidal category*) est une catégorie munie d'un produit tensoriel* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, d'une unité* I*, d'une contrainte d'associativité* a*, d'une contrainte d'unité à gauche* l *par rapport à* I*, et d'une contrainte d'unité à droite* r *par rapport à* I*, t.q. l'*axiome du pentagone *(*Pentagon Axiom*)*

$$\begin{tikzcd}
 & \big((U \otimes V) \otimes W\big) \otimes X \arrow[dl, "a_{U,V,W} \otimes \mathrm{id}_X"'] \arrow[dr, "a_{U \otimes V,W,X}"] & \\
\big(U \otimes (V \otimes W) \otimes X\big) \arrow[d, "a_{U,V \otimes W,X}"'] & & (U \otimes V) \otimes (W \otimes X) \arrow[dl, "a_{U,V,X \otimes X}"] \\
U \otimes \big((V \otimes W) \otimes X\big) \arrow[dr, "\mathrm{id}_U \otimes a_{V,W,X}"'] & & \\
 & U \otimes \big(V \otimes (W \otimes X)\big) &
\end{tikzcd}$$

*et l'*axiome du triangle *(*Triangle Axiom*)*

$$\begin{tikzcd}
(V \otimes I) \otimes W \arrow[rr, "a_{V,I,W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (I \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
 & V \otimes W &
\end{tikzcd}$$

sont satisfaits pour tous les objets U, V, W, X *de* $\mathcal{C}$*.*

Exemple: L'exemple le plus fondamental est la catégorie $\mathrm{Vec}(\mathbb{K})$ des espaces vectoriels sur un corps $\mathbb{K}$. Elle est munie d'un produit tensoriel $\otimes : \mathrm{Vec}(\mathbb{K}) \times \mathrm{Vec}(\mathbb{K}) \ni U \times V \mapsto U \otimes V \in \mathrm{Vec}(\mathbb{K})$, l'unité est le corps $\mathbb{K}$, et les contraintes d'associativité et d'unité sont données par les isomorphismes naturels $a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, $l_V : \mathbb{K} \otimes V \to V$, $r_V : V \otimes \mathbb{K} \to V$,

$$a_{U,V,W}\big((u \otimes v) \otimes w\big) = u \otimes (v \otimes w), \qquad l_V(\lambda \otimes v) = \lambda v = r_V(v \otimes \lambda)$$

où $u \in U$, $v \in V$, $w \in W$, $U, V, W \in \mathrm{Vec}(\mathbb{K})$, $\lambda \in \mathbb{K}$.

Exemple: $\mathcal{A}$ soit une bigèbre. La catégorie des $\mathcal{A}$-modules à gauche ou à droite et la catégorie des $\mathcal{A}$-comodules à droite ou à gauche peuvent être munies d'un produit tensoriel.

Définition 2.3.2 *(a) Un* $\mathcal{A}$-module à gauche (droite) *(*left (right) $\mathcal{A}$-module*) d'une algèbre* $\mathcal{A}$ *est un couple* (M, μ_M) *consistant en un espace vectoriel* M *et une application linéaire* $\mu_M : \mathcal{A} \otimes M \to M$ *(*$\mu_M : M \otimes \mathcal{A} \to M$*, resp.) t.q.* $\mu_M(a \otimes \mu_M(b \otimes u)) = \mu_M(m(a \otimes b) \otimes b)$ *et* $\mu_M(e \otimes u) = u$ *(ou* $\mu_M(\mu_M(u \otimes a) \otimes b) = \mu_M(u \otimes m(a \otimes b))$ *et* $\mu_M(u \otimes 1) = u$*, resp.) pour tous* $a, b \in \mathcal{A}$*,* $u \in M$*.*

(b) Un $\mathcal{C}$-comodule à droite (gauche) *(*right (left) $\mathcal{C}$-comodule*) d'une cogèbre* $\mathcal{C}$ *est un couple* (N, δ_N) *consistant en un espace vectoriel* N *et une application linéaire* $\delta_N : N \to N \otimes \mathcal{C}$ *(*$\delta_N : N \to \mathcal{C} \otimes N$*, resp.) t.q.* $(\delta_N \otimes \mathrm{id}_{\mathcal{C}}) \circ \delta_N = (\mathrm{id}_N \otimes \Delta) \circ \delta_N$ *et* $(\mathrm{id}_N \otimes \epsilon) \circ \delta_N = \mathrm{id}_N$ *(ou* $(\mathrm{id}_{\mathcal{C}} \otimes \delta_N) \circ \delta_N = (\Delta \otimes \mathrm{id}_N) \circ \delta_N$ *et* $(\varepsilon \otimes \mathrm{id}_N) \circ \delta_N = \mathrm{id}_N$*, resp.).*

Soient M, M' des $\mathcal{A}$-modules à gauche (droite), alors $M \otimes M'$ est un $\mathcal{A} \otimes \mathcal{A}$-module à gauche (droite) avec $(\mu_M \otimes \mu_{M'}) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$. Si $\mathcal{A}$ est une bigèbre, alors $M \otimes M'$ devient un $\mathcal{A}$-module, si on pose $\mu_{M \otimes M'} = (\mu_M \otimes \mu_{M'}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \tau \otimes \mathrm{id}'_M) \circ (\Delta \otimes \mathrm{id}_{M \otimes M'})$ (ou $\mu_{M \otimes M'} = (\mu_M \otimes \mu_{M'}) \circ (\mathrm{id}_M \otimes \tau \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{M \otimes M'} \otimes \Delta)$ pour des modules à droite).

De façon analogue on définit une coaction sur le produit tensoriel de deux $\mathcal{A}$-comodules N, N' à gauche (droite) par $\delta_{N \otimes N'} = (\mathrm{id}_{N \otimes N'} \otimes m) \circ (\mathrm{id}_N \otimes \tau \otimes \mathrm{id}_{\mathcal{A}}) \circ (\delta_N \otimes \delta_{N'})$ (ou $\delta_{N \otimes N'} = (m \otimes \mathrm{id}_{N \otimes N'}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \tau \mathrm{id}_{N'}) \circ (\delta_N \otimes \delta_{N'})$, resp.).

Définition 2.3.3 *Soit* $(\mathcal{C}, \otimes, I, a, l, r)$ *une catégorie tensorielle. Un* tressage *(*braiding*)* Ψ *de* $\mathcal{C}$ *est une contrainte de commutativité telle que l'*axiome de l'hexagone *(*Hexagon Axiom*) est satisfait, c-à-d*

$$\begin{array}{ccccc}
 & U \otimes (V \otimes W) & \xrightarrow{\Psi_{U,V \otimes W}} & (V \otimes W) \otimes U & \\
a_{U,V,W} \nearrow & & & & \searrow a_{V,W,U} \\
(U \otimes V) \otimes W & & & & V \otimes (W \otimes U) \\
\Psi_{U,V} \otimes \mathrm{id}_W \searrow & & & & \nearrow \mathrm{id}_V \otimes \Psi_{U,W} \\
 & (V \otimes U) \otimes W & \xrightarrow{a_{V,U,W}} & V \otimes (U \otimes W) &
\end{array}$$

et

$$\begin{array}{ccccc}
 & (U \otimes V) \otimes W & \xrightarrow{\Psi_{U \otimes V,W}} & W \otimes (U \otimes V) & \\
a^{-1}_{U,V,W} \nearrow & & & & \searrow a^{-1}_{W,U,V} \\
U \otimes (V \otimes W) & & & & (W \otimes U) \otimes V \\
\mathrm{id}_U \otimes \Psi_{V,W} \searrow & & & & \nearrow \Psi_{U,W} \otimes \mathrm{id}_V \\
 & U \otimes (W \otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U \otimes W) \otimes V &
\end{array}$$

commute pour tous les objets U, V, W, X *de* $\mathcal{C}$*.*

Exemple : Un exemple trivial d'un tressage est la permutation $\tau_{U,V} : U \otimes V \to V \otimes U$, $\tau(u \otimes v) = v \otimes u$, dans la catégorie des modules à gauche ou à droite d'une bigèbre cocommutative. Au niveau d'espaces vectoriels ceci est évident. Mais on a aussi $\mu_{M' \otimes M} = \tau^{-1}_{M' \otimes M} \circ \mu_{M \otimes M'} \circ (\mathrm{id}_{\mathcal{A}} \otimes \tau_{M' \otimes M})$. En général, la catégorie des modules d'une bigèbre $\mathcal{A}$ admet un tressage si et seulement si elle est *quasi-triangulaire* (ou *tressée* (*braided*)), c-à-d s'il existe un élément inversible $R \in \mathcal{A} \otimes \mathcal{A}$, dite *R-matrice universelle* (*universal R-matrix*) qui 'contrôle' la non-cocommutativité de $\mathcal{A}$, au sens suivant:

$$\tau \circ \Delta(a) = R\Delta(a)R^{-1}, \qquad \text{for all } a \in \mathcal{A}$$

et qui satisfait $(\Delta\otimes\mathrm{id})(R) = R_{13}R_{23}$ et $(\mathrm{id}\otimes\Delta)(R) = R_{13}R_{12}$, voir eg. [Kas95, Proposition XIII.1.4]. Ici $R_{12} = R\otimes 1$, $R_{23} = 1\otimes R$, $R_{13} = (\mathrm{id}\otimes\tau)(R\otimes 1)$ sont des élément de l'algèbre $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$.

Définition 2.3.4 *Une catégorie tensorielle tressée (*braided tensor category*) $(\mathcal{C},\Psi)$ consiste en une catégorie tensorielle $\mathcal{C}$ et un tressage Ψ de $\mathcal{C}$.*

Considérons une catégorie tressée $(\mathcal{C},\Psi)$ dont les objets $A, B, \ldots$ sont des algébres. Ceci signifie que les multiplications $m_A, m_B, \ldots$ et les unités $e_A, e_B, \ldots$ sont des morphismes de $\mathcal{C}$. Alors l'objet $A\underline{\otimes}B$ est aussi une algèbre (*braided tensor product algebra*), avec la multiplication $m_{A\underline{\otimes}B} = (m_A\otimes m_B)\circ(\mathrm{id}\otimes\Psi\otimes\mathrm{id})$ et l'unité $e_{A\underline{\otimes}B} = e_A\otimes e_B$, voir [Maj93a, Lemma 4.1]. On peut donc aussi définir des bigèbres tressées (*braided bialgebras*), ie. des algèbres qui sont des objets d'une catégorie tressée et pour lesquelles il existent des morphismes $\underline{\Delta}_A : A \to A\underline{\otimes}A$, $\underline{\varepsilon} : A \to \mathbb{K}$, t.q. les conditions analogues à celles de la définition 2.2.2 soient satisfaites :

$$(\underline{\Delta}\otimes\mathrm{id})\circ\underline{\Delta} = (\mathrm{id}\otimes\underline{\Delta})\circ\underline{\Delta}, \qquad (\underline{\varepsilon}\otimes\mathrm{id})\circ\underline{\Delta} = \mathrm{id} = (\mathrm{id}\otimes\underline{\varepsilon})\circ\underline{\Delta},$$

et t.q. $\underline{\Delta}$ et $\underline{\varepsilon}$ sont des homomorphismes d'algèbres.

Une *antipode tressée* (*braided antipode*) $\underline{S} : A \to A$ est caractérisée par la condition

$$m_A\circ(\underline{S}\otimes\mathrm{id})\circ\underline{\Delta} = e\circ\underline{\varepsilon} = m_A\circ(\mathrm{id}\otimes\underline{S})\circ\underline{\Delta},$$

elle est toujours unique, mais ce n'est pas un anti-homomorphisme d'algèbres. Elle satisfait $\underline{S}\circ m = m\circ\Psi\circ(\underline{S}\otimes\underline{S})$ et $S(e) = e$. Par rapport au coproduit et à la counité nous avons $\underline{\Delta}\circ\underline{S} = (S\circ S)\circ\Psi\circ\underline{\Delta}$ et $\underline{\varepsilon}\circ\underline{S} = \underline{S}$.

La plupart des notions introduites ci-dessus (par exemple module, comodule, couple dual ou représentations duales) se généralisent immédiatement au cas tressé.

Mais les définitions à prendre pour les *-bigèbres et les *-algèbres de Hopf ne sont pas évidentes, voir [Maj94, Maj95a, Maj95c] pour un choix d'axiomes. Pour l'étude des processus de Lévy ces axiomes ne sont pas appropriés parce que le coproduit n'est pas un homomorphisme de *-algèbres.

Nous proposons une autre définition (voir aussi sous-section 5.2.3).

Définition 2.3.5 *Une* *-bigèbre tressée *(*braided *-bialgebra*) est une bigèbre tressée $(A, m, e, \underline{\Delta}, \underline{\varepsilon})$ (sur un corps avec involution, e.g. $\mathbb{C}$) munie d'une involution $* : A \to A$, t.q.*

*(i) $(A,*)$ est une *-algebra (ie. $*\circ m = m\circ\tau\circ(*\otimes *)$, $*\circ * = \mathrm{id}$, et $e(\lambda)^* = e(\overline{\lambda})$ pour tout $\lambda\in\mathbb{K}$), et*

*(ii) $\underline{\varepsilon} : A \to \mathbb{K}$ *-homomorphisme,*

*(iii) $\underline{\Delta} : A \to A\underline{\otimes}A$ est un homomorphisme de *-algèbre par rapport à l'involution de $A\underline{\otimes}A$ definit par $(a\otimes b)^* = \Psi(b^*\otimes a^*)$.*

Remarque: La définition de * pour $A\underline{\otimes}A$ se déduit immédiatement de la condition imposant que les inclusions canoniques $\mathcal{A} \xrightarrow{i_1} \mathcal{A}\underline{\otimes}\mathcal{A} \xleftarrow{i_2} \mathcal{A}$ soient des homomorphismes de *-algèbres, ie. de $(1 \otimes a)^* = 1 \otimes a^*$ et de $(a \otimes 1)^* = a^* \otimes 1$, $\forall a \in \mathcal{A}$. Cette définition coïncide avec celle de M. Schürmann[Sch93, page 27], si Ψ est défini par un facteur de commutation.

Pour des exemples de *-bigèbres au sens de notre définition voir 5.3.2.

2.4 Indépendance.

Plusieurs définitions inéquivalentes d'indépendance ont été proposées et étudiées en probabilités non-commutatives. Il y a par exemple l'indépendance libre de Voiculescu[VDN92], liée au produit libre d'algèbres, l'indépendance tensorielle de Schürmann[Sch93], liée au produit tensoriel, et l'indépendance booléenne (voir eg. [SW93]). Des travaux récents de M. Schürmann[Sch95a] et de R. Speicher[Spe] indiquent que ce sont les seules définitions, sous certaines hypothèses. L. Accardi et al. ont proposé la notion d'indépendance statistique qui comprend tous les types d'indépendance énumerés ci-dessus [ALV94].

La notion utilisée ici est celle de l'indépendance tensorielle (ou tressée). Dans le cas commutatif c'est la notion classique d'indépendance en probabilité. M. Schürmann a aussi considéré des produits tensoriels non-symétriques, où le tressage est défini par une action α et une co-action γ d'un groupe algèbre $\mathbb{CL}$ comme $\Psi = (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma \otimes \mathrm{id})$ ou $\Psi = (\mathrm{id} \otimes \alpha) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma)$. Nous allons généraliser cette définition au cas d'un tressage quelconque (mais la contrainte d'associativité reste triviale).

Définition 2.4.1 *Soit $(\mathcal{A}, \Phi)$ un espace de probabilité non-commutatif, et $\mathcal{B}$ une *-algèbre dans une catégorie tressée $(\mathcal{C}, \Psi)$. Un n-uplet $(j_1, \ldots, j_n)$ de variables aléatoires non-commutatives $j_i : \mathcal{B} \to \mathcal{A}$, $i = 1, \ldots, n$, est* Ψ-indépendant *(*Ψ-independent *ou* braided independent*), si*

(i) $\Phi\big(j_{\sigma(1)}(b_1) \cdots j_{\sigma(n)}(b_n)\big) = \Phi(j_{\sigma(1)}(b_1)) \cdots \Phi(j_{\sigma(n)}(b_n))$ *pour toute permutation* $\sigma \in \mathcal{S}(n)$ *et tous* $b_1, \ldots, b_n \in \mathcal{B}$, *et*

(ii) $m_{\mathcal{A}} \circ (j_k \otimes j_l) = m_{\mathcal{A}} \circ (j_l \otimes j_k) \circ \Psi$ *pour tout* $1 \leq k < l \leq n$.

Remarques:

1. Dans le cas non-commutatif l'indépendance dépend de l'ordre des v.a. $(j_1, \ldots, j_n)$.

2. Si $\mathcal{A}$ est commutative, alors (i) est équivalent à la condition utilisée en probabilités classiques:

   (i') $\Phi\big(j_1(b_1) \cdots j_n(b_n)\big) = \Phi(j_1(b_1)) \cdots \Phi(j_n(b_n))$ pour tous $b_1, \ldots, b_n \in \mathcal{B}$.

   Nous allons dire que $j_1, \ldots, j_n$ sont *pseudo-(Ψ-)indépendants*, s'ils satisfont seulement (i') et (ii). Cette notion sera aussi utilisée s'il n'y a pas d'involution * sur $\mathcal{A}$ ou $\mathcal{B}$, ou si la fonctionnelle Φ n'est pas positive.

3. Une famille $\{j_\imath | \imath \in I\}$ de variables aléatoire non-commutative indexée par un ensemble partiellement ordonné I est indépendante, si $(j_{\imath_1}, \ldots, j_{\imath_n})$ est indépendant pour tous $(\imath_1, \ldots, \imath_n)$ avec $\imath_1 < \imath_2 < \cdots < \imath_n$).

L'indépendance et aussi la pseudo-indépendance impliquent que $\Phi \circ m_{\mathcal{A}}^{(n-1)} \circ (j_1 \otimes \cdots \otimes j_n)$ factorise comme produit tensoriel des lois marginales $\phi_i = \Phi \circ j_i$, $i = 1, \ldots, n$, et donc que la distribution jointe de $(j_1 \otimes \cdots j_n)$ est déterminée de façon unique par les lois marginales. Mais de la 'vraie' indépendance il suit en plus une condition d'invariance ou de commutativité:

$$\hat{\phi}_1 = \phi_1 \otimes \underbrace{\underline{\varepsilon} \otimes \cdots \otimes \underline{\varepsilon}}_{(n-1) \text{ times}}, \quad \hat{\phi}_2 = \underline{\varepsilon} \otimes \phi_2 \otimes \underbrace{\underline{\varepsilon} \otimes \cdots \otimes \underline{\varepsilon}}_{(n-2) \text{ times}}, \quad \ldots \quad \hat{\phi}_n = \underbrace{\underline{\varepsilon} \otimes \cdots \otimes \underline{\varepsilon}}_{(n-1) \text{ times}} \otimes \phi_n$$

doivent commuter (dans l'algèbre des fonctionnelles sur $\mathcal{B}^{\underline{\otimes} n}$).

On dira qu'une fonctionnelle ϕ sur une algèbre dans une catégorie tressé $(\mathcal{C}, \Psi)$ est Ψ-*invariante*, si

$$(\phi \otimes \theta) \circ \Psi = \theta \otimes \phi.$$

pour toute fonctionnelle $\theta : \mathcal{A} \to \mathbb{K}$,

En général la convolution de deux fonctionnelles positives n'est pas positive, mais si ϕ est une fonctionnelle positive et Ψ-invariante sur une bigèbre tressée $\mathcal{A}$, et ψ est une fonctionnelle positive sur $\mathcal{A}$, alors $\phi \star \theta = (\phi \otimes \theta) \otimes \Delta$ est aussi positive, voir lemme 5.2.4.

2.5 Processus de Lévy sur les bigèbres

Nous introduisons maintenant une autre notion centrale de cette thése, celle de processus de Lévy sur une bigèbre.

Définition 2.5.1 *[Sch93] Soit $\mathcal{B}$ une *-bigèbre (tressée). Un processus stochastique non-commutatif $\{j_{st} : \mathcal{B} \to \mathcal{A} | 0 \leq s \leq t \leq T\}$, $T \in \mathbb{R}_+ \cup \{\infty\}$ sur un espace de probabilité non-commutatif $(\mathcal{A}, \Phi)$ est un* processus de Lévy *s'il satisfait aux conditions suivantes:*

1. *(propriété des accroissements)*

$$\begin{aligned} j_{rs} \star j_{st} &= j_{rt} \quad \textit{pour tout } 0 \leq r \leq s \leq t \leq T, \\ j_{tt} &= e \circ \varepsilon \quad \textit{pour tout } 0 \leq t \leq T, \end{aligned}$$

2. *(indépendance des accroissements) la famille $\{j_{st} | 0 \leq s \leq t \leq T\}$ est indépendante,*

3. *(stationnarité des accroissements) la distribution $\varphi_{st} = \Phi \circ j_{st}$ de j_{st} ne dépend que de la différence $t - s$,*

4. *(continuité faible) j_{st} converge vers j_{ss} $(= e \circ \varepsilon)$ en distribution si $t \searrow s$.*

Remarques :

1. Rappel : La convolution $j_1 \star j_2 : \mathcal{C} \to \mathcal{A}$ de deux applications linéaires d'une cogèbre dans une algèbre est définie par

$$j_1 \star j_2 = m_{\mathcal{A}} \circ (j_1 \otimes j_s) \circ \Delta_{\mathcal{C}}.$$

2. Soit $\{j_t | 0 \leq t \leq T\}$ un processus stochastique non-commutatif sur une *-algèbre de Hopf. Alors on peut définir ses accroissements par

$$j_{st} = (j_s \circ S) \star j_t,$$

ce processus satisfait automatiquement la propriété des accroissements. Nous appelons $\{j_t | 0 \leq t \leq T\}$ un processus de Lévy si $\{j_{st} | 0 \leq s \leq t \leq T\}$ est un processus de Lévy.

Il est bien connu (voir [Sch93]) qu'un processus de Lévy sur une bigèbre est déterminé de façon unique par ses lois marginales $\varphi_t = \Phi \circ j_{0t}$, et qu'il existe une unique fonctionnelle hermitienne conditionnellement positive $L : \mathcal{B} \to \mathbb{C}$, telle que $\varphi_t = \exp_\star tL$. De plus, l'indépendance des accroissements implique $(L \odot L) \circ \Psi = L \odot L$. Si on se restreint aux générateurs Ψ-invariants, alors on a aussi le réciproque. D'après lemme 5.2.4 le semi-groupe engendré par L est positif, et la construction décrite dans la section 4.8 ou dans les pages 38-40 en [Sch93] donne une 'représentation canonique' du processus. Il est intéressant de remarquer que cette construction ne dépend pas de la positivité, on peut aussi l'utiliser pour obtenir une 'représentation canonique' d'un 'pseudo-processus de Lévy' (c-à-d d'un processus dont les accroissement sont seulement pseudo-indépendants, ou dans le cas ou la fonctionnelle n'est pas positive).

Nous résumons ceci dans la proposition suivante, pour les détails consulter [Sch93, Corollary 1.9.7, Theorem 3.2.8].

Proposition 2.5.2 *Soit $\mathcal{B}$ une *-bigèbre dans une catégorie tressée $(\mathcal{C}, \Psi)$.*

(i) Soit $\Psi = \tau$. Alors il y a une correspondence unique entre les (classes d'équivalence de) processus de Lévy $\{j_{st}\}$, les semi-groupes de convolution de fonctionnelles positives hermitiennes $\varphi_t = \Phi \circ j_{0t} = \exp_\star tL$, et les fonctionnelles hermitiennes conditionnellement positives $L = \frac{d}{dt}\varphi_t\Big|_{t=0}$ sur $\mathcal{B}$.

(ii) Cette correspondence existe aussi pour des tressages quelconques, si on se restreint aux générateurs Ψ-invariants, à leurs semi-groupes et à leurs processus stochastiques.

L'ingédient essentiel pour passer des générateurs α-invariants[Sch93] aux générateurs Ψ-invariants est le lemme 5.2.4.

Pour la réalisation des processus de Lévy sur un espace de Fock et pour le lien avec les équations différentielles stochastiques non-commutatives consulter [Sch93].

Chapitre 3

Les résultats principaux

Dans ce chapitre nous présentons un résumé des principaux résultats contenus dans cette thèse, les preuves se trouvent dans la partie II.

3.1 Construction des processus de Lévy et des semi-groupes de convolution sur les bigèbres

Dans la section 2.5 nous avons vu trois façons équivalentes pour présenter un processus de Lévy : le processus lui-même en tant que famille d'homomorphismes d'algèbres, le semi-groupe des distributions uni-dimensionnelles ou le générateur. M. Skeide[Ske94] a montré comment les générateurs des processus de Lévy sur une bigèbre donnée peuvent être caractérisés. M. Schürmann a montré comment le processus peut être reconstruit à partir du générateur ou du semi-groupe des distributions uni-dimensionnelles.

Dans cette thèse deux méthodes pour construire des semi-groupes de convolution sont étudiées, voir chapitres 4 et 5. La première (chapitre 4) s'inspire de l'intégrale multiplicative de McKean. Sur les groupes de Lie (connexes, simplement connexes) on peut utiliser l'exponentielle pour définir une exponentielle stochastique qui associe une semi-martingale sur le groupe à chaque semi-martingale sur son algèbre de Lie. Nous démontrons que pour une classe d'algèbres de Hopf caractérisées par l'hypothèse H (page 29) il existe un élément du produit tensoriel de l'algèbre avec son dual qui a beaucoup de propriétés en commun avec l'exponentielle d'un groupe de Lie. Cet élément, appelé "pairing dual" formel, est calculé, voir équation (4.3), page 32. Notre construction d'un semi-groupe de convolution part d'un processus classique à accroissements indépendants, et utilise d'abord une identification de l'algèbre de Hopf avec l'algèbre des polynômes pour définir une famille de fonctionnelles sur l'algèbre de Hopf, et ensuite une procédure consistant à passer à la limite pour en faire un semi-groupe. Cette limite est l'analogue de l'intégrale multiplicative de McKean au niveau des fonctionnelles. En utilisant le pairing dual formel nous démontrons une formule de Feynman-Kac pour ces semi-groupes, voir théorème 4.6.1, page 36. Nous en déduisons une formule de Trotter pour les produits de q-exponentielles comme corollaire (proposition 4.6.2, page 37).

Le principal désavantage de cette construction est qu'elle ne donne pas automatiquement un semi-groupe positif, comme dans le cas classique, parce que l'exponentielle n'a

pas d'analogue en tant qu'application entre les espaces topologiques sous-jacents.

Ensuite, dans la section 4.8 nous construisons le processus, c-à-d une algèbre avec un état et des homomorphismes d'algèbres, à partir de l'algèbre et du semi-groupe des distributions uni-dimensionnelles. Si l'identification choisie au début de la construction pour la définition des fonctionnnelles sur l'algèbre de Hopf à partir du processus classique conserve la positivité, alors la procédure de passage à la limite donne un semi-groupe positif, et la reconstruction du processus donne un vrai processus stochastique quantique. Dans ce cas on peut appliquer la construction de Gelfand-Naimark-Segal (GNS) pour obtenir une réalisation du processus sur un espace (pré-)hilbertien. Mais les résultats de ce chapitre (par exemple la formule de Feynman-Kac et la partie sur les systèmes d'Appell en section 4.7, page 38) sont valables sans l'hypothèse de positivité.

La deuxième construction est relative aux espaces tressés. Ceci nécessite une généralisation de l'indépendence tensorielle de M. Schürmann aux produits tensoriels tressés, et une nouvelle définition de l'involution * pour les espaces tressés qui se rapproche de celle de M. Schürmann, car avec la définition de S. Majid[Maj94, Maj95a, Maj95c] la convolution de fonctionnelles positives n'est, en général, pas positive. Cette construction est motivée par l'approche basée sur les marches aléatoires de S. Majid et al[Maj93d, MRP94]. Le but n'est pas la construction d'un processus de Lévy quelconque, mais d'un processus qui mérite d'être appelé diffusion. La condition classique de continuité des chemins est remplacée par l'hypothèse que les fonctionnelles peuvent être obtenues par un théorème de la limite centrale (voir théorème 5.3.1, page 55, et définition 5.3.2, page 56). En général, ces fonctionnelles ne sont pas positives, dans ce cas nous les appelons pseudo-diffusions. Pour des exemples d'espaces tressés involutifs et de vraies diffusions (= pseudo-diffusions positives) on pourra consulter subsection 5.3.2, page 58. Nous discutons aussi deux approches pour associér des densités à ces processus, voir section 5.5. La deuxième est celle que nous avons déjà abord'e en section 2.1, et qui est d'ailleurs utilisée couramment en probabilités non-commutatives. La première utilise les fonctionnelles invariantes ou intégrales qui jouent le rôle de mesures de Haar dans la théorie des algèbres de Hopf. Dans ce cas les densités sont certains éléments de l'algèbre de Hopf elle-même.

3.2 Processus stochastiques et équations d'évolution

Soit la fonctionnelle linéaire $L : \mathcal{A} \to \mathbb{C}$ le générateur du semi-groupe $\{\varphi_t; t \in \mathbb{R}_+\}$. Ce semi-groupe definit aussi un semi-groupe d'opérateurs $\{\mathbb{P}_t = (\mathrm{id} \otimes \varphi_t) \circ \Delta = \rho(\varphi_t) : \mathcal{A} \to \mathcal{A}; t \in \mathbb{R}\}$ à l'aide de la représentation duale à droite. Notamment $\mathbb{P}_t = P_{\{0\}} \circ T_{0,t}$ avec la notation de la sous-section 10.2.2, càd ceci est le semi-groupe markovien du processus de Lévy associé (s'il existe, càd si L est positif et Ψ-invariant). Pour tout élément $a \in \mathcal{A}$ nous avons une famille $\{a_t = \mathbb{P}_t(a); t \in \mathbb{R}_+\}$ caractérisée par

$$\partial_t a_t = \rho(L) a_t, \qquad \text{and} \qquad a_0 = a.$$

Nous étudions le lien entre les processus de Lévy quantiques (ou plus généralement des semi-groupes de convolution) et les *équations d'evolution* dans les chapitres 4, 5 et 7. On y trouve la définition des systèmes d'Appell qui sont des solutions polynômiales des équations d'évolution et une étude des leurs propriétés par rapport au coproduit et aux

opérateurs de création et d'annihilation, voir section 4.7 (page 38), section 5.4 (page 60) et section 7.5 (page 99). De nombreux exemples sont traités explicitement (voir aussi chapitre 8).

Au chapitre 7 un autre type d'équations d'évolution associées à des processus stochastiques est également considéré. Nous introduisons les densités de Wigner, ie. densités jointes pas nécessairement positives, d'un ensemble d'opérateurs non-commutatifs, et démontrons que les densités de Wigner d'un processus de Lévy quantiques satisfont à une équation de Fokker-Planck de la forme

$$\frac{\mathrm{d}}{\mathrm{d}t} W^*(\Phi_t) = \tilde{\rho}(L)^* W^*(\Phi_t)$$

voir proposition 7.6.1, page 103.

3.3 Caractérisation

Dans le chapitre 6 nous cherchons des états gaussiens sur des algèbres de Hopf. Le théorème de Bernstein donne une caractérisation des mesures gaussiennes qui n'utilise que la structure de groupe de $\mathbb{R}^n$ et la notion d'indépendence. Si nous considérons les algèbres de Hopf comme 'analogues quantiques' des groupes, alors il est naturel d'essayer d'étendre cette caractérisation. Mais, comme c'est déjà suggéré par les résultats sur les groupes non-abéliens, la classe des fonctionnelles qui satisfont un analogue de la propriété de Bernstein (voir définitions 6.4.4 et 6.4.5) est trop petite pour constituer une classe satisfaisante de fonctionnelles gaussiennes, voir théorèmes 6.4.7, 6.4.10 et 6.4.12. En plus, elles ne forment pas des semi-groupes de convolution. Néanmoins elles définissent des homomorphismes de cogèbres, ce qui nous a amené à définir des semi-groupes de convolution quantiques , voir définition 6.4.18, page 82.

Une autre approche, inspirée des résultats de H. Heyer et W. Hazod, basée systematiquement sur des semi-groupes de convolution, est plus satisfaisante. Un semi-groupe est dit gaussien, si son générateur satisfait la condition de la définition 6.6.2 (page 87). On trouve que les fonctionnelles primitives (ie. $X(fg) = X(f)\varepsilon(g) + \varepsilon(f)X(g)$) ainsi que les expressions au plus quadratiques dans ces fonctionnelles et les fonctionnelles qui sont quadratiques aus sens de Schürmann[Sch93] génèrent des semi-groupes gaussiens, voir propositions 6.6.3 et 6.6.5. Dans les exemples que nous avons étudiés ce sont les seules fonctionnelles possibles, mais nous ne savons pas si c'est vrai en général.

Dans la définition 6.5.1, page 83, nous avons introduit la notion de nilpotence pour les algèbres de Hopf. Sur ces algèbres le plongement d'une fonctionnelle infiniment divisible normalisée dans un semi-groupe continu de convolution est unique, voir théorème 6.5.6 sur page 85. La raison en est que la nilpotence garantit l'existence d'une base ordonnée telle que l'ordre est respecté par le coproduit. Ceci permet de calculer la racine d'une fonctionnelle normalisée par récurrence, voir lemme 6.5.5, page 85.

3.4 Théorèmes limites sur les bigèbres

Ce sujet est traité au chapitre 9. D'abord, les résultats de M. Schürmann[Sch93], Ph. Feinsilver[Fei87], de D. Neuenschwander et R. Schott[NS96], sont énoncés et appliqués à la droite tressée et au groupe de Heisenberg-Weyl tressé. Il s'avère que les lois limites obtenues ici coïncident avec les pseudo-diffusions du chapitre 5 et avec les semi-groupes de convolution (faiblement) gaussiens du chapitre 6.

Dans la section 9.3 nous nous posons la question de savoir quelle forme la loi du logarithme itéré pourrait prendre en probabilités non-commutatives. Nous introduisons le concept de supremum d'un processus non-commutatif lelong d'un chemin, et esquissons une approche pour calculer sa loi.

Ensuite nous considérons une algèbre munie d'une famille de coproduits qui dépendent d'un paramètre réel. Si on prend une fonctionnelle initiale fixe et si on la convole successivement avec elle-même en utilisant un coproduit choisi selon une suite de variables aléatoires réelles i.i.d., alors on obtient une suite de variables aléatoires à valeurs fonctionnelles. Nous démontrons que les moments de ces fonctionnelles, convénablement normalisées, convergent en probabilité pour une certaine fonctionnelle initiale, voir théorème 9.4.1, page 128. Il est évident que ce résultat s'étend à d'autres fonctionnelles initiales. On peut s'attendre à ce qu'une grande classe de lois apparaîsse comme loi limite, en regardant seulement le cas déterministe on trouve déjà les lois marginales de la martingale d'Azéma.

Des compléments aux résultats de ce chapitre suivrons.

3.5 Versions classiques des processus de Lévy quantiques

Dans le dernier chaptire nous indiquons comment des processus de Markov classiques peuvent être obtenus à partir des processus de Lévy sur des bigèbres. Comme en probabilités classiques, les processus de Lévy (quantiques) sont aussi des processus de Markov (quantiques), ceci est démonttré dans la sous-section 10.2.2. Mais il est bien connu que les processus de Markov quantiques sur des algèbres commutatives possèdent des versions classiques, voir par exemple [Küm88, BKS96]. Nous donnons des conditions suffisantes pour que la restriction d'un processus de Lévy quantique à une sous-algèbre reste markovienne (proposition 10.2.3, page 136), ce qui donne donc immédiatement des conditions suffisantes pour l'existence d'une version classique. Celle-ci est de plus markovienne.

Dans la section 10.3 nous étudions quelques exemples. Nous montrons comment les moments et le générateur du processus classique peuvent être calculés à l'aide des représentations duales. La célèbre martingale d'Azéma est incluse dans notre exposition, mais nous introduisons aussi un nouveau processus, qui correspond à un processus de Poisson symétrique dans la 'limite classique', ie. pour le cas co-commutatif $q = 1$. A partir de ce processus on peut construire tout processus de Poisson composé symétrique.

3.6 Recherche future

Nous allons à présent jeter un coup d'œil sur ce que pourrait être la continuation de ce travail.

D'abord les résultats du chapitre 9 doivent être affinés. Le théorème 9.4.1 (page 128) doit être formulé pour des états initiaux plus généraux, et la relation entre la loi des $\{q_i; i \in \mathbb{N}\}$ selon laquelle la convolution est choisie et la fonctionnelle limite φ_∞ pourrait être rendus plus explicite à l'aide du corollaire 9.4.2 (page 130). Ceci permettrait aussi une caractérisation des lois limites dans ce type de théorème. Il serait aussi intéressant de démontrer des résultats avec convergence forte.

On pourrait également chercher d'autres types de théorèmes limites, comme suggéré dans les sections 9.3 et 9.5.

Dans le chapitre 10 nous avons considéré seulement une algèbre de Hopf. Il faudrait étudier s'il y a des processus classiques associés aux processus de Lévy sur d'autres algèbres de Hopf, comme par exemple celles introduites par S.L. Woronowicz[Wor87, Wor91, Wor92, WZ94].

Les calculs des représentations duales ou des systèmes d'Appell sont souvent plutôt longues et techniques. Il serait très utile d'étendre aux algèbres de Hopf les logiciels écrits en Maple par Ph. Feinsilver et R. Schott[FS96a], et M. Giering[Gie95] pour le calcul symbolique sur les groupes de Lie.

Finalement, comme project à long terme, on pourraient essayer d'étendre ce travail à d'autres notions d'indépendence comme l'indépendence libre ou l'indépendence booléenne (pour la définition des processus de Lévy par rapport à ces notions d'indépendence voir [Sch95b]), ou de comparer la théorie des processus de Lévy sur les groupes quantiques avec celle des hypergroupes[BH95].

3.7 Liste de publications

Ce travail a débouché sur plusieurs plublications, nous en donnons ci-dessous une liste[1] :

Articles publiés (ou acceptés pour publication) dans des journaux

- Gauss Laws in the Sense of Bernstein and Uniqueness of Embedding into Convolution Semigroups on Quantum Groups and Braided Groups, (avec D. Neuenschwander et R. Schott), Prépublication Institut Elie Cartan 96/n 13, 1996, accepté pour publication dans Probability Theory and Related Fields.

- Duality and Multiplicative Processes on Quantum Groups, (avec Ph. Feinsilver et R. Schott), Prépublication Institut Elie Cartan 95/n 26, 1995, J. Theor. Prob. 10, No. 3, p. 795-818, 1997.

Notes aux Comptes Rendus de l'Académie des Sciences, Paris

- Gauss Laws in the Sense of Bernstein and Uniqueness of Embedding into Covolution Semigroups on Quantum Groups and Braided Groups, (avec D. Neuenschwander et R. Schott), C. R. Acad. Sci. Paris, t. 324, Série I, p. 827-832, 1997.

- Feynman-Kac Formula and Appell Systems on Quantum Groups, (avec Ph. Feinsilver et R. Schott), C. R. Acad. Sci., Paris, t. 321, Série I, p. 1615-1619, 1995.

Articles acceptés pour présentation à des conférences internationales avec actes et comité de programmme

- On the Computation of Polynomial Representations of Nilpotent Lie Groups: A Symbolic Mathematical Approach, (avec Ph. Feinsilver et R. Schott), Proceedings of the 1997 ACM Symposium on Applied Computing, ACM Publ., p. 537-539, 1997.

- A Stochastic Approach to Evolution Equations on Nilpotent Quantum Groups, (avec R. Schott), Proceedings of the XXI International Colloquium on Group-theoretical Methods in Physics, Goslar, Germany, 15-20. July 1996 (sous presse).

- Operator Calculus and Symbolic Computation on Lie Groups, (avec P. Feinsilver et R. Schott), in GROUP21, Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras, Proceeding of the XXI International Colloquium on Group Theoretical Methods in Physics, 15-20 July 1996, Goslar, Germany, H.-D. Doebner, P. Nattermann, W. Scherer and C. Schulte (Eds.), World Scientific, Singapore, Vol. 1, p. 157-161, à paraître 1997.

- Diffusions on Braided Spaces, (avec R. Schott), Proceedings of IV Wigner Symposium on Group Theory and its Applications, Guadalajara, Mexico, August 7-11,

1. voir aussi http://www.loria.fr/~franzu/liste_de_publications.html

1995, N.M. Atakishiyev, T.H. Seligman, K.B. Wolf, editors, World Scientific, p. 239-242, 1996.

- Duality and Multiplicative Processes on Quantum Groups, (avec Ph. Feinsilver et R. Schott), Proceedings of the International Conference on "Nonlinear, Deformed and Irreversible Quantum Systems", H.D. Doebner, V.K. Dobrev, P. Nattermann, editors, World Scientific, p. 462-468, 1995.

Prépublications

- Limit Theorems on Quantum Groups, Prépublication, 1997.
- Classical Versions of Quantum Lévy Processes, Prépublication, 1997.
- Evolution Equations and Lévy Processes on Quantum Groups, (avec R. Schott), Prépublication Institut Elie Cartan 97/n 2, 1997.
- Diffusions on Braided Spaces, (avec R. Schott), ASI-TPA/28/95, Prépublication Institut Elie Cartan 96/n 4, 1995.
- On the Computation of Polynomial Representations of Nilpotent Lie Groups: A Symbolic Mathematical Approach, (avec Ph. Feinsilver et R. Schott), Prépublication Institut Elie Cartan 94/n 27, 1994.

Deuxième partie

Chapitre 4

Duality and Multiplicative Stochastic Processes on Quantum Groups

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Résumé

An analogue of McKean's stochastic product integral is introduced and used to define stochastic processes with independent increments on quantum groups. The explicit form of the dual pairing (q-analogue of the exponential map) is calculated for a large class of quantum groups. The constructed processes are shown to satisfy generalized Feynman-Kac type formulas, and polynomial solutions of associated evolution equations are introduced in the form of Appell systems. Explicit calculations for Gauss and Poisson processes complete the presentation.

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4.1 Introduction

Stochastic processes have many applications in analysis and physics. Functional integrals can be used to solve partial differential equations, cf., the celebrated Feynman-Kac formula. Wiener integrals are formally close to Feynman path integrals. In this paper we generalize part of the theory of stochastic processes to quantum groups. We propose a method to construct analogues of additive stochastic processes on quantum groups.

We consider here a class of Hopf algebras characterized by the coalgebraic relations and part of the algebraic relations of their generators, cf. Section 4.3. For this class we show that the dual pairing can be expressed as a product of q-exponentials. Section 4.4 shows how this dual pairing can be used formally to calculate the dual representations. In the following section we undertake the first step of the construction of the process. Motivated by McKean's [McK69] stochastic product integrals on Lie groups (see also [FS89a, FS89b]) we define a limiting procedure to obtain functionals, that, if their limit exists, converge to a one-parameter semi-group of functionals that will be considered as the analogous multiplicative process on the quantum group. In Section 4.6 we show an analogue of Trotter's product formula and a Feynman-Kac type formula. Then we define solutions of the associated evolution equations as Appell polynomials. Finally, in Section 4.8 we complete the construction by introducing an algebra that is a candidate for a canonical representation of the process.

For alternative approaches see [Sch93, Maj93d, MRP94, Maj95b].

4.2 Preliminaries

Quantum groups. We briefly recall some definitions concerning quantum groups, for detailed introductions see e. g. [DHL91, Dri87, Jim85, Jim86, Maj90, Maj95b, RTF90, Wor87, Koo94].

Recall that a Hopf algebra is defined as an associative unital algebra (H, m, e) with two homomorphisms $\Delta : H \to H \otimes H$, $\varepsilon : H \to \mathbb{C}$, and an anti-homomorphism $S : H \to H$ that satisfy

$$\begin{aligned}
(\Delta \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta) \circ \Delta, \\
(\varepsilon \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \varepsilon) \circ \Delta \\
m \circ (\mathrm{id} \otimes S) \circ \Delta &= m \circ (S \otimes \mathrm{id}) \circ \Delta = e \circ \varepsilon.
\end{aligned}$$

These maps are called coproduct, counit, and antipode, respectively. Two Hopf algebras $(H_1, m_1, e_1, \Delta_1, \varepsilon_1, S_1)$ and $(H_2, m_2, e_2, \Delta_2, \varepsilon_2, S_2)$ are said to be in duality, if there exists a (non-degenerate) bilinear map $< \cdot, \cdot >: H_1 \times H_2 \to \mathbb{C}$ such that

$$\begin{aligned}
< m_1(a_1 \otimes b_1), c_2 > &= < a_1 \otimes b_1, \Delta_2(c_2) >_{\otimes}, \\
< c_1, m_2(a_2 \otimes b_2) > &= < \Delta_1(c_1), a_2 \otimes b_2 >_{\otimes}, \\
< e_1, a_2 > &= \varepsilon_2(a_2), \\
< a_1, e_2 > &= \varepsilon_1(a_1), \\
< S_1(a_1), a_2 > &= < a_1, S_2(a_2) >,
\end{aligned}$$

for all $a_1, b_1, c_1 \in H_1$, $a_2, b_2, c_2 \in H_2$. Left (resp. right) corepresentations, the dual notion of representations, are maps $\Delta_V : V \to H \otimes V$ (resp. $\Delta_V : V \to V \otimes H$) that satisfy

$$(\Delta \otimes \mathrm{id}_V) \circ \Delta_V = (\mathrm{id}_H \otimes \Delta_V) \circ \Delta_V \text{ and } (\varepsilon \otimes \mathrm{id}_V) \circ \Delta_V = \mathrm{id}_V,$$
$$\Big(\text{resp. } (\Delta_V \otimes \mathrm{id}_H) \circ \Delta_V = (\mathrm{id}_V \otimes \Delta) \circ \Delta_V \text{ and } (\mathrm{id}_V \otimes \varepsilon) \circ \Delta_V = \mathrm{id}_V \Big).$$

The algebra of continuous functions on a Lie group and the universal enveloping algebra of a Lie algebra can be equipped with a natural Hopf algebra structure. By a quantum group, we mean a Hopf algebra that can be considered as a deformation of an algebra of functions on a Lie group and by a quantum algebra, we mean a Hopf algebra that can be considered as a deformation of a universal enveloping algebra.

Multi-indices. We will also use standard multi-index notation. If $n \in \mathbb{Z}^d$ (or $\mathbb{N}^d$) stands for a multi-index, $n = (n_1, \ldots, n_d)$, then we define

$$|n| = \sum_{i=1}^{d} |n_i|, \qquad n! = \prod_{i=1}^{d} n_i!.$$

For a vector $x \in \mathbb{C}^d$ we set $x^n = \prod_{i=1}^{d} x_i^{n_i}$. A partial ordering for multi-indices is defined by

$$n \geq 0 \qquad \Leftrightarrow \qquad \forall i = 1, \ldots, d: \quad n_i \geq 0,$$

and $n \geq m$ if and only if $n - m = (n_1 - m_1, \ldots, n_d - m_d) \geq 0$. We also introduce $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) = (\delta_{ij})_{j=1,\ldots,d}$.

q-Numbers. Here are the definitions of some q-special functions. For $n \in \mathbb{N}$ we set $q_n = \sum_{\nu=0}^{n-1} q^\nu$ for $q \in \mathbb{C}$, i. e., $q_n = n$ for $q = 1$, and $q_n = \frac{1-q^n}{1-q}$ for $q \in \mathbb{C}\backslash\{1\}$. Then the q-factorial is defined by $q_n! = \prod_{\nu=1}^{n} q_n$, $q_0! = 1$. An analogue of the binomial coefficients can be defined by the recurrence relation

$$\begin{bmatrix} m+1 \\ \mu \end{bmatrix}_q = \begin{bmatrix} m \\ \mu \end{bmatrix}_q + q^{m-\mu+1} \begin{bmatrix} m \\ \mu - 1 \end{bmatrix}_q, \qquad m \in \mathbb{N}, \quad \mu = 1, \ldots, m,$$
$$\begin{bmatrix} m \\ m \end{bmatrix}_q = \begin{bmatrix} m \\ 0 \end{bmatrix}_q = 1, \qquad m \in \mathbb{N}.$$

They are also known as Gauss polynomials. If q is not a root of unity one has

$$\begin{bmatrix} m \\ \mu \end{bmatrix}_q = \frac{q_m!}{q_\mu! \cdot q_{m-\mu}!}, \qquad m \in \mathbb{N}, \quad \mu = 0, 1, \ldots, m.$$

The power series $e_q^x = \sum_{n=0}^{\infty} \frac{x^n}{q_n!}$, defined for $q \in \mathbb{C}$ not a root of unity, also plays an important rôle in what follows.

4.3 q-Exponentials

In this section we consider a class of Hopf algebras characterized by the relations of their generators (see below), and calculate the dual algebras and dual pairings. The pairing between such an algebra and its dual can formally be written as a product of q-exponentials, a result that is helpful later when we study multiplicative processes on these algebras.

Let $\mathcal{U}$ be a Hopf algebra with generators $X_1, \ldots, X_{d_X}, H_1, \ldots, H_{d_H}, Y_1, \ldots, Y_{d_Y}$. We add generators $K_1, \ldots, K_{d_H}$ that commute with $H_1, \ldots, H_{d_H}$. We will later see that they play the rôle of $e^{h_1 H_1}, \ldots, e^{h_{d_H} H_{d_H}}$, $h_i \in \mathbb{C}$, and suppose the following conditions are satisfied:

$$\mathbf{H}: \begin{cases} \text{The set } \{\psi_{klmn} = Y^k H^l K^m X^n; k \in \mathbb{N}^{d_Y}, l \in \mathbb{N}^{d_H}, m \in \mathbb{Z}^{d_H}, n \in \mathbb{N}^{d_X}\} \text{ spans } \mathcal{U} \\ \Delta H_i = H_i \otimes 1 + 1 \otimes H_i, \\ \Delta X_k = \prod K_i^{s_{ik}} \otimes X_k + X_k \otimes \prod K_i^{t_{ik}}, \\ \Delta Y_l = \prod K_i^{p_{il}} \otimes Y_l + Y_l \otimes \prod K_i^{q_{il}}, \\ [H_i, H_j] = 0, \quad [H_i, K_j] = 0, \quad [K_i, K_j] = 0 \\ [H_i, X_k] = \chi_{ik} X_k, \quad K_i X_k = e^{\chi_{ik} h_i} X_k K_i \\ [H_i, Y_l] = \eta_{il} Y_l, \quad K_i Y_l = e^{\eta_{il} h_i} Y_l K_i \end{cases}$$

for some constants $s_{ik}, t_{ik}, p_{il}, q_{il} \in \mathbb{Z}$, $\chi_{ik}, \eta_{il} \in \mathbb{C}$.

Remark: Note that we impose no conditions on the relations between the X_k and the Y_l; they are not needed for the calculations in this section. But the condition that the ψ_{klmn} span $\mathcal{U}$ implies that such relations exist, namely that their commutator is a linear combination of the ψ_{klmn}. Furthermore, they are restricted by the condition that $\mathcal{U}$ is a Hopf algebra, but this leaves many possibilities. Just looking at Lie algebras we see among the possibilities that a commutator of X's and/or Y's may be zero, or an element of the Cartan subalgebra (i. e., a linear combination of the H's), or a linear combination of X's and Y's.

The conditions **H** are satisfied for most quantum algebras, in particular for the standard semi-simple quantum groups introduced by Drinfeld and Jimbo[Dri87, Jim85, Jim86], as well as many others, e. g. , that have been considered by physicists.

We also introduce the subspace $\mathcal{U}_0 = \text{span}\{\psi_{kl0r}\}$.

Then the dual $\mathcal{U}^*$ of $\mathcal{U}$ is an algebra with the multiplication defined by

$$(f_1 \cdot f_2)(u) = (f_1 \otimes f_2)(\Delta u). \tag{4.1}$$

We define functionals $A_{kln} \in \mathcal{U}^*$ by

$$A_{k'l'n'}(\psi_{klmn}) = \begin{cases} \delta_{kk'}\delta_{nn'} \prod \frac{(h_i m_i)^{m'_i}}{(l'_i - l_i)!} & \text{if } l'_i \geq l_i \text{ for all } i \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

and set $\mathcal{A}_0 = \text{span}\{A_{kln}; k \in \mathbb{N}^{d_Y}, l \in \mathbb{N}^{d_H}, n \in \mathbb{N}^{d_X}\}$. It turns out that $\mathcal{A}_0$ is a subalgebra of $\mathcal{U}^*$. This definition guarantees that $\sum_{n=0}^{N} \frac{(h_i H_i)^n}{n!}$ will tend to K_i in the weak topology, i. e., that K_i can be considered as $e^{h_i H_i}$, if $\mathcal{U}$ is interpreted as a subspace of $\mathcal{A}_0^*$.

Lemma 4.3.1 *Let $X \in \mathcal{U}$ be (A,B)-primitive, i. e., $\Delta X = A \otimes X + X \otimes B$ and suppose $XA = \alpha AX$, $XB = \beta BX$, $\alpha, \beta \in \mathbb{C}$. Then*

$$\begin{aligned}\Delta X^n &= \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_{\alpha/\beta} \beta^{(n-\nu)\nu} A^{n-\nu} X^{\nu} \otimes B^{\nu} X^{n-\nu} \\ &= \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_{\alpha/\beta} \alpha^{-(n-\nu)\nu} X^{\nu} A^{n-\nu} \otimes X^{n-\nu} B^{\nu}.\end{aligned}$$

Proof: We set $\Delta X^n = \sum_{\nu=0}^{n} C_\nu^n A^{n-\nu} X^\nu \otimes B^\nu X^{n-\nu}$, consider $\Delta X^{n+1} = \Delta X \cdot \Delta X^n$, and find the following recursion relation

$$C_\nu^{n+1} = \beta^\nu C_\nu^n + \alpha^{n-\nu+1} C_{\nu-1}^n.$$

Solving this relation now completes the proof. ∎

We introduce the following constants:

$$\begin{aligned} \alpha_k &= e^{-\sum_i s_{ik}\chi_{ik}h_i}, & \beta_k &= e^{-\sum_i t_{ik}\chi_{ik}h_i}, \\ \gamma_l &= e^{-\sum_i p_{il}\eta_{il}h_i}, & \delta_l &= e^{-\sum_i q_{il}\eta_{il}h_i}, \\ \alpha_{kk'} &= e^{-\sum_i s_{ik}\chi_{ik'}h_i}, & \beta_{kk'} &= e^{-\sum_i t_{ik}\chi_{ik'}h_i}, \\ \gamma_{ll'} &= e^{-\sum_i p_{il}\eta_{il'}h_i}, & \delta_{ll'} &= e^{-\sum_i q_{il}\eta_{il'}h_i}. \end{aligned}$$

WARNING: The double-indexed δ used here is not a Kronecker delta.

Lemma 4.3.2

$$\begin{aligned}&\Delta\psi_{klmn} = \\ &\sum_{(\kappa,\lambda,\nu)\leq(kln)} \begin{bmatrix} k_1 \\ \kappa_1 \end{bmatrix}_{\gamma_1/\delta_1} \cdots \begin{bmatrix} k_{d_Y} \\ \kappa_{d_Y} \end{bmatrix}_{\gamma_{d_Y}/\delta_{d_Y}} \begin{pmatrix} l_1 \\ \lambda_1 \end{pmatrix} \cdots \begin{pmatrix} l_{d_H} \\ \lambda_{d_H} \end{pmatrix} \begin{bmatrix} n_1 \\ \nu_1 \end{bmatrix}_{\alpha_1/\beta_1} \cdots \begin{bmatrix} n_{d_X} \\ \nu_{d_X} \end{bmatrix}_{\alpha_{d_X}/\beta_{d_X}} \\ &\prod_l \gamma^{-(k_l-\kappa_l)\kappa_l} \prod \beta_r^{(n_r-\nu_r)\nu_r} \prod_{l'>l} \gamma_{ll'}^{-(k_l-\kappa_l)\kappa_{l'}} \prod_{l'>l} \delta_{ll'}^{-\kappa_l(k_{l'}-\kappa_{l'})} \prod_{k'<k} \alpha_{kk'}^{(n_k-\nu_k)\nu_{k'}} \prod_{k'<k} \beta_{kk'}^{\nu_k(n_{k'}-\nu_{k'})} \\ &Y_1^{\kappa_1} \cdots Y_{d_Y}^{\kappa_{d_Y}} H_1^{\lambda_1} \cdots H_{d_H}^{\lambda_{d_H}} \prod K_i^{s_{ik}(n_k-\nu_k)+p_{il}(k_l-\kappa_l)+m_i} X_1^{\nu_1} \cdots X_{d_X}^{\nu_{d_X}} \\ &\otimes Y_1^{k_1-\kappa_1} \cdots Y_{d_Y}^{k_{d_Y}-\kappa_{d_Y}} H_1^{l_1-\lambda_1} \cdots H_{d_H}^{l_{d_X}-\lambda_{d_X}} \prod K_i^{t_{ik}\nu_k+q_{il}\kappa_l+m_i} X_1^{n_1-\nu_1} \cdots X_{d_X}^{n_{d_X}-\nu_{d_X}}.\end{aligned}$$

Proof: Apply Lemma 4.3.1 and reorder the terms. ∎

We set $a_l = A_{e_l,0,0}$, $b_i = A_{0,e_i,0}$, $c_k = A_{0,0,e_k}$. The formula of the previous lemma allows us to calculate the algebraic relations on a_l, b_i, c_k.

Lemma 4.3.3

$$b_i \cdot A_{0,m,0} = A_{0,m,0} \cdot b_i = (m_i+1) A_{0,m+e_i,0},$$

$$
\begin{aligned}
a_l \cdot A_{nmr} &= \sum_{m' \le m} \left(\frac{\gamma_l}{\delta_l}\right)_{n_l+1} \gamma_l^{-n_l} \prod_{l'<l} \gamma_{l'l}^{-n_{l'}} \prod_{l'>l} \delta_{ll'}^{-n_l} \prod_{i=1}^{d_H} \frac{q_{il}^{m_i-m'_i}}{(m_i-m'_i)!} A_{n+e_l,m',r} \\
A_{nmr} \cdot a_l &= \sum_{m' \le m} \left(\frac{\gamma_l}{\delta_l}\right)_{n_l+1} \gamma_l^{-n_l} \prod_{l'>l} \gamma_{ll'}^{-n_{l'}} \prod_{l>l'} \delta_{l'l}^{-n_{l'}} \prod_{i=1}^{d} \frac{p_{il}^{m_i-m'_i}}{(m_i-m'_i)!} A_{n+e_l,m',r} \\
c_k \cdot A_{nmr} &= \sum_{m' \le m} \left(\frac{\alpha_k}{\beta_k}\right)_{r_k+1} \beta_k^{r_k} \prod_{k<k'} \alpha_{k'k}^{r_{k'}} \prod_{k'<k} \beta_{kk'}^{r_{k'}} \prod_{i=1}^{d} \frac{t_{ik}^{m_i-m'_i}}{(m_i-m'_i)!} A_{n,m',r+e_k}, \\
A_{nmr} \cdot c_k &= \sum_{m' \le m} \left(\frac{\alpha_k}{\beta_k}\right)_{r_k+1} \beta_k^{r_k} \prod_{k'<k} \alpha_{kk'}^{r_{k'}} \prod_{k<k'} \beta_{k'k}^{r_k} \prod_{i=1}^{d} \frac{s_{ik}^{m_i-m'_i}}{(m_i-m'_i)!} A_{n,m',r+e_k},
\end{aligned}
$$

where $\left(\frac{\gamma_l}{\delta_l}\right)_{n_l+1}$ *and* $\left(\frac{\alpha_k}{\beta_k}\right)_{r_k+1}$ *are the q-numbers introduced in the previous section.*

Proof: Follows from the definition of the multiplication in $\mathcal{U}^*$ in Equation (4.1) and Lemma 4.3.2. ∎

These relations show that $\mathcal{A}_0$ is in fact a subalgebra of $\mathcal{U}^*$.

Proposition 4.3.4 *If* γ_k/δ_k *and* α_k/β_k *are not roots of unity, then the algebra* $\mathcal{A}_0$ *is generated by* $a_1, \ldots, a_{d_Y}, b_1, \ldots, b_{d_H}, c_1, \ldots, c_{d_X}$ *with the relations*

$$
\begin{aligned}
a_l \cdot c_k &= c_k \cdot a_l, & b_i \cdot b_j &= b_j \cdot b_i, \\
\delta_{ll'}\, a_l \cdot a_{l'} &= \gamma_{ll'}\, a_{l'} \cdot a_l, & \beta_{k'k}\, c_k \cdot c_{k'} &= \alpha_{k'k}\, c_{k'} \cdot c_k, \\
[b_i, a_l] &= (p_{il} - q_{il}) a_l, & [b_i, c_k] &= (s_{ik} - t_{ik}) c_k.
\end{aligned}
$$

Proof: Follows directly from Lemma 4.3.3. ∎

If some of the γ_k/δ_k, α_k/β_k are roots of unity, then the algebra generated by $a_1, \ldots, a_{d_Y}$, $b_1, \ldots, b_{d_H}, c_1, \ldots, c_{d_X}$ is a subalgebra of $\mathcal{A}_0$, which we shall denote by $\hat{\mathcal{A}}_0$. For this case we introduce the algebra $\hat{\mathcal{U}} = \mathcal{U}/\mathcal{I}$, where $\mathcal{I} = \{u \in \mathcal{U}; \forall a \in \hat{\mathcal{A}}_0 : \ < u, a >= 0\}$.

For the following calculations we will assume that

$$
q_{il} = 0, \qquad s_{ik} = 0.
$$

This can always be achieved by an appropriate choice of the generators, e. g., set $\tilde{Y}_l = \prod K_i^{-q_{il}} Y_l$, $\tilde{X}_k = \prod K_i^{-s_{ik}} X_k$. Then $\alpha_k = 1$, $\alpha_{kk'} = 1$, $\delta_l = 1$, $\delta_{ll'} = 1$.

We find in this case

$$
\begin{aligned}
a_l \cdot A_{nmr} &= \left(\frac{1}{\gamma_l}\right)_{n_l+1} A_{n+e_l,m,r} && \text{if } n_{l'} = 0 \text{ for } l' < l, \\
A_{nmr} \cdot c_k &= (\beta_k)_{r_k+1} A_{n,m,r+e_k} && \text{if } r_{k'} = 0 \text{ for } k' > k.
\end{aligned}
$$

If we assume also that $1/\gamma_i$ and β_i are not roots of unity, then we have

$$
A_{nmr} = \frac{a_1^{n_1} \cdots a_{d_Y}^{n_{d_H}} b_1^{m_1} \cdots b_{d_H}^{n_{d_H}} c_1^{r_1} \cdots c_{d_X}^{r_{D_X}}}{\left(\frac{1}{\gamma_1}\right)_{n_1}! \cdots \left(\frac{1}{\gamma_{d_Y}}\right)_{n_{d_Y}}! m_1! \cdots m_{d_H}! (\beta_1)_{r_1}! \cdots (\beta_{r_{d_X}})_{r_{d_X}}!}.
$$

We consider the sequence

$$g^{(N)}(A,\psi) = \sum_{nmr}^{N} A_{nmr} \otimes \psi_{nmr}. \tag{4.3}$$

The sequence $(g^{(N)})_{N\in\mathbb{N}} \subset \mathcal{A}\otimes\mathcal{U}_0 \subset \mathcal{U}_0^* \otimes \mathcal{U}_0 \subset \mathrm{End}(\mathcal{U}_0)$ converges weakly towards the identity $\mathrm{id}_{\mathcal{U}_0}$. If we omit the tensor product we can also formally write for the sequence $g = (g^{(N)})_{N\in\mathbb{N}}$,

$$g(\{a\},\{b\},\{c\};\{Y\},\{H\},\{X\}) = e_{1/\gamma_1}^{a_1 Y_1}\cdots e_{1/\gamma_{d_Y}}^{a_{d_Y} Y_{d_Y}} e^{b_1 H_1}\cdots e^{b_{d_H} H_{d_H}} e_{b_1}^{c_1 X_1}\cdots e_{\beta_{d_X}}^{c_{d_X} X_{d_X}},$$

i. e., we have found that we can write the formal pairing as a product of q-exponentials if hypothesis **H** is satisfied and the generators are chosen appropriately. The duality between $\Delta_{\mathcal{A}}$ and $m_{\mathcal{U}}$ implies that we can formally write (see e. g. [BCG+94, FG93])

$$g(\Delta_{\mathcal{A}}(a);\varphi) = g(a';\psi)g(a'';\psi), \tag{4.4}$$

where $a' = a\otimes 1$, $a'' = 1\otimes a$.

With some modifications this result remains valid also for the case where some of the parameters γ_k/δ_k, α_k/β_k are roots of unity. The sum has to be restricted to the terms where all q-factorials are different from zero, and we get a dual pairing between $\hat{\mathcal{U}}_0$ and $\hat{\mathcal{A}}_0$ in this case.

Example: The q-affine algebra is the quantum algebra $\mathcal{U}$ with two generators X, Y and relations

$$\begin{aligned} XY - YX &= \alpha Y, \\ \Delta(X) &= X\otimes 1 + 1\otimes X, \\ \Delta(Y) &= Y\otimes \exp(\beta X) + 1\otimes Y, \\ \varepsilon(X) &= \varepsilon(Y) = 0, \end{aligned}$$

with $\alpha,\beta\in\mathbb{C}$, $q = e^{\alpha\beta}$ not a root of unity. Then we get $\mathcal{A} = \mathrm{span}\{a^n b^m; n,m\in\mathbb{N}\}$, $ba - ab = \beta b$, and

$$g(a,b;X,Y) = \sum_{n,m=0}^{\infty} \frac{a^n b^m}{n! q_m!} X^n Y^m = e^{aX} e_q^{bY}$$

We will also call $\mathcal{A}$ the q-affine group.

4.4 Dual representations

Suppose we are given two Hopf algebras $\mathcal{A}$ and $\mathcal{U}$ that are in duality, and let $g(a;X) = \sum A_n\psi_n$ be their dual pairing. We introduce actions of $\mathcal{U}$ on $\mathcal{A}$ and show how the dual pairing can be used for explicit calculations.

The dual right and left representations $\rho_R^*, \rho_L^* : \mathcal{U}\to \mathrm{Hom}(\mathcal{A},\mathcal{A})$ are defined by

$$\begin{aligned} <\rho_R^*(X)a, u> &= <a, u\cdot X> \\ <\rho_L^*(X)a, u> &= <a, X\cdot u> \end{aligned}$$

for all $a \in \mathcal{A}$, $u \in \mathcal{U}$. This leads to the formulae

$$\begin{aligned} \rho_R^*(X) &= (\mathrm{id} \otimes X) \circ \Delta_A \\ \rho_L^*(X) &= (X \otimes \mathrm{id}) \circ \Delta_A \end{aligned}$$

where the coproduct Δ_A for $\mathcal{A}$ is defined such that $\mathcal{A}$ and $\mathcal{U}$ are dually paired bialgebras and the X appearing on the right-hand-side is interpreted as a functional on $\mathcal{A}$. We will compute the action of $\rho_{R,L}^*(X_i)$ on the basis $\{A_n\}$ with the formal pairing, and use it later to calculate the coproduct on $\mathcal{A}$.

Then ρ_R^* is an algebra homomorphism and ρ_L^* is an algebra anti-homomorphism, i. e.,

$$\begin{aligned} \rho_R^*(XY) &= \rho_R^*(X)\rho_R^*(Y) \\ \rho_L^*(XY) &= \rho_L^*(Y)\rho_L^*(X) \end{aligned}$$

for all $X, Y \in \mathcal{U}$.

The key property of the formal pairing is that multiplying the ψ_n in each term from the right (left) by X_i leads to the same result as applying the right (left) dual representation $\rho_R^*(X_i)$ ($\rho_L^*(X_i)$) to A_n in each term, i. e.,

$$\begin{aligned} \rho_R^*(X_i)g(a;X) &= g(a;X)X_i \\ \rho_L^*(X_i)g(a;X) &= X_i g(a;X). \end{aligned}$$

To compute the dual representations, commute X_i past the factors $e_{q_j}^{a_j X_j}$ in $g(a;X)$ until it is next to $e_{q_i}^{a_i X_i}$, and then replace it by the operator δ_i,

$$\delta_i f(a_i) = \begin{cases} \frac{f(q_i a_i) - f(a_i)}{(q_i - 1)a_i} & \text{if } q_i \neq 1, \\ \frac{\partial f(a_i)}{\partial a_i} & \text{if } q_i = 1, \end{cases}$$

since on the individual factors we have $X_i e_{q_i}^{a_i X_i} = \delta_i e_{q_i}^{a_i X_i}$. For factors where q_j is equal to 1, we can apply the relation $e^{a_j X_j} X_i e^{-a_j X_j} = e^{a_j \,\mathrm{ad} X_j} X_i$ (where $\mathrm{ad} X_j\,(X_i) = [X_j, X_i]$).

Quotient representations of the right and left dual representation can be constructed by factoring out a left or right ideal, respectively. The duals of these representations then give corepresentations of $\mathcal{A}$.

Let $\rho : \mathcal{U} \to \mathrm{Hom}(V, V)$ be a representation of $\mathcal{U}$, and $\{v_n\}$ a basis of V. We define the matrix elements of ρ w.r.t. $\{v_n\}$ by

$$(\mathrm{id}_{\mathcal{A}} \otimes \rho)(g(a;X)))(v_n) = g(a;\rho(X))(v_n) = \sum M_{mn} v_m. \tag{4.5}$$

With the relation $g(\delta(a);\psi) = g(a';\psi)g(a'';\psi)$ one can show that formally

$$\Delta_W(w_n) = M_{nm} \otimes w_n$$

defines a corepresentation on $W = \mathrm{span}\{w_n\}$, where the w_n are functionals on V defined by $w_n(v_{n'}) = \delta_{n,n'}$. Anti-homomorphisms lead to right corepresentations by $\Delta_W(w_n) = w_m \otimes M_{nm}$.

Example: We get for the q-affine algebra

$$\begin{aligned} \rho_L^*(X) &= \partial_a + \alpha b \partial_b, & \rho_L^*(Y) &= \delta_b, \\ \rho_R^*(X) &= \partial_a, & \rho_R^*(Y) &= \exp(\alpha a)\delta_b. \end{aligned}$$

Since the dual of these representations is the coproduct of $\mathcal{A}$ (interpreted as right or left corepresentation of $\mathcal{A}$ on itself), we can use their matrix elements to get the coproduct of $\mathcal{A}$:

$$\begin{aligned} \Delta(a) &= a \otimes 1 + 1 \otimes a, \\ \Delta(b) &= b \otimes 1 + \exp(\alpha a) \otimes b. \end{aligned}$$

We shall use the quotient representation that arises from the relation $X = r$:

$$\rho_r(X) = r - \alpha b \partial_b, \qquad \rho_r(Y) = \delta_b. \tag{4.6}$$

4.5 Construction

Let $(W_t)_{t\in\mathbb{R}_+} = (W_t^1, \cdots, W_t^d)_{t\in\mathbb{R}_+}$ be a stochastic process with values in $\mathbb{R}^d$ with independent increments, and assume that all moments of W_t are finite. Then a functional $\hat{\Phi}_{t_1,t_2}$ on $\mathbb{R}[x_1,\ldots,x_d]$, corresponding to an increment $W_{t_2} - W_{t_1}$, is defined by $\hat{\Phi}_{t_1,t_2}(x_1^{n_1}\cdots x_d^{n_d}) = \mathbb{E}((W_{t_2}^1 - W_{t_1}^1)^{n_1}\cdots(W_{t_2}^d - W_{t_1}^d)^{n_d})$. We can identify $\mathcal{A}$ with $\mathbb{R}[x_1,\ldots,x_d]$ as a vector space, if we fix a Poincaré-Birkhoff-Witt (PBW) basis $\{A_n; n \in \mathbb{N}^d\}$ for $\mathcal{A}$ and set $\imath(A_n) = x^n$. We denote the functional on $\mathcal{A}$ obtained in this way also by $\hat{\Phi}_{t_1,t_2}$.

We suppose that $\imath$ is chosen such that the functionals $\hat{\Phi}_{t_1,t_2}$ are positive. For example for the q-affine group this is the case for $\imath$ defined by $\imath(\frac{a^n b^m}{n!q_m!}) = \frac{x_1^n x_2^m}{n!m!}$ with an appropriate definition of positivity[4]. We do not know how to define $\imath$ in general to guarantee the positivity of $\hat{\Phi}_{t_1,t_2}$. Nevertheless, the results below hold regardless of the positivity hypothesis.

The following construction will start with these functionals, and then recover their properties with respect to the coproduct of $\mathcal{A}$.

We suppose that W_t has independent increments. For the functional $\hat{\Phi}_t$ on $\mathbb{R}[x_1,\ldots,x_d]$ this means that

$$\hat{\Phi}_{t_1,t_2} \otimes \hat{\Phi}_{t_2,t_3}(\Delta z) = \hat{\Phi}_{t_1,t_3}(z), \qquad \text{for } z \in \mathbb{R}[x_1,\ldots,x_d]$$

where the coproduct of $\mathbb{R}[x_1,\ldots,x_d]$ is defined by $\Delta x_i = x_i \otimes 1 + 1 \otimes x_i$. We want to construct a functional Φ_t on $\mathcal{A}$ that satisfies the same relation with respect to the coproduct of $\mathcal{A}$. To this end we define a sequence of functionals $\Phi_t^{(N)}$, and take its limit for Φ_t, if it exists. Let

$$\Phi_{s,t}^{(N)}(a) = \hat{\Phi}_{s,s+(t-s)/N} \otimes \cdots \otimes \hat{\Phi}_{s+(t-s)(N-1)/N,t}\left(\Delta^{N-1}(a)\right) \qquad \text{for } N > 1, \quad a \in \mathcal{A}.$$

4. If we define the positive elements of $\mathcal{A}$ as the inverse image under $\imath$ of the positive elements of $\mathbb{R}[x_1, x_2]$.

Loosely speaking, this corresponds to decomposing the desired process into its increments via the coproduct, and approximating its expected value by the expected value with respect to $\hat{\Phi}$ in each increment. We define

$$\Phi_{s,t}(a) = \lim_{N\to\infty} \Phi_{s,t}^{(N)}(a) \qquad \text{for } a \in \mathcal{A}, \qquad s < t \in \mathbb{R}_+, \tag{4.7}$$

if this limit exists.

Definition 4.5.1 *Let W_t be a stochastic process on $\mathbb{R}^n$ with independent increments $W_{s,t} = W_t - W_s$, and all moments finite. Let further $\mathcal{A}$ be a Hopf algebra and $\imath : \mathcal{A} \to \mathbb{R}[x_1, \ldots, x_n]$ a vector space isomorphism. We call*

$$\Phi_{s,t}(a) = \lim_{N\to\infty} \mathbb{E}(\imath^{\otimes N}(\Delta^{N-1}(a))(W_{s,s+(t-s)/N}, \ldots, W_{s+(t-s)(N-1)/N,t}))$$

the $\imath$-analogue of W_t on $\mathcal{A}$, if this limit exists.

In our applications we will consider processes whose increments are stationary, i. e., the functionals $\hat{\Phi}_{s,t}$ and $\Phi_{s,t}$ depend only on the difference $t - s$, in this case we can also write $\hat{\Phi}_t$ and Φ_t.

In the following section we have a class of examples where this limit exists, and the expected value of the dual pairing, which in this context should be understood as a generating function, is found. For this it is necessary to extend the definition of the functional Φ_t to infinite series thus:

$$\Phi_t(\sum_{n=0}^{\infty} a_n) = \sum_{n=0}^{\infty} \Phi_t(a_n),$$

if the left-hand-side converges absolutely.

4.6 Feynman-Kac formula

On Lie groups one can obtain Feynman-Kac type formulae with Trotter's product formula

$$\lim_{N\to\infty} \left(e^{X_i/N} \cdots e^{X_d/N}\right)^N = e^{X_1+\cdots+X_d}.$$

Let W_t be a stochastic process on $\mathbb{R}^d$ with independent and stationary increments and independent components, and

$$\mathbb{E}\left(e^{W_{s,t}^1 X_1} \cdots e^{W_{s,t}^d X_d}\right) = e^{L_1(X_1)(t-s)} \cdots e^{L_d(X_d)(t-s)}$$

be the expectation value of an increment, i. e., L_i is the generator of $W_i(t)$. Then, if we approximate the process $g(\tilde{W}(t); X)$ on the Lie group defined by McKean's stochastic product integral [McK69] by $g^{(n)}(t)$,

$$\begin{aligned} \mathbb{E}\left(g^{(n)}(t)\right) &= \mathbb{E}\left(\prod e^{w_1 X_i} \cdots e^{w_d X_d}\right) = \prod \mathbb{E}\left(e^{w_1 X_1} \cdots e^{w_d X_d}\right) \\ &= \left(e^{L_1(X_1)\Delta t} \cdots e^{L_d(X_d)\Delta t}\right)^n \underset{n\to\infty}{\longrightarrow} e^{t(L_1(X_1)+\cdots+L_d(X_d))}. \end{aligned}$$

If we take the limit on both sides we get

$$\mathbb{E}\Big(g(\tilde{W}(t);X)\Big) = e^{t(L_1(X_1)+\cdots+L_d(X_d))}.$$

Note that this formula still contains Trotter's product formula for the special case of a deterministic process, i. e., if $L_i(X_i) = X_i$ for $i = 1, \ldots, d$.

Theorem 4.6.1 (Feynman-Kac formula) *Let $\mathcal{A}$ and $\mathcal{U}$ be a quantum group and a quantum algebra that are in duality, with generators $a_1, \ldots, a_d$ and $X_1, \cdots, X_d$, respectively, and dual pairing*

$$g(a;X) = e_{q_1}^{a_1 X_1} \cdots e_{q_d}^{a_d X_d}, \qquad q_i \in \mathbb{C}.$$

Let ρ be a representation of $\mathcal{U}$ by operators that are bounded with respect to some norm $||\cdot||$. We set

$$p_i = \begin{cases} \max\{p; \rho(X_i)^p \neq 0\} & \text{if } \rho(X_i) \text{ is nilpotent, i.e. } \rho(X_i)^{r+1} = 0 \text{ for some integer } r, \\ \infty & \text{otherwise.} \end{cases}$$

Let $\imath : \mathcal{A} \to \mathbb{R}[x_1, \ldots, x_n]$ be defined by $\imath(a^n) = x^n$

Suppose further that we are given a d-dimensional stochastic process $W(t)$ with independent and stationary increments and independent components, and moment generating functions

$$\mathbb{E}(e^{uW^i(t)}) = e^{tL_i(u)}, \qquad L_i(u) = \sum_{k=1}^{\infty} \frac{l_k^{(i)} u^k}{k!} \text{ analytic,}$$

and moments $m_k(W^i(t)) = \left(\frac{\mathrm{d}}{\mathrm{d}u}\right)^k e^{tL_i(u)}|_{u=0}$, such that the series

$$\tilde{L}_i(u) = \sum_{k=1}^{p_i} \frac{l_k^{(i)} u^k}{(q_i)_k!}, \qquad \text{and} \qquad \sum_{k=0}^{p_i} \frac{m_k(W^i(t))\rho(X_i)^k}{(q_i)_k!}$$

are well-defined and converge (in particular, $(q_i)_k \neq 0$, for $1 \leq k \leq p_i$).

Then we have for the $\imath$-analogue of $A(t)$ on the associated quantum group $\mathcal{A}$

$$(\Phi_t \otimes \mathrm{id}) \left(\prod_{i=1}^{d} e_{q_i}^{a_i \rho(X_i)} \right) = \exp\left[t \sum_{i=1}^{d} \tilde{L}_i(\rho(X_i)) \right].$$

Remark: If $\mathcal{U}$ has a sufficiently rich class of representations that satisfy the conditions of the theorem, then this relation allows us to read off all moments of the Brownian motion.
Proof: According to Definition 4.5.1 we have to consider

$$\Phi_t^{(N)}(g(a;\rho(X))) = \hat{\Phi}_{t/N}^{\otimes N}(g(\Delta^{N-1}a;\rho(X)))$$

where $\hat{\Phi}_{t/N}$ is the functional corresponding to an increment of W_t for $\Delta t = t/N$. By the identification procedure outlined in the previous section this gives $\hat{\Phi}_{t/N}(a^n) = \prod_i m_{n_i}(W_i(t/N))$. If we take into account that (cf. Equation (4.4))

$$g(\Delta a; X) = g(a'; X) g(a''; X),$$

where $a' = a \otimes 1$, $a'' = 1 \otimes a$, then we get

$$\Phi_t^{(N)}(g(a;\rho(X))) = \hat{\Phi}_{t/N}^{\otimes N}(g(a^{(1)};X)\cdots g(a^{(N)};X))$$

where $a^{(k)} = 1^{\otimes k-1} \otimes a \otimes 1^{N-k}$, and thus

$$\begin{aligned}\Phi_t^{(N)}(g(a;\rho(X))) &= \hat{\Phi}^{\otimes N}\left(\prod_{\nu=1}^{N}\prod_{i=1}^{d} e_{q_i}^{a_i\rho(X_i)}\right)\\ &= \prod_{\nu=1}^{N}\prod_{i=1}^{d}\left(1+\sum_{k=1}^{p_i}\frac{m_k(W^i(t/N))\rho(X_i)^k}{(q_i)_k!}\right)\end{aligned}$$

Note that for $k \geq 1$ each $m_k(W^i(\tau))$ can be written as

$$m_k(W^i(\tau)) = \tau l_k^{(i)} + \tau^2 R_{ik}(\tau)$$

therefore

$$\sum_{k=1}^{p_i}\frac{m_k(W^i(t/N))\rho(X_i)^k}{(q_i)_k!} = \frac{t}{N}\tilde{L}_i(\rho(X_i)) + \frac{t^2}{N^2}R_i(t/N)$$

where $R_i(t/N)$ is some bounded operator. Taking the product we get

$$\prod_{i=1}^{d}\left(1+\sum_{k=1}^{p_i}\frac{m_k(W^i(t/N))\rho(X_i)^k}{(q_i)_k!}\right) = 1 + \frac{t}{N}\sum_{i=1}^{d}\tilde{L}_i(\rho(X_i)) + \frac{t^2}{N^2}R(t/N),$$

where $R(t/N)$ is also bounded.

Using $\prod_{\nu=1}^{N} a_\nu - \prod_{\nu=1}^{N} b_\nu = \sum_{\nu=1}^{N} a_1\cdots a_{\nu-1}(a_\nu - b_\nu)b_{\nu+1}\cdots b_N$, we have

$$\begin{aligned}&\left\|\prod_{\nu=1}^{N}\prod_{i=1}^{d}\left(1+\sum_{k=1}^{p_i}\frac{m_k(W^i(t/N))\rho(X_i)^k}{(q_i)_k!}\right) - \prod_{\nu=1}^{N}\left(1+\frac{t}{N}\sum_{i=1}^{d}\tilde{L}_i(\rho(X_i))\right)\right\|\\ =&\left\|\sum_{\nu=1}^{N}\left(\prod_{i=1}^{d}1+\sum_{k=1}^{p_i}\frac{m_k(W^i(t/N))\rho(X_i)^k}{(q_i)_k!}\right)^{\nu-1}\frac{t^2R(t/N)}{N^2}\left(1+\frac{t}{N}\sum_{i=1}^{d}\tilde{L}_i(\rho(X_i))\right)^{N-\nu}\right\|\\ \leq&\ \frac{\text{constant}}{N} \to 0\end{aligned}$$

for $N \to \infty$. This concludes the proof, since $\prod_{\nu=1}^{N}\left(1+\frac{t}{N}\sum_{i=1}^{d}\tilde{L}_i(\rho(X_i))\right)$ converges to $\exp\left(t\sum_{i=1}^{d}\tilde{L}_i(\rho(X_i))\right)$. ∎

We can now obtain a Trotter-type product formula for q-exponentials.

Proposition 4.6.2 (Trotter's product formula) *Let X_i, $i = 1,\ldots,d$ be bounded operators and assume that for each $i = 1,\ldots,d$ one of the following two conditions is satisfied:*

1. $q_i \in \{1\} \cup \{z \in \mathbb{C}; |z| \neq 1\}$,
2. *the operator X_i is nilpotent of some order p_i, i. e., $X_i^{p_i+1} = 0$, and $q_i \in \{1\} \cup \{z \in \mathbb{C}; z^\nu \neq 1$ for $\nu = 1,\ldots,p_i\}$.*

Then we have

$$\prod_{\nu=0}^{N}\prod_{i=1}^{d} e_{q_i}^{X_i/N} \underset{N\to\infty}{\longrightarrow} \exp\sum_{i=1}^{d} X_i. \qquad \textit{(norm convergence)}$$

Remark: For the second case, the q-exponential can be defined by the finite sum $e_{q_i}^{X_i} = \sum_{\nu=0}^{p_i} \frac{X_i^\nu}{q_\nu!}$ since $q_\nu! \neq 0$ for $\nu \leq p_i$ and $X_i^\nu = 0$ for $\nu > p_i$.

Proof: Take a deterministic process with $L_i(X_i) = X_i$, $i = 1, \ldots, d$, and apply Theorem 4.6.1. ∎

The following proposition can also be presented as a consequence of Theorem 4.6.1.

Proposition 4.6.3 *The functionals defined as the ι-analogue of a stochastic process with independent and stationary increments and independent components form a convolution semi-group.*

4.7 Appell systems

Appell systems $\{h_n(x); n \in \mathbb{N}\}$ on $\mathbb{R}$ are usually characterized by the two conditions

- $h_n(x)$ is a polynomial of degree n,
- $\frac{\mathrm{d}}{\mathrm{d}x} h_n(x) = n h_{n-1}(x)$.

Interesting examples are furnished by the shifted moment sequences

$$h_n(x) = \int_{-\infty}^{\infty} (x+y)^n p(\mathrm{d}y),$$

where p is a probability measure on $\mathbb{R}$ with all moments finite. This includes the Hermite moment polynomials

$$h_n(x;t) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} (x+y)^n e^{-y^2/2t} \mathrm{d}y, \qquad t > 0,$$

for the Gaussian case. These polynomials are the solutions of the heat equation

$$\partial_t f(x,t) = \frac{1}{2} \partial_x^2 f(x,t)$$

with $f(x, t \to 0) = x^n$. We will define Appell "polynomials" (or better Appell functions) on quantum groups in this section. It turns out that in general they are only polynomials for $t = 0$. They provide polynomial solutions of evolution equations of the form $\partial_t f = Lf$, where L is an operator consisting of differential and difference operators.

On Lie groups P. Feinsilver and R. Schott [FS92] have defined Appell systems as shifted moment sequences. They are introduced as the expectation of coordinate functions shifted by right or left multiplication of the argument by a group-valued random variable. Let

$f : G \to \mathbb{C}$ be a function on a Lie group G, $g(a)$ an element of G, and $g(Z)$ a G-valued random variable, then we can evaluate

$$\mathbb{E}(f(g(a)g(Z))) \text{ and } \mathbb{E}(f(g(Z)g(a)))$$

We choose for f a set of basis elements $\{A_m; m \in \mathbb{N}^d\}$ of the algebra of functions on G, then the right and left Appell systems (w. r. t. Z) are defined as

$$\begin{aligned} h_n^R(a) &= \mathbb{E}(A_n(g(a)g(Z))), \\ h_n^L(a) &= \mathbb{E}(A_n(g(Z)g(a))). \end{aligned}$$

The Appell functions are again functions on G. We rewrite these equations in terms of the coproduct (note $f(g_1g_2) = \Delta f(g_1, g_2)$).

Definition 4.7.1 *The Appell systems on $\mathcal{A}$ with respect to the basis $\{A_m\}$ and the random variable Z are defined as*

$$\begin{aligned} h_n^R(a) &= (\mathrm{id} \otimes Z) \circ \Delta A_n(a), \\ h_n^L(a) &= (Z \otimes \mathrm{id}) \circ \Delta A_n(a). \end{aligned}$$

Remark: Note that for quantum groups Z exists only as a functional on $\mathcal{A}$.
If we use the parametrization introduced by the formal pairing then we can express the Appell systems in terms of the basis $\{A_n\}$ and the matrix elements (cf. Equation (4.5)) with respect to this basis:

$$\begin{aligned} h_n^R(a) &= (\mathrm{id} \otimes Z)\left(\sum_m M_{nm}^L(a) \otimes A_m\right) = \sum_m M_{nm}^L(a) Z(A_m) \\ &= (\mathrm{id} \otimes Z)\left(\sum_m A_m \otimes M_{nm}^R(a)\right) = \sum_m A_m Z\left(M_{nm}^R(a)\right) \\ h_n^L(a) &= \sum_m Z\left(M_{nm}^L(a)\right) A_m = \sum_m Z(A_m) M_{nm}^R(a). \end{aligned}$$

The Appell systems have the following transformation property

$$\begin{aligned} \Delta h_n^R(a) &= \sum_r M_{nr}^L(a) \otimes h_r^R(a), \\ \Delta h_n^L(a) &= \sum_r h_r^L(a) \otimes M_{nr}^R(a). \end{aligned}$$

Interesting relations to evolution equations arise if we consider the Appell systems with respect to multiplicative stochastic processes. Suppose now that Z_t has generator L, i. e.

$$(Z_t \otimes \mathrm{id})(g(a; X)) = \exp tL.$$

Then

$$\begin{aligned} \partial_t h_n^R(a;t) &= \rho_R^*(L) h_n^R(a;t) \\ \partial_t h_n^L(a;t) &= \rho_L^*(L) h_n^L(a;t) \end{aligned}$$

i. e., h_n^R, h_n^L provide solutions to evolution equations (where $h_n^R(a;t=0) = h_n^L(a;t=0) = A_n$).

Let R_i, L_i be the raising and lowering operators with respect to the basis $\{A_n\}$, with

$$R_i A_n = A_{n+e_i}, \qquad L_i A_n = c_{ni} A_{n-e_i}$$

for some coefficients c_{ni}, then

$$\tilde{R}_i = e^{tL} R_i e^{-tL}, \qquad \tilde{L}_i = e^{tL} L_i e^{-tL}$$

are raising and lowering operators for the Appell system.

We also define analogues of relative Appell systems, i. e., Appell systems on homogeneous spaces. Let $(V, \Delta_V : V \to \mathcal{A} \otimes V)$ (or $(V, \Delta_V : V \to V \otimes \mathcal{A})$) be a left (or right) corepresentation, then the Appell system on V with respect to a basis $\{v_n\}$ of V and a functional Z on $\mathcal{A}$ are defined by

$$\begin{aligned} h_n^V &= (Z \otimes \mathrm{id}_V) \circ \Delta_V v_n, \\ (\text{ or } h_n^V &= (\mathrm{id}_V \otimes Z) \circ \Delta_V v_n). \end{aligned}$$

Examples: We start with a very simple example. Consider the process on the q-affine group (with $\alpha = -1$) obtained from the one-dimensional Brownian motion $(0, B_t)$ on $\mathbb{R}^2$. Then

$$\Phi_t(e^{a\rho(X)} e_q^{b\rho(Y)}) = \exp \frac{t\rho(Y)^2}{1+q}.$$

for all representations where $\rho(Y)$ satisfies the conditions of Theorem 4.6.1. Take e. g. $V^{(N)} = \mathrm{span}\{b^n; n = 0, \ldots, N\}$ and $\rho(X) = b\partial_b$, $\rho(Y) = \delta_b$ (cf. Equation (4.6)). The moments of this process are

$$\mathbb{E}(\tilde{B}_t^n) = \begin{cases} 0 & n \text{ odd} \\ \frac{t^m q_{2m}!}{(1+q)^m m!} & n \text{ even}, n = 2m \end{cases}$$

and the right Appell polynomials are

$$h_{nm}^R(a,b;t) = \delta_{n,0} \sum_{\mu=0}^{[m/2]} \frac{b^{m-2\mu} t^\mu}{(1+q)^\mu \mu! q_{m-2\mu}!}.$$

These polynomials are solutions of

$$\partial_f f(a,b;t) = \frac{\delta_b^2}{1+q} f(a,b;t)$$

In the limit $q \to 1$ they tend to Hermite polynomials. We can also try to associate a density to this process by requiring $\Phi_t(p(b)) = \int p(x)\mu(\mathrm{d}x)$ for all polynomials $p(b)$. For $0 < q < 1$ such a density exists and can be characterized with the help of the random variable $Y = (1-q)\sum_{i=0}^{\infty} X_i$, where the X_i are independent, and exponentially distributed with respective means q^i. The characteristic function of Y is $\mathbb{E}(e^{itY}) = e_q^{it}$, see [Fei87].

Thus $\Phi_t \mathbb{E}_Y(e^{bYu}) = \Phi_t(e_q^{bu}) = \exp \frac{tu^2}{1+q}$, i. e., bY is Gauss-distributed with mean zero and variance $\sigma^2 = \frac{2t}{1+q}$. By [Fei87, Theorem 3] this characterizes the distribution uniquely.

Now look at $X_t = (0, N_t)$, where N_t is a Poisson process, $\mathbb{E}(e^{uN_t}) = \exp(t\lambda(e^u - 1))$. By the Feynman-Kac formula we have

$$\Phi_t(e^{a\rho(X)}e^{b\rho(Y)}) = \exp(t\lambda(e_q^{\rho(Y)} - 1)).$$

We take again $\rho(Y) = \delta_b$, then $(e_q^{\rho(Y)} - 1)b^n = \sum_{k=1}^n \begin{bmatrix} n \\ k \end{bmatrix}_q b^{n-k}$, this yields the associated Appell polynomials $h_n(b;t) = \exp(t\lambda(e_q^{\rho(Y)} - 1))b^n$.

We look now at the stochastic process obtained from the two-dimensional Brownian motion (A_t, B_t) on $\mathbb{R}^2$. We calculate the Appell system corresponding to this process and the corepresentation that is obtained as the dual of the representation ρ_r on the polynomials in one variable, $\rho_r(X) = x\partial_x + r$, $\rho_r(Y) = i\delta_x$ (cf. Equation 4.6). We choose the functions $C_{n;q}^r(x)$ given by

$$C_{n;q}^r(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k (r)_{n-k} x^{n-2k}}{(1+q)^k 2^k k! q_{n-2k}!}$$

where $(r)_m = r(r+1)\cdots(r+m-1)$. For $q = 1$ these are Gegenbauer polynomials. They satisfy

$$\tilde{L}C_{n;q}^r(x) = \frac{(n+r)^2}{2} C_{n;q}^r(x)$$

with $\tilde{L} = \frac{1}{2}(x\partial_x + r)^2 - \frac{\delta_x^2}{1+q}$. For the Appell system one obtains

$$h_n^r(x;t) = \exp\left(\frac{(n+r)^2}{2}t\right) C_{n;q}^r(x).$$

They are solutions of $\partial_t u = \tilde{L}u$. The $C_{n;q}^r(x)$, $n \in \mathbb{N}$ form a basis for $\mathbb{R}[x]$, if q is not a root of unity, and $r \neq -k$ for $k \in \mathbb{N}$, and the Appell polynomials corresponding to other choices of the basis can be computed from those defined above.

4.8 Extension of the construction

From a probabilistic point of view the preceding construction is far from satisfactory. We can only evaluate the process at one fixed time. But the interesting feature of a stochastic process is exactly that it is a parametrized family of random variables, i. e., that it is defined for all values of the parameter. In this section we will construct a larger algebra that can be understood as the tensor product of $\mathcal{A}$ over $t \in \mathbb{R}_+$. On this algebra we can define a functional that corresponds to the distribution of the process (see also [ASW88, Sch93]).

Let $T_n = \{t = (t_1, \ldots, t_n); t_i \in \mathbb{R}_+, t_1 < t_2 < \cdots < t_n\} \subset \mathbb{R}_+^n$, $T_0 = \{\emptyset\}$, $T = \bigcup_{n=0}^\infty T_n$, i. e., T_n contains all subsets of $\mathbb{R}_+$ with n elements, and T all finite subsets of $\mathbb{R}_+$. The

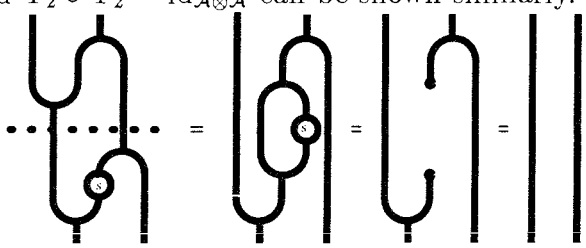
set T is partially ordered by inclusion. For each $t = (t_1, \ldots, t_n) \in T$ we set $\mathcal{A}_t = \mathcal{A}^{\otimes|t|}$ as a vector space, where $|t| = n$ is the length of t. Let $\psi_t : \mathcal{A}_t \to \mathcal{A}^{\otimes|t|}$ be defined by

$$\psi_t(a_1 \otimes \cdots \otimes a_{|t|}) = (a_1 \otimes 1 \otimes \cdots \otimes 1)(\Delta a_2 \otimes 1 \otimes \cdots \otimes 1) \cdots (\Delta^{|t|-1} a_{|t|}) \tag{4.8}$$

where the multiplication on the right hand side is the usual multiplication $m^{\otimes|t|}$ in the tensor product algebra $\mathcal{A}^{\otimes|t|}$.

Lemma 4.8.1 *These maps are vector space isomorphisms.*

Proof: Let $\Psi_2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be defined by $\Psi_2(a \otimes b) = (a \otimes 1)(\Delta b)$, i. e., $\Psi_2 = (m \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta)$. Ψ_2 is an isomorphism, and its inverse is given by $\Phi_2 = (m \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes S \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta)$. Here is a diagrammatic proof of the relation $\Psi_2 \circ \Phi_2 = \mathrm{id}_{\mathcal{A}\otimes\mathcal{A}}$, and $\Phi_2 \circ \Psi_2 = \mathrm{id}_{\mathcal{A}\otimes\mathcal{A}}$ can be shown similarly.



For the interpretation of this diagram see [Maj93a, Maj95b]. Not using the diagram technique we can write the proof as $\Psi_2 \circ \Phi_2 = (m \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes S \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta) \circ (m \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta) = (m \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes m \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \mathrm{id}_{\mathcal{A}} \otimes S \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta) = (m \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes e \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \varepsilon \otimes \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Delta) = \mathrm{id}_{\mathcal{A}\otimes\mathcal{A}}$

Note now that ψ_t can be written as

$$\psi_t = (\Psi_2 \otimes \mathrm{id}_{\mathcal{A}} \otimes \cdots \mathrm{id}_{\mathcal{A}}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \Psi_2 \otimes \mathrm{id}_{\mathcal{A}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{A}}) \circ \cdots \circ (\mathrm{id}_{\mathcal{A}} \otimes \cdots \otimes \mathrm{id}_{\mathcal{A}} \otimes \Psi_2). \tag{4.9}$$

This completes the proof. ∎

This lemma allows us to equip $\mathcal{A}_t$ with an algebra structure by the definition

$$m_t(a, b) = \psi_t^{-1}(m^{\otimes|t|}(\psi_t(a), \psi_t(b))). \tag{4.10}$$

Define $i_{t',t} : \mathcal{A}_{t'} \to \mathcal{A}_t$ for $t' \in t$ by inserting the unit element 1 in the factors corresponding to the indices of t that are missing in t', e. g. for $t = (t_1, \ldots, t_n)$ and $t' = (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n)$ we have $i_{t',t}(a_1 \otimes \cdots \otimes a_{n-1}) = a_1 \otimes \cdots a_{j-1} \otimes 1 \otimes a_{j+1} \otimes \cdots a_{n-1}$. Note $i_{t',t} \circ i_{t'',t'} = i_{t'',t}$ and that these mappings are injective. Related to these are the maps $j_{t',t} = \psi_t \circ i_{t',t} \circ \psi_{t'}^{-1} : \mathcal{A}^{\otimes|t'|} \to \mathcal{A}^{\otimes|t|}$. One verifies that for these maps also $j_{t',t} \circ j_{t'',t'} = j_{t'',t}$, and that for $t = (t_1, \ldots, t_n)$ and $t' = (t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_n)$ we have $j_{t',t}(a_1 \otimes \cdots \otimes a_{n-1}) = a_1 \otimes \cdots a_{j-1} \otimes \Delta a_j \otimes a_{j+1} \otimes \cdots a_{n-1}$, i. e., $j_{t',t}$ decomposes the increment from t_{j-1} to t_{j+1} into the increments from t_{j-1} to t_j and from t_j to t_{j+1} via the coproduct, whereas for $t = (t_1, \ldots, t_n)$ and $t' = (t_1, \ldots, t_{n-1})$ we have $j_{t',t}(a_1 \otimes \cdots \otimes a_{n-1}) = a_1 \otimes \cdots a_{n-1} \otimes 1$, i. e., for increments corresponding to times larger than the maximal time represented in t' the unit element 1 is adjoined.

Proposition 4.8.2 *The algebras $(\mathcal{A}_t, m_t)$ and the mappings $i_{t,t'}$ form an inductive system and thus define an algebra $\tilde{\mathcal{A}}$ with injective algebra homomorphisms $i_t : \mathcal{A}_t \to \tilde{\mathcal{A}}$ such that $i_t \circ i_{t',t} = i_{t'}$.*

Proof: To prove the proposition we have to show $m_t \circ (i_{t',t} \otimes i_{t',t}) = i_{t',t} \circ m_{t'}$, i. e., that the maps $i_{t',t}$ are algebra homomorphisms with respect to the multiplications m_t, $m_{t'}$. It is clear that $j_{t',t}$ is an algebra homomorphism, since the coproduct Δ is one, and adjoining the unit element is one, too. Now

$$\begin{aligned} m_t \circ (i_{t',t} \otimes i_{t',t}) &= \psi_t^{-1} \circ m^{\otimes|t|} \circ (\psi_t \otimes \psi_t) \circ ((\psi_t^{-1} \circ j_{t',t} \circ \psi_{t'}) \otimes (\psi_t^{-1} \circ j_{t',t} \otimes \psi_{t'})) \\ &= \psi_t^{-1} \circ m^{\otimes|t|} \circ (j_{t',t} \otimes j_{t',t}) \circ (\psi_{t'} \otimes \psi_{t'}) = \psi_t^{-1} \circ j_{t',t} \circ m^{\otimes|t'|} \circ (\psi_{t'} \otimes \psi_{t'}) \\ &= i_{t',t} \circ m_{t'}. \end{aligned}$$

∎

This algebra plays the rôle of the canonical representation of the process. If $\mathcal{A}$ is generated by $a_1, \ldots, a_n$, then $\tilde{\mathcal{A}}$ is spanned by the polynomials in $\{(a_1, t), \ldots, (a_n, t); t \in \mathbb{R}_+\}$ (we also write a_t for a pair (a, t), $a \in \mathcal{A}$, $t \in \mathbb{R}_+$). For the Hopf algebras $\mathcal{A}$ the algebraic relations are sufficient to order the factors such that the index corresponding to time increases from the left to the right, i. e., $\tilde{\mathcal{A}}$ is spanned by the monomials

$$(a_1, t_1)^{k_{11}} \cdots (a_n, t_1)^{k_{1n}} \cdots (a_1, t_m)^{k_{m1}} \cdots (a_n, t_m)^{k_{mn}} \tag{4.11}$$

where $t = (t_1, \cdots, t_m)$ varies in T.

The functional on $\tilde{\mathcal{A}}$ can now be defined with the help of the maps ψ_t. Let $(\Phi_t)_{t \in \mathbb{R}_+}$ be the family of functionals defined in Section 4.5. Then we define

$$\tilde{\Phi}_t(a) = \Phi_{t_1} \otimes \Phi_{t_2 - t_1} \otimes \cdots \otimes \Phi_{t_n - t_{n-1}}(\psi_t(a)) \tag{4.12}$$

for $a \in \mathcal{A}_t$, $t \in T$.

Proposition 4.8.3 *The functionals $\{\tilde{\Phi}_t; t \in T\}$ determine a unique functional $\tilde{\Phi}$ on $\tilde{\mathcal{A}}$ such that $\tilde{\Phi} \circ i_t = \tilde{\Phi}_t$.*

Proof: We have to show that $\tilde{\Phi}_{t'} = \tilde{\Phi}_t \circ i_{t',t}$ for all $t' \subseteq t$. The functionals Φ_t in Section 4.5 form a convolution semi-group, i. e., $(\Phi_t \otimes \Phi_s) \circ \Delta = \Phi_{t+s}$, and, more generally, for $t = (t_1, \ldots, t_n)$ and $t' = (t_{\nu_1}, \cdots, t_{\nu_k}) \subseteq t$,

$$\Phi_{t_{\nu_1}} \otimes \cdots \otimes \Phi_{t_{\nu_k} - t_{\nu_{k-1}}} = \Phi_{t_1} \otimes \Phi_{t_2 - t_1} \otimes \cdots \otimes \Phi_{t_n - t_{n-1}} \circ j_{t',t}.$$

The proposition now follows from the definition of $\tilde{\Phi}_t = (\Phi_{t_1} \otimes \cdots \otimes \Phi_{t_n - t_{n-1}}) \circ \psi_t$ and of $i_{t',t}$. ∎

We define a family of maps $(\varphi_t)_{t \in \mathbb{R}_+}$, $\varphi_t : \mathcal{A} \to \tilde{\mathcal{A}}$ by

$$\varphi_t(a) = \pi((a, t)) = a_t. \tag{4.13}$$

We have not considered *-structures here, but one immediately sees that $(\varphi_t)_{t \in \mathbb{R}_+}$ is a quantum stochastic process with respect to all other requirements, i. e., it is an algebra homomorphism from algebra $\mathcal{A}$ to the algebra $\tilde{\mathcal{A}}$ with the unital functional $\tilde{\Phi}$. The one-dimensional distributions of this process are exactly the functionals $\Phi_t = \tilde{\Phi} \circ \varphi_t$.

The following theorem summarizes the main results of this section.

Theorem 4.8.4 *$\{\varphi_t : \mathcal{A} \to (\tilde{\mathcal{A}}, \tilde{\Phi}); t \in \mathbb{R}_+\}$ is a quantum process with independent and stationary increments, i. e., it satisfies*

1. *$\varphi_t : \mathcal{A} \to \tilde{\mathcal{A}}$ is a unital algebra homomorphism for all $t \in \mathbb{R}_+$,*
2. *the functionals $\Phi_t = \tilde{\Phi} \circ \varphi_t$ form a convolution semi-group, i. e.,*

$$(\Phi_t \star \Phi_s)(a) = (\Phi_t \otimes \Phi_s)(\Delta a) = \Phi_{t+s}(a)$$

for all $a \in \mathcal{A}$, $t, s \in \mathbb{R}_+$.

Example: For the q-affine group $\tilde{\mathcal{A}}$ is spanned by

$$\{a_{t_1}^{n_1} b_{t_1}^{m_1} \cdots a_{t_p}^{n_p} b_{t_p}^{m_p} : p \in \mathbb{N}, t_1 < t_2 < \ldots < t_p \in \mathbb{R}_+, n, m \in \mathbb{N}^p\}$$

The commutation relations between elements corresponding to the same time t remain unchanged, those between elements corresponding to different times $s < t$ are determined by

$$\begin{aligned} a_t \cdot a_s &= a_s \cdot a_t, & b_t \cdot a_s &= a_s \cdot b_t + \beta b_s, \\ a_t \cdot b_s &= b_s \cdot a_t - \beta b_s, & b_t \cdot b_s &= q^{-1} b_s \cdot b_t + (1 - q^{-1}) b_s^2. \end{aligned}$$

4.9 Conclusion

We have shown how to construct stochastic multiplicative processes on quantum groups. For the analogue of processes with independent and stationary increments we proved a Feynman-Kac type formula. Appell systems have been defined and shown to be solutions of evolution equations.

The positivity of the constructed process is not *a priori* clear, but depends on the choice of the identification of $\mathcal{A}$ and $\mathbb{R}[x_1, \ldots, d]$. Nevertheless, the proof of the Feynman-Kac formula doesn't require positivity, while the other results presented do not depend on it. Finding good criteria for which cases the obtained process will be positive is an open problem, for some results about positive functionals on quantum groups see [Koo91].

Chapitre 5

Diffusions on Braided Spaces

Uwe Franz **René Schott**[5]

Résumé

The notion of q-Brownian motion introduced by Majid is extended to braided spaces corresponding to a generic R-matrix, and combined with the theory of quantum probability. This leads to a definition of diffusions on these spaces. The corresponding heat equations (difference-differential equations) are solved in terms of Appell polynomials (i.e. shifted moment systems). Some examples of interest for applications are given.

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5.1 Introduction

The applications of diffusions in physics go far beyond the description of the physical phenomenon they are named after. Functional integrals can be used to solve partial differential equations, cf. the celebrated Feynman-Kac formula. Wiener integrals are very close to Feynman path integrals. Another interesting application of diffusions is the stochastic mechanics of Nelson [Nel66].

We will present an approach to diffusions on braided spaces here. Diffusions on manifolds are characterized by two properties, the first being their Markov property, i.e. that at every instant t they start again, and their evolution does not depend on their history, but only on their distribution at time t. This gives rise to a semi-group and this property will also play a central role for the study of diffusions on braided spaces. The other property, i.e. that they have continuous sample paths, does not have a direct counterpart on braided spaces. We replace it by the condition that diffusions can be obtained as the limit of a (simple) random walk.

The motivations for our approach come from two directions. First, there is Majid's random walk approach to Brownian motion on the braided line and on anyspaces [Maj93d, MRP94]. We extend his definition to (pseudo-) diffusions on multi-dimensional braided spaces and, using coalgebraic limit theorems due to Schürmann [Sch93], give their explicit form. This allows to consider semigroups of functionals and Markovian transition operators, as well as the associated heat equations, and to introduce Appell systems as their polynomial solutions. Heat kernels, i.e. the densities of the functionals are also considered.

The second ingredient, M. Schürmann's theory of quantum Lévy processes, comes into play to assure the existence of the associated processes, e.g. as operators on a Hilbert space. We propose a definition of *-structures for braided Hopf algebras (differing from those due to S. Majid), and give several examples that satisfy our axioms. Then we generalize M. Schürmann's definitions and a result assuring the positivity of the convolution of positive functionals under certain conditions to braided groups.

To distinguish the two levels of the constructions, we call the functionals and semigroups obtained via S. Majid's construction pseudo-diffusions, and reserve the name diffusion for the case where the processes can be realised as operators and are Lévy processes (i.e. independent increment processes) in the sense of M. Schürmann.

This paper starts with a brief summary of the definition of braided spaces and their main properties in Section 5.2. We also introduce some basic notions of quantum probability such as independence and Lévy processes, including a few generalizations that are necessary to apply the existing theory to braided spaces.

The next section (Section 5.3) shows how (pseudo-) diffusion processes can be constructed on generic braided spaces associated to a pair of matrices (R, R'). The approach is based on the idea that diffusions can be approximated by random walks on lattices. This replaces the classical condition of the continuity of sample paths. We obtain the explicit form of the limiting process as the chosen time interval and the lattice parameters tend to zero.

Section 5.4 investigates the relation between the diffusion processes and associated evolution equations. These equations are of the form $(\partial_t - L)u = 0$, where L is an ope-

rator consisting of linear and quadratic terms in the braided-partials ∂^i. Appell systems (introduced as shifted moment polynomials) are shown to be solutions of the evolution equation. We investigate their properties.

In Section 5.5 we discuss briefly how density functions can be introduced.

We close with a few final remarks in Section 5.6.

5.2 Preliminaries

5.2.1 Braided spaces

Let us recall the notion of a braided group. This is $(B, m, 1, \underline{\Delta}, \epsilon, \underline{S}, \Psi)$ where $(B, m, 1)$ is a unital associative algebra, $(B, \underline{\Delta}, \epsilon)$ a coalgebra, Ψ a braiding (satisfies the braid relations), and $\underline{S}$ an antipode (i.e. $m \circ (\mathrm{id} \otimes \underline{S}) \circ \underline{\Delta} = m \circ (\underline{S} \otimes \mathrm{id}) \circ \underline{\Delta} = 1 \circ \epsilon$). The difference that distinguishes braided groups from quantum groups (Hopf algebras) is that now $\underline{\Delta}$ is braided-multiplicative, i.e.

$$\underline{\Delta} \circ m = m_\psi^\otimes \circ \underline{\Delta},$$

where $m_\psi^\otimes$ is defined by $m_\psi^\otimes((a \otimes c)(b \otimes d)) = a\Psi(c \otimes b)d$. For a more detailed and elegant introduction see e.g. [Maj93a, Maj96].

Braided spaces are braided groups with a particular form of the product and coproduct, see below. They are a generalisation of superspaces, there are braided lines, braided planes, braided matrices, etc., all in analogy with superlines, superplanes, etc. Braided spaces are the elementary building blocks of the 'braided world' or braided linear algebra.

For a pair of invertible matrices (R, R') that satisfy

(i) $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ (i.e. the Quantum-Yang-Baxter equation),

(ii) $R_{12}R_{13}R'_{23} = R'_{23}R_{13}R_{12}$, $R_{23}R_{13}R'_{12} = R'_{12}R_{13}R_{23}$,

(iii) $(PR + 1)(PR' - 1) = 0$,

(iv) $R_{21}R'_{12} = R'_{21}R_{12}$,

where P is the permutation matrix ($\mathbf{x}_2 \otimes \mathbf{x}_1 P = \mathbf{x}_1 \otimes \mathbf{x}_2$), and $R_{12} = R \otimes \mathrm{id}$, $R_{23} = \mathrm{id} \otimes R$, etc., there exist two braided Hopf algebras $V(R, R')$ and $V^{\vee}(R, R')$ with generators $v_1, \ldots, v_n$ and $x_1, \ldots, x_n$ and relations

- for $V(R, R')$:

$$\begin{aligned}
\underline{\Delta} v^i &= v^i \otimes 1 + 1 \otimes v^i, \\
\Psi(v^i \otimes v^j) &= R^i{}_k{}^j{}_l v^l \otimes v^k, \\
v^i v^j &= R'^i{}_k{}^j{}_l v^l v^k, \\
\underline{S}(v^i) &= -v_i, \\
\epsilon(v^i) &= 0,
\end{aligned}$$

- for $V^{\vee}(R,R')$:

$$\begin{aligned}
\underline{\Delta} x_i &= x_i \otimes 1 + 1 \otimes x_i, \\
\Psi(x_i \otimes x_j) &= x_l \otimes x_k R^{k}{}_{i}{}^{l}{}_{j}, \\
x_i x_j &= x_l x_k R'^{k}{}_{i}{}^{l}{}_{j}, \\
\underline{S}(x_i) &= -x_i, \\
\epsilon(x_i) &= 0,
\end{aligned}$$

see [Maj93b, Theorem 1]. Here the first two indices (i,k) correspond to the first factor in the tensor product, and the last two (j,l) to the second factor. We use the summation convention, i.e. indices that appear twice (once as subscript and once as superscript) are summed over. In a shorter notation this becomes

- for $V(R,R')$: $\underline{\Delta}\mathbf{v} = \mathbf{v} \otimes 1 + 1 \otimes \mathbf{v}$, $\Psi(\mathbf{v}_1 \otimes \mathbf{v}_2) = R_{12}\mathbf{v}_2 \otimes \mathbf{v}_1$, $R'_{12}\mathbf{v}_2\mathbf{v}_1 = \mathbf{v}_1\mathbf{v}_2$, $\underline{S}(\mathbf{v}) = -\mathbf{v}$, $\epsilon(\mathbf{v}) = 0$,
- for $V^{\vee}(R,R')$: $\underline{\Delta}\mathbf{x} = \mathbf{x} \otimes 1 + 1 \otimes \mathbf{x}$, $\Psi(\mathbf{x}_1 \otimes \mathbf{x}_2) = \mathbf{x}_2 \otimes \mathbf{x}_1 R_{12}$, $\mathbf{x}_2\mathbf{x}_1 R'_{12} = \mathbf{x}_1\mathbf{x}_2$, $\underline{S}(\mathbf{x}) = -\mathbf{x}$, $\epsilon(\mathbf{x}) = 0$.

The braided Hopf algebras $V(R,R')$ and $V^{\vee}(R,R')$ are called the *braided vector space* and the *braided covector space* associated to (R,R'). For a detailed development of their properties see also [Maj93c, Maj93a, Maj96]. We will summarize here only some facts that are used in this presentation.

Let $\mathcal{S}\{1,\ldots,n\}$ be the set of all finite sequences composed of the elements $1,\ldots,n$, i.e. $\mathcal{S}\{1,\ldots,n\} = \{\emptyset; 1,\ldots,n; 11, 12, \ldots, nn; 111, \ldots\}$. Set $\mathbf{v}^a = v^{a_k}\cdots v^{a_1}$ if $a = (a_1 \cdots a_k) \in \mathcal{S}\{1,\ldots,n\}\backslash\{\emptyset\}$, and $\mathbf{v}^{\emptyset} = 1$, and analogously, but in reversed order, for $\mathbf{x}_a = x_{a_1}\cdots x_{a_k}$. Then $\{\mathbf{v}^a; a \in \mathcal{S}\{1,\ldots,n\}\}$, $\{\mathbf{x}_a; a \in \mathcal{S}\{1,\ldots,n\}\}$ span $V(R,R')$ and $V^{\vee}(R,R')$, respectively. Notice that in general these sets do not form a basis. But a basis can easily be extracted from them, if one orders them, and eliminates all elements that are linearly dependent on those that precede them.

The algebras $V(R,R')$ and $V^{\vee}(R,R')$ are $\mathbb{N}$-graded with

$$\begin{aligned}
V(R,R')^{(r)} &= \text{span } \{\mathbf{v}^a; a \in \mathcal{S}^r\{1,\ldots,n\}\} \\
V^{\vee}(R,R')^{(r)} &= \text{span } \{\mathbf{x}_a; a \in \mathcal{S}^r\{1,\ldots,n\}\},
\end{aligned}$$

where $\mathcal{S}^r\{1,\ldots,n\} \subseteq \mathcal{S}\{1,\ldots,n\}$ contains only the sequences of length r. This allows us to define a scaling $s(\lambda) : V^{\vee}(R,R') \to V^{\vee}(R,R')$ by $s(\lambda)a = \lambda^{\deg(a)}a$ for homogeneous elements $a \in V^{\vee}(R,R')$ or $V(R,R')$.

$V(R,R')$ and $V^{\vee}(R,R')$ are mutually dual with the definition

$$< \mathbf{v}^a, \mathbf{x}_a >= \delta_{|a|,|b|}[|a|, R]^{!a}_{b}$$

where $|a|$, $|b|$ is the length of $a, b \in \mathcal{S}\{1,\ldots,n\}$ and

$$[m;R] = 1 + (PR)_{12} + (PR)_{12}(PR)_{23} + \cdots + (PR)_{12}\cdots(PR)_{m-1,m}, \tag{5.1}$$

$$[m;R]! = [2;R]_{m-1,m}[3;R]_{m-2,m-1,m}\cdots[m;R]_{1,\ldots,m} \tag{5.2}$$

(cf. [Maj96, Section 5]). This defines also an action of $V(R,R')$ on $V^{\vee}(R,R')$ (and vice versa) by

$$\begin{aligned}\rho(v^i) &= \partial^i = (< v^i, \cdot > \otimes \mathrm{id}) \circ \underline{\Delta},\\ \rho(x_i) &= \hat{\partial}_i = (< \cdot, x_i > \otimes \mathrm{id}) \circ \underline{\Delta}.\end{aligned}$$

Braided exponentials $\exp(\mathbf{x}|\mathbf{v})$ are defined as solutions of

$$\begin{aligned}\partial^i \exp(\mathbf{x}|\mathbf{v}) &= \exp(\mathbf{x}|\mathbf{v})v^i, \quad & (\epsilon \otimes \mathrm{id})\exp(\mathbf{x}|\mathbf{v}) &= 1\\ \exp(\mathbf{x}|\mathbf{v})\hat{\partial}_i &= x_i \exp(\mathbf{x}|\mathbf{v}), \quad & (\mathrm{id} \otimes \epsilon)\exp(\mathbf{x}|\mathbf{v}) &= 1.\end{aligned}$$

If the braided integers $[m;R]$ are all invertible, the $\exp(\mathbf{x}|\mathbf{v})$ is given by

$$\exp(\mathbf{x}|\mathbf{v}) = \sum \mathbf{x}_a([m;R]!^{-1})^a_b \mathbf{v}^b.$$

We will assume that this is the case in what follows, or rather that there exists a braided-exponential $\exp(\mathbf{x}|\mathbf{v}) = \sum \mathbf{x}_a F(m;R)^a_b \mathbf{v}^b$, see [Maj96, Section 5.4].

5.2.2 Quantum probability and quantum Lévy processes

A quantum probability space is defined as a pair $(\mathcal{A},\Phi)$ consisting of a *-algebra $\mathcal{A}$ and a state (i.e. a normed positive linear functional) Φ on $\mathcal{A}$, and a quantum random variable j over a quantum probability space $(\mathcal{A},\Phi)$ on a *-algebra $\mathcal{B}$ is a *-algebra homomorphism $j : \mathcal{B} \to \mathcal{A}$. In particular, every element a of $\mathcal{A}$ defines a random variable on the free algebra $\mathbb{C} < z, z^* >$ with two generators z, z^* (with the obvious *-structure), simply set $j(z) = a$, $j(z^*) = a^*$, and extend as a *-algebra homomorphism. If a is self-adjoint, then we can take $\mathcal{B} = \mathbb{C}[z]$, with the *-structure defined by $z^* = z$, i.e. the algebra of complex-valued polynomials on the real line.

A central notion to this work is that of independence. Independence in quantum probability is closely related to the notion of products of (*-)algebras, as e.g. the tensor product or the free product. The kind of independence we use here is related to tensor product of algebras in braided categories, it generalizes the tensorial independence introduced by M. Schürmann for braidings that arise from an action α and a coaction γ of some group algebra $\mathbb{C}\mathbb{L}$ as $\Psi = (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma \otimes \mathrm{id})$ or $\Psi = (\mathrm{id} \otimes \alpha) \circ (\tau \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \gamma)$. For a more general definition of independence see [ALV94], for a general analysis of the notion of independence we refer to [Sch95a, Spe]

Definition 5.2.1 *The quantum random variables $j_1, j_2, \cdots j_n : \mathcal{B} \to \mathcal{A}$ are called Ψ-independent (w.r.t. the state Φ), if*

(i)

$$\Phi\Big(j_{\sigma(1)}(b_1)\cdots j_{\sigma(n)}(b_n)\Big) = \Phi(j_{\sigma(1)}(b_1))\cdots\Phi(j_{\sigma(n)}(b_n))$$

for all permutations $\sigma \in \mathcal{S}(n)$ and all $b_1,\ldots,b_n \in \mathcal{B}$, and

(ii)

$$m_{\mathcal{A}} \circ (j_k \otimes j_l) = m_{\mathcal{A}} \circ (j_l \otimes j_k) \circ \Psi$$

for all $1 \leq k < l \leq n$.

Remark: If $\mathcal{A}$ is commutative, then (i) is equivalent to the condition used in classical probability theory:

(i')

$$\Phi\Big(j_1(b_1) \cdots j_n(b_n)\Big) = \Phi(j_1(b_1)) \cdots \Phi(j_n(b_n))$$

for all $b_1, \ldots, b_n \in \mathcal{B}$.

We will call $j_1, \ldots, j_n$ pseudo-(Ψ-)independent, if they satisfy only (i') and (ii). For many of the algebraic aspects this behaves like 'true' independence.

Let us investigate, when two quantum random variables on a braided space $V^{\vee}$ are (pseudo-) independent. Condition (ii) implies that $j_1(x_i)$ and $j_2(x_j)$ satisfy the braid relations. Therefore we can assume $\mathcal{A} = V^{\vee} \underline{\otimes} V^{\vee}$, and $j_1(u) = u \otimes 1$, $j_2(u) = 1 \otimes u$ (take $\hat{\Phi} = \Phi \circ (j_1 \otimes j_2)$. Condition (i'), i.e. pseudo-independence, is now equivalent to the fact that $\hat{\Phi}$ can be factorized into $\hat{\Phi} = \phi^{(2)} \otimes \phi^{(1)}$, where $\phi^{(i)} = \Phi \circ j_i$, $i = 1, 2$. Thus the joint distribution of j_1 and j_2 is uniquely determined by the marginal distributions.

Condition (i) is stronger, for independence in the sense of the preceding definition we need a kind of invariance or commutativity condition in addition to the factorisation, namely $\phi^{(1)} \otimes \phi^{(2)}(u \otimes v) = \phi^{(2)} \otimes \phi^{(1)}\Big(\Psi(v \otimes u)\Big)$, or $\phi^{(1)} \otimes \phi^{(2)} = \Psi\Big(\phi^{(2)} \otimes \phi^{(1)}\Big)$, i.e. $\underline{\varepsilon} \otimes \phi^{(2)}$ and $\phi^{(1)} \otimes \underline{\varepsilon}$ have to commute.

We will study here time and space homogeneous Markov processes, i.e. Lévy processes.

Definition 5.2.2 *A quantum Lévy process on a (braided) *-Hopf algebra $\mathcal{B}$ is a quantum stochastic process $\{j_t : \mathcal{B} \to (\mathcal{A}, \Phi) | t \in \mathbb{R}_+\}$, such that*

(i) $\{j_t | t \in \mathbb{R}_+\}$ *has independent increments, i.e.* $j_{s_1 t_1}, \ldots, j_{s_n t_n}$ *are (Ψ-)independent for all* $0 \leq s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n$, *where the increments are defined by* $j_{st} = (j_s \circ S) \star j_t = m_{\mathcal{A}} \circ ((j_s \circ S) \otimes j_t) \circ \Delta$,

(ii) the distribution $\Phi \circ j_{st}$ *of an increment* j_{st} *depends only on* $t - s$ *(stationarity),*

(iii) j_{st} *converges to* j_{ss} *in distribution for* $t \searrow s$ *(weak continuity),*

(iv) $j_{su} = j_{st} \star j_{tu} \Big(= m_{\mathcal{A}} \circ (j_{st} \otimes j_{tu}) \circ \Delta_{\mathcal{B}} \Big)$ *for all* $0 \leq s \leq t \leq u$.

We shall speak of pseudo-Lévy processes, if the increments are only pseudo-independent. In this case we will not insist on the positivity of the state either, we study pseudo-Lévy processes (and pseudo-diffusions, see below) only to illustrate the algebraic part of the theory.

It follows from the general theory (cf. [Sch93]) that a Lévy process is uniquely determined by its marginal distributions $\varphi_t = \Phi \circ j_t$, that the marginal distribution form a

convolution semigroup, i.e. $\varphi_0 = \underline{\varepsilon}$, $\varphi_s \star \varphi_t = (\varphi_s \otimes \varphi_t) \circ \Delta = \varphi_{s+t}$, and that there exists a unique functional L, called generator, such that $\varphi_t = \exp(tL)$. Vice versa, the semigroup, and thus the process, is also uniquely determined by its generator. All this is also true for pseudo-Lévy processes.

Let us now look at the differences. The independence of the increments implies $\Psi(e^{tL} \otimes e^{sL}) = e^{sL} \otimes e^{tL}$, and thus $\Psi(L \otimes L) = L \otimes L$, i.e. $L \otimes \underline{\varepsilon}$ and $\underline{\varepsilon} \otimes L$ commute (in the convolution algebra of functionals on $V\check{}\underline{\otimes}V\check{}$). Also, since the semigroup is positive, L is conditionally positive (i.e. positive on the kernel of $\underline{\varepsilon}$), to see this just differentiate w.r.t. t, and hermitian. In the sequel, when we speak of Lévy processes, we will always assume that our states ϕ (on $\mathcal{A}$) are Ψ-invariant in the sense that $\Psi(\phi \otimes u) = u \otimes \phi$ for all functionals $u : \mathcal{B} \to \mathbb{C}$, i.e. that we can pull it across braidings.

If L is a hermitian, conditionally positive, and Ψ-invariant functional, then all the convolution powers of L are positive (see Lemma 5.2.4), and the semigroup associated to L is positive. Thus there exists a Lévy process with $\varphi_t = \exp(tL)$ as marginal distributions.

We summarize this in the following proposition.

Proposition 5.2.3 *Let $\mathcal{A}$ be a (braided) (*-)Hopf algebra (cf. Subsection 5.2.3).*

(i) There is a one-to-one correspondence between pseudo-Lévy processes $\{j_t | t \in \mathbb{R}_+\}$, normed convolution semi-groups $\{\varphi_t | t \in \mathbb{R}_+\}$, $\varphi_t = \exp(tL) = \Phi \circ j_t$, and functionals $L = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\Big|_{t=0}$ with $L(1) = 0$ on $\mathcal{A}$.

(ii) There is also a one-to-one correspondence between Lévy processes, normed positive convolution semigroups of Ψ-invariant states, and Ψ-invariant, hermitian, conditionally positive functionals.

5.2.3 A remark on braided *-Hopf algebras

S. Majid has studied *-structures on braided Hopf algebras in a series of articles[Maj94, Maj95a, Maj95c]. An essential result of his work is a new set of axioms for the braided case. But we found it necessary to use different axioms here, in particular, it is important for our purposes that the coproduct is a *-algebra homomorphism.

We fix the *-structure in the tensor product $\mathcal{A}\underline{\otimes}\mathcal{A}$ by the condition that the canonical inclusions $\mathcal{A} \xrightarrow{i_1} \mathcal{A}\underline{\otimes}\mathcal{A} \xleftarrow{i_2} \mathcal{A}$ are *-algebra homomorphism, i.e. $(1 \otimes a)^* = 1 \otimes a^*$ and $(a \otimes 1)^* = a^* \otimes 1$ for all $a \in \mathcal{A}$. This implies $(a \otimes b)^* = \psi(b^* \otimes a^*)$ for all $a, b \in \mathcal{A}$. Then we say that $\mathcal{A}$ is a *(braided) *-Hopf algebra* if $(\mathcal{A}, *)$ is a *-algebra, and $\underline{\Delta}$ and $\underline{\varepsilon}$ are *-algebra homomorphism (with the *-structure of $\mathcal{A}\underline{\otimes}\mathcal{A}$ defined as above!).

For examples of braided *-Hopf algebras satisfying these conditions see Subsection 5.3.2.

Let us outline the main steps and ideas of their construction. We start with a braided-covector space $V\check{}(R, R')$ with R, R' of real type I, i.e. $\overline{R^i{}_j{}^k{}_l} = R^l{}_k{}^j{}_i$. S. Majid has defined a map $*$ for this case that goes from $V\check{}(R, R')$ to $V(R, R')$, and vice versa, cf. [Maj95a, Lemma 2.1]. The free product $V(R, R') * V\check{}(R, R')$ thus becomes a *-algebra with this map, and the coproducts of $V(R, R')$ and $V\check{}(R, R')$ define a *-algebra homomorphism $\underline{\Delta}_0 : V(R, R') * V\check{}(R, R') \to (V(R, R')\underline{\otimes}V(R, R')) * (V\check{}(R, R')\underline{\otimes}V\check{}(R, R'))$. Now it turns

out that $(V(R,R') * V^{\check{}}(R,R'))\underline{\otimes}(V(R,R') * V^{\check{}}(R,R'))$ can be obtained as a quotient of $(V(R,R')\underline{\otimes}V(R,R')) * (V^{\check{}}(R,R')\underline{\otimes}V^{\check{}}(R,R'))$ in such a way that the canonical projection is a *-algebra homomorphism, and that $V(R,R') * V^{\check{}}(R,R')$ is a braided *-Hopf algebra in the sense of our definition with $\underline{\Delta} = \pi \circ \underline{\Delta}_0$. Set $x_{1i} = x_i \otimes 1$, $v_1^i = v^i \otimes 1$, $x_{2i} = 1 \otimes x_i$, and $v_2^i = 1 \otimes v^i$, then the relations defining the braided *-Hopf algebra structure of $V(R,R') * V^{\check{}}(R,R')$ are

$$
\begin{array}{rclrcll}
x_i x_j & = & x_l x_k R'^{k}{}_{i}{}^{l}{}_{j}, & v^j v^i & = & R'^{j}{}_{l}{}^{i}{}_{k} v^k v^l, & \\
x_i^* & = & v^i, & (v^i)^* & = & x_i, & \\
\underline{\Delta}(x_i) & = & x_{1i} + x_{2i}, & \underline{\Delta}(v^i) & = & v_1{}^i + v_2{}^i, & \\
\multicolumn{7}{l}{\underline{S}(x_i) = -x_i, \quad \underline{S}(v^i) = -v^i, \quad \underline{\varepsilon}(x_i) = \underline{\varepsilon}(v^i) = 0} \\
x_{\alpha,i} x_{\alpha,j} & = & x_{\alpha,l} x_{\alpha,k} R'^{k}{}_{i}{}^{l}{}_{j}, & v_\alpha{}^j v_\alpha{}^i & = & R'^{j}{}_{l}{}^{i}{}_{k} v_\alpha{}^k v_\alpha{}^l, & \alpha = 1,2, \\
(x_{\alpha,i})^* & = & v_\alpha{}^i, & (v_\alpha{}^i)^* & = & x_{\alpha,i}, & \alpha = 1,2, \\
x_{2i} x_{1j} & = & R^{k}{}_{i}{}^{l}{}_{j} x_{1l} x_{2k}, & v_2{}^i x_{1j} & = & R^{l}{}_{j}{}^{i}{}_{k} x_{1l} v_2{}^k & \\
x_{2i} v_1{}^j & = & \tilde{R}^{k}{}_{i}{}^{j}{}_{l} v_1{}^l x_{2k}, & v_1{}^j v_2{}^i & = & (R^{-1})^{j}{}_{k}{}^{i}{}_{l} v_2{}^k v_i{}^l &
\end{array}
$$

where $\tilde{R}$ is the second inverse of R, i.e. the solution of $\sum \tilde{R}^{i}{}_{k}{}^{j}{}_{l} R^{k}{}_{n}{}_{j}^{m} = \sum R^{i}{}_{k}{}^{j}{}_{l} \tilde{R}^{k}{}_{n}{}^{m}{}_{j} = \delta^i{}_n \delta^m{}_l$. We observe that, with the definitions $x_i^* = v_i$, $x_{\alpha,i}^* = v_\alpha{}^i$, the 'v-v' relations follow from the corresponding 'x-x' relations, and the second line of the braid relations from the first (where we use that R, R' are real type I).

Note also that $\Psi(x_k \otimes \sum x_{1i} v_1{}^i) = \sum x_{1i} v_1{}^i \otimes x_k$, $\Psi(v^k \otimes \sum x_{1i} v_1{}^i) = \sum x_{1i} v_1{}^i \otimes v^k$. Thus the functionals that are non-zero only on $\sum x_i v^i$ are Ψ-invariant.

We prove the following.

Lemma 5.2.4 *Let $\mathcal{A}$ be a braided *-bialgebra (in the sense of the preceding definition, i.e. the *-structure in the braided product algebra $\mathcal{A}\underline{\otimes}\mathcal{A}$ is given by $\Psi \circ (* \otimes *) \circ \tau$, and $\underline{\Delta}$ is a *-algebra homomorphism), and let ϕ a Ψ-invariant positive functional on $\mathcal{A}$, and θ a positive functional on $\mathcal{A}$. Then $\phi \star \theta = (\phi \otimes \theta) \circ \underline{\Delta}$ is also positive.*

Proof: Let $a \in \mathcal{A}$, and $\underline{\Delta}(a) = \sum_i a_i^{(1)} \otimes a_i^{(2)}$ (Sweedler's notation). Then $\underline{\Delta}(a^*) = \sum_j \Psi\left(\left(a_j^{(2)}\right)^* \otimes \left(a_j^{(1)}\right)^*\right)$ and therefore $(\phi \otimes \theta) \circ \underline{\Delta}(a^* a) = (\phi \otimes \theta) \circ (m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes \mathrm{id}) \circ (\Psi \otimes \mathrm{id} \otimes \mathrm{id}) \left(\sum_{i,j} \left(a_j^{(2)}\right)^* \otimes \left(a_j^{(1)}\right)^* \otimes a_i^{(1)} \otimes a_i^{(2)}\right) = \theta \circ (\phi \otimes m) \circ (\Psi \circ \mathrm{id}) \circ (\mathrm{id} \otimes m \otimes \mathrm{id}) \left(\sum_{i,j} \left(a_j^{(2)}\right)^* \otimes \left(a_j^{(1)}\right)^* \otimes a_i^{(1)} \otimes a_i^{(2)}\right) = \sum_{i,j} \theta\left(\left(a_j^{(2)}\right)^* a_i^{(2)}\right) \phi\left(\left(a_j^{(1)}\right)^* a_i^{(1)}\right)$ is positive, since it is the Schur product of two positive definite matrices. ∎

5.3 A construction of (pseudo-) diffusions on braided spaces

Following Majid we first define a quantum random walk. Consider a classical random walk on the lattice. We assume that the process can only jump from a lattice site x to a finite number of other sites $x + d_i$, $i = 1, \ldots, k$. If at time t the particle is located at x,

then it will be at $x + d_i$ at time $t + \Delta t$ with a probability p_i, $0 \leq p_i \leq 1$, and naturally the probabilities add to one,

$$\sum_{i=1}^{k} p_i = 1.$$

In classical probability theory there corresponds an evolution operator to this, that maps a function (i.e. an observable) at time t to the corresponding function one time step later

$$f_t(x) \mapsto f_{t+\Delta t}(x) = \sum_{i=1}^{k} p_i f_t(x + d_i)$$

Let $\phi^{(d_i)}$ be the functional corresponding to evaluation of a function at d_i, i.e. $\phi^{(d_i)}$: $f(x) \mapsto f(d_i)$, then the above equation can be written as

$$f_t(x) \mapsto f_{t+\Delta t}(x) = (\phi \otimes \mathrm{id}) \circ \Delta f_t(x)$$

where $\phi = \sum p_i \phi^{(d_i)}$ and Δ stands for the usual cocommutative coproduct on $\mathbb{R}[x_1, \ldots, x_n]$. This can be done in the same manner on braided spaces and quantum groups, if one chooses an appropriate functional ϕ. The problem here (in dimension > 1) is that one can not define the $\phi^{(d_i)}$ simply as replacing the variables x_i by the components of the vectors $d_j = (d_{j1}, \ldots, d_{jn})$ due to the non-commutativity. One possibility is to bring the variables into some fixed order (i.e. fix a basis of monomials) and replace them then by the d_{ji}. We shall see later that in the limit only the choice on the quadratic and linear terms matters.

A functional $\phi^{(d)}$ that corresponds to evaluation at $d = (d_1, \ldots, d_n)$ should satisfy

F1: $\phi^{(d)}(1) = 1$, or equivalently $\phi^{(d)}|_{V^{\vee}(R,R')^{(0)}} = \epsilon$,

F2: evaluation on the generators gives the components of d, i.e.

$$\phi^{(d)}(x_i) = d_i,$$

for $i = 1, \ldots, n$,

F3: it has the correct behavior with respect to a rescaling of d, more precisely,

$$\phi^{(\lambda d)}(a) = \phi^{(d)}(s(\lambda)a) = \lambda^{\deg(a)} \phi^{(d)}(a)$$

for homogeneous $a \in V^{\vee}(R, R')$.

But these conditions do not determine $\phi^{(d)}$ uniquely.

A possible choice for $\phi^{(d)}$ with $d = (d_1, \cdots, d_n)$ is to use the braided-exponential

$$\phi^{(d)} = \exp(d|\mathbf{v}) = \sum_{a \in \mathcal{S}\{1,\ldots,n\}} d_a ([m; R]!^{-1})^a_b \mathbf{v}^b. \tag{5.3}$$

This definition makes sense for all cases where the braided-integers $[m; R]$ are invertible, and can also be used if this is not the case, if an appropriate braided-exponential is chosen. For the commutative case we have $\phi^{(d)} f(\mathbf{x}) = f(d)$ by Taylor's theorem, and similarly for

the braided line $\mathbb{R}_q$ by a q-Taylor theorem. On anyspace this agrees with definition of the Dirac δ-function, see [MRP94].

Another possibility is to use (positive) Dirac functionals, defined as functionals φ for which there exists a (*-)algebra $\mathcal{A}$, a state Φ on $\mathcal{A}$, and a (*-)algebra homomorphism $j : V^{\vee}(R, R') \to \mathcal{A}$ such that $\varphi = \Phi \circ j$ and the pair (j, j) is independent. Typically these Dirac functionals replace one generator by a real or complex number, and all other generators by zero, and lead to the same class of diffusions as the functional presented above.

Assume now that we have defined the functionals $\phi^{(d_i)}$, and chosen values for the transition probabilities p_i, i.e. fixed ϕ. Then this defines an evolution for functions, i.e. observables on our space by

$$f_t \mapsto f_{t+\Delta t} = (\phi \otimes \mathrm{id}) \circ \underline{\Delta} f_t$$

as before. As explained in [Maj93d] there is a one-to-one correspondence between functionals ϕ and transition operators[6] T, to pass from one to the other take

$$\begin{aligned} \phi &\mapsto T_\phi = (\phi \otimes \mathrm{id}) \circ \underline{\Delta}, \\ T &\mapsto \phi_T = \epsilon \circ T. \end{aligned}$$

The transition operators form a (discrete) semi-group with respect to composition, and the functionals with respect to convolution. The passage from one to the other is in fact a homomorphism, i.e. $T_\phi \circ T_\psi = T_{\phi\psi}$ and $\phi_T \phi_S = \phi_{TS}$.

We will give a limit theorem that arises when the lattice parameters d_i and the time step Δt go to zero. The discrete semi-groups (indexed by $\mathbb{N}$) will then converge to continuous ones (indexed by $\mathbb{R}_+$).

Theorem 5.3.1 *Let $d_1, \ldots, d_k$ and $p_1^0, \ldots, p_k^0$ be chosen such that $\sum_{i=1}^k p_i^0 = 1$ and $\phi = \sum_{i=1}^k p_i^0 \phi^{(d_i)}$ vanishes on $V^{\vee}(R, R')^{(1)}$, i.e. $\sum_{i=1}^k p_i^0 d_{ij} = 0$ for all $j = 1, \ldots, n$.*

Choose furthermore $p_1^1, \ldots, p_k^1$ such that $p_1(\mu) = p_1^0 + \mu p_1^1, \ldots, p_k(\mu) = p_k^0 + \mu p_k^1$ satisfy $0 \le p_i \le 1$ and $\sum_{i=1}^k p_i = 1$ for sufficiently small μ (this implies in particular $\sum_{i=1}^k p_i^1 = 0$).

Then for $\phi_N = \sum_{i=1}^k p_i (1/\sqrt{N}) \phi^{(d_i/\sqrt{N})}$ we have

$$\lim_{N\to\infty} (\phi_N)^N = \exp\Big(\sum_{k,l=1}^n a_{kl} v^k v^l + \sum_{j=1}^n b_i v^i\Big)$$

where $a_{kl} = \sum_{i=1}^k p_i^0 \sum_{r,s=1}^n d_{ir} d_{is} ([2; R]^{-1})_{kl}^{rs}$ and $b_j = \sum_{i=1}^k p_i^1 d_{ij}$.

Proof: We apply [Sch93, Theorem 6.1.1]. Let $k_N = N$, $\varphi_{Nk} = \phi_N$ for $k = 1, \ldots, N$. We have to check the conditions of the theorem.

(i) $\varphi_{N1}, \ldots, \varphi_{NN}$ commute since they are identical.

6. The (Markov) transition operators are here characterised by their left invariance, i.e. by $(T \otimes \mathrm{id}) \circ \underline{\Delta} = \underline{\Delta} \circ T$.

(ii) Let $c = \sum_{a\in\mathcal{S}\{1,\dots,n\}} c^a \mathbf{x}_a$ with some coefficients $c^a \in \mathbb{C}$. Then

$$\max_{1\le k\le N} |(\varphi_{Nk} - \epsilon)(c)| = \phi_N(c) - c^{\emptyset} = O(1/\sqrt{N})$$

goes to zero as $N \to \infty$.

(iii)

$$\begin{aligned}\sum_{k=1}^{N}(\varphi_{Nk} - \epsilon)(c) &= N(\phi_N(c) - c^{\emptyset}) \\ &= \sum_{i=1}^{k} p_i^1 \sum_{j=1}^{n} d_{ij} c^{\{j\}} + \sum_{i=1}^{k} p_i^0 \sum_{r,s=1}^{n} d_{ir} d_{is} c^{rs} + O(1/\sqrt{N})\end{aligned}$$

is bounded and tends to $(\sum_{k,l=1}^{n} a_{kl} v^k v^l + \sum_{j=1}^{n} b_i v^i)(c)$.

Thus $\lim_{N\to\infty} \left(\prod_{k=1}^{N} \varphi_{Nk}\right)(c) = \exp(\sum_{k,l=1}^{n} a_{kl} v^k v^l + \sum_{j=1}^{n} b_i v^i)(c)$. ∎

We will consider functionals that can be obtained in this way as diffusions on braided spaces and call $L = \sum_{k,l=1}^{n} a_{kl} v_k v_l + \sum_{j=1}^{n} b_i v^i$ their generator.

Definition 5.3.2 *We will call a semi-group of functionals* $\exp(tL)$ *on* $V^{\vee}(R, R')$ *a* (homogeneous) pseudo-diffusion, *if there exist parameters* $d_i = (d_{i1}, \dots, d_{in})$, p_i^0, p_i^1 $(i = 1, \dots, k)$ *such that the conditions of Theorem 5.3.1 are satisfied, and such that* $L = \sum_{k,l=1}^{n} a_{kl} v^k v^l + \sum_{j=1}^{n} b_i v^i$ *with* $a_{kl} = \sum_{i=1}^{k} p_i^0 \sum_{r,s=1}^{n} d_{ir} d_{is} ([2; R]^{-1})_{kl}^{rs}$ *and* $b_j = \sum_{i=1}^{k} p_i^1 d_{ij}$.

If the process associated to $\exp(tL)$ *is also a Lévy process, then we will call it a* diffusion.

Remarks:

1. We have formulated the limit theorem in terms of the functionals, analogous theorems for the associated transition operators hold also.

2. In the statements of Theorem 5.3.1 we assumed that the braided-integers are invertible. The theorem remains valid if this is not the case, as long as a braided-exponential $\exp(\mathbf{x}|\mathbf{v}) = \sum \mathbf{x}_a F(m; R)_b^a \mathbf{v}^b$ exists, the inverted braided-exponentials just have to be replaced by the $F(m; R)$.

5.3.1 Examples of (pseudo-) diffusions

Dimension $n = 1$: The braided line $\mathbb{R}_q$

Here $R = (q)$ and $R' = (1)$. This leads to the braiding $\Psi(x^m \otimes x^p) = q^{mp} x^p \otimes x^m$ (i.e. $x'x = qxx'$, where $x = x \otimes 1$, $x' = 1 \otimes x$). The diffusions on the braided line have the form

$$\phi_t = \exp\left[t\left(\frac{a^2}{1+q}v^2 + bv\right)\right].$$

Using the expansion $e^{-tu^2/2+xu} = \sum_{p=0}^{\infty} \frac{u^p}{p!} H_p(x,t)$ with the Hermite polynomials $H_p(x,t) = \sum_{k=0}^{[\frac{p}{2}]} \begin{pmatrix} p \\ 2k \end{pmatrix} \frac{(2k)!}{2^k k!} x^{p-2k}(-t)^k$ we find for the moments

$$\phi_t\left(\frac{x^p}{[p;q]!}\right) = \frac{H_p\left(bt, -\frac{2a^2t}{1+q}\right)}{p!}.$$

Dimension $n = 2$: The quantum plane $\mathbb{C}_q^{2|0}$

Here

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2-1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix}, \qquad R' = q^{-2}R.$$

This is the standard two-dimensional quantum plane. It has two generators x, y with relations $yx = qxy$ and braid statistics $x'x = q^2xx'$, $x'y = qyx'$, $y'y = q^2yy'$, $y'x = qxy' + (q^2-1)yx'$. For the second braided-integer we get

$$[2;R] = \begin{pmatrix} 1+q^2 & 0 & 0 & 0 \\ 0 & 1 & q & 0 \\ 0 & q & q^2 & 0 \\ 0 & 0 & 0 & 1+q^2 \end{pmatrix},$$

i.e. it is not invertible. There exists nonetheless a braided-exponential [Maj96, Example 5.4]

$$\exp(\mathbf{x}|\mathbf{v}) = \sum_{a\in\mathcal{S}\{1,\dots,n\}} \frac{\mathbf{x}_a\mathbf{v}^a}{[|a|;q^2]!} = \sum_{p=0}^{\infty} \frac{(\mathbf{x}\cdot\mathbf{v})^p}{[p;q^{-2}]!},$$

where $[m;q^2] = \frac{1-q^{2m}}{1-q^2}$ and $[m;q^2]! = \prod_{\mu=1}^{m}[\mu;q^2]$. For the generator we get

$$L = \frac{1}{1+q^2}\sum a_{ij}\partial_i\partial_j + \sum b_i\partial_i.$$

The free braided-space

Here $R' = P$, there are no relations in the algebras $V(R,R')$ and $V^{\vee}(R,R')$ since the ideal generated by $R'_{12}\mathbf{v}_2\mathbf{v}_1 = \mathbf{v}_1\mathbf{v}_2$ (respectively $\mathbf{x}_1\mathbf{x}_2R'_{12} = \mathbf{x}_2\mathbf{x}_1$) is equal to $\{0\}$, i.e. we have the free algebra with n generators. $\{\mathbf{v}^a; a \in \mathcal{S}\{1,\dots,n\}\}$ (respectively $\{\mathbf{x}_a; a \in \mathcal{S}\{1,\dots,n\}\}$) is thus a basis of $V(R,R')$ (respectively $V^{\vee}(R,R')$).

We will assume that R is also equal to P, i.e. that the braiding is given by $\Psi(v^i \otimes v^j) = (v^i \otimes v^j)$ (respectively $\Psi(x_i \otimes x_j) = (x_i \otimes x_j)$). In this case the braided-integers $[m;P]^a_b = m\delta_{ab}$, $m = |a| = |b|$, are invertible and thus

$$\exp(\mathbf{x}|\mathbf{v}) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{a\in\mathcal{S}^k\{1,\dots,n\}} \mathbf{x}_a\mathbf{v}^a,$$

The dual action is

$$\rho(v^i)\mathbf{x}_a = \begin{cases} |a|\,\mathbf{x}_{a_2\cdots a_{|a|}} & \text{if } a_1 = i, \\ 0 & \text{else} \end{cases}$$

on the basis elements. Generators of diffusions have again the form

$$L = \sum_{k,l=1}^{n} a_{kl}\partial_k\partial_l + \sum_{j=1}^{n} b_j\partial_j \tag{5.4}$$

with symmetric a_{kl}.

5.3.2 Examples of true diffusions

A free two-dimensional braided plane: $V^{\check{}} = \mathbb{R}_q * \check{\mathbb{R}}_q$

The algebra underlying this example can be considered as the free product of the braided line with its dual. As an algebra it is the free unital algebra $\mathbb{C} < x, p >$ with two generators. Let x be the generator of the braided-covector space $\mathbb{R}_q$, and p that of the braided-vector space dual to it (i.e. another copy of $\mathbb{R}_q$). Let $q \in \mathbb{R}\backslash\{0\}$, so that $R = (q)$ is of real type I in the sense of [Maj95a]. Following Majid's definitions we have the braid relations

$$\begin{aligned} x'x &= qxx & x'p &= q^{-1}px' \\ p'x &= q^{-1}xp' & p'p &= qpp', \end{aligned}$$

and

$$\begin{aligned} *_{\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q} &: \mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q \to \mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q, & x^* &= p \\ *_{\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q\underline{\otimes}\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q} &: \mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q\underline{\otimes}\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q \to \mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q\underline{\otimes}\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q, & x^* &= p',\ (x')^* = p, \end{aligned}$$

turn $\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q$ and $\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q\underline{\otimes}\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q$ in into *-algebras such that $\underline{\Delta} : \mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q \to \mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q\underline{\otimes}\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q$ satisfies $\underline{\Delta} \circ *_{\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q} = *_{\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q\underline{\otimes}\mathbb{R}_q\underline{\otimes}\check{\mathbb{R}}_q} \circ \tau \circ \underline{\Delta}$. Even though this *-structure does not suit our purposes, at least not in this form, it does tell how we can guess a 'better' one.

The coproduct on $\mathbb{R}_q$ and its dual extends to a map $\underline{\Delta}_0 : \mathbb{R}_q * \check{\mathbb{R}}_q \to (\mathbb{R}_q\underline{\otimes}\mathbb{R}_q) * (\check{\mathbb{R}}_q\underline{\otimes}\check{\mathbb{R}}_q) = \mathbb{C} < p, p', x, x' > /\mathcal{I}_0$, where $\mathcal{I}_0$ is the ideal generated by $x'x - qxx'$ and $p'p - qpp'$, and satisfies $\underline{\Delta} \circ *_{\mathbb{R}_q * \check{\mathbb{R}}_q} = *_{(\mathbb{R}_q\underline{\otimes}\mathbb{R}_q)*(\check{\mathbb{R}}_q\underline{\otimes}\check{\mathbb{R}}_q)} \circ \underline{\Delta}$, if we define the *-structures in those algebras by

$$\begin{aligned} *_{\mathbb{R}_q * \check{\mathbb{R}}_q} &: \mathbb{R}_q * \check{\mathbb{R}}_q \to \mathbb{R}_q * \check{\mathbb{R}}_q, & x^* &= p, \\ *_{(\mathbb{R}_q\underline{\otimes}\mathbb{R}_q)*(\check{\mathbb{R}}_q\underline{\otimes}\check{\mathbb{R}}_q)} &: (\mathbb{R}_q\underline{\otimes}\mathbb{R}_q) * (\check{\mathbb{R}}_q\underline{\otimes}\check{\mathbb{R}}_q) \to (\mathbb{R}_q\underline{\otimes}\mathbb{R}_q) * (\check{\mathbb{R}}_q\underline{\otimes}\check{\mathbb{R}}_q), & x^* &= p',\ (x')^* = p. \end{aligned}$$

We now note that we get $(\mathbb{R}_q * \check{\mathbb{R}}_q)\underline{\otimes}(\mathbb{R}_q * \check{\mathbb{R}}_q)$ if we quotient by the relations $qxp = px$ and $qp'x' = x'p'$, i.e. $(\mathbb{R}_q * \check{\mathbb{R}}_q)\underline{\otimes}(\mathbb{R}_q * \check{\mathbb{R}}_q) = (\mathbb{R}_q\underline{\otimes}\mathbb{R}_q) * (\check{\mathbb{R}}_q\underline{\otimes}\check{\mathbb{R}}_q)/\{qxp = pxp, qp'x' = x'p'\} = \mathbb{C} < p', x, p, x' > /\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $x'x - qxx'$, $qxp - px$, $qp'x' - x'p'$, and $p'p - qpp'$. The canonical projection $\pi : (\mathbb{R}_q\underline{\otimes}\mathbb{R}_q) * (\check{\mathbb{R}}_q\underline{\otimes}\check{\mathbb{R}}_q \to (\mathbb{R}_q * \check{\mathbb{R}}_q)\underline{\otimes}(\mathbb{R}_q * \check{\mathbb{R}}_q)$ is a *-algebra homomorphism, and $\underline{\Delta} = \pi \circ \underline{\Delta}_0$ turns $V^{\check{}} = \mathbb{R}_q * \check{\mathbb{R}}_q$ into a braided *-Hopf algebra in the sense we need. In particular, the coproduct is a *-algebra homomorphism.

Note that we now have $p_1 = p'$ and $x_1 = x$ in the first factor of the tensor product, and $p_2 = p$ and $x_2 = x'$ in the second, i.e. the braid relations take the form

$$\begin{aligned} x_2x_1 &= qx_1x_2, & p_2x_1 &= q^{-1}x_1p_2, \\ x_2p_1 &= qp_1x_2, & p_2p_1 &= q^{-1}p_1p_2 \end{aligned}$$

and the *-structure becomes $x_i^* = p_i$, $i = 1, 2$.

The braiding of the real line, and thus also that of $V^\vee$, are induced by an action and coaction of the group algebra $\mathbb{C}\mathbb{Z}$. $V^\vee$ is the (free) braided plane associated to the matrices

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

The possible generators of pseudo-diffusions are of the same form as in Equation (5.4), but now we are interested in generators of 'true' diffusions, i.e. Ψ-invariant, hermitian, conditionally positive generators. Choose the set of all words $\{1 = \emptyset, x, p, xx, xp, px, pp, \ldots\}$ as basis of $V^\vee$, then ϕ_{xp} and ϕ_{px} with

$$\phi_{xp}(u) = \begin{cases} 1 & \text{if } u = xp, \\ 0 & \text{else,} \end{cases} \qquad \phi_{px}(u) = \begin{cases} 1 & \text{if } u = px, \\ 0 & \text{else,} \end{cases}$$

on basis elements u are such generators. The processes associated to these generators have been studied by M. Schürmann[Sch93], we find the processes that have the Azéma martingales as classical version.

Let us now consider an example whose braiding does not come from a group, but from a non-commutative and non-cocommutative bialgebra.

Diffusions on the four-dimensional braided-covector space $V^\vee = \mathbb{C}_q^{2|0} * \check{\mathbb{C}}_q^{2|0}$

Let again q in $\mathbb{R}\backslash\{0\}$, and set $V^\vee = \mathbb{C}_q^{2|0} * \check{\mathbb{C}}_q^{2|0} = \mathbb{C} < v, w, x, y > / \{yx - qxy, vw - qwv\}$. A similar construction as in the preceding example leads to a four-dimensional braided-covector space, where the braid relations between x_1, y_1 and x_2, y_2 are described by the R-matrix of the braided plane,

$$R = \begin{pmatrix} q^2 & 0 & 0 & 0 \\ 0 & q & q^2 - 1 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & q^2 \end{pmatrix},$$

those between x_1, y_1 and v_2, w_2 by its second inverse,

$$R^{-1} = \begin{pmatrix} q^{-2} & 0 & 0 & 0 \\ 0 & q^{-1} & q^{-4} - q^{-2} & 0 \\ 0 & 0 & q^{-1} & 0 \\ 0 & 0 & 0 & q^{-2} \end{pmatrix},$$

those between v_1, w_2 and x_2, y_2 again by R, and, finally, those between v_2, w_2 by R^{-1}, see also subsection 5.2.3. The *-structure is defined by $x^* = v$ and $y^* = w$, compare also [Maj95a].

$\{x^n y^m | n, m \in \mathbb{N}\}$ and $\{w^m v^n | n, m \in \mathbb{N}\}$ are bases of the two components of the free product defining $V^{\vee}$, and therefore we can take the finite, alternating sequences of elements of these bases as a basis of $V^{\vee}$:
$\{1 = \emptyset, x^{n_1} y^{m_1}, w^{m_1} v^{n_1}, x^{n_1} y^{m_1} w^{m_2} v^{n_2}, w^{m_1} v^{n_1} x^{n_2} y^{m_2}, \ldots | n_1, n_2, \ldots, m_1, m_2, \ldots \in \mathbb{N}\}$.
The functional defined on this basis by

$$\phi_{xv}(u) = \begin{cases} 1 & \text{if } u = xv, yw \\ 0 & \text{else,} \end{cases}$$

generates a diffusion.

5.4 Appell systems

Appell systems $\{h_k; k \in \mathbb{N}\}$ can be defined by the conditions

A1: h_k is a polynomial of degree k,

A2: $Dh_k = kh_{k-1}$,

where $D = \mathrm{d}/\mathrm{d}x$. Examples are furnished by shifted moment sequences of the form

$$h_k(x) = \int_{\mathbb{R}} (x + y)^k p(\mathrm{d}x),$$

where p is any probability measure on $\mathbb{R}$ with all moments finite. For example in the Gaussian case this gives the Hermite polynomials $H_n(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (x + iy)^n e^{-y^2/2} \mathrm{d}y$. Recently analogous classes of 'moment polynomials' on Lie groups [FS92] and quantum groups [FFS95] have been introduced. They cease to be polynomials in general.

The definition on quantum groups is easily adapted to braided groups.

Definition 5.4.1 *We define the Appell polynomials on the braided covector space $V^{\vee}(R, R')$ with respect to the functional ϕ by*

$$h_a = (\phi \otimes \mathrm{id}) \circ \underline{\Delta} \mathbf{x}_a,$$

i.e. $h_a = T_\phi(\mathbf{x}_a)$, for $a \in \mathcal{S}\{1, \ldots, n\}$.

We are in particular interested in the Appell systems with respect to diffusions.

Proposition 5.4.2 *Let ϕ_t be a diffusion on $V^{\vee}(R, R')$ with generator L and $\{h_a(t); a \in \mathcal{S}\{1, \ldots, n\}\}$ the associated Appell system. Then we have the following:*

1. *$h_a(t) = \mathbf{x}_a + p_a(t)$, where $p_a(t) \in \bigoplus_{\alpha=0}^{|a|-1} V^{\vee}(R, R')^{(\alpha)}$,*
2. *$D^i h_a = \sum_{b \in \mathcal{S}^{|a|-1}\{1,\ldots,n\}} [|a|; R]_a^{i.b} h_b$, where $D^i = \exp(tL)\partial^i \exp(-tL)$ and $i.b$ the concatenation of i with b, i.e. $i.b = ib_1 \cdots b_{|a|-1}$,*

3. *the $h_a(t)$ satisfy the evolution equation $\partial_t u = Lu$.*

Proof: Note that 1. and 2. are the analogues of A1 and A2.

1. The braided-binomial theorem [Maj96, Equation (91)] tells us that

$$\underline{\Delta}\mathbf{x}_a = \sum_{i=0}^{|a|} \sum_{b\in\mathcal{S}^i\{1,\dots,n\},c\in\mathcal{S}^{|a|-i}\{1,\dots,n\}} \left[\begin{matrix} |a| \\ i \end{matrix}; R\right]_a^{b.c} \mathbf{x}_b \otimes \mathbf{x}_c.$$

where the braided-binomial coefficients are defined by

$$\begin{aligned}
\left[\begin{matrix} m \\ r \end{matrix}; R\right] &= (PR)_{r,r+1}\cdots(PR)_{m-1,m}\left[\begin{matrix} m-1 \\ r-1 \end{matrix}; R\right] + \left[\begin{matrix} m-1 \\ r \end{matrix}; R\right], \\
\left[\begin{matrix} m \\ 0 \end{matrix}; R\right] &= 1, \\
\left[\begin{matrix} m \\ r \end{matrix}; R\right] &= 0 \qquad \text{for } r > m.
\end{aligned}$$

The statement follows now since $\phi(t)|_{V^*(R,R')^{(0)}} = \epsilon$, i.e. $h_a(t) = \mathbf{x}_a + p_a(t)$ with

$$p_a(t) = \sum_{i=0}^{|a|-1} \sum_{b\in\mathcal{S}^{|a|-i}\{1,\dots,n\},c\in\mathcal{S}^{i}\{1,\dots,n\}} \left[\begin{matrix} |a| \\ i \end{matrix}; R\right]_a^{b.c} \phi_t(\mathbf{x}_b)\mathbf{x}_c.$$

Note that this formula can also be used to calculate the Appell systems, if the moments of the process are known.

2.

$$\begin{aligned}
D^i h_a &= \exp(tL)\partial^i \exp(-tL)\exp(tL)\mathbf{x}_a \\
&= \exp(tL)\partial^i \mathbf{x}_a \\
&= \exp(tL) \sum_{b\in\mathcal{S}^{|a|-1}\{1,\dots,n\}} [|a|; R]_a^{i.b}\mathbf{x}_b \\
&= \sum_{b\in\mathcal{S}^{|a|-1}\{1,\dots,n\}} [|a|; R]_a^{i.b} h_b
\end{aligned}$$

with [Maj96, Equation (77)].

3.

$$\begin{aligned}
\partial_t h_a(t) &= \partial_t \exp(tL)\mathbf{x}_a = L\exp(tL)\mathbf{x}_a \\
&= Lh_a(t).
\end{aligned}$$

∎

Property 2 shows that the D^i act as lowering operators. We also define raising operators.

Proposition 5.4.3 *Let $X_i = \exp(tL)x_i\exp(-tL)$, then*

$$X_i h_a(t) = h_{i.a}(t).$$

Examples

5.4.1 The braided line $\mathbb{R}_q$

The Appell polynomials associated to the functional $\phi_t = \exp(tL)$ where $L = \frac{a^2}{1+q}v^2 + bv$ are

$$\begin{aligned} h_k(x;t) &= (\phi_t \otimes \mathrm{id}) \circ \underline{\Delta} x^k \\ &= \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}; q \Bigg] \phi_t(x^\nu) x^{k-\nu} \\ &= \sum_{\nu=0}^{K} \frac{[k;q]! H_\nu\left(bt, -\frac{2a^2 t}{1+q}\right)}{[k-\nu;q]!\nu!} x^{k-\nu}. \end{aligned}$$

These polynomials are solutions of

$$\partial_t u = \frac{a^2}{1+q}\delta^2 u + b\delta u$$

with $u(x, t=0) = x^k$ (where δ is the q-difference operator $(\delta f)(x) = \frac{f(qx)-f(x)}{x(q-1)}$).

For $b = 0$, $a = \sqrt{(1+q)/2}$ the Appell polynomials simplify to

$$h_k(x;t) = \sum_{\nu=0}^{[\frac{k}{2}]} \frac{[k;q]! t^\nu x^{k-2\nu}}{[k-2\nu;q]! 2^\nu \nu!},$$

these polynomials are a q-analogue of the Hermite polynomials (see also [FFS95]).

5.4.2 The quantum plane $\mathbb{C}_q^{2|0}$

We choose $\phi_t = \exp tL$ with $L = \frac{1}{1+q^2}(\partial_1^2 + \partial_2^2)$. We get

$$\phi_t = \sum_{r=0}^{\infty} \frac{t^r}{(1+q^2)^r r!} \sum_{\nu=0}^{r} \begin{bmatrix} r \\ \nu \end{bmatrix}; q^{-4} \Bigg] \partial_2^{2\nu} \partial_1^{2(r-\nu)}$$

since $\partial_2\partial_1 = q^{-1}\partial_1\partial_2$. This leads to the following formula for the Appell polynomials:

$$\begin{aligned} h_{rm}(x,y;t) &= \exp(tL) x^r y^m \\ &= \sum_{\nu=0}^{[\frac{r}{2}]} \sum_{\mu=0}^{[\frac{m}{2}]} \frac{\begin{bmatrix} \mu+\nu \\ \nu \end{bmatrix}; q^{-4} \Bigg] [m;q^2]![r;q^2]! q^{2\mu(r-2\nu)} t^{\mu+\nu}}{[m-2\mu-1;q^2]![r-2\nu-1;q^2]!(1+q^2)^{\mu+\nu}(\mu+\nu)!} x^{r-2\nu} y^{m-2\mu}, \end{aligned}$$

where $\begin{bmatrix} r \\ \nu \end{bmatrix}; q^{-4} \Bigg]$ are the braided-binomial coefficients with respect to $R = (q^{-4}) \in M(1) \otimes M(1)$. These polynomials solve the evolution equation

$$\partial_t u = \frac{1}{1+q^2}\left(\partial_1^2 + \partial_2^2\right) u$$

where ∂_1, ∂_2 can be defined by

$$\partial_1 f(x,y) = \frac{f(q^2x,y) - f(x,y)}{x(q^2-1)}, \qquad \partial_2 f(x,y) = \frac{f(qx,q^2y) - f(qx,y)}{y(q^2-1)}.$$

5.4.3 The free braided-space

We consider again the free braided-space associated to $R = R' = P$. It is straightforward to calculate the action of the generator of a diffusion on a basis element:

$$L\mathbf{x}_k = r(r-1)a_{k_1k_2}\mathbf{x}_{k_3\cdots k_r} + rb_{k_1}\mathbf{x}_{k_2\cdots k_r},$$

for $L = \sum_{k,l=1}^n a_{kl}\partial_k\partial_l + \sum_{j=1}^n b_j\partial_j$ and $k = (k_1,\ldots,k_r) \in \mathcal{S}\{1,\ldots,n\}$. Thus we get for the Appell polynomials

$$\begin{aligned} h_k &= \exp(tL)\mathbf{x}_k \\ &= \mathbf{x}_k + t(r(r-1)a_{k_1k_2}\mathbf{x}_{k_3\cdots k_r} + rb_{k_1}\mathbf{x}_{k_2\cdots k_r}) \\ &\quad + \frac{t^2}{2}\Big(r(r-1)(r-2)(r-3)a_{k_1k_2}a_{k_3k_4}\mathbf{x}_{k_5\cdots k_r} + r(r-1)(r-2)(b_{k_1}a_{k_2k_3} + a_{k_1k_2}b_{k_3})\mathbf{x}_{k_4\cdots k_r} \\ &\quad + r(r-1)b_{k_1}b_{k_2}\mathbf{x}_{k_3\cdots k_r}\Big) + \cdots, \end{aligned}$$

the drift term vanishes, i.e. $b_j = 0$ for $j = 1,\ldots,n$, then

$$h_k = \sum_{\rho=0}^{[\frac{r}{2}]} \frac{t^\rho}{\rho!} r(r-1)\cdots(r-2\rho+1)a_{k_1k_2}\cdots a_{k_{2\rho-1}k_{2\rho}}\mathbf{x}_{k_{2\rho+1}\cdots k_r}.$$

5.5 Densities

There are two ways in which densities can be associated to the diffusions constructed here.

This first uses the (right or left) invariant integrals that are defined on Hopf algebras. Fix such a functional $\int : V^\vee(R,R') \to \mathbb{C}$. One can look for an element $\rho_t \in V^\vee(R,R')$ such that

$$\phi_t(a) = \int(\rho_t a), \qquad \Big(\text{or } \phi_t(a) = \int(a\rho_t)\Big)$$

is satisfied for all $a \in V^\vee(R,R')$ (where $\phi_t(a) =< \phi_t, a >$ denotes the evaluation map between $V^\vee(R,R')$ and $V(R,R')$). This defines a (non-commutative) left (or right) density. For e.g. anyspace this leads to the results exhibited in [MRP94]. To calculate it the Fourier transform (w.r.t. to the integral considered) can be used. For a braided-Hopf algebra B with dual B^*, exponential exp and invariant integral $\int : B \to \mathbb{C}$ the Fourier transform $\mathcal{F} : B \to B^*$ is defined by $\mathcal{F} = (\int \otimes \mathrm{id}) \circ (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \exp)$, see [KM94]. Let now $\int$ be a left invariant integral on $V^\vee(R,R')$ and $\int^*$ a right invariant integral on $V(R,R')$. Then, since $(\mathrm{id}\otimes < \cdot,\cdot >) \circ (\exp\otimes\mathrm{id}) = \mathrm{id}$,

$$\int(\rho a) =< \mathcal{F}(\rho), a >$$

for all $a \in V^{\vee}(R, R')$ and $\rho \in V(R, R')$. Thus we need to invert the Fourier transform $\mathcal{F}$ to find the density corresponding to the functional ϕ_t. By [KM94, Proposition 6.4], $\mathcal{F}^* \circ \mathcal{F} = \mathrm{vol}\,\underline{S}$, where vol is a constant and $\mathcal{F}^*$ the Fourier transform w.r.t. $\int^*$, i.e. $\mathcal{F}^* = (\mathrm{id} \otimes \int^* \circ m) \circ (\Psi \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \exp)$. Thus

$$\rho_t = \frac{1}{\mathrm{vol}} \underline{S}^{-1} \circ \mathcal{F}^*(\phi_t).$$

Take, e.g., the braided-line and $\phi_t = \exp(\frac{tv^2}{2})$. We shall use the Jackson integral $\int_{-\gamma.\infty}^{+\gamma.\infty} f(x)\mathrm{d}_q x$ where γ is a parameter. For $f(x) = \sum f_n x^n$ we set $f(\gamma) = \sum f_n \gamma^n$ and

$$\int_{-\gamma.\infty}^{+\gamma.\infty} f(x)\mathrm{d}_q x = (1-q) \sum_{k=-\infty}^{\infty} \left(f(q^k\gamma) + f(-q^k\gamma)\right) q^k \gamma$$

if the right-hand-side converges. We will express the density in terms of the discrete q-Hermite II polynomials

$$\tilde{h}_n(x;q) = (1-q)^n [n;q]! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{-2nk} q^{k(2k+1)} x^{n-2k}}{(1-q^2)^k [k;q^2]! (1-q)^{n-2k} [n-2k;q]!}.$$

With the orthogonality relations for the discrete q-Hermite II polynomials [Koo95, Equation (8.14)] and

$$\phi_t(\tilde{h}_n(x;q)) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ (1-q)^n [n;q]! \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k q^{-2nk} q^{k(2k+1)} t^{\frac{n}{2}-k}}{(1-q^2)^k [k;q^2]! (1-q)^{n-2k} 2^{\frac{n}{2}-k}} & \text{if } n \text{ is even,} \end{cases}$$

we get

$$\rho_t = \frac{1}{c_q(\gamma)} \sum_{m=0}^{\infty} q^{4m^2} \sum_{k=0}^{m} \frac{(-1)^k q^{-4mk} q^{k(2k+1)} t^{m-k}}{(1-q^2)^k [k;q^2]! 2^{m-k}} \tilde{h}_{2m}(x;q) e_{q^2}(-(1-q^2)x^2), \tag{5.5}$$

where $e_{q^2}(z) = \sum_{n=0}^{\infty} \frac{z^n}{[n;q^2]!}$, and $c_q(\gamma)$ is a constant depending only on q and γ (for its value see [Koo95, Equation (8.15)]).

The other way, commonly used in quantum probability (see e.g. [Mey93]) leads to a density on the usual real line. Consider for example a diffusion $\phi_t = \exp(tv^2/2)$ on the braided line, and the generator x, then $\phi_t(e^{itx})$ can be interpreted as the Fourier transform of the density of x in the state ϕ_t. Let $0 < q < 1$ and $Y = (1-q)\sum_{i=0}^{\infty} X_i$, where the X_i are independent, and exponentially distributed with respective means q^i (see [Fei87]). Then $E(e^{iuY}) = \prod_{i=0}^{\infty} (1 - iu(1-q)q^i)^{-1} = \sum_{n=0}^{\infty} (iu)^n/[n;q]! = e_q^{iu}$. Thus $\phi_t(E(e^{iuYx})) = \phi_t(e_q^i ux) = e^{-tu^2/2}$, and it follows that Yx has a Gauss distribution. This determines the distribution of x uniquely, see [Fei87, Theorem 3].

5.6 Conclusion

The notion of q-Brownian motion introduced by Majid is extended in this paper to braided spaces. The resulting processes differ from the ones obtained in [BS91, MM93]

by more algebraic considerations. To obtain these (for fixed parameter t=1, say), we can consider the semi-direct cross product of the braided line with its dual. This algebra has basis $\{v^n \otimes x^m; n, m \in \mathbb{N}\}$ and the defining relations are $vx - qxv = 1$ (where $v = v \otimes 1$, $x = 1 \otimes x$). Take the functional ϕ defined by $\phi(\sum a_{nm} v^n \otimes x^m) = a_{00}$, then the distribution of $x + v$ in the state ϕ is equal to that of [BS91] for $t = 1$, for a study of its density see [LM95]. This density is concentrated on the bounded interval $[-1/\sqrt{1-q}, 2/\sqrt{1-q}]$ (for $-1 < q < 1$). The q-deformed Brownian motion of [MM93] can be obtained similarly. Obviously, there is still some more non-trivial construction required to pass from the random variable presented here to the full process.

Chapitre 6

Gauss Laws in the Sense of Bernstein and Uniqueness of Embedding into Convolution Semigroups on Quantum Groups and Braided Groups

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Résumé

The goal of this paper is to characterise certain probability laws on a class of quantum groups or braided groups that we will call nilpotent. First we introduce a braided analogue of the Heisenberg-Weyl group, which shall serve as standard example. We introduce Gaussian functionals on quantum groups or braided groups as functionals that satisfy an analogue of the Bernstein property, i.e. that the sum and difference of independent random variables are also independent. The corresponding functionals on the braided line, braided plane and a braided q-Heisenberg-Weyl group are determined. Section 6.5 deals with continuous convolution semigroups on nilpotent quantum groups and braided groups. We extend recent results proving the uniqueness of the embedding of an infinitely divisible probability law into a continuous convolution semigroup for simply connected nilpotent Lie groups to nilpotent quantum groups and braided groups. Finally, in Section 6.6 we give some indications how the semigroup approach of Heyer and Hazod to the Bernstein theorem on groups can be extended to quantum groups and braided groups.

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6.1 Introduction

Quantum probability is now a quite active research area motivated by applications in physics. Recent books of Biane [Bia93], Meyer [Mey93], and Parthasarathy [Par92] are recommended as introductions to this field. For quantum probability on Hopf algebras see Majid [Maj95b] and Schürmann [Sch93]. In this paper we continue the investigation of probability measures on quantum groups and braided groups initiated in [FFS97, FS96b]. Our goal is to characterise certain probability laws on a class of quantum groups or braided groups that we will call nilpotent. We determine the functionals on these structures which satisfy an analogue of the Bernstein property, i.e. that the sum and difference of independent random variables are also independent. In doing so we extend results obtained previously on Lie groups by D. Neuenschwander et al., cf. [Neu93, NRS96, NRS97]. Then we turn to convolution semigroups. We show the uniqueness of embedding into continuous convolution semigroups on nilpotent quantum groups and braided groups, and calculate the generators of continuous convolutions semigroups on several nilpotent braided groups that satisfy an analogue of the Bernstein property.

The organization of the paper is as follows: Section 2 provides some preliminaries on quantum groups and quantum probability. A braided Heisenberg-Weyl group is introduced in Section 3 and the corresponding Gaussian functionals in the sense of Bernstein are given in Section 4. The embedding problem is investigated in Section 5, while Section 6 gives some hints on Gaussian semigroups in the sense of Bernstein on quantum groups and braided groups. Concluding remarks and further research aspects are mentioned in Section 7.

6.2 Preliminaries

6.2.1 Quantum groups

We briefly recall some definitions concerning quantum groups and braided groups, see also [Maj95b] and references therein.

Recall that a quantum group is a Hopf algebra, i.e. an associative unital algebra (H, m, e) with two homomorphisms $\Delta : H \to H \otimes H$, $\varepsilon : H \to \mathbb{C}$, and an anti-homomorphism $S : H \to H$ that satisfy

$$\begin{aligned}
(\Delta \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \Delta) \circ \Delta, \\
(\varepsilon \otimes \mathrm{id}) \circ \Delta &= (\mathrm{id} \otimes \varepsilon) \circ \Delta \\
m \circ (\mathrm{id} \otimes S) \circ \Delta &= m \circ (S \otimes \mathrm{id}) \circ \Delta = e \circ \varepsilon.
\end{aligned}$$

These maps are called coproduct, counit, and antipode, respectively. Here the tensor product $H \otimes H$ is considered as an algebra with the multiplication defined by $m_{\otimes}((a \otimes b) \otimes (c \otimes d)) = ac \otimes bd$, i.e. $m_{\otimes} = (m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$, where $\tau : H \otimes H \to H \otimes H$ is the flip automorphism, $\tau(a \otimes b) = b \otimes a$. Placing ourselves in a braided category and substituting τ by the braiding Ψ we are naturally led to braided Hopf algebras or braided groups. The axioms above remain unchanged, but now m, e, Δ, ε, and S have to satisfy certain compatibility conditions with respect to the braiding Ψ. The comultiplication Δ has to

be an algebra homomorphism from H to $H\tilde{\otimes}H$, i.e. the algebra structure in the tensor product is defined by $m_\Psi = (m\otimes m)\circ(\mathrm{id}\otimes\Psi\otimes\mathrm{id})$, or $m_\Psi((a\otimes b)\otimes(c\otimes d)) = a\Psi(b\otimes c)d$. And the antipode is no longer an anti-homomorphism, but satisfies $S\circ m = m\circ\Psi\circ(S\otimes S)$.

Example: The braided line $\mathbb{R}_q$. Let $q\in\mathbb{C}\backslash\{0\}$, q not a root of unity. As an algebra $\mathbb{R}_q$ is isomorphic to the ring of polynomials in one variable $\mathbb{R}[x]$. The braiding used here is defined by $\Psi(x^n\otimes x^m) = q^{nm}x^m x^n$. On the generator the coproduct, counit, and antipode are defined by

$$\Delta(x) = x + x', \qquad \varepsilon(x) = 0, \quad S(x) = -x,$$

where we used $x = x\otimes 1$, $x' = 1\otimes x$. From this one computes

$$\Delta(x^n) = \sum_{\nu=0}^{n}\begin{bmatrix} n \\ \nu \end{bmatrix}_q x^\nu (x')^{n-\nu}$$

and $S(x^n) = (-1)^n q^{\frac{n(n-1)}{2}} x^n$. Recall that the q-numbers q_n, the q-factorial $q_n!$ and the q-binomial coefficients $\begin{bmatrix} n \\ \nu \end{bmatrix}_q$ are defined as

$$q_n = \sum_{\nu=0}^{n-1} q^\nu = \frac{q^n-1}{q-1}, \qquad q_n! = \prod_{\nu=1}^{n} q_\nu, \qquad \begin{bmatrix} n \\ \nu \end{bmatrix}_q = \frac{q_n!}{q_\nu! q_{n-\nu}!}.$$

Example: The braided plane $\mathbb{C}_q^{2|0}$. Again $q\in\mathbb{C}\backslash\{0\}$, q not a root of unity. The braided plane is the braided Hopf algebra with two generators x, y and defining relations

$$\begin{aligned} yx &= qxy, \\ x'x = q^2xx', &\qquad x'y = qyx', \\ y'x = qxy' + (q^2-1)yx', &\qquad y'y = q^2yy', \\ \Delta x = x + x', &\qquad \Delta y = y + y' \\ \varepsilon(x) = \varepsilon(y) = 0, &\qquad S(x) = -x, \quad S(y) = -y \end{aligned}$$

We will present another example of a braided group in Section 6.3.

6.2.2 Quantum probability

We summarize the basic definitions of quantum probability or non-commutative probability. A quantum probability space is usually defined as a pair $(\mathcal{A},\Phi)$ consisting of a *-algebra $\mathcal{A}$ and a state (i.e. a normalizeded positive linear functional) Φ on $\mathcal{A}$. A classical probability space $(\Omega,\mathcal{F},P)$ gives rise to a quantum probability space by taking a *-algebra of complex-valued integrable functions on Ω, e.g. $L^\infty(\Omega,\mathcal{F},P)$, and the functional defined by $\Phi : f\mapsto\int_\Omega f dP$.

A quantum random variable j over a quantum probability space $(\mathcal{A},\Phi)$ on a *-algebra $\mathcal{B}$ is simply a *-algebra homomorphism $j:\mathcal{B}\to\mathcal{A}$. A classical random variable $X:\Omega\to E$ over a probability space $(\Omega,\mathcal{F},P)$ with values in a measurable space $(E,\mathcal{E})$ defines a quantum random variable via $j_X(f) = f\circ X$ for $f\in\mathcal{B}$ (where $\mathcal{B}$ is an appropriately

chosen algebra of functions on E). For a quantum random variable $j : \mathcal{B} \to \mathcal{A}$ we will call $\varphi_j = \Phi \circ j$ its distribution in the state Φ.

In this article we will often use the terminology introduced above but neglect the *-structure. The reason for this is that the algebras we consider have natural *-structures only for rather special values of the parameters. We will for example characterize Gaussian functionals in the sense of Bernstein using only their algebraic properties. Once this is done one can without much difficulty check positivity for those cases where a natural *-structure is defined.

6.3 A braided Heisenberg-Weyl group

In this section we introduce a braided Hopf algebra structure for the q-oscillator algebra HW_q defined by

$$ac - qca = (1-q)b, \qquad ab = ba, \qquad cb = bc, \tag{6.1}$$

where $q \in \mathbb{C}\backslash\{0\}$, q not a root of unity. For other quantum or braided groups related to the Heisenberg group see [BM93] and the references therein.

We can also regard this algebra as generated by two generators a, c with the cubic relations

$$\begin{aligned} aac + qcaa &= (1+q)aca \\ acc + qcca &= (1+q)cac \end{aligned}$$

with $q \in \mathbb{C}$, q not a root of unity. If we define the braiding by

$$\begin{aligned} a'a &= qaa' & a'c &= ca' \\ c'a &= \frac{1}{q}ac' & c'c &= qcc' \end{aligned} \tag{6.2}$$

then

$$\Delta a = a + a', \qquad \Delta c = c + c'$$

defines an algebra homomorphism from HW_q to $\mathrm{HW}_q \tilde{\otimes} \mathrm{HW}_q$. The counit is defined by $\varepsilon(a) = \varepsilon(c) = 0$ (extend as a unital algebra homomorphism). The braided bialgebra defined in this way also admits an antipode. Define S by

$$S(a) = -a, \qquad S(c) = -c,$$

and extend by $S \circ m = m \circ \Psi \circ (S \otimes S)$.

The advantage of introducing a third generator, e.g. by $ac - qca = (1-q)b$ or $d = ac - ca$, is that $\{a^n b^m c^r; n, m, r \in \mathbb{N}\}$ and $\{a^n d^m c^r; n, m, r \in \mathbb{N}\}$ are bases (Poincaré-Birkhoff-Witt bases) of HW_q.

One can calculate the coproduct on a general basis element

$$\begin{aligned} \Delta(a^n b^m c^r) &= \sum_{\nu=0}^{n}\sum_{\mu=0}^{m}\sum_{\sigma=0}^{\mu}\sum_{\rho=0}^{r} \begin{bmatrix} n \\ \nu \end{bmatrix}_q \begin{bmatrix} m \\ \mu \end{bmatrix}_q \begin{bmatrix} \mu \\ \sigma \end{bmatrix}_q \begin{bmatrix} r \\ \rho \end{bmatrix}_q q^{\rho(m-\mu)} \\ & q^{\sigma(n-\nu)} a^\nu b^\sigma c^{\rho+\mu-\sigma} a'^{n-\nu+\mu-\sigma} b'^{m-\mu} c'^{r-\rho}. \end{aligned} \tag{6.3}$$

We will construct the dual of HW_q. The coproduct of HW_q defines an algebra structure in its dual. Let $X, Y : \mathrm{HW}_q \to \mathbb{C}$ be two functionals on HW_q, then their product is defined by

$$(X \star Y)(u) = (X \otimes Y)\Delta(u) = \sum X(u_{(2)})Y(u_{(1)}) \qquad \forall u \in \mathrm{HW}_q,$$

where we used Sweedler's notation, $\Delta(u) = \sum u_{(1)} \otimes u_{(2)}$. For the functionals defined by

$$X_{ijk}(a^n b^m c^r) = \delta_{in}\delta_{jm}\delta_{kr}$$

we see that they span an algebra, denoted by $\mathrm{HW}_q^\vee$, generated by the three elements $x = X_{100}$, $y = X_{010}$, $z = X_{001}$ such that $X_{ijk} = \frac{z^k y^j x^i}{q_k! q_j! q_i!}$ and

$$xy = qyx, \quad yz = qzy, \quad xz - zx = y.$$

We see that this algebra is isomorphic to HW_q, the isomorphism being $x \mapsto a$, $z \mapsto c$.

The following formula is useful to compute products of general basis elements of the HW_q algebra, it can be derived by induction.

$$c^r a^n = \sum_{k=0}^{r\wedge n} \frac{(q^{-1})_r!(q^{-1})_n!(-1)^k}{(q^{-1})_{r-k}!(q^{-1})_{n-k}!(q^{-1})_k!} a^{n-k} d^k c^{r-k}. \tag{6.4}$$

If q is real, then the involution $* : \mathrm{HW}_q \to \mathrm{HW}_q$ defined by

$$a^* = c, \qquad b^* = b, \qquad c^* = a.$$

satisfies

$$\begin{aligned}
* \circ * &= \mathrm{id} \qquad (\text{i.e. } (a^*)^* = a \quad \forall a \in \mathcal{A}),\\
* \circ m &= m \circ \tau \circ (* \otimes *) \qquad \text{i.e. } * \text{ is an anti-homomorphism},\\
(* \otimes *) \circ \Delta &= \tau \circ \Delta \circ *,\\
\varepsilon \circ * &= \bar{\ } \circ \varepsilon,\\
S \circ * &= * \circ S,
\end{aligned}$$

i.e. turns HW_q into a braided *-Hopf algebra. For the origin of the last three conditions see e.g. [Maj95a].

6.4 Gaussian functionals in the sense of Bernstein

The Gauss law plays a central role in probability theory on Euclidean spaces. Bernstein's theorem gives a characterisation in terms of the group law of the underlying space.

Theorem 6.4.1 *Let P be a probability measure on $\mathbb{R}^n$. P is a Gauss law if and only if for all pairs of independent random variables X, Y with distribution P the random variables $X + Y$ and $X - Y$ are also independent.*

We want to remark here that it is not essential that the two random variables X and Y have the same law, we also have the following version of the above theorem.

Theorem 6.4.2 *Let P_X and P_Y be two probability measures on $\mathbb{R}^n$. If there exists a pair of independent random variables X, Y with distribution P_X, P_Y, respectively, such that the random variables $X+Y$ and $X-Y$ are also independent, then P_X and P_Y are Gauss laws (with identical covariance matrices).*

This characterisation can be used as a definition of Gaussian functionals on groups in general, if some care is taken concerning the order of the factors in non-commutative groups.

Definition 6.4.3 *Let G be a (locally compact topological) group. A probability measure μ on G is called Gaussian in the sense of Bernstein if there exists a probability space $(\Omega, \mathcal{B}, P)$ and a pair of G-valued r.v. X, Y, such that $\mu = X(P) = Y(P)$, and $(X \cdot Y, Y \cdot X)$ and $(X \cdot Y^{-1}, Y^{-1} \cdot X)$ are independent.*

D. Neuenschwander showed recently [Neu93] that the Gauss measures in the sense of Bernstein on the Heisenberg-Weyl group are exactly the Gauss measures concentrated on Abelian subspaces. In fact, this result holds on connected simply-connected nilpotent Lie groups in general [NRS96, NRS97].

We will now give a definition of Gaussian functionals on (braided) Hopf algebras.

6.4.1 Definition of Gaussian functionals in the sense of Bernstein on (braided) Hopf algebras

We have to translate $X+Y$ and $X-Y$ into Hopf algebra language. Replacing the group operation by the coproduct and the inverse by the antipode we write $j_1 \star j_2 = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta_{\mathcal{B}}$ and $j_1 \star (j_2 \circ S) = m_{\mathcal{A}} \circ (j_1 \otimes (j_2 \circ S)) \circ \Delta_{\mathcal{B}}$ for the sum and difference of two quantum random variables $j_1, j_2 : \mathcal{B} \to \mathcal{A}$, see also [ASW88, Sch93].

Definition 6.4.4 *A functional φ on the (braided) Hopf algebra $\mathcal{B}$ is called* right (left) Gaussian in the sense of Bernstein *if there exists a quantum probability space $(\mathcal{A}, \Phi)$ and two independent quantum random variables $j_1, j_2 : \mathcal{B} \to \mathcal{A}$ such that $\varphi = \Phi \circ j_1 = \Phi \circ j_2$ and the applications*

$$J = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta_{\mathcal{B}} \quad \textit{and} \quad K_r = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ (\mathrm{id} \otimes S) \circ \Delta_{\mathcal{B}}$$
$$\Big(J = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta_{\mathcal{B}} \quad \textit{and} \quad K_l = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ (S \otimes \mathrm{id}) \circ \Delta_{\mathcal{B}}\Big)$$

form also a pair of independent quantum random variables.

A Gaussian functional in the sense of Bernstein *is a functional that is right and left Gaussian in the sense of Bernstein.*

Here j_1 and j_2 are required to have the same law. We shall see that in the examples we consider this condition is too strong. We propose therefore also a weaker definition that is motivated by Theorem 6.4.2.

Definition 6.4.5 *A functional φ on a (braided) Hopf algebra $\mathcal{B}$ is called weakly right (left) Gaussian in the sense of Bernstein if there exists a quantum probability space $(\mathcal{A}, \Phi)$ and a pair of independent quantum random variables $j_1, j_2 : \mathcal{B} \to \mathcal{A}$ such that $\varphi = \Phi \circ j_1$ ($\varphi = \Phi \circ j_2$) and the applications*

$$J = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta_{\mathcal{B}} \quad \text{and} \quad K_r = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ (\mathrm{id} \otimes S) \circ \Delta_{\mathcal{B}}$$
$$\Big(J = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta_{\mathcal{B}} \quad \text{and} \quad K_l = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ (S \otimes \mathrm{id}) \circ \Delta_{\mathcal{B}}\Big)$$

form also a pair of independent quantum random variables.

6.4.2 Independence

The notion of independence used in the above definitions is that of tensor independence, see e.g. Schürmann [Sch93]. It consists of two parts, a factorisation property and a condition on the commutation relations. Like in classical probability theory the q.r.v. $j_1, \dots j_n : \mathcal{B} \to \mathcal{A}$ have to satisfy the factorisation property

$$\Phi\Big(j_{\sigma(1)}(b_1) \cdots j_{\sigma(n)}(b_n)\Big) = \varphi_{\sigma(1)}(b_1) \cdots \varphi_{\sigma(n)}(b_n) \tag{6.5}$$

for all $b_1, \dots, b_n \in \mathcal{B}$ and $\sigma \in \mathcal{S}_n$ in order to be independent. In the commutative case this gives one condition for each choice of arguments $b_1, \dots, b_n$, but here the different permutations can give rise to different equations.

Let us now come to the commutation relations. Schürmann supposes that $\mathcal{B}$ is graded by some group $\mathbb{L}$ and postulates for left (right) independence that

$$\begin{aligned} j_k(b) j_l(c) &= j_l(\mathrm{d}_b.c) j_k(b) \\ \Big(j_k(b) j_l(c) &= j_l(b) j_k(b.\mathrm{d}_c)\Big) \end{aligned}$$

for all homogeneous elements $b, c \in \mathcal{B}$ and for all $k < l$. Here $\mathrm{d}_b.c$ ($b.\mathrm{d}_c$) stands for a left (right) action of $\mathbb{L}$ on $\mathcal{B}$. For the trivial action $\mathrm{d}_b.c = c$ (or $b.\mathrm{d}_c = b$) we also speak of Bose or symmetric independence.

Notice that if $\gamma : \mathcal{B} \to \mathbb{C}\mathbb{L} \otimes \mathcal{B}$ denotes the coaction defined by $\gamma(b) = \mathrm{d}_b \otimes b$ on the homogeneous elements of $\mathcal{B}$ and α the action $\alpha(\mathrm{d}_b \otimes c) = \mathrm{d}_b.c$ (extended linearly to the group algebra $\mathbb{C}\mathbb{L}$), then the commutation relations can also be written as

$$m_{\mathcal{A}} \circ (j_k \otimes j_l) = m_{\mathcal{A}} \circ (j_l \otimes j_k) \circ (\alpha \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \tau) \circ (\gamma \otimes \mathrm{id}), \qquad \text{for } k < l, \tag{6.6}$$

where $m_{\mathcal{A}}$ is the multiplication in $\mathcal{A}$ and τ the flip automorphism.

We will also speak of Ψ-independence or braided independence, without checking if Ψ originates from an action and coaction as in Equation (6.6), if we have

$$m_{\mathcal{A}} \circ (j_k \otimes j_l) = m_{\mathcal{A}} \circ (j_l \otimes j_k) \circ \Psi$$

for all $k < l$.

6.4.3 Preliminary results

Before presenting explicit results on particular (braided) Hopf algebras, we give a lemma showing a link between left and right (weakly) Gaussian functionals in the sense of Bernstein.

Taking a Hopf algebra H whose antipode S is invertible, we can define three Hopf algebras H^{op}, H^{cop}, H^{opcop}, if we replace the multiplication or the comultiplication by their opposites $m^{\mathrm{op}} = m \circ \tau$ or $\Delta^{\mathrm{cop}} = \tau \circ \Delta$, and invert the antipode each time we take an opposite. Similarly, we can take opposites or co-opposites of braided Hopf algebras using the braiding or its inverse and inverting the braiding, i.e. the opposites lie in the braided category that has the opposite braiding Ψ^{-1}.

In particular, let $(\mathcal{B}, m, 1, \Delta, \varepsilon, S, \Psi)$ be a braided Hopf algebra with invertible antipode, and define $\Delta^{\mathrm{cop}} = \Psi^{-1} \circ \Delta$. Then $\mathcal{B}^{\mathrm{cop}} = (\mathcal{B}, m, 1, \Delta^{\mathrm{cop}}, \varepsilon, S^{-1}, \Psi^{-1})$ is also a braided Hopf algebra [Maj93a, Lemma 4.6]. Notice that we have to use the inverse of the braiding, $\tilde{\Delta}^{\mathrm{cop}} = \Psi \circ \Delta$ does not lead to a braided Hopf algebra. We can also define a braided Hopf algebra with opposite multiplication. Set $m^{\mathrm{op}} = m \circ \Psi^{-1}$, then $\mathcal{B}^{\mathrm{op}} = (\mathcal{B}, m^{\mathrm{op}}, 1, \Delta, \varepsilon, S^{-1}, \Psi^{-1})$ is a braided Hopf algebra. Combining the above operations we get the braided Hopf algebras $(\mathcal{B}^{\mathrm{cop}})^{\mathrm{op}} = (\mathcal{B}, m \circ \Psi, 1, \Psi^{-1} \circ \Delta, \varepsilon, S, \Psi)$ and $(\mathcal{B}^{\mathrm{op}})^{\mathrm{cop}} = (\mathcal{B}, m \circ \Psi^{-1}, 1, \Psi \circ \Delta, \varepsilon, S, \Psi)$, and, more generally, $\mathcal{B}^{k \times \mathrm{cop}\,\mathrm{op}} = (\mathcal{B}, m \circ \Psi^{k}, 1, \Psi^{-k}, \varepsilon, S, \Psi)$ for $k \in \mathbb{Z}$.. All these braided Hopf algebras are isomorphic, since the antipode is an isomorphism from $\mathcal{B}^{k \times \mathrm{cop}\,\mathrm{op}}$ to $\mathcal{B}^{(k+1) \times \mathrm{cop}\,\mathrm{op}}$. In the same way we can construct a chain of isomorphic braided Hopf algebras starting from $\mathcal{B}^{\mathrm{op}}$ or $\mathcal{B}^{\mathrm{cop}}$, in particular, $S : \mathcal{B}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{cop}}$ is an isomorphism.

We have the following lemma.

Lemma 6.4.6 *Let $j_1, j_2 : \mathcal{B} \to \mathcal{A}$ be a pair of Ψ-independent quantum random variables on $\mathcal{B}$ over some quantum probability space $(\mathcal{A}, \Phi)$. Then the following statements are equivalent.*

1. *$J(j_1, j_2) = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta$ and $K_r(j_1, j_2) = m_{\mathcal{A}} \circ (j_1 \otimes (j_2 \circ S)) \circ \Delta$ are independent (and thus $\varphi = \Phi \circ j_1$ is a (weakly) right Gaussian functional in the sense of Bernstein on $\mathcal{B}$),*

2. *$J^{\mathrm{cop}}(j_2, j_1) = m_{\mathcal{A}} \circ (j_2 \otimes j_1) \circ \Delta^{\mathrm{cop}}$ and $K_l^{\mathrm{cop}}(j_2, j_1) = m_{\mathcal{A}} \circ ((j_2 \circ S) \otimes j_1) \circ \Delta^{\mathrm{cop}}$ are independent,*

3. *$J^{\mathrm{op}}(j_2 \circ S, j_1 \circ S) = m_{\mathcal{A}} \circ ((j_2 \circ S) \otimes (j_1 \circ S)) \circ \Delta$ and $K_r^{\mathrm{op}}(j_2 \circ S, j_1 \circ S) = m_{\mathcal{A}} \circ ((j_2 \circ S \circ S) \otimes (j_1 \circ S)) \circ \Delta$ are independent.*

Proof: The equivalence of 2. and 3. is clear since $S : \mathcal{B}^{\mathrm{op}} \to \mathcal{B}^{\mathrm{cop}}$ is an isomorphism.

Let us show now that 1. and 2. are equivalent. The pair (j_1, j_2) is Ψ-independent if and only if (j_2, j_1) is Ψ^{-1}-independent.

We find

$$\begin{aligned} J(j_1, j_2) &= m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta \\ &= m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Psi \circ \Psi^{-1} \Delta \\ &= m_{\mathcal{A}} \circ (j_2 \otimes j_1) \circ \Delta^{\mathrm{cop}} \\ &= J^{\mathrm{cop}}(j_2, j_1), \end{aligned}$$

and

$$\begin{aligned} K_r(j_1, j_2) &= m_{\mathcal{A}} \circ (j_1 \otimes (j_2 \circ S)) \circ \Delta \\ &= m_{\mathcal{A}} \circ ((j_2 \circ S) \otimes j_1) \circ \Delta^{\mathrm{cop}} \\ &= K_l^{\mathrm{cop}}(j_2, j_1), \end{aligned}$$

so that $J(j_1, j_2)$ and $K_r(j_1, j_2)$ are independent exactly when $J^{\mathrm{cop}}(j_2, j_1)$ and $K_l^{\mathrm{cop}}(j_2, j_1)$ are independent. $\diamondsuit$

To use this lemma to relate (weakly) left Gaussian functionals in the sense of Bernstein to (weakly) right Gaussian functionals in the sense of Bernstein on the opposite or co-opposite quantum/braided group we have to deal with two technical difficulties. First, we have to assume that j_1, j_2 are Ψ-independent. In the examples considered we see that the condition that $j_1 \star j_2$ is also an algebra homomorphism is sufficient to show that j_1, j_2 satisfy the commutation relations for Ψ-independence.

Second, in the definition of $K_l^{\mathrm{op}}, K_l^{\mathrm{cop}}$ we have the original antipode. We should use the antipode of the opposite or co-opposite quantum/braided group though, i.e. S^{-1}. But in the applications we will see that the condition that $j_1 \star (j_2 \circ S)$ is an algebra homomorphism implies that there is no difference between $j_2 \circ S$ and $j_2 \circ S^{-1}$.

6.4.4 The braided line

If we look for left or right Gaussian functionals in the sense of Bernstein on the braided line $\mathbb{R}_q$ we find only the trivial solution $\varphi = \varepsilon$ (the counit). But, as the following theorem shows, we can find non-trivial weakly Gaussian functionals in the sense of Bernstein.

Theorem 6.4.7 *Let φ be a weakly right Gaussian functional in the sense of Bernstein on the braided line $\mathbb{R}_q$, $q \neq 0, 1$, and q not a root of unity. Then there exists a constant $a \in \mathbb{C}$ such that φ is given by*

$$\varphi(x^n) = a^n.$$

Conversely, if φ is of this form, then it is a weakly right Gaussian functional in the sense of Bernstein.

Proof: We shall determine the pairs of functionals (φ, ψ) on $\mathbb{R}_q$ such that there exists a quantum probability space $(\mathcal{A}, \Phi)$ and independent quantum random variables $j_1, j_2 : \mathbb{R}_q \to \mathcal{A}$ such that $\varphi = \Phi \circ j_1$ and $\psi = \Phi \circ j_2$ and the applications

$$J = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ \Delta_{\mathbb{R}_q} \text{ and } K_r = m_{\mathcal{A}} \circ (j_1 \otimes j_2) \circ (\mathrm{id} \otimes S) \circ \Delta_{\mathbb{R}_q} \tag{6.7}$$

form a pair of independent quantum random variables. This way we will also see that there are no right Gaussian functionals in the sense of Bernstein.

Step 1. We can assume that $\mathcal{A}$ is of the form $\mathbb{R}_q \tilde{\otimes} \mathbb{R}_q / \{(x')^2 = 0\} = \mathbb{C} << x, x' >> / \{x'x = qxx', (x')^2 = 0\}$, where $\mathbb{C} << x, x' >>$ denotes the free complex algebra with two generators.

Lemma 6.4.8 *Suppose we are given an algebra $\mathcal{C}$ and two algebra homomorphisms $k_1, k_2 : \mathbb{R}_q \to \mathcal{C}$ such that $k_1 \star k_2$ and $k_1 \star (k_2 \circ S)$ are also algebra homomorphisms. Then there exists an algebra homomorphism $\pi : \mathbb{R}_q \tilde{\otimes} \mathbb{R}_q / \{(x')^2 = 0\} \to \mathcal{C}$ and $j_1, j_2 : \mathbb{R}_q \to \mathbb{R}_q \tilde{\otimes} \mathbb{R}_q / \{(x')^2 = 0\}$ with $j_1(x) = x$, $j_2(x) = x'$, and $k_i = \pi \circ j_i$, $i = 1, 2$.*

Proof of Lemma. Set $\pi(x) = k_1(x)$ and $\pi(x') = k_2(x)$. We can directly check the relations to verify that π is an algebra homomorphism. $\pi(x')\pi(x) = q\pi(x)\pi(x')$ follows since $k_1 \star k_2$ is an algebra homomorphism:

$$\begin{aligned} 0 &= k_1 \star k_2(x^2) - (k_1 \star k_2(x))^2 \\ &= \pi(x)^2 + (1+q)\pi(x)\pi(x') + \pi(x')^2 - (\pi(x) + \pi(x'))^2 \\ &= q\pi(x)\pi(x') - \pi(x')\pi(x), \end{aligned}$$

Similarly, checking on $x^2 = x \cdot x$ that $k_1 \star (k_2 \circ S)$ is an algebra homomorphism yields $\pi(x')^2 = 0$.

Step 2. If Φ is a functional on $\mathbb{R}_q \tilde{\otimes} \mathbb{R}_q / \{(x')^2 = 0\}$ such that j_1 and j_2 are independent, then Φ is of the form $\varphi \otimes \psi$, $\varphi : \mathbb{R}_q \to \mathbb{C}$, $\psi : \mathbb{R}_q / \{(x')^2 = 0\} \to \mathbb{C}$.

This follows from the factorisation property

$$\Phi(j_{\sigma(1)}(y) j_{\sigma(2)}(z)) = \Phi \circ j_{\sigma(1)}(y) \Phi \circ j_{\sigma(2)}(z)$$

(where $y, z \in \mathbb{R}_q$, $\sigma \in \mathcal{S}_2$), i.e. taking $\sigma = \mathrm{id}$ we can set $\varphi = \Phi \circ j_1$, $\psi = \Phi \circ j_2$. Furthermore, setting $y = x$ and $z = x^n$ we see

$$\begin{aligned} \psi(x)\varphi(x^n) &= \Phi(j_2(x) j_1(x^n)) = \Phi(x' x^n) = q^n \Phi(x^n x') \\ &= q^n \psi(x) \varphi(x^n). \end{aligned}$$

Since we suppose that q is not a root of unity this means $\varphi(x^n) = 0$ for all $n \geq 1$ if $\psi(x) \neq 1$. Vice versa, if $\varphi(x^n) \neq 0$ for some $n \geq 1$, then $\psi(x) = 0$.

Step 3. The only pair (φ, ψ) such that $J = j_1 \star j_2$ and $K_r = j_1 \star (j_2 \circ S)$ are independent is φ as given in Theorem 6.4.7, and $\psi = \varepsilon$.

a. Let us first consider the case $\psi(x) = c \neq 0$. Then $\varphi = \varepsilon$. We check the factorisation property for J and K_r on $y = z = x$:

$$\Phi(J(x) K_r(x)) = \Phi((x + x')(x - x')) = \Phi(x^2 + (q-1)xx' + (x')^2) = 0,$$

but

$$\Phi(J(x))\Phi(K_r(x)) = \Phi(x + x')\Phi(x - x') = -c^2$$

so that the factorisation property is only satisfied for $c = 0$.

b. So let now $\psi(x) = 0$, $\varphi(x^n) = c_n$. We check the factorisation property on $y = x^n$, $z = x^m$. We have $J(x^n) = j_1 \otimes j_2 \left(\sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_q x^\nu (x')^{n-\nu} \right) = x^n + q_n x^{n-1} x'$ and $K_r(x^m) = x^m - q_m x^{m-1} x'$, since $(x')^2 = 0$. For the product we get $J(x^n) K_r(x^m) = x^{n+m} + (q_n q^m - q_m) x^{n+m-1} x'$. This implies

$$c_{n+m} = \Phi(J(x^n) K_r(x^m)) = \Phi(J(x^n))\Phi(K_r(x^m)) = c_n c_m,$$

and thus $c_n = a^n$ with $a = c_1$.

Step 4. One checks without difficulty that J and K_r verify the factorisation property for all $\sigma \in \mathcal{S}_2$. But on $\mathbb{R}_q \tilde{\otimes} \mathbb{R}_q / \{(x')^2 = 0\}$ they don't satisfy the right commutation relations. To remedy this we quotient by $x' = 0$, this is possible without changing the

functionals since $\psi(x) = 0$. We get thus $\tilde{\mathcal{A}} = \mathbb{R}_q$, $\tilde{j}_1 = \mathrm{id} = J = K_r$, $\tilde{j}_2 = e \circ \varepsilon$ and $\varphi(x^n) = a^n$ for all n, and (J, K_r) satisfy (trivially) the conditions of Bose or symmetric independence.

Corollary 6.4.9 *The weakly left Gaussian functionals on the braided line $\mathbb{R}_q$ are exactly the functionals that can be written as*

$$\varphi(x^n) = a^n$$

for some constant $a \in \mathbb{C}$.

Proof: Note that the first step of the proof of Theorem 6.4.7 shows that the quantum random variables j_1 and j_2 have to be Ψ-independent and that $j_2 \circ S$ and $j_2 \circ S^{-1}$ are identical.

We show that $\mathbb{R}_{q^{-1}}$ and $\mathbb{R}_q^{\mathrm{cop}}$ are isomorphic, and then apply Lemma 6.4.6.

The opposite coproduct gives $\Delta x = x + x'$, if we extend (using the inverse braiding) as an algebra homomorphism we get $\Delta x^n = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_{q^{-1}} x^\nu (x')^{n-\nu}$. We see that the identity map is a braided Hopf algebra isomorphism between $\mathbb{R}_{q^{-1}}$ and $\mathbb{R}_q^{\mathrm{cop}}$. $\diamondsuit$.

6.4.5 The braided plane

Theorem 6.4.10 *The weakly right Gaussian functionals in the sense of Bernstein on the braided plane $\mathbb{C}_q^{2|0}$ are exactly the functionals of the form*

$$\varphi(x^n y^m) = \delta_{m,0} z^n \quad \text{or} \quad \varphi(x^n y^m) = \delta_{n,0} z^m$$

where $z \in \mathbb{C}$ is some constant.

Proof: The proof is based on the same ideas as in the case of the braided line, we indicate only the results of each step.

Step 1. We can suppose $\mathcal{A} = \mathbb{C}_q^{2|0} \tilde{\otimes} \mathbb{C}_q^{2|0} / \{(x')^2 = 0, x'y' = 0, (y')^2 = 0\}$.

Step 2. We can suppose $\Phi = \varphi \otimes \psi$ and

$$\psi(x) \neq 0 \;\Rightarrow\; \varphi(x^n y^m) = 0 \text{ if } n + m > 0$$
$$\psi(y) \neq 0 \;\Rightarrow\; \varphi(x^n y^m) = 0 \text{ if } n + m > 0$$

Step 3. We can show $\psi = \varepsilon$, and φ is of the form given in the theorem.

Step 4. We get $\tilde{\mathcal{A}} = \mathbb{C}_q^{2|0}$, $\tilde{j}_1 = \mathrm{id}$, $\tilde{j}_2 = 1 \circ \varepsilon$. $\diamondsuit$

Corollary 6.4.11 *A functional on the braided plane $\mathbb{C}_q^{2|0}$ is a weakly left Gaussian functional in the sense of Bernstein if and only if it is a weakly right Gaussian functional in the sense of Bernstein on $\mathbb{C}_q^{2|0}$.*

Proof: We use Lemma 6.4.6, as for the braided line. The isomorphism between $\left(\mathbb{C}_q^{2|0}\right)^{\mathrm{cop}}$ and $\mathbb{C}_{q^{-1}}^{2|0}$ is defined by $x^n y^m \mapsto y^n x^m$. $\diamondsuit$

6.4.6 The braided q-Heisenberg algebra

Theorem 6.4.12 *Let $q \in \mathbb{C}\backslash\{0\}$, $q \neq 1$, and q not a root of unity.*

The weakly right Gaussian functionals in the sense of Bernstein on the braided q-Heisenberg-Weyl algebra are exactly the functionals which can be defined by

$$\varphi(a^n d^r c^s) = m_a^n m_c^s \delta_{r0}$$

with constants $m_a, m_c \in \mathbb{C}$.

Proof: We proceed as for the braided line.

Step 1. We can assume that $\mathcal{A}$ is of the form $\mathrm{HW}_q \tilde{\otimes} \mathrm{HW}_q / \{c'a' - 2b' = 0, (a')^2 = 0, (c')^2 = 0\}$

Lemma 6.4.13 *Suppose we are given an algebra $\mathcal{C}$ and two algebra homomorphisms $k_1, k_2 : \mathrm{HW}_q \to \mathcal{C}$ such that $k_1 \star k_2$ and $k_1 \star (k_2 \circ S)$ are also algebra homomorphisms. Then (in addition to the HW_q-relations) $k_i(a), k_i(b), k_i(c)$, $i = 1, 2$, satisfy the following relations:*

$$\begin{array}{rrr}
k_2(a)k_1(a) = qk_1(a)k_2(a) & k_2(a)k_1(b) = qk_1(b)k_2(a) & k_2(a)k_1(c) = k_1(c)k_2(a) \\
k_2(b)k_1(a) = k_1(a)k_2(b) & k_2(b)k_1(b) = qk_1(b)k_2(b) & k_2(b)k_1(c) = qk_1(c)k_2(b) \\
k_2(c)k_1(a) = \frac{1}{q}k_1(a)k_2(c) & k_2(c)k_1(b) = k_1(b)k_2(c) & k_2(c)k_1(c) = qk_1(c)k_2(c) \\
k_2(a)^2 = 0 & k_2(c)k_2(a) = 2k_2(b) & k_2(c)^2 = 0.
\end{array}$$

Proof of Lemma: The first nine relations are exactly the braid relations, they follow from the condition that $k_1 \star k_2$ is an algebra homomorphism.

The last three relations follow if we also require $k_1 \star (k_2 \circ S)$ to be an algebra homomorphism. $\diamondsuit$

We shall from now on simply write a, b, c for $j_1(a), j_1(b), j_1(c)$ and a', b', c' for $j_2(a), j_2(b), j_2(c)$.

Step 2. If j_1 and j_2 are independent, then Φ can be written as the tensor product of two functionals, i.e. $\Phi = \varphi \otimes \psi$. Furthermore, we have the following restrictions

$$\begin{array}{lll}
\psi(a) \neq 0 & \Rightarrow & \varphi(a^n b^m c^r) = 0 \text{ if } n \neq 0 \text{ or } m \neq 0 \\
\psi(b) \neq 0 & \Rightarrow & \varphi(a^n b^m c^r) = 0 \text{ if } m \neq 0 \text{ or } r \neq 0 \\
\psi(c) \neq 0 & \Rightarrow & \varphi(a^n b^m c^r) = 0 \text{ if } n \neq r.
\end{array}$$

To show this we apply the factorisation property twice, e.g.

$$\psi(c)\varphi(a^n b^m c^r) = \Phi(c' a^n b^m c^r) = q^{r-n}\Phi(a^n b^m c^r c') = q^{r-n}\varphi(a^n b^m c^r)\psi(c).$$

Step 3. $J = j_1 \star j_2$ and $K_r = j_1 \star (j_2 \circ S)$ only satisfy the factorisation property if $\psi = \varepsilon$ and φ has the form given in Theorem 6.4.12.

First we show $\psi = \varepsilon$. Suppose $\psi(a) \neq 0$. Then φ is of the form $\varphi(a^n b^m c^r) = \delta_{n0}\delta_{m0} m_r$. Checking the factorisation property for J and K_r on a leads to a contradiction:

$$\begin{aligned}
0 &= \Phi((a + a')(a - a')) = \Phi(J(a)K_r(a)) = \Phi(J(a))\Phi(K_r(a)) \\
&= \Phi(a + a')\Phi(a - a') = -\psi(a)^2,
\end{aligned}$$

and therefore $\psi(a) = 0$. In the same way we obtain $\psi(b) = \psi(c) = 0$.

So we have seen that all non-trivial moments of ψ vanish. Let $\varphi(a^n d^m c^r) = m_{nmr}$, then

$$\begin{aligned} \varphi_J(a^n d^m c^r) &= \Phi \circ J(a^n d^m c^r) = m_{nmr} \\ \varphi_{K_r}(a^n d^m c^r) &= \Phi \circ K_r(a^n d^m c^r) = m_{nmr}, \end{aligned}$$

and the factorisation property implies

$$\begin{aligned} \varphi_{K_r}(a^{n_1} d^{m_1} c^{r_1}) \varphi_J(a^{n_2} d^{m_2} c^{r_2}) &= \Phi(K_r(a^{n_1} d^{m_1} c^{r_1}) J(a^{n_2} d^{m_2} c^{r_2})) \\ &= \Phi(a^{n_1} d^{m_1} c^{r_1} a^{n_2} d^{m_2} c^{r_2}). \end{aligned}$$

which leads to

$$\begin{aligned} m_{n_1 m_1 r_1} m_{n_2 m_2 r_2} &= \sum_{k=0}^{r_1 \wedge n_2} \frac{(q^{-1})_{r_1}!(q^{-1})_{n_2}!(-1)^k}{(q^{-1})_{r_1-k}!(q^{-1})_{n_2-k}!(q^{-1})_k!} \\ &\quad \times q^{-m_1(n_2-k)} q^{m_2(r_1-k)} m_{n_1+n_2-k, m_1+m_2+k, r_1+r_2-k} \end{aligned}$$

Taking only the n's different from zero (or the m's, or the r's), we immediately see that all moments are completely determined by the first order moments, $m_{n00} = (m_{100})^n$ ($m_{0m0} = (m_{010})^m$, $m_{00r} = (m_{001})^r$), and $m_{nmr} = m_{100}^n m_{010}^m m_{001}^r$.

Furthermore, choosing (100) and (001) we get

$$m_{101} = m_{100} m_{001} = m_{001} m_{100} = m_{101} - m_{010},$$

i.e. $m_{010} = 0$.

The general solution is thus given by

$$\varphi(a^n d^m c^r) = \delta_{m0} m_a^n m_c^r,$$

(one checks that this really defines a solution).

Step 4. We can now take $\mathcal{A} = \mathrm{HW}_q/\{d = 0\} = \mathbb{C}[a, c]$, i.e. the algebra of polynomials in two variables, and $j_1 = \pi = J = K_r$ and $j_2 = \pi \circ 1 \circ \varepsilon$ with out changing the functionals. π is here the canonical projection $\pi : \mathrm{HW}_q \to \mathrm{HW}_q/\{d = 0\}$. $\diamondsuit$

Corollary 6.4.14 *The sets of weakly left and weakly right Gaussian functionals in the sense of Bernstein on the braided q-Heisenberg-Weyl group* HW_q *are identical.*

Proof: Notice that $a^n d^m c^r \mapsto (-1)^m c^n d^m a^r$ defines a braided Hopf algebra isomorphism between $\mathrm{HW}_q^{\mathrm{cop}}$ and $\mathrm{HW}_{q^{-1}}$, and apply Lemma 6.4.6. $\diamondsuit$

6.4.7 Positivity

The braided line. Let $q \in \mathbb{R}$. Then $x^* = x$ defines a *-structure on $\mathbb{R}_q$. One easily verifies the following result.

Proposition 6.4.15 *Let* $q \in \mathbb{R}$, $q \notin \{-1, 0, 1\}$.

The positive weakly right Gaussian functionals in the sense of Bernstein on the braided line $\mathbb{R}_q$ *are exactly the linear functionals* φ *such that*

$$\varphi(x^n) = c^n \qquad \textit{for all } n \in \mathbb{N},$$

for some real constant $c \in \mathbb{R}$.

The braided plane. If q is real then we can define a *-structure on the braided plane $\mathbb{C}_q^{2|0}$ by $x^* = y$, $y^* = x$. The only weakly right Gaussian functional in the sense of Bernstein that is positive with respect to this *-structure is the counit ε. To see this evaluate a weakly right Gaussian functional in the sense of Bernstein φ_z (where $\varphi_z(x^n y^m) = z^n \delta_{m,0}$ or $z^m \delta_{n,0}$, $z \in \mathbb{C}$) on a^*a for $a = x + y$ and $a = x - y$. We get

$$\begin{aligned} \varphi_z\big((x+y)^*(x+y)\big) &= \varphi_z(x^2 + (1+q)xy + y^2) = z^2, \\ \varphi_z\big((x-y)^*(x-y)\big) &= \varphi_z(-x^2 + (1+q)xy - y^2) = -z^2, \end{aligned}$$

so that φ_z can only be positive, if $z = 0$.

The braided q-Heisenberg-Weyl group. Let q be real, and consider the *-structure defined by $a^* = c$, $c^* = a$, $d^* = d$.

Proposition 6.4.16 *Let $q \in \mathbb{R}$, $q \notin \{-1, 0, 1\}$.*

The weakly right Gaussian functionals in the sense of Bernstein on the braided q-Heisenberg-Weyl group HW_q *are positive if and only if $m_a = \overline{m_c}$.*

Proof: In the proof of Theorem 6.4.12 we saw that we can assume that j_1, j_2, J, K_r take values in the algebra of polynomials in two variables, denoted e.g. by $\alpha = j_1(a), \gamma = j_1(c)$. For j_1, j_2, J, K_r to be *-algebra homomorphisms we need $\alpha^* = \gamma$ and $\gamma^* = \alpha$. Introduce $x = \alpha + \gamma$, $y = i(\alpha - \gamma)$, then $x^* = x$, $y^* = y$. In these coordinates φ is given by $\varphi(x^n y^m) = (m_a + m_c)^n i^m (m_a - m_c)^m$. We see that φ is positive if and only if $m_a + m_c$ and $i(m_a - m_c)$ are real, i.e. if $m_a = \overline{m_c}$. $\diamondsuit$

6.4.8 Remarks

The result of Section 6.4 has to be considered as a negative result. The only Gaussian functionals in the sense of Bernstein that we found, can be considered as deterministic or Dirac laws, in the sense of the following definition.

Definition 6.4.17 *A state ϕ on $\mathcal{B}$ is called* deterministic, *or a* Dirac law, *if there exists a quantum probability space $(\mathcal{A}, \Phi)$ and a quantum random variable $j : \mathcal{B} \to \mathcal{A}$ such that the pair (j, j) is independent and $\Phi \circ j = \phi$.*

The Bernstein functionals obtained here do not form convolution semigroups. It is interesting to remark that they can be considered as quantum convolution semigroups, though. Note that a continuous convolution semigroup (ccs) $\{\mu_t; t \in \mathbb{R}_+\}$ (see Section 6.5 for their definition) defines a map from $\mathcal{A}$ to the algebra of continuous functions on $\mathbb{R}_+$ via $f_a(t) = \mu_t(a)$ such that

$$f_a(t + t') = \sum f_{a^{(1)}}(t) f_{a^{(2)}}(t') \qquad \text{and} \qquad f_a(0) = \varepsilon(a),$$

where $\Delta a = \sum a^{(1)} \otimes a^{(2)}$ (Sweedler's notation). But $f(t) \mapsto f(t + t')$ and $f(t) \mapsto f(0)$ are nothing else but the coproduct and counit of the algebra of functions on the semigroup $(\mathbb{R}_+, +)$, i.e. the map $a \mapsto f_a(t)$ is a coalgebra homomorphism.

This suggests the following definition, if one wants to "deform" the t-dependence of convolution semigroups.

Definition 6.4.18 *Let $\mathcal{C}$ be a coalgebra. A $\mathcal{C}$-*quantum convolution semigroup *($\mathcal{C}$-qcs) on $\mathcal{A}$ is a coalgebra homomorphism from $\mathcal{A}$ to $\mathcal{C}$.*

The family of weakly right Gaussian functionals in the sense of Bernstein on the braided plane $\mathbb{C}_q^{2|0}$ e.g. gives two families of $\mathbb{R}_q$-qcs $x^n y^m \mapsto \alpha^n z^n \delta_{m0}$ and $x^n y^m \mapsto \alpha^m z^m \delta_{n0}$, where z denotes now the generator of $\mathbb{R}_q$, and $\alpha \in \mathbb{C}$ is some constant.

The weakly right Gaussian functionals in the sense of Bernstein on the braided q-Heisenberg-Weyl group also give rise to $\mathbb{R}_q$-qcs if one of the constants is replaced by the generator of $\mathbb{R}_q$. The original functional can be recuperated, if the $\mathbb{R}_q$-qcs are composed with weakly right Gaussian functionals in the sense of Bernstein on $\mathbb{R}_q$.

The family of one-dimensional distributions of the quantum Brownian motion defined by Bozejko and Speicher [BS91] can also be interpreted as a $\mathbb{R}_q$-qcs (for $0 < q < 1$). These functionals can be represented on the braided q-Heisenberg-Weyl group as

$$\varphi_t(a^n b^m c^r) = t^m \delta_{n0} \delta_{r0}, \qquad \forall n, m, r \in \mathbb{N}.$$

To see this consider the unitary representation of HW_q on the Hilbert space $H = \overline{\mathrm{span}\{\psi_n; n \in \mathbb{N}\}}$ with inner product $< \psi_n | \psi_m >= \delta_{nm} q_n!$, defined by

$$\begin{aligned} \rho_t(a)\psi_n &= t\psi_{n+1} \\ \rho(c)\psi_n &= \begin{cases} 0 & \text{if } n = 0 \\ q_n \psi_{n-1} & \text{if } n > 0 \end{cases} \\ \rho(b)\psi_n &= t\psi_n. \end{aligned}$$

Then in the state $\Phi =< \psi_0 | \cdot \psi_0 >$ we get $\Phi \circ \rho_t = \varphi_t$.

Interpreting t as the generator of $\mathbb{R}_q$ we have a $\mathbb{R}_q$-qcs: $a^n b^m c^r \mapsto t^m \delta_{n0} \delta_{r0}$.

6.5 Uniqueness of embedding

A *continuous convolution semi-group* (ccs) of functionals on a (braided) Hopf algebra $\mathcal{A}$ is a family of functionals $\{\varphi_t : \mathcal{A} \to \mathbb{C}; t \in \mathbb{R}_+\}$ indexed by $\mathbb{R}_+$ such that

1. $\varphi_0 = \varepsilon$ (the counit).
2. $\varphi_s \star \varphi_t = \varphi_{s+t}$ for all $s, t \in \mathbb{R}_+$. Here the convolution is defined by $\phi \star \psi = (\phi \otimes \psi) \circ \Delta$,
3. $\varphi_t(a)$ is continuous as a function of t for all $a \in \mathcal{A}$.

M. Schürmann [Sch93] has shown that every ccs on a (braided) Hopf algebra has a generator ψ and can be expressed as a (convolution) exponential $\varphi_t = \exp_\star(t\psi) = \varepsilon + t\psi + \frac{t^2}{2}\psi \star \psi + \cdots + \frac{t^n}{n!}\psi^{\star n} + \cdots$. Here we are interested in the question whether such a ccs is already uniquely determined by its value for one fixed value $t \neq 0$. D. Neuenschwander and others [Neu96, Bal85, Pap94] have shown on connected simply-connected nilpotent Lie

groups that if a probability measure is embeddable, then this embedding is unique (under some regularity conditions that guarantee the existence of moments). In this section we will introduce a notion of nilpotence and show that the above uniqueness result can be extended to nilpotent (braided) Hopf algebras.

For Hopf algebras in general this result is not true, we can e.g. translate the counter-example for ccs on $\mathbb{Z}_3$ from Heyer's book [Hey77] to a counter-example on the Hopf algebra of functions on $\mathbb{Z}_3$. This counter-example also shows that our notion of nilpotence can not include all algebras of functions on nilpotent groups.

6.5.1 Definition of nilpotence

Nilpotent connected simply-connected Lie groups are diffeomorphic to some $\mathbb{R}^n$, and the group law can be brought into the following form (e.g. with the Campbell-Hausdorff formula)

$$(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (z_1, \ldots, z_n)$$

where the z_i are of the form

$$z_i = x_i + y_i + p_i(x_1, \ldots, x_{i-1}, y_1, \ldots, y_{i-1})$$

where the p_i are polynomials. We will use the analogue of this property as a definition of nilpotence for (braided) Hopf algebras.

Definition 6.5.1 *Let $\mathcal{A}$ be a (braided) Hopf algebra with unit element 1. We say that $\mathcal{A}$ is* nilpotent *if there exists a well-ordered set $\mathcal{I}$ and a basis $\mathcal{V} = \{a_\imath; \imath \in \mathcal{I}\}$ indexed by $\mathcal{I}$ such that for all $a_\imath \in \mathcal{V}$*

$$\Delta(a_\imath) = a_\imath \otimes 1 + 1 \otimes a_\imath + \sum_{\imath', \imath'' < \imath} C^{\imath}_{\imath', \imath''} a_{\imath'} \otimes a_{\imath''} \tag{6.8}$$

where (for fixed $\imath$) only finitely many $C^{\imath}_{\imath', \imath''} \neq 0$.

The above discussion shows that the Hopf algebras of polynomials on connected simply-connected nilpotent Lie groups are nilpotent in the sense of our definition.

For the "unbraided" or quantum group case (i.e. $\Psi = \tau$) there exists a nilpotent Lie algebra that is naturally associated to a nilpotent quantum group. It is obtained in the same way that can be used to reconstruct the Lie algebra of a Lie group from the Hopf algebra of smooth functions on the groups, i.e. as the space of derivations at the neutral element.

Proposition 6.5.2 *Let $\mathcal{A}$ be a nilpotent quantum group and suppose that*

$$L(\mathcal{A}) = \{f : \mathcal{A} \to \mathbb{C} \text{ linear} | f(ab) = f(a)\varepsilon(b) + \varepsilon(a)f(b)\}$$

is finite-dimensional. Then $L(\mathcal{A})$ is a nilpotent Lie algebra with the commutator $[f, g] = m(f \otimes g) - m(g \otimes f)$, $m = \Delta^*_{\mathcal{A}}$.

Proof: $L(\mathcal{A})$ is the set of primitive elements of the dual of $\mathcal{A}$, and thus a Lie algebra. Set $L^\imath = \{f \in L(\mathcal{A}) | f(a_{\imath'}) = 0 \text{ for } \imath' < \imath\}$. We have $[L(\mathcal{A}), L^\imath] \subseteq L^{\imath+1}$, since

$$\begin{aligned}[f,g](a_{\imath'}) &= (f \otimes g - f \otimes g)\left(\sum_{\imath'',\imath''<\imath'} C^{\imath}_{\imath'',\imath'''} a_{\imath''} \otimes a_{\imath'''}\right) \\ &= 0\end{aligned}$$

for $f \in L$, $g \in L^\imath$, $\imath' \leq \imath$. But since $L(\mathcal{A})$ is finite-dimensional, this implies that we actually have a finite sequence $L_0 = L(\mathcal{A}) \supset L^1 \supset \cdots \supset L^{r+1} = \{0\}$ of ideals, and thus that $L_0 = L(\mathcal{A})$ is nilpotent. $\diamondsuit$

There does not seem to be a natural generalisation of this result for braided groups, because the primitive elements of a braided Hopf algebra do not have an obvious Lie algebra structure.

6.5.2 Poincaré-Birkhoff-Witt property and nilpotence

We say that a (braided) Hopf algebra $\mathcal{A}$ has the Poincaré-Birkhoff-Witt (PBW) property if there exists a set of elements (called generators) $\{a_1, \ldots, a_d\}$ such that $\{a_1^{k_1} \cdots a_d^{k_d}; k = (k_1, \ldots, k_d) \in \mathbb{N}^d\}$ is a basis of $\mathcal{A}$. We shall also use the multi-index notation $a^k = a_1^{k_1} \cdots a_d^{k_d}$.

Proposition 6.5.3 *Let $\mathcal{A}$ have PBW property with generators $a_1, \ldots, a_d$. If for all $i, j = 1, \ldots, d$*

$$[a_i, a_j] = \sum_{k \in \mathbb{N}^{\max(i,j)-1}} P_{i,j,k} a^k, \tag{6.9}$$

$$\Delta(a_i) = a_i \otimes 1 + 1 \otimes a_i + \sum_{k,k' \in \mathbb{N}^{i-1}} D^i_{k,k'} a^k \otimes a^{k'} \tag{6.10}$$

then $\mathcal{A}$ is nilpotent.

Proof: We are going to use the index set $\mathcal{I} = \mathbb{N}^d$ with the lexicographic order and the PBW basis $\mathcal{V} = \{a_1^{k_1} \cdots a_d^{k_d}; k = (k_1, \ldots, k_d) \in \mathbb{N}^d\}$. Let $0 \leq \delta \leq d$ and define $\mathcal{A}^{(\delta)}$ to be the linear span of $\{a_1^{k_1} \cdots a_\delta^{k_\delta}; k = (k_1, \ldots, k_\delta) \in \mathbb{N}^\delta\}$. By the condition on the product, $\mathcal{A}^{(\delta)}$ is a subalgebra. Furthermore, we even have $[a_\delta, \mathcal{A}^{(\delta-1)}] \subseteq \mathcal{A}^{(\delta-1)}$. Thus

$$\begin{aligned}\Delta(a_1^{k_1} \cdots a_\delta^{k_\delta}) &= \Delta(a_1)^{k_1} \cdots \Delta(a_\delta)^{k_\delta} \\ &= (a_1 \otimes 1 + 1 \otimes a_1)^{k_1} \left(a_2 \otimes 1 + 1 \otimes a_2 + \sum_{r,s} D^2_{r,s} a_1^r \otimes a_1^s\right)^{k_2} \times \cdots \\ &\quad \times \left(a_\delta \otimes 1 + 1 \otimes a_\delta + \sum_{\imath',\imath'' \in \mathbb{N}^{\delta-1}} D^\delta_{\imath',\imath''} a^{\imath'} \otimes a^{\imath''}\right)^{k_\delta} \\ &= \sum_{k'+k''=k} \prod_{i=1}^{\delta} \binom{k_i}{k_i'} a^{k'} \otimes a^{k''} + K,\end{aligned}$$

where $K \in \mathcal{A}^{(\delta-1)} \otimes \mathcal{A}^{(\delta-1)}$, so that $\Delta(a^k)$ is of the form required in Definition 6.5.1. $\diamondsuit$

Remark: A similar result for braided Hopf algebras with PBW property can be formulated if we require in addition that the braiding $a_i' a_j = \psi(a_i \otimes a_j) = \sum_{kl} R_{ij}^{kl} a_l \otimes a_k$ doesn't involve generators that come "after" a_i and a_j, i.e.

$$R_{ij}^{kl} = 0 \qquad \text{if} \qquad k, l \geq \max(i, j). \tag{6.11}$$

The following result is immediate from the formula for the coproduct of the HW_q algebra, see Equation (6.3).

Proposition 6.5.4 *The braided q-Heisenberg-Weyl algebra* HW_q *is nilpotent.*

6.5.3 Uniqueness of embedding

Lemma 6.5.5 *Let $\mathcal{A}$ be a nilpotent (braided) Hopf algebra and $\Phi : \mathcal{A} \to \mathbb{C}$ a normalized functional on $\mathcal{A}$. Then there exists a unique normalized functional $\Xi : \mathcal{A} \to \mathbb{C}$ such that*

$$\Phi = \Xi \star \Xi. \tag{6.12}$$

Proof: We can directly calculate the action of Ξ on the basis $\mathcal{V}$. On 1 we have

$$1 = \Phi(1) = (\Xi \otimes \Xi)(\Delta 1) = \Xi(1)^2. \tag{6.13}$$

so that Ξ can be chosen to be normalized, i.e. $\Xi(1) = 1$. Now Ξ is uniquely determined by

$$\Phi(a_\imath) = (\Xi \otimes \Xi)(\Delta(a_\imath)). \tag{6.14}$$

In fact, suppose we have calculated $\Xi(a_\imath)$ already for all $\imath \in M \subseteq \mathcal{I}$. Then

$$\Xi(a_\kappa) = \frac{1}{2}\left\{\Phi(a_\kappa) - \sum_{\imath',\imath''<\kappa} C_{\imath',\imath''}^{\kappa} \Xi(a_{\imath'})\Xi(a_{\imath''})\right\} \tag{6.15}$$

allows to calculate $\Xi(a_\kappa)$ for $\kappa = \min \mathcal{I} \backslash M$. $\diamondsuit$

Theorem 6.5.6 *If ϕ is normalized, infinitely divisible, and embeddable, then the embedding is unique.*

Proof: Lemma 6.5.5 implies that the embedding is unique on the dyadic numbers. The rest follows from continuity. $\diamondsuit$

6.5.4 Remark

Note that e.g. on $\mathbb{R}$ we can also find a "root" for functionals that are not divisible (as probability measures). Take e.g. a symmetric Bernoulli law (concentrated on ± 1). The moments are

$$\Phi(x^n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases} \tag{6.16}$$

Then (by the lemma in Subsection 3) there exists a normalized functional Ξ such that $\Phi = \Xi \star \Xi$. But Ξ does not correspond to a probability measure.

We can calculate $\Xi_n = \Xi(x^n)$ recursively by

$$\Xi_n = \frac{1}{2}\left\{1 - \sum_{\nu=1}^{n-1} \binom{n}{\nu} \Xi_\nu \Xi_{n-\nu}\right\} \qquad \text{if } n \text{ is even,} \tag{6.17}$$

$$\Xi_n = -\frac{1}{2}\sum_{\nu=1}^{n-1} \binom{n}{\nu} \Xi_\nu \Xi_{n-\nu} \qquad \text{if } n \text{ is odd.} \tag{6.18}$$

All odd moments vanish, and $\Xi_0 = 1$, $\Xi_2 = \frac{1}{2}$, $\Xi_4 = -1$, etc. We see that Ξ is not positive.

6.6 Gaussian semigroups in the sense of Bernstein

Since the results of Section 6.4 must be interpreted as a rather "negative" result, the question arises if there are "better" extensions of the Bernstein property to quantum groups and braided groups.

Let us consider the situation on Lie groups. If from the beginning one suggests that the measure is embeddable into a continuous convolution semigroup (ccs) of probability measures on the Lie group G, then a natural version of the Bernstein theorem on G is given in Hazod ([Haz77], Satz III.3.3) and Heyer ([Hey77], Theorem 8.4) by considering the generating distribution of the ccs. A ccs $\{\mu_t\}_{t\geq 0} \subset M^1(G)$ is a topological monoid homomorphism

$$[0,\infty[\ni t \mapsto \mu_t \in (M^1(G), \delta_e, \xrightarrow{w}).$$

The generating distribution of $\{\mu_t\}_{t\geq 0}$ is given by

$$\mathcal{A}(f) := \lim_{t\to 0+} \frac{1}{t}\int_G [f(x) - f(0)]\mu_t(dx)$$

for bounded complex-valued C^∞-functions f on G (cf. Siebert [Sie81, p. 119]). We write $\mu_t = \exp t\mathcal{A}$. Then $\mathcal{A}$ has always the form

$$\mathcal{A}(f) = \sum_{i=1}^{d} a_i X_i f(0) + \sum_{i,j=1}^{d} b_{i,j} X_i X_j f(0) + \int_{G\setminus\{0\}} [f(x) - f(0) - \Phi(f,x)]\eta(dx)$$

where $a = (a_1, a_2, \ldots, a_d) \in I\!R^d$, $M = (b_{i,j})_{1\leq i,j\leq d}$ is a symmetric positive semidefinite $d \times d$-matrix, η is a Lévy measure (i.e. a non-negative measure on $G\setminus\{0\}$ such that $\int_{G\setminus\{0\}} x^2/(1+x^2)\eta(dx) < \infty$),

$$\Phi(f,x) := \begin{cases} \langle x, \nabla\rangle f(0) & : \quad ||x|| \leq 1, \\ \langle \frac{x}{||x||}, \nabla\rangle f(0) & : \quad ||x|| > 1 \end{cases}$$

and $X_1, X_2, \ldots, X_d$ is a fixed basis of the Lie algebra $\mathcal{G}$ of G. (Cf. Siebert [Sie73, Satz 1]) For short, we will write $\mathcal{A} = (a, M, \eta)$. The ccs $\{\exp t\mathcal{A}\}_{t\geq 0}$ is called Gaussian if

$\mathcal{A} = (a, M, 0)$. Now the "semigroup version" of the Bernstein theorem reads as follows: Let $H := G \times G$ and define the map $\varphi : H \to H$ by $\varphi(x, y) := (x \cdot y, x \cdot y^{-1})$.

Theorem 6.6.1 *Let $\mathcal{A}_0$ be a generating distribution on G and consider the generating distribution $\mathcal{A} := \mathcal{A}_0 \otimes \delta_0 + \delta_0 \otimes \mathcal{A}_0$ on H. Furthermore let $\mathcal{B}(f) := \mathcal{A}(f \circ \varphi^{-1})$ and define, for any generating distributions $\mathcal{B}_1, \mathcal{B}_2$ on G, the generating distribution $\mathcal{B}_3$ on H by $\mathcal{B}_3 := \mathcal{B}_1 \otimes \delta_0 + \delta_0 \otimes \mathcal{B}_2$. There exist $\mathcal{B}_1, \mathcal{B}_2$ such that $\mathcal{B} = \mathcal{B}_3$ iff $\mathcal{A}_0, \mathcal{A}, \mathcal{B}, \mathcal{B}_i$ $(1 \leq i \leq 3)$ are Gaussian.*

6.6.1 Definition of Gaussian convolution semigroups in the sense of Bernstein

Schürmann [Sch93] tells us that every convolution semigroup on a quantum or braided group has a unique generator. We will use it to define Gaussian convolution semigroups. The map $\varphi : (x, y) \mapsto (x + y, x - y)$ becomes $\Phi = (m \otimes m) \circ (\mathrm{id} \otimes \Psi \otimes S) \circ (\Delta \otimes \Delta)$ on (braided) Hopf algebras.

Definition 6.6.2 *The semigroup μ_t on $\mathcal{A}$ with generator A is called* Gaussian in the sense of Bernstein *if $(A \otimes \varepsilon + \varepsilon \otimes A) \circ \Phi^{-1}$ can be written in the form $B_1 \otimes \varepsilon + \varepsilon \otimes B_2$. We will call μ_t* weakly Gaussian in the sense of Bernstein *if there exists another semigroup μ'_t with generator A' such that $(A \otimes \varepsilon + \varepsilon \otimes A') \circ \Phi^{-1}$ can be written in the form $B_1 \otimes \varepsilon + \varepsilon \otimes B_2$.*

6.6.2 General results

Since Φ is linear, it follows directly form the definition that the generators of Gaussian semigroups in the sense of Bernstein and those of weakly Gaussian semigroups in the sense of Bernstein form vector spaces. If $\mathcal{A}$ has a involution $*$ then we want to consider only semigroups of positive functionals. If we restrict ourselves to α-invariant functionals, then $\mu_t = \exp_\star tA$ is positive for all $t \geq 0$, if and only if A is hermitian and conditionally positive, i.e. positive on the kernel of the counit ε, cf. [Sch93, Theorem 3.2.7].

The following proposition is the analogue of one direction of Theorem 6.6.1, if A is the sum of first and second order terms of primitive elements, then A is the generator of a weakly Gaussian convolution semigroup in the sense of Bernstein.

Proposition 6.6.3 *Let $\mathcal{A}$ be a (braided) Hopf algebra, and let $X_1, X_2 : \mathcal{A} \to \mathbb{C}$ be primitive linear functionals on $\mathcal{A}$, i.e. $X_i(ab) = X_i(a)\varepsilon(b) + \varepsilon(a)X_i(b)$, $i = 1, 2$, for all $a, b \in \mathcal{A}$. Then X_1, and X_2 are generators of Gaussian convolution semigroups in the sense of Bernstein, and X_1X_2 is a generator of a weakly Gaussian convolution semigroup in the sense of Bernstein.*

Proof: To show that $A \in \{X_1, X_2, X_1X_2\}$ is (weakly) Gaussian in the sense of Bernstein we show that there exist B_1, B_2 such that $(B_1 \otimes \varepsilon + \varepsilon \otimes B_2) \circ \Phi = A \otimes \varepsilon + \varepsilon \otimes A'$.

It suffices to use that $S(X_i) = -X_i$ since the X_i are primitive and $(\mathrm{id} \otimes S) \circ \Psi(X_2 \otimes X_1) = \Psi(S(X_2) \otimes X_1) = -\Psi(X_2 \otimes X_1)$ since we can pull the antipode across a braiding,

and to take

$$(B_1, B_2) = \begin{cases} (X_1, 0) & \text{for } A = X_1, \\ (X_2, 0) & \text{for } A = X_2, \\ \left(\frac{1}{2}X_1X_2, \frac{1}{2}X_1X_2\right) & \text{for } A = X_1X_2. \end{cases}$$

E.g. for the last case we get

$$\begin{aligned} (B_1 \otimes \varepsilon + \varepsilon \otimes B_2) \circ \Phi &= \Phi^\vee(B_1 \otimes \varepsilon + \varepsilon \otimes B_2) \\ &= \frac{1}{2}\Delta(X_1X_2) + \frac{1}{2}(\mathrm{id} \otimes S) \circ \Delta(X_1X_2) \\ &= X_1X_2 \otimes \varepsilon + \varepsilon \otimes \frac{1}{2}(X_1X_2 + S(X_1X_2)). \end{aligned}$$

$\diamondsuit$

Remark: We see that $S(X_1X_2) = m \circ \Psi(X_1 \otimes X_2)$ is also generator of a weakly Gaussian convolution semigroup.

M. Schürmann has proposed an alternative definition, see [Sch93, Proposition 5.1.1].

Definition 6.6.4 *A functional A on a bialgebra $\mathcal{A}$ is called* quadratic, *or a* Gaussian generator in the sense of Schürmann, *if for all $a, b, c \in \mathcal{A}$ with $\varepsilon(a) = \varepsilon(b) = \varepsilon(c) = 0$ follows*

$$A(abc) = 0.$$

This definition is more general in that it requires less structure on $\mathcal{A}$, only the counit and the multiplication are needed. But, if $\mathcal{A}$ is actually a Hopf algebra (or a braided Hopf algebra), then we can show that this property implies that A is a generator of a weakly Gaussian convolution semigroup in the sense of Bernstein.

Proposition 6.6.5 *Let $\mathcal{A}$ be a (braided) Hopf algebra, and $A : \mathcal{A} \to \mathbb{C}$ a quadratic functional. Then A generates a weakly Gaussian convolution semigroup in the sense of Bernstein.*

Proof: Let $K = \ker\varepsilon$. We can write $\mathcal{A}$ as a direct sum $\mathcal{A} = K \oplus \mathbb{C}1$, just set $a = a' + \varepsilon(a)1$, with $a' = a - \varepsilon(a)1$. The coproduct of an element $a \in K$ is of the form $\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a_i^{(1)} \otimes a_i^{(2)}$, with $a_i^{(1)}, a_i^{(2)} \in K$, and therefore $S(a)$ is equal to $-a$ plus some term in K^2, more precisely,

$$S(a) = -a + \sum a_i^{(1)} S(a_i^{(2)}). \tag{6.19}$$

Set $A' = \frac{1}{2}(A + A \circ S)$, $B_1 = B_2 = \frac{1}{2}A$, then A', B_1, B_2 are also quadratic functionals. We verify $A \otimes \varepsilon + \varepsilon \otimes A' = (B_1 \otimes \varepsilon + \varepsilon \otimes B_2) \circ \Phi$ with these definitions. On $1 \otimes 1$, $1 \otimes a$, or $a \otimes 1$, with $a \in K$, the equality is obvious. This leaves the case $a \otimes b$, $a, b \in K$, to check. In this case both sides vanish, $A(a)\varepsilon(b) + \varepsilon(a)(A(b) + A(S(b))/2 = 0$, and

$$\begin{aligned} (B_1 \otimes \varepsilon + \varepsilon \otimes B_2) \circ \Phi(a \otimes b) &= B_1(ab) + B_2(aS(b)) \\ &= \frac{1}{2}\left\{A(ab) - A(ab) + \sum A\left(ab_i^{(1)} S(b_i^{(2)})\right)\right\} \\ &= 0, \end{aligned}$$

where we used Equation (6.19) for $S(b)$. $\diamondsuit$

6.6.3 The braided line

Theorem 6.6.6 *Let $q \in \mathbb{C}\backslash\{0\}$, q not a root of unity. The weakly Gaussian convolution semigroups in the sense of Bernstein on the braided line $\mathbb{R}_q$ are exactly the convolution semigroups whose generator is of the form*

$$A\left(\sum_{n=0}^{\infty} c_n x^n\right) = a_0 c_0 + a_1 c_1 + a_2 c_2$$

with some constants $a_0, a_1, a_2 \in \mathbb{C}$. For $q \neq 1$ they are Gaussian in the sense of Bernstein iff $a_2 = 0$.

Remark: Using the duality of $\mathbb{R}_q$ with another copy of $\mathbb{R}_q$ we can write A as $A = a_0\varepsilon + a_1 p + \frac{a_2}{1+q}p^2$, where p denotes the generator of the dual copy.

Proof: Following Definition 6.6.2 we have to look for functionals A on $\mathbb{R}_q$ such that $(A \otimes \varepsilon + \varepsilon \otimes A') \circ \Phi^{-1}$ is of the form $B_1 \otimes \varepsilon + \varepsilon \otimes B_2$. We will do the calculation in the "opposite direction", i.e. we will start with two functionals B_1, B_2, and check under what conditions we can write $(B_1 \otimes \varepsilon + \varepsilon \otimes B_2) \circ \Phi$ in the form $A \otimes \varepsilon + \varepsilon \otimes A'$.

A functional on $\mathbb{R}_q$ is characterized by its values on the monomials x^n, so that using a dual copy of $\mathbb{R}_q$ with generator p, we can write $B_i = \sum_{n=0}^{\infty} b_n^{(i)} p^n$, where $b_n^{(i)} = \frac{B_i(x^n)}{q_n!}$, $i = 1, 2$. Composing $B_1 \otimes \varepsilon + \varepsilon \otimes B_2$ with Φ is equivalent to taking $\Phi^\vee(B_1 \otimes \varepsilon + \varepsilon \otimes B_2)$, where $\Phi^\vee$ is formally identical to Φ, but the coproduct, antipode, and multiplication are now those of the dual copy of $\mathbb{R}_q$.

One calculates $\Phi^\vee(p^n \otimes 1) = \Delta p^n = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_q p^\nu \otimes p^{n-\nu}$ and $\Phi^\vee(1 \otimes p^n) = (\mathrm{id} \otimes S) \circ \Delta p^n = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_q (-1)^{n-\nu} q^{\frac{(n-\nu)(n-\nu-1)}{2}} p^\nu \otimes p^{n-\nu}$, and thus

$$\Phi^\vee(B_1 \otimes \varepsilon + \varepsilon \otimes B_2) = \sum_{n,\nu} \begin{bmatrix} n \\ \nu \end{bmatrix}_q (b_n^{(1)} + (-1)^{n-\nu} q^{\frac{(n-\nu)(n-\nu-1)}{2}} b_n^{(2)}) p^\nu \otimes p^{n-\nu}.$$

This expression has to be of the form $A \otimes \varepsilon + \varepsilon \otimes A'$, i.e. the coefficients of the terms $p^\nu \otimes p^{n-\nu}$ have to vanish for $0 < \nu < n$. This is the case if and only if all coefficients except $b_0^{(i)}$, $b_1^{(i)}$, $b_2^{(i)}$ are zero. For A this gives the form given in the theorem.

A is the generator of a Gaussian semigroup in the sense of Bernstein if we have $A = A'$. This implies that the coefficients of $p^n \otimes 1$ and $1 \otimes p^n$ have to be identical. This is the case if and only if all coefficients except $b_0^{(1)}$, $b_1^{(1)}$, and $b_0^{(2)}$ vanish. It follows that $a_2 = 0$. $\diamondsuit$

6.6.4 The braided plane

Theorem 6.6.7 *Let $q \in \mathbb{C}\backslash\{0\}$, q not a root of unity. The weakly Gaussian convolution semigroups in the sense of Bernstein on the braided plane $\mathbb{C}_q^{2|0}$ are exactly the convolution semigroups whose generator is of the form*

$$A\left(\sum_{n,m=0}^{\infty} c_{nm} x^n y^m\right) = a_{00}c_{00} + a_{10}c_{10} + a_{01}c_{01} + a_{20}c_{20} + a_{11}c_{11} + a_{02}c_{02},$$

with some constants $a_{00}, a_{10}, a_{01}, a_{20}, a_{11}, a_{02} \in \mathbb{C}$.

Proof: We proceed as for the braided line. This gives the following conditions:

$$b^{(1)}_{nm} + (-1)^{n-\nu+m-\mu} q^{(n-\nu+m-\mu)(n-\nu+m-\mu-1)} b^{(2)}_{nm} = 0,$$

if $1 \leq \nu \leq n-1$ or $1 \leq \mu \leq m-1$. $\diamondsuit$

6.6.5 The braided q-Heisenberg-Weyl group

Proposition 6.6.8 *Let $q \in \mathbb{C}\backslash\{0\}$, $q \neq 1$, and q not a root of unity. The generators of weakly Gaussian convolution semigroups in the sense of Bernstein on the braided q-Heisenberg-Weyl group are of the form:*

$$A\left(\sum_{n,m,p=0}^{\infty} c_{nmp} a^n b^m c^p\right) = a_{000}c_{000} + a_{100}c_{100} + a_{010}c_{010} + a_{001}c_{001} + a_{200}c_{200} + a_{101}c_{101} + a_{002}c_{002}$$

with some constants $a_{000}, a_{100}, a_{010}, a_{001}, a_{110}, a_{020}, a_{011}$.

Remark: Using the dual of HW_q constructed in Section 6.3, we see that the space of weakly Gaussian generators in the sense of Bernstein is spanned by $\{x, y, z, x^2, z^2, xz\}$. But for $q \neq 1$, xy, y^2, and yz do not generate weakly Gaussian convolution semigroups, unlike in the classical case. This is due to the fact that y is no longer primitive.

Proof: By Proposition 6.6.3 we know that x, $y = xz - zx$,z, x^2, z^2, and xz are generators of weakly Gaussian semigroups, since x and z are primitive. The hard part of the proof is to show that none of the higher order terms generate Gaussian semigroups. This involves rather tedious calculations, and we will only outline the major steps.

We take again two general functionals $B_i = \sum_{n,m,r=0}^{\infty} b^{(i)}_{nmr}$, $i = 1, 2$, and check when $\Phi^\vee(B_1 \otimes \varepsilon + \varepsilon \otimes B_2) = \Delta(B_1) + (\mathrm{id} \otimes S) \circ \Delta(B_2)$ is of the form $A \otimes \varepsilon + \varepsilon \otimes A'$, i.e. under what conditions on the coefficients $b^{(i)}_{nmr}$ the mixed terms vanish. We use the fact that $\mathrm{HW}_q^\vee$ is $\mathbb{Z}^2$-graded with $\deg(x) = (1,0)$ and $\deg(z) = (0,1)$ to control which terms in the B_i contribute to a given term $x^{n_1}y^{m_1}z^{r_1} \otimes x^{n_2}y^{m_2}z^{r_2}$.

We check that x and z are indeed primitive, and use $u = \frac{xz - qzx}{1-q}$ instead of y, so that we can use Formula (6.3). The antipode of $x^n u^m z^r$ can be written as

$$S(x^n u^m z^r) = \sum_{k=-m}^{n \wedge r} D_{nmrk} x^{n-k} u^{m+r} z^{r-k}, \tag{6.20}$$

where the coefficients D_{nmrk} are defined via certain recurrence relations. No other terms can appear since S has to respect the grading. We will only need the explicit form of the antipode for a few lower order terms: $S(x) = -x$, $S(z) = -z$, $S(x^2) = qx^2$, $S(xz) = m \circ \Psi(x \otimes z) = zx = \frac{xz}{q} + \left(1 - \frac{1}{q}\right) u$, etc.

Using Equations (6.3) and (6.20), and the values of $D_{\nu\mu\rho k}$ for small ν, μ, ρ, we can write down the conditions on the $b^{(i)}_{n,m,r}$ that are equivalent to the vanishing of the coefficient of $x^{n_1}u^{m_1}z^{r_1} \otimes x^\nu u^\mu z^\rho$ in $\Phi^\vee(B_1 \otimes \varepsilon + \varepsilon \otimes B_2)$. We get for $(\nu, \mu, \rho) = (1, 0, 0)$ and $n_1 + m_1 + r_1 > 0$:

$$q_{n_1+1} q^{m_1} b^{(1)}_{n_1+1,m_1,r_1} + q_{m_1+1} b^{(1)}_{n_1,m_1+1,r_1-1} - q_{n_1+1} q^{m_1} b^{(2)}_{n_1+1,m_1,r_1} + q_{m_1+1} b^{(2)}_{n_1,m_1+1,r_1-1} = 0,$$

for $(\nu,\mu,\rho) = (0,1,0)$ and $n_1 + m_1 + r_1 > 0$:

$$q_{m_1+1}q^{r_1}b^{(1)}_{n_1,m_1+1,r_1} - q_{m_1+1}q^{r_1-1}b^{(2)}_{n_1,m_1+1,r_1} + \left(1 - \frac{1}{q}\right) q_{n_1+1}q_{r_1+1}q^{m_1}b^{(2)}_{n_1+1,m_1,r_1+1} = 0,$$

for $(\nu,\mu,\rho) = (0,0,1)$ and $n_1 + m_1 + r_1 > 0$:

$$q_{r_1+1}b^{(1)}_{n_1,m_1,r_1+1} - q_{r_1+1}b^{(2)}_{n_1,m_1,r_1+1} = 0,$$

and for $(\nu,\mu,\rho) = (0,0,2)$ and $n_1 + m_1 + r_1 > 0$:

$$\begin{bmatrix} r_1+2 \\ 2 \end{bmatrix}_q b^{(1)}_{n_1,m_1,r_1+2} + q \begin{bmatrix} r_1+2 \\ 2 \end{bmatrix}_q b^{(2)}_{n_1,m_1,r_1+2} = 0.$$

Combining these equations we can show that $b^{(1)}_{nmr} = b^{(2)}_{nmr}$ for $n+m+r \geq 2$, $b^{(i)}_{nmr} = 0$ for $n+m+r > 2$, and, by looking at the cases for small n_1, m_1, r_1 in detail, $b^{(i)}_{110} = b^{(i)}_{020} = b^{(i)}_{011} = 0$, $i = 1,2$. This implies that A and A' have the form stated in the theorem. $\diamondsuit$

6.6.6 $\mathbb{R}_q$-quantum convolution semigroups

We can also define the notion of em (weakly) Gaussian $\mathbb{R}_q$-quantum convolution semigroups on quantum/braided groups. Let p be the functional on $\mathbb{R}_q$ defined by $p(x^n) = \delta_{n0}$ and Φ a $\mathbb{R}_q$-quantum convolution semigroup. Then we define the generator of Φ as $A_\Phi = \Phi^*(p) = p \circ \Phi$, and call a $\mathbb{R}_q$-quantum convolution semigroup (weakly) Gaussian in the sense of Bernstein if its generator is the generator of a (weakly) Gaussian ccs in the sense of Bernstein. With this definition we can immediately translate the previous results into characterisations of (weakly) Gaussian $\mathbb{R}_q$-quantum convolution semigroups.

Theorem 6.6.9 *The normalized (i.e. $A_\Phi(1) = 0$) weakly Gaussian $\mathbb{R}_{q'}$-quantum convolution semigroups in the sense of Bernstein on the braided line $\mathbb{R}_q$ are of the form*

$$\Phi : x^k \mapsto \sum_{n=\lceil \frac{k}{2} \rceil}^{k} \binom{n}{2n-k} a_1^{2n-k} a_2^{k-n} \frac{q_k!}{q'_n!} (x')^n,$$

where $a_1, a_2 \in \mathbb{C}$ are constants, and x, x' denote the generators of $\mathbb{R}_q$ and $\mathbb{R}_{q'}$, respectively.

Proof: The possible generators were given in Theorem 6.6.6. This allows to construct Φ^* and thus Φ. $\diamondsuit$

Remark: The $\mathbb{R}_q$-qcs associated to Bozejko and Speicher's quantum Brownian motion is also weakly Gaussian in the sense of Bernstein (as a qcs!).

6.7 Conclusion

We have given a definition of Gaussian functionals on quantum and braided groups based on the Bernstein property and presented a method that allows to calculate these

functionals explicitely. The results in this section confirm that the Bernstein property gives a useful characterisation of the Gauss law only on commutative groups, as was already indicated by the results on nilpotent Lie groups.

We have proposed a definition of nilpotence for quantum groups and braided groups, and shown that the embedding of functionals into continuous convolution semigroups on these structures is unique. Nilpotent quantum groups and nilpotent braided groups could also turn out to be of interest for the study of analogues of the heat equation on quantum groups and braided groups.

We have also introduced a definition of Gaussian convolution semigroups on quantum groups and braided groups following the approach of Heyer and Hazod to the Bernstein theorem on Lie groups. The results obtained here are more satisfactory than those of Section 6.4, e.g. on the braided line and braided plane we rediscovered the functionals that were already in introduced in [FS96b].

It is an open problem if the converse of Proposition 6.6.5 is also true.

Chapitre 7

Evolution Equations and Lévy Processes on Quantum Groups

Uwe Franz **René Schott**[9]

Résumé

Evolution equations like the heat or diffusion equation or the Schrödinger equation can be associated with stochastic processes. In this paper we study the relation between equations of the form $\partial_t u = Lu$ and Lévy processes (i.e. quantum stochastic processes with independent and stationary increments) on quantum groups and braided groups. Solutions of such equations are calculated as Appell systems. Wigner distributions of these processes are defined and it is proven that they satisfy a Fokker-Planck equation.

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7.1 Introduction

Quantum groups are unital associative algebras, equipped with an additional structure, the coalgebra structure, that allows to define notions like increments, translations, etc., and thus allows to define analogues of many (physically!) important concepts for stochastic processes on vector spaces or groups, as, e.g., the notion of stationary and independent increments that is at the basis of Brownian motion and diffusions, or the various characterisations of Gauss distributions. We recommend recent books by Majid[Maj95b] (in particular Chapter 5) and Schürmann[Sch93] as introduction to this field.

In this article we consider stochastic processes on quantum groups that are related to evolution equations of the form

$$\partial_t u = Lu,$$

with some difference-differential operator L. For the equations considered in Section 7.5, u is an element of a quantum or braided group $\mathcal{A}$. We recall that solutions of these equations can be given as Appell systems or shifted moments of the associated process, and show how these can be calculated explicitely on the q-affine group, the braided line and plane, and a braided analogue of the Heisenberg-Weyl group. These calculations are original.

In Section 7.6, which is the main contribution of this paper, we define a Wigner map from functionals on a quantum group or braided group to a "Wigner" density on the undeformed space. We prove that the densities associated in this way to Lévy processes (i.e. processes with independent and stationnary increments) satisfy a Fokker-Planck type equation. In the one-dimensional case these coincide with the evolution equations of Section 8.6.3, but in the general case we get new equations.

We close with a few final remarks in Section 7.7.

7.2 Preliminaries

Notation. Let $q \in \mathbb{C}$. We set $q_n = \sum_{\nu=0}^{n-1} q^\nu$, i.e. $q_n = n$ if $q = 1$, and $q_n = \frac{q^n-1}{q-1}$ else. We will also use $q_n! = \prod_{\nu=1}^n q_\nu$, and $q_{2n}!! = \prod_{\nu=1}^n q_{2\nu}$, $q_{2n-1}!! = \prod_{\nu=1}^n q_{2\nu-1}$, where $q_0! = q_0!! = 1$. A q-exponential is defined by $e_q^x = \sum_{n=0}^\infty \frac{x^n}{q_n!}$ (e.g., as a formal power series), if q is not a root of unity.

An analogue of the binomial coefficients can be defined by the recurrence relation

$$\begin{bmatrix} m+1 \\ \mu \end{bmatrix}_q = \begin{bmatrix} m \\ \mu \end{bmatrix}_q + q^{m-\mu+1} \begin{bmatrix} m \\ \mu - 1 \end{bmatrix}_q,$$

$\begin{bmatrix} m \\ 0 \end{bmatrix}_q = \begin{bmatrix} m \\ m \end{bmatrix}_q = 1$. If q is not a root of unity one has

$$\begin{bmatrix} m \\ \mu \end{bmatrix}_q = \frac{q_m!}{q_\mu! \cdot q_{m-\mu}!}, \qquad m \in \mathbb{N}, \quad \mu = 0, 1, \ldots, m.$$

For the definition of quantum groups and braided groups (or Hopf algebras and braided Hopf algebras) see [Maj95b]. We recall a few examples which we shall use to illustrate

our approach. For the origin of the first two examples see [Maj92, Maj93b], the third is well known in the quantum group literature, while it seems that the last was first studied (as a braided Hopf algebra) by the present authors [FNS96] (but see also the appendix of [ADT94]).

Example: The braided line. Let $q \in \mathbb{C}$, q not a root of unity. As an algebra $\mathbb{R}_q$ is isomorphic to the ring of polynomials in one variable $\mathbb{R}[x]$. The braiding used here is defined by $\Psi(x^n \otimes x^m) = q^{nm} x^m x^n$. On the generator the coproduct, counit, and antipode are defined by

$$\Delta(x) = x + x', \qquad \varepsilon(x) = 0, \qquad S(x) = -x,$$

where we used the notation $x = x \otimes 1$, $x' = 1 \otimes x$.

$\mathbb{R}_q$ can be dually paired with another copy of $\mathbb{R}_q$ by

$$< p^n, x^m >= q_n! \delta_{nm},$$

where p denotes the generator of the dual copy.

The dual copy acts on $\mathbb{R}_q$ via

$$\rho(u)a = u \triangleright a = \sum < u, a^{(1)} > a^{(2)},$$

where $\Delta a = \sum a^{(1)} \otimes a^{(2)}$ (Sweedler's notation). This gives $p \triangleright x^n = q_n x^{n-1}$, i.e. $\rho(p)$ is the q-difference operator δ_q,

$$\delta_q : f(x) \mapsto \frac{f(qx) - f(x)}{x(q-1)}.$$

For $q \in \mathbb{R}$ a *-structure is defined by $x^* = x$.

Example: The braided plane. Again $q \in \mathbb{C}$, q not a root of unity. The braided plane is the braided Hopf algebra with two generators x, y and defining relations $yx = qxy$,

$$\begin{array}{rclrcl} x'x &=& q^2 xx', & x'y &=& qyx', \\ y'x &=& qxy' + (q^2-1)yx', & y'y &=& q^2 yy', \\ \Delta x &=& x + x', & \Delta y &=& y + y', \end{array}$$

and $\varepsilon(x) = \varepsilon(y) = 0$, $S(x) = -x$, $S(y) = -y$.

We can dually pair $\mathbb{C}_q^{2|0}$ with a copy of $\mathbb{C}_{q^{-1}}^{2|0}$ and thus define partial derivatives. Denote the generators of the dual by ∂_1 and ∂_2, they satisfy $\partial_2 \partial_1 = q^{-1} \partial_1 \partial_2$, and act as

$$\partial_1 x^n y^m = (q^2)_n x^{n-1} y^m, \qquad \partial_2 x^n y^m = (q^2)_m q^n x^n y^{m-1},$$

For real q a *-structure is defined by $x^* = y$.

Example: The q-affine group. Let $\alpha, \beta \in \mathbb{C}$, s.t. $q = e^{\alpha\beta}$ is not a root of unity. The q-affine group Aff_q is the Hopf algebra with two generators a, b and relations

$$\begin{aligned} ba &= (a+\beta)b, \\ \Delta a &= a + a', \\ \Delta b &= b + e^{\alpha a} b', \end{aligned}$$

$\varepsilon(a) = \varepsilon(b) = 0$, $S(a) = -a$, $S(b) = -e^{-\alpha a} b$, and trivial braiding, i.e. we use the usual tensor product and the trivial twist map $\Psi = \tau$. We can dually pair Aff_q with $U_q(\mathrm{Aff})$, where $U_q(\mathrm{Aff})$ is generated by X, Y with the relations

$$\begin{aligned} XY &= Y(X+\alpha) \\ \Delta(X) &= X + X' \\ \Delta(Y) &= Y e^{\beta X} + Y', \end{aligned}$$

$\varepsilon(X) = \varepsilon(Y) = 0$, $S(X) = -X$, $S(Y) = -Y e^{-\beta X}$. The dual pairing is given by

$$< X^n Y^m, a^{n'} b^{m'} >= n! q_m! \delta_{nn'} \delta_{mm'},$$

or formally by $g = e^{aX} e_q^{bY}$. For the dual action we get

$$\rho(X) a^n b^m = n a^{n-1} b^m, \qquad \rho(Y) a^n b^m = q_m a^n b^{m-1}.$$

Example: A braided q-Heisenberg-Weyl group. We will give a braided Hopf algebra structure for the algebra known as q-oscillator algebra or q-Heisenberg-Weyl algebra, i.e. the algebra defined by $ac - qca = \mathbb{1}$, $\mathbb{1}$ central. We can regard this algebra as generated by two generators a, c with the cubic relations

$$aac + qcaa = (1+q)aca, \quad acc + qcca = (1+q)cac,$$

with $q \in \mathbb{C}$, q not a root of unity. If we define the braiding by $a'a = qaa'$, $a'c = ca'$, $c'a = \frac{1}{q} ac'$, $c'c = qcc'$, then $\Delta a = a + a'$, $\Delta c = c + c'$ defines an algebra homomorphism from HW_q to $\mathrm{HW}_q \tilde{\otimes} \mathrm{HW}_q$. The counit is defined by $\varepsilon(a) = \varepsilon(c) = 0$ (extend as a unital algebra homomorphism). The braided bialgebra defined in this way also admits an antipode. Define S by $S(a) = -a$, $S(c) = -c$, and extend by $S \circ m = m \circ \Psi \circ (S \otimes S)$.

We can introduce a third generator, e.g. by $ac - qca = (1-q)b$ or $d = ac - ca$, then $\{a^n b^m c^r; n, m, r \in \mathbb{N}\}$ and $\{a^n d^m c^r; n, m, r \in \mathbb{N}\}$ are bases (Poincaré-Birkhoff-Witt bases) of HW_q (b is central, and d satisfies $ad = qda$ and $dc = qcd$).

The following formula is useful to compute the product of general basis elements of the HW_q algebra, it can be derived by induction.

$$c^r a^n = \sum_{k=0}^{r \wedge n} \frac{(q^{-1})_r! (q^{-1})_n! (-1)^k}{(q^{-1})_{r-k}! (q^{-1})_{n-k}! (q^{-1})_k!} a^{n-k} d^k c^{r-k}.$$

A *-structure is defined by $a^* = c$, $c^* = a$, if q is real.

If we define partial differential operators on HW_q via

$$\begin{array}{ll} \rho(x)a = 1 & \rho(x)c = 0 \\ \rho(z)a = 0 & \rho(z)c = 1 \end{array}$$

and extend with the Leibnitz rules

$$\begin{array}{rclrcl} \rho(x)(au) & = & u + qa\rho(x)u, & \rho(x)(cu) & = & \frac{1}{q}c\rho(x)u, \\ \rho(z)(au) & = & a\rho(z)u, & \rho(z)(cu) & = & u + qc\rho(z)u, \end{array}$$

for $u \in \mathrm{HW}_q$, then $\rho(x)$ and $\rho(z)$ satisfy again the HW_q-relations, and $< X, u >= \varepsilon(\rho(X)u)$ defines a dual pairing.

7.3 Evolution equations

We will consider equations of the form

$$\partial_t u = Lu \tag{7.1}$$

where L is a differential operator, independent of t, e.g.

$$\begin{array}{rcll} \partial_t u & = & (a\delta_q^2 + b\delta_q)u & \text{on } \mathbb{R}_q \\ \partial_t u & = & \left(\partial_1^2 + \partial_2^2\right) u & \text{on } \mathbb{C}_q^{2|0} \\ \partial_t u & = & \left(\rho(X)^2 + \rho(Y)^2\right) u & \text{on } \mathrm{Aff}_q \\ \partial_t u & = & \left(\rho(x)^2 + \rho(z)^2\right) u & \text{on } \mathrm{HW}_q \end{array}$$

In the first equation, L is a general second order q-difference operator, but for the explicit calculations we shall assume that a and b are constants.

In the second equation we have an analogue of the Laplacian, the operator in the third equation is related to the Gegenbauer or ultraspherical polynomials, see e.g. [FF]. In the fourth equation we have an analogue of the Kohn-Laplacian on the Heisenberg group.

An equation of the form (7.1) gives rise to a transition operator, formally written as e^{tL}.

7.4 Quantum stochastic processes

A quantum probability space is usually defined as a pair $(\mathcal{A}, \Phi)$ consisting of a *-algebra $\mathcal{A}$ and a state (i.e. a normed positive linear functional) Φ on $\mathcal{A}$. A quantum random variable j over a quantum probability space $(\mathcal{A}, \Phi)$ on a *-algebra $\mathcal{B}$ is simply a *-algebra homomorphism $j : \mathcal{B} \to \mathcal{A}$. For a quantum random variable $j : \mathcal{B} \to \mathcal{A}$ the functional $\varphi_j = \Phi \circ j$ is called its distribution in the state Φ.

A quantum stochastic process is a family of quantum random variables over the same quantum probability space and taking values in the same algebra, indexed by some set, e.g., $\mathbb{R}_+$.

Here we focus on processes with independent increments indexed by $\mathbb{R}_+$, they are characterized by their one-dimensional distributions $\{\varphi_t; t \in \mathbb{R}_+\}$. The notion of independence to be used in this context is that of tensorial independence, cf. [Sch93]. If the increments are also stationary, then the one-dimensional distributions form a convolution semi-group, i.e. $\varphi_0 = \varepsilon$ and $\varphi_s \star \varphi_t = \varphi_{s+t}$. Such processes are called Lévy processes.

In this case there is a transition operator associated to the process, defined by

$$U_t(\varphi) : a \mapsto \sum \varphi_t(a^{(1)})a^{(2)}$$

where we used Sweedler's notation $\Delta a = \sum a^{(1)} \otimes a^{(2)}$. These operators form also a semi-group, $U_s(\varphi) \circ U_t(\varphi) = U_{s+t}(\varphi)$, and $U_0(\varphi) = \mathrm{id}$.

We will say that φ is associated to the equation $(\partial_t - L)u = 0$, if $U_t(\varphi) = e^{tL}$, and call L the generator of φ in this case.

7.5 Appell systems

We recall the definition of Appell system, see [FFS97, FS96b].

Definition 7.5.1 *We define the (left) Appell polynomials on a braided group $\mathcal{A}$ with respect to a semi-group of functionals $\{\varphi_t\}$ by*

$$h_k = (\varphi_t \otimes \mathrm{id}) \circ \Delta a_k,$$

i.e. $h_k = U_t(\varphi)(a_k)$, for a fixed basis $\{a_k\}$ of $\mathcal{A}$.

If L is the generator of $\{\varphi_t\}$, then h_k solves

$$\partial_t h_k = L h_k.$$

For other interesting properties, e.g., raising operators, or relation to matrix elements, see [FFS97, FS96b].

7.5.1 Example: The braided line

The Appell polynomials associated to the functional $\varphi_t = \exp(tL)$ where $L = \frac{a^2}{1+q}p^2 + bp$ are

$$\begin{aligned} h_k(x;t) &= (\varphi_t \otimes \mathrm{id}) \circ \Delta x^k \\ &= \sum_{\nu=0}^{k} \begin{bmatrix} k \\ \nu \end{bmatrix}_q \varphi_t(x^\nu) x^{k-\nu} \\ &= \sum_{\nu=0}^{k} \frac{q_k! H_\nu\left(bt, -\frac{2a^2 t}{1+q}\right)}{q_{k-\nu}!\nu!} x^{k-\nu}, \end{aligned}$$

where H_ν denotes the Hermite polynomials, defined by

$$H_p(x,t) = \sum_{k=0}^{[\frac{p}{2}]} \binom{p}{2k} \frac{(2k)!}{2^k k!} x^{p-2k}(-t)^k.$$

These Appell polynomials are solutions of

$$\partial_t u = \frac{a^2}{1+q}\delta^2 u + b\delta u$$

with $u(x,t=0) = x^k$ (where δ is the q-difference operator $(\delta f)(x) = \frac{f(qx)-f(x)}{x(q-1)}$).

For $b=0$, $a=\sqrt{(1+q)/2}$ the Appell polynomials simplify to

$$h_k(x;t) = \sum_{\nu=0}^{[\frac{k}{2}]} \frac{q_k! t^\nu x^{k-2\nu}}{q_{k-2\nu}! 2^\nu \nu!},$$

these polynomials are a q-analogue of the Hermite polynomials.

7.5.2 Example: The braided plane

We choose $\phi_t = \exp tL$ with $L = \frac{1}{1+q^2}(\partial_1^2 + \partial_2^2)$. We get

$$\phi_t = \sum_{r=0}^{\infty} \frac{t^r}{(1+q^2)^r r!} \sum_{\nu=0}^{r} \begin{bmatrix} r \\ \nu \end{bmatrix}_{q^{-4}} \partial_2^{2\nu}\partial_1^{2(r-\nu)}$$

since $\partial_2\partial_1 = q^{-1}\partial_1\partial_2$. This leads to the following formula for the Appell polynomials:

$$h_{rm}(x,y;t) = \exp(tL)x^r y^m = \sum_{\nu,\mu=0}^{[\frac{r}{2}],[\frac{m}{2}]} \frac{\begin{bmatrix} \mu+\nu \\ \nu \end{bmatrix}_{q^{-4}} (q^2)_r!(q^2)_m! q^{2\mu(r-2\nu)} t^{\mu+\nu}}{(q^2)_{r-2\nu-1}!(q^2)_{m-2\mu-1}!(1+q^2)^{\mu+\nu}(\mu+\nu)!} x^{r-2\nu} y^{m-2\mu}.$$

These polynomials solve the evolution equation

$$\partial_t u = \frac{1}{1+q^2}\left(\partial_1^2 + \partial_2^2\right) u$$

where ∂_1, ∂_2 can be defined by

$$\begin{aligned} \partial_1 f(x,y) &= \frac{f(q^2x,y) - f(x,y)}{x(q^2-1)}, \\ \partial_2 f(x,y) &= \frac{f(qx,q^2y) - f(qx,y)}{y(q^2-1)}. \end{aligned}$$

7.5.3 Example: The q-affine group

We can use the generalized Gegenbauer polynomials defined in [FF]

$$C_n^h(x) = \sum_{\nu=0}^{[\frac{n}{2}]} \frac{(h)_{n-\nu}}{q_{n-2\nu}!} \frac{(-1)^\nu}{\nu!} (2x)^{n-2\nu},$$

and the representation $\rho_h(X) = \alpha(x\partial_x + h)$, $\rho_h(Y) = i\alpha\delta_x$ to calculate the moments of $\Phi_t = \exp\left(\frac{t}{2}(X^2 + Y^2)\right)$. These polynomials are eigenfunctions of $S_h = (x\partial_x + h)^2 - \delta_x^2 = \rho_h(X)^2 + \rho_h(Y)^2$, i.e. $S_h C_n^h(x) = (n+h)^2 C_n^h(x)$, and their inversion formula is

$$x^n = \frac{q_n!}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \frac{h+n-2k}{(h)_{n-k+1}k!} C_{n-2k}^h(x).$$

Using the Feynman-Kac type formula (cf. [FFS97])

$$\Phi_t\left(e^{a\rho_h(X)} e_q^{b\rho_h(Y)} x^n\right) = e^{\frac{t}{2}(\rho_h(x)^2 + \rho_h(Y)^2)} x^n,$$

and comparing the coefficients of x^{n-2r} we get

$$\begin{aligned} \Phi_t\left(e^{(n-2r+h)\alpha a} b^{2r}\right) &= \frac{q_{2r}}{\alpha^{2r}} \sum_{k=0}^{r} \frac{(h+n-2k)(h)_{n-r-k}(-1)^k}{4^r (h)_{n-k+1} k!(r-k)!} e^{(n-2k+h)^2\alpha^2 t/2} \qquad \text{for } n \geq 2r, \\ \Phi_t\left(e^{(n-2r+h)\alpha a} b^{2r+1}\right) &= 0 \qquad \text{for } n \geq 2r+1. \end{aligned}$$

Differentiating ν times w.r.t. h and setting $h = m - \mu + 2r - n$, we get all moments that are needed to calculate the Appell functions

$$h_{nm}(a,b,t) = \Phi_t\left(\Delta(a^n b^m)\right) = \sum_{\nu=0}^{n}\sum_{\mu=0}^{m} \binom{n}{\nu} \begin{bmatrix} m \\ \mu \end{bmatrix}_q \Phi_t\left(a^\nu e^{(m-\mu)\alpha a} b^\mu\right) a^{n-\nu} b^{m-\mu}.$$

7.5.4 Example: The braided q-Heisenberg-Weyl group

Consider $L = x^2 + z^2$. Then

$$e^{tL} = \sum_{\nu=0}^{n} \sum_{\kappa=0}^{2(n-\nu)\wedge 2\nu} \frac{C_{\nu,\kappa}^n t^n}{n!} z^{2(n-\nu)-\kappa} y^\kappa x^{2\nu-\kappa}$$

where the coefficients $C_{\nu,\kappa}^n$ are determined by the recurrence relations

$$C_{\nu,\kappa}^{n+1} = C_{\nu-1,\kappa}^n + q^{2\kappa} C_{\nu,\kappa}^n + q_{2\nu-\kappa+1} q_2 q^{\kappa-1} C_{\nu,\kappa-1}^n + q_{2\nu-\kappa+2} q_{2\nu-\kappa+1} C_{\nu,\kappa-2}^n.$$

For $\kappa = 0$ we have the binomial coefficients $C_{\nu,0}^n = \binom{n}{\nu}$. This allows to calculate

$$h_{nmr}(t) = e^{tL}(a^n b^m c^r),$$

using e.g. the dual pairing

$$< z^r y^m x^n, a^{n'} b^{m'} c^{r'} >= \delta_{nn'}\delta_{mm'}\delta_{rr'} q_n! q_m! q_r!.$$

7.6 Densities

For one single variable, or in the commutative case, one can use Bochner's theorem to associate a density to a quantum random variable, cf. [Mey93].

We now want to associate joint densities to several non-commutating variables, along the line of Wigner distributions [Wig32]. We will map functionals on an algebra with n generators to measures on $\mathbb{R}^n$. Equivalently, we can ask for a map from functions on $\mathbb{R}^n$ (e.g., polynomials) to elements of the algebra.

Consider the following diagram:

$$\begin{array}{ccccc} & & \text{Wigner} & & \\ & \text{QS} & \longrightarrow & \text{CS} & \\ \text{Duality} & \updownarrow & & \updownarrow & \text{Duality} \\ & \text{QO} & \longleftarrow & \text{CO} & \\ & & \text{Weyl} & & \end{array}$$

where
QS: (linear span of) the set of quantum states
CS: (linear span of) set of classical states
QO: set of quantum observables
CO: set of classical observables
We want the following similar diagram:

$$\begin{array}{ccccc} & & \text{“Wigner”} & & \\ & \mathcal{U} & \longrightarrow & \mathcal{M}(X) & \\ \text{q-Fourier} & \updownarrow & & \updownarrow & \text{Fourier} \\ & \mathcal{A} & \longleftarrow & C(X) & \\ & & \text{“Weyl”} & & \end{array}$$

where X is the undeformed space or group, and $\mathcal{M}(X)$ denotes the (convolution) algebra of (signed) measures on X, $C(X)$ the algebra of continuous functions on X, and $\mathcal{A}$ and $\mathcal{U}$ the quantum group and algebra. The q-Fourier transformation (with respect to an integral $\int$) is defined by

$$\mathcal{F}(u) = \int (u \exp)$$

where exp is the exponential or coevalution map, see also [KM94, Koo95]. We also have $\int(ab) =< \mathcal{F}(a), b >$. and thus in this setting a density (w.r.t. $\int_{\mathcal{A}}$) of a functional $\Phi \in \mathcal{A}^*$ can, at least in principle, be calculated with the inverse Fourier transform, $\rho_\Phi = \mathcal{F}_{\mathcal{A}}^{-1}(\Phi)$. A more detailed discussion, including an explicite example on $\mathbb{R}_q$, can be found in [FS96b].

Here we shall use the right-hand-side of the diagram to introduce densities that 'live' on the classical, i.e. undeformed, group or space. Following Anderson[And69], we fix a set of generators $x_1, \ldots, x_n$ of $\mathcal{A}$ and define the *Weyl map* [Wey31] on polynomials $u_1^{k_1} \cdots u_n^{k_n}$ by

$$W(u_1^{k_1} \cdots u_n^{k_n}) = \frac{k_1! \cdots k_n!}{k!} \sum_{\pi \in S_k} x_{\pi(1)} \cdots x_{\pi(k)}$$

where $k = k_1 + \cdots + k_n$. Other definitions are possible, e.g. $W_W : u_1^{k_1} \cdots u_n^{k_n} \mapsto x_1^{k_1} \cdots x_n^{k_n}$ ("Wick"), or $W_{AW} : u_1^{k_1} \cdots u_n^{k_n} \mapsto x_n^{k_n} \cdots x_1^{k_1}$ ("Anti-Wick"), or also $W_{q-\exp}$ defined by $e^{u \cdot v} \mapsto \exp(x|v)$. But $W_{q-\exp}$ will not leave the marginal distributions unchanged. In fact, W is uniquely determined by the conditions $W(u_i) = x_i$ and $W\left((a_1 u_1 + \cdots + a_n u_n)^k\right) = (a_1 W(u_1) + \cdots + a_n W(u_n))^k$, and thus only W^* will give the correct marginal distributions for all linear combinations of the generators. Ordered monomials like the "Wick" or "Anti-Wick" map still lead to the right marginal distributions for the generators.

The Wigner map W^* is defined by the condition

$$< \Phi, W(u) >= \int u \mathrm{d}W^*(\Phi),$$

i.e. as the dual of the Weyl map. $W^*(\Phi)$ is called the *Wigner density* of Φ. The Fourier transform of the measure $W^*(\Phi)$ is

$$\begin{aligned} g_\Phi(v) &= \mathcal{F}(W^*(\Phi))(v) = \int e^{iu \cdot v} \mathrm{d}W^*(\Phi) \\ &= < \Phi, W(e^{iu \cdot v}) >, \end{aligned}$$

where we assumed that we can interchange the limits involved, and that this series defines an analytic function.

If the functionals Φ_t form a convolution semigroup, then the associated Wigner densities satisfy an evolution equation or *Fokker-Planck-type equation.*

Proposition 7.6.1 *Let $\{\Phi_t; t \in \mathbb{R}_+\}$ be a convolution semigroup with generator L, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t = L\Phi_t = \Phi_t L$. Suppose further that W is invertible. Then the Wigner distribution of Φ_t satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} W^*(\Phi_t) = \tilde{\rho}(L)^* W^*(\Phi_t)$$

with $\tilde{\rho}(X) = W^{-1} \circ \rho(X) \circ W$ and $\tilde{\rho}(X)^*$ defined by duality.

Proof: Differentiate

$$< \Phi_t, W(u) >= \int u \mathrm{d}W^*(\Phi_t)$$

with respect to t, on the right hand side we get $\int u \mathrm{d}\frac{\mathrm{d}W^*(\Phi_t)}{\mathrm{d}t}$, while the left hand side gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} < \Phi_t, W(u) > &= < L\Phi_t, W(u) >=< \Phi_t, \rho(L)W(u) >=< \Phi_t, W(\tilde{\rho}(L)u) > \\ &= \int \tilde{\rho}(L) u \mathrm{d}W^*(\Phi_t) = \int u \mathrm{d}\left(\tilde{\rho}(L)^* W^*(\Phi_t)\right). \end{aligned}$$

7.6.1 Example: The braided line

Here we have only one variable, and the algebra is commutative, so the Weyl map is just $W : u^n \mapsto x^n$. The Fourier transform of the functional $\Phi = \sum_{n=0}^{\infty} a_n p^n$ is thus

$$g_\Phi(u) =< \Phi, e^{iux} >= \sum_{n=0}^{\infty} \frac{a_n q_n!(iu)^n}{n!}$$

where we assumed that the regularity conditions necessary for interchanging the limits are satisfied. E.g. for $\Phi_a = e^{ap}$, i.e. the functional determined by $\Phi_a(e_q^{x|p}) = e^{ap}$ (where $e_q^{x|p} = \sum_{n=0}^{\infty} \frac{x^n p^n}{q_n!}$) we get

$$g_{\Phi_a}(u) = \sum_{n=0}^{\infty} \frac{q_n!(iau)^n}{(n!)^2}.$$

We need $\tilde{\rho}(p)^*$ to be able to give the form of the Fokker-Planck equation. Because of the simple form of W we have $\tilde{\rho}(p) = \delta$. The adjoint of the q-difference operator is a multiple of the q-difference with q replaced by q^{-1},

$$\delta^* f(x) = -\frac{1}{q}\frac{f(q^{-1}x) - f(x)}{x(q^{-1}-1)} = -\frac{1}{q}\delta_{1/q} f(x)$$

so that the Wigner density $\eta_t(\mathrm{d}x) = \mathrm{d}W^*(\Phi_t)$ of the semigroup with generator $L = \sum c_n p^n$ satisfies

$$\partial_t \eta_t = \sum \frac{(-1)^n c_n}{q^n} \delta_{1/q}^n \eta_t.$$

7.6.2 Example: The braided plane

In order to determine W we calculate $(a_1 x + a_2 y)^N$, the coefficient of $a_1^n a_2^m$ is the image of $u_1^n u_2^m$. We get $W : u_1^n u_2^m \mapsto \frac{\begin{bmatrix} n+m \\ n \end{bmatrix}_q}{\begin{pmatrix} n+m \\ n \end{pmatrix}} x^n y^m$, and thus

$$g_\Phi(v_1, v_2) = \sum_{n,m=0}^{\infty} \frac{\begin{bmatrix} n+m \\ n \end{bmatrix}_q}{(n+m)!} i^{n+m} < \Phi, x^n y^m > v_1^n v_2^m.$$

For the Gauss functionals in the sense of Bernstein ($\Phi(x^n y^m) = z^n \delta_{m,0}$ or $z^m \delta_{n,0}$, see [FNS96]) we get $g_\Phi(v) = e^{izv_1}$ or $g_\Phi(v) = e^{izv_2}$, i.e. $W^*(\Phi)$ is a Dirac mass in $(z, 0)$ or $(0, z)$.

To write down the Fokker-Planck equations for Wigner distributions we need the representation $\tilde{\rho}$. For the two generators we get

$$\begin{aligned} \tilde{\partial}_1 u_1^n u_2^m &= n\frac{q_{n+m}}{n+m} u_1^{n-1} u_2^m \\ \tilde{\partial}_2 u_1^n u_2^m &= m q^n \frac{q_{n+m}}{n+m} u_1^n u_2^{m-1}. \end{aligned}$$

7.6.3 Example: The q-affine group

We get $W : u_1^n u_2^m \mapsto \sum_{\nu=0}^{n} \frac{n!m!}{(n+m)!} A_{n,\nu}^{n+m} \beta^{n-\nu} a^\nu b^m$, where the $A_{n\nu}^N$ are determined by $A_{n\nu}^N = 0$ if $n > N$ or $\nu > n$ or $n < 0$, $A_{00}^N = 1$, $A_{N\nu}^N = \delta_{N\nu}$, and

$$A_{n\nu}^{N+1} = A_{n-1,\nu-1}^N + (N+1-n) A_{n-1,\nu}^N + A_{n\nu}^N.$$

For the special case where the two lower indices are identical, we get $A^N_{nn} = \begin{pmatrix} N \\ n \end{pmatrix}$. The Weyl map and its inverse are characterized by

$$\begin{aligned} W &: \quad e^{a_1u_1+a_2u_2} \quad \mapsto \quad e^{a_1a+a_2b} = e^{a_1a}e^{\frac{e^{a_1\beta}-1}{a_1\beta}a_2b} \\ W^{-1} &: \quad e^{b_1a}e^{b_2b} = e^{b_1a+\frac{\beta b_1b_2}{e^{b_1\beta}-1}b} \quad \mapsto \quad e^{b_1u_1+\frac{\beta b_1b_2}{e^{b_1\beta}-1}u_2} \end{aligned}$$

This allows to write down $\tilde{\rho}$ and thus the Fokker-Planck equation for any Lévy process. For X we simply get $\tilde{\rho}(X) = \frac{\partial}{\partial u_1}$, the expression for $\tilde{\rho}(Y)$ is more complicated.

7.6.4 Example: The q-Heisenberg-Weyl algebra

The q-Heisenberg-Weyl algebra HW_q can be treated in the same way. Choose a, b, c as generators of HW_q, then the Weyl map is given by $W : u_1^n u_2^m u_3^r \mapsto \sum_{k=0}^{n\wedge r} \frac{n!m!r!D^{n+m+r}_{n,m,k}}{(n+m+r)!} a^{n-k}b^{m+k}c^{r-k}$, where the coefficients $D^{n+m+r}_{n,r,k}$ are defined by

$$\begin{aligned} D^N_{n,r,k} &= 0 \qquad \text{if } n<0 \text{ or } r<0 \text{ or } n+r>N \text{ or } k<0 \text{ or } k>n\wedge r, \\ D^N_{n,0,0} &= \begin{pmatrix} N \\ n \end{pmatrix}, \qquad \text{if } 0 \le n \le N \\ D^N_{0,r,0} &= \begin{pmatrix} N \\ r \end{pmatrix}, \qquad \text{if } 0 \le r \le N \end{aligned}$$

and the recurrence relation

$$D^{N+1}_{n,r,k} = D^N_{n,r,k} + D^N_{n,r-1,k} + q^{k-m}D^N_{n-1,r,k} + (q^{-1})_{r-k+1}\left(1-\frac{1}{q}\right)D^N_{n-1,r,k-1}.$$

7.7 Conclusion

We gave a relation between stochastic processes and evolution equations on quantum / braided groups, and illustrated it on several examples. Two types of evolution equation were considered. First, we let the generator of the process act via the dual representation on the quantum or braided group itself. Solutions to these evolution equations are given in terms of Appell systems. We showed on several examples how they can be calculated. To get the second kind of evolution equation we associated Wigner-type distributions to the processes and functionals. These distributions are distributions on the undeformed space, and we thus get evolution equations for ordinary functions. If we replace the Weyl map by other maps, e.g. those corresponding to normal ordering ("Wick" or "Anti-Wick") or to q-exponentials, then we can hope to get simpler expression for $\tilde{\rho}$ and thus for the Fokker-Planck equation, but the relation between the moments of the functionals and their "Wigner" distributions becomes more complicated.

Chapitre 8

Brownian Motion on the Affine Group and Generalized Gegenbauer Polynomials

Philip Feinsilver[10] Uwe Franz

Work in Progress

Résumé

Polynomial eigenfunctions of the generator of Brownian motion on the affine group are studied in a representation using a general shift operator. This yields a generalization of Gegenbauer polynomials. We determine the matrix elements of the group element and a variant, derive addition formulae, and give the conditions these polynomials to form a family of orthogonal polynomials.

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8.1 Introduction

Let $\xi_1, \xi_2, \ldots, \xi_d$ denote a basis for a Lie algebra of finite dimension d. Using McKean's product integral ([McK69], see also [FFS97]) to construct Brownian motion on the corresponding Lie group leads to the generator is $S = \xi_1^2 + \xi_2^2 + \cdots + \xi_d^2$. Here we study the two-dimensional non-abelian Lie algebra corresponding to the affine group acting on the real line. We may take as basis $\xi_1 = X, \xi_2 = L$ acting on x^n with $\xi_1 = xD + \alpha$, i.e., $\xi_1 x^n = (n+\alpha)\, x^n$, for some constant α, and L is a general form of lowering operator acting as $Lx^n = b_n x^{n-1}$, with $b_0 = 0$ and $b_n \neq 0$ for $n \geq 1$. Defining $\gamma_n = b_1 b_2 \cdots b_n$ for $n \geq 1$, with $\gamma_0 = 1$, on the basis $X_n = x^n/\gamma_n$ we have $\xi_1 X_n = (n+\alpha)X_n$ and $LX_n = X_{n-1}$, $n \geq 1$, $LX_0 = 0$

We will find polynomial eigenfunctions of the generator S. For $L = \frac{\mathrm{d}}{\mathrm{d}x}$, these are the Gegenbauer polynomials. We thus dub our polynomials generalized Gegenbauer polynomials. Matrix elements for the affine group and variants, in this polynomial basis are found. Then we consider conditions for these to be a family of orthogonal polynomials. Some remarks about Appell sequences conclude the study.

8.2 Brownian motion on the affine group

Let G be the two-dimensional Lie Group $G = \{(a,b); a \in \mathbb{R}_+, b \in \mathbb{R}\}$ with multiplication $(a,b) \odot (a',b') = (aa', ba' + b')$, this is the group of affine transformations on the real line, $(a,b) : x \mapsto ax + b$. Let $\mathbb{L}$ be the associated Lie algebra, i.e. $\mathbb{L}$ has basis $\{X, L\}$ with commutation relations $[L, X] = L$. If we require X to be diagonal and suppose that there is a countable basis $\{e_n\}$ that can be labelled by its X-eigenvalues λ_n, then $[L,\rho]e_n = \sum(\rho_m - \rho_n)L_{nm}e_m = \sum L_{nm}e_m = Le_n$ shows that L can only map the basis element with eigenvalue λ_n to one with eigenvalue $\lambda_n - 1$. Excluding the case that the representation space is the direct sum of several invariant subspaces we thus get $Xe_n = (\alpha + n)e_n$ and $Le_n = b_n e_n$ for some constant α and some sequence $\{b_n; n \in \mathbb{N}\}$, $b_0 = 0$, $b_n \in \mathbb{C}^\times = \mathbb{C}/\{0\}$ for $n \geq 1$. L is a generalized shift operator, such operators have been studied in umbral calculus, cf. [BL96]. We will assume $e_n = x^n$, then we can write $X = xD + \alpha$ and $L = \sum_{k=1}^{\infty} l_k x^{k-1} D^k$, where $l_n = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{k!(n-k)!} b_k$.

We define a generalized factorial $\gamma_0 = 1$, $\gamma_n = b_1 b_2 \cdots b_n$, and a generalized exponential series $e(x) = \sum_{n=0} \frac{x^n}{\gamma_n}$, it is an eigenfunction of L.

We use McKean's product integral ([McK69], see also [FFS97]) to construct Brownian motion on the affine group. Its generator is $S = X^2 - L^2$.

If we look for polynomials eigenfunctions of S,

$$Sp_n(x) = \lambda_n p_n(x), \text{ where } p_n(x) = \sum_{\nu=0}^{n} p_{n\nu} x^\nu,$$

we find by comparing the coefficients of x^n that the eigenvalue has to be $\lambda_n = (n+\alpha)^2$. Comparing the coefficients of x^{n-1} we see $p_{n,n-1} = 0$. Looking at x^ν we get the recurrence relation

$$b_{\nu+2} b_{\nu+1} p_{n,\nu+2} = [\lambda_n - (\nu+\alpha)^2] p_{n\nu}.$$

Solving this recurrence we find that the polynomial eigenfunctions of S are generalized Gegenbauer polynomials,

$$\phi_n^\alpha(x) = \sum \frac{\gamma_n}{\gamma_{n-2k}} \frac{(\alpha)_{n-k}}{(\alpha)_n} \frac{(-1)^k}{4^k k!} x^{n-2k},$$

or, in a different normalization,

$$\tilde{C}_n^\alpha(x) = \frac{2^n (\alpha)_n}{\gamma_n} \phi_n^\alpha(x) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(\alpha)_{n-k}}{\gamma_{n-2k}} \frac{(-1)^k}{k!} (2x)^{n-2k}.$$

The second normalization has the advantage that we get the usual Gegenbauer polynomials $C_n^\alpha(x)$ in the conventional normalization as special case $b_n = n$; the ϕ_n^α are determined by the condition that the highest coefficient is equal to one, this form can also be used for $\alpha = 0$. We will from now on assume $\alpha > -1$.

Proposition 8.2.1 *We have the inversion formulas*

$$\begin{aligned} x^n &= \sum_{k=0}^{[\frac{n}{2}]} \frac{\gamma_n}{\gamma_{n-2k}} \frac{(\alpha)_{n-2k+1}}{(\alpha)_{n-k+1}} \frac{1}{4^k k!} \phi_{n-2k}^\alpha(x) \\ &= \frac{\gamma_n}{2^n} \sum_{k=0}^{[\frac{n}{2}]} \frac{\alpha + n - 2k}{(\alpha)_{n-k+1} k!} \tilde{C}_{n-2k}^\alpha(x). \end{aligned}$$

Proof:

$$\begin{aligned} & \sum_{k=0}^{[\frac{n}{2}]} \frac{\gamma_n}{\gamma_{n-2k}} \frac{(\alpha)_{n-2k+1}}{(\alpha)_{n-k+1}} \frac{1}{4^k k!} \phi_{n-2k}^\alpha(x) \\ = & \sum_{k=0}^{[\frac{n}{2}]} \frac{\gamma_n}{\gamma_{n-2k}} \frac{(\alpha)_{n-2k+1}}{(\alpha)_{n-k+1}} \frac{1}{4^k k!} \sum_{l=0}^{[\frac{n}{2}]-k} \frac{\gamma_{n-2k}}{\gamma_{n-2k-2l}} \frac{(\alpha)_{n-2k-l}}{(\alpha)_{n-2k}} \frac{(-1)^l}{4^l l!} x^{n-2k-2l} \\ = & \sum_{m=0}^{[\frac{n}{2}]} \frac{\gamma_n}{\gamma_{n-2m}} \frac{x^{n-2m}}{4^m} \sum_{\mu=0}^{m} \frac{(-1)^{m-\mu}}{\mu!(m-\mu)!} \frac{(\alpha+n-2\mu)(\alpha)_{n-m-\mu}}{(\alpha)_{n-\mu+1}} \\ = & \sum_{m=0}^{[\frac{n}{2}]} \frac{\gamma_n}{\gamma_{n-2m}} \frac{x^{n-2m}}{4^m m!} \underbrace{\sum_{\mu=0}^{m} \binom{m}{\mu} \frac{(-1)^\mu (\alpha+n-2m+2\mu)}{(\alpha+n-2m+\mu)\cdots(\alpha+n-m+\mu)}}_{=:\, \Sigma_m(\alpha+n-2m)} \end{aligned}$$

if we substitute $\mu \mapsto m - \mu$. For $m = 0$ we get x^n, i.e. the desired result. We will now show by induction that $\Sigma_m(\alpha + n - 2m)$ gives zero for $m \geq 1$ and α arbitrary. For $m = 1$ we have

$$\Sigma_1(\beta) = \frac{\beta}{\beta(\beta+1)} - \frac{\beta+2}{(\beta+1)(\beta+2)} = 0.$$

Assume now that $\Sigma_m(\beta) = 0$ for all β, then we get

$$0 = \frac{1}{\beta+m+1} (\Sigma_m(\beta) - \Sigma_m(\beta+2))$$

$$
\begin{aligned}
&= \frac{\left\{\sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-1)^{\mu}(\beta+2\mu)}{(\beta+\mu)\cdots(\beta+m+\mu)} - \sum_{\mu=0}^{m}\binom{m}{\mu}\frac{(-1)^{\mu}(\beta+2\mu+2)}{(\beta+2+\mu)\cdots(\beta+2+m+\mu)}\right\}}{\beta+m+1} \\
&= \sum_{\mu=0}^{m+1}\frac{\left\{\binom{m}{\mu}(\beta+m+1+\mu)+\binom{m}{\mu-1}(\beta+\mu)\right\}(-1)^{\mu}(\beta+2\mu)}{(\beta+m+1)(\beta+\mu)\cdots(\beta+m+1+\mu)} \\
&= \Sigma_{m+1}(\beta),
\end{aligned}
$$

since $\binom{m}{\mu}(\beta+m+1+\mu)+\binom{m}{\mu-1}(\beta+\mu) = \binom{m+1}{\mu}(\beta+m+1)$. The inversion formula for the $\tilde{C}_n^{\alpha}(x)$ follows immediately. ∎

8.3 Group elements and matrix elements

Using coordinates of the second kind we can write a group element of the affine group as

$$g(A,B) = e^{AX}e^{BL},$$

in these coordinates the group law has the form

$$g(A,B)g(A',B') = g(A+A', Be^{A'}+B').$$

It is also interesting to consider the 'group-like' element defined by

$$\tilde{g}(A,B) = e^{AX}e(BL),$$

in the following section we will investigate for which cases their is some kind of group law or multiplication rule for $\tilde{g}(A,B)$.

We define the matrix elements of $g(A,B)$ and $\tilde{g}(A,B)$ w.r.t. to the $\tilde{C}_n^{\alpha}$ by

$$
\begin{aligned}
g(A,B)C_n^{\alpha}(x) &= \sum M_{mn}(A,B)\tilde{C}_m^{\alpha}(x), \\
\tilde{g}(A,B)C_n^{\alpha}(x) &= \sum \tilde{M}_{mn}(A,B)\tilde{C}_m^{\alpha}(x).
\end{aligned}
$$

Proposition 8.3.1 *The matrix elements are*

$$
\begin{aligned}
M_{pn}(A,B) &= \sum_{\mu} C_{n-\mu}^{\alpha+\mu}(B)\frac{(\alpha+p)}{(\alpha+\mu)_{\frac{\mu-p}{2}+1}}\frac{(\alpha)_{\mu+1}a^{\mu}}{(\frac{\mu-p}{2})!} \\
\tilde{M}_{pn}(A,B) &= \sum_{\mu} \tilde{C}_{n-\mu}^{\alpha+\mu}(B)\frac{(\alpha+p)}{(\alpha+\mu)_{\frac{\mu-p}{2}+1}}\frac{a^{\mu}}{(\frac{\mu-p}{2})!},
\end{aligned}
$$

for $0 \le p \le n$, and where $a = e^A$ and the summation index μ takes the values $p, p-2, p-4, \ldots, 0$ or 1 according to whether p is even or odd.

Proof: Let $\hat{g}(A,B)$ stand for either $g(A,B)$ or $\tilde{g}(A,B)$, and $[n]!$ for $n!$ or γ_n, respectively.

$$\begin{aligned}
\hat{g}(A,B)\tilde{C}_n^\alpha(x) &= \sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k(\alpha)_{n-k}2^{n-2k}}{\gamma_{n-2k}k!} \sum_{l=0}^{n-2k} \frac{a^{n-2k-l}B^l\gamma_{n-2k}}{[l]!\gamma_{n-2k-l}} \\
&\quad \times \frac{\gamma_{n-2k-l}}{2^{n-2k-l}} \sum_{s=0}^{[\frac{n-l}{2}]-k} \frac{\alpha+n-2k-l-2s}{(\alpha)_{n-l-2k-s+1}s!}\tilde{C}^\alpha_{n-2k-l-2s}(x) \\
&= \sum \frac{(-1)^k(\alpha)_{n-k}2^l a^{n-2k-l}B^l(\alpha+n-2k-l-2s)}{[l]!k!s!(\alpha)_{n-l-2k-s+1}}\tilde{C}^\alpha_{n-2k-l-2s}(x) \\
&= \sum \frac{(-1)^k(\alpha)_{n-k}(2B)^{n-\mu-2k}a^\mu(\alpha+p)}{[n-\mu-2k]!k!(\frac{\mu-p}{2})!(\alpha)_{\frac{3\mu-p}{2}+1}}\tilde{C}_p^\alpha(x) \\
&= \sum \left(\sum \frac{(-1)^k(\alpha+\mu)_{n-\mu-k}(2B)^{n-\mu-2k}}{[n-\mu-2k]!k!}\right) \frac{(\alpha+p)}{(\alpha+\mu)_{\frac{\mu-p}{2}+1}} \frac{a^\mu}{(\frac{\mu-p}{2})!}\tilde{C}_p^\alpha(x)
\end{aligned}$$

where we recognize the (generalized) Gegenbauer polynomials insides the brackets for the matrix elements of $g(A,B)$ (or $\tilde{g}(A,B)$, respectively). ∎

Remark: The fact that the matrix elements of $g(A,B)$ contain the usual Gegenbauer polynomials and not the generalized ones follows from the relation $L = U^{-1}DU$, where U is defined by $Ux^n = \frac{\gamma_n}{n!}$, $D = \frac{\mathrm{d}}{\mathrm{d}x}$, if we note $e^{AX}e^{BD} = Ug(A,B)U^{-1}$, and $C_n^\alpha(x) = U\tilde{C}_n^\alpha(x)$.

8.4 Multiplication rules and addition formulae

The group law of the affine group leads immediately to an addition formula for the associated matrix elements,

$$M_{pn}(aa', Ba'+B') = \sum_{r=0}^{n}\sum_{r} M_{rn}(a,B)M_{pr}(a',B').$$

Proposition 8.4.1 *We have*

$$C_{n-\mu}^{\alpha+\mu}(Ba'+B') = \sum_{r=0}^{n}\sum_{\mu'} C_{n-\mu}^{\alpha+\mu}(B)C_{r-\mu'}^{\alpha+\mu'}(B')\frac{(\alpha+r)((a')^{\mu'-\mu}}{(\alpha+\mu')_{\frac{\mu'-r}{2}+1}\left(\frac{\mu-r}{2}\right)!}$$

Proof: Applying $g(aa', Ba'+b') = g(a,B)\odot g(a',B')$ to $\tilde{C}_n^\alpha(x)$ and using Proposition 8.3.1 we get

$$\begin{aligned}
&\sum_{p=0}^{n}\sum_{\mu} C_{n-\mu}^{\alpha+\mu}(Ba'+B')\frac{(\alpha+p)}{(\alpha+\mu)_{\frac{\mu-p}{2}+1}}\frac{(aa')^\mu}{(\frac{\mu-p}{2})!}\tilde{C}_p^\alpha(x) = \\
&\sum_{r=0}^{n}\sum_{\mu'} C_{n-\mu'}^{\alpha+\mu'}(B')\frac{(\alpha+r)}{(\alpha+\mu')_{\frac{\mu'-r}{2}+1}}\frac{(a')^{\mu'}}{(\frac{\mu'-r}{2})!}\sum_{p=0}^{r}\sum_{\mu} C_{r-\mu}^{\alpha+\mu}(B)\frac{(\alpha+p)}{(\alpha+\mu)_{\frac{\mu-p}{2}+1}}\frac{a^\mu}{(\frac{\mu-p}{2})!}\tilde{C}_p^\alpha(x).
\end{aligned}$$

Comparing the coefficients of $a^\mu\tilde{C}_p^\alpha(x)$ yields the proposition. ∎

Remark: This follows also, if we apply the group element $a^X e^{BD}$ to $C_n^\alpha(x)$.

To get a summation formula for the matrix elements $\tilde{M}_{pn}$ of the group-like element we first need a multiplication law for $\tilde{g}(A,B)$. For this we will introduce non-commutative variables for A and B. What commutation relations do A, B, A', and B' have to satisfy so that $\tilde{g}(A,B)\tilde{g}(A',B')$ is again of the form $\tilde{g}(f,h)$, with $f = f(A,A',B,B')$, $h = h(A, A'B, B')$?

Writing this as a power series in X and L we get the condition

$$\sum_{n,m,s,r\geq\rho} \frac{m^{r-\rho}A^n B^m (A')^r (B')^s}{n!\gamma_m \rho!(n-\rho)!\gamma_s} X^{n+\rho} L^{m+s} = \sum_{kl} \frac{f^k h^l}{k!\gamma_l} X^k L^l.$$

Comparing the coefficients of X and L yields

$$f = A + A', \qquad h = Be^{A'} + B'.$$

From the coefficients of X^2 and XL we get

$$AA' = A'A, \qquad A'B = BA'.$$

The coefficients of L^2 lead to the most interesting commutation relation, namely

$$B'Be^{A'} = \left(\frac{\gamma_2}{\gamma_1^2} - 1\right) Be^{A'}B',$$

i.e. B' and $Be^{A'}$ q-commute with $q = \frac{\gamma_2}{\gamma_1^2} - 1 = \frac{b_2}{b_1} - 1$. The commutation relation between B' and A is not used, we can safely assume that they commute.

The commutation relations are now fixed, and we know that we get q-exponentials, and the generalized coefficients b_k have to be q-numbers (up to a factor). For $q = 0$ we get $b_k = b =$ constant, and $e(x) = \frac{b}{b-x}$.

This allows to get summations theorems for the matrix elements of $\tilde{g}(A,B)$, but we have to be careful about the order of the A's and B's in the matrix elements

Proposition 8.4.2 *We have*

$$\tilde{C}_{n-\mu}^{\alpha+\mu}(Ba' + B') = \sum_{r=0}^{n} \sum_{\mu'} (a')^{\mu'-\mu} \tilde{C}_{n-\mu}^{\alpha+\mu}(B) \tilde{C}_{r-\mu'}^{\alpha+\mu'}(B') \frac{(\alpha+r)}{(\alpha+\mu')_{\frac{\mu'-r}{2}+1} \left(\frac{\mu-r}{2}\right)!}$$

also for the q-case, i.e. for $b_n = cq_n$ for some $c \in \mathbb{C}$ and $q \in \mathbb{C}$, q not a root of unity, if the variables satisfy the commutation relations

$$\begin{aligned} Ba &= qaB, & Ba' &= a'B, \\ B'a &= aB', & B'a' &= qa'B', \\ aa' &= a'a, & BB' &= B'B. \end{aligned}$$

Proof: As for Proposition 8.4.1, but taking into account the commutation relations and the preceding discussion. ∎

8.5 Orthogonality

We want to find out, if there exists a measure with respect to which the $\tilde{C}_n^\alpha(x)$ are orthogonal. This is the case, iff they satisfy a recurrence relation of the form

$$x\tilde{C}_n^\alpha(x) = A_n\tilde{C}_{n+1}^\alpha(x) + B_n\tilde{C}_n^\alpha(x) + C_n\tilde{C}_{n-1}^\alpha(x), \tag{8.1}$$

with $A_nC_{n+1} > 0$ for $n \geq 1$.

Proposition 8.5.1 *Let $\alpha \in \mathbb{R}_+^*$. There exists a measure μ on $\mathbb{R}$ such that the C_n^α are orthogonal w.r.t. μ if and only if*

$$\begin{aligned} b_{2m} &= mb_2, \\ b_{2m+1} &= b_1 + m(b_3 - b_1) \end{aligned}$$

for all $m \geq 1$, and either $b_2 > 0$, $b_3 > \frac{\alpha+2}{\alpha+1}b_1 > 0$ or $b_2 < 0$, $b_3 < \frac{\alpha+2}{\alpha+1}b_1 < 0$.

In this case they satisfy a recurrence relation of the form (8.1) with

$$\begin{aligned} A_n &= \frac{b_{n+1}}{2(\alpha+n)} \\ B_n &= 0 \\ C_n &= \frac{(\alpha+n-1)b_{n+1} - (\alpha+n)b_{n-1}}{2(\alpha+n)}. \end{aligned}$$

Proof: (i) Suppose the $\tilde{C}_n^\alpha$ do satisfy a three-term recurrence relation. It follows immediately that $B_n = 0$ for all n. Looking at the coefficient of x^{n+1} in $x\tilde{C}_n^\alpha(x) - A_n\tilde{C}_{n+1}^\alpha(x) - C_n\tilde{C}_{n-1}^\alpha(x)$ we get

$$A_n = \frac{b_{n+1}}{2(\alpha+n)}.$$

The coefficient of x^{n-1} is

$$-\frac{(\alpha)_{n-1}}{\gamma_{n-2}}2^{n-2} + 2^{n-1}A_n\frac{(\alpha)_n}{\gamma_{n-1}} - 2^{n-1}C_n\frac{(\alpha)_{n-1}}{\gamma_{n-1}}$$

i.e. we get

$$C_n = \frac{1}{2}(b_{n+1} - b_{n-1}) - \frac{b_{n+1}}{2(\alpha+n)},$$

for $n \geq 2$. Looking at the coefficient of x^{n-3}

$$\frac{(\alpha)_{n-2}}{\gamma_{n-4}2}2^{n-4} - 2^{n-3}A_n\frac{(\alpha)_{n-3}}{\gamma_{n-3}2} + 2^{n-3}C_n\frac{(\alpha)_{n-2}}{\gamma_{n-3}}$$

we get the following expression for C_n:

$$C_n = \frac{1}{4}(b_{n+1} - b_{n-3}) - \frac{b_{n+1}}{2(\alpha+n)},$$

for $n \geq 4$. Equating those two expressions we find

$$b_{n+1} - b_{n-1} = b_{n-1} - b_{n-3}$$

for $n \geq 4$.

For $n = 0, 1, 2$ the recurrence is satisfied with $A_0 = \frac{b_1}{2\alpha}$, $A_1 = \frac{b_2}{2(\alpha+1)}$, $C_1 = \frac{\alpha b_2}{2(\alpha+1)}$, $A_2 = \frac{b_3}{2(\alpha+2)}$, $C_2 = \frac{b_3-b_1}{2} - \frac{b_3}{2(\alpha+2)}$.

For $n = 3$ (and $\alpha \neq 0$) we find that the recurrence is only satisfied if $b_4 = 2b_2$, and that in this case $A_3 = \frac{b_4}{2(\alpha+3)} = \frac{b_2}{\alpha+3}$ and $C_3 = \frac{\alpha+2}{\alpha+3}b_2$.

Checking $A_1C_2 > 0$ we see that either $b_2 > 0$ and $b_3 > \frac{\alpha+2}{\alpha+1}b_1$ or $b_2 < 0$ and $b_3 < \frac{\alpha+2}{\alpha+1}b_1$. $A_0C_1 = \frac{b_1b_2}{2(\alpha+1)} > 0$ implies that b_1 and b_2 have the same sign, $A_2C_3 = \frac{b_2b_3}{2(\alpha+3)}$ implies that b_2 and b_3 have the same sign.

(ii) We will now show that these are the only restrictions on the b_n necessary in order to get a three term recurrence relation. Let b_1, b_2, b_3 be arbitrary, and set ($b_0 = 0$)

$$\begin{aligned} b_{2m+1} &= b_1 + m(b_3 - b_1), \\ b_{2m} &= mb_2. \end{aligned}$$

Then with

$$\begin{aligned} A_n &= \frac{b_{n+1}}{2(\alpha+n)} \\ C_{2m+1} &= \frac{b_2}{2} - A_{2m+1} = \frac{\alpha+m}{2(\alpha+2m+1)}b_2 \\ C_{2m} &= \frac{b_3-b_1}{2} - A_{2m} = \frac{(\alpha+m)(b_3-b_1)-b_1}{2(\alpha+2m)}, \end{aligned}$$

we get (for $n \geq 4$)

$$\begin{aligned} & A_n\tilde{C}^{\alpha}_{n+1}(x) + C_n\tilde{C}^{\alpha}_{n-1}(x) \\ = \; & A_n \sum_{k=0}^{[\frac{n+1}{2}]} \frac{(\alpha)_{n+1-k}(-1)^k}{\gamma_{n+1-2k}k!}(2x)^{n+1-2k} \\ & + C_n \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(\alpha)_{n-1-k}(-1)^k}{\gamma_{n-1-2k}k!}(2x)^{n-1-2k} \\ = \; & A_n\frac{(\alpha)_{n+1}}{\gamma_{n+1}}(2x)^{n+1} \\ & + \sum_{k=1}^{[\frac{n+1}{2}]} \frac{(\alpha)_{n-k}(-1)^k(2x)^{n-2k}x}{\gamma_{n-2k}k!} \frac{(A_n2(\alpha+n-k) - C_n2k)}{b_{n+1-2k}} \\ = \; & x\tilde{C}^{\alpha}_n(x) \end{aligned}$$

since $A_n2(\alpha+n-k) - C_n2k = b_{n+1} - k(b_{n+1} - b_{n-1}) = b_{n+1-2k}$.

If in addition $b_2 > 0$ and $b_3 > \frac{\alpha+2}{\alpha+1}b_1 > 0$ or $b_2 < 0$ and $b_3 < \frac{\alpha+2}{\alpha+1}b_1 < 0$ then one can check that $A_nC_{n+1} > 0$ for all n. E.g. on the odd numbers, $n = 2m+1$, we find that

$b_3 - \frac{\alpha+m+2}{\alpha+m+1}b_1$ has to have the same sign as b_2 for all m. ∎

Proposition 8.5.2 *The measure of orthogonality is*

$$\frac{2\Gamma(\alpha+1)}{\gamma^\alpha\Gamma(\alpha-\beta+1)\Gamma(\beta)}|x|^{2\beta-1}(\gamma-x^2)^{\alpha-\beta}\mathbb{1}_{[-\gamma,\gamma]}(x)\mathrm{d}x$$

where $\gamma = \frac{b_2(b_3-b_1)}{4}$, $\beta = \frac{b_1}{b_3-b_1}$.

Remark: The conditions on b_1, b_2, b_3 are equivalent to $\gamma > 0$ and $0 < \beta < \alpha+1$ For the special case $b_n = n$ (i.e. $\gamma = 1$, $\beta = \frac{1}{2}$, and therefore $\alpha > -\frac{1}{2}$) this reduces to the measure of orthogonality of the Gegenbauer polynomials

$$\mu(\mathrm{d}x) = \frac{\Gamma(\alpha+1)}{4^{\alpha-1/2}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}(1-x^2)^{\alpha-\frac{1}{2}}\mathbb{1}_{[-1,1]}(x)\mathrm{d}x.$$

Proof: Using the inversion formula (Proposition 8.2.1) and orthogonality we get

$$\begin{aligned}\int x^n \mathrm{d}\mu(x) &= \int \frac{\gamma_n}{2^n}\sum_{k=0}^{[\frac{n}{2}]}\frac{\alpha+n-2k}{k!(\alpha)_{n-k+1}}C^\alpha_{n-2k}(x)\mathrm{d}\mu \\ &= \begin{cases} \frac{\alpha\gamma_{2m}}{4^m(\alpha)_{m+1}m!} & \text{if } n \text{ even}, n = 2m, \\ 0 & \text{if } n \text{ odd}. \end{cases}\end{aligned}$$

Rewriting γ_{2m} as

$$\gamma_{2m} = m!b_2^m(b_3-b_1)^m\left(\frac{b_1}{b_3-b_1}\right)_m$$

we get

$$\tilde{\mu}(t) = \int e^{itx}\mathrm{d}\mu(x) = {}_1F_2\left(\begin{matrix}\frac{b_1}{b_3-b_1} \\ \frac{1}{2}, \alpha+1\end{matrix}\middle| -\frac{b_2(b_3-b_1)}{16}t^2\right).$$

for the Fourier transform of μ. Now we can use e.g. [Obe90, I.2.23] to conclude. ∎

8.6 Appell Systems

Definition 8.6.1 *We call a family of polynomials* $\{p_n; n \in \mathbb{N}\}$ *a* L-$\{b_n\}$-*Appell system, if*

(i) p_n *is a polynomial of degree* n,

(ii) $Lp_n = b_n p_{n-1}$.

Remark: The sequence $\{b_n\}$ can be replaced by any other sequence $\{\tilde{b}_n\}$, if the p_n are rescaled accordingly:

$$\left\{\tilde{p}_n = \frac{b_1b_2\cdots b_n}{\tilde{b}_1\tilde{b}_2\cdots\tilde{b}_n}p_n; n \in \mathbb{N}\right\}$$

is a L-$\{\tilde{b}_n\}$-Appell system if $\{p_n\}$ is a L-$\{b_n\}$-Appell system.

Proposition 8.6.2 $\{\phi_n^{\alpha-n}; n \in \mathbb{N}\}$ *is a* L-$\{\tilde{b}_n\}$-*Appell system.*

An L-$\{b_n\}$-Appell system is completely determined by the sequence $\{p_n = p_n(0); n \in \mathbb{N}\}$, just like usual Appell systems (see e.g. [FS92]). We define a generating function

$$F(x,t) = \sum_{n=0}^{\infty} p_n(x) \frac{t^n}{\gamma_n}.$$

This function satisfies $LF(x,t) = tF(x,t)$, i.e. it is of the form

$$F(x,t) = e(tx)G(t)$$

where $e(tx)$ is the generalized exponential series, $e(tx) = \sum_{n=0}^{\infty} \frac{(tx)^n}{\gamma_n}$, and $G(t) = F(0,t) = \sum_{n=0}^{\infty} p_n \frac{t^n}{\gamma_n}$.

Proposition 8.6.3 *The L-$\{b_n\}$-Appell polynomials have the form*

$$p_n(x) = \sum_{\nu=0}^{n} \frac{\gamma_n}{\gamma_\nu \gamma_{n-\nu}} p_{n-\nu} x^\nu.$$

Proof: Apply L ν times to $p_n(x) = \sum_{\nu=0}^{n} p_{nm} x^m$ and set x equal to zero. ∎

For the Appell system $\{\phi_n^{\alpha-n}(x)\}$ we have

$$p_n = \phi_n^{\alpha-n}(0) = \begin{cases} 0 & \text{if} \quad n \text{ is odd}, \\ \frac{1}{4^m m!(-\alpha+1)_m} & \text{if} \quad n \text{ is even}, n = 2m. \end{cases}$$

Rewriting the equation from Proposition 8.6.3 as

$$\frac{x^n}{\gamma_n} = \frac{p_n(x)}{\gamma_n} - \sum_{\nu=0}^{n-1} \frac{p_{n-\nu}}{\gamma_{n-\nu}} \frac{x^\nu}{\gamma_\nu},$$

and using it recursively to replace the powers of x, we get an inversion formula for L-$\{b_n\}$-Appell polynomials.

Proposition 8.6.4

$$x^n = \gamma_n \sum_{k=0}^{n} \sum_{n_1=1}^{n} \sum_{n_2=1}^{n-n_1} \cdots \sum_{n_k=1}^{n-n_1-\cdots-n_{k-1}} \frac{(-1)^k p_{n_1} \cdots p_{n_k}}{\gamma_{n1} \cdots \gamma_{n_k}} \frac{p_{n-n_1-\cdots-n_k}(x)}{\gamma_{n-n_1-\cdots n_k}}.$$

Chapitre 9

Limit Theorems on Quantum Groups

This chapter describes work in progress. The problems are formulated, approaches are suggested, conjectures or partial answers are given, but the results are still incomplete.

Résumé

We examine the possibilities of giving limit theorems on bialgebras, including precise characterisations of the limit laws. First, we examine what can be obtained from results due to Schürmann [Sch93] and D. Neuenschwander and R. Schott (cf. [NS96]) in particular cases, focussing on explicit characterisations of the limit distributions.

Then we look for law of iterated logarithm type results. We formulate the problem, suggest a definition replacing that of the path-wise supremum in the classical law of iterated logarithm for quantum stochastic processes, and propose an approach for calculating the law of this supremum.

In Section 9.4, we study a different kind of limit theorem, combining classical random variables and quantum random variables. Here the convolution of the functionals is chosen according to a sequence of classical i.i.d. random variables.

9.1 Introduction

We want to get limit theorems on bialgebras, and as explicit as possible. The limit theorems by M. Schürmann [Sch93, Theorem 6.1.1, 6.1.2, and 6.1.3] apply in a very general context, using only the coalgebra structure, but do not give much information about the explicit form of the limit distributions.

Feinsilver [Fei87] and Neuenschwander and Schott [NS96] follow a different approach. On the braided line the q-convolution can be defined for usual (signed) measures on the real line, and thus we are not restricted to moment methods. This convolution does not conserve positivity, so in general the limit distributions are not positive. Also, even though [Fei87, Theorem 3] states that the q-characteristic function $\psi(t) = \int e_q^{itx} \mathrm{d}\mu(x)$ characterizes uniquely the measure μ, we do not know how to explicitely calculate μ from ψ.

In the first part (Section 9.2) we will look for analogues of the law of large numbers, central limit theorem, etc. Here our results mostly follow from previous work, in particular M. Schürmann[Sch93], Ph. Feinsilver[Fei87], and D. Neuenschwander and R. Schott[NS96].

In the second part (Section 9.3) we attack the question of the existence of limit theorems of the form of the law of the iterated logarithm.

The study of analogues of trajectorial properties (as e.g. the law of the iterated logarithm) of quantum processes is related to the question of the existence of classical versions, see e.g. [BKS96]. Thus the definition of the supremum of quantum random variables should coincide with the corresponding supremum of their classical versions, if these exist. Since the quantum process only determines the finite joint distributions of the classical process, we also have to address the question which classical version (left or right continuous, càdlàg, etc., if they exist) should be taken.

In the next section we prove a limit theorem that combines classical random variables and quantum random variables. The limit theorem in this section is motivated by R. Speicher's article[Spe92].

In Section 9.5, we replace the scaling in the normalization by more general operators.

9.2 Analogues of the law of large numbers and the central limit theorem

9.2.1 General results for limit theorems on bialgebras

The most general result is probably the following theorem due to M. Schürmann.

Theorem 9.2.1 ([Sch93, Theorem 6.1.1] *Let φ_{nk}, $n \in \mathbb{N}$, $k = 1, \ldots, k_n$ ($k_n \in \mathbb{N}$), be linear functionals on a coalgebra $\mathcal{C}$ satisfying*

(i) $\varphi_{n1}, \ldots, \varphi_{nk_n}$ *commute for each $n \in \mathbb{N}$ with respect to convolution,*

(ii) $\lim_{n\to\infty} \lim_{1\le k\le k_n} |(\varphi_{nk} - \varepsilon)(c)| = 0$ *for all $c \in \mathcal{C}$,*

(iii) $\sup_{n\in\mathbb{N}} \sum_{1\le k\le k_n} |(\varphi_{nk} - \epsilon)(c)| < \infty$ *for all $c \in \mathcal{C}$.*

Furthermore, suppose that there is a linear functional ψ on $\mathcal{C}$ such that for all $c \in \mathcal{C}$

$$\lim_{n\to\infty}\left(\sum_{1\le k\le k_n}(\varphi_{nk}-\varepsilon)(c)\right)=\psi(c).$$

Then

$$\lim_{n\to\infty}\left(\prod_{1\le k\le k_n}^{\star}\varphi_{nk}\right)(c)=(\exp_\star\psi)(c)$$

for all $c \in \mathcal{C}$ (where the product $\prod^\star$ is the convolution product).

We will use a corollary of this theorem for $\mathbb{N}$-graded coalgebras. Let $\mathcal{C}$ be an $\mathbb{N}$-graded coalgebra, i.e. $\mathcal{C}=\bigoplus_{n\in\mathbb{N}}\mathcal{C}^{(n)}$ and $\Delta\mathcal{C}^{(n)}\subseteq\bigoplus_{n_1+n_2=n}\mathcal{C}^{(n_1)}\otimes\mathcal{C}^{(n_2)}$. Then we can define a scaling $s:\mathbb{C}\times\mathcal{C}\to\mathcal{C}$ by setting

$$s(z)c=z^{-\deg(c)}c,$$

for $c\in\mathcal{C}^{(\deg(c))}$, and extending linearly in the second argument. With this definition we have $(\varphi\star\psi)\circ s(z)=(\varphi\circ s(z))\star(\psi\circ s(z))$ for all linear functionals φ,ψ on $\mathcal{C}$ and all $z\in\mathbb{C}$.

Corollary 9.2.2 ([Sch93, Theorem 6.1.3]) *Let $\mathcal{C}$ be an $\mathbb{N}$-graded coalgebra and let $\kappa\in\mathbb{N}$. If a linear functional φ on $\mathcal{C}$ satisfies*

(i) $\varphi|_{\mathcal{C}^{(l)}}=0$ *for* $0<l<\kappa$,

(ii) $\varphi|_{\mathcal{C}^{(0)}}=\varepsilon|_{\mathcal{C}^{(0)}}$,

then for all $c\in\mathcal{C}$

$$\lim_{n\to\infty}\left(\varphi^{\star n}\circ s(n^{1/\kappa})\right)(c)=(\exp_\star g_\varphi)(c),$$

where g_φ denotes the linear functional on $\mathcal{C}$ with

$$\begin{aligned} g_\varphi|_{\mathcal{C}^{(l)}} &= 0 \text{ for all } l\neq\kappa,\\ g_\varphi|_{\mathcal{C}^{(\kappa)}} &= \varphi|\mathcal{C}^{(\kappa)}. \end{aligned}$$

9.2.2 The braided line

The study of the 'q-convolution' and related limit theorems was initiated by Ph. Feinsilver [Fei87]. In this case the convolution can be defined via a q-characteristic function, and it is therefore possible to consider also measures that are not characterized by their moments. Thus we can look e.g. for analogues of stable laws (cf. [NS96]).

Application of Corollary 9.2.2 gives the following result

Proposition 9.2.3 *Let $\kappa\in\mathbb{N}$, $\varphi:\mathbb{R}_q\to\mathbb{C}$ normed, and $\varphi(x)=\cdots\varphi(x^{\kappa-1})=0$. Set $\varphi_n=\varphi^{\star n}\circ s(n^{1/\kappa})$, then*

$$\lim_{n\to\infty}\varphi_n=e^{cp^\kappa},$$

where $c=\varphi(x^\kappa)/q_\kappa!$

For $|q|<1$, Neuenschwander and Schott's result [NS96] allows to extend the previous proposition to $\kappa\in]0,2[$, if e^{cp^κ} is interpreted as the q-characteristic function of the limit distribution.

9.2.3 The braided q-Heisenberg-Weyl group

We take the grading that assigns degree one to the two primitive generators. Then, with Corollary 9.2.2, we find exactly the functionals that belong to Gaussian convolution semigroups in the sense of Subsection 6.6.1 (cf. Proposition 6.6.8) as limit distributions of the law of large numbers and the central limit theorem (i.e. for $\kappa = 1$, and $\kappa = 2$, respectively).

9.3 Law of iterated logarithm type results

Let $(B_t)_{t\in\mathbb{R}_+}$ be standard Brownian motion on $\mathbb{R}$. We have the following classical descriptions of the asymptotic behavior of B_t.

Theorem 9.3.1

$$\begin{aligned}\limsup_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} &= 1 \quad p.s.\\ \liminf_{t\to\infty} \frac{B_t}{\sqrt{2t\log\log t}} &= -1 \quad p.s.\end{aligned}$$

Theorem 9.3.2

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} \sup_{1\le s\le t} |B_s| = \frac{\pi}{\sqrt{8}} \quad p.s.$$

Is it possible to state and prove similar results for quantum stochastic processes?

For classical processes the supremum $\sup_{s\in[0,t]} X_s$ is taken for a fixed element ω of the underlying probability space Ω, i.e. for each $\omega \in \Omega$ we take the supremum of the trajectory $\varphi_\omega = X_\bullet(\omega) : s \mapsto X_s(\omega)$.

Our first step will be to define an analogue of the supremum for quantum stochastic processes. In order to be meaningful such a definition should lead to the same distributions as the classical one, if the process has a classical version. This problem is treated in Subsection 9.3.1.

The final goal should be to construct $\sup_{s\in[0,t]} X_s$ or $\limsup_{t\to\infty} X_t$, $\liminf_{t\to\infty} X_s$ as an operator, in the same operator algebra as $(X_s)_{s\in I}$, e.g. acting on some Fock space. But for a first approach we satisfy ourselves with a construction of the law of $\sup_{s\in[0,t]} X_s$. In Subsection 9.3.2 we devise an approach for calculating the law of finite suprema of (quantum or classical) random variables using their characteristic function and the Hilbert transform.

9.3.1 Definition of the supremum for quantum stochastic variables

In this section quantum random variables will be self-adjoint operators (say bounded) on a Hilbert space, and quantum stochastic processes will be families of such operators. Thus $|X_t|$, $0\wedge X_t$, $0\vee X_t$ can be defined via spectral calculus, as well as $X_s\vee X_t$, as long as $X_s\vee X_t$ commute. Let $X = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda$ be the spectral resolution of X, and $\Phi = \sum < v_i, \cdot v_i >$

a state. Then $F_X(c) = \int_{-\infty}^{c} \mathrm{d}\sum < v_i, E_\lambda v_i >$ is the distribution function of the classical density of X in the state Φ. Let $\tilde{X}$ be a classical random variable, then it is clear that for all functions (say bounded measurable) h the density of $h(\tilde{X})$ and that of $h(X) = \int_{\mathbb{R}} h(\lambda)\mathrm{d}E_\lambda$ (in the state Φ) agree. This generalize also quantum stochastic processes $(X_t)_{t\in I}$ (with some index set I), as long as it is commutative.

Let now $(X_t)_{t\in I}$ (with I an interval) be a quantum stochastic process such that there exists a classical version, i.e. a classical process $(\tilde{X}_t)_{t\in I}$, that has the same time-ordered moments,

$$\Phi(X_{t_1}^{k_1}\cdots X_{t_n}^{k_n}) = \mathbb{E}(\tilde{X}_{t_1}^{k_1}\cdots\tilde{X}_{t_n}^{k_n}),$$

for all $n, k_1, \ldots k_2 \in \mathbb{N}$, $t_1, \ldots, k_n \in I$. Let $X_t = \int_{\mathbb{R}} \lambda \mathrm{d}E_\lambda^t$ be the spectral resolution of X_t, $t \in I$. Then the operator that has the same density as $h(\tilde{X}_{t_1}, \ldots, \tilde{X}_{t_n})$ for $n \in \mathbb{N}$ and some measurable function h (suppose $h(\tilde{X}_{t_1}, \ldots, \tilde{X}_{t_n})$ is bounded, i.e. h is bounded on the Cartesian product of the spectra of the operators $X_{t_1}, \ldots, X_{t_n}$ to keep everything nice and bounded) is

$$X_h = \int_{\mathbb{R}^n} h(\lambda_1, \ldots, \lambda_n)\mathrm{d}E_{\lambda_1}^{t_1}\cdots E_{\lambda_n}^{t_n},$$

i.e. we have to keep the projection operators in the integrand time-ordered. Thus, if we want the supremum of the quantum stochastic process to correspond to the supremum of the classical process, then we have to define

$$X_s \vee X_t = \int_{\mathbb{R}^2} (\lambda_1 \vee \lambda_2)\mathrm{d}E_{\lambda_1}^s E_{\lambda_2}^t,$$

for $s < t$. This extends to finite suprema $X_{t_1} \vee \cdots \vee X_{t_n}$. We define

$$\sup_{s\in[a,b]} = \lim_{|pi|\to 0} X_{t_1} \vee \cdots \vee X_{t_n},$$

where $\pi = (a = t_0 < t_1 < \cdots < t_n = b)$, and $|\pi| = \max\{t_i - t_{i-1} | i = 1, \ldots, n\}$.

9.3.2 Calculation of the distribution of $\sup X_1, \ldots, X_n$

First we reduce $\sup X_{t_1}, \ldots, X_{t_n}$ to an iteration of suprema of two random variables:

$$\begin{aligned}\sup X_1, \ldots, X_n &= X_1 \vee \cdots \vee X_n = X_1 + 0 \vee (X_2 - X_1) \vee \cdots \vee (X_n - X_{n-1}) \\ &= X_1 + 0 \vee ((X_1 - X_2) + 0 \vee ((X_3 - X_2) + 0 \vee (\cdots 0 \vee (X_n - X_{n-1})\cdots))).\end{aligned}$$

Thus all we need to know is how to take the supremum of a random variable with 0, i.e. to construct its positive part. Let μ be the law of X, then the law of $0 \vee X$ is $\mu_+(\mathrm{d}x) = \mathbb{1}_{]0,+\infty[}(x)\mu(\mathrm{d}x) + P(X < 0)\delta_0(\mathrm{d}x)$. Denote the operator $\mu \mapsto \mu_+$ by P. Obviously $P : \mathcal{M}^1(\mathbb{R}) \to \mathcal{M}^1(\mathbb{R})$ and $P^2 = P$. Let H be the Hilbert transform,

$$Hf(x) = \frac{1}{\pi}\lim_{\varepsilon\searrow 0}\int_{\varepsilon<|y|<1/\varepsilon} \frac{f(x-y)}{y}\mathrm{d}y,$$

(for its properties see e.g. [BN71, EMO+54]), and $\mathcal{F}$ the Fourier transform. With H we can write $\hat{P} = \mathcal{F} \circ P \circ \mathcal{F}^{-1}$ as

$$\hat{P}\varphi_X(x) = \frac{\varphi_X(x) + \varphi_X(0)}{2} + \frac{iH\varphi_X(x) - iH\varphi_X(0)}{2} \qquad \big(= \varphi_{0\vee X}(x)\big).$$

Passing from (Y, X_1, X_2) to $(Y, X_1 \vee X_2)$ can be decomposed into the following steps, all of which can be translated into operations on the joint characteristic function:

$$\begin{array}{rl}
(Y, X_1, X_2): & \varphi_{(Y,X_1,X_2)}(u, v_1, v_2) \\
\downarrow & \\
(Y, X_1, X_2 - X_1): & \varphi_{(Y,X_1,X_2-X_1)}(u, v_1, v_2) = \varphi_{(Y,X_1,X_2)}(u, v_1 - v_2, v_2) \\
\downarrow & \\
(Y, X_1, 0 \vee (X_2 - X_1)): & \varphi_{(Y,X_1,0\vee(X_2-X_1))}(u, v_1, v_2) \\
& = \hat{P}_{v_2}\varphi_{(Y,X_1,X_2-X_1)}(u, v_1, v_2) \\
\downarrow & \\
(Y, X_1 \vee X_2) = (Y, X_1 + 0 \vee (X_2 - X_1)): & \varphi_{(Y,X_1\vee X_2)}(u, v) = \varphi_{(Y,X_1,0\vee(X_2-X_1))}(u, v, v),
\end{array}$$

where the subscript of $\hat{P}_{v_2}$ indicates that it acts on the variable v_2. We thus get a new operator $S = S_{v_1,v_2\to v} : \varphi_{(Y,X_1,X_2)}(u, v_1, v_2) \mapsto \varphi_{(Y,X_1\vee X_2)}(u, v)$,

$$\begin{aligned}
S\varphi_{(Y,X_1,X_2)}(u, v) &= \varphi_{(Y,X_1\vee X_2)}(u, v) \\
&= \frac{\varphi_{(Y,X_1,X_2)}(u, v, 0) + \varphi_{(Y,X_1,X_2)}(u, 0, v)}{2} \\
&\quad + \frac{i}{2\pi}\mathrm{vp}\text{–}\!\!\int_{\mathbb{R}} \varphi_{(Y,X_1,X_2)}(u, v, v - w)\left(\frac{1}{v - w} + \frac{1}{w}\right) \mathrm{d}w.
\end{aligned}$$

where vp–$\int$ indicates that we have to take the Cauchy principal value of the integral at the singularities.

From its definition it is clear that S is 'associative', i.e. the order does not matter if we iterate S, since $(X_1 \vee X_2) \vee X_3 = X_1 \vee (X_2 \vee X_3)$. This implies that there is a well-defined operator $S^{(n)}$, corresponding to n-fold suprema, and that $S^{(n)}$ can be calculated by iterating S in any way we want.

Example: Processes with independent (and stationary) increments, If $(X_t)_{t\in\mathbb{R}_+}$ is a process with independent and stationary increments, then the characteristic function of $(X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}})$ factorizes,

$$\varphi_{(X_{t_1},X_{t_2}-X_{t_1},\ldots,X_{t_n}-X_{t_{n-1}})}(u_1, \ldots, u_n) = e^{t_1 L(u_1)} e^{(t_2-t_1)L(u_2)} \cdots e^{(t_n-t_{n-1})L(u_n)},$$

and the expression for the n-fold suprema simplifies considerably. We get

$$\varphi_{\sup\{X_{t_1},\ldots,X_{t_n}\}}(u) = e^{t_1 L(u)} \hat{P}\left(e^{(t_2-t_1)L(u)} \hat{P}\left(\cdots \hat{P}\left(e^{(t_n-t_{n-1})L(u)}\right)\cdots\right)\right).$$

In the limit $n \to \infty$, $\max_{\nu=1,\ldots,n}(t_\nu - t_{\nu-1}) \to 0$ we should expect

$$\varphi_{\sup_{s\in[0,t]} X_s}(u) = e^{t\hat{P}\circ L(u)\circ\hat{P}} 1.$$

Note the reversal of the time-order, i.e. $\hat{P} \circ e^{(t_n-t_{n-1})L(u)}$ is the first to act on $1 = \hat{P}1$. If the increments are independent, but not stationary, then the preceding discussion suggests

$$\varphi_{\sup_{s\in[0,t]} X_s}(u) = G(0, t)1,$$

where $G(t,T)$ is defined as the limit $|\pi| = \sup_{i=1,\dots,T}(t_i - t_{i-1}) \to 0$ (where $\pi = (t_0 = t < t_1 \cdots < t_{n-1} < t_n = T)$ is a partition of the interval $[t,T]$) of

$$\hat{P} \circ e^{L(t_0,t_1;u)} \circ \hat{P} \circ \cdots \hat{P} \circ e^{L(t_{n-1},t_n;u)}.$$

Here $e^{L(t_i,t_{i-1};u)}$ is the characteristic function of the increment $X_{t_i} - X_{t_{i-1}}$, interpreted as an operator (acting simply by pointwise multiplication). Heuristically, we see that, if

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,u) &= L(t)\varphi(t,u) \\ \varphi(0,\cdot) &= \delta_0 \end{aligned}$$

is the evolution equation of the characteristic function of the stochastic process $(X_t)_{t\in\mathbb{R}_+}$ at t, then the characteristic function of

$$X_t^* = \sup_{s\in[0,t]} X_s$$

is determined by the evolution equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t,u) &= \hat{P} \circ L(t) \circ \hat{P}\varphi(t,u) \\ \varphi(0,\cdot) &= \delta_0, \end{aligned}$$

with the 'boundary condition'

$$(\mathrm{id} - \hat{P})\varphi(t,u) = 0.$$

We will not try to make the speculations above rigorous, prove existence of the limit, or check under what conditions on the process the limit is really the characteristic function of X_t^*, but consider the more general case of a Markov process.

9.3.3 Calculation of the supremum of a Markov chain

Let $(X)_{n\in\mathbb{N}}$ be a Markov chain (e.g. extracted from a Markov process $(Y_t)_{t\in\mathbb{R}_+}$ by setting $X_n = Y_{t_n}$ for some increasing sequence $0 \le t_0 < t_1 < \cdots$), and let $T_{n,n+1}$ be the operator that maps the characteristic function[11] of X_n to that of the pair (X_n, X_{n+1}). Then the joint characteristic function of $(X_0, X_1, \dots, X_n)$ is obtained by applying $T_{n,n-1} \circ \cdots \circ T_{0,1}$, i.e. we apply $n-1$ times the operator T. Next, we apply $S^{(n)} = S_{n,n-1} \circ \cdots \circ S_{1,0}$. The two operations of transition and taking the supremum can also be reordered to give

$$\varphi_{X_n,X_{n-1}^*} = S_{n-1,n-2} \circ T_{n,n-1} \circ S_{n-2,n-3} \circ \cdots S_{1,0} \circ T_{2,1} \circ T_{1,0}\varphi_{X_0}$$

where $X_n^* = \sup_{\nu=0,1,\dots,n} X_\nu$. We see that all we have to do is consider the two-dimensional Markov chain $\tilde{X} = (X_{n+1}, X_n^*)$ with transition operator $\mathbb{T}_{n,n+1} = S_{n+1,n} \circ T_{n+1,n+2}$ starting from $\tilde{X}_0 = (X_1, X_0)$.

11. For discrete Markov chains, i.e. if X_n has only values in $\mathbb{Z}$ or $\mathbb{N}$, it might be simpler to work with the generating function.

Example: Simple random walk. Let $(X_n)_{n\in\mathbb{N}}$ be the simple (non-symmetric) random walk on $\mathbb{Z}$, $X_0 = 1$, $X_n = \sum Y_\nu$, where the Y_ν, $\nu \in \mathbb{N}$ are i.i.d. Bernoulli random variables, $P(Y_\nu = 1) = p$, $P(Y_\nu = -1) = 1 - p$ for all $\nu \in \mathbb{N}$. We will use generating functions $g_{(X_n,X^*_{n-1})}(v,w) = \sum_{\nu,\mu\in\mathbb{Z}} P(X_n = \nu, X^*_{n-1} = \mu)v^\nu w^\mu$ instead of characteristic functions. The transition operator $T_{n,n+1}$ of $(X_n)_{n\in\mathbb{N}}$ maps v^ν to $pu^{\nu+1}v^\nu + (1-p)u^{\nu-1}v^\nu$, while S acts on the terms of the generating function as $S : v^\nu w^\mu \mapsto v^{\nu\vee\mu}$. Combining these two operators we obtain the new generator $\mathbb{T} : v^\nu w^\mu \mapsto pu^{\nu+1}v^{\nu\vee\mu} + (1-p)u^{\nu-1}v^{\nu\vee\mu}$. This leads to the recurrence relation

$$a^n_{\nu,\mu} = p \sum_{\mu':\mu=\mu'\vee(\nu-1)} a^{n-1}_{\nu-1,\mu'} + (1-p) \sum_{\mu':\mu=\mu'\vee(\nu+1)} a^{n-1}_{\nu+1,\mu'}$$

for the coefficients of the generating function $g_{(X_n,X^*_{n-1})}(u,v) = \sum a^n_{\nu,\mu} u^\nu v^\mu$ of (X_n, X^*_{n-1}).

9.3.4 Continuous time limit

In the preceeding section we have seen that we get a Markov chain with transition operator $\mathbb{T}$. Can we compute the generator of the process that is obtained in the continuous time limit? Or can we at least get some asymptotic information?

9.4 A mixed classical-quantum limit theorem

In this section we will consider an algebra on which a one-parameter family of coproducts $\{\Delta_q : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}; q \in \mathbb{R}\}$ can be defined, and investigate what distributions appear in the limit it the convolution is chosen randomly, i.e. we study the sequence

$$\begin{aligned}\varphi_N &= \underbrace{(\varphi\otimes\cdots\otimes\varphi)}_{N \text{ times}} \circ (\Delta_{q_1}\otimes \mathrm{id}\otimes\cdots\otimes \mathrm{id}) \circ \cdots \circ (\Delta_{q_{N-2}}\otimes \mathrm{id}) \circ \Delta_{q_{N-1}} \circ s(\sqrt{N}) \\ &= \left(\cdots(\varphi *_{q_1} \varphi *_{q_2})\cdots *_{q_{N-1}} \varphi\right) \circ s(\sqrt{N})\end{aligned}$$

where $q_1, q_2, \ldots$ are some i.i.d. real-valued random variables and φ is some appropriately chosen state on $\mathcal{A}$. We will further suppose that $q_n \neq 0$ for all $n \in \mathbb{N}$ and that $\mathbb{E}(|q_1^m|) < \infty$ for all $m \in \mathbb{Z}$.

Let $\mathcal{A}$ be the (unital, associative) *-algebra generated by x, x^*, and $\{y_\alpha; \alpha \in \mathbb{R}\}$ with the relations $y^*_\alpha = y_\alpha$, and

$$xy_\alpha = \alpha y_\alpha x, \qquad \alpha x^* y_\alpha = y_\alpha x^*, \qquad y_\alpha y_\beta = y_{\alpha\beta} \text{ for } \alpha,\beta \in \mathbb{R}, \qquad y_1 = 1.$$

Note that $\deg x = \deg x^* = 1$, $\deg y_\alpha = 0$ for all $\alpha \in \mathbb{R}$ defines a grading, and so we can introduce a scaling map $s(r) : \mathcal{A} \to \mathcal{A}$, for $r \in \mathbb{R}$, by setting $s(r)a = r^{-\deg a}a$ on homogeneous elements.

A basis of $\mathcal{A}$ is given by

$$\mathcal{B} = \{y_\alpha w; w \text{ a word in the two letters } x, x^*, \text{ and } \alpha \in \mathbb{R}\backslash\{0\}\}.$$

On this algebra we can define a whole family of coalgebras, depending on one parameter $q \in \mathbb{R}$, namely,

$$\Delta_q x = x \otimes 1 + y_q \otimes x, \qquad \Delta_q x^* = x \otimes 1 + y_q \otimes x^*, \qquad \Delta_q(y_\alpha) = y_\alpha \otimes y_\alpha \text{ for } \alpha \in \mathbb{R},$$

and $\varepsilon_q(x) = \varepsilon_q(x^*) = 0$, $\varepsilon_q(y_\alpha) = 1$ for all $\alpha \in \mathbb{R}$. In fact, for $q \neq 0$, the subalgebra $\mathcal{A}_0$ generated by $\{x, x^*, y_\alpha; \alpha \neq 0\}$ is a Hopf algebra with the antipode $S_q : \mathcal{A}_0 \to \mathcal{A}_0$,

$$S_q(y_\alpha) = y_{1/\alpha}, \qquad S_q(x) = -y_{1/q}x, \qquad S_q(x^*) = -y_{1/q}x^*.$$

Using these different coalgebra structures we obtain different multiplications for functionals on $\mathcal{A}$, i.e. a one-parameter family of convolutions,

$$\varphi *_q \psi = (\varphi \otimes \psi) \circ \Delta_q$$

for linear functionals $\varphi, \psi : \mathcal{A} \to \mathbb{C}$.

We state now the main result of this section (in a preliminary form).

Theorem 9.4.1 *Let $(q_n)_{n\in\mathbb{N}}$ be i.i.d. random variables with values in $\mathbb{R}\backslash\{0\}$ such that $\mathbb{E}(|q_1^m|) < \infty$ for all $m \in \mathbb{Z}$, and let $\varphi : \mathcal{A} \to \mathbb{C}$ be the state defined on the basis $\mathcal{B}$ by*

$$\varphi(y_\alpha w) = \begin{cases} 1 & \text{if } w = 1, \\ 1 & \text{if } w = x^*x, \\ 0 & \text{else.} \end{cases}$$

Then the moments of $\varphi_N = (\cdots(\varphi *_{q_1} \varphi) *_{q_2} \cdots *_{q_{N-1}} \varphi) \circ s(\sqrt{N})$ converge for $N \to \infty$ in probability to those of some functional φ_∞.

Remark: We give the explicite form of φ_∞ in Corollary 9.4.2.

Before we prove this theorem we will have to study the algebraic structure of the dual of $\mathcal{A}_0$.

9.4.1 The algebra $\mathcal{U}$

Let $\chi(w)$ be the functional defined by

$$\chi(w)(y_\beta w') = \begin{cases} 1 & \text{if } w = w', \\ 0 & \text{else.} \end{cases}$$

Then $\mathcal{U} = \text{span}\,\{\chi(w); w \in \mathbb{C} < x, x^* >\}$ is a subalgebra of the dual of $\mathcal{A}_0$ (with the multiplication $m_q = \Delta_q^*$). Let $a \in \mathcal{A}_0$, $\Delta_q a = \sum a_i^{(1)} \otimes a_i^{(2)}$, then the product $\chi(w) *_q \chi(w')$ is defined by $(\chi(w) \cdot \chi(w'))(a) = \sum \chi(w)(a_i^{(1)})\chi(w')(a_i^{(2)})$. We get by induction

$$\begin{aligned} \Delta_q(x^n) &= \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_q y_{q^{n-\nu}} x^\nu \otimes x^{n-\nu} \\ \Delta_q\left((x^*)^n\right) &= \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_q (x^*)^\nu y_{q^{n-\nu}} \otimes (x^*)^{n-\nu} = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix}_{1/q} y_{q^{n-\nu}} (x^*)^\nu \otimes (x^*)^{n-\nu} \end{aligned}$$

and from this we can also calculate the coproduct of any element of $\mathcal{A}_0$.

To calculate the product $\chi(w) *_q \chi(w')$ we have to see what elements of $\mathcal{A}_0$ have a term $y_\alpha w \otimes y_\beta w'$ in their coproduct. Since the coproduct does not change the total number of x's and x^*'s, but just splits a word into two, the product has to do the inverse. We get

$$\chi(w) *_q \chi(w') = \sum c^v_{w,w'}(q)\chi(v), \tag{9.1}$$

where v runs over all words that can be obtained by shuffling w and w', and the $c^v_{w,w'}$ are polynomials in q and q^{-1}. To get the explicite expression, use the following procedure.

In the first term v is simply the concatenation of w and w', and the coefficient is equal to one. Then move the letters of w to the right, without changing their order, and multiply by q every time an x is moved to the right, and by q^-1 every time an x^* is moved to the right. Thus e.g. $\chi(x) *_q \chi(x) = \chi(xx) + q\chi(xx) = (1+q)\chi(xx)$ or $\chi(x^*x) *_q \chi(x) = \chi(x^*xx) + q\chi(x^*xx) + \chi xx^*x = (1+q)\chi(x^*xx) + \chi(xx^*x)$.

9.4.2 Proof of Theorem 9.4.1

Let

$$\tilde{\varphi}_N = \left(\cdots!(\varphi *_{q_1} \varphi *_{q_2}) \cdots *_{q_{N-1}} \varphi\right)$$

be the unscaled N^{th} convolution, it can be written as

$$\tilde{\varphi}_N = \sum_v f^N_v(q_1, \ldots, q_{N-1})\chi(v)$$

where the summation is only over words that have the same number of x's and x^*'s. The scaled convolution product is then

$$\varphi_N = \sum_v \frac{f^N_v(q_1, \ldots, q_{N-1})}{N^{\frac{|v|}{2}}}\chi(v),$$

where $|v|$ denotes the length of v. To prove the theorem it is sufficient to show that $\mathbb{E}\left(\frac{f^N_v(q_1,\ldots,q_{N-1})}{N^{\frac{|v|}{2}}}\right)$ converges and that $\mathrm{Var}\left(\frac{f^N_v(q_1,\ldots,q_{N-1})}{N^{\frac{|v|}{2}}}\right)$ goes to zero for $N \to \infty$ (and fixed v). We will do this by induction over the length of v. The coefficient of 1 (the empty word) is constant and the coefficient of the only word with $|v| < 4$ that occurs, i.e. $v = x^*x$, is equal to N, so that the induction hypothesis is satisfied for $|v| < 4$.

Let $c^v_{w,w'}(q)$ denote the coefficients of the multiplication, as in Equation (9.1). We have the following relations for $f^{N+1}_v(q_1, \ldots, q_N)$,

$$f^{N+1}_v(q_1, \ldots, q_N) = f^N_v(q_1, \ldots, q_{N-1}) + \sum_{v' : |v'| = |v|-2} c^v_{v',x^*x}(q_N) f^N_{v'}(q_1, \ldots, q_{N-1}),$$

or

$$f^N_v(q_1, \ldots, q_{N-1}) = \sum_{k=\frac{|v|}{2}}^{N-1} \sum_{v' : |v'| = |v|-2} c^v_{v',x^*x}(q_{k-1}) f^{k-1}_{v'}(q_1, \ldots, q_{k-2}),$$

and therefore

$$\begin{aligned}
\mathbb{E}\left(\frac{f_v^N(q_1,\dots,q_{N-1})}{N^{\frac{|v|}{2}}}\right) &= \frac{1}{N}\sum_{k=\frac{|v|}{2}}^{N-1}\sum_{v':|v'|=|v|-2}\mathbb{E}\left(c^v_{v',x^*x}(q_{k-1})\right)\mathbb{E}\left(\frac{f_{v'}^{k-1}(q_1,\dots,q_{k-2})}{N^{\frac{|v'|}{2}}}\right)\\
&= \frac{1}{N}\sum_{v':|v'|=|v|-2}\mathbb{E}\left(c^v_{v',x^*x}(q_{k-1})\right)\sum_{k=\frac{|v|}{2}}^{N-1}\frac{(k-1)^{\frac{|v'|}{2}}}{N^{\frac{|v'|}{2}}}\mathbb{E}\left(\frac{f_{v'}^{k-1}(q_1,\dots,q_{k-2})}{(k-1)^{\frac{|v'|}{2}}}\right)\\
&\xrightarrow{N\to\infty}\frac{2}{|v|}\sum_{v':|v'|=|v|-2}\mathbb{E}\left(c^v_{v',x^*x}(q_1)\right)\lim_{N\to\infty}\mathbb{E}\left(\frac{f_{v'}^N(q_1,\dots,q_{N-1})}{N^{\frac{|v'|}{2}}}\right) \qquad (9.2)
\end{aligned}$$

The same technique works for the limit of the variance. Suppose $\mathrm{Var}\left(f_v^N(q_1,\dots,q_{N-1})\right)$ is bounded by $K_vN^{|v|-1}$, this is obviously true for $|v|=0,1,2,3$. Then, by

$$\begin{aligned}
\mathrm{Var}\left(f_v^N(q_1,\dots,q_{N-1})\right) &= \mathrm{Var}\left(\sum_{k=\frac{|v|}{2}}^{N-1}\sum_{v':|v'|=|v|-2}c^v_{v',x^*x}(q_{k-1})f_{v'}^{k-1}(q_1,\dots,q_{k-2})\right)\\
&\le N\sum_{k=\frac{|v|}{2}}^{N-1}\sum_{v':|v'|=|v|-2}\mathbb{E}\left((c^v_{v',x^*x}(q_{k-1}))^2\right)\mathrm{Var}\left(f_{v'}^{k-1}(q_1,\dots,q_{k-2})\right)\\
&\le N^{|v|-1}\sum_{v':|v'|=|v|-2}\mathbb{E}\left((c^v_{v',x^*x}(q_1))^2\right)K_{v'}
\end{aligned}$$

it is true for all v, and therefore $\mathrm{Var}\left(\frac{f_v^N(q_1,\dots,q_{N-1})}{N^{\frac{|v|}{2}}}\right)\xrightarrow{N\to\infty}0$.

We have actually shown more than Theorem 9.4.1, Equation (9.2) allows to calculate the limit φ_∞. Define a new binary operation $\overline{*}:\mathcal{U}\otimes\mathcal{U}\to\mathcal{U}$ as the 'average' of the convolutions $*_q$ w.r.t. to the distribution of the $\{q_i\}$, i.e.

$$u\,\overline{*}\,v=\mathbb{E}(u*_{q_1}v).$$

Then the 'averaged convolution' of two basis elements is

$$\chi(w)\,\overline{*}\,\chi(w')=\sum\mathbb{E}\left(c^v_{w,w'}(q_1)\right)\chi(v),$$

and we have the following result.

Corollary 9.4.2 *We have*

$$\varphi_\infty=\exp_{\overline{*}}\chi(x^*x),$$

and

$$\varphi_\infty\left((x+x^*)^n\right)=\begin{cases}\frac{k_{2m}k_{2m-2}\cdots k_2}{m!} & \text{if } n \text{ is even, } n=2m,\\ 0 & \text{if } n \text{ is odd,}\end{cases}$$

where $k_n=\sum_{\nu=1}^{n-1}\nu\mathbb{E}(q_i^{n-\nu})$.

Examples:

- In the deterministic case, $P(q_1=q)=1$, we get the marginal distribution of the Azéma martingale.

– For Bernoulli, $P(q_1 = 1) = p$, $P(q_1 = -1) = 1 - p$, $p \in [0, 1]$,

$$\varphi_\infty\Big((x + x^*)^{2m}\Big) = \prod_{\mu=0}^{m-1} (1 + 2\mu p)$$

i.e. the moments of $\mu = c_p |x|^{\frac{1}{p}-1} e^{-x^2/2p} \mathrm{d}x$.

9.5 Operator-limit theorems on bialgebras

One can also try to formulate limit theorems where the normalization is done by linear operators, e.g. algebra or coalgebra homomorphism, instead of scalars, along the lines of [JM93].

Chapitre 10

Classical Markov Processes from Quantum Lévy Processes

Work in Progress

Résumé

We show how classical Markov processes can be obtained from quantum Lévy processes. It is shown that quantum Lévy processes are quantum Markov processes, and sufficient conditions for restrictions to subalgebras to remain quantum Markov processes are given. Then there exists a classical Markov process whenever we can restrict to a commutative subalgebra without loosing the quantum Markov property[Küm88]. Several examples, including the Azéma martingale, with explicit calculations are presented. In particular, the action of the generator of the classical Markov process on polynomials and the moments of the classical process can be calculated using Hopf algebra duality.

10.1 Introduction

It is an interesting question which quantum stochastic processes admit classical versions, i.e. for what families of operators $\{X_t = j_t(x) | t \in I\}$ there exists a classical stochastic process $\{\tilde{X}_t | t \in I\}$ on some probability space $(\Omega, \mathcal{F}, P)$ such that all time-ordered moments agree, i.e.

$$\Phi(X_{t_1}^{k_1} \cdots X_{t_n}^{k_n}) = \mathbb{E}(\tilde{X}_{t_1}^{k_1} \cdots \tilde{X}_{t_n}^{k_n}), \quad \text{for all } n, k_1, \ldots, k_n \in \mathbb{N}, \quad t_1 \leq \ldots \leq t_n \in I,$$

for some fixed state Φ.

A famous example of a classical version of a quantum Lévy process is the the Azéma martingale [Azé85, Eme89, Par90, Sch91, Sch93]. In this case there exists an element x such that the process $\{X_t = j_t(x)\}$ is actually commutative, and thus it is clear that it is equivalent to a classical process.

We will use the theory of quantum Markov processes to give weaker conditions that guarantee the existence of classical versions of Lévy process, and study examples.

In Section 10.2 we recall the definitions of quantum Lévy processes and quantum Markov processes, show that every quantum Lévy process is a quantum Markov process, and give conditions under which the restriction of a quantum Lévy process to a subalgebra remains a Markov process.

In the following section (Section 10.3) we study several examples, including the Azéma martingale and a symmetrized Poisson process, and show how Hopf algebra duality can be used for explicite calculations.

10.2 Classical versions of quantum Lévy processes

10.2.1 Quantum Lévy processes

The definition of a Lévy process $\{j_{st} | 0 \leq s \leq t \leq T\}$, $T \in \mathbb{R}_+ \cup \{\infty\}$ on a (braided) *-bialgebra $\mathcal{B}$ over a quantum probability space $(\mathcal{A}, \Phi)$ is presented in Definition 2.5.1. For simplicity we will assume that $\mathcal{B}$ is a Hopf algebra, and that the process is given by $\{j_t = j_{0t} | 0 \leq t \leq T\}$. Let us also assume that $(\mathcal{A}, \Phi)$ is the quantum probability space obtained as inductive limit from finite tensor products on $\mathcal{B}$ and the marginal distribution $\varphi_t = \Phi \circ j_t$ by the construction described in [Sch93, pp. 38-40] and in Section 4.8. For an interval $I \subseteq [0, T]$ we denote by $\mathcal{A}_I$ the *-subalgebra of $\mathcal{A}$ generated by $\bigcup_{\tau \in I} j_\tau(\mathcal{B})$. For singletons $\{t\} = [t, t]$ the algebra $\mathcal{A}_t = \mathcal{A}_{[t,t]}$ is isomorphic to $\mathcal{B}$, and we have the isomorphisms $T_{s,t} : \mathcal{A}_s \to \mathcal{A}_t$, $T_{s,t} = j_t \circ j_s^{-1}$.

10.2.2 From quantum Lévy to quantum Markov

We want to define conditional expectations $P_I : \mathcal{A} \to \mathcal{A}_I$ and show that a Lévy process satisfies $P_{[0,s]}(\mathcal{A}_t) \subseteq \mathcal{A}_s$ for all $0 \leq s \leq t \leq T$, i.e. that it is a quantum Markov process. Remember that in the finite tensor product via which the inductive limit is defined the factors correspond to increments. Consider an interval of the form $[0, t]$. Let $\mathcal{B}^{\otimes n}$ be the 'approximation' of $\mathcal{A}$ corresponding to $(s_1, \ldots, s_k, \ldots, s_n)$, $s_1 < \cdots < s_n$, i.e. the entries

of the ith factor in the tensor product represent the increment during the interval $[s_{i-1}, s_i]$ (set $s_0 = 0$). Let $s_k = t$. Then set

$$\begin{aligned} P_{[0,t]}^{(s_1,\ldots,s_k,\ldots,s_n)} &= \underbrace{\mathrm{id} \otimes \cdots \mathrm{id}}_{k \text{ times}} \otimes (e \circ \varphi_{s_{k+1}-s_k}) \otimes (e \circ \varphi_{s_n - s_{n-1}}) \end{aligned}$$

In the inductive limit this defines an operator $P_{[0,t]} : \mathcal{A} \to \mathcal{A}_{[0,t]}$.

Proposition 10.2.1 *The operator $P_{[0,t]}$ is completely positive and satisfies $P_{[0,t]}^2 = P_{[0,t]}$ and $P_{[0,t]}(xyz) = xP_{[0,t]}(y)z$ for all $x, z \in \mathcal{A}_{[0,t]}$, $y \in \mathcal{A}$, i.e. it is a* conditional expectation.

Proof: The positivity follows immediately from the way $P_{[0,t]}^{(s_1,\ldots,s_k,\ldots,s_n)}$ is defined as a combination of the identity map and the positive functionals $\varphi_{s_i - s_{i-1}}$. And the compatibility with the inductive limit, i.e. the compatibility with the maps $j_{\cdot,\cdot}$ follows from the fact that the functionals form a convolution semigroup. ∎

Proposition 10.2.2 *We have $P_{[0,s]}(\mathcal{A}_t) \subseteq \mathcal{A}_s$ for all $0 \leq s \leq t \leq T$, i.e. $\{j_t | 0 \leq t \leq T\}$ is a* quantum Markov process.

Proof: Since the one-dimensional distributions form a convolution semigroup, all $P_{[0,s]}^{(s_1,\ldots,s_k,\ldots,s_n)}$ for $n \geq 2$, $0 < s_k = s < s_n = t$ act in the same way on elements of $\mathcal{A}_t$, namely $P_{[0,s]}^{(s_1,\ldots,s_k,\ldots,s_n)}\Big|_{\mathcal{A}_t} = j_s \circ (\mathrm{id}_{\mathcal{B}} \otimes \varphi_{t-s}) \circ \Delta \circ j_t^{-1}$, and therefore this is also true for $P_{[0,s]}\Big|_{\mathcal{A}_t}$. In particular, it follows $P_{[0,s]}(\mathcal{A}_t) \subseteq \mathcal{A}_s$ ∎

10.2.3 From quantum Markov to classical Markov

We know (see e.g. [BKS96]) that for every quantum Markov process on a commutative *-algebra (e.g. the subalgebra generated by one self-adjoint element) there exists a classical version, i.e. a classical stochastic process that has the same (time-ordered) joint moments as the quantum Markov process. The quantum Markov property is sufficient (not necessary) for the joint density associated to the joint time-ordered moments to be positive, and then the classical process exists by Kolmogorov's construction.

Thus we need to look for self-adjoint elements of $\mathcal{B}$ who generate a subalgebra such that the restriction of the quantum Lévy process to this algebra remains Markov. Take $x \in \mathcal{B}$, $x^* = x$, and denote the *-algebra generated by x by $\mathcal{B}^x$.

We give two criteria that guarantee that the restriction of $\{j_t\}$ to $\mathcal{B}^x$ remains Markov.

Proposition 10.2.3

*(a) Let $\mathcal{B}_0$ be a *-subalgebra of $\mathcal{B}$. If $\Delta(\mathcal{B}_0) \subseteq \mathcal{B}_0 \otimes \mathcal{B}$ (i.e. $\mathcal{B}_0$ is a right coideal), then the restriction $\{j_t|_{\mathcal{B}_0}\}$ to $\mathcal{B}_0$ of every Lévy process $\{j_t\}$ on $\mathcal{B}$ is a quantum Markov process.*

(b) Let L be the generator of $\{j_t\}$, $x \in \mathcal{B}$ self-adjoint. Then the restriction of $\{j_t\}$ to $\mathcal{B}^x$ is a quantum Markov process if and only if $\rho(L) = (\mathrm{id} \otimes L) \circ \Delta$ leaves $\mathcal{B}^x$ invariant, i.e. if $\rho(L)(\mathcal{B}^x) \subseteq \mathcal{B}^x$.

10.3 Examples of classical versions of Lévy processes on $\mathbb{R}_q * \mathbb{R}_{1/q}$

Let us first consider the bialgebra that leads to the Azéma martingale (cf. [Sch91, Sch93]). It can be viewed as the free product of the braided line with its dual. As an algebra it is generated by one element a and its adjoint a^*, with no relations between a and a^*. Thus the set $\mathcal{B} = \{1, a, a^*, aa, aa^*, a^*a, a^*a^*, \ldots\}$ of all words in the two letters a, a^* forms a basis of $\mathbb{R}_q * \mathbb{R}_{1/q}$. The coproduct and counit are defined as

$$\Delta(a) = a + a', \qquad \Delta(a^*) = a^* + a^{*\prime}, \qquad \varepsilon(a) = \varepsilon(a^*) = 0$$

on the generators and extended as algebra homomorphisms (Notation: $a^{(*)} = a^{(*)} \otimes 1$, $a^{(*)\prime} = 1 \otimes a^{(*)}$). Here the algebra structure of $(\mathbb{R}_q * \mathbb{R}_{1/q})\hat{\otimes}(\mathbb{R}_q * \mathbb{R}_{1/q})$ is determined by the braid relations

$$\begin{aligned} a'a &= qaa', & a^{*\prime}a &= qaa^{*\prime}, \\ a'a^* &= q^{-1}a^*a & a^{*\prime}a^* &= q^{-1}a^*a^{*\prime}, \end{aligned}$$

where we assume $q \in \mathbb{R}\backslash\{0\}$.

We need to construct the dual $\mathcal{U}$ of $\mathbb{R}_q * \mathbb{R}_{1/q}$. For $q = 1$ this is the shuffle algebra, see e.g. [SS93, Section 3.8]. In the general case the dual might be called a q-shuffle algebra, the formulas of the shuffle algebra only have to modified by some q-dependent combinatorial coefficients.

A functional on $\mathbb{R}_q * \mathbb{R}_{1/q}$ is determined by its action on the basis chosen above, i.e. on the 'words' in a, a^*. Thus it can be written as

$$u = \sum_{\chi \in \mathcal{X}} c_\chi \chi,$$

where $\mathcal{X}$ is the dual basis of $\mathcal{B}$, i.e. the set of all words in two letters (x, x^*), and $c_\chi \in \mathbb{C}$. As dual of a coalgebra this is an algebra, and if we restrict to finite linear combinations, then it is also a coalgebra, and the dual pairing

$$< \chi, \beta >= \begin{cases} 1 & \text{if } \chi, \beta \text{ are identical modulo the substitution } a \leftrightarrow x,\ a^* \leftrightarrow x^*, \\ 0 & \text{else,} \end{cases}$$

$\chi \in \mathcal{X}$, $\beta \in \mathcal{B}$, is still non-degenerate.

The coproduct of a word in x, x^* is just the sum over the different ways to split the word in two, i.e. for a word with n letters there are exactly $n+1$ terms. Thus we have e.g. $\Delta x = x \otimes 1 + 1 \otimes x$, $\Delta x^* = x^* \otimes 1 + 1 \otimes x^*$, or $\Delta x x^* x = x x^* x \otimes 1 + x x^* \otimes x + x \otimes x^* x + 1 \otimes x x^* x$.

To compute the product [12] of two basis elements χ_1, χ_2 in the q-shuffle algebra, we have to look at all the ways in which the first word χ_1 can be 'shuffled' into the second. The first term is simply the concatenation $\chi_1\chi_2$. Now we form all combinations of the letters of χ_1 and χ_2 where the order of the letters of χ_1 (and χ_2, resp.) remains unchanged, and

12. Here we used the convention $< u \cdot v, \beta >= \sum < u, \beta^{(1)} >< v, \beta^{(2)} >$, i.e. we defined the dual pairing of the tensor product algebra with the flip automorphism τ: $< \cdot, \cdot >_\otimes = (< \cdot, \cdot > \otimes < \cdot, \cdot >) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})$.

add a factor q every time an x (from χ_1) is moved to the right across a letter of χ_2, and a factor q^{-1} every time an x^* (from χ_1) is moved to the right across a letter of χ_2. Thus, e.g.

$$\begin{aligned} x\cdot x &= xx+qxx=(1+q)xx, & x\cdot x^* &= xx^*+qx^*x,\\ x^*\cdot x &= x^*x+q^{-1}xx^*, & x^*\cdot x^* &= x^*x^*+q^{-1}x^*x^*=(1+q^{-1})x^*x^*, \end{aligned}$$

or

$$\begin{aligned} \widehat{xx^*}\cdot\underbrace{xx^*} &= \widehat{xx^*}\,\underbrace{xx^*}+q^{-1}\,\widehat{x\,\underbrace{xx^*}\,x^*}+\underbrace{x\,\widehat{xx^*}\,x^*}+q^{-2}\,\widehat{x\,\underbrace{xx^*}\,x^*}+q^{-1}\,\underbrace{x\,\widehat{xx^*}\,x^*}+\underbrace{xx^*}\,\widehat{xx^*}\\ &= 2xx^*xx^*+(1+2q^{-1}+q^{-2})xxx^*x^*. \end{aligned}$$

WARNING: xx, xx^*, xx^*xx^*, etc. means concatenation, in this section the multiplication in $\mathcal{U}$ is **always** indicated by a dot.

The dual action or right regular representation[13] $\rho(\chi)=(\mathrm{id}\otimes\chi)\circ\Delta$ of $\mathcal{U}$ eliminates the letters corresponding to those of χ, if they are in the same order, with possibly additional letters in between, and adds a factor q (q^{-1}, resp.) for every letter to the left of an a (a^*, resp.) that is suppressed. E.g., on an element $a_1a_2\cdots a_n\in\mathbb{R}_q*\mathbb{R}_{1/q}$, with $a_i\in\{a,a^*\}$ for $i=1,\dots,n$,

$$\begin{aligned} \rho(x):a_1\cdots a_n &\mapsto \sum_{i:\,a_i=a} q^{i-1}a_1\cdots\check{a}_i\cdots a_n\\ \rho(x^*):a_1\cdots a_n &\mapsto \sum_{i:\,a_i=a^*} q^{-i+1}a_1\cdots\check{a}_i\cdots a_n\\ \rho(xx^*):a_1\cdots a_n &\mapsto \sum_{i<j:\,a_i=a,\,a_j=a^*} q^{i-j+1}a_1\cdots\check{a}_i\cdots\check{a}_j\cdots a_n. \end{aligned}$$

10.3.1 The Azéma martingale

If we choose the generator $L=c_1xx^*+c_2x^*x$, with $c_1,c_2\in\mathbb{R}_+$, then the classical version of $j_t(a+a^*)$ is the Azéma martingale M_t (cf. [Sch91, Sch93]), i.e. the (finite) joint moments of $\{j_t(a+a^*);t\in\mathbb{R}_+\}$ agree with those of the Azéma martingale: $\mathbb{E}(M_{t_1}^{n_1}\cdots M_{t_k}^{n_k})=\Phi\left((j_{t_1}(a+a^*))^{n_1}\cdots(j_{t_k}(a+a^*))^{n_k}\right)$ for all $t_1,\dots,t_k\in\mathbb{R}_+$, $n_1,\dots,n_k\in\mathbb{N}$, $k\in\mathbb{N}$. In order to find the generator of the Azéma martingale (as a classical Markov process) and to compute the moments, we need to know how L acts on the subalgebra generated by $z=a+a^*$.

Proposition 10.3.1 (Leibnitz formulas). *Let $f=\sum_{k=0}^n f_kz^k$ with $f_k\in\mathbb{C}$ and $z=a+a^*$. Then*

$$\begin{aligned} \rho(x)(zf) &= f+qz\rho(x)f,\\ \rho(x^*)(zf) &= f+q^{-1}z\rho(x^*)f,\\ \rho(xx^*)(zf) &= \rho(x^*)f+z\rho(xx^*)f,\\ \rho(x^*x)(zf) &= \rho(x)f+z\rho(x^*x)f. \end{aligned}$$

13. i.e. $\rho(\chi)\beta=(<\cdot,\cdot>\otimes\mathrm{id})\circ(\mathrm{id}\otimes\tau)\circ(\mathrm{id}\otimes\Delta)(\chi\otimes\beta)$ for all $\chi\in\mathcal{U}$, $\beta\in\mathcal{A}$.

Proof: This follow from the braided version of Equation (2.1). We have $\rho(u)(\alpha\beta) = \sum \alpha^{(1)} \cdot (\mathrm{id} \otimes u^{(1)})(\Psi(\alpha^{(2)} \otimes \rho(u^{(2)})\beta))$ (where we used Sweedler's notation for the coproduct). Applying the braid relations and recalling that z, x, x^* are primitive, while $\Delta(xx^*) = xx^* \otimes 1 + x \otimes x^* + 1 \otimes xx^*$, $\Delta(x^*x) = x^*x \otimes 1 + x^* \otimes x + 1 \otimes x^*x$, we get the desired formulas. ∎

Setting $f = z^n$, we obtain recurrence relations that allow us to determine the operators on polynomials in z,

$$\begin{array}{lllllll}
\rho(x) : z^n & \mapsto & q_n z^n & \Rightarrow & (\rho(x)f)(z) & = & \delta_q f(z) = \frac{f(qz)-f(z)}{z(q-1)} \\
\rho(x^*) : z^n & \mapsto & (q^{-1})_n z^n & \Rightarrow & (\rho(x)f)(z) & = & \delta_{1/q} f(z) = \frac{f(q^{-1}z)-f(z)}{z(q^{-1}-1)} \\
\rho(xx^*) : z^n & \mapsto & \left\{ \begin{array}{ll} \frac{(q^{-1})_n - n}{q^{-1}-1} z^{n-2} & n \geq 2, \\ 0 & n = 0,1 \end{array} \right. & \Rightarrow & (\rho(xx^*)f)(z) & = & \frac{\delta_{1/q} f(z) - f'(z)}{z(q^{-1}-1)} \\
\rho(x^*x) : z^n & \mapsto & \left\{ \begin{array}{ll} \frac{q_n - n}{q-1} z^{n-2} & n \geq 2, \\ 0 & n = 0,1 \end{array} \right. & \Rightarrow & (\rho(xx^*)f)(z) & = & \frac{\delta_q f(z) - f'(z)}{z(q-1)}
\end{array}$$

for $q \neq 1$, and $\rho(x)f = \rho(x^*)f = f'$, $\rho(xx^*)f = \rho(x^*x)f = \frac{1}{2}f''$ for $q = 1$.

Using these operators to calculate the Appell polynomials

$$h_n(z;t) = e^{t\rho(L)} z^n,$$

we can also calculate the moments of the Azéma martingale, $\mathbb{E}(M_t^n) = h_n(0,t)$. We summarize the results in the following theorem.

Theorem 10.3.2 *Let $(j_t)_{t \in \mathbb{R}_+}$ be the quantum Lévy process on $\mathbb{R}_q * \mathbb{R}_{1/q}$ with generator $L = c_1 xx^* + c_2 x^*x$, $c_1, c_2 \in \mathbb{R}_+$, and $(M_t)_{t \in \mathbb{R}_+}$ the classical version of $j_t(a + a^*)$ (i.e. the Azéma martingale). Then the generator L_{a+a^*} of M_t (as a classical Markov process) is given by*

$$L_{a+a^*} f(z) = \left\{ \begin{array}{ll} \frac{c_2 \delta_q f(z) - q c_1 \delta_{1/q} f(z) - (c_2 - q c_1) f'(z)}{z(q-1)} & q \neq 1 \\ \frac{c_1 + c_2}{2} f''(z) & q = 1 \end{array} \right.$$

on polynomials $f(z) = \sum_{k=0}^n f_k z^n$, $f_0, f_1, \ldots, f_n \in \mathbb{C}$. The moments of this process are

$$\mathbb{E}(M_t^n) = \left\{ \begin{array}{ll} \frac{k_{2m} k_{2m-2} \cdots k_2}{m!} t^m & n = 2m \text{ even}, \\ 0 & n \text{ odd}, \end{array} \right.$$

where $k_n = c_1 \frac{(q^{-1})_n - n}{q^{-1}-1} + c_2 \frac{q_n - n}{q-1}$ for $q \neq 1$, and $k_n = (c_1 + c_2)\frac{n(n-1)}{2}$ for $q = 1$.

10.3.2 Other processes on $\mathbb{R}_q * \mathbb{R}_{1/q}$.

If we want to obtain other classical processes we can either change the quantum process, i.e. choose a different generator, or use a different commutative subalgebra of $\mathbb{R}_q * \mathbb{R}_{1/q}$ that satisfies the conditions discussed in Subsection 10.2.3. Let us briefly look at the second possibility. We need an element of $u \in \mathcal{B} = \mathbb{R}_q * \mathbb{R}_{1/q}$ such that $\Delta \mathcal{B}^u \subseteq \mathcal{B}^{\otimes}(\mathbb{R}_q * \mathbb{R}_{1/q})$. A possible choice is $u = a^*a + qaa^*$. In fact, $\Delta u^n = \sum_{\nu=0}^n \binom{n}{\nu} u^\nu (u')^{n-\nu}$, so that

u actually generates a Hopf subalgebra. But, since this Hopf subalgebra is isomorphic to the Hopf algebra of polynomials in $\mathbb{R}$, we find exactly the classical Lévy processes whose moments are finite.

Let us now look at other generators. Since no algebraic relations are imposed on a and a^*, we can take any operator X acting on some Hilbert space $\mathcal{H}$ to define a representation ρ_X of $\mathbb{R}_q * \mathbb{R}_{1/q}$. To get a positive functional on $\mathbb{R}_q * \mathbb{R}_{1/q}$ we now simply fix an element $h \in \mathcal{H}$ and set $\tilde{\psi}_{h,X}(u) =< h, \rho_X(u)h >$. Then $\psi_{h,X} = \tilde{\psi}_{h,X} - \tilde{\psi}_{h,X}(1)\varepsilon$ is conditionally positive, and, if $\psi_{h,X}$ also satisfies the invariance condition, then there exists a Lévy process with generator $\psi_{h,X}$. The invariance condition in this case means simply that $\psi_{h,X}$ vanishes on 'words' that do not have the same number of a's and a^*'s. Let $\mathcal{H} = \mathbb{C}^2$,

$$X_\alpha = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \qquad h_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad h_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

then $\psi_{i,\alpha} = \psi_{h_i,X_\alpha}$ are generators of Lévy processes on $\mathbb{R}_q * \mathbb{R}_{1/q}$. One verifies that they can be written as

$$\begin{aligned} \psi_{1,\alpha} &= |\alpha|^2 xx^* + |\alpha|^4 xx^*xx^* + |\alpha|^6 xx^*xx^*xx^* + \cdots \\ \psi_{2,\alpha} &= |\alpha|^2 x^*x + |\alpha|^4 x^*xx^*x + |\alpha|^6 x^*xx^*xx^*x + \cdots, \end{aligned}$$

i.e. $\psi_{1,\alpha}$ ($\psi_{2,\alpha}$, resp.) is the sum over all concatenations of xx^* (x^*x, resp.) with itself, with coefficient $|\alpha|^l$, where l is the length of the 'word'. Introducing

$$\begin{aligned} \phi_{1,\alpha} &= |\alpha|^2 x^* + |\alpha|^4 x^*xx^* + |\alpha|^6 x^*xx^*xx^* + \cdots = |\alpha|^2 x^*(1 + \tilde{\psi}_{1,\alpha}) \qquad \text{(Concatenation!)} \\ \phi_{2,\alpha} &= |\alpha|^2 x + |\alpha|^4 xx^*x + |\alpha|^6 xx^*xx^*x + \cdots = |\alpha|^2 x(1 + \tilde{\psi}_{2,\alpha}) \qquad \text{(Concatenation!)}, \end{aligned}$$

we can state the Leibnitz formulas that allow to determine the action of $\psi_{1,\alpha}$ and $\psi_{2,\alpha}$ on polynomials in $z = a + a^*$.

Proposition 10.3.3 *Let $f = \sum_{k=0}^n f_k z^k$. We have*

$$\begin{aligned} \rho(\psi_{1,\alpha})(zf) &= \rho(\phi_{1,\alpha})f + z\rho(\psi_{1,\alpha})f \\ \rho(\phi_{1,\alpha})(zf) &= |\alpha|^2\rho(1 + \psi_{1,\alpha})f + q^{-1}z\rho(\phi_{1,\alpha})f \\ \rho(\psi_{2,\alpha})(zf) &= \rho(\phi_{2,\alpha})f + z\rho(\psi_{2,\alpha})f \\ \rho(\phi_{2,\alpha})(zf) &= |\alpha|^2\rho(1 + \psi_{2,\alpha})f + qz\rho(\phi_{2,\alpha})f \end{aligned}$$

Proof: Similarly as in the proof of Proposition 10.3.1. We only need the first two terms of $\Delta(\psi_{i,\alpha})$, since the others vanish when applied to 1 or z. ∎

We set $P_n = \rho(\psi_{1,\alpha})z^n$, and combine the recurrence relations above to

$$P_{n+1}(z) = (1 + q^{-1})zP_n(z) + (|\alpha|^2 - q^{-1}z^2)P_{n-1} + |\alpha|^2 z^{n-1},$$

for $n \geq 1$, $P_1(z) = P_0(z) = 0$. $Q_n = \rho(\psi_{2,\alpha})z^n$ satisfies the same relation with q instead of q^{-1}.

This recurrence relation allows to calculate the Q_n, for the first ten we get (with Maple)

$$Q(0,x):=0$$

$$Q(1,x):=0$$

$$Q(2,x):=|\alpha|^2$$

$$Q(3,x):=\left(2\,|\alpha|^2+q\,|\alpha|^2\right)x$$

$$Q(4,x):=\left(3\,|\alpha|^2+2\,q\,|\alpha|^2+q^2\,|\alpha|^2\right)x^2+|\alpha|^4$$

$$Q(5,x):=\left(4\,|\alpha|^2+2\,q^2\,|\alpha|^2+q^3\,|\alpha|^2+3\,q\,|\alpha|^2\right)x^3+\left(3\,|\alpha|^4+2\,q\,|\alpha|^4\right)x$$

$$Q(6,x):=\left(3\,q^2\,|\alpha|^2+5\,|\alpha|^2+4\,q\,|\alpha|^2+q^4\,|\alpha|^2+2\,q^3\,|\alpha|^2\right)x^4+\left(6\,|\alpha|^4+6\,q\,|\alpha|^4+3\,q^2\,|\alpha|^4\right)x^2+|\alpha|^6$$

$$\begin{aligned}Q(7,x):=&\left(6\,|\alpha|^2+5\,q\,|\alpha|^2+4\,q^2\,|\alpha|^2+3\,q^3\,|\alpha|^2+q^5\,|\alpha|^2+2\,q^4\,|\alpha|^2\right)x^5\\&+\left(12\,q\,|\alpha|^4+4\,q^3\,|\alpha|^4+10\,|\alpha|^4+9\,q^2\,|\alpha|^4\right)x^3+\left(3\,q\,|\alpha|^6+4\,|\alpha|^6\right)x\end{aligned}$$

$$\begin{aligned}Q(8,x):=&\left(q^6\,|\alpha|^2+6\,q\,|\alpha|^2+3\,q^4\,|\alpha|^2+2\,q^5\,|\alpha|^2+4\,q^3\,|\alpha|^2+5\,q^2\,|\alpha|^2+7\,|\alpha|^2\right)x^6\\&+\left(5\,q^4\,|\alpha|^4+15\,|\alpha|^4+12\,q^3\,|\alpha|^4+18\,q^2\,|\alpha|^4+20\,q\,|\alpha|^4\right)x^4\\&+\left(6\,q^2\,|\alpha|^6+12\,q\,|\alpha|^6+10\,|\alpha|^6\right)x^2+|\alpha|^8\end{aligned}$$

$$\begin{aligned}Q(9,x):=&\left(5\,q^3\,|\alpha|^2+8\,|\alpha|^2+4\,q^4\,|\alpha|^2+3\,q^5\,|\alpha|^2+2\,q^6\,|\alpha|^2+6\,q^2\,|\alpha|^2+q^7\,|\alpha|^2+7\,q\,|\alpha|^2\right)x^7\\&+\left(21\,|\alpha|^4+6\,q^5\,|\alpha|^4+15\,q^4\,|\alpha|^4+30\,q\,|\alpha|^4+24\,q^3\,|\alpha|^4+30\,q^2\,|\alpha|^4\right)x^5\\&+\left(30\,q\,|\alpha|^6+24\,q^2\,|\alpha|^6+20\,|\alpha|^6+10\,q^3\,|\alpha|^6\right)x^3+\left(4\,q\,|\alpha|^8+5\,|\alpha|^8\right)x\end{aligned}$$

$$\begin{aligned}Q(10,x):=&\left(8\,q\,|\alpha|^2+4\,q^5\,|\alpha|^2+6\,q^3\,|\alpha|^2+5\,q^4\,|\alpha|^2+2\,q^7\,|\alpha|^2+9\,|\alpha|^2+3\,q^6\,|\alpha|^2+7\,q^2\,|\alpha|^2+q^8\,|\alpha|^2\right)\\&x^8+\left(28\,|\alpha|^4+18\,q^5\,|\alpha|^4+40\,q^3\,|\alpha|^4+30\,q^4\,|\alpha|^4+42\,q\,|\alpha|^4+7\,q^6\,|\alpha|^4+45\,q^2\,|\alpha|^4\right)x^6\\&+\left(60\,q^2\,|\alpha|^6+40\,q^3\,|\alpha|^6+15\,q^4\,|\alpha|^6+35\,|\alpha|^6+60\,q\,|\alpha|^6\right)x^4\\&+\left(15\,|\alpha|^8+10\,q^2\,|\alpha|^8+20\,q\,|\alpha|^8\right)x^2+|\alpha|^{10}\end{aligned}$$

For $q=1$ these polynomials are $\frac{1}{2}\{(x+|\alpha|)^n+(x-|\alpha|)^n-2x^n$, i.e. the generator is a difference operator, $Lf(x)=\frac{1}{2}\{f(x+|\alpha|)+f(x-|\alpha|)-2f(x)\}$, and the process is a difference of two Poisson processes with jumps of size $|\alpha|$ and same intensity. From these we can build any symmetric compound Poisson process by taking the generator

$L_\mu = \int_{\mathbb{R}} \psi_{1,\alpha} \mathrm{d}\mu(\alpha)$, and thus we have 'q-analogues' of symmetric compound Poisson processes at hand.

If we replace P_n by $\tilde{P}_n = P_n + z^n$, then we get a homogeneous recurrence relation of the form $P_{n+1} = aP_n + bP_{n-1}$, with $a = (1+q^{-1})z$ and $b = (|\alpha|^2 - q^{-1}z^2)$. The solution of this relation with $P_0 = 1$, $P_1 = z$, is given by

$$P_n = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} a^{n-2k} b^k,$$

and similarly for the Q_n, just replace q by q^{-1}.

Theorem 10.3.4 *Let $(j_t)_{t\in\mathbb{R}_+}$ be the Lévy process with generator $\psi_{2,\alpha}$. Then the generator of the associated classical Markov process acts on polynomials as*

$$L_{2,\alpha,q} : z^n \mapsto \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} (1+q)^{n-2k} z^{n-2k} (|\alpha|^2 - qz^2)^k - z^n.$$

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Résumé

Ce mémoire est rélatif à l'étude des processus stochastiques sur les algèbres de Hopf. Ces algèbres jouent un rôle important en physique mathématique sous le nom groupes quantiques.

Une grande partie de cette thèse est consacrée à l'étude des processus de Lévy, c-à-d des processus à accroissements indépendants et stationnaires, sur ces algèbres. Deux constructions, soit à partir d'un processus de Lévy classique, soit à partir d'une marche aléatoire quantique, sont proprosées. Ces processus sont ensuite étudiés à l'aide des représentations duales et de leurs systèmes d'Appell. En particulier, ceci a permis de démontrer une formule de Feynman-Kac et d'établir un lien étroit entre ces processus et des équations d'évolution sur les groupes quantiques.

Les représentations duales sont également utilisées pour donner des conditions suffisantes pour l'existence de versions classiques des processus de Lévy sur des bigèbres et pour les caractériser. Plusieurs exemples, y compris la martingale d'Azéma, sont traités en détail.

Un autre thème central de ce travail est la caractérisation des lois de Gauss au sens de Bernstein. Il est montré comment les fonctionnelles ainsi que les semi-groupes de convolution sur des algèbres de Hopf qui satisfont l'analogue de la propriété de Bernstein peuvent être calculés. Il est aussi démontré que le plongement d'une fonctionnelle normée infiniment divisible dans un semi-groupe de convolution continu sur un groupe quantique nilpotent ou sur un groupe tressé nilpotent est unique.

Finalement, plusieurs théorèmes limites (loi des grands nombres, théorème de la limite centrale, etc.) sur les groupes quantiques sont présentés.

Mots-clés: Processus de Lévy sur des bigèbres, théorèmes limites

Abstract

In this thesis stochastique processus on Hopf algebras are studied. These algebras, also known under the name quantum group, play in important role in mathematical physics.

A major part of this work is concerned with Lévy processes, i.e. processes with stationary and independent increments. Two construction are proposed, one starts from a classical Lévy process, the other one uses quantom random walks. These processes are then studied with the help of dual representations and Appell systems. This has allowed us to prove an analogue of the Feynman-Kac formula, and to study the relation between the processes and their evolution equations.

Dual representations are also used to give sufficient conditions for the existence of classical versions of Lévy processes on bialgebras, and to calculate the classical generators. Several examples including the Azéma martingale are treated in detail.

Another central theme is the classification of Gaussian laws. It is shown, how the functionals and convolution semigroups that satisfy an analogue of the Bernstein property can be determined. We also prove the the embedding of a normalized functional into a continuous convolution semigroup on a nilpotent quantum group or nilpotent braided group is unique.

Finally, there are also several limit theorems (law of large numbers, central limit theorem, etc.) presented in this work.

Mathematics Subject Classifications (1991): *60B99 Probability theory on general structures, 16W30 Hopf algebras (assoc. rings and algebras), 60F05 Weak limit theorems.