Champs de vecteurs, flots et géodésiques sur les supervariétés

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Résumé

Le résultat principal de cette thèse est de donner une définition de géodésique sur les supervariétés riemanniennes $(\mathcal{M}, g)$ paires (et aussi impaires) et de la justifier par un théorème reliant les courbes géodésiques avec le flot géodésique sur $\mathrm{T}^*\mathcal{M}$. Pour ce faire, nous construisons la 2-forme symplectique canonique sur $\mathrm{T}^*\mathcal{M}$ et l'analogue H de la fonctionnelle énergie dans le contexte des supervariétés. Nous prenons ainsi le flot du champ de vecteurs hamiltonien associé à H que nous nommons “flot géodésique”. Alors, nous relions les supergéodésiques, que nous définissons à l'aide de la dérivée covariante comme des courbes à vitesse auto-parallèle, avec le flot géodésique via des conditions initiales adaptées aux supervariétés. Une autre définition de géodésique a été proposée en 2006 par O. Goertsches mais ces courbes ne sont pas en bijection avec les courbes intégrales du flot géodésique que nous construisons. Notre définition de géodésique semble donc présenter plus d'avantages. Par ailleurs, nous pouvons, à l'aide du flot, construire l'application exponentielle. Nous en profitons pour démontrer le résultat, bien connu au cas de cadre des variétés classiques (non-graduées), de linéarisation des isométries en utilisant l'exponentielle. Dans la dernière partie, nous redémontrons un résultat de J. Monterde et O.M. Sánchez-Valenzuela concernant l'intégration des champs de vecteur pairs, impairs et aussi non homogènes dans le but d'éviter d'utiliser un modèle de Batchelor. Ceci permet par exemple, de généraliser leurs résultats aux supervariétés holomorphes.

Abstract

We give a natural definition of geodesics on a Riemannian supermanifold $(\mathcal{M}, g)$ and extend the usual geodesic flow on T^*M associated to the underlying Riemannian manifold (M, g) to a geodesic “superflow” on $T^*\mathcal{M}$. Integral curves of this flow turn out to be in natural bijection with geodesics on $\mathcal{M}$. We also construct the corresponding exponential map and generalize the well-known faithful linearization of isometries to Riemannian supermanifolds. We give also a new proof of the Monderde et al. result about flows of non-homogeneous supervector fields. We give a treatment which allows extensions for instance to the holomorphic category. The original proof given by Monderde et al. is only applicable to split supermanifolds, since their proofs relied on Batchelor's Theorem. Finally, we reproves a characterization of vector fields whose flows are local $\mathbb{R}^{1|1}$-actions of an appropriate Lie supergroups structure.

Table des matières

Introduction

La géométrie des supervariétés, qui ont été définies dans les années 70-80, n'est pas aussi finement connue que celle des variétés classiques. De nombreux mathématiciens étudient désormais cette géométrie et ses applications. Notre travail sur les supervariétés est au moins motivé par deux développements, l'un provenant de la physique et l'autre de la géométrie Riemannienne étendue à la catégorie des supervariétés. Premièrement, nous sommes intéressés par les aspects du "programme de Stolz-Teichner" sur la géométrisation de la cohomologie elliptique qui est liée au cobordisme des supervariétés Riemanniennes et aux supervariétés en générale (voir [22] pour plus de détails). Bien sûr, étendre certains résultats de cette thèse aux super espaces de lacets (voir [26] pour le cas non-gradué) donnerait des liens directs avec la littérature en physique dont la cohomologie elliptique est sous-jacente, où le rôle des espaces de lacets est central (voir, par exemple, [25]). Deuxièmement, le travail de Zirnbauer et ses collaborateurs (voir, par exemple, [27]) montre l'importance de la notion d'espaces supersymétriques en physique, notamment en physique mésoscopique. Dans ce contexte, il est très naturel qu'une théorie générale des supervariétés Riemanniennes et ses objets remplacent les espaces symétriques de la géométrie Riemannienne classique. Les espaces supersymétriques étaient, en fait, la principale motivation du travail de Goertsches ([12]) sur les supervariétés Riemanniennes.

La pierre angulaire de la géométrie Riemannienne est -après la métrique elle-même- la notion de courbes géodésiques. Il paraît raisonnable de demander que les notions de métriques et de géodésiques (au moins dans le cas de métriques paires) généralisées aux supervariétés remplissent les conditions suivantes:

(i) les supercourbes géodésiques devraient se projeter, par réduction, sur les géodésiques usuelles dans le cas non-gradué des variétés Riemanniennes,

(ii) une isométrie qui fixe un point ayant une linéarisation triviale en ce point devrait être le morphisme identité,

(iii) les géodésiques devraient être étroitement liées aux courbes intégrales du flot ("géodésique") d'un champ de vecteurs approprié sur le fibré (co-)tangent de la supervariété Riemannienne donnée.

La définition d'une métrique Riemannienne semble être généralement acceptée dans la littérature (voir définition 1.7 ci-dessous). La notion de courbe géodésique proposée par Goertsches vérifie (i) et (ii), mais malheureusement, nous ne pouvons pas établir (iii) avec cette approche. Puisque le flot géodésique est un outil d'une très grande importance en géométrie et en analyse sur les variétés Riemanniennes, nous trouvons qu'il est crucial de ne pas renoncer à la condition (iii). De plus, notre recours à une définition d'une "condition initiale" comme morphisme aproprié (voir Theorem 2.14) ne semble pas souffrir des défauts que Goertsches attribue au début de la sous-section 4.4 (de [12]) à des systèmes de second ordre. Soulignons que les équations choisies ici pour définir les géodésiques se traduisent en coordonnées locales par un système d'équations du second ordre, tandis que la définition des géodésiques de Goertsches donne lieu à des équations du second ordre, respectivement du premier ordre, pour les coordonnées paires, respectivement impaires. Ceci explique, au moins intuitivement, pourquoi on n'a pas pu définir un flot géodésique lié à la notion de géodésiques de Goertsches.

Donnons maintenant une brève description du contenu de ce texte.

En premier, nous donnons la définition de supervariété, avec laquelle nous avons choisi de travailler. Notre approche est celle remontant à Kostant et à l'école russe de Berezin, Bernstein, Leites et Manin.

Nous donnons aussi la définition des superfibrés vectoriels et le détail de la construction du superfibré vectoriel associé à un faisceau de module sur les superfonctions localement libre de rang fini. Nous définissons alors le (co-)tangent d'une supervariété ainsi que l'application tangent d'un morphisme de supervariétés dans la catégorie des superfibrés. Nous utilisons dans la suite ces objets pour donner une condition initiale aux courbes géodésiques que nous voyons comme des supercourbes "autoparallèles":

$$0 = \frac{\nabla}{dt}\big(d\Phi^*(\partial_t)\big) = (\Phi^*\nabla)_{\partial_t}(\partial_t \circ \Phi^*),$$

où $\Phi^*\nabla$ est le tiré en arrière de la connection de Levi-Civita, ∂_t est $\frac{\partial}{\partial t}$ et $\partial_t \circ \Phi^*$ est le champ de vecteur canonique le long de Φ. L'équation de géodésique donné ci-dessus a déjà été formulée par J. Monterde et J. Muñoz Masqué par un principe variationnel (voir Thm. 7.1 et Cor. 7.1 dans [16]). Apparemment, les solutions de cette équation n'ont jamais été étudié en détails, seule l'observation que (i) est satisfait a déjà été faite dans [16].

Le théorème 2.14 donne l'existence et l'unicité de géodésiques sur une supervariété Riemannienne $\mathcal{M}$, avec pour condition initial généralisé $\mathcal{J}$ dans $\mathrm{Mor}(\mathcal{S}, T\mathcal{M})$, où $\mathcal{S}$ est une supervariété arbitraire.

Puis, nous rappelons la classe de superfonctions sur le superfibré vectoriel qui sont polynomiales dans la fibre, nous permettant ainsi de définir la fonctionnelle énergie naturelle H sur le (co-)tangent $T^*\mathcal{M}$, respectivement $\Pi T^*\mathcal{M}$, d'une supervariété Riemannienne paire, respectivement impaire. Nous donnons intrinsèquement (et en coordonnées) la forme symplectique canonique sur $T^*\mathcal{M}$, respectivement $\Pi T^*\mathcal{M}$, et explicitons le champ de vecteur Hamiltonien X_H associé à H. Nous montrons ensuite que les "supercourbes intégrales" de ce champ de vecteurs (i.e. courbes données par le flot géodésique) sont en bijection avec les géodésiques de $(\mathcal{M}, g)$, voir théorèmes 2.28, 2.29 et 2.36. Ce résultat à fait l'objet d'une publication dans la revue "Journal of Geometry and Physics" [10].

Par la suite, nous donnons des applications du morphisme de flot géodésique. Premièrement, nous montrons l'existence de l'application exponentielle qui généralise le cas non-gradué et montrons que $\exp_q : T_q\mathcal{M} \to \mathcal{M}$, pour q un point de M, la variété (classique) sous-jacente à $\mathcal{M}$, est un difféomorphisme autour de 0 (Théorème 2.40). Puis, nous démontrons qu'une isométrie d'une supervariété Riemannienne connexe qui fixe un point et dont l'application tangent en ce point est l'identité, est déjà le morphisme identité sur la supervariété. Ceci implique immédiatement l'unicité des "symétries géodésiques" dans le sens des espaces symétriques Riemannien fixant un point et ayant moins l'identité pour application tangente en ce point.

Dans la dernière partie, nous exposons un résultat de J. Monterde et O.M. Sánchez-Valenzuela et donnons une nouvelle preuve. Le notre ne fait pas intervenir de modèle de Batchelor ce qui, par exemple, la rend directement valable aussi dans le contexte holomorphe. Ce résultat concerne l'intégration des champs de vecteur pairs, impairs et aussi non homogènes. Il donne un moyen d'intégrer un superchamp de vecteurs quelconque sur une supervariété $\mathcal{M}$ et donne des conditions portant sur les composantes homogènes du champ pour que le flot définisse une action locale de $\mathbb{R}^{1|1}$ sur la supervariété $\mathcal{M}$. Ce n'est effectivement pas toujours le cas contrairement au cas classique: Un champ de vecteur sur une variété M donne toujours une action locale de $\mathbb{R}$ sur M.

Chapitre 1

Catégories des supervariétés et des superfibrés vectoriels

1.1 Supervariétés et métriques Riemanniennes

Dans cette partie, nous rappelons brièvement plusieurs notions fondamentales en donnant la preuve de certains faits folkloriques si nous n'avons pas trouver de référence dans la littérature.

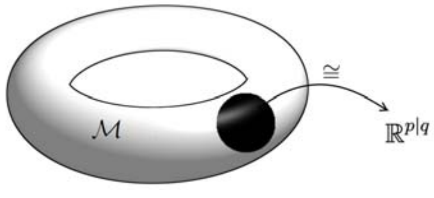
Nous utilisons, dans tout le texte qui suit, l'approche faisceautique des supervariétés données par Kostant [13] et Leites [14]. Les supervariétés peuvent aussi être étudiées dans d'autres formalismes comme celui de DeWitt [8] et Tuynman [23] ou celui de Molotkov–Sachse [19]. Il s'avère que toutes ces catégories de supervariétés sont isomorphes dans le cas réel et holomorphe en dimension fini (voir [3], [4] et [2]).

Nous basons donc notre travail sur la définition qui suit:

Définition 1.1. *Une* ***supervariété*** $\mathcal{M}$ *de* ***dimension*** $m|n$ *est un espace localement annelé* $(M, \mathcal{O}_\mathcal{M})$*, où* $\mathcal{O}_\mathcal{M}$ *est un faisceau de algèbres* $\mathbb{Z}_2$*-graduées commutatives sur* M*, tel que*

(i) M *est un espace topologique de Hausdorff, à base dénombrable,*

(ii) tout point $m \in M$ *possède un voisinage ouvert* U *tel que l'espace annelé* $(U, \mathcal{O}_{\mathcal{M}|U})$ *est isomorphe à un* ***superdomaine*** $\mathcal{V}^{m|n} = (V, \mathcal{O}_{\mathbb{R}^{m|n}|V})$*, où* V *est un ouvert de* $\mathbb{R}^m$*.*



Remarques/Rappels.

(i) Rappelons qu'un superdomaine est la restriction sur un ouvert du modèle local suivant:
$\mathbb{R}^{m|n} = (\mathbb{R}^m, \mathcal{O}_{\mathbb{R}^{m|n}})$ est le faisceau d'anneaux qui à tout ouvert U de $\mathbb{R}^m$ associe

$$\mathcal{O}_{\mathbb{R}^{p|q}}(U) = \mathcal{C}^\infty(U; \mathbb{R})[\xi_1, \xi_2, \dots, \xi_n],$$

la $\mathcal{C}^\infty(U; \mathbb{R})$-superalgèbre commutative unitale librement engendrée par les éléments impairs $\xi_1, \xi_2, \dots, \xi_n$. Elle est évidemment isomorphe à $\mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \Lambda(\mathbb{R}^*)^n$.

(ii) Si $\mathcal{X} = (X, \mathcal{O}_\mathcal{X})$ et $\mathcal{Y} = (Y, \mathcal{O}_\mathcal{Y})$ sont des espaces annelés, un **morphisme** $\Phi : \mathcal{X} \to \mathcal{Y}$ est un couple $(\widetilde{\Phi}, \Phi^*)$, où $\widetilde{\Phi} : X \to Y$ est une application continue et $\Phi^* : \mathcal{O}_\mathcal{Y} \to \widetilde{\Phi}_* \mathcal{O}_\mathcal{X}$ une collection d'homomorphismes d'anneaux $\Phi^*(U) : \mathcal{O}_\mathcal{Y}(U) \to \mathcal{O}_\mathcal{X}(\widetilde{\Phi}^{-1}(U))$, pour chaque ouvert U de Y, compatibles avec les applications de restrictions. Si l'espace annelé est local, i.e., pour tout $x \in X$, l'espace $\mathcal{O}_{\mathcal{X},x}$ des germes en x possède un unique idéal maximal $\mathfrak{m}_x$, le morphisme $\Phi^*_x : \mathcal{O}_{\mathcal{Y},\widetilde{\Phi}(x)} \to \mathcal{O}_{\mathcal{X},x}$ est supposé local pour tout $x \in X$, i.e., $\Phi^*_x(\mathfrak{m}_{\widetilde{\Phi}(x)}) \subset \mathfrak{m}_x$.

On remarque que la donnée d'une telle collection Φ^* d'homomorphisme est équivalent à $\Phi^* : \widetilde{\Phi}^{-1}\mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$, la donnée d'une collection d'homomorphisme d'anneaux $\Phi^*(V) : (\widetilde{\Phi}^{-1}\mathcal{O}_\mathcal{Y})(V) \to \mathcal{O}_\mathcal{X}(V)$, pour tout ouvert V de X, compatibles avec les applications de restriction. Rappelons que l'image inverse $\widetilde{\Phi}^{-1}\mathcal{O}_\mathcal{Y}$ de $\mathcal{O}_\mathcal{Y}$ par l'application continue $\widetilde{\Phi}$ est définie par

$$(\widetilde{\Phi}^{-1}\mathcal{O}_\mathcal{Y})(V) := \varinjlim_{\substack{W \supset \widetilde{\Phi}(V) \\ W \text{ ouvert de } Y}} \mathcal{O}_\mathcal{Y}(W).$$

Cette équivalence est due au fait que $\widetilde{\Phi}_*$ et $\widetilde{\Phi}^{-1}$ sont des foncteurs co-adjoints, i.e.

$$\mathrm{Hom}(\mathcal{O}_\mathcal{Y}, \widetilde{\Phi}_*\mathcal{O}_\mathcal{X}) \xrightarrow{\cong} \mathrm{Hom}(\widetilde{\Phi}^{-1}\mathcal{O}_\mathcal{Y}, \mathcal{O}_\mathcal{X}).$$

(iii) Si M est une variété de dimension m alors $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ est une supervariété de dimension $m|m$ si $\mathcal{O}_\mathcal{M}$ est défini comme suit:

$$\mathcal{O}_\mathcal{M}(U) = \bigoplus_{k \geq 0} \Omega^k_M(U).$$

Plus généralement, si $E \to M$ est un fibré vetoriel réel de rang n, on lui associe le faisceau $(M, \Gamma^\infty_{\wedge E^*})$ des sections lisses dans $\wedge E^*$ qui est une supervariété de dimension $m|n$. Réciproquement, nous avons, dans le cadre réel lisse, le théorème suivant, dit de Batchelor :

Théorème 1.2. *(voir, par exemple, [6] section 4.4) Soit $\mathcal{M}$ une supervariété de dimension $m|n$. Il existe un fibré vectoriel $E \to M$ de rang n, unique à isomorphisme près, tel que $\mathcal{O}_\mathcal{M}$ soit isomorphe au faisceau des sections du fibré extérieur de E^*:*

$$\mathcal{M} \cong (M, \Gamma^\infty_{\wedge E^*})\,.$$

Cet isomorphisme n'est ni canonique ni unique, il résulte d'un choix de représentants, dans une certaine cohomologie, des fonctions de changement de coordonnées. Le théorème de Batchelor ne dit pas que la catégorie des supervariétés se réduit à celle des fibrés vectoriels: Tout morphisme de fibrés vectoriels induit naturellement un morphisme entre les supervariétés associées aux fibrés vectoriels. Mais la réciproque est fausse. La catégorie des supervariétés sur une variété M donnée contient “beaucoup plus“ de morphisme que celle des fibrés vectoriels sur M.

(iv) Dans le cadre réel lisse, un morphisme de supervariétés est entièrement déterminé par le morphisme entre les sections globales, i.e. la donnée de $\Phi : \mathcal{X} \to \mathcal{Y}$ est équivalente à celle de $\Phi^*(Y) : \mathcal{O}_\mathcal{Y}(Y) \to \mathcal{O}_\mathcal{X}(X)$. Ce résultat est faux dans le cadre holomorphe, car, par exemple, l'ensemble des sections globales d'un tore compacte complexe, vu comme supervariété purement paire, est trop “pauvre” puisqu'il ne contient que les fonctions constantes.

Théorème 1.3. *([6], Satz 4.5) Etant données deux supervariétés $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ et $\mathcal{N} = (N, \mathcal{O}_\mathcal{N})$,*

(i) il existe un morphisme de faisceaux $\beta : \mathcal{O}_\mathcal{M} \to \mathcal{C}^\infty_M$, appelé morphisme de réduction, tel que la suite

$$0 \longrightarrow \mathcal{O}^{nil}_\mathcal{M} \longrightarrow \mathcal{O}_\mathcal{M} \xrightarrow{\beta} \mathcal{C}^\infty_M \longrightarrow 0$$

soit exacte, où $\mathcal{O}^{nil}_\mathcal{M}$ est le faiseau des éléments nilpotents de $\mathcal{O}_\mathcal{M}$.

(ii) Si $\Phi : \mathcal{N} \to \mathcal{M}$ est un morphisme de supervariétés, alors nous avons le diagramme commutatif suivant

$$\begin{array}{ccc} \mathcal{O}_\mathcal{M} & \xrightarrow{\Phi^*} & \widetilde{\Phi}_*\mathcal{O}_\mathcal{N} \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ \mathcal{C}^\infty_M & \xrightarrow{(\widetilde{\Phi})^*} & \widetilde{\Phi}_*\mathcal{C}^\infty_N \end{array}\,.$$

Pour toutes supervariétés $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$, on définit $\mathcal{T}_\mathcal{M}(U) := \mathrm{Der}_\mathbb{R}(\mathcal{O}_\mathcal{M}(U))$ pour tout ouvert U de M. Ici $\mathrm{Der}_\mathbb{R}(\mathcal{O}_\mathcal{M}(U))$ signifie le $\mathcal{O}_\mathcal{M}(U)$-module des dérivations $\mathcal{O}_\mathcal{M}(U)$, où une dérivation homogène φ est par définition un élément de $\mathrm{End}_\mathbb{R}(\mathcal{O}_\mathcal{M}(U))_0 \cup \mathrm{End}_\mathbb{R}(\mathcal{O}_\mathcal{M}(U))_1$ tel que

$$\varphi(ab) = \varphi(a)b + (-1)^{|\varphi||a|}a\varphi(b) \text{ pour tout } a, b \in \mathcal{O}_\mathcal{M}(U).$$

$\mathcal{T}_{\mathcal{M}}$ est appelé le **faisceau tangent** et possède la structure suivante (voir, e.g. [6], Satz 4.36):

Théorème 1.4. *$\mathcal{T}_{\mathcal{M}}$ est un faisceau localement libre de $\mathcal{O}_{\mathcal{M}}$-modules de rang (m, n).*

Les sections de $\mathcal{T}_M$ sont appelées **champs de vecteurs**. Le $\mathbb{R}$-espace vectoriel $\mathcal{T}_{\mathcal{M}}(U)$ a naturellement une structure de superalgèbre de Lie donnée par le **crochet** $[.,.]$

$$[X, Y]f := X(Yf) - (-1)^{|X||Y|}Y(Xf),$$

qui vérifie, dans le sens gradué, l'identité de Jacobi: Pour tout triplet X, Y, Z de champs de vecteurs:

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]].$$

Sur une supervariété, on a aussi le **tangent numérique**, définit comme suit:
Pour tout point p de M, l'espace tangent numérique $T_p^{\mathrm{num}}\mathcal{M}$ de $\mathcal{M}$ en p est le $\mathbb{R}$-espace vectoriel engendré par les dérivations homogènes $\varphi : \mathcal{O}_{\mathcal{M},p} \to \mathbb{R}$ vérifiant

$$\varphi(fg) = \varphi(f)\widetilde{g}(p) + (-1)^{|\varphi||f|}\widetilde{f}(p)\varphi(g),$$

où $\mathcal{O}_{\mathcal{M},p} := \varinjlim_{U \ni p} \mathcal{O}_{\mathcal{M}}(U)$ est l'espace des germes de $\mathcal{O}_{\mathcal{M}}$ en p et $\widetilde{f}(p) = \beta(f)(p)$.

Pour tout $p \in U$, il existe une application naturelle $\mathcal{T}_{\mathcal{M}}(U) \to T_p^{\mathrm{num}}\mathcal{M}$ appliquant le champ de vecteurs $X \in \mathcal{T}_{\mathcal{M}}(U)$ sur sa valeur $X^{\mathrm{num}}(p)$ en p comme suit:
Soit $g \in \mathcal{O}_{\mathcal{M},p}$. Il existe $f \in \mathcal{O}_{\mathcal{M}}(W)$, où W est un ouvert de U contenant p, tel que $[f]_p = g$. Alors, on définit

$$X^{\mathrm{num}}(p)(g) := \beta(X_{|W}(f))(p)$$

où $X_{|W}$ est la restriction de X sur l'ouvert W. Ceci est bien défini puisque les dérivations sont des applications locales.

En pratique, si (q_i) sont des coordonnées sur U et $X = \sum_i a_i \partial_{q_i}$ avec $a_i \in \mathcal{O}_{\mathcal{M}}(U)$ alors

$$X^{\mathrm{num}}(p) = \sum_i \widetilde{a}_i(p)\partial_{q_i|p}.$$

Remarquons que, contrairement au cas classique, un champ de vecteur n'est pas déterminé par la donnée de ses valeurs en tout point.

Il existe un fibré vectoriel réel naturel de rang $m + n$ sur M, ayant les espaces tangents numériques pour fibres, qui se décompose en la somme direct de $(T^{\mathrm{num}}\mathcal{M})_0 \to M$ et

$(T^{\mathrm{num}}\mathcal{M})_1 \to M$. Alors X^{num} est une section lisse de $T^{\mathrm{num}}\mathcal{M} \to M$. Si $\mathcal{M} = (M, \Gamma^\infty_{\Lambda E^*})$ avec E un fibré vectoriel lisse au-dessus de M, la "partie impaire" du fibré tangent numérique, $(T^{\mathrm{num}}\mathcal{M})_1$, est en faite isomorphe à E.

Nous utilisons encore la lettre β pour l'application de réduction sur le fibré tangent numérique composée avec la projection sur la partie paire de l'espace tangent:

$$\beta : \mathcal{T}_\mathcal{M} \to \left(M, \Gamma^\infty_{(T^{\mathrm{num}}\mathcal{M})_0}\right),$$

où $\left(M, \Gamma^\infty_{(T^{\mathrm{num}}\mathcal{M})_0}\right)$ est naturellement isomorphe au faisceau (M, Γ^∞_{TM}) des champs de vecteurs sur M.

Pour $f \in \mathcal{O}_\mathcal{M}(U)$ et $X \in \mathcal{T}_\mathcal{M}(U)$, nous noterons souvent $\widetilde{f}$ et $\widetilde{X}$ au lieu de $\beta(f)$ et $\beta(X)$, respectivement.

Définition 1.5. *Le **faisceau cotangent** d'une supervariété $\mathcal{M}$ est le dual $\Omega^1_\mathcal{M} := Hom_{\mathcal{O}_\mathcal{M}}(\mathcal{T}_\mathcal{M}, \mathcal{O}_\mathcal{M})$ de $\mathcal{T}_\mathcal{M}$.*

Proposition 1.6. *Le faisceau de $\mathcal{O}_\mathcal{M}$-modules $\Omega^1_\mathcal{M}$ est localement libre de rang (m, n).*

Preuve. Le faisceau $\mathcal{T}_\mathcal{M}$ est lui-même localement libre de rang (m, n). □

Définition 1.7. *Une **métrique Riemannienne graduée paire, resp. impaire**, sur une supervariété $\mathcal{M}$ est un morphisme de faisceaux $\mathcal{O}_\mathcal{M}$-linéaire, gradué paire, resp. impaire, symétrique non-dégénéré*

$$g : \mathcal{T}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{T}_\mathcal{M} \to \mathcal{O}_\mathcal{M}.$$

C'est à dire

(i) $g(X, Y) = (-1)^{|X||Y|} g(Y, X)$,

(ii) $|g(X, Y)| + |X| + |Y| = 0$, *resp.* 1, *pour tout champ de vecteurs homogènes* $X, Y \in \mathcal{T}_\mathcal{M}(U)$,

(iii) l'application $X \mapsto g(X, .)$ *est un isomorphisme* $\mathcal{O}_\mathcal{M}$*-linéaire de* $\mathcal{T}_\mathcal{M}$ *dans* $\Omega^1_\mathcal{M}$.

Une supervariété équipée d'une métrique Riemannienne graduée paire, resp. impaire, est appelée une **supervariété Riemannienne paire**, resp. **impaire**.

Remarques.

(1) Les adjectifs "paire" et "impaire" sont ajoutés au terme "supervariété Riemannienne" pour spécifier la nature de la métrique et, bien entendu, pas celle de la supervariété.

(2) Soit g une métrique Riemannienne paire sur une supervariété $\mathcal{M}$. Soit $q \in M$, le morphisme g définit un superproduit scalaire g_q sur le super espace vectoriel réel $T_q^{\mathrm{num}}\mathcal{M}$ comme suit: Soient u, v deux vecteurs tangents numériques de $T_q^{\mathrm{num}}\mathcal{M}$. Choisissons deux vecteurs tangents $X, Y \in \mathcal{T}_{\mathcal{M}}(U)$, où U est un voisinage ouvert de q dans M, vérifiant $X^{\mathrm{num}}(q) = u$ et $Y^{\mathrm{num}}(q) = v$. On pose alors $g_q^{\mathrm{num}}(u, v) := \widetilde{g(X,Y)}(q)$. En général, cette famille de superproduits scalaires ne déterminent pas la métrique Riemannienne initiale g sur $\mathcal{M}$. Par restriction sur la partie paire du tangent numérique, ce superproduit scalaire induit une métrique pseudo-Riemannienne $\widetilde{g}$ sur M, tandis que la restriction sur la partie impaire donne une structure symplectique, notée ω^E, sur le fibré vectoriel réel $(T^{\mathrm{num}}\mathcal{M})_1 \to M$. En conséquence, la dimension impaire n d'une supervariété Riemannienne paire est toujours paire.

(3) Si g est une métrique Riemannienne impaire, les dimensions pair m et impaire n de la supervariété doivent être égales puisque g est non-dégénéré si et seulement si g^{num} est non-dégénéré. De plus, pour q dans M et pour $u, v \in T_q^{\mathrm{num}}\mathcal{M}$, on a toujours $g_q^{\mathrm{num}}(u, v) = 0$ si $|u| = |v|$.

Exemple. Etant donnés M une variété et E un fibré vectoriel sur M, donnons nous une métrique Riemannienne $g^M : \Gamma_{TM} \otimes_{\mathcal{C}^\infty_M} \Gamma_{TM} \to \mathcal{C}^\infty_M$ sur M et une deux forme $\omega^E : \Gamma_E \otimes_{\mathcal{C}^\infty_M} \Gamma_E \to \mathcal{C}^\infty_M$ sur E non-dégénéré. De plus, supposons que l'on ait une connexion $\nabla : \Gamma_E \to \Omega^1_M \otimes_{\mathcal{C}^\infty_M} \Gamma_E$ sur E quelconque (pas nécessairement respectant ω). Alors, il existe une métrique Riemannienne paire naturelle sur $\mathcal{M} = (M, \Gamma_{\Lambda E^*})$ la supervariété associé au fibré vectoriel E. En effet, les dérivations de $\Gamma_{\Lambda E^*}$ se décomposent, via la connexion ∇, en deux types de dérivations auxquelles nous pouvons naturellement appliquer g et ω. Cette décomposition est donnée par le théorème suivant:

Théorème 1.8. *(voir [15]) Soit $\mathcal{M} = (M, \Gamma_{\Lambda E^*})$, la supervariété associée à un fibré vectoriel $E \xrightarrow{\pi} M$, alors, si on munit E d'une connexion ∇, le faisceau tangent $\mathcal{T}_{\mathcal{M}} = Der(\Gamma_{\Lambda E^*})$ est isomorphe à la somme direct $\Gamma_{\Lambda E^* \otimes TM} \oplus \Gamma_{\Lambda E^* \otimes E} \cong \Gamma_{\Lambda E^*} \otimes_{\mathcal{C}^\infty_M} \Gamma_{T^{num}\mathcal{M}}$.*

Preuve. Observons d'abord qu'on a un morphisme $(M, \Gamma_{\Lambda E^*}) \to (M, \mathcal{C}^\infty_M = \Gamma_{\Lambda^0 E^*})$ que nous notons π. Soit $D \in \mathrm{Der}(\Gamma_{\Lambda E^*})$, alors la restriction de D à $\mathcal{C}^\infty_M$:

$$\begin{tikzcd}
\Gamma_{\Lambda E^*} \arrow[r, "D"] & \Gamma_{\Lambda E^*} \\
\mathcal{C}^\infty_M \arrow[u, "\pi^*"] \arrow[ur, "D\circ\pi^*"'] &
\end{tikzcd}$$

est une dérivation le long de π^*. Or, l'espace des dérivations le long de π^* est isomorphe à $\Gamma_{\Lambda E^* \otimes TM}$ (voir la Proposition 2.8 au prochain chapitre pour la description, en toute généralité, des dérivations le long d'un morphisme de supervariétés). Notons D^{TM} l'élément de $\Gamma_{\Lambda E^* \otimes TM}$ correspondant à $D \circ \pi^*$. Remarquons que la connexion ∇ induit

naturellement une connexion sur Γ_{E^*} et par suite sur $\Gamma_{\Lambda E^*}$. Nous pouvons alors, pour tout $K \in \Gamma_{\Lambda E^* \otimes TM}$, définir un endomorphisme $\nabla_K : \Gamma_{\Lambda E^*} \to \Gamma_{\Lambda E^*}$ tel que pour tout $a, k \in \Gamma_{\Lambda E^*}$ et $X \in \Gamma_{TM}$ on ait $\nabla_{k \otimes X}(a) = k \cdot \nabla_X(a)$.
On montre alors, sans difficulté, que l'endomorphisme $D' := D - \nabla_{D^{TM}}$ est une dérivation de $\Gamma_{\Lambda E^*}$ qui agit trivialement sur le fonction de $\mathcal{C}^\infty_M$. Donc la restriction de D' à Γ_{E^*} est un morphisme $\mathcal{C}^\infty_M$-linéaire:

$$\begin{tikzcd}
\Gamma_{\Lambda E^*} \arrow[r, "D'"] & \Gamma_{\Lambda E^*} \\
\Gamma_{E^*} \arrow[u, hook] \arrow[ur, "D'_{|\Gamma_{E^*}}"'] &
\end{tikzcd}$$

Puisque l'espace des endomorphismes $\mathcal{C}^\infty_M$-linéaires de Γ_{E^*} dans $\Gamma_{\Lambda E^*}$ est isomorphe à $\Gamma_{\Lambda E^* \otimes (E^*)^*}$ et que nous avons un isomorphisme naturel de E dans $(E^*)^*$ alors $D'_{|\Gamma_{E^*}}$ peut être vu comme un élément de $\Gamma_{\Lambda E^* \otimes E}$. Notons D^E cet élément.
Remarquons que, pour tout élément ϕ de $\Gamma_{\Lambda E^* \otimes E}$, nous pouvons définir un endomorphisme i_ϕ de $\Gamma_{\Lambda E^*}$ telle que si $a, k \in \Gamma_{\Lambda E^*}$ et $e \in \Gamma_E$ alors $i_{k \otimes e}(a) = k \cdot i(e)(a)$, où i est la "contraction" ou "produit intérieur". On montre aisément que, pour tout $\phi \in \Gamma_{\Lambda E^* \otimes E}$, i_ϕ est une dérivation de $\Gamma_{\Lambda E^*}$ qui agit trivialement sur les fonctions de $\mathcal{C}^\infty_M$.
Pour finir, on peut montrer que les deux dérivations D' et i_{D^E} de $\Gamma_{\Lambda E^*}$ coïncident sur $\Gamma_{E^*} \oplus \mathcal{C}^\infty_M$ et qu'elles ont le même degré. Elles sont donc égales, ce qui donne:

$$D = \nabla_{D^{TM}} + i_{D^E}.$$

Pour finir, cette construction réalise bien l'isomorphisme $\Gamma_{\Lambda E^*}$-linéraire recherché si on munit la somme directe $\Gamma_{\Lambda E^* \otimes TM} \oplus \Gamma_{\Lambda E^* \otimes E}$ de la bonne stucture $\mathbb{Z}_2$-gradué (la parité des sections de ΛE^* est donnée par la $\mathbb{Z}_2$-graduation usuelle, les sections de TM sont paires et celles de E sont impaires). □

Alors, on définit g une métrique Riemannienne paire sur $\mathcal{M} = (M, \Gamma_{\Lambda E^*})$ comme suit:

$$g(D_1, D_2) := \widehat{g^M}(D_1^{TM}, D_2^{TM}) + \widehat{\omega^E}(D_1^E, D_2^E), \quad \forall D_i \in \mathrm{Der}(\Gamma_{\Lambda E^*}),$$

où les D_i^{TM}, D_i^E sont donnés par le théorème précédent et $\widehat{g^M}$, $\widehat{\omega^E}$ sont tels que, pour $k_i \in \Gamma_{\Lambda E^*}$, $X_i \in \Gamma_{TM}$ et $e_i \in \Gamma_E$, on a

$$\widehat{g^M}(k_1 \otimes X_1, k_2 \otimes X_2) = k_1 \cdot k_2 \cdot g^M(X_1, X_2) \text{ et } \widehat{\omega^E}(k_1 \otimes e_1, k_2 \otimes e_2) = (-1)^{|k_2|} k_1 \cdot k_2 \cdot \omega^E(e_1, e_2).$$

1.2 Superfibrés vectoriels et faisceaux de $\mathcal{O}_{\mathcal{M}}$-modules localement libres

Dans cette partie, nous allons donner quelques détails concernant les superfibrés vectoriels sur une supervariété $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$.

Définition 1.9. *Soient $\mathcal{V} = (V, \mathcal{O}_{\mathcal{V}})$ et $\mathcal{M}$ deux supervariétés et soit $\pi : \mathcal{V} \to \mathcal{M}$ un morphisme de supervariétés de $\mathcal{V}$ dans $\mathcal{M}$. $\mathcal{V}$ est appelé **superfibré vectoriel** de base $\mathcal{M}$ et de fibre typique $\mathbb{R}^{m|n}$ si*

(i) pour tout $x \in M$, il existe une trivialisation locale (U, φ), i.e. il existe un voisinage ouvert U de x et un isomorphisme $\varphi : \mathbb{R}^{m|n} \times \mathcal{M}_{|U} \to \mathcal{V}_{|\widetilde{\pi}^{-1}(U)}$ tel que le diagramme suivant commute

$$\begin{tikzcd}
\mathcal{V}_{|\widetilde{\pi}^{-1}(U)} \arrow[d, "\pi"'] & \mathbb{R}^{m|n} \times \mathcal{M}_{|U} \arrow[l, "\varphi"'] \arrow[dl, "proj_2"] \\
\mathcal{M}_{|U} &
\end{tikzcd}$$

(ii) et il existe un recouvrement de M par des trivialisations locales $\{(U_\alpha, \varphi_\alpha)\}$, comme celles décritent en (i), telles que pour tout (α, β), avec $U_{\alpha\beta} := U_\alpha \cap U_\beta$ non vide, la fonction de transition $\varphi_{\alpha\beta} := (\varphi_\beta)^{-1} \circ \varphi_\alpha : \mathbb{R}^{m|n} \times \mathcal{M}_{|U_{\alpha\beta}} \to \mathbb{R}^{m|n} \times \mathcal{M}_{|U_{\alpha\beta}}$ est déterminée par un élément $\phi_{\alpha\beta} \in GL_{(p,q)}(\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta}))$, à savoir

$$\begin{array}{rcl}
(\varphi_{\alpha\beta})^*(f) & = & f \quad \forall f \in \mathcal{O}_{\mathcal{M}}(U_{\alpha\beta}), \\
(\varphi_{\alpha\beta})^*(v_i) & = & \sum_j v_j \cdot (\phi_{\alpha\beta})_{ji} \quad \forall i,
\end{array}$$

où (v_i) sont les coordonnées canoniques sur $\mathbb{R}^{m|n}$.

Dans cette définition et souvent par la suite, nous notons f, resp. v_i, au lieu de $\mathrm{proj}_2^*(f)$, resp. $\mathrm{proj}_1^*(v_i)$.

Remarque. Par réduction des conditions (i) et (ii), $V \xrightarrow{\widetilde{\pi}} M$ est un fibré vectoriel réel de rang m dont les fonctions de transitions sont les $\widetilde{(\phi_{\alpha\beta})}_{ij}$ pour $1 \leq i, j \leq m$.

On peut aussi définir la notion de morphisme de superfibrés vectoriels. On sait (voir [21]) que la catégorie des superfibrés vectoriels sur $\mathcal{M}$ est équivalente à la catégorie des faisceaux de $\mathcal{O}_{\mathcal{M}}$-modules localement libres de rang fini. Dans ce qui suit, nous allons donner les détails de la construction du superfibré vectoriel partant d'un faisceau de $\mathcal{O}_{\mathcal{M}}$-modules localement libre de rang fini.

Soit $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ une supervariété et $\mathcal{F}$ un faisceau de $\mathcal{O}_{\mathcal{M}}$-modules localement libre de rang fini (m, n). Par définition de la liberté locale, il existe un recouvrement $(U_\alpha)_{\alpha \in A}$ de M par des ouverts tels que $\mathcal{F}_{|U_\alpha}$ est un $\mathcal{O}_{\mathcal{M}|U_\alpha}$-module libre de rang (m, n). Pour chaque $\alpha \in A$, nous avons donc $(f_\alpha^i)_{i=1,\dots,m+n}$ une base de $\mathcal{F}(U_\alpha)$ où les $f_\alpha^i \in \mathcal{F}(U_\alpha)$ sont répartis comme suit: $|f_\alpha^i| = 0$ si $i = 1, \dots, m$ et $|f_\alpha^i| = 1$ si $i = m+1, \dots, m+n$. Par la suite, nous noterons toujours $|i|$ au lieu de $|f_\alpha^i|$. Maintenant, soient $\alpha, \beta \in A$ deux indices tels que $U_{\alpha\beta}$ est non vide. On définit les éléments $(\phi_{\alpha\beta})_{ij}$ de $\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta})$ tels qu'on ait l'égalité $f_\alpha^i = \sum_j (\phi_{\alpha\beta})_{ij} \cdot f_\beta^j$, où les éléments f_α^i et f_β^j sont vus dans $\mathcal{F}(U_{\alpha\beta})$ par restriction sur $U_{\alpha\beta}$. La donnée des $((\phi_{\alpha\beta})_{ij})_{i,j=1,\dots,m+n}$ est équivalente à celle d'un élément $\phi_{\alpha\beta}$ de $GL_{(m,n)}(\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta}))$, l'ensemble des matrices paires inversibles de rang (m, n) à coefficients dans $\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta})$. Automatiquement, $\{\phi_{\alpha\beta}\}_{\alpha,\beta \in A}$ forme un cocycle, c'est à dire $\forall \alpha, \beta, \gamma$, on a $(\phi_{\alpha\gamma})_{ij} = \sum_k (\phi_{\alpha\beta})_{ik} \cdot (\phi_{\beta\gamma})_{kj}$ dans $\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta\gamma})$.
Rappelons que nous avons une injection $M \hookrightarrow \mathcal{M}$ donnée par le morphisme de faisceaux d'anneaux $\mathcal{O}_{\mathcal{M}} \to \mathcal{C}_M^\infty : f \mapsto \widetilde{f}$, appelé réduction. Celui-ci nous donne alors des éléments $\widetilde{\phi_{\alpha\beta}}$ de $\mathrm{GL}_m(\mathcal{C}_M^\infty(U_{\alpha\beta}))$, par réduction de chacun des coefficients $(\phi_{\alpha\beta})_{ij}$ pour $|i| = |j| = 0$. La famille $\{\widetilde{\phi_{\alpha\beta}}\}_{\alpha,\beta \in A}$ forme, elle aussi, un cocyle. Il existe donc, à isomorphisme près, un unique fibré vectoriel $V \xrightarrow{p} M$ de rang m et des isomorphismes $\eta_\alpha : \mathbb{R}^m \times U_\alpha \to p^{-1}(U_\alpha)$ tels que le diagramme

$$\begin{tikzcd} V_\alpha \arrow[d, "p"'] & \mathbb{R}^m \times U_\alpha \arrow[l, "\eta_\alpha"'] \arrow[dl, "\mathrm{proj}_2"] \\ U_\alpha & \end{tikzcd},$$

où V_α désigne $p^{-1}(U_\alpha)$, soit commutatif et tels que $\theta_{\alpha\beta} := \eta_\alpha^{-1} \circ \eta_\beta : \mathbb{R}^m \times U_{\alpha\beta} \to \mathbb{R}^m \times U_{\alpha\beta}$ soit donnée par $\theta_{\alpha\beta}(v, x) = (\widetilde{\phi_{\alpha\beta}}(x) \cdot v, x)$ pour tout $x \in U_{\alpha\beta}$ et $v \in \mathbb{R}^m$.

Nous allons maintenant construire un faisceau $\mathcal{O}_{\mathcal{V}}$ de base V qui se révèlera être le faisceau de structure du superfibré vectoriel recherché. Pour cela, nous allons coller les faisceaux $\mathcal{O}_{\mathbb{R}^{m|n} \times \mathcal{M}_{|U_\alpha}}$ entre eux en utilisant la donnée des $\phi_{\alpha\beta}$ grâce au théorème suivant:

Théorème 1.10. *[20] Soient X un espace topologique et $(X_\alpha)_{\alpha \in A}$, un recouvrement ouvert de X, et, pour chaque $\alpha \in A$, soit $\mathcal{G}_\alpha$ un faisceau sur X_α; pour tout couple (α, β), soit $\psi_{\alpha\beta}$ un isomorphisme de $\mathcal{G}_{\beta|X_{\alpha\beta}}$ dans $\mathcal{G}_{\alpha|X_{\alpha\beta}}$; supposons que l'on ait $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$ sur $X_{\alpha\beta\gamma}$ pour tout système (α, β, γ). Il existe alors un faisceau $\mathcal{G}$ sur X, et, pour chaque $\alpha \in A$, un isomorphisme ψ_α de $\mathcal{G}_{|X_\alpha}$ dans $\mathcal{G}_\alpha$, tels que $\psi_{\alpha\beta} = \psi_\alpha \circ \psi_\beta^{-1}$ sur $X_{\alpha\beta}$. De plus, $\mathcal{G}$ et les ψ_α sont déterminés à un isomorphisme près par la condition précédente.*

En premier lieu, nous avons le recouvrement $(V_\alpha)_{\alpha\in A}$ d'ouverts de V et les faisceaux $\mathcal{G}_\alpha := (\eta_\alpha)_*\mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}_{|U_\alpha}}$ de base V_α, où

$$\left((\eta_\alpha)_*\mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}_{|U_\alpha}}\right)(W) := \mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}_{|U_\alpha}}(\eta_\alpha^{-1}(W))$$

pour tout W ouvert de V_α. Maintenant, soient α, β deux indices de A. Nous allons construire $\psi_{\alpha\beta} : \mathcal{G}_{\beta|V_{\alpha\beta}} \to \mathcal{G}_{\alpha|V_{\alpha\beta}}$, un morphisme de faisceaux de base $V_{\alpha\beta}$. Soit W un ouvert de $V_{\alpha\beta}$, on définit l'homomorphisme $\psi_{\alpha\beta}(W) : \mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}}(W_\beta) \to \mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}}(W_\alpha)$, avec $W_\alpha = \eta_\alpha^{-1}(W)$ et $W_\beta = \eta_\beta^{-1}(W)$, comme suit:

$$\begin{array}{rcll} \psi_{\alpha\beta}(W)(\mathrm{proj}_2^*(W_\beta)(f)) & = & \mathrm{proj}_2^*(W_\alpha)(f) & \forall f \in \mathcal{O}_{\mathcal{M}}(U), \\ \psi_{\alpha\beta}(W)(\mathrm{proj}_1^*(W_\beta)(v_{i|R_\beta})) & = & \displaystyle\sum_j \mathrm{proj}_1^*(W_\alpha)(v_{j|R_\alpha}) \cdot \mathrm{proj}_2^*(W_\alpha)\left((\phi_{\alpha\beta})_{ji|U}\right) & \forall i = 1, \ldots, m+n, \end{array}$$

où les v_i sont les coordonnées canoniques de $\mathcal{O}_{\mathbb{R}^{m|n}}(\mathbb{R}^m)$, $R_\beta := \mathrm{proj}_1(W_\beta) \subset \mathbb{R}^m$, $\mathrm{proj}_1^* : (\mathrm{proj}_1)^{-1}\mathcal{O}_{\mathbb{R}^{m|n}} \to \mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}}$, $U := \mathrm{proj}_2(W) \subset U_{\alpha\beta}$, $\mathrm{proj}_2^* : (\mathrm{proj}_2)^{-1}\mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}}$ sont donnés par les deux projections $\mathrm{proj}_1 : \mathbb{R}^{m|n}\times\mathcal{M} \to \mathbb{R}^{m|n}$ et $\mathrm{proj}_2 : \mathbb{R}^{m|n}\times\mathcal{M} \to \mathcal{M}$. Ces conditions définissent complètement l'homomorphisme $\psi_{\alpha\beta}(W)$ sur $\mathcal{O}_{\mathbb{R}^{m|n}\times\mathcal{M}}(W_\beta)$. Par ailleurs, les $\psi_{\alpha\beta}(W)$ sont compatibles avec les restrictions. Nous avons donc construit un morphisme de faisceaux $\psi_{\alpha\beta} : \mathcal{G}_{\beta|V_{\alpha\beta}} \to \mathcal{G}_{\alpha|V_{\alpha\beta}}$ qui, de surcroît, se révèle être un isomorphisme puisque la matrice $\phi_{\alpha\beta}$ est inversible. Maintenant, vérifions formellement que $\psi_{\alpha\beta}\circ\psi_{\beta\gamma} = \psi_{\alpha\gamma}$:

$$\begin{array}{rcl} \psi_{\alpha\beta}\circ\psi_{\beta\gamma}(v_i) & = & \psi_{\alpha\beta}\left(\displaystyle\sum_k v_k \cdot (\phi_{\beta\gamma})_{ki}\right) \\ & = & \displaystyle\sum_k \psi_{\alpha\beta}(v_k) \cdot (\phi_{\beta\gamma})_{ki} \\ & = & \displaystyle\sum_{k,j} v_j \cdot (\phi_{\alpha\beta})_{jk} \cdot (\phi_{\beta\gamma})_{ki} \\ & = & \displaystyle\sum_j v_j \cdot (\phi_{\alpha\gamma})_{ji} \\ & = & \psi_{\alpha\gamma}(v_i) \quad \forall i. \end{array}$$

Nous pouvons maintenant appliquer le théorème 1.10 qui nous donne un faisceau de base V, que l'on note $\mathcal{O}_{\mathcal{V}}$, et des isomorphismes $\psi_\alpha : \mathcal{O}_{\mathcal{V}|V_\alpha} \to \mathcal{G}_\alpha$ tels que $\psi_{\alpha\beta} = \psi_\alpha\circ\psi_\beta^{-1}$ sur $V_{\alpha\beta}$. Il est clair que $\mathcal{V} = (V, \mathcal{O}_{\mathcal{V}})$ est une supervariété et que dim $\mathcal{V}$ = dim $\mathcal{M}$+dim $\mathbb{R}^{m|n}$.

Maintenant, nous allons construire le morphisme de supervariété $\pi = (\widetilde{\pi}, \pi^*)$ de $\mathcal{V}$ dans $\mathcal{M}$. Pour cela, nous devons donner une application continue $\widetilde{\pi} : V \to M$ et pour chaque W ouvert de V, un morphisme $\pi^*(W) : \widetilde{\pi}^{-1}\mathcal{O}_{\mathcal{M}}(W) \to \mathcal{O}_{\mathcal{V}}(W)$. Mais, il n'est pas nécessaire de définir tout les $\pi^*(W)$. Il suffit de donner tout les $\pi^*(W)$ seulement pour des ouverts W de V entièrement contenus dans un des V_α, comme l'affirme le lemme suivant:

Lemme 1.11. *Soient $\mathcal{F}$ et $\mathcal{G}$ deux faisceaux sur X, un espace topologique. Soit $\Omega = (U_i)_{i\in I}$ une famille d'ouvert de X telle que tout ouvert de X s'écrit comme une réunion d'ouvert de Ω. On suppose cette famille munie d'une collection d'homomorphismes $f_i : \mathcal{F}(U_i) \to \mathcal{G}(U_i)$ tels que, si $U_j \subset U_i$,*

$$f_i(u)_{|U_j} = f_j(u_{|U_j}) \quad \forall u \in \mathcal{F}(U_i).$$

Alors, il existe un unique homomorphisme de faisceaux $f : \mathcal{F} \to \mathcal{G}$ tel que $f(U_i) = f_i$.

Preuve. Soient U un ouvert quelconque de X et $u \in \mathcal{F}(U)$. Considérons J la partie de I tel que $\{U_i | i \in J\}$ est l'ensemble des ouverts de Ω contenus dans U. Alors, on a nécessairement $U = \bigcup_{i\in J} U_i$ par hypothèse. Posons $u_i := u_{|U_i} \in \mathcal{F}(U_i)$ et $v_i := f_i(u_i) \in \mathcal{G}(U_i)$ pour tout $i \in J$. Montrons que $v_{i|U_i\cap U_j} = v_{j|U_i\cap U_j}$ pour tout $i, j \in J$. Puisqu'il existe une partie K de I telle que $U_i \cap U_j = \bigcup_{k\in K} U_k$, il suffit de montrer que $v_{i|U_k} = v_{j|U_k}$ pour $k \in K$.

$$\begin{aligned} v_{i|U_k} &= f_i(u_i)_{|U_k} \\ &= f_k(u_{i|U_k}) \\ &= f_k(u_{|U_k}) \\ &= f_k(u_{j|U_k}) \\ &= f_j(u_j)_{|U_k} \\ &= v_{j|U_k}. \end{aligned}$$

Alors, étant donné que $\mathcal{F}$ est un faisceau, il existe un unique élément $v \in \mathcal{F}(U)$ tel que $v_{|U_i} = v_i = f_i(u_i)$ pour tout $i \in J$. Posons maintenant $f(U)(u) := v$. L'application $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ ainsi définie est bien un homomorphisme. En effet, $f(U)$ hérite de la propriété d'homomorphisme des f_i par l'axiom "d'unicité des recollements" que satisfait le faisceau $\mathcal{G}$. Maintenant, soient U et W deux ouverts de X tels que $U \subset W$. Le procédé ci-dessus nous donne deux homomorphismes $f(U)$ et $f(W)$. Vérifions la compatibilité avec la restriction de W dans U. Soient J, resp. K, la partie de I telles que $\{U_j | j \in J\}$, resp. $\{U_k | k \in K\}$, soit l'ensemble des ouverts de Ω contenus dans U, resp. W. Soit $w \in \mathcal{F}(W)$, montrons que $f(W)(w)_{|U} = f(U)(w_{|U})$. Soit $j \in J$, on a

$$f(U)(w_{|U})_{|U_j} = f_j(w_{|U_j}) = f(W)(w)_{|U_j},$$

où la dernière égalité est justifiée par le fait que $U \subset W$ implique $J \subset K$. Enfin, on a trivialement $f(W)(w)_{|U_j} = f(W)(w)_{|U|U_j}$, ce qui signifie que $f(U)(w_{|U})$ et $f(W)(w)_{|U}$ sont identiques sur U_j pour tout j. Finalement $f(U)(w_{|U}) = f(W)(w)_{|U}$ par les axioms de faisceau. Pour conclure, $f : \mathcal{F} \to \mathcal{G}$ est un morphisme de faisceaux qui satisfait évidemment $f(U_j) = f_j$. Par ailleurs, on voit que cette condition détermine f de manière unique. $\square$

Compte tenu de ce lemme, on définit $\pi = (\widetilde{\pi}, \pi^*) : \mathcal{V} \to \mathcal{M}$ tel que $\widetilde{\pi} = p : V \to M$ et tel que pour tout ouvert W de V contenu dans un V_α,

$$\pi^*(W) := (\psi_\alpha(W))^{-1} \circ \mathrm{proj}_2^*(W_\alpha) : \mathcal{O}_\mathcal{M}(U) \to \mathcal{O}_\mathcal{V}(W),$$

avec $W_\alpha := \eta_\alpha^{-1}(W)$ et $U := \mathrm{proj}_2(W) \subset U_\alpha$, satisfaisant à $U = \mathrm{proj}_2(W_\alpha)$. Le lemme précédent s'applique bel et bien car on a formellement

$$\begin{aligned}(\psi_\alpha)^{-1}(f) &= (\psi_\beta)^{-1} \circ \psi_{\beta\alpha}(f) \\ &= (\psi_\beta)^{-1}(f) \quad \forall f \in \mathcal{O}_\mathcal{M}(U).\end{aligned}$$

Enfin, on peut vérifier que $\mathcal{V} \xrightarrow{\pi} \mathcal{M}$ est un fibré vectoriel de fibre typique $\mathbb{R}^{m|n}$.

Les sections du dual du faisceau $\mathcal{F}$ peuvent être vues comme des superfonctions du superfibré vectoriel $\mathcal{V}$ "linéaires dans les directions de la fibre". Pour nous, le dual $\mathcal{F}^*$ désigne $\mathrm{Mor}_{\mathcal{O}_\mathcal{M}}(\mathcal{F}, \mathcal{O}_\mathcal{M})$. En effet, nous avons la proposition suivante:

Proposition 1.12. *Soient $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ une supervariété et $\mathcal{F}$ un faisceau de $\mathcal{O}_\mathcal{M}$-module localement libre de rang fini et soit $\mathcal{V} = (V, \mathcal{O}_\mathcal{V}) \xrightarrow{\pi} \mathcal{M}$ le superfibré vectoriel associé à $\mathcal{F}$. Alors il existe un morphisme de faisceaux naturel $\widetilde{\pi}^{-1}\mathcal{F}^* \longrightarrow \mathcal{O}_\mathcal{V}$ sur V ou de manière équivalente un morphisme de faisceaux $\mathcal{F}^* \longrightarrow \widetilde{\pi}_*\mathcal{O}_\mathcal{V}$ sur M.*

Preuve. En utilisant les notations de la construction du superfibré $\mathcal{V}$ donnée plus haut, on pose $\chi(U) := (\psi_\alpha(p^{-1}(U)))^{-1} \circ \chi_\alpha(U) : \mathcal{F}^*(U) \to \mathcal{O}_\mathcal{V}(p^{-1}(U))$ pour tout U ouvert de M contenu dans un U_α, où $\chi_\alpha(U) : \mathcal{F}^*(U) \to \mathcal{O}_{\mathbb{R}^{m|n} \times \mathcal{M}}(\mathbb{R}^m \times U)$ est donnée par

$$\chi_\alpha(U)(\omega) = \sum_i v_i \cdot \langle f_\alpha^i, \omega \rangle \quad \forall \omega \in \mathcal{F}^*(U),$$

où $\langle f_\alpha^i, \omega \rangle := (-1)^{|i||\omega|}\omega(f_\alpha^i)$. Ici, on a volontairement omis les "proj*" pour plus de légèreté. Cette expression ne dépend pas du α choisi et on définit alors le morphisme de faisceaux $\chi : \mathcal{F}^* \to (p^{-1})_* \mathcal{O}_{\mathcal{V}}$. En effet, on a formellement

$$\begin{aligned}
\chi(\omega) &= \psi_\alpha^{-1} \circ \chi_\alpha(\omega) \\
&= \psi_\beta^{-1} \circ \psi_{\beta\alpha} \circ \chi_\alpha(\omega) \\
&= \psi_\beta^{-1} \circ \psi_{\beta\alpha} \left(\sum_i v_i \cdot \langle f_\alpha^i, \omega \rangle \right) \\
&= \psi_\beta^{-1} \left(\sum_i \psi_{\beta\alpha}(v_i) \cdot \langle f_\alpha^i, \omega \rangle \right) \\
&= \psi_\beta^{-1} \left(\sum_{i,j} v_j \cdot (\phi_{\beta\alpha})_{ji} \cdot \langle f_\alpha^i, \omega \rangle \right) \\
&= \psi_\beta^{-1} \left(\sum_j v_j \cdot \langle f_\beta^j, \omega \rangle \right) \\
&= \psi_\beta^{-1} \circ \chi_\beta(\omega) \quad \forall \omega \in \mathcal{F}^*(U).
\end{aligned}$$

□

Maintenant, donnons nous $\mathcal{F}_1$, $\mathcal{F}_2$ deux faisceaux de $\mathcal{O}_\mathcal{M}$-modules localement libres de rang fini et un morphisme $\gamma : \mathcal{F}_1 \to \mathcal{F}_2$ de $\mathcal{O}_\mathcal{M}$-modules pair, i.e. $|\gamma(U)(f)| = |f|$ et $\gamma(U)(a \cdot f) = a \cdot \gamma(U)(f)$ pour tout ouvert U de M, pour tout $a \in \mathcal{O}_\mathcal{M}(U)$ et $f \in \mathcal{F}_1(U)$. Nous allons construire le morphisme de superfibrés vectoriels $\Gamma := \mathcal{V}_1 \to \mathcal{V}_2$ associé à γ, où $\mathcal{V}_a = (V_a, \mathcal{O}_{\mathcal{V}_a}) \xrightarrow{\pi_a} \mathcal{M}$ est le superfibré vectoriel associé au faisceau $\mathcal{F}_a$ pour $a = 1, 2$. Nous allons utiliser les notations de la partie où l'on construit le superfibré vectoriel $\mathcal{V}$ à partir du faisceau $\mathcal{F}$ et l'on distinguera les objets provenant de la construction de $\mathcal{O}_{\mathcal{V}_1}$ ou de $\mathcal{O}_{\mathcal{V}_2}$ en ajoutant un 1 ou un 2 aux symboles correspondant. Il existe un recouvrement $(U_\alpha)_{\alpha \in A}$ de M par des ouverts tels que $\mathcal{F}_{a|U_\alpha}$ est un $\mathcal{O}_{\mathcal{M}|U_\alpha}$-module libre de rang (m_a, n_a) pour $a = 1, 2$. Pour chaque $\alpha \in A$, nous avons donc $(f_\alpha^{ai})_{i=1,\ldots,m_a+n_a}$, une base de $\mathcal{F}_a(U_\alpha)$, où les $f_\alpha^{ai} \in \mathcal{F}_a(U_\alpha)$ sont répartis comme suit: $|f_\alpha^{ai}| = 0$ si $i = 1, \ldots, m_a$ et $|f_\alpha^{ai}| = 1$ si $i = m_a + 1, \ldots, m_a + n_a$ pour $a = 1, 2$. On définit les éléments $(\gamma_\alpha)_{ij}$ de $\mathcal{O}_\mathcal{M}(U_\alpha)$ tels qu'on ait l'égalité $\gamma(f_\alpha^{1i}) = \sum_j (\gamma_\alpha)_{ij} \cdot f_\alpha^{2j}$. Soient $\alpha, \beta \in A$ deux indices tels que $U_{\alpha\beta}$ est non vide. On définit les éléments $(\phi_{\alpha\beta}^a)_{ij}$ de $\mathcal{O}_\mathcal{M}(U_{\alpha\beta})$ tels qu'on ait l'égalité $f_\alpha^{ai} = \sum_j (\phi_{\alpha\beta}^a)_{ij} \cdot f_\beta^{aj}$. Pour tout i, j, on a la relation suivante

$$\sum_k (\phi_{\alpha\beta}^1)_{ik} \cdot (\gamma_\beta)_{kj} = \sum_k (\gamma_\alpha)_{ik} \cdot (\phi_{\alpha\beta}^2)_{kj}. \tag{1.1}$$

En effet, on a d'une part

$$\begin{aligned}
\gamma(f_\alpha^{1i}) &= \sum_k (\phi_{\alpha\beta}^1)_{ik} \cdot \gamma(f_\beta^{1k}) \\
&= \sum_{k,j} (\phi_{\alpha\beta}^1)_{ik} \cdot (\gamma_\beta)_{kj} \cdot f_\beta^{2j}
\end{aligned}$$

et d'autre part

$$\begin{aligned}\gamma(f_\alpha^{1i}) &= \sum_k (\gamma_\alpha)_{ik} \cdot f_\alpha^{2k} \\ &= \sum_{j,k} (\gamma_\alpha)_{ik} \cdot (\phi^2_{\alpha\beta})_{kj} \cdot f_\beta^{2j}.\end{aligned}$$

La théorie classique des fibrés vectoriels, nous donne, à partir de la donnée des $\widetilde{(\gamma_\alpha)_{ij}}$ pour $|i| = |j| = 0$, un morphisme de fibrés vectoriels que l'on note $\widetilde{\Gamma} : V_1 \to V_2$. Soient $\alpha \in A$ et O un ouvert de V_2 contenu dans $(\widetilde{\pi_2})^{-1}(U_\alpha)$. Posons $U := \widetilde{\pi_2}(O)$ et définissons le morphisme $\Gamma^*(O) : \mathcal{O}_{\mathcal{V}_2}(O) \to \mathcal{O}_{\mathcal{V}_1}((\widetilde{\Gamma})^{-1}(O))$ tel que le diagramme suivant

$$\begin{tikzcd} \mathcal{O}_{\mathcal{V}_2}(O) \arrow[r, "\Gamma^*(O)"] \arrow[d, "\psi^2_\alpha(O)"'] & \mathcal{O}_{\mathcal{V}_1}(\widetilde{\Gamma}^{-1}(O)) \arrow[d, "\psi^1_\alpha(\widetilde{\Gamma}^{-1}(O))"] \\ \mathcal{O}_{\mathbb{R}^{m_2|n_2}\times\mathcal{M}}((\eta^2_\alpha)^{-1}(O)) \arrow[r, "\Gamma^*_\alpha"] & \mathcal{O}_{\mathbb{R}^{m_1|n_1}\times\mathcal{M}}((\eta^1_\alpha)^{-1}(\widetilde{\Gamma}^{-1}(O))) \end{tikzcd}$$

soit commutatif, avec Γ^*_α défini par

$$\begin{aligned}\Gamma^*_\alpha(f) &= f && \forall f \in \mathcal{O}_{\mathcal{M}}(U), \\ \Gamma^*_\alpha(v^2_{i|U}) &= \sum_{j=1}^{m_1+n_1} v^1_{j|U} \cdot (\gamma_\alpha)_{ji|U} && \forall i = 1, \dots, m_2+n_2,\end{aligned}$$

où (v_i^a) sont les coordonnées canoniques de $\mathbb{R}^{m_a|n_a}$. On peut définir $\Gamma : \mathcal{O}_{\mathcal{V}_2} \to (\widetilde{\Gamma})_*\mathcal{O}_{\mathcal{V}_1}$ si et seulement si on a formellement $(\psi^1_\alpha)^{-1} \circ \Gamma^*_\alpha \circ \psi^2_\alpha = (\psi^1_\beta)^{-1} \circ \Gamma^*_\beta \circ \psi^2_\beta$ ou de manière équivalente $\Gamma^*_\alpha \circ \psi^2_{\alpha\beta} = \psi^1_{\alpha\beta} \circ \Gamma^*_\beta$ pour tout α, β dans A. C'est bien le cas, puisqu'on a

$$\begin{aligned}\Gamma^*_\alpha \circ \psi^2_{\alpha\beta}(v^2_j) &= \Gamma^*_\alpha\left(\sum_k v^2_k \cdot (\phi^2_{\alpha\beta})_{kj}\right) \\ &= \sum_{k,i} v^1_i \cdot (\gamma_\alpha)_{ik} \cdot (\phi^2_{\alpha\beta})_{kj} \\ &= \sum_{k,i} v^1_i \cdot (\phi^1_{\alpha\beta})_{ik} \cdot (\gamma_\beta)_{kj}, \quad \text{par l'équation (1.1),} \\ &= \sum_k \psi^1_{\alpha\beta}(v^1_k) \cdot (\gamma_\beta)_{kj} \\ &= \psi^1_{\alpha\beta} \circ \Gamma^*_\beta(v^2_j) \quad \forall j.\end{aligned}$$

Nous pouvons vérifier que Γ est bien un morphisme de superfibrés vectoriels:

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, "\Gamma=(\widetilde{\Gamma},\Gamma^*)"] \arrow[d, "\pi_1"'] & \mathcal{V}_2 \arrow[d, "\pi_2"] \\ \mathcal{M} \arrow[r, equal] & \mathcal{M}\ . \end{tikzcd}$$

Maintenant, étant donnée une supervariété $\mathcal{M}$, nous avons $\mathcal{T}_{\mathcal{M}} := \mathrm{Der}(\mathcal{O}_{\mathcal{M}})$ et $\Omega^1_{\mathcal{M}} := \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}})$ les faisceaux de $\mathcal{O}_{\mathcal{M}}$-modules localement libres de rang (m,n), où $\dim\mathcal{M} = (m|n)$. Naturellement, nous notons $T\mathcal{M}$ et $T^*\mathcal{M}$ les superfibrés vectoriels associés aux faisceaux $\mathcal{T}_{\mathcal{M}}$ et $\Omega^1_{\mathcal{M}}$ respectivement. Dans le cas du tangent, si les (q_i^α) sont des coordonnées de $\mathcal{M}$ sur U_α, nous prenons $f_\alpha^i = \partial_{q_i^\alpha}$ et, par conséquent, $(\phi_{\alpha\beta})_{ij} = \partial_{q_i^\alpha}(q_j^\beta)$. Dans le cas du cotangent, nous prenons $f_\alpha^i = d(q_i^\alpha)$, et, par conséquent, $(\phi_{\alpha\beta})_{ij} = (-1)^{|j|(|j|+|i|)}\partial_{q_j^\beta}(q_i^\alpha)$ (l'inverse de la supertransposé de la matrice $(\partial_{q_i^\alpha}(q_j^\beta))_{ij}$).

Maintenant, si $\mathcal{M}$ est muni d'une métrique Riemannienne paire g, alors, nous avons les morphismes de faisceaux $g^\natural : \mathcal{T}_{\mathcal{M}} \to \Omega^1_{\mathcal{M}}$ et $g^\flat : \Omega^1_{\mathcal{M}} \to \mathcal{T}_{\mathcal{M}}$. Donnons l'expression locale de $g^\flat$ et $g^\natural$ en terme de (∂_i) et $(d(q_i))$ bases locales de $\mathcal{T}_{\mathcal{M}}$ de $\Omega^1_{\mathcal{M}}$ respectivement:

$$\begin{aligned} g^\flat(\partial_i) &= \sum_j a_{ij} \cdot d(q_j) \\ g^\natural(d(q_i)) &= \sum_i b_{ij} \cdot \partial_j \end{aligned}$$

où a_{ij} et b_{ij} sont des éléments de $\mathcal{O}_{\mathcal{M}}$. On obtient les premiers coefficients en considérant, d'une part,

$$g^\flat(\partial_i)(\partial_j) = g(\partial_{q_i}, \partial_{q_j}) =: g_{ij}$$

et d'autre part

$$\begin{aligned} g^\flat(\partial_i)(\partial_j) &= \sum_k a_{ik} \cdot d(q_k)(\partial_j) \\ &= (-1)^{|j|} a_{ij}. \end{aligned}$$

Les coefficients b_{ij} sont obtenus grâce aux calculs suivant

$$g^\flat \circ g^\natural(d(q_i))(\partial_{q_j}) = d(q_i)(\partial_{q_j}) = (-1)^{|i|}\delta_{i,j},$$

et

$$g^\flat\left(\sum_k b_{ik} \cdot \partial_k\right)(\partial_i) = \sum_k b_{ik} \cdot g_{kj}.$$

Finalement,

$$g^\natural(\partial_i) = \sum_j (-1)^{|j|} g_{ij} \cdot d(q_j) \quad \text{et} \quad g^\flat(d(q_i)) = \sum_j (-1)^{|i|} g^{ij} \cdot \partial_j,$$

où les g^{ij} sont les éléments de $\mathcal{O}_{\mathcal{M}}$ tels que $\sum_k g_{ik} \cdot g^{kj} = \delta_{i,j}$.

Alors, par ce qui précède, nous avons les morphismes de superfibrés associés, que nous allons noter $G^\natural : T\mathcal{M} \to T^*\mathcal{M}$ et $G^\flat : T^*\mathcal{M} \to T\mathcal{M}$, qui sont donnés en coordonnées par

$$(G^\natural)^*(p_i) = \sum_j (-1)^{|i|} v_j \cdot g_{ji} \quad \text{et} \quad (G^\flat)^*(v_i) = \sum_j (-1)^{|j|} p_j \cdot g^{ji}. \tag{1.2}$$

Maintenant, nous allons donner les détails de la construction du superfibré vectoriel $\Pi\mathcal{V}$ associé à $\mathcal{V}$. Cette construction nous est utile pour le cas des métriques impaires. En effet, si g est un morphisme impaire, il n'existe pas de morphisme naturel de $T\mathcal{M}$ dans $T^*\mathcal{M}$ induit par g, en revanche, il en existe un de $T\mathcal{M}$ dans $\Pi T^*\mathcal{M}$.

Si $\mathcal{V}$ est le superfibré vectoriel associé au faisceau $\mathcal{F}$ de $\mathcal{O}_\mathcal{M}$-modules localement libres de rang (m, n), alors $\Pi\mathcal{V}$ est le superfibré vectoriel associé au faisceau $\Pi\mathcal{F}$ de $\mathcal{O}_\mathcal{M}$-modules localement libres de rang (n, m). En premier lieu, rappelons ce qu'est le changement de parité ΠN de N un module $\mathbb{Z}_2$-gradué.

Soient A un anneau unital $\mathbb{Z}_2$-gradué et N un $\mathbb{K}$-espace vectoriel $\mathbb{Z}_2$-gradué. Notons $|.|$ l'application qui donne la parité des éléments homogènes de A et de N. Supposons que N est un A-module à gauche. Notons "$\cdot$" la multiplication à gauche d'un élément de A avec un élément de N. Dans le contexte des espaces $\mathbb{Z}_2$-gradués, nous imposons une condition supplémentaire de compatibilité: Pour tout $a \in A$ et $n \in N$ homogènes, on a $|a \cdot n| = |a| + |n|$. Naturellement, N possède aussi une structure de A-module à droite qui est donnée par $n \cdot_d a := (-1)^{|a||n|} a \cdot n$. Nous dirons d'un $\mathbb{K}$-espace vectoriel $\mathbb{Z}_2$-gradué qu'il est un A-module si c'est un bimodule tel que les multiplications à gauche et à droite satisfont l'égalité précédente. Dans ce qui suit, nous noterons toujours "$\cdot$" au lieu de "$\cdot_d$", le produit à droite. Maintenant, si on se focalise sur le produit à gauche et l'application qui donne la graduation, on peut noter N le A-module $\mathbb{Z}_2$-gradué de la façon suivante: $(N, \cdot, |.|)$. Donnons maintenant la définition du renversement de parité d'un A-module $\mathbb{Z}_2$-gradué en utilisant la notation que nous venons d'introduire.

Définition 1.13. *Soit* $(N, \cdot, |.|)$ *un* A*-module* $\mathbb{Z}_2$*-gradué. Notons* ΠN*, le* A*-module* $\mathbb{Z}_2$*-gradué* $(N, \cdot_\Pi, |.|_\Pi)$ *avec* $a \cdot_\Pi n := (-1)^{|a|} a \cdot n$ *et* $|n|_\Pi := |n| + 1$ *pour tout* $a \in A$ *et* $n \in N$ *homogènes.*

Remarques.

(1) Nous avons

$$a \cdot_\Pi n = (-1)^{|a|} a \cdot n = (-1)^{|a|+|a||n|} n \cdot a = (-1)^{|a||n|_\Pi} n \cdot a.$$

La multiplication à droite n'est donc pas modifiée dans cette construction:

$$n \cdot_{\Pi} a = n \cdot a.$$

(2) En oubliant toutes les autres structures, N et ΠN sont les mêmes ensembles et les mêmes groupes abéliens. Il n'y a que la structure de A-module et la graduation qui diffèrent entre N et ΠN.

(3) Si N est un A-module libre de rang (m, n), il est clair que ΠN est un A-module libre de rang (n, m).

L'opération de changement de parité s'étend tout naturellement aux faisceaux de $\mathcal{O}_{\mathcal{M}}$-modules puisque les multiplications et les applications donnant la parité sont, par définition, compatibles avec les restrictions.

Revenons à la construction du superfibré $\Pi\mathcal{V}$. Si $\mathcal{F}$ est un faisceau de $\mathcal{O}_{\mathcal{M}}$-modules localement libre de rang (m, n) alors $\Pi\mathcal{F}$ est un faisceau de $\mathcal{O}_{\mathcal{M}}$-modules localement libre de rang (n, m). Par la construction d'écrite plus haut, nous obtenons un superfibré vectoriel de fibre typique $\mathbb{R}^{n|m}$ que nous notons $\Pi\mathcal{V} \xrightarrow{\pi} \mathcal{M}$. Les bases locales du faisceau $\Pi\mathcal{F}$ sont toujours notées (f^i_α) et sont donc telles que $|i|_\Pi := |f^i_\alpha|_\Pi = 1$, resp. 0, si $1 \leq i \leq m$, resp. $m + 1 \leq i \leq m + n$. Nous pouvons noter les coordonnées canoniques de la fibre $\mathbb{R}^{n|m}$ comme précédemment mais nous allons les réindexer de telle sorte que les coordonnées impaires, resp. paires, soient celles correspondant à $i = 1, \ldots, m$, resp. $i = m+1, \ldots, m+n$. Nous rajouterons le symbole Π en exposant pour signaler ce changement d'indexation. Ainsi $|v^\Pi_i| = 1$, resp. 0, si $1 \leq i \leq m$, resp. $m+1 \leq i \leq m+n$. Notons que $|v^\Pi_i| = |i| + 1$. Cette convention se révèle être très pratique pour des calculs en coordonnées.

Maintenant, on note $(\phi^\Pi_{\alpha\beta})_{ij}$ les éléments de $\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta})$ tels qu'on ait l'égalité

$$f^i_\alpha = \sum_j (\phi^\Pi_{\alpha\beta})_{ij} \cdot_\Pi f^j_\beta.$$

Puisqu'on a $f^i_\alpha = \sum (\phi_{\alpha\beta})_{ij} \cdot f^j_\beta = \sum (-1)^{|i|+|j|} (\phi_{\alpha\beta})_{ij} \cdot_\Pi f^j_\beta$, alors on a la relation suivante:

$$(\phi^\Pi_{\alpha\beta})_{ij} = (-1)^{|i|+|j|} (\phi_{\alpha\beta})_{ij}.$$

Remarques.

(i) Si i, j sont des éléments de $[\![m+1, m+n]\!]$, on a $\widetilde{(\phi^{\Pi}_{\alpha\beta})_{ij}} = \widetilde{(\phi_{\alpha\beta})_{ij}} \in \mathcal{C}^{\infty}_{M}(U_{\alpha\beta})$ puisque $|i| = |j| = 1$. Alors la base de la supervariété $\Pi\mathcal{V}$ est le fibré vectoriel réel de rang n au-dessus de M obtenu en recollant les fonctions de transition $(\widetilde{(\phi_{\alpha\beta})_{ij}})_{i,j=m+1,\dots,m+n}$ car les coordonnées des fibres de $\Pi\mathcal{V}$ qui correspondent à de tels i, j sont les coordonnées paires ($|i|_{\Pi} = |j|_{\Pi} = 0$).
Dans le cas $\mathcal{F} = \mathcal{T}_{\mathcal{M}}$ (et donc $\mathcal{V} = T\mathcal{M}$) rappelons que $(\phi_{\alpha\beta})_{ij} = \partial_{q_i^{\alpha}}(q_j^{\beta})$. Alors la base de la supervariété $\Pi T\mathcal{M}$ (c'est à dire la variété lisse sous-jacente ou "body" en anglais) est isomorphe au fibré vectoriel réel $\widetilde{(T^{\mathrm{num}}\mathcal{M})_1} \to M$ de rang n puisque les fonctions de transitions de ce fibré sont les $\widetilde{\partial_{q_i^{\alpha}}(q_j^{\beta})}$ pour $i, j = m+1, \dots, m+n$.

(ii) Si $E \to M$ est un fibré vectoriel réel que l'on voit comme un superfibré vectoriel purement pair au-dessus de la supervariété M purement paire, alors ΠE est isomorphe à $(M, \Gamma_{\Lambda E^*})$.

La proposition 1.12 implique la proposition suivante:

Proposition 1.14. *Il existe un morphisme (pair) de faisceaux* $Mor_{\mathcal{O}_{\mathcal{M}}}(\Pi\mathcal{F}, \mathcal{O}_{\mathcal{M}}) \xrightarrow{\chi^{\Pi}} \widetilde{\pi}_*\mathcal{O}_{\Pi\mathcal{V}}$.

Comme d'habitude, $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\Pi\mathcal{F}, \mathcal{O}_{\mathcal{M}})$ est naturellement un $\mathcal{O}_{\mathcal{M}}$-module $\mathbb{Z}_2$-gradué. Ces structures sont induites par celles de $\Pi\mathcal{F}$ et celles de $\mathcal{O}_{\mathcal{M}}$. Les ensembles $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\Pi\mathcal{F}, \mathcal{O}_{\mathcal{M}})$ et $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{F}, \mathcal{O}_{\mathcal{M}})$ sont les mêmes et les multiplications à gauche avec un élément de $\mathcal{O}_{\mathcal{M}}$ le sont aussi. En revanche, la graduation naturelle de $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\Pi\mathcal{F}, \mathcal{O}_{\mathcal{M}})$, que l'on note $|.|_{\Pi}$, n'est pas la même que celle de $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{F}, \mathcal{O}_{\mathcal{M}})$, noté $|.|$. En effet, si $\omega \in \mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{F}, \mathcal{O}_{\mathcal{M}})$ et $f \in \mathcal{F}$, on a

$$|\omega(f)| = |\omega| + |f| = |\omega| + 1 + |f|_{\Pi}.$$

Donc $|\omega|_{\Pi} = |\omega| + 1$.

Le morphisme χ^{Π} est localement donné par

$$\chi^{\Pi}_{\alpha}(U)(\omega) = \sum_i v_i^{\Pi} \cdot \langle f^i_{\alpha}, \omega\rangle_{\Pi} \quad \forall \omega \in \mathcal{F}^*(U),$$

où $\langle f^i_{\alpha}, \omega\rangle_{\Pi} := (-1)^{|i|_{\Pi}|\omega|_{\Pi}}\omega(f^i_{\alpha})$.

Une fois de plus, donnons nous $\mathcal{F}_1$ et $\mathcal{F}_2$ deux faisceaux de $\mathcal{O}_{\mathcal{M}}$-modules localement libres de rang fini. Soit $\gamma : \mathcal{F}_1 \to \mathcal{F}_2$ un morphisme de $\mathcal{O}_{\mathcal{M}}$-modules impair, i.e. $|\gamma(U)(f)| =$

$1+|f|$ et $\gamma(U)(a\cdot f)=(-1)^{|a|}a\cdot\gamma(U)(f)$ pour tout ouvert U de M, pour tout $a\in\mathcal{O}_{\mathcal{M}}(U)$ et $f\in\mathcal{F}_1(U)$. Alors, il n'est pas possible de construire un morphisme de superfibrés vectoriels associé à γ tel quel. Pour utiliser la construction précédente, il nous faut modifier la parité de γ pour obtenir un morphisme pair. Pour cela, il nous suffit de considérer γ comme un élément de $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{F}_1,\Pi\mathcal{F}_2)$ ou de $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\Pi\mathcal{F}_1,\mathcal{F}_2)$. Dans ces deux cas, γ est alors un morphisme pair. En effet,

$$\gamma(a\cdot f)=(-1)^{|a|}a\cdot\gamma(f)=a\cdot_\Pi\gamma(f) \text{ et } |\gamma(f)|_\Pi=1+|\gamma(f)|=|f|,$$

dans le premier cas et dans le deuxième

$$\gamma(a\cdot_\Pi f)=(-1)^{|a|}\gamma(a\cdot f)=a\cdot\gamma(f) \text{ et } |\gamma(f)|=1+|f|=|f|_\Pi.$$

En utilisant les notations introduites plus haut, nous définissions alors les éléments $(\gamma_\alpha^\Pi)_{ij}$ et $({}^\Pi\gamma_\alpha)_{ij}$ de $\mathcal{O}_{\mathcal{M}}(U_{\alpha\beta})$ comme suit

$$\gamma(f_\alpha^{1i})=\sum_j(\gamma_\alpha^\Pi)_{ij}\cdot_\Pi f_\alpha^{2j} \text{ avec } f_\alpha^{1j} \text{ vu dans } \mathcal{F}_1 \text{ et } f_\alpha^{2j} \text{ dans } \Pi\mathcal{F}_2,$$

et

$$\gamma(f_\alpha^{1i})=\sum_j({}^\Pi\gamma_\alpha)_{ij}\cdot f_\alpha^{2j} \text{ avec } f_\alpha^{1j} \text{ vu dans } \Pi\mathcal{F}_1 \text{ et } f_\alpha^{2j} \text{ dans } \mathcal{F}_2.$$

Nous avons alors les relations suivantes: $(\gamma_\alpha^\Pi)_{ij}=(-1)^{|i|+|j|+1}(\gamma_\alpha)_{ij}$ et $({}^\Pi\gamma_\alpha)_{ij}=(\gamma_\alpha)_{ij}$. Alors, nous avons les morphismes de superfibrés vectoriels $\Gamma^\Pi:\mathcal{V}_1\to\Pi\mathcal{V}_2$ et ${}^\Pi\Gamma:\Pi\mathcal{V}_1\to\mathcal{V}_2$ associés à γ vu dans $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\mathcal{F}_1,\Pi\mathcal{F}_2)$ et $\mathrm{Mor}_{\mathcal{O}_{\mathcal{M}}}(\Pi\mathcal{F}_1,\mathcal{F}_2)$ respectivement, qui localement sont donnés par

$$(\Gamma^\Pi)^*(v_i^{\Pi 2})=\sum_j v_j^1\cdot(\gamma_\alpha^\Pi)_{ji} \quad\text{et}\quad ({}^\Pi\Gamma)^*(v_i^2)=\sum_j v_j^{1\Pi}\cdot({}^\Pi\gamma_\alpha)_{ji}.$$

Maintenant, si $\mathcal{M}$ est munie d'une métrique Riemannienne impaire g, alors, nous avons les morphismes de faisceaux $g^\natural:\mathcal{T}_{\mathcal{M}}\to\Omega^1_{\mathcal{M}}$ et $g^\flat:\Omega^1_{\mathcal{M}}\to\mathcal{T}_{\mathcal{M}}$. L'expression locale de ces morphismes est alors donnée par:

$$g^\natural(\partial_i)=\sum_j(-1)^{|j|}g_{ij}\cdot d(q_j) \quad\text{et}\quad g^\flat(d(q_i))=\sum_j(-1)^{|j|+1}g^{ij}\cdot\partial_j,$$

car, dans ce cas, $|g_{ij}|=|i|+|j|+1$.

Alors, par ce qui précède, nous avons les morphismes de superfibrés associés, que nous allons simplement noter $G^\natural:T\mathcal{M}\to\Pi T^*\mathcal{M}$ et $G^\flat:\Pi T^*\mathcal{M}\to T\mathcal{M}$, qui sont donnés en coordonnées par

$$(G^\natural)^*(p_i^\Pi)=\sum_j(-1)^{|j|+1}v_j\cdot g_{ji} \quad\text{et}\quad (G^\flat)^*(v_i)=\sum_j(-1)^{|i|+1}p_j^\Pi\cdot g^{ji}.$$

Pour finir ce chapitre, nous allons construire l'application tangente associée à un morphisme de supervariétés $\Phi : \mathcal{N} \to \mathcal{M}$.

Proposition 1.15. *Tout morphisme de supervariétés $\Phi : \mathcal{N} \to \mathcal{M}$ induit un morphisme de superfibrés vectoriels $T\Phi : T\mathcal{N} \to T\mathcal{M}$ appelé **l'application tangente** de Φ et tel que le diagramme suivant commute:*

$$\begin{tikzcd} T\mathcal{N} \arrow[r, "T\Phi"] \arrow[d, "\pi^{T\mathcal{N}}"'] & T\mathcal{M} \arrow[d, "\pi^{T\mathcal{M}}"] \\ \mathcal{N} \arrow[r, "\Phi"] & \mathcal{M}. \end{tikzcd}$$

Preuve. Soient (q_i^α) et (q_j^μ) des coordonnées sur $\mathcal{M}$ et $\mathcal{N}$ respectivement. Celles-ci induisent des coordonnées naturelles $(q_i^\alpha, v_i^{\mathcal{M}})$ et $(q_j^\mu, v_j^{\mathcal{N}})$ qui trivialisent $T\mathcal{M}$ et $T\mathcal{N}$. Dans ces coordonnées, nous définissons l'application tangente de Φ comme suit:

$$\begin{array}{rcl} (T\Phi)^*(q_j^\alpha) & = & \Phi^*(q_j^\alpha) \quad \forall j, \\ (T\Phi)^*(v_j^{\mathcal{M}}) & = & \displaystyle\sum_i v_i^{\mathcal{N}} \cdot \partial_{q_i^\mu} \Phi^*(q_j^\alpha) \quad \forall j. \end{array}$$

Cette écriture locale définit bien un morphisme de $T\mathcal{N}$ dans $T\mathcal{M}$ car si $(q_i^\beta, v_i^{\mathcal{M}})$ et $(q_j^\delta, v_j^{\mathcal{N}})$ sont d'autres coordonnées naturelles (qui trivialisent $T\mathcal{M}$ et $T\mathcal{N}$), en utilisant les notations adoptées plus haut pour la construction du superfibré associé à un faisceau de module localement libre, on a,

$$\begin{array}{rcl} (T\Phi)^* \circ \psi_{\alpha\beta}^{\mathcal{M}}(v_j^{\mathcal{M}}) & = & (T\Phi)^* \left(\displaystyle\sum_k v_k^{\mathcal{M}} \cdot (\phi_{\alpha\beta}^{\mathcal{M}})_{kj} \right) \\ & = & (T\Phi)^* \left(\displaystyle\sum_k v_k^{\mathcal{M}} \cdot \partial_{q_k^\alpha}(q_j^\beta) \right) \\ & = & \displaystyle\sum_{k,i} v_i^{\mathcal{N}} \cdot \partial_{q_i^\mu} \Phi^*(q_k^\alpha) \cdot (\Phi^* \circ \partial_{q_k^\alpha})(q_j^\beta) \\ & = & \displaystyle\sum_i v_i^{\mathcal{N}} \cdot \partial_{q_i^\mu} \Phi^*(q_j^\beta) \\ & = & \displaystyle\sum_{i,k} v_i^{\mathcal{N}} \cdot \partial_{q_i^\mu}(q_k^\delta) \cdot \partial_{q_k^\delta} \Phi^*(q_j^\beta) \\ & = & \displaystyle\sum_{i,k} v_i^{\mathcal{N}} \cdot (\phi_{\mu\delta}^{\mathcal{N}})_{ik} \cdot \partial_{q_k^\delta} \Phi^*(q_j^\beta) \\ & = & \psi_{\mu\delta}^{\mathcal{N}} \left(\displaystyle\sum_k v_k^{\mathcal{N}} \cdot \partial_{q_k^\delta} \Phi^*(q_j^\beta) \right) \\ & = & \psi_{\mu\delta}^{\mathcal{N}} \circ (T\Phi)^*(v_j^{\mathcal{M}}) \quad \forall j. \end{array}$$

La commutativité du diagramme est bien sûr assurée par l'expression locale de $T\Phi$ sur les coordonnées q_j^α. $\square$

Remarques. Cette construction est une généralisation de l'application tangente entre deux variétés classiques puisqu'on a $\widetilde{T\Phi} = T\widetilde{\Phi}$.

Chapter 2

Riemannian Supergeodesics

2.1 Riemannian metrics and Levi-Civita connections on a supermanifold

Our goal is to define the notion of supergeodesics, as "autoparallel curves" with respect to a connection. Let thus $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a supermanifold and $\mathcal{E}$ a locally free sheaf of $\mathcal{O}_\mathcal{M}$-modules of finite rank on M.

Definition 2.1. *A* ***connection*** *on $\mathcal{E}$ is an even morphism $\nabla : \mathcal{E} \to \Omega^1_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{E}$ of sheaves of $\mathbb{R}$-supervector spaces that satisfies the Leibniz rule*

$$\nabla(f \cdot v) = d(f) \otimes v + f \cdot \nabla(v)$$

for all sections $f \in \mathcal{O}_\mathcal{M}(U)$ and $v \in \mathcal{E}(U)$ where $d(f)(X) := (-1)^{|X||f|}X(f)$ for all $X \in \mathcal{T}_\mathcal{M}(U)$, and on all open subsets U of M.

Given a vector field X on $\mathcal{M}$, we have a map $\nabla_X : \mathcal{E} \to \mathcal{E}$ defined via the canonical pairing $\langle ., . \rangle$ between the tangent and the cotangent sheaf, $\nabla_X(v) = \langle X, \nabla(v) \rangle$. Then ∇_X satisfies:

$$(\nabla_X)(f \cdot v) = X(f) \cdot v + (-1)^{|X||f|} f \cdot (\nabla_X)(v) \quad \text{and} \quad |(\nabla_X)(v)| = |X| + |v|.$$

In the case $\mathcal{E} = \mathcal{T}_M$, we speak of a **connection on $\mathcal{M}$**.

Definition 2.2. *We define the* ***torsion*** *of a connection ∇ on $\mathcal{M}$ by*

$$T_\nabla(X, Y) := \nabla_X Y - (-1)^{|X||Y|}\nabla_Y X - [X, Y].$$

Definition/Proposition 2.3. *If $\mathcal{M}$ is equipped with a graded Riemannian metric g of degree $|g|$, a connection ∇ is called* ***metric*** *if the connection respects the metric in the* *following sense*

$$\nabla g = 0, \text{ i.e. } (-1)^{|X||g|}X(g(Y\otimes Z)) = g(\nabla_X Y\otimes Z)+(-1)^{|X||Y|}g(Y\otimes\nabla_X Z) \quad \forall X, Y, Z \in \mathcal{T}_{\mathcal{M}}(U).$$

Proof. First of all, remark that the metric g can be viewed as an element of $\Omega^1_{\mathcal{M}}\otimes\Omega^1_{\mathcal{M}}$. As in the case of ungraded sheaves and classical manifolds, a connection ∇ on $\mathcal{E}$ induces a connection ∇^* on $\mathcal{E}^*$ as follows

$$\nabla^*_X(\omega)(s)+(-1)^{|\omega||X|}\omega(\nabla_X(s)) = X(\omega(s)) \text{ for all } X \in \mathcal{T}_{\mathcal{M}}(U), s \in \mathcal{E}(U) \text{ and } \omega \in \mathcal{E}^*(U),$$

where U is an open set of M. Moreover, a connection ∇ on $\mathcal{E}$ induces a connection $\nabla^{\otimes}$ on $\mathcal{E}\otimes\mathcal{E}$ as follows

$$\nabla^{\otimes}_X(s_1\otimes s_2) = \nabla_X(s_1)\otimes s_2+(-1)^{|X||s_1|}s_1\otimes\nabla_X(s_2) \text{ for all } X \in \mathcal{T}_{\mathcal{M}}(U) \text{ and } s_1, s_2 \in \mathcal{E}(U).$$

Combining these facts, we show that $\nabla(g) = 0$ can be rewritten as the equation given above. $\square$

We also have a natural graded version of **Christoffel symbols**: If (q_i) is a system of coordinates on $U \subset M$, the expansion

$$\nabla_{\partial_{q_i}}\partial_{q_j} = \sum_k \Gamma^k_{ij}\partial_{q_k}$$

gives elements $\Gamma^k_{ij} \in \mathcal{O}_M(U)$ of parity $|\Gamma^k_{ij}| = |q_i| + |q_j| + |q_k|$.

Theorem 2.4. *On a supermanifold $\mathcal{M}$ with a graded Riemannian metric g of degree $|g|$, there exists a unique torsion free and metric connection ∇, which will be called the* ***Levi-Civita connection*** *of the metric.*

The proof follows immediately upon interpreting the usual six-term-formula in a graded sense and sheaf-theoretically:

$$\begin{aligned} 2g(\nabla_X Y\otimes Z) = (-1)^{|X||g|}Xg(Y\otimes Z) - (-1)^{|Z||g|+|Z|(|X|+|Y|)}Zg(X\otimes Y) \\ +(-1)^{|Y||g|+|X|(|Y|+|Z|)}Yg(Z\otimes X) + g([X,Y]\otimes Z) \\ -(-1)^{|X|(|Y|+|Z|)}g([Y,Z]\otimes X)+(-1)^{|Z|(|X|+|Y|)}g([Z,X]\otimes Y) \end{aligned}$$

for all X, Y and Z homogeneous vector fields.

Remarks.

(1) A connection on $\mathcal{M}$ induces a connection on the vector bundle $TM \to M$ by reduction. Indeed, there exists a unique connection $\widetilde{\nabla}$ on M such that

$$\widetilde{\nabla}_{\beta(X)}(\beta(Y)) = \beta(\nabla_X Y) \quad \forall X, Y \in \mathcal{T}_M(U) \text{ and for all } U \text{ open of } M.$$

In fact, $\widetilde{\nabla}$ is the pullback connection of ∇ via β. We will discuss in 2.9 pullback connections sheaf-theoretically.
In the case of even metric the Levi-Civita connection $\widetilde{\nabla}$ induced by ∇ is the usual Levi-Civita connection of the induced metric $\widetilde{g}$ on M. Moreover, in coordinates (q_i), the reduction $\beta(\Gamma_{ij}^k)$ of a Christoffel symbol Γ_{ij}^k is the Christoffel symbol of the Levi-Civita connection on M with respect to the corresponding coordinates on M.

(2) In addition, ∇ induces a connection ∇^E on the vector bundle $E := (T^{\mathrm{num}}\mathcal{M})_1$ over M. If the Riemannian metric g is even, ∇^E is then a "symplectic connection" on E, i.e. a connection such that the 2-form ω^E (defined in the remark (2) given just after the Définition 1.7) is covariant constant.

(3) We then observe that an even Riemannian metric g on $\mathcal{M}$ induces a triple $(\widetilde{g}, \omega^E, \nabla^E)$. But going back via the construction explained after Théorème 1.8 does, in general, not give back the original metric g. In fact, there are many Riemannian metrics g that induce the same triple $(\widetilde{g}, \omega^E, \nabla^E)$.

In the classical case, we have an explicit formula for Christoffel symbols. Adding appropriate signs this formula holds true in the super context:

Proposition 2.5. *In coordinates (q_i) on U, the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection on a Riemannian supermanifold are given by*

$$(-1)^{(|i|+|j|+|k|)|g|}\Gamma_{ij}^k = \frac{1}{2}\sum_l \left[(-1)^{|i||g|}\partial_i g_{jl} + (-1)^{|j||g|+|i||j|}\partial_j g_{il} - (-1)^{|l||g|+|l|(|i|+|j|)}\partial_l g_{ij}\right] g^{lk}, \tag{2.1}$$

where $g_{ij} := g(\partial_{q_i}, \partial_{q_j}) \in \mathcal{O}_{\mathcal{M}}(U)$ and $g^{lk} \in \mathcal{O}_{\mathcal{M}}(U)$ are defined by $\sum_k g^{ik} g_{kj} = \delta_{i,j}$.

We note here and in the sequel the parity $|q_i|$ and the derivation ∂_{q_i} often by $|i|$ and ∂_i respectively.

Proof. We are going to do a direct computation, in the case of $|g| = 0$, using the six-term-formula that defines the Levi-Civita connection. The odd case can be treated in a similar fashion.
Let (q_i) be coordinates on U. Using the six-term-formula with $X = \partial_{q_i}$ and $Y = \partial_{q_j}$ we

get:

$$\begin{aligned}
2g(\nabla_{\partial_i}\partial_j, Z) &= \partial_i g(\partial_j, Z) && (2.2)\\
&\quad -(-1)^{|Z|(|i|+|j|)} Z(g_{ij}) && (2.3)\\
&\quad +(-1)^{|i|(|j|+|Z|)} \partial_j g(Z, \partial_i) && (2.4)\\
&\quad +g([\partial_i, \partial_j], Z) && (2.5)\\
&\quad -(-1)^{|i|(|j|+|Z|)} g([\partial_j, Z], \partial_i) && (2.6)\\
&\quad +(-1)^{|Z|(|i|+|j|)} g([Z, \partial_i], \partial_j). && (2.7)
\end{aligned}$$

The vector field Z can be written in the form

$$Z = \sum_l \partial_l . Z_l \text{ with } Z_l \in \mathcal{O}_{\mathcal{M}}(U).$$

Let us calculate term by term:

(2.3) : $-(-1)^{|Z|(|i|+|j|)} Z(g_{ij}) = -\sum_l (-1)^{|l|(|i|+|j|)} \partial_l(g_{ij}) Z_l$ since $|Z_l| = |l| + |Z|$ and $|g_{ij}| = |i| + |j|$.

(2.5) : $g([\partial_i, \partial_j], Z) = 0$ since $[\partial_i, \partial_j] = 0$.

(2.6) : At first, we need to develop $[\partial_j, Z]$.

$$\begin{aligned}
[\partial_j, Z] &= \sum_l [\partial_j, \partial_l . Z_l]\\
&= \sum_l \Big[\partial_j((-1)^{|l||Z_l|} Z_l \partial_l) - (-1)^{|j|(|l|+|Z_l|)+|l||Z_l|} Z_l . \partial_l \partial_j\Big]\\
&= \sum_l (-1)^{|l||Z_l|} \partial_j(Z_l)\partial_l + \sum_l (-1)^{|l||Z_l|+|Z_l||j|} Z_l \underbrace{\Big[\partial_j \partial_l - (-1)^{|j||l|} \partial_l \partial_j\Big]}_{[\partial_j, \partial_l]=0}
\end{aligned}$$

Therefore the fifth term is $-(-1)^{|i|(|j|+|Z|)} g([\partial_j, Z], \partial_i) = -(-1)^{|i|(|j|+|Z|)} \sum_l (-1)^{|l||Z_l|} \partial_j(Z_l) . g_{li}$.

(2.7) : Using the previous calculations, we obtain

$$\begin{aligned}
(-1)^{|Z|(|i|+|j|)} g([Z, \partial_i], \partial_j) &= -(-1)^{|Z|(|i|+|j|)+|i||Z|} g([\partial_i, Z], \partial_j)\\
&= -(-1)^{|Z||j|} \sum_l (-1)^{|l||Z_l|} \partial_i(Z_l) . g_{lj}
\end{aligned}$$

We now fix k corresponding to the coordinate (q_k), and set $Z_l = g^{lk}$ where $\sum_l g^{il} g_{lj} = \delta_{i,j}$.

Remark that $|Z_l| = |l| + |k|$ and $|Z| = |k|$ since $|g^{lk}| = |l| + |k|$.

We interrupt the proof in order to give a useful algebraic fact about the inverse of supersymmetric matrices.

Lemma 2.6. *Let $B = (b_{ij})$ be an homogeneous invertible matrix of order (m, n) over a superalgebra $\mathcal{A}$. Assume that B is supersymmetric, i.e. $b_{ij} = (-1)^{|i||j|} b_{ji}$, then we have*

$$b^{ij} = (-1)^{|B|+|i||j|+|i|+|j|} b^{ji},$$

where (b^{ij}) are the coefficients of the matrix B^{-1}.

Proof. By the definition of the matrix B^{-1}, we have $\sum_l b^{il} b_{lj} = \delta_{i,j}$, where $|b^{il}| = |i| + |l| + |B|$ since the degrees of B and B^{-1} are necessarily equal. This implies that

$$\begin{aligned}
\sum_l (-1)^{(|i|+|l|+|B|)(|j|+|l|+|B|)} b_{lj} b^{il} &= \delta_{i,j} \\
\sum_l (-1)^{(|i|+|l|+|B|)(|j|+|l|+|B|)+|j||l|} b_{jl} b^{il} &= \delta_{i,j} \\
\sum_l (-1)^{|B|+|l||i|+|l|} b_{jl} b^{il} &= (-1)^{|i||j|+|B|(|i|+|j|)} \delta_{i,j} = (-1)^{|i|} \delta_{i,j} \\
\sum_l b_{jl} ((-1)^{|B|+|l||i|+|l|+|i|} b^{il}) &= \delta_{i,j} \quad \forall i, j.
\end{aligned}$$

The left and the right inverse matrix of B are equal by the very definition of the inverse matrix. Then we have the intended formula. □

Proof of proposition 2.5(resumed):
It follows that $g^{ij} = (-1)^{|i||j|+|i|+|j|} g^{ji}$ since g is non-degenerated, symmetric and even.

We next calculate each summand of the six-term-formula in the case that $Z_l = g^{lk}$.

First, we have

$$\begin{aligned}
2g(\nabla_{\partial_i} \partial_j, Z) &= 2 \sum_{u,l} g(\Gamma^u_{ij} \partial_u, \partial_l g^{lk}) \\
&= 2\Gamma^k_{ij}
\end{aligned}$$

and the right-hand side is given by:

$$(2.2) : \partial_i g(\partial_j, Z) = \partial_i(\sum_l g(\partial_j, \partial_l.g^{lk})) = \partial_i(\delta_{j,k}) = 0$$

$$(2.3) : -\sum_l (-1)^{|l|(|i|+|j|)}\partial_l(g_{ij})Z_l = -\sum_l (-1)^{|l|(|i|+|j|)}\partial_l(g_{ij})g^{lk}$$

$$\begin{aligned}(2.4) : (-1)^{|i|(|j|+|Z|)}\partial_j g(Z, \partial_i) &= (-1)^{|i|(|j|+|k|)}\partial_j\left(\sum_l (-1)^{|l|(|k|+|l|)}g(g^{lk}\partial_l, \partial_i)\right) \\ &= (-1)^{|i|(|j|+|k|)}\partial_j\left(\sum_l (-1)^{|k|}g^{kl}g_{li}\right) \\ &= (-1)^{|i|(|j|+|k|)}\partial_j\left((-1)^{|k|}\delta_{l,i}\right) = 0.\end{aligned}$$

$$\begin{aligned}(2.6) : -(-1)^{|i|(|j|+|Z|)}\sum_l (-1)^{|l||Z_l|}\partial_j(Z_l).g_{li} &= -(-1)^{|i|(|j|+|k|)}\sum_l (-1)^{|l|(|l|+|k|)}\partial_j(g^{lk}).g_{li} \\ &= -(-1)^{|i|(|j|+|k|)}\sum_l (-1)^{|k|}\partial_j(g^{kl}).g_{li}\end{aligned}$$

To continue we need to remark that the following equality

$$\partial_j(g^{kl}g_{li}) = \partial_j(g^{kl})g_{li} + (-1)^{|j|(|k|+|l|)}g^{kl}\partial_j(g_{li})$$

summed over l gives

$$\partial_j(\delta_{k,i}) = 0 = \sum_l \partial_j(g^{kl})g_{li} + \sum_l (-1)^{|j|(|k|+|l|)}g^{kl}\partial_j(g_{li}).$$

Thus we have

$$\begin{aligned}-(-1)^{|i|(|j|+|k|)}\sum_l (-1)^{|k|}\partial_j(g^{kl}).g_{li} &= (-1)^{|i|(|j|+|k|)}\sum_l (-1)^{|k|+|j|(|k|+|l|)}g^{kl}\partial_j(g_{li}) \\ &= (-1)^{|i|(|j|+|k|)}\sum_l (-1)^{|k|+(|l|+|i|)(|k|+|l|)}\partial_j(g_{li})g^{kl} \\ &= (-1)^{|i|(|j|+|k|)}\sum_l (-1)^{|i||k|}\partial_j(g_{il})g^{lk} \\ &= \sum_l (-1)^{|i||j|}\partial_j(g_{il})g^{lk}\end{aligned}$$

$$\begin{aligned}
(2.7):\ -(-1)^{|Z||j|}\sum_l(-1)^{|l||Z_l|}\partial_i(Z_l).g_{lj} &= -(-1)^{|k||j|}\sum_l(-1)^{|l|(|k|+|l|)}\partial_i(g^{lk}).g_{lj}\\
&= -(-1)^{|k||j|}\sum_l(-1)^{|k|}\partial_i(g^{kl}).g_{lj}\\
&= (-1)^{|k||j|}\sum_l(-1)^{(|i|(|k|+|l|)+|k|}g^{kl}.\partial_i(g_{lj})\\
&= (-1)^{|k||j|}\sum_l(-1)^{(|l|+|j|)(|k|+|l|)+|k|}\partial_i(g_{lj})g^{kl}\\
&= \sum_l \partial_i(g_{jl})g^{lk}
\end{aligned}$$

Consequently, $\Gamma^k_{ij}=\dfrac{1}{2}\big((2.2)+\ldots+(2.7)\big)$ gives

$$\Gamma^k_{ij}=\frac{1}{2}\sum_l\left[\partial_i g_{jl}+(-1)^{|i||j|}\partial_j g_{il}-(-1)^{|l|(|i|+|j|)}\partial_l g_{ij}\right]g^{lk}. \qquad \square$$

A connection on $\mathcal{M}$ allows to define a covariant derivative of vector fields along a curve. We develop these notions here in some detail for supermanifolds.

Definition 2.7. *Let $\Phi=(\widetilde{\Phi},\Phi^*):\mathcal{N}\to\mathcal{M}$ be a morphism of supermanifolds and U an open set of M. A **vector field along** $\Phi:\mathcal{N}\to\mathcal{M}$ on U is a morphism of supervector spaces*

$$X:\mathcal{O}_{\mathcal{M}}(U)\to(\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}})(U)$$

such that its homogeneous components X_0, X_1 satisfy the following derivation property

$$X_\alpha(fg)=X_\alpha(f)\Phi^*(g)+(-1)^{\alpha\cdot|f|}\Phi^*(f)X_\alpha(g)\quad \textit{for } \alpha=0,1 \textit{ and } f,g\in\mathcal{O}_{\mathcal{M}}(U).$$

The set $\mathrm{Der}_{\mathbb{R}}\left(\mathcal{O}_{\mathcal{M}}(U),(\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}})(U)\right)$ of all vector fields along Φ will also be denoted by $\mathcal{T}^{\Phi}_M(U)$. Remark that $\mathcal{T}^{\Phi}_M$ is naturally a sheaf of $\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}$-modules over M.

Recall that a morphism of supermanifolds can be viewed as a morphism $\Phi^*:\mathcal{O}_{\mathcal{M}}\to\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}$ of sheaves over M or as a morphism $\Phi^*:\widetilde{\Phi}^{-1}\mathcal{O}_{\mathcal{M}}\to\mathcal{O}_{\mathcal{N}}$ of sheaves over N (see [6] for more details). Then vector fields along Φ can also be viewed as a sheaf of $\mathcal{O}_{\mathcal{N}}$-modules over N: $\mathcal{T}^{\Phi}_N:=\mathrm{Der}_{\mathbb{R}}\left(\widetilde{\Phi}^{-1}\mathcal{O}_{\mathcal{M}},\mathcal{O}_{\mathcal{N}}\right)$.

There are two standard ways of constructing vector fields along Φ. If X is a vector field on $\mathcal{M}$, then

$$\widehat{X}:=\Phi^*\circ X$$

is a vector field along Φ, and if Y is a vector field on $\mathcal{N}$, we get the following vector field along Φ:

$$(d\Phi^*)(Y) := Y \circ \Phi^*.$$

We thus obtain two morphisms of sheaves over N, namely $\widetilde{\Phi}^{-1}\mathcal{T}_{\mathcal{M}} \xrightarrow{\Phi^*} \mathcal{T}_N^{\Phi}$ and $\mathcal{T}_{\mathcal{N}} \xrightarrow{d\Phi^*} \mathcal{T}_N^{\Phi}$, and their analogues over M. Since knowledge of the sheaves of vector fields along Φ over M and over N will be useful in the sequel, we give a more explicit description.

Proposition 2.8.

(i) $\mathcal{T}_M^{\Phi}$ *is a locally free sheaf of* $\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}$*-modules over* M *of the same rank as* $\mathcal{T}_{\mathcal{M}}$*. More precisely, we have an isomorphism*

$$\mathcal{T}_M^{\Phi} \xrightarrow{\cong} \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}}.$$

(ii) $\mathcal{T}_N^{\Phi}$ *is a locally free sheaf of* $\mathcal{O}_{\mathcal{N}}$*-modules over* N *of the same rank as* $\mathcal{T}_{\mathcal{M}}$*. More precisely, we have an isomorphism*

$$\mathcal{T}_N^{\Phi} \xrightarrow{\cong} \mathcal{O}_{\mathcal{N}} \otimes_{\widetilde{\Phi}^{-1}\mathcal{O}_{\mathcal{M}}} \widetilde{\Phi}^{-1}\mathcal{T}_{\mathcal{M}}.$$

Proof. We will prove only *(i)*. Part *(ii)* then follows from the fact that $\widetilde{\Phi}_*$ and $\widetilde{\Phi}^{-1}$ are adjoint functors.
We will construct two isomorphisms Q and S as below:

$$\mathrm{Der}_{\mathbb{R}}\left(\mathcal{O}_{\mathcal{M}}, \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}\right) \xrightarrow[\cong]{Q} \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}}\left(\Omega^1_{\mathcal{M}}, \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}\right) \xrightarrow[\cong]{S} \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}}.$$

Step 1:
Let D be a vector field along Φ on U. Observe that for $f \in \mathcal{O}_{\mathcal{M}}(U)$, $d(f) = 0$ implies $D(f) = 0$. We then define the $\mathcal{O}_{\mathcal{M}}(U)$-module morphism $Q(U)(D)$ as follows:

$$Q(U)(D)(d(f)) := D(f), \quad \forall f \in \mathcal{O}_{\mathcal{M}}(U).$$

This defines $Q(U)(D)$ on $\Omega^1_{\mathcal{M}}(U)$ since every one-form ω can be written locally in the form $\omega = \sum_i \omega_i \cdot d(q_i)$ where (q_i) are coordinates on $\mathcal{M}$. In this case, we have $Q(U)(D)(\omega) = \sum_i (-1)^{|D||\omega_i|}\Phi^*(\omega_i) \cdot D(q_i)$. It follows that $Q(U)$ is a morphism of $(\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}})(U)$-modules and Q is a $\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}$-module morphism.

Step 2:
Here, we define S on a chart domain U_α. Let $\varphi \in \mathrm{Hom}_{\mathcal{O}_{\mathcal{M}}(U_\alpha)}\left(\Omega^1_{\mathcal{M}}(U_\alpha), \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}(U_\alpha)\right)$, we define

$$S(U_\alpha)(\varphi) := \sum_i \varphi(d(q_i^\alpha)) \otimes_{\mathcal{O}_{\mathcal{M}}(U_\alpha)} \partial_{q_i^\alpha}, \tag{2.8}$$

where (q_i^α) are coordinates on $\mathcal{M}$.

Let us check that this is independent of the (q_i^α) coordinates. Let (q_i^β) be coordinates on U_β. We know that $d(q_j^\beta)$ can be written as

$$d(q_j^\beta) = \sum_i d(q_i^\alpha) \cdot c_i^j \text{ on } U_{\alpha\beta} = U_\alpha \cap U_\beta,$$

where (c_i^j) are elements of $\mathcal{O}_\mathcal{M}(U_{\alpha\beta})$ of parity $|i| + |j|$.
Evaluating on $\partial_{q_i^\alpha}$ this equality, the left side gives

$$d(q_j^\beta)(\partial_{q_i^\alpha}) \quad = \quad (-1)^{|i||j|}\partial_{q_i^\alpha}(q_j^\beta),$$

and the right side

$$\begin{aligned} \sum_k [d(q_k^\alpha) \cdot c_k^j](\partial_{q_i^\alpha}) &= \sum_k (-1)^{(|k|+|j|)|i|+|k||i|}\partial_{q_i^\alpha}(q_k^\alpha) \cdot c_k^j \\ &= (-1)^{|i||j|}a_i^j. \end{aligned}$$

Then

$$d(q_j^\beta) = \sum_i d(q_i^\alpha) \cdot \partial_{q_i^\alpha}(q_j^\beta).$$

Now, we can check that the equation 2.8 does not depend on the α's coordinates:

$$\begin{aligned} \sum_j \varphi(d(q_j^\beta)) \otimes \partial_{q_j^\beta} &= \sum_j \varphi(\sum_i d(q_i^\alpha) \cdot \partial_{q_i^\alpha}(q_j^\beta)) \otimes \partial_{q_j^\beta} \\ &= \sum_{i,j} \varphi(d(q_i^\alpha)) \cdot \Phi^*(\partial_{q_i^\alpha}(q_j^\beta)) \otimes \partial_{q_j^\beta} \\ &= \sum_i \left[\varphi(d(q_i^\alpha)) \otimes \sum_j \partial_{q_i^\alpha}(q_j^\beta) \cdot \partial_{q_j^\beta} \right] \end{aligned}$$

Then we conclude our calculation using the chain rule:

$$\partial_{q_i^\alpha} = \sum_j \partial_{q_i^\alpha}(q_j^\beta)\partial_{q_j^\beta} \text{ on } U_{\alpha\beta}.$$

A direct computation in local coordinates shows that both maps Q and S are isomorphisms of $\widetilde{\Phi}_*\mathcal{O}_\mathcal{N}$-modules. $\square$

Remark. The proof of Proposition 2.8 shows that the isomorphism $L := S \circ Q$ looks as follows in local coordinates on U, for a $D \in \mathcal{T}_M^\Phi(U)$:

$$\begin{aligned} L(D) &= \sum_i Q(D)(d(q_i)) \otimes \partial_{q_i} \\ &= \sum_i D(q_i) \otimes \partial_{q_i}. \end{aligned}$$

Furthermore, if (q_i) are coordinates for $\mathcal{M}$ on U, then $(\widehat{\partial_{q_i}})$ is a basis of $\mathcal{T}_M^\Phi(U)$ and we have

$$T\Phi^*(Y) = Y \circ \Phi^* = \sum_i Y(\Phi^*(q_i))\widehat{\partial_{q_i}}$$

for all vector fields Y on $\mathcal{N}$.

Given a connection on $\mathcal{M}$ and a morphism $\Phi : \mathcal{N} \to \mathcal{M}$, we introduce the associated pull-back connection in order to get a covariant derivative along a curve. One has, in fact, the statement below:

Proposition/Definition 2.9. *There exists a unique connection* $\widehat{\nabla}$ *on* $\mathcal{T}_M^\Phi$ *such that the diagram*

$$\begin{tikzcd} \mathcal{T}_\mathcal{M} \arrow[r, "\nabla"] \arrow[d, "\Phi^*"'] & \Omega^1_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{T}_\mathcal{M} \arrow[d, "\Phi^*"] \\ \mathcal{T}_M^\Phi \arrow[r, "\widehat{\nabla}"] & \widetilde{\Phi}_*\Omega^1_\mathcal{N} \otimes_{\widetilde{\Phi}_*\mathcal{O}_\mathcal{N}} \mathcal{T}_M^\Phi \end{tikzcd}$$

commutes. This connection is usually called the ***pullback of*** ∇ ***via*** Φ *and is denoted by* $\Phi^*\nabla$.

Remark. The right vertical arrow is defined as follows. First of all, there is a natural map from $\Omega^1_\mathcal{M}$ to $\widetilde{\Phi}_*\Omega^1_\mathcal{N}$ induced by Φ^* which is also denoted by Φ^*. Given U an open set in M and $\omega \in \Omega^1_\mathcal{M}(U)$, we get an element υ of $\widetilde{\Phi}_*\Omega^1_\mathcal{N}(U)$:

$$\upsilon(X) := \widehat{\omega}(d\Phi^*(X)) \text{ for } X \in (\widetilde{\Phi}_*\mathcal{T}_\mathcal{N})(U),$$

where $\widehat{\omega}$ is defined by

$$\begin{aligned} \widehat{\omega}: \quad \widetilde{\Phi}_*\mathcal{O}_\mathcal{N}(U) \otimes_{\mathcal{O}_\mathcal{M}(U)} \mathcal{T}_\mathcal{M}(U) &\longrightarrow \widetilde{\Phi}_*\mathcal{O}_\mathcal{N}(U) \\ f \otimes Y &\longmapsto (-1)^{|f||\omega|} f \cdot \Phi^*(\omega(Y)) \end{aligned}$$

using the isomorphism $\mathcal{T}_M^\Phi \cong \widetilde{\Phi}_*\mathcal{O}_\mathcal{N} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{T}_\mathcal{M}$. The map $\Phi^* : \Omega^1_\mathcal{M} \longrightarrow \widetilde{\Phi}_*\Omega^1_\mathcal{N}$ turns out to be a morphism of $\mathcal{O}_\mathcal{M}$-modules, that can be tensored with $\Phi^* : \mathcal{T}_\mathcal{M} \to \mathcal{T}_M^\Phi$ to obtain the arrow

$$\Phi^* : \Omega^1_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{T}_\mathcal{M} \longrightarrow \widetilde{\Phi}_*\Omega^1_\mathcal{N} \otimes_{\widetilde{\Phi}_*\mathcal{O}_\mathcal{N}} \mathcal{T}_M^\Phi.$$

Proof of Proposition 2.9. The proof follows immediately upon interpreting the usual proof in the classical case of ungraded manifolds sheaf-theoretically. Let us nevertheless give some details:
Remark that we have the isomorphisms

$$\widetilde{\Phi}_*\Omega^1_{\mathcal{N}} \otimes_{\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}} \mathcal{T}^{\Phi}_{M} \cong \widetilde{\Phi}_*\Omega^1_{\mathcal{N}} \otimes_{\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}} \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \cong \widetilde{\Phi}_*\Omega^1_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}}.$$

The sum of the following maps defines the required connection $\widetilde{\nabla}$:

$$\begin{aligned} d \otimes \mathrm{id} \quad &: \quad \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \longrightarrow \widetilde{\Phi}_*\Omega^1_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \\ \Phi^*(\mathrm{id} \otimes \nabla) \quad &: \quad \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \longrightarrow \widetilde{\Phi}_*\Omega^1_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \end{aligned}$$

where the second map is defined as follows:

$$\begin{tikzcd}
\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \arrow[r, "\mathrm{id}\otimes\nabla"'] \arrow[ddr, "\Phi^*(\mathrm{id}\otimes\nabla)"'] & \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \Omega^1_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \arrow[d, "\mathrm{id}\otimes\Phi^*\otimes\mathrm{id}"] \\
& \widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}} \otimes_{\widetilde{\Phi}_*\mathcal{O}_{\mathcal{N}}} \widetilde{\Phi}_*\Omega^1_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \arrow[d, "\cong"] \\
& \widetilde{\Phi}_*\Omega^1_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}}
\end{tikzcd}$$

The uniqueness of the pullback connection is due to the condition

$$\widehat{\nabla}(\widehat{\partial_{q_i}}) = \widehat{\nabla}(\Phi^* \circ \partial_{q_i}) = \Phi^*\big(\nabla(\partial_{q_i})\big),$$

where (q_i) are local coordinates on $\mathcal{M}$. □

Remark. The diagram in Proposition/Definition 2.9 can be equivalently replaced by the following diagram:

$$\begin{tikzcd}
\widetilde{\Phi}^{-1}\mathcal{T}_{\mathcal{M}} \arrow[r, "\nabla"] \arrow[d, "\Phi^*"'] & \widetilde{\Phi}^{-1}\Omega^1_{\mathcal{M}} \otimes_{\widetilde{\Phi}^{-1}\mathcal{O}_{\mathcal{M}}} \widetilde{\Phi}^{-1}\mathcal{T}_{\mathcal{M}} \arrow[d, "\Phi^*"] \\
\mathcal{T}^{\Phi}_{N} \arrow[r, "\widehat{\nabla}"] & \Omega^1_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{N}}} \mathcal{T}^{\Phi}_{N}
\end{tikzcd}$$

We can now define the covariant derivative along a supercurve $\Phi : \mathbb{R}^{1|1} \to \mathcal{M}$. Let $(\mathcal{M}, g)$ be a Riemannian supermanifold and ∇ its Levi-Civita connection. Consider $\Phi^*\nabla$, the pullback connection of ∇ under Φ.

Definition 2.10. *The* ***even covariant derivative along*** $\Phi : \mathbb{R}^{1|1} \to \mathcal{M}$ *is defined as*

$$\frac{\nabla}{dt} := (\Phi^*\nabla)_{\partial_t} : \mathcal{T}_M^\Phi \longrightarrow \mathcal{T}_M^\Phi,$$

induced by the duality pairing of ∂_t *with elements of* $\Omega^1_{\mathbb{R}^{1|1}}$ *where* (t, θ) *are the canonical coordinates on* $\mathbb{R}^{1|1} \cong \mathbb{R} \times \mathbb{R}^{0|1}$.

Note that if (q_i) are local coordinates on $\mathcal{U} = (U, \mathcal{O}_{\mathcal{M}|U}) \subset \mathcal{M}$, we have for all vector fields $X \in \mathcal{T}_M^\Phi(U)$

$$\begin{aligned}
\frac{\nabla}{dt}(X) &= (\Phi^*\nabla)_{\partial_t}(X) \\
&= \langle \partial_t, \Phi^*\nabla(X)\rangle \\
&= \langle \partial_t, \Phi^*\nabla(\sum_i X(q_i) \otimes \partial_i)\rangle \\
&= \langle \partial_t, (d \otimes \mathrm{id} + \Phi^*(\mathrm{id} \otimes \nabla))(\sum_i X(q_i) \otimes \partial_i)\rangle \\
&= \sum_i \langle \partial_t, dX(q_i) \otimes \partial_i\rangle + \sum_i \langle \partial_t, \Phi^*(X(q_i) \otimes \nabla(\partial_i))\rangle \\
&= \sum_i \partial_t X(q_i) \otimes \partial_i + \sum_i X(q_i)\langle \partial_t, \Phi^*(\nabla(\partial_i))\rangle
\end{aligned}$$

We must express $\nabla(\partial_i)$ in terms of Christoffel symbols in order to calculate the term $\Phi^*(\nabla(\partial_i))$. By definition we have

$$\nabla(\partial_i) = \sum_{j,k} d(q_j) \otimes \Gamma^k_{ji}\partial_k.$$

Remark that the operator d and the morphism Φ^* in the term $\Phi^*(d(q_j))$ commutes. Indeed, we have

$$\begin{aligned}
\Phi^*(d(q_j))(Y) &= \widehat{d(q_j)}(d\Phi^*(Y)) \\
&= \widehat{d(q_j)}\left(\sum_k Y(\Phi^*(q_k)) \otimes \partial_{q_k}\right) \\
&= \sum_k (-1)^{|j|(|k|+|Y|)} Y(\Phi^*(q_k)) \cdot d(q_j)(\partial_{q_k}) \\
&= (-1)^{|j||Y|} Y(\Phi^*(q_j)) \\
&= d(\Phi^*(q_j))(Y),
\end{aligned}$$

for all $Y \in \widetilde{\Phi}_*\Omega^1_{\mathbb{R}^{1|1}}(U)$. Then we have

$$\begin{aligned}
\Phi^*(\nabla(\partial_i)) &= \sum_{j,k} \Phi^*\left(d(q_j) \otimes \Gamma^k_{ji}\partial_k\right) \\
&= \sum_{j,k} d(\Phi^*(q_j)) \otimes \Phi^*(\Gamma^k_{ji}) \otimes \partial_k
\end{aligned}$$

Returning now to our local calculation of $\frac{\nabla}{dt}(X)$, we get

$$\begin{aligned}\frac{\nabla}{dt}(X) &= \sum_i \partial_t X(q_i) \otimes \partial_i + \sum_i X(q_i)\langle \partial_t, \Phi^*(\nabla(\partial_i))\rangle \\ &= \sum_k \partial_t X(q_k) \otimes \partial_k + \sum_i X(q_i) \sum_{j,k} \langle \partial_t, d(\Phi^*(q_j)) \otimes \Phi^*(\Gamma^k_{ji}) \otimes \partial_k\rangle \\ &= \sum_k \left[\partial_t X(q_k) + \sum_{i,j} X(q_i) \cdot \partial_t \Phi^*(q_j) \cdot \Phi^*(\Gamma^k_{ji})\right] \otimes \partial_k\end{aligned}$$

Analogously, we can define the covariant derivative along a curve with respect to the odd coordinate θ on $\mathbb{R}^{1|1}$:

$$\frac{\nabla}{d\theta} := (\Phi^*\nabla)_{\partial_\theta} : \mathcal{T}^\Phi_M \longrightarrow \mathcal{T}^\Phi_M.$$

In local coordinates, we have a similar formula with an additional sign:

$$\frac{\nabla}{d\theta}(X) = \sum_k \left[\partial_\theta X(q_k) + \sum_{i,j} (-1)^{|X|+|q_i|} X(q_i) \partial_\theta \Phi^*(q_j) \Phi^*(\Gamma^k_{ji})\right] \otimes \partial_k$$

2.2 Riemannian supergeodesics on a supermanifold

We can now give a natural definition of supergeodesics, already given by DeWitt (compare [8]) and Monterde - Muñoz Masqué (compare [16]).

Definition 2.11. *A* ***supergeodesic*** *on* $(\mathcal{M}, \nabla)$*, a supermanifold with a connection* ∇*, is a supercurve* $\Phi : I^{1|1} \cong I \times \mathbb{R}^{0|1} \to \mathcal{M}$ *such that*

$$\frac{\nabla}{dt}(d\Phi^*(\partial_t)) = 0,$$

where I *is an open interval in* $\mathbb{R}$*.*

The preceding condition is locally equivalent to

$$\partial_t^2 \Phi^*(q_k) + \sum_{i,j} \partial_t \Phi^*(q_i) \partial_t \Phi^*(q_j) \Phi^*(\Gamma^k_{ji}) = 0 \quad \forall k, \tag{2.9}$$

where (q_k) are local coordinates on $\mathcal{M}$. Note that the equation (2.9) restricted to those k corresponding to even coordinates gives the usual geodesic equation of the underlying classical manifold equipped with the reduced connection $\widetilde{\nabla}$ on M.

Remarks. (1) Of course, in general a supergeodesic is only defined on an open subsupermanifold of $\mathbb{R}^{1|1}$ with body an open interval in $\mathbb{R}$. Slightly abusing, we will often write $\mathbb{R}^{1|1}$ meaning such an open subsupermanifold of it.

(2) One can easily generalize Definitions 2.10 and 2.11 to the case $\mathbb{R} \times \mathcal{S}$, $\mathcal{S}$ an arbitrary supermanifold replacing $\mathbb{R}^{0|1}$.

Let $(\mathcal{M}, g)$ be an even Riemannian supermanifold. A different definition was given by Goertsches ([12]): a **supergeodesic in the sense of Goertsches** is a supercurve $\Phi : \mathbb{R}^{1|1} \to \mathcal{M}$ such that

$$\left(\frac{\nabla}{dt}(d\Phi^*(\partial_t))\right)^{\mathrm{num}} = \left(\frac{\nabla}{dt}(d\Phi^*(\partial_\theta))\right)^{\mathrm{num}} = 0. \tag{2.10}$$

Remark that such a curve automatically satisfies the following equations:

$$\left(\frac{\nabla}{d\theta}(d\Phi^*(\partial_t))\right)^{\mathrm{num}} = \left(\frac{\nabla}{d\theta}(d\Phi^*(\partial_\theta))\right)^{\mathrm{num}} = 0.$$

Locally, (2.10) is equivalent to

$$f_k'' + \sum_{\substack{i \text{ even,} \\ j \text{ even}}} f_i' \cdot f_j' \cdot \Phi^*(\Gamma_{ji}^k) = 0 \quad \forall k \text{ even,}$$

and

$$f_\delta' + \sum_{\substack{i \text{ even,} \\ \beta \text{ odd}}} f_\beta \cdot f_i' \cdot \Phi^*(\Gamma_{i\beta}^\delta) = 0 \quad \forall \delta \text{ odd,}$$

where $f_i, f_\delta \in \mathcal{C}^\infty_{\mathbb{R}}(\widetilde{\Phi}^{-1}(U))$ are functions defined as follows:

$$\Phi^*(q_i) = f_i \quad \forall i \text{ even and } \Phi^*(q_\beta) = f_\beta \cdot \theta \quad \forall \beta \text{ odd.}$$

We call here and in the sequel an index $(i, \delta, k, \beta, \ldots)$ "even" resp. "odd" if the corresponding coordinate has the property.

In particular, if $\mathcal{M} = \mathbb{R}^{1|2}$ is equipped with a flat even metric (i.e., $\Gamma_{ij}^k = 0$), (2.10) is equivalent to

$$f_k'' = 0 \quad \forall k \text{ even, and } f_\delta' = 0 \quad \forall \delta \text{ odd.}$$

On the other hand, the geodesic equation (2.9) is equivalent to

$$f_k'' = 0 \quad \forall k \text{ even, and } f_\delta'' = 0 \quad \forall \delta \text{ odd.}$$

The equations (2.9) and (2.10) are thus obviously non-equivalent, moreover initial conditions for these two notions of geodesics are not the same.

Let us explain in some detail what an initial condition is in the super context.

Reminder 2.12. *([21]) Let $\mathcal{M}$ be a supermanifold. The category of locally free sheaves of $\mathcal{O}_{\mathcal{M}}$-modules of finite rank is equivalent to the category of supervector bundles over $\mathcal{M}$.*

Now let $T\mathcal{M} = (TM, \mathcal{O}_{T\mathcal{M}})$ and $T^*\mathcal{M} = (T^*M, \mathcal{O}_{T^*\mathcal{M}})$ be the supervector bundles associated to the locally free sheaves $\mathcal{T}_{\mathcal{M}}$ and $\Omega^1_{\mathcal{M}}$ respectively. If $\mathcal{M}$ is of dimension $n|m$, then the total spaces of these bundles both have dimension $2n|2m$, which fits with the naïve idea of position and velocity resp. position and momentum variables. Of course, this point of view is also crucial for the existence of a canonical even symplectic form on $T^*\mathcal{M}$ (see below in section 3; compare also [13]). Denoting the projection from $T\mathcal{M}$ to $\mathcal{M}$ by π, one observes that $\Gamma(\mathcal{M}, T\mathcal{M}) := \{\Phi \in \mathrm{Mor}(\mathcal{M}, T\mathcal{M}) \,|\, \pi \circ \Phi = id_{\mathcal{M}}\}$ is the space of even vector fields or derivations on $\mathcal{M}$. Instead of going to a "bigger tangent bundle" of dimension $2m+n|2n+m$ (compare, e.g., [18]), we prefer to stick to the above tangent bundle and note that $\Gamma(\mathcal{M}, T\mathcal{M}) \oplus \Gamma(\mathcal{M}, \Pi T\mathcal{M})$, with $\Pi T\mathcal{M}$ the vector bundle associated to the module sheaf $\Pi\mathcal{T}_{\mathcal{M}}$ obtained by applying parity change to $\mathcal{T}_{\mathcal{M}}$, is the space of all vector fields (or derivations) on $\mathcal{M}$ (compare, e.g., the sections 7 and 8 of [21]). In more modern language, we can formulate this via relative morphisms (see [1] for a recent and detailed account of this point of view). Let $p : \mathcal{V} \to \mathcal{M}$ be a fibre bundle in the category **Sman** of supermanifolds, then we have a functor to the category **Set** of sets:

$$\mathbf{Sman} \to \mathbf{Set},\, \mathcal{S} \mapsto \mathrm{Mor}_{\mathcal{M}}(\mathcal{M} \times \mathcal{S}, \mathcal{V}) := \{\Phi \in \mathrm{Mor}(\mathcal{M} \times \mathcal{S}, \mathcal{V}) \,|\, p \circ \Phi = \mathrm{proj}_1\},$$

and vector fields on $\mathcal{M}$ are given by taking $\mathcal{V} = T\mathcal{M}$ and $\mathcal{S} = \mathbb{R}^{0|1}$. This formalizes, in fact, the classical point of view of physicists saying that given an odd vector field X, multiplication by an "odd constant" (realized by the canonical odd coordinate τ on $\mathbb{R}^{0|1}$) yields an even "vector field" $\tau{\cdot}X$.

Reminder 2.13. *Each morphism of supermanifolds $\Phi : \mathcal{N} \to \mathcal{M}$ induces a morphism of supervector bundles $T\Phi : T\mathcal{N} \to T\mathcal{M}$ called* ***the tangent map*** *of Φ.*

Let $(q_i^{\mathcal{M}})$ and $(q_j^{\mathcal{N}})$ be coordinates on $\mathcal{M}$ and $\mathcal{N}$. They induce natural coordinates $(q_i^{\mathcal{M}}, v_i^{\mathcal{M}})$ and $(q_j^{\mathcal{N}}, v_j^{\mathcal{N}})$ which trivialize $T\mathcal{M}$ and $T\mathcal{N}$, respectively. In these coordinates, the tangent map of Φ is given as follows

$$\begin{array}{rcl} (T\Phi)^*(q_j^{\mathcal{M}}) &=& \Phi^*(q_j^{\mathcal{M}}) \quad \forall j, \\ (T\Phi)^*(v_j^{\mathcal{M}}) &=& \sum_i v_i^{\mathcal{N}} \cdot \partial_{q_i^{\mathcal{N}}} \Phi^*(q_j^{\mathcal{M}}) \quad \forall j. \end{array}$$

Let $\Phi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ be a morphism of supermanifolds. Using the tangent map $T\Phi : T\mathbb{R} \times T\mathcal{S} \to T\mathcal{M}$, the section $s_1 : t \mapsto (1,t)$ of $T\mathbb{R}$ and the canonical zero section $s_0 : \mathcal{S} \to T\mathcal{S}$ of $T\mathcal{S}$, we define $\partial_t\Phi : \mathbb{R} \times \mathcal{M} \to T\mathcal{M}$, the **partial derivative of Φ in**

the direction of $\mathbb{R}$, as follows:

$$\partial_t \Phi := T\Phi \circ (s_1 \times s_0) : \mathbb{R} \times \mathcal{S} \to T\mathcal{M}.$$

In local coordinates, this amounts to

$$\begin{array}{rcl} (\partial_t \Phi)^*(q_i^{\mathcal{M}}) & = & \Phi^*(q_i^{\mathcal{M}}) \\ (\partial_t \Phi)^*(v_i^{\mathcal{M}}) & = & \partial_t \Phi^*(q_i^{\mathcal{M}}). \end{array}$$

We define for $\mathcal{X} = (X, \mathcal{O}_\mathcal{X})$, $\mathcal{Y} = (Y, \mathcal{O}_\mathcal{Y})$ supermanifolds and $x_0 \in X$, $\mathrm{inj}_{x=x_0} : \mathcal{Y} \to \mathcal{X} \times \mathcal{Y}$ by $\widetilde{\mathrm{inj}_{x=x_0}}(y) = (x_0, y)$ and $(\mathrm{inj}_{x=x_0})^*(f \otimes g) = \widetilde{f}(x_0) \cdot g$ for $f \in \mathcal{O}_\mathcal{X}$ and $g \in \mathcal{O}_\mathcal{Y}$. The pullback operation $(\mathrm{inj}_{x=x_0})^*$ is often denoted by $\mathrm{ev}_{x=x_0}$.

Theorem 2.14. *Let $\mathcal{J} \in Mor(\mathbb{R}^{0|1}, T\mathcal{M})$. Then there exists a unique geodesic $\Phi : \mathbb{R}^{1|1} \cong \mathbb{R} \times \mathbb{R}^{0|1} \to \mathcal{M}$ such that $\partial_t \Phi \circ inj_{t=0} = \mathcal{J}$.*

Proof. The "initial condition" $\mathcal{J}$ renders the solution of the system of ODEs (2.9) unique, since $\partial_t \Phi \circ \mathrm{inj}_{t=0} = \mathcal{J}$ reads in the local coordinates discussed before the theorem as follows:

$$\mathrm{inj}_{t=0}^* \circ \Phi^*(q_i) = \Phi^*(q_i)_{|t=0} = \mathcal{J}^*(q_i) \text{ and } \partial_t \Phi^*(q_i)_{|t=0} = \mathcal{J}^*(v_i). \qquad \square$$

Remark. (1) Of course, the morphism Φ is, in general, only defined on an open connected subsupermanifold of $\mathbb{R}^{1|1}$, whose body contains the point 0.
(2) The preceding theorem can easily be generalized to cover the situation discussed in the remark after Definition 2.11: given a supermanifold $\mathcal{S}$ and a morphism $\mathcal{J} \in \mathrm{Mor}(\mathcal{S}, T\mathcal{M})$, there exists a unique geodesic $\Phi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ such that $\partial_t \Phi \circ \mathrm{inj}_{t=0} = \mathcal{J}$.

2.3 Geodesic flow

In this section we get a natural "energy function" H on the (co-)tangent bundle of a supermanifold and an associated Hamiltonian vector field X_H. The flow and integral curves of the latter even vector field exist by results of Monterde and Montesinos resp. Dumitrescu. We then prove that integral curves of X_H on $T^*\mathcal{M}$ or $\Pi T^*\mathcal{M}$, depending on whether the metric g is even or odd, are in natural bijection with geodesics on $\mathcal{M}$.

2.3.1 Even Riemannian metrics

Reminder 2.15. *Let $\mathcal{M} = (M, \mathcal{O}_\mathcal{M})$ be a supermanifold and $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_\mathcal{M}$-modules and let $\mathcal{V} = (V, \mathcal{O}_\mathcal{V}) \xrightarrow{\pi} \mathcal{M}$ be the supervector bundle associated to $\mathcal{F}$. Then there exists a natural sheaf morphism $\widetilde{\pi}^{-1}\mathcal{F}^* \longrightarrow \mathcal{O}_\mathcal{V}$ over V or equivalently a sheaf morphism $\mathcal{F}^* \longrightarrow \widetilde{\pi}_*\mathcal{O}_\mathcal{V}$ over M.*

Remark. Sections of $\widetilde{\pi}^{-1}\mathcal{F}^*$ over on open subset in V are of course local superfunctions of $\mathcal{O}_{\mathcal{V}}(V)$ that are "linear in the fiber directions".

Corollary 2.16. *Let $\pi : \mathcal{V} \to \mathcal{M}$ be the canonical projection. Then we have a natural morphism of sheaves*

$$\begin{tikzcd}
\widetilde{\pi}^{-1}\left(Sym\left(\mathcal{F}^*\right)\right) \arrow[rr, "\chi"] \arrow[dr] & & \mathcal{O}_{\mathcal{V}}, \arrow[dl] \\
& V &
\end{tikzcd}$$

where $Sym\left(\mathcal{F}^*\right)$ is the supersymmetric tensor algebra of the $\mathcal{O}_{\mathcal{M}}$-module $\mathcal{F}^*$.

Proof. By the preceding proposition we have a morphism of $\widetilde{\pi}^{-1}\mathcal{O}_{\mathcal{M}}$-module sheaves $\widetilde{\pi}^{-1}\mathcal{F}^* \longrightarrow \mathcal{O}_{\mathcal{V}}$. Since $\mathcal{O}_{\mathcal{V}}$ is a sheaf of unital supercommutative superalgebras, we have a canonical extension to $\widetilde{\pi}^{-1}\left(\mathrm{Sym}(\mathcal{F}^*)\right) \cong \mathrm{Sym}(\widetilde{\pi}^{-1}\mathcal{F}^*) \longrightarrow \mathcal{O}_{\mathcal{V}}$, denoted by χ. ☐

In the case that $\mathcal{F} = \Omega^1_{\mathcal{M}}$, we have notably the following morphism of $\mathcal{O}_{\mathcal{M}}(M)$-modules:

$$\left(\mathrm{Sym}^2\left((\Omega^1_{\mathcal{M}})^*\right)\right)(M) \xrightarrow{\chi(T^*M)} \mathcal{O}_{T^*\mathcal{M}}(T^*M).$$

If $\mathcal{M}$ is equipped with an even Riemannian metric $g : \mathcal{T}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$, we have the isomorphisms $g^\flat : \mathcal{T}_{\mathcal{M}} \xrightarrow{\cong} \Omega^1_{\mathcal{M}}$ and $g^\natural : \Omega^1_{\mathcal{M}} \xrightarrow{\cong} \mathcal{T}_{\mathcal{M}}$ induced by the metric, where

$$g^\flat(X)(Y) := g(X,Y) \quad \forall X, Y \in \mathcal{T}_{\mathcal{M}}(U).$$

Using the above identification and the metric we have

$$\Omega^1_{\mathcal{M}}(M) \otimes \Omega^1_{\mathcal{M}}(M) \underset{\cong}{\xrightarrow{g^\natural \otimes g^\natural}} \mathcal{T}_{\mathcal{M}}(M) \otimes \mathcal{T}_{\mathcal{M}}(M) \xrightarrow{\quad g \quad} \mathcal{O}_{\mathcal{M}}(M),$$

which determines an element h of $\left(\Omega^1_{\mathcal{M}}(M) \otimes \Omega^1_{\mathcal{M}}(M)\right)^*$. The standard theory of linear superalgebra gives the following canonical isomorphism

$$\left(\Omega^1_{\mathcal{M}}(M) \otimes \Omega^1_{\mathcal{M}}(M)\right)^* \cong \left(\Omega^1_{\mathcal{M}}(M)\right)^* \otimes \left(\Omega^1_{\mathcal{M}}(M)\right)^*.$$

Let us now define $H := \frac{1}{2}\chi(T^*M)(h) \in \mathcal{O}_{T^*\mathcal{M}}(T^*M)$, where h is viewed as an element of $\left(\Omega^1_{\mathcal{M}}(M)\right)^* \otimes \left(\Omega^1_{\mathcal{M}}(M)\right)^*$.

Lemma 2.17. *We obtain locally*

$$H = \frac{1}{2}\sum_{i,j}(-1)^{|i|+|j|}p_i \cdot g^{ij} \cdot p_j,$$

where (q_i, p_i) are coordinates on $T^*\mathcal{M}$ induced by coordinates (q_i) on $\mathcal{M}$.

Proof. Recall that we have (see page 15)

$$g^\flat(\partial_i) = \sum_j (-1)^{|j|} g_{ij} \cdot d(q_j) \text{ and } g^\natural(d(q_i)) = \sum_j (-1)^{|i|} g^{ij} \cdot \partial_j.$$

Let us denote the one-form $d(q_i)$ by d_i in the rest of this proof. Then $\{d^*_{ij} | 1 \leq i, j \leq m+n\}$ is a local basis of $(\Omega^1_{\mathcal{M}} \otimes \Omega^1_{\mathcal{M}})^*$ where $d^*_{ij} := (d_i \otimes d_j)^*$ is defined as follows

$$(d_i \otimes d_j)^*(d_k \otimes d_l) = (-1)^{|i||j|} \delta_{i,k} \cdot \delta_{j,l}.$$

Using this basis, we have

$$\begin{aligned}
h &= g \circ (g^\natural \otimes g^\natural) \\
&= \sum_{i,j} g \circ (g^\natural \otimes g^\natural)(d_i \otimes d_j)(-1)^{|i||j|} d^*_{ij} \\
&= \sum_{i,j} (-1)^{|i||j|} g(g^\natural(d_i) \otimes g^\natural(d_j)) d^*_{ij} \\
&= \sum_{\substack{i,j \\ k,l}} (-1)^{|i||j|} g((-1)^{|i|} g^{ik} \partial_k, (-1)^{|j|} g^{jl} \partial_l) d^*_{ij} \\
&= \sum_{\substack{i,j \\ k,l}} (-1)^{|i||j|+|i|+|j|+|l|(|j|+|l|)} g^{ik} g_{kl} g^{jl} d^*_{ij} \\
&= \sum_{i,j} (-1)^{|i||j|+|i|+|j|+|i|(|j|+|i|)} g^{ji} d^*_{ij} \\
&= \sum_{i,j} (-1)^{|j|} g^{ji} d^*_{ij}
\end{aligned}$$

Now we have the following natural morphism

$$\left(\Omega^1_{\mathcal{M}}(M)\right)^* \otimes \left(\Omega^1_{\mathcal{M}}(M)\right)^* \xrightarrow{A} \left(\Omega^1_{\mathcal{M}}(M) \otimes \Omega^1_{\mathcal{M}}(M)\right)^*$$

defined as follows

$$A(\varphi_1 \otimes \varphi_2)(\omega_1 \otimes \omega_2) := (-1)^{|\omega_1||\varphi_2|} \varphi_1(\omega_1) \otimes \varphi_2(\omega_2)$$

where φ_i and ω_i are homogeneous elements of $(\Omega^1_{\mathcal{M}}(M))^*$ and $\Omega^1_{\mathcal{M}}(M)$ respectively. Remark that A is an isomorphism since $\Omega^1_{\mathcal{M}}$ is a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-module of finite rank (see theorem 1.6). The inverse of A is locally given by

$$B(d^*_{ij}) = d^*_i \otimes d^*_j$$

where $d^*_i = (d_i)^*$ is defined by $(d_i)^*(d_j) = \delta_{ij}$. Indeed,

$$A(d^*_i \otimes d^*_j)(d_k \otimes d_l) = (-1)^{|j||k|} d^*_i(d_k) d^*_j(d_l) = (d^*_{ij})(d_k \otimes d_l).$$

Now the element h viewed in $(\Omega^1_{\mathcal{M}}(M))^* \otimes (\Omega^1_{\mathcal{M}}(M))^*$ is $B(h) = \sum_{i,j}(-1)^{|j|}g^{ji}d_i^* \otimes d_j^*$. Finally, in coordinates the energy function is given by

$$
\begin{aligned}
H &= \frac{1}{2}\chi(B(h)) \\
&= \frac{1}{2}\sum_{i,j}(-1)^{|j|}g^{ji}\chi\left(d_i^* \otimes d_j^*\right) \\
&= \frac{1}{2}\sum_{i,j}(-1)^{|j|}g^{ji}\chi(d_i^*) \otimes \chi(d_j^*) \\
&= \frac{1}{2}\sum_{\substack{i,j\\k,l}}(-1)^{|j|}g^{ji}p_k\langle d_k, d_i^*\rangle p_l\langle d_l, d_j^*\rangle \\
&= \frac{1}{2}\sum_{\substack{i,j\\k,l}}(-1)^{|j|}g^{ji}p_k(-1)^{|i||k|}\delta_{i,k}p_l(-1)^{|l||j|}\delta_{l,j} \\
&= \frac{1}{2}\sum_{i,j}(-1)^{|i|}g^{ji}p_ip_j \\
&= \frac{1}{2}\sum_{i,j}(-1)^{|i|+|j|}p_ig^{ij}p_j. \qquad \square
\end{aligned}
$$

Recall that in the super context we have a natural notion of graded differential k-forms $\Omega^k_{\mathcal{M}}$, defined, e.g., in [13]. Moreover, we have a canonical exterior differentiation d of bidegree $(1,0)$ on the $(\mathbb{Z} \times \mathbb{Z}_2)$-graded commutative algebra $\Omega_{\mathcal{M}}$ of graded differential forms.

Definition 2.18. *A* ***symplectic supermanifold*** *is a supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ together with a closed non-degenerated two-form $\omega \in \Omega^2_{\mathcal{M}}(M)$. We call the symplectic form and the corresponding symplectic manifold* ***even (resp. odd)*** *if $\omega \in \Omega^2_{\mathcal{M}}(M)_0$ resp. $\omega \in \Omega^2_{\mathcal{M}}(M)_1$.*

Proposition 2.19. *Let $\mathcal{M}$ be a supermanifold. Then the cotangent bundle $T^*\mathcal{M}$ of $\mathcal{M}$ is equipped with a canonical even symplectic form $\omega \in \Omega^2_{T^*\mathcal{M}}(T^*M)_0$. As in the classical case of ungraded manifold, $\omega = -d\alpha$ with a canonical potential $\alpha \in \Omega^1_{T^*\mathcal{M}}(T^*M)_0$.*

Proof. Using the notations of Proposition 2.8 and the remark after this proposition, we can define the canonical even one-form α on $T^*\mathcal{M}$ as follows:

$$\begin{array}{ccc}
\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{T^*\mathcal{M}}(T^*M)) & \xrightarrow{d(\pi^{T^*\mathcal{M}})^*} & \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(M), \mathcal{O}_{T^*\mathcal{M}}(T^*M)) \\
 & & \Big\downarrow L \; \cong \\
 & & \mathcal{O}_{T^*\mathcal{M}}(T^*M) \otimes_{\mathcal{O}_{\mathcal{M}}(M)} \mathcal{T}_{\mathcal{M}}(M) \\
 & & \Big\downarrow \mathrm{id}\otimes C \\
 & & \mathcal{O}_{T^*\mathcal{M}}(T^*M) \otimes_{\mathcal{O}_{\mathcal{M}}(M)} (\Omega^1_{\mathcal{M}}(M))^* \\
 & & \Big\downarrow \mathrm{id}\otimes\chi \\
 & & \mathcal{O}_{T^*\mathcal{M}}(T^*M).
\end{array}$$

Here $T^*\mathcal{M} \overset{\pi^{T^*\mathcal{M}}}{\longrightarrow} \mathcal{M}$ is the canonical projection, L is the map defined in the Proposition 2.8, B the natural morphism from $\mathcal{T}_{\mathcal{M}}(M)$ to $(\Omega^1_{\mathcal{M}}(M))^*$ and χ is the morphism given by the Proposition 1.12. The projection $\pi^{T^*\mathcal{M}}$ will simply be denoted by π in the rest of this proof.

We obtain an element $\alpha \in \Omega^1_{T^*\mathcal{M}}(T^*M)_0$ described as follows in the local coordinates (q_i, p_i):

$$\begin{aligned}
\alpha(X) &= (\mathrm{id}\otimes\chi)\circ(\mathrm{id}\otimes C)\circ L\circ d\pi^*(X) \\
&= (\mathrm{id}\otimes\chi)\circ(\mathrm{id}\otimes C)\circ L(X\circ\pi^*) \\
&= (\mathrm{id}\otimes\chi)\circ(\mathrm{id}\otimes C)\left(\sum_i X\circ\pi^*(q_i)\otimes\partial_{q_i}\right) \\
&= (\mathrm{id}\otimes\chi)\left(\sum_i X(q_i)\otimes C(\partial_{q_i})\right), \text{ by identifying } \pi^*(q_i) \text{ with } q_i,
\end{aligned}$$

$$\begin{aligned}
\alpha(X) &= \sum_i X(q_i)\cdot\chi(C(\partial_{q_i})) \\
&= \sum_{i,j} X(q_i)\cdot p_j\cdot\langle d(q_j), C(\partial_{q_i})\rangle \\
&= \sum_{i,j} X(q_i)\cdot p_j\cdot d(q_j)(\partial_{q_i}) \\
&= \sum_i (-1)^{|i|} X(q_i)\cdot p_i \\
&= \sum_i (-1)^{|i||X|} p_i\cdot X(q_i) \\
&= \sum_i p_i\cdot d(q_i)(X) \quad \forall X\in\mathcal{T}_{T^*\mathcal{M}}(T^*M).
\end{aligned}$$

We thus locally have $\alpha = \sum_i p_i \cdot d(q_i)$.

Now we set $\omega := -d\alpha \in \Omega^2_{T^*\mathcal{M}}(T^*M)_0$. To see that ω is non-degenerated, consider its local expression:

$$\begin{aligned}\omega = -d\alpha &= -\left[\sum_i d(p_i) \wedge d(q_i) + \sum_i (-1)^{0\cdot 1+|i|\cdot 0} p_i \cdot d^2(q_i)\right] \\ &= -\sum_i (-1)^{1\cdot 1+|i||i|} d(q_i) \wedge d(p_i) \\ &= \sum_i (-1)^{|i|} d(q_i) \wedge d(p_i). \end{aligned}$$ □

The non-degeneracy of a symplectic form ω allows us to give the following definition:

Definition 2.20. *Let $(\mathcal{M}, \omega)$ be a symplectic supermanifold. For any $f \in \mathcal{O}_{\mathcal{M}}(M)$ let $X_f \in \mathcal{T}_{\mathcal{M}}(M)$ be the* ***Hamiltonian vector field*** *defined by the relation:*

$$i(X_f)\omega = d(f),$$

where $(i(X)\omega)(Y) := (-1)^{|X||\omega|}\omega(X, Y)$.

Note that X_f has necessarily the same parity as f if ω is even.

Proposition 2.21. *Let $\mathcal{M}$ be a supermanifold and $(T^*\mathcal{M}, \omega)$ be its cotangent bundle equipped with the canonical symplectic form. The Hamiltonian vector field $X_f \in \mathcal{T}_{T^*\mathcal{M}}(T^*M)$ associated to an even function $f \in \mathcal{O}_{T^*\mathcal{M}}(T^*M)_0$ is an even vector field and it is locally given by the following equations:*

$$\begin{aligned} X_f(q_i) &= \partial_{p_i} f \\ X_f(p_i) &= -(-1)^{|i|}\partial_{q_i} f. \end{aligned}$$

Proof. Let X and Y be vector field on $\mathrm{T}^*\mathcal{M}$ we have

$$\begin{aligned}\omega(X, Y) &= \sum_j (-1)^{|j|} d(q_j) \wedge d(p_j)(X, Y) \\ &= \sum_j (-1)^{|j|}\left[(-1)^{|X||j|} d(q_j)(X) \cdot d(p_j)(Y) + (-1)^{1\cdot 1+|X||Y|+|Y||j|} d(q_j)(Y) \cdot d(p_j)(X)\right] \\ &= \sum_j (-1)^{|j|}\left[(-1)^{|Y||j|} X(q_j) \cdot Y(p_j) - (-1)^{|X||Y|+|X||j|} Y(q_j) \cdot X(p_j)\right] \end{aligned}$$

In the case $Y = \dfrac{\partial}{\partial p_i}$, we get

$$\begin{aligned}\omega(X_f, \frac{\partial}{\partial p_i}) &= \sum_j (-1)^{|j|}\left[(-1)^{|i||j|} X_f(q_j) \cdot \partial_{p_i}(p_j) - (-1)^{|X_f||i|+|X_f||j|}\partial_{p_i}(q_j) \cdot X_f(p_j)\right] \\ &= X_f(q_i). \end{aligned}$$

And on the other hand we have

$$\begin{aligned}\omega(X_f, \frac{\partial}{\partial p_i}) &= i(X_f)\omega(\partial_{p_i}) \\ &= d(f)(\partial_{p_i}) \\ &= (-1)^{|f||i|}\partial_{p_i} f.\end{aligned}$$

Similarly, the second formula is obtained from:

$$\begin{aligned}\omega(X_f, \frac{\partial}{\partial q_i}) &= \sum_j (-1)^{|j|}\Big[(-1)^{|i||j|}X_f(q_j)\cdot\partial_{q_i}(p_j) - (-1)^{|X_f||i|+|X_f||j|}\partial_{q_i}(q_j)\cdot X_f(p_j)\Big] \\ &= -(-1)^{|i|}X_f(p_i) \\ &= (-1)^{|f||i|}\partial_{q_i} f,\end{aligned}$$

finishing the proof of the proposition. □

Corollary 2.22. *In the situation of the preceding proposition, we have for H, the energy function, the following equations:*

$$X_H(q_i) = \sum_j (-1)^{|j|} p_j g^{ji} \tag{2.11}$$

$$X_H(p_i) = -\frac{1}{2}\sum_{k,j}(-1)^{|i|+|k|+|j|+|i||k|} p_k \partial_{q_i}(g^{kj})p_j. \tag{2.12}$$

Proof. We calculate $X_H(q_i)$ and $X_H(p_i)$ using the local expression of H given by Lemma 2.17. Then we get

$$\begin{aligned}\partial_{p_i} H &= \frac{1}{2}\sum_{k,j}(-1)^{|k|+|j|}\partial_{p_i}(p_k g^{kj} p_j) \\ &= \frac{1}{2}\sum_{k,j}(-1)^{|k|+|j|}\Big[\partial_{p_i}(p_k)g^{kj}p_j + (-1)^{|k||i|}p_k\partial_{p_i}(g^{kj}p_j)\Big] \\ &= \frac{1}{2}\sum_{k,j}(-1)^{|k|+|j|}\Big[\partial_{p_i}(p_k)g^{kj}p_j + (-1)^{|k||i|+|i|(|k|+|j|)}p_k g^{kj}\partial_{p_i}(p_j)\Big] \\ &= \frac{1}{2}\sum_j(-1)^{|i|+|j|}g^{ij}p_j + \frac{1}{2}\sum_k(-1)^{|k|+|i|+|k||i|+|i|(|k|+|i|)}p_k g^{ki} \\ &= \frac{1}{2}\sum_j(-1)^{|j|}p_j g^{ji} + \frac{1}{2}\sum_k(-1)^{|k|}p_k g^{ki} \\ &= \sum_j(-1)^{|j|}p_j g^{ji}\end{aligned}$$

and

$$\begin{aligned}\partial_{q_i} H &= \frac{1}{2}\sum_{k,j}(-1)^{|k|+|j|}\partial_{q_i}(p_k g^{kj} p_j)\\ &= \frac{1}{2}\sum_{k,j}(-1)^{|k|+|j|+|i||k|}p_k\partial_{q_i}(g^{kj})p_j.\end{aligned}$$

We conclude from the preceding proposition that the equations 2.11 and 2.12 hold for all i. □

Definition 2.23. *Let $\mathcal{M}$ be a supermanifold and X be an even vector field on $\mathcal{M}$. A **flow of** X is a morphism $F : \mathbb{R}\times\mathcal{M}\to\mathcal{M}$ of supermanifolds such that:*

$$\widehat{\partial_t}\circ F^* = F^*\circ X \text{ and } F\circ inj_{t=0} = id_{\mathcal{M}}, \tag{2.13}$$

where t is the canonical coordinate on $\mathbb{R}$, $\widehat{\partial_t}$ the lift of the derivation ∂_t on $\mathbb{R}$ to a derivation on $\mathbb{R}\times\mathcal{M}$ acting just on the first factor, and $inj_{t=0} : \mathcal{M}\cong\{0\}\times\mathcal{M}\overset{0\times id_{\mathcal{M}}}{\longrightarrow}\mathbb{R}\times\mathcal{M}$ is the injection map at $t=0$.

Remark. Of course, the domain of the flow map F is in general only an open subsupermanifold $\mathcal{V}$ of $\mathbb{R}\times\mathcal{M}$, containing $\{0\}\times\mathcal{M}$. Abusively, we will often write $\mathbb{R}\times\mathcal{M}$ instead of $\mathcal{V}$.

The following result is due to Monterde and Montesinos [15]:

Theorem 2.24. *Let X be an even vector field on a supermanifold $\mathcal{M}$, $\widetilde{X}$ the reduced vector field on M, and $V_{\widetilde{X}}\subset\mathbb{R}\times M$ the flow domain of $\widetilde{X}$. Then there exists a unique map $F : \mathcal{V}_X\to\mathcal{M}$, where $\mathcal{V}_X$ is the open subsupermanifold of $\mathbb{R}\times\mathcal{M}$ having as body $V_{\widetilde{X}}$, satisfying (2.13).*

We will give in the next chapter a new proof of this theorem which can be extended to supervector fields on holomorphic supermanifolds. Indeed, the proof given in [15] is based on Batchelor's theorem that do not generally holds true in the holomorphic case (see [11] for more details).

Given the flow F of an even vector field X, we want to use an appropriate notion of initial condition to get "integral supercurves". If we choose a point q_0 in M, we get a curve as follows:

$$\begin{tikzcd} \mathbb{R}\cong\mathbb{R}\times\{q_0\} \arrow[r, "\mathrm{inj}_{q=q_0}"] \arrow[dr, "\Psi"'] & \mathbb{R}\times\mathcal{M} \arrow[d, "F"] \\ & \mathcal{M} \end{tikzcd}$$

This type of initial condition leads to a poor notion of "integral supercurves", since Ψ factorizes over M, the body of the supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$. Thus we need a notion of initial condition that is more elaborated than simply choosing a point in the body of $\mathcal{M}$.

Following Dumitrescu ([9]) we give

Definition 2.25. *Let $\mathcal{M}$, $\mathcal{S}$ be supermanifolds, X an even vector field on $\mathcal{M}$ and $\mathcal{I} \in Mor(\mathcal{S}, \mathcal{M})$. A morphism $\Psi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ such that*

$$\widehat{\partial_t} \circ \Psi^* = \Psi^* \circ X \text{ and } \Psi \circ inj_{t=0} = \mathcal{I}, \tag{2.14}$$

*where t is the canonical even coordinate on $\mathbb{R}$ and $inj_{t=0} : \mathcal{S} \longrightarrow \mathbb{R} \times \mathcal{S}$ the evaluation map at $t = 0$, is called an "**integral curve** for X, parametrized by $\mathcal{S}$ and satisfying the initial condition $\mathcal{I}$".*

The case we are mostly concerned with will be $\mathcal{S} = \mathbb{R}^{0|1}$ but we need this more general "family notion" for integral curves notably in the proof of Theorem 2.45 below. We note that one can give the following geometrical description of the set of initial conditions (for $\mathcal{S} = \mathbb{R}^{0|1}$):

Proposition 2.26. *Let $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ be a supermanifold and E be a real vector bundle over M such that $\mathcal{M} \cong (M, \Gamma^{\infty}_{\wedge E^*})$. Then there is a 1:1 correspondence between E and $Mor(\mathbb{R}^{0|1}, \mathcal{M})$, the set of initial conditions for $\mathcal{S} = \mathbb{R}^{0|1}$.*

Proof. We use local coordinates given by local trivializations of E to construct a bijection between E and $\mathrm{Mor}(\mathbb{R}^{0|1}, \mathcal{M})$. Let v be an element of E in the fiber over a point $p \in M$. Let U_α be an open set of M containing p such that there exists an isomorphism

$$E_{|U_\alpha} \xrightarrow[\cong]{\phi_\alpha} U_\alpha \times \mathbb{R}^n.$$

Let $\{e^i_\alpha | 1 \leq i \leq n\}$ be the sections of E over U_α given by the trivialization ϕ_α such that for all x in U_α, $(e^i_\alpha(x))_i$ is a basis of the $\mathbb{R}$-vector space E_x.

Then the element v can be written as

$$v = \sum_i v^\alpha_i e^i_\alpha(p) \text{ where } v^\alpha_i \in \mathbb{R}.$$

Let us define Ψ_α the unique morphism of supermanifolds from $\mathbb{R}^{0|1}$ to $\mathcal{M}$ such that

$$\widetilde{\Psi_\alpha}(0) = p \text{ and } \Psi^*_\alpha(U_\alpha)(\xi^i_\alpha) = v^\alpha_i \cdot \theta,$$

where θ is the canonical coordinate on $\mathbb{R}^{0|1}$ and $(\xi_\alpha^i)_i$ are the odd coordinates of $\mathcal{O}_\mathcal{M}(U_\alpha) = \Gamma^\infty(U_\alpha, \wedge E^*)$ given by $\xi_\alpha^i = (e_\alpha^i)^*$. Actually, the above conditions fixe the morphism $\Psi_\alpha : \mathbb{R}^{0|1} \to \mathcal{M}$ since the following diagram is commutative

$$\begin{tikzcd}
\mathcal{O}_\mathcal{M}(M) \arrow[r, "\text{restriction}"] \arrow[d, "\Psi_\alpha^*(M)"'] & \mathcal{O}_\mathcal{M}(U_\alpha) \arrow[d, "\Psi_\alpha^*(U_\alpha)"] \\
\mathcal{O}_{\mathbb{R}^{0|1}}(\{0\}) \arrow[r, "\text{id}"] & \mathcal{O}_{\mathbb{R}^{0|1}}(\{0\}),
\end{tikzcd}$$

and then we have $\Psi_\alpha^*(M)(f) = \Psi_\alpha^*(U_\alpha)(f_{|U_\alpha})$ for all $f \in \mathcal{O}_\mathcal{M}(M)$.

Now the assignment from E to $\mathrm{Mor}(\mathbb{R}^{0|1}, \mathcal{M})$ is independent of the choice of trivializations. Actually, if U_β is an open set of M containing p as well such that there exists a basis $(e_\beta^j)_j$ of $E_{|U_\beta}$ as for $E_{|U_\alpha}$ above then we have elements $(\phi_{\alpha\beta})_{ij} \in \mathcal{C}^\infty(U_{\alpha\beta}, \mathbb{R})$ such that

$$e_\alpha^i = \sum_j (\phi_{\alpha\beta})_{ij} e_\beta^j \text{ on } U_{\alpha\beta} = U_\alpha \cap U_\beta.$$

We have the following relations:

$$v_j^\beta = \sum_i v_i^\alpha (\phi_{\alpha\beta})_{ij}(p) \quad \text{and} \quad \xi_\beta^j = \sum_i \xi_\alpha^i (\phi_{\alpha\beta})_{ij} \text{ over } U_{\alpha\beta}$$

which imply that

$$\begin{aligned}
\Psi_\alpha^*(U_{\alpha\beta})(\xi_\beta^j) &= \sum_i \Psi_\alpha^*(U_{\alpha\beta})(\xi_\alpha^i (\phi_{\alpha\beta})_{ij}) \\
&= \sum_i v_i^\alpha (\phi_{\alpha\beta})_{ij}(p) \cdot \theta \\
&= v_j^\beta \\
&= \Psi_\beta^*(U_{\alpha\beta})(\xi_\beta^j).
\end{aligned}$$

Consequently the relation $\Psi_\alpha = \Psi_\beta$ holds for all α, β.
Finally one easily get from its very definition that the map between E and $\mathrm{Mor}(\mathbb{R}^{0|1}, \mathcal{M})$ constructed above is a bijection. $\square$

Similarly to the standard ungraded case, integral curves are given by factorization over the flow map:

Proposition 2.27. *([9]) Let X be an even vector field on $\mathcal{M}$ and F be the flow of X. Let $\mathcal{I} \in Mor(\mathcal{S}, \mathcal{M})$ be an initial condition. There exists an unique integral curve $\Psi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ of X with initial condition $\mathcal{I}$. Precisely, it is given by $\Psi = F \circ (id_\mathbb{R} \times \mathcal{I})$.*

Proof. Let $\Psi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ be the morphism of supermanifold defined by $\Psi := F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})$.
Recall that $F : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ is the unique morphism such that

$$\widehat{\partial_t} \circ F^* = F^* \circ X \text{ and } (\mathrm{inj}_{t=0})^* \circ F^* = \mathrm{id}_{\mathcal{M}}{}^*.$$

We have the following equations:

$$(\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^* \circ \widehat{\partial_t} = \partial_t \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^* \text{ and } \mathcal{I}^* \circ (\mathrm{inj}_{t=0})^* = (\mathrm{inj}_{t=0})^* \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^*$$

given by the calculations

$$\begin{aligned}
\big((\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^* \circ \widehat{\partial_t}\big)(f \otimes g) &= (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^*((\partial_t f) \otimes g) \\
&= (\partial_t f) \cdot \mathcal{I}^*(g) = \partial_t(f \cdot \mathcal{I}^*(g)) \\
&= \partial_t\big((\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^*(f \otimes g)\big) \quad \forall f \in \mathcal{O}_{\mathbb{R}}(\mathbb{R}) \text{ and } \forall g \in \mathcal{O}_{\mathcal{M}}(M),
\end{aligned}$$

and

$$\begin{aligned}
\big(\mathcal{I}^* \circ (\mathrm{inj}_{t=0})^*\big)(f \otimes g) &= \mathcal{I}^*(f(0) \cdot g) \\
&= f(0) \cdot \mathcal{I}^*(g) = (\mathrm{inj}_{t=0})^*(f \otimes \mathcal{I}^*(g)) \\
&= \big((\mathrm{inj}_{t=0})^* \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^*\big)(f \otimes g) \quad \forall f \in \mathcal{O}_{\mathbb{R}}(\mathbb{R}) \text{ and } \forall g \in \mathcal{O}_{\mathcal{M}}(M).
\end{aligned}$$

Applying $(\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^*$ to the first equation satisfied by the flow F and $\mathcal{I}^*$ to the second one, the preceding equations yield

$$\widehat{\partial_t} \circ \Psi^* = \Psi^* \circ X \text{ and } (\mathrm{inj}_{t=0})^* \circ \Psi^* = \mathcal{I}^*.$$

It follows that Ψ satisfies the condition (2.14), i.e. Ψ is an integral curve of X subject to the initial condition $\mathcal{I}$. Now the uniqueness is proved by regarding locally the equations (2.14) as in the proof of the uniqueness of the flow F of X given in the next chapter. □

We can now give the first fundamental result of this thesis, relating the integral curves of the geodesic flow on $T^*\mathcal{M}$ to the geodesics of a Riemannian supermanifold $\mathcal{M}$.

Theorem 2.28. *Let $(\mathcal{M}, g)$ be a even Riemannian supermanifold and $(T^*\mathcal{M}, \omega, H)$ be the Hamiltonian dynamical system associated to $\mathcal{M}$, and let $\mathcal{S}$ be a supermanifold. Furthermore, let X_H be the Hamiltonian vector field associated to H, $\mathcal{I}$ an initial condition in Mor$(\mathcal{S}, T^*\mathcal{M})$ and Ψ the associated integral curve of X_H ($\Psi = F \circ (id_{\mathbb{R}} \times \mathcal{I})$). Then the supercurve $\Phi := \pi^{T^*\mathcal{M}} \circ \Psi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ satisfies the geodesic equation:*

$$\frac{\nabla}{dt}(d\Phi^*(\partial_t)) = 0,$$

subject to the initial condition $\partial_t \Phi \circ inj_{t=0} = G^\natural \circ \mathcal{I} \in Mor(\mathcal{S}, T\mathcal{M})$ *, where* $G^\natural : T^*\mathcal{M} \xrightarrow{\cong} T\mathcal{M}$ *is the isomorphism induced by the metric* g.

Proof. Let Ψ be an integral curve of X_H. Then Ψ is given by $\Psi = F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})$ using Proposition (2.27) where $F : \mathbb{R} \times T^*\mathcal{M} \to T^*\mathcal{M}$ is the flow of X_H and $\mathcal{I} \in \mathrm{Mor}(\mathcal{S}, T^*\mathcal{M})$ is the initial condition.

Recall that F is the unique morphism such that

$$\widehat{\partial_t} \circ F^* = F^* \circ X_H \text{ and } F \circ \mathrm{inj}_{t=0} = \mathrm{id}_{T^*\mathcal{M}}.$$

In local coordinates F is uniquely determined by the following system of equations (compare Corollary 2.22):

$$\begin{aligned}
\widehat{\partial_t}\left(F^*(q_i)\right) &= \sum_j (-1)^{|j|} F^*(p_j g^{ji}) \\
\widehat{\partial_t}\left(F^*(p_i)\right) &= -\frac{1}{2}\sum_{k,j} (-1)^{|i|+|k|+|j|+|i||k|} F^*\left(p_k \partial_{q_i}(g^{kj}) p_j\right) \\
(\mathrm{inj}_{t=0})^* \circ F^*(q_i) &= q_i \\
(\mathrm{inj}_{t=0})^* \circ F^*(p_i) &= p_i
\end{aligned}$$

We note here and in the sequel the parity $|p_i| = |q_i|$ often by $|i|$.

Applying $(\mathrm{id}_{\mathbb{R}} \times \mathcal{I})^*$ to the two first equations of the above system and $\mathcal{I}^*$ to the remaning equations, we get

$$\partial_t\left(\Psi^*(q_i)\right) = \sum_j (-1)^{|j|} \Psi^*(p_j g^{ji}) \tag{2.15}$$

$$\partial_t\left(\Psi^*(p_i)\right) = -\frac{1}{2}\sum_{k,j} (-1)^{|i|+|k|+|j|+|i||k|} \Psi^*\left(p_k \partial_{q_i}(g^{kj}) p_j\right) \tag{2.16}$$

$$(\mathrm{inj}_{t=0})^* \circ \Psi^*(q_i) = \mathcal{I}^*(q_i) \tag{2.17}$$

$$(\mathrm{inj}_{t=0})^* \circ \Psi^*(p_i) = \mathcal{I}^*(p_i) \tag{2.18}$$

Equation (2.15) gives now

$$\sum_i \partial_t \Psi^*(q_i) \cdot \Psi^*(g_{ij}) = (-1)^{|j|} \Psi^*(p_j). \tag{2.19}$$

Derivation along the "even time direction" t yields:

$$\begin{aligned}
(-1)^{|j|} \partial_t \Psi^*(p_j) &= \sum_i \left[\partial_t^2 \Psi^*(q_i) \cdot \Psi^*(g_{ij}) + \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(g_{ij})\right] \\
&= \sum_i \partial_t^2 \Psi^*(q_i) \cdot \Psi^*(g_{ij}) + \sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \cdot \Psi^*(\partial_{q_k} g_{ij}).
\end{aligned}$$

Let us first consider the last summand of the last RHS:

$$\begin{aligned}
&\sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \cdot \Psi^*(\partial_{q_k} g_{ij}) \\
&\qquad = \frac{1}{2}\left[\sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \cdot \Psi^*(\partial_{q_k} g_{ij}) + \sum_{i,k} \partial_t \Psi^*(q_k) \cdot \partial_t \Psi^*(q_i) \cdot \Psi^*(\partial_{q_i} g_{kj})\right] \\
&\qquad\qquad = \frac{1}{2}\sum_{k,i} \partial_t \Psi^*(q_i) \cdot \partial_t \Psi^*(q_k) \left(\Psi^*(\partial_{q_k} g_{ij}) + (-1)^{|i||k|}\Psi^*(\partial_{q_i} g_{kj})\right).
\end{aligned}$$

After relabelling indices we arrive at

$$(-1)^{|l|}\partial_t \Psi^*(p_l) = \sum_j \partial_t^2 \Psi^*(q_j)\cdot\Psi^*(g_{jl}) + \frac{1}{2}\sum_{i,j} \partial_t \Psi^*(q_j)\cdot\partial_t \Psi^*(q_i)\cdot\left(\Psi^*(\partial_{q_i} g_{jl}) + (-1)^{|j||i|}\Psi^*(\partial_{q_j} g_{il})\right). \tag{2.20}$$

Using equation (2.19), equation (2.16) yields:

$$\begin{aligned}
(-1)^{|i|}\partial_t\left(\Psi^*(p_i)\right) &= -\frac{1}{2}\sum_{k,j}(-1)^{|k|+|j|+|i||k|}\Psi^*(p_k)\cdot\Psi^*(\partial_{q_i} g^{kj})\cdot\Psi^*(p_j) \\
&= -\frac{1}{2}\sum_{\substack{k,j\\ l,s}}(-1)^{|i||k|}\partial_t\Psi^*(q_l)\cdot\Psi^*(g_{lk})\cdot\Psi^*(\partial_{q_i} g^{kj})\cdot\partial_t\Psi^*(q_s)\cdot\Psi^*(g_{sj}).
\end{aligned}$$

Summing the following equality over k

$$\partial_{q_i}(g_{lk}\cdot g^{kj}) = \partial_{q_i} g_{lk}\cdot g^{kj} + (-1)^{|i|(|l|+|k|)} g_{lk}\cdot\partial_{q_i} g^{kj}$$

gives

$$\sum_k (-1)^{|i||l|}\partial_{q_i} g_{lk}\cdot g^{kj} + \sum_k (-1)^{|i||k|} g_{lk}\cdot\partial_{q_i} g^{kj} = 0.$$

Then

$$\begin{aligned}
(-1)^{|i|}\partial_t\left(\Psi^*(p_i)\right) &= \frac{1}{2}\sum_{\substack{k,j\\ l,s}}(-1)^{|i||l|}\partial_t\Psi^*(q_l)\cdot\Psi^*(\partial_{q_i} g_{lk})\cdot\Psi^*(g^{kj})\cdot\partial_t\Psi^*(q_s)\cdot\Psi^*(g_{sj}) \\
&= \frac{1}{2}\sum_{\substack{k,j\\ l,s}}(-1)^{|i||l|+|s|}\partial_t\Psi^*(q_l)\cdot\Psi^*(\partial_{q_i} g_{lk})\cdot\Psi^*(g^{kj})\Psi^*(g_{js})\cdot\partial_t\Psi^*(q_s) \\
&= \frac{1}{2}\sum_{k,l}(-1)^{|i||l|+|k|}\partial_t\Psi^*(q_l)\cdot\Psi^*(\partial_{q_i} g_{lk})\cdot\partial_t\Psi^*(q_k) \\
&= \frac{1}{2}\sum_{k,l}(-1)^{|k|(|i|+|l|)+|i||l|}\partial_t\Psi^*(q_l)\cdot\partial_t\Psi^*(q_k)\cdot\Psi^*(\partial_{q_i} g_{lk}).
\end{aligned}$$

After again relabelling indices, we have

$$(-1)^{|l|}\partial_t\Psi^*(p_l) = \frac{1}{2}\sum_{j,i}(-1)^{|j|(|l|+|i|)+|l||i|}\partial_t\Psi^*(q_i)\cdot\partial_t\Psi^*(q_j)\cdot\Psi^*(\partial_{q_l}g_{ij})$$

$$= \frac{1}{2}\sum_{j,i}(-1)^{|l|(|i|+|j|)}\partial_t\Psi^*(q_j)\cdot\partial_t\Psi^*(q_i)\cdot\Psi^*(\partial_{q_l}g_{ij}). \tag{2.21}$$

Multiplying (2.20) with $\Psi^*(g^{lk})$ and summing over l gives:

$$\sum_l(-1)^{|l|}\partial_t\Psi^*(p_l)\cdot\Psi^*(g^{lk}) = \partial_t^2\Psi^*(q_k)+\frac{1}{2}\sum_{i,j,l}\partial_t\Psi^*(q_j)\cdot\partial_t\Psi^*(q_i)\cdot\Psi^*\left(\partial_{q_i}g_{jl}+(-1)^{|j||i|}\partial_{q_j}g_{il}\right)\cdot\Psi^*(g^{lk}). \tag{2.22}$$

Applying the same operations to (2.21) yields

$$\sum_l(-1)^{|l|}\partial_t\Psi^*(p_l)\cdot\Psi^*(g^{lk}) = \frac{1}{2}\sum_{j,i,l}(-1)^{|l|(|i|+|j|)}\partial_t\Psi^*(q_j)\cdot\partial_t\Psi^*(q_i)\cdot\Psi^*(\partial_{q_l}g_{ij})\cdot\Psi^*(g^{lk}). \tag{2.23}$$

Now, using the equality $\Phi^*(q_i) = \Psi^*\circ(\pi^{T^*\mathcal{M}})^*(q_i) = \Psi^*(q_i)$ and equating the RHS of the equalities (2.22) and (2.23), we get exactly (2.9) since the Christoffel symbols are given by (2.1).

Let us check that $\partial_t\Phi\circ\mathrm{inj}_{t=0} = G^\natural\circ\mathcal{I}$.

Recall the canonical sections s_0, s_1 defined before Theorem 2.14. Since $\Phi = \pi^{T^*\mathcal{M}}\circ F\circ(\mathrm{id}_\mathbb{R}\times\mathcal{I})$, we can, in fact, show that $\partial_t\Phi = G^\natural\circ F\circ(\mathrm{id}_\mathbb{R}\times\mathcal{I})$.
Firstly, we have

$$\begin{aligned}\partial_t\Phi &= T\Phi\circ(s_1\times s_0)\\ &= T\pi^{T^*\mathcal{M}}\circ TF\circ(\mathrm{id}_{T\mathbb{R}}\times T\mathcal{I})\circ(s_1\times s_0)\\ &= T\pi^{T^*\mathcal{M}}\circ TF\circ(s_1\times(T\mathcal{I}\circ s_0)) : \mathbb{R}\times\mathcal{S}\to T\mathcal{M}.\end{aligned}$$

If (q_i, p_i, v_i, r_i) are the usual local coordinates on $T(T^*\mathcal{M})$ and (q_i, v_i) local coordinates on $T\mathcal{M}$, identify with the image of the tangent map associated to the zero section of $T^*\mathcal{M}\to\mathcal{M}$. Let further denote (v_t, t) the point $v_t\frac{d}{dt}_{|t}$ of $T\mathbb{R}$. We then have

$$\begin{aligned}(\partial_t\Phi)^*(q_i) &= (s_1^*\times(s_0^*\circ(T\mathcal{I})^*))\circ(TF)^*\circ(T\pi^{T^*\mathcal{M}})^*(q_i)\\ &= (s_1^*\times(s_0^*\circ(T\mathcal{I})^*))\circ(TF)^*(q_i)\\ &= (s_1^*\times(s_0^*\circ(T\mathcal{I})^*))(F^*(q_i))\\ &= (\mathrm{id}_\mathbb{R}^*\times\mathcal{I}^*)(F^*(q_i))\\ &= (\mathrm{id}_\mathbb{R}^*\times\mathcal{I}^*)\circ F^*\circ(G^\natural)^*(q_i)\\ &= \left(G^\natural\circ F\circ(\mathrm{id}_\mathbb{R}\times\mathcal{I})\right)^*(q_i),\end{aligned}$$

and

$$\begin{aligned}
(\partial_t\Phi)^*(v_i) &= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \circ (TF)^* \circ (T\pi^{T^*\mathcal{M}})^*(v_i) \\
&= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \circ (TF)^*(v_i) \\
&= (s_1^* \times (s_0^* \circ (T\mathcal{I})^*)) \left(\sum_j \left(v_j.\partial_{q_j}F^*(q_i) + r_j.\partial_{p_j}F^*(q_i)\right) + v_t.\partial_t F^*(q_i)\right) \\
&= (\mathrm{id}_{\mathbb{R}}^* \times \mathcal{I}^*)\,(\partial_t F^*(q_i)) \\
&= (\mathrm{id}_{\mathbb{R}}^* \times \mathcal{I}^*)\,((F^* \circ X_H)(q_i))\,, \text{ since } F \text{ is the flow of } X_H, \\
&= (\mathrm{id}_{\mathbb{R}}^* \times \mathcal{I}^*)\left(F^*((-1)^{|j|}\textstyle\sum_j p_j g^{ji})\right) \\
&= (\mathrm{id}_{\mathbb{R}}^* \times \mathcal{I}^*)\left(F^* \circ (G^\natural)^*(v_i)\right), \text{ see formula (1.2)} \\
&= \left(G^\natural \circ F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})\right)^*(v_i).
\end{aligned}$$

Since we now have $\partial_t\Phi = G^\natural \circ F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})$, we can easily conclude the proof of the theorem:

$$\partial_t\Phi \circ \mathrm{inj}_{t=0} = G^\natural \circ F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I}) \circ \mathrm{inj}_{t=0} = G^\natural \circ \mathcal{I}. \qquad \square$$

Reciprocally, we have the following statement:

Theorem 2.29. *Let $\Phi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ be a supergeodesic with initial condition $\partial_t\Phi \circ inj_{t=0} = \mathcal{J}$ in $Mor(\mathcal{S}, T\mathcal{M})$. Then $G^\flat \circ \partial_t\Phi : \mathbb{R} \times \mathcal{S} \to T^*\mathcal{M}$ is the integral curve of X_H with initial condition $G^\flat \circ \mathcal{J} \in Mor(\mathcal{S}, T^*\mathcal{M})$, where $G^\flat : T\mathcal{M} \xrightarrow{\cong} T^*\mathcal{M}$ is the isomorphism induced by the metric g.*

Proof. Let $\Phi : \mathbb{R} \times \mathcal{S} \to \mathcal{M}$ be the supercurve such that $\frac{\nabla}{dt}(d\Phi^*(\partial_t)) = 0$ and $\partial_t\Phi \circ \mathrm{inj}_{t=0} = \mathcal{J}$ in $\mathrm{Mor}(\mathcal{S}, T\mathcal{M})$. We define $\mathcal{I} := G^\flat \circ \mathcal{J}$ and $\Psi := F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})$. By the previous theorem, we know that $\pi^{T^*\mathcal{M}} \circ \Psi$ is the geodesic of $\mathcal{M}$ with initial condition $G^\natural \circ \mathcal{I} = \mathcal{J}$. By uniqueness, we have $\Phi = \pi^{T^*\mathcal{M}} \circ \Psi$. Now, using a calculation made in the proof of the preceding theorem, we have $\partial_t\Phi = G^\natural \circ F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})$. Then $G^\flat \circ \partial_t\Phi = F \circ (\mathrm{id}_{\mathbb{R}} \times \mathcal{I})$, i.e., $G^\flat \circ \partial_t\Phi$ is the integral curve of X_H with initial condition $\mathcal{I} = G^\flat \circ \mathcal{J}$. $\square$

2.3.2 Odd Riemannian metrics

We treat here the case of odd Riemannian metrics.

In order to construct the "geodesic flow" corresponding to supergeodesic on an odd Riemannian supermanifold $(\mathcal{M}, g)$, one first has to observe that an odd metric canonically induces an isomorphism of super vectorbundles $G^\flat : T\mathcal{M} \to \Pi T^*\mathcal{M}$, where $\Pi T^*\mathcal{M}$ is the super vectorbundle associated to the $\mathcal{O}_{\mathcal{M}}$-module sheaf $\Pi\Omega^1_{\mathcal{M}}$. We observe furthermore that the supermanifold $\Pi T^*\mathcal{M}$ has a canonical odd 1-form $\alpha^{\Pi T^*\mathcal{M}}$ whose exterior derivative $\omega^{\Pi T^*\mathcal{M}} := -d(\alpha^{\Pi T^*\mathcal{M}})$ is an odd symplectic form on $\Pi T^*\mathcal{M}$. The construction of $\alpha^{\Pi T^*\mathcal{M}}$ closely parallels the construction of the 1-form $\alpha = \alpha^{T^*\mathcal{M}}$. Let us give some details of this construction. Recall the following proposition (See proposition 1.14).

Reminder 2.30. *There exists a natural (even) morphism of sheaves*

$$Mor_{\mathcal{O}_{\mathcal{M}}}(\Pi\Omega^1_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}}) \xrightarrow{\chi} \widetilde{\pi}_* \mathcal{O}_{\Pi T^*\mathcal{M}}.$$

Proposition 2.31. *The supervector bundle $\Pi T^*\mathcal{M}$ is equipped with a canonical odd symplectic form $\omega \in \Omega^2_{\Pi T^*\mathcal{M}}(E)_1$, where E denotes the body of $\Pi T^*\mathcal{M}$. As in the case of $T^*\mathcal{M}$, $\omega = -d\alpha$ with a canonical potential $\alpha \in \Omega^1_{\Pi T^*\mathcal{M}}(E)_1$.*

Proof. Using the notations of Proposition 2.8 and the remark after this proposition, we can define the canonical even one-form α on $\Pi T^*\mathcal{M}$ as follows:

$$\begin{tikzcd}
\mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{\Pi T^*\mathcal{M}}(E)) \arrow[r, "d(\pi^{\Pi T^*\mathcal{M}})^*"] \arrow[ddrr, "\alpha"'] & \mathrm{Der}_{\mathbb{R}}(\mathcal{O}_{\mathcal{M}}(M), \mathcal{O}_{\Pi T^*\mathcal{M}}(E)) \arrow[r, "L", "\cong"'] & \mathcal{O}_{\Pi T^*\mathcal{M}}(E) \otimes_{\mathcal{O}_{\mathcal{M}}(M)} \mathcal{T}_{\mathcal{M}}(M) \arrow[d, "\mathrm{id}\otimes C"] \\
 & & \mathcal{O}_{\Pi T^*\mathcal{M}}(E) \otimes_{\mathcal{O}_{\mathcal{M}}(M)} (\Pi\Omega^1_{\mathcal{M}}(M))^* \arrow[d, "\mathrm{id}\otimes\chi"] \\
 & & \mathcal{O}_{\Pi T^*\mathcal{M}}(E).
\end{tikzcd}$$

Here $\Pi T^*\mathcal{M} \overset{\pi^{\Pi T^*\mathcal{M}}}{\longrightarrow} \mathcal{M}$ is the canonical projection, L is the map defined in the Proposition 2.8, C the natural odd morphism from $\mathcal{T}_{\mathcal{M}}(M)$ to $(\Pi\Omega^1_{\mathcal{M}}(M))^*$ and χ is the morphism given by the proposition above. Of course, $(\mathrm{id} \otimes \chi)(f \otimes \varphi) = f \cdot \chi(\varphi)$ for all $f \in \mathcal{O}_{\Pi T^*\mathcal{M}}(E)$ and $\varphi \in (\Pi\Omega^1_{\mathcal{M}}(M))^*$. The projection $\pi^{\Pi T^*\mathcal{M}}$ will simply be denoted by π in the rest of this proof.

If $(q_1, \ldots, q_{m+n})$ are local coordinates on $\mathcal{M}$ and $(p_1, \ldots, p_{m+n})$ are the associated fiber coordinates on $T^*\mathcal{M}$ with $|q_i| = |p_i| = 0$ resp. 1 for $1 \leq i \leq m$ resp. $m < i \leq m+n$, then $\Pi T^*\mathcal{M}$ has local coordinates (q_i, p_i^{Π}), where p_i^{Π} is odd resp. even for $1 \leq i \leq m$ resp. $m < i \leq m+n$. We obtain an element $\alpha \in \Omega^1_{\Pi T^*\mathcal{M}}(E)_1$ described as follows in the local coordinates (q_i, p_i^Π):

$$
\begin{aligned}
\alpha(X) &= (\mathrm{id} \otimes \chi) \circ (\mathrm{id} \otimes C) \circ L \circ d\pi^*(X) \\
&= (\mathrm{id} \otimes \chi) \circ (\mathrm{id} \otimes C) \circ L(X \circ \pi^*) \\
&= (\mathrm{id} \otimes \chi) \circ (\mathrm{id} \otimes C) \left(\sum_i (X \circ \pi^*)(q_i) \otimes \partial_{q_i} \right) \\
&= (\mathrm{id} \otimes \chi) \left(\sum_i (-1)^{|X|+|i|} X(q_i) \otimes C(\partial_{q_i}) \right), \text{ by identifying } \pi^*(q_i) \text{ with } q_i, \\
&= \sum_i (-1)^{|X|+|i|} X(q_i) \cdot \chi(C(\partial_{q_i})) \\
&= \sum_{i,j} (-1)^{|X|+|i|} X(q_i) \cdot p_j^\Pi \cdot \langle d(q_j), C(\partial_{q_i}) \rangle_\Pi \\
&= \sum_{i,j} (-1)^{|X|+|i|+|i|_\Pi |j|_\Pi} X(q_i) \cdot p_j^\Pi \cdot C(\partial_{q_i})(d(q_j)) \\
&= \sum_{i,j} (-1)^{|X|+|i|+|i|_\Pi |j|_\Pi + |i||j|_\Pi} X(q_i) \cdot p_j^\Pi \cdot d(q_j)(\partial_{q_i}) \\
&= \sum_i (-1)^{|X|+|i|+|i|_\Pi |i|_\Pi + |i||i|_\Pi + |i|} X(q_i) \cdot p_i^\Pi \\
&= \sum_i (-1)^{|X|+|i|+1} X(q_i) \cdot p_i^\Pi \\
&= \sum_i (-1)^{|i|+1+|i||X|} p_i^\Pi \cdot X(q_i) \\
&= \sum_i (-1)^{|i|+1} p_i^\Pi \cdot d(q_i)(X) \quad \forall X \in \mathcal{T}_{\Pi T^*\mathcal{M}}(E).
\end{aligned}
$$

We thus locally have $\alpha = \sum_i (-1)^{|i|+1} p_i^\Pi \cdot d(q_i)$.

Now we set $\omega := -d\alpha \in \Omega^2_{\Pi T^*\mathcal{M}}(E)_1$. Its local expression is:

$$
\begin{aligned}
\omega &= \sum_i (-1)^{|i|} d(p_i^\Pi) \wedge d(q_i) + \sum_i (-1)^{|i|+0\cdot 1+|i|_\Pi \cdot 0} p_i^\Pi \cdot d^2(q_i) \\
&= \sum_i (-1)^{|i|+1\cdot 1+|i|_\Pi |i|} d(q_i) \wedge d(p_i^\Pi) \\
&= \sum_i (-1)^{|i|+1} d(q_i) \wedge d(p_i^\Pi).
\end{aligned}
$$
□

We define the energy function H as in the even case (compare the text after Corollary 2.16) but it should be immediately stressed that H is an odd global superfunction on $\Pi T^*\mathcal{M}$.

If $\mathcal{M}$ is equipped with an odd Riemannian metric $g : \mathcal{T}_\mathcal{M} \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{T}_\mathcal{M} \to \mathcal{O}_\mathcal{M}$, we have the even isomorphisms $g^\flat : \mathcal{T}_\mathcal{M} \xrightarrow{\cong} \Pi\Omega^1_\mathcal{M}$ and $g^\natural : \Pi\Omega^1_\mathcal{M} \xrightarrow{\cong} \mathcal{T}_\mathcal{M}$ (see the discussions after Définition 1.13). Thus we have

$$
\Pi\Omega^1_\mathcal{M}(M) \otimes_{\mathcal{O}_\mathcal{M}} \Pi\Omega^1_\mathcal{M}(M) \xrightarrow[\cong]{g^\natural \otimes g^\natural} \mathcal{T}_\mathcal{M}(M) \otimes_{\mathcal{O}_\mathcal{M}} \mathcal{T}_\mathcal{M}(M) \xrightarrow{\quad g \quad} \mathcal{O}_\mathcal{M}(M),
$$

which determines an odd element h of $(\Pi\Omega^1_{\mathcal{M}}(M) \otimes_{\mathcal{O}_{\mathcal{M}}} \Pi\Omega^1_{\mathcal{M}}(M))^*$. The standard theory of linear superalgebra gives the following canonical even isomorphism

$$\left(\Pi\Omega^1_{\mathcal{M}}(M) \otimes_{\mathcal{O}_{\mathcal{M}}} \Pi\Omega^1_{\mathcal{M}}(M)\right)^* \cong \left(\Pi\Omega^1_{\mathcal{M}}(M)\right)^* \otimes_{\mathcal{O}_{\mathcal{M}}} \left(\Pi\Omega^1_{\mathcal{M}}(M)\right)^*.$$

This allows us to define $H := \frac{1}{2}\chi(E)(h) \in \mathcal{O}_{\Pi T^*\mathcal{M}}(E)$. Observe that the reduction $\widetilde{H}$ of the odd function H is the zero function on the body of $\Pi T^*\mathcal{M}$.

Lemma 2.32. *We obtain the following local identity*

$$H = \frac{1}{2}\sum_{i,j} p_i^{\Pi} \cdot g^{ij} \cdot p_j^{\Pi},$$

where (q_i, p_i^{Π}) are the coordinates on $\Pi T^*\mathcal{M}$ that are induced by coordinates (q_i) on $\mathcal{M}$.

Proof. At first, we express locally $g^\flat$ and $g^\natural$ in terms of (∂_i) and $(d(q_i))$, local basis of $\mathcal{T}_{\mathcal{M}}$ and $\Omega^1_{\mathcal{M}}$ respectively:

$$\begin{aligned} g^\flat(\partial_i) &= \sum_j (-1)^{|j|} g_{ij} \cdot d(q_j) = \sum_j (-1)^{|i|+1} g_{ij} \cdot_{\Pi} d(q_j) \\ g^\natural(d(q_i)) &= \sum_i (-1)^{|j|+1} g^{ij} \cdot \partial_j \end{aligned}$$

Let us denote $d(q_i)$ by d_i in the rest of this proof. Then $\left(d^*_{ij}\right)_{i,j}$ is a local basis of $(\Pi\Omega^1_{\mathcal{M}} \otimes \Pi\Omega^1_{\mathcal{M}})^*$ where $d^*_{ij} := (d_i \otimes d_j)^*$ is defined as follows

$$(d_i \otimes d_j)^*(d_k \otimes d_l) = (-1)^{|i|_\Pi |j|_\Pi} \delta_{i,k} \cdot \delta_{j,l}.$$

Using this basis, we have

$$\begin{aligned} h &= g \circ (g^\natural \otimes g^\natural) \\ &= \sum_{i,j} g \circ (g^\natural \otimes g^\natural)(d_i \otimes d_j)(-1)^{|i|_\Pi |j|_\Pi} d^*_{ij} \\ &= \sum_{i,j} (-1)^{|i|_\Pi |j|_\Pi} g(g^\natural(d_i) \otimes g^\natural(d_j)) \cdot d^*_{ij} \\ &= \sum_{\substack{i,j \\ k,l}} (-1)^{|i|_\Pi |j|_\Pi} g((-1)^{|k|+1} g^{ik} \cdot \partial_k, (-1)^{|l|+1} g^{jl} \cdot \partial_l) \cdot d^*_{ij} \\ &= \sum_{\substack{i,j \\ k,l}} (-1)^{|i|_\Pi |j|_\Pi + |i| + |l| + 1 + |l|(|j|+|l|+1)} g^{ik} \cdot g_{kl} \cdot g^{jl} \cdot d^*_{ij} \\ &= \sum_{i,j} (-1)^{|i|+|j|} g^{ji} \cdot d^*_{ij} \end{aligned}$$

Now we have the following natural morphism

$$\left(\Pi\Omega^1_{\mathcal{M}}(M)\right)^* \otimes \left(\Pi\Omega^1_{\mathcal{M}}(M)\right)^* \xrightarrow{A} \left(\Pi\Omega^1_{\mathcal{M}}(M) \otimes \Pi\Omega^1_{\mathcal{M}}(M)\right)^*$$

defined as follows

$$A(\varphi_1 \otimes \varphi_2)(\omega_1 \otimes \omega_2) := (-1)^{|\omega_1|_\Pi |\varphi_2|_\Pi} \varphi_1(\omega_1) \otimes \varphi_2(\omega_2)$$

where φ_i and ω_i are homogeneous elements of $(\Omega^1_{\mathcal{M}}(M))^*$ and $\Omega^1_{\mathcal{M}}(M)$ respectively. Remark that $\Omega^1_{\mathcal{M}}$ is a locally free sheaf of $\mathcal{O}_{\mathcal{M}}$-module of finite rank (see Proposition 1.6) and thus A is an isomorphism. The inverse of A is locally given by

$$B(d^*_{ij}) = d^*_i \otimes d^*_j$$

where $d^*_i = (d_i)^*$ is defined by $(d_i)^*(d_j) = \delta_{ij}$. Indeed,

$$A(d^*_i \otimes d^*_j)(d_k \otimes d_l) = (-1)^{|j|_\Pi |k|_\Pi} d^*_i(d_k) d^*_j(d_l) = (d^*_{ij})(d_k \otimes d_l).$$

Now the element h, viewed in $(\Pi\Omega^1_{\mathcal{M}}(M))^* \otimes (\Pi\Omega^1_{\mathcal{M}}(M))^*$, is

$$B(h) = \sum_{i,j} (-1)^{|i|+|j|} g^{ji} \cdot d^*_i \otimes d^*_j.$$

Finally, in coordinates the energy function is given by

$$\begin{aligned}
H &= \frac{1}{2}\chi(B(h)) \\
&= \frac{1}{2}\sum_{i,j}(-1)^{|i|+|j|} g^{ji} \chi\left(d^*_i \otimes d^*_j\right) \\
&= \frac{1}{2}\sum_{i,j}(-1)^{|i|+|j|} g^{ji} \chi(d^*_i) \otimes \chi(d^*_j) \\
&= \frac{1}{2}\sum_{\substack{i,j\\k,l}}(-1)^{|i|+|j|} g^{ji} \cdot p^\Pi_k \cdot \langle d_k, d^*_i\rangle_\Pi \cdot p^\Pi_l \cdot \langle d_l, d^*_j\rangle_\Pi \\
&= \frac{1}{2}\sum_{\substack{i,j\\k,l}}(-1)^{|i|+|j|} g^{ji} \cdot p^\Pi_k (-1)^{|i|_\Pi |k|_\Pi} \delta_{i,k} \cdot p^\Pi_l (-1)^{|l|_\Pi |j|_\Pi} \delta_{l,j} \\
&= \frac{1}{2}\sum_{i,j} g^{ji} \cdot p^\Pi_i \cdot p^\Pi_j \\
&= \frac{1}{2}\sum_{i,j} p^\Pi_i \cdot g^{ij} \cdot p^\Pi_j \text{ since } g^{ji} = (-1)^{|i||j|+|j|+|i|+1} g^{ij} \text{ by the lemma 2.6} \qquad \square
\end{aligned}$$

Since ω and H both are odd, the Hamiltonian vector field X_H on $\Pi T^*\mathcal{M}$ fulfilling $d(H) = i(X_H)\omega$ is even.

Proposition 2.33. *The Hamiltonian vector field $X_f \in \mathcal{T}_{\Pi T^*\mathcal{M}}(E)$ associated to an odd function $f \in \mathcal{O}_{\Pi T^*\mathcal{M}}(E)_1$ is an even vector field and it is locally given by the following equations:*

$$\begin{aligned}
X_f(q_i) &= (-1)^{|i|+1} \partial_{p^\Pi_i} f \\
X_f(p^\Pi_i) &= (-1)^{|i|} \partial_{q_i} f.
\end{aligned}$$

Proof. Let X and Y be vector fields on $\Pi T^*\mathcal{M}$. We have

$$\begin{aligned}
\omega(X,Y) &= \sum_j (-1)^{|j|+1} d(q_j) \wedge d(p_j^\Pi)(X,Y) \\
&= \sum_j (-1)^{|j|+1} \Big[(-1)^{|X||j|_\Pi} d(q_j)(X) \cdot d(p_j^\Pi)(Y) + \\
&\qquad\qquad (-1)^{1\cdot 1+|X||Y|+|Y||j|_\Pi} d(q_j)(Y) \cdot d(p_j^\Pi)(X) \Big] \\
&= \sum_j (-1)^{|j|+1} \Big[(-1)^{|X|+|Y||j|_\Pi} X(q_j) \cdot Y(p_j^\Pi) - \\
&\qquad\qquad (-1)^{|X||Y|+|Y|+|X||j|_\Pi} Y(q_j) \cdot X(p_j^\Pi) \Big]
\end{aligned}$$

Then we get $\omega(X_f, \frac{\partial}{\partial p_i^\Pi}) = X_f(q_i)$ and $\omega(X_f, \frac{\partial}{\partial q_i}) = X(p_i^\Pi)$. And on the other hand we have $\omega(X_f, \frac{\partial}{\partial p_i^\Pi}) = (-1)^{|f||i|_\Pi} \partial_{p_i^\Pi} f$ and $\omega(X_f, \frac{\partial}{\partial q_i}) = (-1)^{|f||i|} \partial_{q_i} f$. □

Corollary 2.34. *In the situation of the preceding proposition, we have for H, the energy function, the following equations:*

$$X_H(q_i) = \sum_j (-1)^{|i|+1} p_j^\Pi \cdot g^{ji} \tag{2.24}$$

$$X_H(p_i^\Pi) = \frac{1}{2} \sum_{k,j} (-1)^{|i||k|} p_k^\Pi \cdot \partial_{q_i}(g^{kj}) \cdot p_j^\Pi. \tag{2.25}$$

Proof. Let us calculate $X_H(q_i)$ and $X_H(p_i^\Pi)$ using the local expression of H given by the lemma 2.32. Then we get

$$\begin{aligned}
\partial_{p_i^\Pi} H &= \frac{1}{2} \sum_{k,j} \partial_{p_i^\Pi} (p_k^\Pi \cdot g^{kj} \cdot p_j^\Pi) \\
&= \frac{1}{2} \sum_{k,j} \Big[\partial_{p_i^\Pi}(p_k^\Pi) \cdot g^{kj} \cdot p_j^\Pi + (-1)^{|k|_\Pi |i|_\Pi + |i|_\Pi(|k|+|j|+1)} p_k^\Pi \cdot g^{kj} \cdot \partial_{p_i^\Pi} p_j^\Pi \Big] \\
&= \frac{1}{2} \sum_j g^{ij} \cdot p_j^\Pi + \frac{1}{2} \sum_k p_k^\Pi \cdot g^{ki} \\
&= \sum_j p_j^\Pi \cdot g^{ji}
\end{aligned}$$

and

$$\begin{aligned}
\partial_{q_i} H &= \frac{1}{2} \sum_{k,j} \partial_{q_i} (p_k^\Pi \cdot g^{kj} \cdot p_j^\Pi) \\
&= \frac{1}{2} \sum_{k,j} (-1)^{(|k|+1)|i|} p_k^\Pi \cdot \partial_{q_i}(g^{kj}) \cdot p_j^\Pi.
\end{aligned}$$

We conclude from the preceding proposition that for all i the equations (2.24) and (2.25) hold for all i. □

Using Theorem 2.24 and Proposition 2.27 we immediately get following

Proposition 2.35. *Let $(\mathcal{M}, g)$ be an odd Riemannian supermanifold and X_H the Hamiltonian vector field associated to the energy function H on the (odd) symplectic supermanifold $(\Pi T^*\mathcal{M}, \omega^{\Pi T^*\mathcal{M}})$. Then*

- *(i) there exists a unique flow for X_H, $F : \mathbb{R} \times \Pi T^*\mathcal{M} \to \Pi T^*\mathcal{M}$,*
- *(ii) given a supermanifold $\mathcal{S}$ and an element $\mathcal{I} \in Mor(\mathcal{S}, \Pi T^*\mathcal{M})$, there is a unique integral curve $\Psi : \mathbb{R} \times \mathcal{S} \to \Pi T^*\mathcal{M}$ of X_H such that the initial condition $\Psi \circ inj_{t=0} = \mathcal{I}$ is satisfied.*

As in the case of even metrics we finally obtain a bijection between integral curves of the geodesic flow and geodesics. Since the proof is very similar to the proof of Theorem 2.28 we only announce the result without giving the proof.

Theorem 2.36. *Let $(\mathcal{M}, g)$ be an odd Riemannian supermanifold and X_H the Hamiltonian vector field associated to the energy function H on $\Pi T^*\mathcal{M}$. Let furthermore $\mathcal{S}$ be an arbitrary supermanifold. Then*

- *(i) given $\mathcal{I} \in Mor(\mathcal{S}, \Pi T^*\mathcal{M})$, the integral curve Ψ of X_H with initial condition $\mathcal{I}$ projects to the geodesic $\Phi := \pi^{\Pi T^*\mathcal{M}} \circ \Psi$ with initial condition $G^\natural \circ \mathcal{I}$, and*
- *(ii) given $\mathcal{J} \in Mor(\mathcal{S}, T\mathcal{M})$, the supergeodesic Φ with initial condition $\mathcal{J}$ yields the integral curve $\Psi := G^\flat \circ \partial_t \Phi$ with initial condition $G^\flat \circ \mathcal{J}$.*

2.4 The exponential map of a Riemannian supermanifold

In this section we study, for q in the body of a supermanifold $\mathcal{M}$, the fiber $T_q\mathcal{M}$ of the tangent bundle over q, the restriction of the tangent map of a morphism to this fiber, and its relation to the "numerical tangent map" (Lemma 2.38). The main result of this section is that a Riemannian supermanifold possesses a (Riemannian) exponential map and that $\exp_q : T_q\mathcal{M} \to \mathcal{M}$ is a diffeomorphism near 0 (Theorem 2.40). As an example of an application we show that linearization in a fixed point of an isometry of a Riemannian supermanifold is faithful (Corollary 2.46).

2.4.1 Definition and Properties of the exponential map

Let $(\mathcal{M}, g)$ be an even Riemannian supermanifold, and X_H be the even Hamiltonian vector field on $T^*\mathcal{M}$ associated to the energy function H. Let furthermore $F : \mathbb{R} \times T^*\mathcal{M} \to T^*\mathcal{M}$ be the flow of X_H and $G^\flat : T\mathcal{M} \to T^*\mathcal{M}$ be the isomorphism of supermanifolds induced by the metric g.

Consider the following composition of applications:

$$\mathbb{R}\times T\mathcal{M} \xrightarrow{id_{\mathbb{R}}\times G^{\flat}} \mathbb{R}\times T^{*}\mathcal{M} \xrightarrow{\ \ F\ \ } T^{*}\mathcal{M} \xrightarrow{\ \ \pi^{T^{*}\mathcal{M}}\ \ } \mathcal{M}.$$

We want to take the fibre of $T\mathcal{M}$ above $q\in M$:

$$\begin{tikzcd} & T\mathcal{M} \arrow[d, "\pi^{T\mathcal{M}}"] \\ \{q\} \arrow[r, "q"] & \mathcal{M} \end{tikzcd}$$

Let us recall the morphism of rings of global sections $(\pi^{T\mathcal{M}})^{*}(TM) : \mathcal{O}_{\mathcal{M}}(M) \to \mathcal{O}_{T\mathcal{M}}(TM)$, associated to the sheaf morphism $(\pi^{T\mathcal{M}})^{*} : (\widetilde{\pi^{T\mathcal{M}}})^{-1}(\mathcal{O}_{\mathcal{M}}) \to \mathcal{O}_{T\mathcal{M}}$ over TM.

Let $\mathcal{J}_q$ be the ideal of $\mathcal{O}_{\mathcal{M}}(M)$ defined by

$$\mathcal{J}_q := \{f\in\mathcal{O}_{\mathcal{M}}(M) \mid q^{*}(f)=0\},$$

where $q^{*}(f)=\widetilde{f}(q)$. We denote the ideal in $\mathcal{O}_{T\mathcal{M}}(TM)$ generated by $(\pi^{T\mathcal{M}})^{*}(TM)(\mathcal{J}_q)$ by

$$\mathcal{J}_q^{\pi^{T\mathcal{M}}} := \langle(\pi^{T\mathcal{M}})^{*}(TM)(\mathcal{J}_q))\rangle_{\mathcal{O}_{T\mathcal{M}}(TM)}.$$

Then (compare [14] for this construction in general), there exists a subsupermanifold $\mathcal{S}\overset{i_q}{\hookrightarrow} T\mathcal{M}$ of $T\mathcal{M}$ such that

$$\mathcal{J}_{\mathcal{S}} := \{f\in\mathcal{O}_{T\mathcal{M}}(TM) \mid i_q^{*}(f)=0\} = \mathcal{J}_q^{\pi^{T\mathcal{M}}}.$$

The supermanifold $\mathcal{S}$ will be denoted by $T_q\mathcal{M}$.

Remarks.

(1) Let f be an element of $\mathcal{O}_{\mathcal{M}}(M)$ viewed as an element of $\mathcal{O}_{T\mathcal{M}}(TM)$ via $(\pi^{T\mathcal{M}})^{*}$, we have $i_q^{*}(f)=\widetilde{f}(q)$. Indeed, $(\pi^{T\mathcal{M}})^{*}(f-\widetilde{f}(q))$ is an element of $\mathcal{J}_q^{\pi^{T\mathcal{M}}}$ then $i_q^{*}(f-\widetilde{f}(q))=0$.

(2) If (q_i, v_i) are local coordinates on $T\mathcal{M}$ then $(v_{i,q}) := ((i_q)^{*}(v_i))$ are coordinates on $T_q\mathcal{M}$.

(3) If $\mathcal{M}=\mathbb{R}^{m|n}$ and q is an element of $\mathbb{R}^{m}$, we have $T\mathcal{M}=\mathbb{R}^{m|n}\times\mathbb{R}^{m|n}$ and $T_q\mathcal{M}=\mathbb{R}^{m|n}\times\{q\}\cong\mathbb{R}^{m|n}$ (Compare the Remark after Theorem 2.12). In general, given

a finite dimensional $\mathbb{Z}_2$-graded $\mathbb{R}$-vector space $V = V_0 \oplus V_1$, we have of course an associated "linear supermanifold". Then $T_q\mathcal{M}$ is the supermanifold associated to the $\mathbb{Z}_2$-graded $\mathbb{R}$-vector space $T_q^{\mathrm{num}}\mathcal{M}$.

We now define the map $\exp_q : T_q\mathcal{M} \to \mathcal{M}$ via the following diagram:

$$\begin{tikzcd}
\mathbb{R}\times T\mathcal{M} \arrow[r, "id_{\mathbb{R}}\times G^{\flat}"] & \mathbb{R}\times T^*\mathcal{M} \arrow[r, "F"] & T^*\mathcal{M} \arrow[r, "\pi^{T^*\mathcal{M}}"] & \mathcal{M} \\
\mathbb{R}\times T_q\mathcal{M} \arrow[u, "id_{\mathbb{R}}\times i_q"] & & & \\
\{1\}\times T_q\mathcal{M}\cong T_q\mathcal{M} \arrow[u, "1\times id_{T_q\mathcal{M}}"] \arrow[uurrr, "\exp_q"'] & & &
\end{tikzcd}$$

Recall that the flow F is only defined on an open subsupermanifold of $\mathbb{R}\times T^*\mathcal{M}$ containing $\{0\}\times T^*\mathcal{M}$. Then $\exp_q$ is in general only defined on an open subsupermanifold of $T_q\mathcal{M}$ containing in its body $(0, q)$.

Remark. The exponential map of an odd Riemannian supermanifold is constructed similarly. Recall that in the case of an odd Riemannian supermanifold $(\mathcal{M}, g)$, the flow of X_H, the Hamiltonian vector field associated to the odd energy function H induced by the metric g, is given by a morphism $F : \mathbb{R}\times \Pi T^*\mathcal{M} \to \Pi T^*\mathcal{M}$. We then define

$$\begin{tikzcd}
\mathbb{R}\times T\mathcal{M} \arrow[r, "id_{\mathbb{R}}\times G^{\flat}"] & \mathbb{R}\times \Pi T^*\mathcal{M} \arrow[r, "F"] & \Pi T^*\mathcal{M} \arrow[r, "\pi^{\Pi T^*\mathcal{M}}"] & \mathcal{M} \\
\mathbb{R}\times T_q\mathcal{M} \arrow[u, "id_{\mathbb{R}}\times i_q"] & & & \\
\{1\}\times T_q\mathcal{M}\cong T_q\mathcal{M} \arrow[u, "1\times id_{T_q\mathcal{M}}"] \arrow[uurrr, "\exp_q"'] & & &
\end{tikzcd}$$

for all $q \in M$.

We can now show that $\exp_q$ is a diffeomorphism near $0 \in T_qM$. First of all, let us explain the process of taking the restriction of the tangent map of a morphism to a fiber.

Lemma 2.37. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds. Let $q \in M$, then we have a unique morphism of supermanifolds $T_q\Phi : T_q\mathcal{M} \to T_{\widetilde{\Phi}(q)}\mathcal{N}$ such that $i_{\widetilde{\Phi}(q)} \circ T_q\Phi = T\Phi \circ i_q$.*

Proof. We have the following commutative diagram:

$$\begin{tikzcd} T_q\mathcal{M} \arrow[r, "i_q"] \arrow[d] & T\mathcal{M} \arrow[r, "T\Phi"] \arrow[d, "\pi^{T\mathcal{M}}"] & T\mathcal{N} \arrow[d, "\pi^{T\mathcal{N}}"] & T_p\mathcal{N} \arrow[l, "i_p"'] \arrow[d] \\ \{q\} \arrow[r] & \mathcal{M} \arrow[r, "\Phi"] & \mathcal{N} & \{p\}, \arrow[l] \end{tikzcd}$$

where $p := \widetilde{\Phi}(q) \in N$. We will show that the morphism $T\Phi \circ i_q$ factorizes over the morphism i_p.

Recall that a morphism of real smooth supermanifolds is completely determined by the superalgebra morphism given by pulling-back global sections.

Let us define the morphism $T_q\Phi : T_q\mathcal{M} \to T_p\mathcal{N}$ by setting its body map equal to $\widetilde{T_q\Phi} := T_q\widetilde{\Phi}$ and on the level of global sections by

$$\begin{aligned} (T_q\Phi)^* : \mathcal{O}_{T_p\mathcal{N}}(T_pN) &\longrightarrow \mathcal{O}_{T_q\mathcal{M}}(T_qM) \\ f &\longmapsto i_q^* \circ (T\Phi)^*(\hat{f}\,), \end{aligned}$$

where $\hat{f}$ is an element of $\mathcal{O}_{T\mathcal{N}}(TN)$ such that $i_p^*(\hat{f}\,) = f$.

For all $\hat{f} \in \mathcal{O}_{T\mathcal{N}}(TN)$ such that $i_p^*(\hat{f}\,) = 0$, we have $\hat{f} \in \mathcal{J}_{T_p\mathcal{N}} = \mathcal{J}_p^{\pi^{T\mathcal{N}}}$ by the very construction of $T_p\mathcal{N}$ and then by definition of $\mathcal{J}_p^{\pi^{T\mathcal{N}}}$:

$$\hat{f} \in \langle(\pi^{T\mathcal{N}})^*(\mathcal{J}_p)\rangle_{\mathcal{O}_{T\mathcal{N}}(TN)}.$$

Thus it is enough to show that $i_q \circ (T\Phi)^*(\hat{f}\,) = 0$ whenever $\hat{f} = (\pi^{T\mathcal{N}})^*(g)$ with $g \in \mathcal{J}_p$ in order to assure that $(T_q\Phi)^*$ is well-defined.
For $g \in \mathcal{J}_p \lhd \mathcal{O}_{\mathcal{N}}(N)$ we have

$$(T\Phi)^*\left((\pi^{T\mathcal{N}})^*(g)\right) = (\pi^{T\mathcal{M}})^*(\Phi^*(g)) \in \langle(\pi^{T\mathcal{M}})^*(\mathcal{J}_q)\rangle_{\mathcal{O}_{T\mathcal{M}}(TM)},$$

since the identity $\widetilde{\Phi^*(h)}(q) = \widetilde{h}(p)$ for $h \in \mathcal{O}_{\mathcal{N}}(N)$ shows that $g \in \mathcal{J}_p$ implies that $\Phi^*(g) \in \mathcal{J}_q$.
Moreover, since $\mathcal{J}_{T_q\mathcal{M}} = \langle(\pi^{T\mathcal{M}})^*(\mathcal{J}_q)\rangle_{\mathcal{O}_{T\mathcal{M}}(TM)}$ we have $i_q^*\left((\pi^{T\mathcal{M}})^*(\Phi^*(g))\right) = 0$, i.e., $i_q^* \circ (T\Phi)^*(\hat{f}\,) = 0$. $\square$

Let us recall that for a morphism of supermanifolds $\Phi : \mathcal{M} \to \mathcal{N}$, and a point $q \in M$, we have a numerical analogue of the tangent map, i.e. a linear map $T_q^{\mathrm{num}}\Phi : T_q^{\mathrm{num}}\mathcal{M} \to$

$T^{\mathrm{num}}_{\widetilde{\Phi}(q)}\mathcal{N}$ defined by

$$T^{\mathrm{num}}_q\Phi(X)\big([f]_{\widetilde{\Phi}(q)}\big) := X\big([\Phi^*(f)]_q\big), \quad \forall X \in T^{\mathrm{num}}_q\mathcal{M}, \quad \forall [f]_{\widetilde{\Phi}(q)} \in \mathcal{O}_{\mathcal{N},\widetilde{\Phi}(q)}.$$

It turns out that this map encodes exactly the same information as the morphism $T_q\Phi$.

Lemma 2.38. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism of supermanifolds, $q \in M$ and $p := \widetilde{\Phi}(q) \in N$. Then the following holds:*

(i) $T_q\Phi : T_q\mathcal{M} \to T_p\mathcal{N}$ is the morphism of (linear) supermanifolds associated to the even $\mathbb{R}$-linear map $T^{num}_q\Phi : T^{num}_q\mathcal{M} \to T^{num}_p\mathcal{N}$, and

(ii) the numerical tangent map $T^{num}_q\Phi$ is uniquely determined by the morphism of supermanifolds $T_q\Phi : T_q\mathcal{M} \to T_p\mathcal{N}$.

Proof. A calculation in local coordinates on $T\mathcal{M}$ and $T\mathcal{N}$, that are induced from local coordinates on $\mathcal{M}$ and $\mathcal{N}$, directly shows that both maps $T_q\Phi$ and $T^{\mathrm{num}}_q\Phi$ are uniquely determined by the evaluation of the super Jacobi matrix of Φ in the point q of the body of $\mathcal{M}$. Indeed, we have

$$\begin{aligned}
(T_q\Phi)^*(v^{\mathcal{N}}_{i,p}) &= i_q^* \circ (T\Phi)^*(\widehat{v^{\mathcal{N}}_{i,p}}) \\
&= i_q^* \circ (T\Phi)^*(v^{\mathcal{N}}_i) \\
&= i_q^*\left(\sum_j v^{\mathcal{M}}_j \cdot \partial_{q^{\mathcal{M}}_j}\Phi^*(q^{\mathcal{N}}_i)\right) \\
&= \sum_j v^{\mathcal{M}}_{j,q} \cdot \Big(\widetilde{\partial_{q^{\mathcal{M}}_j}\Phi^*(q^{\mathcal{N}}_i)}\Big)(q),
\end{aligned}$$

by the remark given page 58, and

$$\begin{aligned}
T^{\mathrm{num}}_q\Phi(\partial_{q^{\mathcal{M}}_i,q})\big([q^{\mathcal{N}}_j]_p\big) &= \partial_{q^{\mathcal{M}}_i,q}\big([\Phi^*(q^{\mathcal{N}}_j)]_q\big) \\
&= \Big(\widetilde{\partial_{q^{\mathcal{M}}_i}\Phi^*(q^{\mathcal{N}}_j)}\Big)(q).
\end{aligned}$$

□

It is well-known that a morphism $\Phi : \mathcal{M} \to \mathcal{N}$ is a diffeomorphism near q if and only if $T^{\mathrm{num}}_q\Phi$ is an isomorphism of $\mathbb{Z}_2$-graded vector space. (see, e.g., [24], Theorem 4.4.1)

Using the preceding lemma, we thus immediately get:

Proposition 2.39. *The morphism $\Phi : \mathcal{M} \to \mathcal{N}$ is locally a diffeomorphism near $q \in M$ if and only if $T_q\Phi : T_q\mathcal{M} \to T_{\widetilde{\Phi}(q)}\mathcal{N}$ is an isomorphism of supermanifolds.*

Theorem 2.40. *Let $(\mathcal{M}, g)$ be an even Riemannian supermanifold, the map $exp_q : T_q\mathcal{M} \to \mathcal{M}$ constructed above is a diffeomorphism near 0.*

Proof. We will show that the map $T_0(\exp_q) : T_0(T_q\mathcal{M}) \to T_q\mathcal{M}$ is an isomorphism. More precisely, after identifying $T_0(T_q\mathcal{M})$ with $T_q\mathcal{M}$ as in the classical, ungraded, case, we will see that $T_0(\exp_q) = \mathrm{id}_{T_q\mathcal{M}}$.

Let us focus on the map:

$$\begin{array}{rcl} T_0^{\mathrm{num}}(\exp_q) : T_0^{\mathrm{num}}(T_q\mathcal{M}) = \mathrm{Der}\left((\mathcal{O}_{T_q\mathcal{M},0}), \mathbb{R}\right) & \longrightarrow & \mathrm{Der}\left((\mathcal{O}_{\mathcal{M},q}), \mathbb{R}\right) = T_q^{\mathrm{num}}\mathcal{M} \\ X & \longmapsto & X \circ \exp_q^* \end{array}.$$

If (q_i) is a system of local coordinates on $\mathcal{M}$ around q, we note (q_i, v_i) the induced local coordinates on $T\mathcal{M}$. Let us denote the natural coordinates on $T_q\mathcal{M}$ by (v_i) as well. Then $T_0^{\mathrm{num}}(T_q\mathcal{M})$ is the $\mathbb{Z}_2$-graded $\mathbb{R}$-vector space spanned by (∂_{v_i}) where $|\partial_{v_i}| = |v_i|$.

Les us first consider $X = \partial_{v_j} \in \mathrm{Der}\left((\mathcal{O}_{T_q\mathcal{M},0}), \mathbb{R}\right)_0 \cong T_qM$ with $|v_j| = 0$. Then the morphism $\gamma : \mathbb{R} \to T_q\mathcal{M}$ given by

$$\gamma^*(v_i) = t \cdot \delta_{i,j}$$

is an integral curve of X, where t is the canonical coordinate of $\mathbb{R}$. Indeed we have

$$X(f) = \frac{d}{dt}_{|t=0} \gamma^*(f) \quad \forall f \in \mathcal{O}_{T_q\mathcal{M}}.$$

Using this integral curve we get

$$\begin{array}{rcl} T_0^{\mathrm{num}}(\exp_q)(X) & = & \dfrac{d}{dt}_{|t=0} \gamma^* \circ \exp_q^* \\ & = & \dfrac{d}{dt}_{|t_2=0} \gamma^* \circ \left(\mathrm{inj}^*_{t_1=1} \times \left(i_q^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^* \\ & = & \dfrac{d}{dt}_{|t_2=0} \left(\mathrm{inj}^*_{t_1=1} \times \left(\gamma^* \circ i_q^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^* \end{array}$$

Remark that the map γ factorizes as follow

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\gamma"] \arrow[dr, "\iota"'] & T_q\mathcal{M} \\ & \mathbb{R} \times T_q\mathcal{M} \arrow[u, "\overline{m}_q"'] \end{tikzcd}$$

where $\iota : \mathbb{R} \to \mathbb{R} \times T_q\mathcal{M}$ has $t_2 \mapsto (t_2, v_j)$ as underlying map and $\overline{m}_q : \mathbb{R} \times T_q\mathcal{M} \to T_q\mathcal{M}$ is defined by $(\overline{m}_q)^*(v_i) = t_2.v_i$ for all i. Then we have

$$T_0^{\mathrm{num}}(\exp_q)(X) \;\; = \;\; \frac{d}{dt}_{|t_2=0} \left(\mathrm{inj}^*_{t_1=1} \times \left(\iota^* \circ (\overline{m}_q)^* \circ i_q^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^*.$$

Recall that in the case of classical (ungraded) manifolds, the flow $\widetilde{F}$ of a vector field $\widetilde{X}$ satisfies $\widetilde{F}(s, t\cdot v) = \widetilde{F}(s\cdot t, v)$ for all $s, t \in \mathbb{R}$ and $v \in T_qM$ provided that it is defined. Before proceeding further we need to verify that the flow F of X has a similar behavior with $\overline{m}$.

Lemma 2.41. *If $m : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the map defined by $m^*(t) = t_1 \cdot t_2$, we have*

$$\pi^{T^*\mathcal{M}} \circ F \circ \Big(id_{\mathbb{R}} \times (G^\flat \circ i_q \circ \overline{m}_q)\Big) = \pi^{T^*\mathcal{M}} \circ F \circ \Big(m \times (G^\flat \circ i_q)\Big)$$

as morphisms from $\mathbb{R} \times (\mathbb{R} \times T_q\mathcal{M}) \cong (\mathbb{R}\times\mathbb{R}) \times T_q\mathcal{M}$ to $\mathcal{M}$.

Proof. Let us consider the following supercurve on $\mathcal{M}$ parametrized by $\mathbb{R} \times T_q\mathcal{M}$:

$$\Gamma := \pi^{T^*\mathcal{M}} \circ F \circ \Big(m \times (G^\flat \circ i_q)\Big) : (\mathbb{R}\times\mathbb{R}) \times T_q\mathcal{M} \cong \mathbb{R} \times (\mathbb{R} \times T_q\mathcal{M}) \to \mathcal{M}.$$

Then we have

$$\frac{\nabla}{dt_1}(d\Gamma^*(\partial_{t_1})) = t_2^2 \cdot m^* \circ \frac{\nabla}{dt}(d\Psi^*(\partial_t)),$$

where $\Psi := \pi^{T^*\mathcal{M}} \circ F \circ \Big(\mathrm{id}_{\mathbb{R}} \times (G^\flat \circ i_q)\Big)$.

By Theorem 2.28, Ψ is the geodesic on $\mathcal{M}$ subject to the condition $\partial_t\Psi \circ \mathrm{inj}_{t=0} = i_q$. It follows that Γ is a geodesic as well subject to the initial condition:

$$\begin{aligned}
(\partial_{t_1}\Gamma \circ \mathrm{inj}_{t_1=0})^*(q_i) &= (\mathrm{inj}_{t_1=0})^* \circ \Gamma^*(q_i) \\
&= (((\mathrm{inj}_{t_1=0})^* \circ m^*) \times \mathrm{id}^*_{T\mathcal{M}}) \circ \Psi^*(q_i) \\
&= ((\mathrm{inj}_{t=0})^* \times \mathrm{id}^*_{T\mathcal{M}}) \circ \Psi^*(q_i) \\
&= i_q^*(q_i) = (\overline{m}_q)^* \circ i_q^*(q_i)
\end{aligned}$$

and

$$\begin{aligned}
(\partial_{t_1}\Gamma \circ \mathrm{inj}_{t_1=0})^*(v_i) &= (\mathrm{inj}_{t_1=0})^* \circ \partial_{t_1}(\Gamma^*(q_i)) \\
&= (\mathrm{inj}_{t_1=0})^* \left(t_2 \cdot (m^* \times \mathrm{id}^*_{T\mathcal{M}}) \circ \partial_t(\Psi(q_i))\right) \\
&= t_2 \cdot (\mathrm{inj}_{t=0})^* \circ \partial_t(\Psi(q_i)) \\
&= t_2 \cdot i_q^*(v_i) = (\overline{m}_q)^* \circ i_q^*(v_i),
\end{aligned}$$

that is $\partial_{t_1}\Gamma \circ \mathrm{inj}_{t=0} = i_q \circ \overline{m}_q$.

We conclude by the theorem 2.29 that $\Gamma = \pi^{T^*\mathcal{M}} \circ F \circ \Big(\mathrm{id}_{\mathbb{R}} \times (G^\flat \circ i_q \circ \overline{m}_q)\Big)$. □

Proof of Theorem 2.40 (resumed).
Returning to our calculation of $T_0^{\mathrm{num}}(\exp_q)(\partial_{v_i})$, where $|v_i| = 0$, we get

$$\begin{aligned} T_0^{\mathrm{num}}(\exp_q)(X) &= \frac{d}{dt}_{|t_2=0} \left(\mathrm{inj}^*_{t=1} \times \left(\iota^* \circ i_q^* \circ (\overline{m}_q)^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^* \\ &= \frac{d}{dt}_{|t_2=0} \left(\mathrm{inj}^*_{t=1} \circ m^* \times \left(\iota^* \circ i_q^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^* \\ &= \frac{d}{dt}_{|t_2=0} \left(\mathrm{id}^*_{\mathbb{R}} \times (\iota^* \circ i_q^* \circ G^{\flat *})\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^*, \end{aligned}$$

since $m \circ \mathrm{inj}_{t_1=1} = \mathrm{id}_{\mathbb{R}}$.
Finally, the derivation $T_0^{\mathrm{num}}(\exp_q)(X)$ in $\mathrm{Der}\left((\mathcal{O}_{\mathcal{M},q}), \mathbb{R}\right)$ is given by

$$T_0^{\mathrm{num}}(\exp_q)(X)(q_i) = \iota^* \circ i_q^*(v_i) = \delta_{i,j} \quad \forall i,$$

meaning that $T_0^{\mathrm{num}}(\exp_q)(X) = \partial_{q_j} = v_j$.
Then we have

$$T_0^{\mathrm{num}}(\exp_q)(\partial_v) = v \quad \forall v \in (T_q^{\mathrm{num}}\mathcal{M})_0.$$

Of course, this result is trivially predicted by the classical theory of exponential maps on ungraded Riemannian manifolds. Indeed, the morphism $T_0^{\mathrm{num}}(\exp_q)$, restricted to even tangent vector, is nothing more than the differential at 0 of the exponential map on $(M, \widetilde{g})$ the underlying Riemannian manifold of $(\mathcal{M}, g)$. But rewritting the classical proof sheaf theoretically allows to treat the case of odd Riemannian metrics in the sequel without any serious change.

Let us now deal with odd tangent vectors. If $X = \partial_{v_j}$ with $|v_j| = 1$, we have an "integral curve" $\gamma : \mathbb{R}^{0|1} \to T_q\mathcal{M}$ of X defined by $\gamma^*(v_i) = \delta_{i,j} \cdot \theta$, where θ is the canonical coordinate of $\mathbb{R}^{0|1}$, that is

$$X(f) = \frac{d}{d\theta}\gamma^*(f) \quad \forall f \in \mathcal{O}_{T_q\mathcal{M}}.$$

We then have

$$T_0^{\mathrm{num}}(\exp_q)(X) = \frac{d}{d\theta}\gamma^* \circ \exp_q^*,$$

where $\gamma^* \circ \exp_q^* = \gamma^* \circ \left(\mathrm{inj}^*_{t=1} \times \left(i_q^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^* = \left(\mathrm{inj}^*_{t=1} \times \left(\gamma^* \circ i_q^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^*$.
Let us consider the map $\Lambda : \mathbb{R}^{1|1} \to \mathcal{M}$ defined by:

$$\Lambda^* := \left(\mathrm{id}^*_{\mathbb{R}} \times \left(\gamma^* \circ i_q^* \circ G^{\flat *}\right)\right) \circ F^* \circ \left(\pi^{T^*\mathcal{M}}\right)^*.$$

Then, using Theorem 2.28, Λ is the unique geodesic of $\mathcal{M}$ with initial condition $\mathcal{J} = i_q \circ \gamma \in \mathrm{Mor}\left(\mathbb{R}^{0|1}, T\mathcal{M}\right)$, i.e. Λ is the supercurve such that

$$\begin{cases} \partial_t^2\Lambda^*(q_k) + \sum_{|u|=|l|=0} \partial_t\Lambda^*(q_u)\partial_t\Lambda^*(q_l)\Lambda^*(\Gamma^k_{lu}) = 0 & \text{if } |q_k| = 0, \\ \partial_t^2\Lambda^*(q_k) + \sum_{\substack{|u|=0\\|l|=0}} \partial_t\Lambda^*(q_u)\partial_t\Lambda^*(q_l)\Lambda^*(\Gamma^k_{lu}) + 2\sum_{\substack{|u|=0\\|l|=1}} \partial_t\Lambda^*(q_u)\partial_t\Lambda^*(q_l)\Lambda^*(\Gamma^k_{lu}) = 0 & \text{if } |q_k| = 1, \end{cases}$$

satisfying the initial condition

$$\begin{cases} \Lambda^*(q_k)_{|t=0} = \widetilde{q_k}(q) & \text{if } |q_k| = 0, \\ \Lambda^*(q_k)_{|t=0} = 0 & \text{if } |q_k| = 1, \end{cases} \qquad \begin{cases} \partial_t\Lambda^*(q_k)_{|t=0} = \mathcal{J}^*(v_k) = 0 & \text{if } |q_k| = 0, \\ \partial_t\Lambda^*(q_k)_{|t=0} = \mathcal{J}^*(v_k) = \delta_{k,i}\cdot\theta & \text{if } |q_k| = 1. \end{cases}$$

Then the unique solution $\Lambda : \mathbb{R}^{1|1} \to \mathcal{M}$ of this system is

$$\begin{cases} \Lambda^*(q_k) = \widetilde{q_k}(q) & \text{if } |q_k| = 0 \\ \Lambda^*(q_k) = t\cdot\theta\cdot\delta_{k,j} & \text{if } |q_k| = 1. \end{cases}$$

Now, we have

$$T_0^{\mathrm{num}}(\exp_q)(X) = \frac{d}{d\theta}\mathrm{inj}^*_{t=1}\circ\Lambda^* \in \mathrm{Der}\left(\mathcal{O}_{\mathcal{M},q}, \mathbb{R}\right),$$

and then

$$T_0^{\mathrm{num}}(\exp_q)(X)(q_i) = \frac{d}{d\theta}\Lambda^*(q_i)_{|t=1} = \begin{cases} 0 & \text{if } |q_i| = 0 \\ \delta_{i,j} & \text{if } |q_i| = 1, \end{cases}$$

that is

$$T_0^{\mathrm{num}}(\exp_q)(X) = \partial_{q_j} = v_j.$$

This finishes the proof that $T_0(\exp_q)$ is the identity map. Therefore $\exp_q$ is a diffeomorphism near 0. □

In the case of odd Riemannian metric, the exponential map satisfies the following analogue of the preceding statement.

Theorem 2.42. *Let $(\mathcal{M}, g)$ be an odd Riemannian supermanifold, the map $\exp_q$: $T_q\mathcal{M} \to \mathcal{M}$ constructed above is a diffeomorphism near* 0*.*

Proof. It suffices to replace $T^*\mathcal{M}$ by $\Pi T^*\mathcal{M}$ in the proof of Theorem 2.40. □

Remark. Recall the following standard statement, usually called Gauss's lemma, valid in context of the classical Riemannian geometry:

Theorem 2.43. *(Folklore) Let (M, g) be a Riemannian manifold. Let q be a point of M and $\exp_q : U \to M$ be the exponential map at q, where U is an open subset of T_qM. Let*

u, w be tangent vectors in T_qM. Then, after identifying $T_u(T_qM)$ with T_qM, we have

$$g_{exp_q(u)}(d_u exp_q(u), d_u exp_q(w)) = g_q(u, w),$$

where $d_u exp_q(u) = (T_u exp_q)(u)$.

Calculations with explicit Riemannian metrics on $\mathcal{M} = \mathbb{R}^{1|2}$ show that the most direct extension of the ungraded Gauss's lemma can not hold true. I.e., given a Riemannian supermanifold $\mathcal{M}$, then, in general, for $q \in M$, $u = u_0 + u_1 \in T_q^{\mathrm{num}}\mathcal{M}$, and $w \in T_{u_0}^{\mathrm{num}}(T_q\mathcal{M}) \cong T_q^{\mathrm{num}}\mathcal{M}$,

$$g_{\widetilde{\exp}_q(u_0)}\left((T_{u_0}^{\mathrm{num}}\exp_q)(u), (T_{u_0}^{\mathrm{num}}\exp_q)(w)\right) \neq g_q\left(u, w\right).$$

2.4.2 Gauss's superlemma - counterexample

In order to give a counterexample to the most direct extension of the ungraded Gauss's lemma recalled just above, we treat, in detail, the case $\mathcal{M} = \mathbb{R}^{1|2}$ equipped with the even metric

$$g = \begin{pmatrix} 1 & 0 & \xi_1 \\ 0 & 0 & -1 \\ \xi_1 & 1 & 0 \end{pmatrix},$$

where $(q, \xi_1, \xi_2) = (q_i)$ are the canonical coordinates on $\mathbb{R}^{1|2}$.

Let $(T^*\mathcal{M}, \omega, H)$ be the canonical Hamiltonian dynamical system associated to $(\mathcal{M}, g)$ and X_H be the Hamiltonian vector field associated to H, the energy function. Recall that the flow $F : \mathbb{R} \times T^*\mathcal{M} \to T^*\mathcal{M}$ of X_H is determined by the following equations:

$$\begin{aligned}
\widehat{\partial}_t\left(F^*(q_i)\right) &= \sum_j (-1)^{|j|} F^*(p_j g^{ji}) \\
\widehat{\partial}_t\left(F^*(p_i)\right) &= -\frac{1}{2}\sum_{k,j}(-1)^{|i|+|k|+|j|+|i||k|} F^*\left(p_k \partial_{q_i}(g^{kj}) p_j\right) \\
(\mathrm{inj}_{t=0})^* \circ F^*(q_i) &= q_i \\
(\mathrm{inj}_{t=0})^* \circ F^*(p_i) &= p_i
\end{aligned}$$

where $(q_i, p_i) = (q, \xi_1, \xi_2, p, \eta_1, \eta_2)$ are the canonical coordinates of $T^*\mathcal{M} \cong \mathbb{R}^{2|4}$. The inverse matrix of g is given by

$$g^{-1} = \begin{pmatrix} 1 & \xi_1 & 0 \\ -\xi_1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

Then we have

$$\begin{aligned}
\widehat{\partial}_t\left(F^*(q)\right) &= F^*(p)+F^*(\eta_1)F^*(\xi_1)\\
\widehat{\partial}_t\left(F^*(\xi_1)\right) &= F^*(p)\cdot F^*(\xi_1)+F^*(\eta_2)\\
\widehat{\partial}_t\left(F^*(\xi_2)\right) &= -F^*(\eta_1)\\
\widehat{\partial}_t\left(F^*(p)\right) &= 0\\
\widehat{\partial}_t\left(F^*(\eta_1)\right) &= -F^*(p)\cdot F^*(\eta_1)\\
\widehat{\partial}_t\left(F^*(\eta_2)\right) &= 0.
\end{aligned}$$

Obviously, these equations and the initial condition imply that $F^*(p)=p$ and $F^*(\eta_2)=\eta_2$. We thus have

$$F^*(\eta_1)=\sum_{k=0}^{\infty}\frac{(-tp)^k}{k!}\eta_1 \text{ and } F^*(\xi_1)=\sum_{k=0}^{\infty}\frac{(tp)^k}{k!}\xi_1+\sum_{k=1}^{\infty}\frac{t^kp^{k-1}}{k!}\eta_2.$$

It follows that $F^*(\xi_2)$ and $F^*(q)$ are given by

$$F^*(\xi_2)=\xi_2+\sum_{k=1}^{\infty}\frac{(-t)^kp^{k-1}}{k!}\eta_1 \text{ and } F^*(q)=q+tp-t\xi_1\eta_1+\sum_{k=2}^{\infty}\frac{(-t)^kp^{k-2}}{k!}\eta_1\eta_2.$$

Let us now explicite the exponential map at $q_0\in M=\mathbb{R}$. Recall that $\exp_{q_0}=\pi^{T^*\mathcal{M}}\circ F\circ(\mathrm{id}_{\mathbb{R}}\times(G^\flat\circ i_{q_0})):T_{q_0}\mathcal{M}\to\mathcal{M}$. The map $G^\flat:T\mathcal{M}\to T^*\mathcal{M}$ induced by the metric is given by $G^{\flat *}(q_i)=q_i$ and $G^{\flat *}(p_i)=\sum_j(-1)^{|i|}v_j\cdot g_{ji}$ where $(q_i,v_i)=(q,\xi_1,\xi_2,v,\zeta_1,\zeta_2)$ are the canonical coordinates on $T\mathcal{M}\cong\mathbb{R}^{2|4}$. Then the exponential map at q_0 is given by

$$\begin{aligned}
(\exp_{q_0})^*(q) &= (i_{q_0}^*\circ G^{\flat *})(F^*(q)_{|t=1})\\
&= (i_{q_0}^*\circ G^{\flat *})\left(q+p-\xi_1\eta_1+\sum_{k=2}^{\infty}\frac{(-1)^kp^{k-2}}{k!}\eta_1\eta_2\right)\\
&= q_0+i_{q_0}^*(v\cdot 1+\zeta_1\cdot 0+\zeta_2\cdot\xi_1)+(i_{q_0}^*\circ G^{\flat *})\left(-\xi_1\eta_1+\sum_{k=2}^{\infty}\frac{(-1)^kp^{k-2}}{k!}\eta_1\cdot\eta_2\right)\\
&= q_0+v+i_{q_0}^*\left(\sum_{k=2}^{\infty}\frac{(-1)^k(v+\zeta_2\cdot\xi_1)^{k-2}}{k!}(-\zeta_2)(-v\cdot\xi_1+\zeta_1)\right)\\
&= q_0+v+\sum_{k=2}^{\infty}\frac{(-1)^kv^{k-2}}{k!}\zeta_1\cdot\zeta_2,
\end{aligned}$$

where (v, ζ_1, ζ_2) also denote the canonical coordinates on $T_{q_0}\mathcal{M} \cong \mathbb{R}^{1|2}$,

$$\begin{aligned}
(\exp_{q_0})^*(\xi_1) &= (i_{q_0}^* \circ G^{\flat *})(F^*(\xi_1)_{|t=1}) \\
&= (i_{q_0}^* \circ G^{\flat *})\left(\sum_{k=0}^{\infty} \frac{p^k}{k!}\xi_1 + \sum_{k=1}^{\infty} \frac{p^{k-1}}{k!}\eta_2\right) \\
&= i_{q_0}^*\left(\sum_{k=0}^{\infty} \frac{(v+\zeta_2\cdot\xi_1)^k}{k!}\xi_1 + \sum_{k=1}^{\infty} \frac{(v+\zeta_2\cdot\xi_1)^{k-1}}{k!}(-v\cdot\xi_1+\zeta_1)\right) \\
&= \sum_{k=1}^{\infty} \frac{v^{k-1}}{k!}\zeta_1,
\end{aligned}$$

and

$$\begin{aligned}
(\exp_{q_0})^*(\xi_2) &= (i_{q_0}^* \circ G^{\flat *})(F^*(\xi_2)_{|t=1}) \\
&= (i_{q_0}^* \circ G^{\flat *})\left(\xi_2 + \sum_{k=1}^{\infty} \frac{(-1)^k p^{k-1}}{k!}\eta_1\right) \\
&= i_{q_0}^*\left(-\sum_{k=1}^{\infty} \frac{(-1)^k (v+\zeta_2\xi_1)^{k-1}}{k!}\zeta_2\right) \\
&= \sum_{k=1}^{\infty} \frac{(-v)^{k-1}}{k!}\zeta_2.
\end{aligned}$$

Now let $u_0 = 1\cdot\partial_v \in T_{q_0}M \cong \mathbb{R}$, we have

$$\begin{aligned}
T_{u_0}^{\text{num}}\exp_{q_0} = \left(\left(\partial_{\zeta_i}\exp_{q_0}^*(q_j)\right)^{\sim}(u_0)\right)_{ij} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sum_{k=1}^{\infty}\frac{1}{k!} & 0 \\ 0 & 0 & \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & e-1 & 0 \\ 0 & 0 & 1-\frac{1}{e} \end{pmatrix}.
\end{aligned}$$

Then

$$\begin{aligned}
g_{\widetilde{\exp_{q_0}}(u_0)}\left(T_{u_0}^{\text{num}}\exp_{q_0}(\partial_v+\partial_{\zeta_1}), T_{u_0}^{\text{num}}\exp_{q_0}(\partial_{\zeta_2})\right) &= g_{\widetilde{\exp_{q_0}}(u_0)}\left(\partial_v+(e-1)\partial_{\xi_1}, (1-\frac{1}{e})\partial_{\xi_2}\right) \\
&= e+\frac{1}{e}-2
\end{aligned}$$

On the other hand, we have

$$g_{q_0}(\partial_v+\partial_{\zeta_1}, \partial_{\zeta_2}) = -1.$$

We conclude that for $u = u_0 + u_1 = \partial_v + \partial_{\xi_1} \in T^{\mathrm{num}}_{q_0}\mathcal{M}$ and $w = \partial_{\xi_2} \in T^{\mathrm{num}}_{u_0}(T_{q_0}\mathcal{M}) \cong T^{\mathrm{num}}_{q_0}\mathcal{M}$, we have

$$g_{\widetilde{\exp_{q_0}}(u_0)}\left(T^{\mathrm{num}}_{u_0}\exp_{q_0}(u), T^{\mathrm{num}}_{u_0}\exp_{q_0}(w)\right) \neq g_{q_0}(u,w).$$

2.4.3 Linearization of isometries

We apply in this section the properties of the above constructed super exponential map to generalize a fundamental linearization result on Riemannian isometries to the category of supermanifolds.

Let $(\mathcal{M}, g^{\mathcal{M}})$ and $(\mathcal{N}, g^{\mathcal{N}})$ be Riemannian supermanifolds and $\Phi : \mathcal{M} \to \mathcal{N}$ a morphism. Recall that $g^{\mathcal{N}}$ can be viewed as an element of $\Omega^1_{\mathcal{N}} \otimes_{\mathcal{O}_{\mathcal{N}}} \Omega^1_{\mathcal{N}}$, moreover $\Phi^* : \widetilde{\Phi}^{-1}\mathcal{O}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{M}}$ induces a morphism of sheaves

$$\begin{tikzcd} \widetilde{\Phi}^{-1}\Omega^1_{\mathcal{N}} \arrow[rr, "\Phi^*"] \arrow[dr] & & \Omega^1_{\mathcal{M}} \arrow[dl] \\ & M & \end{tikzcd}$$

Thus the expression $\Phi^*(\widetilde{\Phi}^{-1}g^{\mathcal{N}})$ makes sense as an element of $\Omega^1_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \Omega^1_{\mathcal{M}}$.

Definition 2.44. *Let $(\mathcal{M}, g^{\mathcal{M}})$ and $(\mathcal{N}, g^{\mathcal{N}})$ be even or odd Riemannian supermanifolds. An **isometry** is an isomorphism of supermanifolds $\Phi : \mathcal{M} \to \mathcal{N}$ such that the following equality is satisfied*

$$\Phi^*(\widetilde{\Phi}^{-1}g^{\mathcal{N}}) = g^{\mathcal{M}}.$$

If $(q_i^{\mathcal{M}})$ and $(q_j^{\mathcal{N}})$ are local coordinates on $\mathcal{M}$ and $\mathcal{N}$ respectively, the above condition means that

$$g_{ij}^{\mathcal{M}} = \sum_{k,l}(-1)^{(|q_i^{\mathcal{M}}|+|q_k^{\mathcal{N}}|)|g|+(|q_k^{\mathcal{N}}|+|g|)(|q_j^{\mathcal{M}}|+|q_l^{\mathcal{N}}|)}\partial_i^{\mathcal{M}}(\Phi^*(q_k^{\mathcal{N}}))\cdot\partial_j^{\mathcal{M}}(\Phi^*(q_l^{\mathcal{N}}))\cdot\Phi^*\left(g_{kl}^{\mathcal{N}}\right),$$

where $\partial_i^{\mathcal{M}}$, $\partial_j^{\mathcal{N}}$ and $|g|$ denote $\partial_{q_i^{\mathcal{M}}}$, $\partial_{q_j^{\mathcal{N}}}$ and $|g^{\mathcal{M}}|(=|g^{\mathcal{N}}|)$ respectively.

Theorem 2.45. *Let $(\mathcal{M}, g^{\mathcal{M}})$ and $(\mathcal{N}, g^{\mathcal{N}})$ be even Riemannian supermanifolds. Let $\Phi : \mathcal{M} \to \mathcal{N}$ be an isometry and $q \in M$, then we have*

$$\Phi \circ exp_q^{\mathcal{M}} = exp_{\widetilde{\Phi}(q)}^{\mathcal{N}} \circ T_q\Phi.$$

Proof. Let us define the following supercurves

$$\Lambda := \pi^{T^*\mathcal{M}} \circ F^{\mathcal{M}} \circ (\mathrm{id}_{\mathbb{R}} \times (G^{\mathcal{M}\flat} \circ i_q)) : \mathbb{R} \times T_q\mathcal{M} \to \mathcal{M}$$

and

$$\Gamma := \Phi \circ \Lambda : \mathbb{R} \times T_q\mathcal{M} \to \mathcal{N}, \tag{2.26}$$

where $F^{\mathcal{M}}$ is the geodesic flow of the Hamiltonian dynamical system $(T^*\mathcal{M}, \omega, H)$.

We want to show that Γ is a geodesic of $\mathcal{N}$ with initial condition $i_{\widetilde{\Phi}(q)} \circ T_q\Phi \in \mathrm{Mor}(T_q\mathcal{M}, T\mathcal{N})$. By the unicity of geodesics, it will then follow

$$\Gamma = \pi^{T^*\mathcal{N}} \circ F^{\mathcal{N}} \circ (\mathrm{id}_{\mathbb{R}} \times (G^{\mathcal{N}\flat} \circ i_{\widetilde{\Phi}(q)} \circ T_q\Phi)). \tag{2.27}$$

The statement of the theorem will then be obtained by evaluating both expressions (2.26) and (2.27) for Γ at $t = 1$.

One can show that $\nabla := \left(\mathrm{id}_{\Omega^1_{\mathcal{M}}} \otimes (d\Phi^*)^{-1}\right) \circ \left(\Phi^*\nabla^{\mathcal{N}}\right) \circ d\Phi^* : \mathcal{T}_{\mathcal{M}} \to \Omega^1_{\mathcal{M}} \otimes \mathcal{T}_{\mathcal{M}}$ is a torsion free connection since the connection $\nabla^{\mathcal{N}}$ has the same property. Moreover, ∇ respects the metric $g^{\mathcal{M}}$ since $\nabla^{\mathcal{N}}$ respects the metric $g^{\mathcal{N}}$ which is related to $g^{\mathcal{M}}$ by the following formula $\Phi^*(\widetilde{\Phi}^{-1}g^{\mathcal{N}}) = g^{\mathcal{M}}$. Then the connection ∇ is the Levi-Civita connection on $(\mathcal{M}, g^{\mathcal{M}})$, i.e. $\nabla = \nabla^{\mathcal{M}}$ which we rewrite in the following way

$$\left(\Phi^*\nabla^{\mathcal{N}}\right)(d\Phi^*) = d\Phi^*(\nabla^{\mathcal{M}}).$$

We thus obtain

$$\frac{\nabla^{\mathcal{N}}}{dt}(d\Gamma^*(\partial_t)) = d\Phi^*\left(\frac{\nabla^{\mathcal{M}}}{dt}(d\Lambda^*(\partial_t))\right) = 0$$

because Λ is a geodesic of $\mathcal{M}$. It follows that Γ is a geodesic.

Furthermore

$$\begin{aligned} \partial_t\Gamma \circ \mathrm{inj}_{t=0} &= \partial_t(\Phi \circ \Lambda) \circ \mathrm{inj}_{t=0} \\ &= T\Phi \circ \partial_t\Lambda \circ \mathrm{inj}_{t=0} \\ &= T\Phi \circ i_q \\ &= i_{\widetilde{\Phi}(q)} \circ T_q\Phi. \end{aligned}$$

We thus have shown that (2.27) holds and the claim of the theorem follows by evaluating both sides in $t = 1$. □

Now, we can give the announced linearization result.

Corollary 2.46. *Let $\mathcal{M}$ be a connected even Riemannian supermanifold and let $\Phi : \mathcal{M} \to \mathcal{M}$ be an isometry such that there exists a point $q_0 \in M$ with*

$$\widetilde{\Phi}(q_0) = q_0 \text{ and } T_{q_0}\Phi = id_{T_{q_0}\mathcal{M}}.$$

Then Φ is the identity morphism of $\mathcal{M}$.

Proof. Let U be the subset of M defined by

$$U := \{q \in M \mid \widetilde{\Phi}(q) = q \text{ and } T_q\Phi = \mathrm{id}_{T_q\mathcal{M}}\}.$$

Remark that if $\widetilde{\Phi}(q) = q$ the equality $T_q\Phi = \mathrm{id}_{T_q\mathcal{M}}$ is equivalent to the numerical version $T_q^{\mathrm{num}}\Phi = \mathrm{id}_{T_q^{\mathrm{num}}\mathcal{M}}$. Thus U is a closed subset of M.

Obviously, the set U is non-empty since $q_0 \in U$. Let q be an element of U. By the preceding theorem we have

$$\Phi \circ \exp_q = \exp_{\widetilde{\Phi}(q)} \circ T_q\Phi = \exp_q.$$

By Theorem 2.40, the exponential map is a local diffeomorphism and we thus have an open V in M containing q such that $\Phi_{|V} = \mathrm{id}_{\mathcal{M}_{|V}}$, implying that U is open. By the connectedness of M, it follows that U equals the whole body M of the supermanifold $\mathcal{M}$.

Now we know that $\widetilde{\Phi}(q) = q$ and $T_q\Phi = \mathrm{id}_{T_q\mathcal{M}}$ for all $q \in M$ and consequently we have $\Phi \circ \exp_q = \exp_q$ for all $q \in M$. The morphism Φ is then the identity near each $q \in M$. This implies, of course, that Φ is the identity morphism of $\mathcal{M}$. □

Remark. Theorem 2.45 and Corollary 2.46 hold also true when $(\mathcal{M}, g^{\mathcal{M}})$ and $(\mathcal{N}, g^{\mathcal{N}})$ are odd Riemannian supermanifolds. One can replace everywhere in this proof $T^*\mathcal{M}$ by $\Pi T^*\mathcal{M}$ to obtain proofs in the case of $|g| = 1$.

Chapter 3

Flows of supervector fields and local actions

3.1 Introduction

The natural problem of integrating vector fields on supermanifolds was considered in many articles but a “complete answer” was to our knowledge not published before the work of J. Monterde and co-workers (see [15] and [17]). Let us consider a supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ together with a vector field $X \in \mathcal{T}_{\mathcal{M}}(M)$, and an initial condition $\phi \in \mathrm{Mor}(\mathcal{S}, \mathcal{M})$, where $\mathcal{S} = (S, \mathcal{O}_{\mathcal{S}})$ is an arbitrary supermanifold. The case of ungraded, classical, manifolds leads one to consider the following question: does there exist a “flow map” F defined on an open subsupermanifolds $\mathcal{V} \subset \mathbb{R}^{1|1} \times \mathcal{S}$ and having values in $\mathcal{M}$ such that, if $D = \partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)$ with a, b real-valued functions of t is an appropriate derivation on $\mathbb{R}^{1|1}$, then

$$\begin{cases} D \circ F^* &= F^* \circ X \text{ and} \\ F \circ \mathrm{inj}_{\{0\}\times\mathcal{S}}^{\mathcal{V}} &= \phi. \end{cases} \tag{3.1}$$

Of course, $\mathcal{V}$ should be a “flow domain”, i.e. an open subsupermanifold of $\mathbb{R}^{1|1} \times \mathcal{S}$ such that $\{0\} \times S$ is in the body V of $\mathcal{V}$ and for $x \in S$, the set $I_x \subset \mathbb{R}$ defined by $I_x \times \{x\} = (\mathbb{R} \times \{x\}) \cap V$ is an open interval.

Of course, we could concentrate on the case $\mathcal{S} = \mathcal{M}$ and $\phi = \mathrm{id}_{\mathcal{M}}$, but it will be useful for our later arguments to state all results in this (formally) more general setting.

Though for homogeneous vector fields ($X = X_0$ or $X = X_1$) system (3.1) is in fact solvable, in the mixed case ($X = X_0 + X_1$ with $X_0 \neq 0$ and $X_1 \neq 0$) the system is overdetermined and thus not always solvable, as can be shown by easy examples. The groundbreaking novelty of [17] is to consider instead of (3.1) the modified, weakened, problem

$$\begin{cases} (\mathrm{inj}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ D \circ F^* &= (\mathrm{inj}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X \text{ and} \\ F \circ \mathrm{inj}_{\{0\}\times\mathcal{S}}^{\mathcal{V}} &= \phi, \end{cases} \tag{3.2}$$

where $\mathrm{inj}_{\mathbb{R}}^{\mathbb{R}^{1|1}}$ is the natural injection (and where the above more general derivation D could be replaced by $\partial_t + \partial_\tau$ since $(\mathrm{inj}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^*$ annihilates superfunctions germs of the type $\tau \cdot f, f \in \mathcal{O}_{\mathbb{R}\times\mathcal{S}}$).

In [17] (making indispensable use of [15]) they show that (3.2) has a unique maximal solution F, defined on the flowdomain $\mathcal{V} = (V, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}})$, where $V \subset \mathbb{R}\times S$ is the maximal flow domain for the flow of the reduced vector field $\widetilde{X} = \widetilde{X_0}$ on M with initial condition $\widetilde{\phi}$. Since the results of [15] are obtained by the use of a Batchelor model for $\mathcal{M}$, i.e. a real vector bundle $E \to M$ such that $\mathcal{M} \cong (M, \Gamma^\infty_{\Lambda E^*})$, and a connection on E, we follow here another road, more close to the classical, ungraded case and directly applicable in the case of complex supermanifolds and holomorphic vector fields.

Our calculations are first local and here inspired by the approach of [9], where the case of homogeneous super vector fields on compact supermanifolds is treated. Then we show unicity of solutions of (3.2) on supermanifolds and easily deduce the results of [17] from our Lemmata 3.1 and 3.2.

A second beautiful result of [17] (more precisely Theorem 3.6 of that reference) concerns the question if the flow F solving (3.2) fulfills "flow equations", as in the ungraded case. Hereby, we mean the existence of a Lie supergroup structure on $\mathbb{R}^{1|1}$ such that F is a local action of $\mathbb{R}^{1|1}$ on $\mathcal{M}$ (in case $\mathcal{S} = \mathcal{M}, \phi = \mathrm{id}_{\mathcal{M}}$). Again, the answer is a little bit unexpected: in general, given X and its flow $F : \mathbb{R}^{1|1} \times \mathcal{M} \supset \mathcal{V} \to \mathcal{M}$, there is no Lie supergroup structure on $\mathbb{R}^{1|1}$ such that F is a local $\mathbb{R}^{1|1}$-action (with regard to this structure). The condition for the existence of such a structure on $\mathbb{R}^{1|1}$ is equivalent to the condition that (3.2) holds without the pre-composition of $(\mathrm{inj}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^*$, i.e. the overdetermined system (3.1) is solvable (and then, of course, uniquely). Furthermore, both conditions cited are equivalent to the condition that $\mathbb{R}X_0 \oplus \mathbb{R}X_1$ is a Lie subsuperalgebra of $\mathcal{T}_{\mathcal{M}}(M)$, the Lie superalgebra of all vector fields on $\mathcal{M}$.

We prove the equivalence of the three conditions, slightly generalizing the result in [17] since we do not ask for any normalization of the supercommutators between X_1 and X_0 resp. X_1. Thus our criterion is a bit more flexible in applications but of course this is only a minor improvement. Our proof is similar but not identical to the proof of the analoguous result in [17], but we give -at least in our conviction- an easier derivation, especially concerning multiplications and right-invariant vector fields on $\mathbb{R}^{1|1}$.

3.2 Flow of a supervector field

For the sake of readability we will use in the sequel the following shorthand: if $\mathcal{P}$ is a supermanifold, we write $\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}}$ for $\operatorname{inj}{}_{\mathbb{R}\times\mathcal{P}}^{\mathbb{R}^{1|1}\times\mathcal{P}}$.

Lemma 3.1. *Let $\mathcal{U} \subset \mathbb{R}^{m|n}$ and $\mathcal{W} \subset \mathbb{R}^{p|q}$ be superdomains, $X \in \mathcal{T}_{\mathcal{W}}(W)$ be a super vector field on $\mathcal{W}$ (not necessarily homogeneous) and $\phi \in Mor(\mathcal{U},\mathcal{W})$, and $t_0 \in \mathbb{R}$. Let furthermore $H : V \to W$ be the maximal flow of $\widetilde{X} \in \mathcal{X}(W)$, i.e. $\partial_t \circ H^* = H^* \circ \widetilde{X}$, subject to the initial condition $H(t_0,\cdot) = \widetilde{\phi} : U \to W$. Let now $\mathcal{V}$ be $(V, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{U}|V})$ and (t,τ) the canonical coordinates on $\mathbb{R}^{1|1}$, then there exists a unique $F : \mathcal{V} \to \mathcal{W}$ such that*

$$(inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* = (inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X \text{ and} \tag{3.3}$$

$$F \circ inj\,{}_{\{t_0\}\times\mathcal{U}}^{\mathcal{V}} = \phi. \tag{3.4}$$

Moreover, $\widetilde{F} : V \to W$ equals the underlying classical flow map H of the vector field $\widetilde{X}$ with initial condition $\widetilde{\phi}$.

Proof. Let $(u_i) = (x_i, \xi_r)$ and $(w_j) = (y_j, \eta_s)$ denote the canonical coordinates on $\mathbb{R}^{m|n}$ and $\mathbb{R}^{p|q}$, respectively. Then there exist smooth functions $a_J^j \in \mathcal{C}^\infty_{\mathbb{R}^p}(W)$ such that

$$X = \sum_{j=1}^{p+q} \left(\sum_J a_J^j(y)\eta^J \right) \partial_{w_j},$$

where $J = (\beta_1, \dots, \beta_q)$ runs over the index set $\{0,1\}^q$ and $\eta^J = \prod_{s=1}^{q} \eta_s^{\beta_s}$. We then have, of course,

$$X_0 = \sum_j \left(\sum_{|J|=|w_j|} a_J^j(y)\eta^J \right) \partial_{w_j}$$

resp.

$$X_1 = \sum_j \left(\sum_{|J|=|w_j|+1} a_J^j(y)\eta^J \right) \partial_{w_j}.$$

The morphism F determines and is uniquely determined by functions $f_I^j \in \mathcal{C}^\infty_{\mathbb{R}\times\mathbb{R}^m}(V)$ fulfilling for each $j \in \{1, \dots, p+q\}$

$$F^*(w_j) = \sum_{|I|=|w_j|} f_I^j(t,x)\xi^I + \sum_{|I|=|w_j|+1} g_I^j(t,x)\tau\xi^I$$

(and $f_I^j = 0$ if $|I| \neq |w_j|$, $g_I^j = 0$ if $|I| \neq |w_j| + 1$) as is well-known from the standard theory of supermanifolds (compare, e.g., Thm. 4.3.1 in [24]). Here and in the sequel $I = (\alpha_1, \dots, \alpha_n)$ is an element of the set $\{0,1\}^n$ and ξ^I stands for the product $\xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n}$.

Equation (3.3) is equivalent to the following equations:

$$
\begin{aligned}
(\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_t \circ F^* &= (\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_0 && (3.5)\\
(\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \circ F^* &= (\operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1 && (3.6)
\end{aligned}
$$

Applying (3.5) to the canonical coordinate functions on $\mathcal{W}$, we get the following system, which is equivalent to (3.5):

$$
\sum_{|I|=|w_j|} \partial_t f_I^j \cdot \xi^I = \sum_{|J|=|w_j|} \check{F}^*(a_J^j)\check{F}^*(\eta^J) \qquad \forall j \in \{1,\dots,p+q\}, \tag{3.7}
$$

and (3.6) is equivalent to

$$
\sum_{|I|=|w_j|+1} g_I^j \cdot \xi^I = \sum_{|J|=|w_j|+1} \check{F}^*(a_J^j)\check{F}^*(\eta^J) \qquad \forall j \in \{1,\dots,p+q\}, \tag{3.8}
$$

where $\check{F} := F \circ \operatorname{inj}{}_{\mathbb{R}}^{\mathbb{R}^{1|1}} : \check{\mathcal{V}} := (V, \mathcal{O}_{\mathbb{R}\times\mathcal{U}|V}) \to \mathcal{W}$. Let us immediately observe that the underlying smooth map of $\check{F}$ equals $\tilde{F}$, the smooth map underlying the morphism F.

Moreover the initial condition (3.4) is equivalent to

$$
\sum_{|I|=|w_j|} f_I^j(t_0,x)\xi^I = \phi^*(w_j) \qquad \forall j \in \{1,\dots,p+q\}. \tag{3.9}
$$

We are going to show that (3.7) and (3.9) uniquely determine the functions f_I^j on V, i.e. the morphism $\check{F}$. Then the functions g_I^j are unambiguously given by (3.8) on V, and then the morphism F is determined.

Let us develop Equation (3.7) for a fixed j:

$$
\sum_{|I|=|w_j|} \partial_t f_I^j \cdot \xi^I = \sum_{\substack{|J|=|w_j|\\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^q \check{F}^*(\eta_s)^{\beta_s} \tag{3.10}
$$

$$
\text{and thus} \quad \sum_{|I|=|w_j|} \partial_t f_I^j \cdot \xi^I = \sum_{\substack{|J|=|w_j|\\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^q \left(\sum_{|L|=1} f_L^{p+s}\xi^L \right)^{\beta_s}. \tag{3.11}
$$

For fixed j this is an equation of Grassmann algebra-valued maps in the variables t and x that can be split in a system of scalar equations as follows. For $K = (\alpha_1,\dots,\alpha_n) \in \{0,1\}^n$, we will denote the coefficient h_K in front of ξ^K of the superfunction $h = \sum_M h_M(t,x)\xi^M \in \mathcal{O}_{\mathbb{R}^{m+1|n}}$ compactly by $(h|\xi^K)$ in the sequel of this proof.

Let us first describe the coefficients for $\check{F}^*(a_J^j)$ in (3.11):

$$(\check{F}^*(a_J^j)|\xi^K) = 0 \quad \text{if } |K| = 1 \tag{3.12}$$

and, if $|K| = 0$,

$$(\check{F}^*(a_J^j)|\xi^K) = a_J^j \circ \widetilde{F} \quad \text{if } K = (0, \dots, 0),$$

and

$$(\check{F}^*(a_J^j)|\xi^K) = \sum_{\mu=1}^{p} (\partial_{y_\mu} a_J^j)(\widetilde{F}(t,x)) \cdot f_K^\mu + R\left(a_J^j, (f_I^\nu)_{\nu,\deg(I)<\deg(K)}\right) \text{ if } \deg(K) > 0. \tag{3.13}$$

Here for $I = (\alpha_1, \dots, \alpha_n)$, $\deg(I) = \alpha_1 + \dots + \alpha_n$, and -more importantly- $R = R_{j,J,K}$ is a polynomial function in a_J^j and its derivatives in the y-variables up to order q included, and in the functions $\{f_I^\nu | 1 \leq \nu \leq p+q, 0 \leq \deg(I) < \deg(K)\}$. Equation (3.12) is obvious since a_J^j is an even function, whereas equation (3.13) can be deduced from standard analysis on superdomains. More precisely, let a be a smooth function on $\mathbb{R}^p$ and $\psi : \mathbb{R}^{m+1|n} \to \mathbb{R}^{p|q}$ morphism (of course to be applied to $a = a_J^j, \psi = \check{F}$). Then we can develop $\psi^*(a)$ as follows (compare the proof of Theorem 4.3.1 in [24]):

$$\begin{aligned}
\psi^*(a) &= \sum_\gamma \frac{1}{\gamma!} (\partial^\gamma a)(\widetilde{\psi}^*(y_1), \dots, \widetilde{\psi}^*(y_p)) \cdot \prod_{\mu=1}^{p} (\psi^*(y_\mu) - \widetilde{\psi}^*(y_\mu))^{\gamma_\mu} \\
&= a(\widetilde{\psi}(t,x)) + \sum_{\mu=1}^{p} (\partial_{y_\mu} a)(\widetilde{\psi}(t,x)) \cdot \left(\sum_{M \neq 0} f_M^\mu \cdot \xi^M \right) + \\
&\quad \frac{1}{2} \sum_{\mu',\mu''=1}^{p} (\partial_{y_{\mu'}} \partial_{y_{\mu''}} a)(\widetilde{\psi}(t,x)) \cdot \left(\sum_{M' \neq 0} f_{M'}^{\mu'} \cdot \xi^{M'} \right) \cdot \left(\sum_{M'' \neq 0} f_{M''}^{\mu''} \cdot \xi^{M''} \right) + \dots,
\end{aligned}$$

where $\sum_{M \neq 0} f_M^\mu \cdot \xi^M = \psi^*(y_\mu) - \widetilde{\psi}^*(y_\mu)$ with f_M^μ depending on t and x. We observe that the last RHS is a finite sum since we work in the framework of finite dimensional supermanifolds.

In order to get a contribution to $(\psi^*(a)|\xi^K)$ we can either extract f_K^μ from the "linear term" or from products coming from the higher order terms in the above development. Thus

$$(\psi^*(a)|\xi^K) = \sum_{\mu=1}^{p} (\partial_{y_\mu} a)(\widetilde{\psi}(t,x)) \cdot f_K^\mu + R(a, (f_I^\nu)_{\nu,\deg(I)<\deg(K)}),$$

where R is a polynomial as described after Equation (3.13).

Furthermore, for an element $J = (\beta_1, \dots, \beta_q)$ with $|J| = 0$ we have for $\deg(K) > 0$

$$\left(\prod_{s=1}^{q} \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \middle| \xi^K \right) = R\left((f_I^j)_{j,\deg(I)<\deg(K)} \right). \tag{3.14}$$

And for an element $J = (\beta_1, \ldots, \beta_q)$ with $|J| = 1$ we get for $\deg(K) > 0$

$$\left(\prod_{s=1}^{q} \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \Bigg| \xi^K \right) = \begin{cases} f_K^{p+l} + R\left((f_I^j)_{j,\deg(I)<\deg(K)} \right) & \text{if } \deg(J) = 1 \text{ and } l \in \{1, \ldots, q\} \\ & \text{such that } \beta_s = \delta_{s,l} \quad \forall s, \\ R\left((f_I^j)_{j,\deg(I)<\deg(K)} \right) & \text{if } \deg(J) > 1. \end{cases}$$

Obviously, the coefficient of ξ^K of the LHS of Equation (3.7) is given by

$$\left(\sum_{|I|=|w_j|} \partial_t f_I^j \cdot \xi^I \Bigg| \xi^K \right) = \begin{cases} \partial_t f_K^j & \text{if } |K| = |w_j| \\ 0 & \text{if } |K| = |w_j| + 1 \end{cases} \quad \forall j.$$

Taking into account the above descriptions of the ξ^K-coefficients, we can show the existence (and uniqueness) of the solution functions $\{f_I^j | 1 \leq j \leq p+q, I \in \{0,1\}^n\}$ for $(t,x) \in V$ by induction on $\deg(I)$ and upon observing that all ordinary differential equations occuring are (inhomogeneous) linear equations for the unknown functions.

Let us start with $\deg(I) = 0$ that is $I = (0, \cdots, 0)$. The "0-level" of the equations (3.11) and (3.9) is

$$\partial_t f^j_{(0,\cdots,0)} = a^j_{(0,\cdots,0)} \circ \widetilde{F} \text{ and } f^j_{(0,\cdots,0)}(t_0, x) = y_j \circ \widetilde{\phi}(x) \quad \forall j \text{ such that } |w_j| = 0.$$

We remark that $f^j_{(0,\cdots,0)}$ is simply $y_j \circ \widetilde{F}$ and $a^j_{(0,\cdots,0)}$ is $\widetilde{X}(y_j)$. Thus $\widetilde{F}$ is the flow of $\widetilde{X}$ with initial condition $\widetilde{\phi}$ at $t = t_0$, i.e., $\widetilde{F} = H$ on V. Thus the claim is true for $I = (0, \cdots, 0)$.

Suppose $k > 0$ and that the functions f_I^j are uniquely defined on V for all j and all I such that $\deg(I) < k$. Let K be such that $\deg(K) = k$. Let us distinguish the two possible parities of k in order to determine f_K^j for all j. Recall that $f_K^j = 0$ if the parities of K and j are different.

If k is even, i.e., $|K| = 0$, we only have to consider j such that $|w_j| = 0$. Putting (3.13) and (3.14) together, we find in this case

$$\begin{aligned}
\partial_t f_K^j &= \left(\sum_{\substack{|J|=0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^{q} \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \left(\sum_{\substack{\deg(J)=0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^{q} \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} + \sum_{\substack{|J|=0 \\ \deg(J)>0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^{q} \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \left(\check{F}^*(a^j_{(0,\cdots,0)}) + \sum_{\substack{|J|=0 \\ \deg(J)>0 \\ J=(\beta_1,\ldots,\beta_q)}} \check{F}^*(a_J^j) \prod_{s=1}^{q} \left(\sum_{|L|=1} f_L^{p+s} \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \sum_{\mu=1}^{p} \left(\partial_{y_\mu} a^j_{(0,\cdots,0)} \circ \widetilde{F} \right) f_K^\mu + R\left((a_J^j)_J, (f_I^\nu)_{\nu, \deg(I)<\deg(K)} \right).
\end{aligned}$$

Moreover, the initial condition gives

$$f_K^j(t_0, x) \;=\; (\phi^*(y_j)|\xi^K), \quad \forall j = 1, \ldots, p.$$

Since the a_J^j are the (given) coefficients of the vector field X and the functions f_I^ν with $\deg(I) < k$ are known by the induction hypothesis, we have a unique local solution function f_K^j. Since the ordinary differential equation for f_K^j is linear its solution is already defined for all $(t, x) \in V$. Thus in the case that k is even f_K^j is unambiguously defined on V for all $j \in \{1, \ldots, p+q\}$ and for all K with $\deg(K) = k$.

Now, if k is odd,i.e., $|K| = 1$, we only have to consider j such that $|w_j| = 1$. Using (3.12) and (3.14), we find in this case:

$$\begin{aligned}
\partial_t f^j_K &= \left(\sum_{\substack{|J|=1\\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \left(\sum_{\substack{\deg(J)=1\\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} + \sum_{\substack{|J|=1\\ \deg(J)>1\\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \left(\sum_{s=1}^{q} \check{F}^*(a^j_{(\delta_{1s},\cdots,\delta_{qs})}) \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right) + \sum_{\substack{|J|=1\\ \deg(J)>1\\ J=(\beta_1,\dots,\beta_q)}} \check{F}^*(a^j_J) \prod_{s=1}^{q} \left(\sum_{|L|=1} f^{p+s}_L \xi^L \right)^{\beta_s} \middle| \xi^K \right) \\
&= \sum_{s=1}^{q} \left(a^j_{(\delta_{1s},\cdots,\delta_{qs})} \circ \widetilde{F} \right) f^{p+s}_K + R\left((a^j_J)_J, (f^\nu_I)_{\nu,\deg(I)<\deg(K)} \right).
\end{aligned}$$

Moreover, the initial condition gives

$$f^j_K(t_0, x) \;=\; (\phi^*(w_j)|\xi^K), \quad \forall j = p+1, \dots, p+q.$$

It follows as in the case of $|K| = 0$, that f^j_K exists uniquely for all $(t, x) \in V$, for all $j \in \{1, \dots, p+q\}$ and for all K with $\deg(K) = k$.

We conclude that the functions $\{f^j_I|1 \le j \le p+q, I \in \{0,1\}^n\}$ are uniquely defined on the whole of V. Since the $\{g^j_I|1 \le j \le p+q, I \in \{0,1\}^n\}$ are determined by Equation (3.8) from the $\{f^j_I|1 \le j \le p+q\}$ via comparison of coefficients, the morphism $F : \mathcal{V} \to \mathcal{W}$ is uniquely determined. $\square$

We now consider the global problem of integrating a vector field on a supermanifold. In order to prove that there exists a unique maximal flow of a supervector field, the following lemma will be crucial.

Lemma 3.2. *Let $\mathcal{S}$ and $\mathcal{M}$ be supermanifolds, X be a supervector field in $\mathcal{T}_{\mathcal{M}}(M)$ and let $\phi \in Mor(\mathcal{S}, \mathcal{M})$. Let $F_1 : \mathcal{V}_1 \to \mathcal{M}$ and $F_2 : \mathcal{V}_2 \to \mathcal{M}$ be maps satisfying*

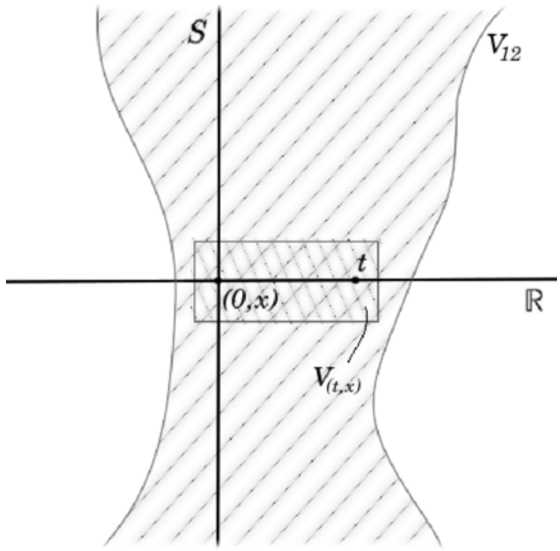
$$\begin{aligned}
(inj\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ (\partial_t + \partial_\tau) \circ F_i^* &= (inj\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ F_i^* \circ X \;\textit{ and} \qquad (3.15)\\
F_i \circ inj\,{}^{\mathcal{V}_i}_{\{0\}\times\mathcal{S}} &= \phi, \qquad (3.16)
\end{aligned}$$

where $\mathcal{V}_i = (V_i, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}})$ *with* V_i *open in* $\mathbb{R}\times S$ *such that* $\{0\}\times S \subset V_i$ *and* $\forall x \in S$, $V_i \cap (\mathbb{R}\times\{x\})$ *is an interval,* $\forall i = 1, 2$. *Then* $F_{1|\mathcal{V}_{12}} = F_{2|\mathcal{V}_{12}}$ *on* $\mathcal{V}_{12} = (V_{12}, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}})$, *where* $V_{12} = V_1 \cap V_2$.

Proof.

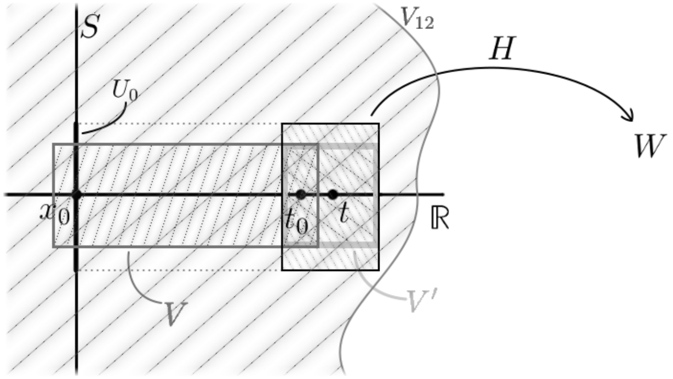
We define A as the set of points $(t,x) \in V_{12}$ such that there exists $\epsilon = \epsilon_{(t,x)} > 0$ and $\mathcal{U} = \mathcal{U}_{(t,x)}$ an open subsupermanifold of $\mathcal{S}$, such that its body U contains x and for $\mathcal{V} = \mathcal{V}_{(t,x)} = (]-\epsilon, t+\epsilon[\times U, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{S}})$ we have $F_{1|\mathcal{V}} = F_{2|\mathcal{V}}$. The claim of the Lemma is now equivalent to $A = V_{12}$.

The set A is obviously open. By an easy application of Lemma 3.1, A contains $\{0\}\times S$. The assumptions imply that for all $x \in S$, the set $I_x \subset \mathbb{R}$, defined by $V_{12} \cap (\mathbb{R}\times\{x\}) = I_x \times \{x\}$, is an open interval containing 0. The definition of A implies that the set $J_x \subset I_x$ such that $A \cap (\mathbb{R}\times\{x\}) = J_x \times \{x\}$ is an open interval containing 0 as well.



Assuming now that $A \neq V_{12}$, then there exists a point $(t, x_0) \in V_{12}\backslash A$ such that $(t', x_0) \in A$ for $0 \leq t' < t$. Let U_0 be an open coordinate neighborhood of x_0 in S and $\delta > 0$ such that, with $V_0 :=]t-\delta, t+\delta[\times U_0 \subset V_{12}$, $H(V_0) \subset W$, where $\mathcal{W} = (W, \mathcal{O}_{\mathcal{M}})$ is a coordinate patch of $\mathcal{M}$. Choose $t_0 \in]t-\delta, t[$. Then $(t_0, x_0) \in A$ and thus there exists $\epsilon > 0$ and $\mathcal{U}$ an open subsupermanifold of $\mathcal{U}_0 = (U_0, \mathcal{O}_{\mathcal{S}})$ containing x_0 such that

$$F_{1|\mathcal{V}} = F_{2|\mathcal{V}}, \quad \text{where } \mathcal{V} =]-\epsilon, t_0+\epsilon[\times\mathbb{R}^{0|1}\times\mathcal{U} \subset \mathcal{V}_{12}. \tag{3.17}$$



On $\mathcal{V}' =]t-\delta, t+\delta[\times\mathbb{R}^{0|1}\times\mathcal{U} \subset \mathcal{V}_{12}$, F_1 and F_2 are defined and for $i = 1, 2$ the maps $F_i \circ \mathrm{inj}^{\mathcal{V}}_{\{t_0\}\times\mathcal{U}}$ coincide by (3.17). By Lemma 3.1 we have $F_{1|\mathcal{V}'} = F_{2|\mathcal{V}'}$. Thus $F_1 = F_2$ on $\mathcal{V}\cup\mathcal{V}'$, and we conclude that $(t, x_0) \in A$. This contradiction shows that $V_{12} = A$. □

Remarks. (1) Lemma 3.1 shows that there exists at least one flow map F as in the preceding lemma defined near $\{0\}\times S$. The above lemma allows to consider the maximal flow of X by taking the union of all flow domains and flows.
(2) Obviously, Lemma 3.2 holds true for an arbitrary $t_0 \in \mathbb{R}$ replacing $t_0 = 0$.

Theorem 3.3. *Let $\mathcal{S}$ and $\mathcal{M}$ be supermanifolds, X be a supervector field in $\mathcal{T}_{\mathcal{M}}(M)$, $\phi \in Mor(\mathcal{S},\mathcal{M})$ and $t_0 \in \mathbb{R}$. Then there exists a unique map $F : \mathcal{V} \to \mathcal{M}$ such that*

$$\begin{aligned}(inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* &= (inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X \ \ and\\ F \circ inj\,{}_{\{t_0\}\times\mathcal{S}}^{\mathcal{V}} &= \phi\end{aligned}$$

where $\mathcal{V} = (V, \mathcal{O}_{\mathbb{R}^{1|1}\times\mathcal{U}|V})$ is the maximal flow domain of X with the given initial condition.
Moreover, $\widetilde{F} : V \to M$ is the maximal flow of $\widetilde{X} \in \mathcal{X}(M)$ subject to the initial condition $\widetilde{\phi}$ at $t = t_0$.

Proof. The proof of the theorem follows immediately from the Lemmata 3.1 and 3.2 and the preceding remarks. □

3.3 Supervector fields and local $\mathbb{R}^{1|1}$-actions

Given a vector field on a classical, ungraded manifold, the flow map $\widetilde{F}$ (for $S = M$, $\widetilde{\phi} = \mathrm{id}_M$) is always a local action of $\mathbb{R}$ with its usual addition. The flow maps for supervector fields described in the preceding section (taking here $\mathcal{S} = \mathcal{M}$, $\phi = \mathrm{id}_{\mathcal{M}}$), do not always have the analogous property of being local actions of $\mathbb{R}^{1|1}$ with an appropriate Lie supergroup structure. Two characterizations of those supervector fields $X = X_0+X_1$ that generate a local $\mathbb{R}^{1|1}$-action were found by J. Monterde and J. Sánchez-Valenzuela. We will give here a short proof of a slightly more general result, whose conditions seem to be more easily verified in practice than those given in [17] (compare Thm. 3.6 and its proof).

Theorem 3.4. *Let $\mathcal{M}$ be a supermanifold, X a supervector field on $\mathcal{M}$ and $\mathcal{V} \subset \mathbb{R}^{1|1}\times\mathcal{M}$ the domain of the maximal flow $F : \mathcal{V} \to \mathcal{M}$ satisfying*

$$\begin{aligned}(inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* &= (inj\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X \ \ and\\ F \circ inj\,{}_{\{0\}\times\mathcal{M}}^{\mathcal{V}} &= id_{\mathcal{M}}.\end{aligned}$$

Then the following assertions are equivalent:

(i) there exists $a, b \in \mathbb{R}$ such that

$$(\partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)) \circ F^* = F^* \circ X,$$

(ii) the flow map F is a local $\mathbb{R}^{1|1}$-action on $\mathcal{M}$ for an appropriate Lie supergroup structure on $\mathbb{R}^{1|1}$,

(iii) $\mathbb{R}X_0 \oplus \mathbb{R}X_1$ is a Lie subsuperalgebra of $\mathcal{T}_{\mathcal{M}}(M)$.

Remark. Given a supermanifold $\mathcal{M}$ and a Lie supergroup $\mathcal{G} = (G, \mathcal{O}_{\mathcal{G}})$ with neutral element e and multiplication μ, a morphism $F : \mathcal{V} \to \mathcal{M}$, where $\mathcal{V} \subset \mathcal{G} \times \mathcal{M}$ is an open supsupermanifold containing $\{e\} \times M$ in its body, is called a "local $\mathcal{G}$-actions" if $F \circ (\mathrm{inj}_e \times \mathrm{id}_{\mathcal{M}}) = \mathrm{id}_{\mathcal{M}}$ on $\mathcal{M}$ and $F \circ (\mu \times \mathrm{id}_{\mathcal{M}}) = F \circ (\mathrm{id}_{\mathcal{G}} \times F)$ on the open subsupermanifold of $\mathcal{G} \times \mathcal{G} \times \mathcal{M}$, where both sides are defined. Let us observe that if either X_0 or X_1 (or both) are zero the three conditions continue to be equivalent.

Before proving the theorem, we give a useful description of the right-invariant vector fields on $\mathbb{R}^{1|1}$.

Lemma 3.5. *Let a and b be real numbers such that $a \cdot b = 0$ and $\mu_{a,b} = \mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ be defined by*

$$\begin{aligned} \widetilde{\mu}(t_1, t_2) &= t_1 + t_2, \\ \mu^*(t) &= t_1 + t_2 + a\tau_1\tau_2, \\ \mu^*(\tau) &= \tau_1 + e^{bt_1}\tau_2 \quad . \end{aligned}$$

Then

(i) there exists a unique Lie supergroup structure on $\mathbb{R}^{1|1}$ such that the multiplication morphism is given by $\mu_{a,b}$,

(ii) the right-invariant supervector fields on $(\mathbb{R}^{1|1}, \mu_{a,b})$ are given by the graded vector space $\mathbb{R}D_0 \oplus \mathbb{R}D_1$, where

$$D_0 := \partial_t + b \cdot \tau\partial_\tau \text{ and } D_1 := \partial_\tau + a \cdot \tau\partial_t.$$

Remark. The case $a = 1, b = 0$ corresponds to the "supersymmetry" very often encountered in theoretical physics. In this case, $D_1 = \partial_\tau + \tau\partial_t$ is the odd vector field fundamental to all consideration in "supersymmetric field theories" (compare the chapter "supersolutions" in [7])

Proof. The first assertion follows by an easy direct verification.
Let now Z be a vector field on $\mathbb{R}^{1|1}$, then there exist $\alpha, \beta, \gamma, \eta \in \mathcal{C}^\infty_{\mathbb{R}}(\mathbb{R})$ such that $Z = Z_0 + Z_1$ where $Z_0 = \alpha\partial_t + \beta\tau\partial_\tau$ and $Z_1 = \gamma\partial_\tau + \eta\tau\partial_t$.
Recall that Z is a right-invariant vector field if Z fulfills the following condition:

$$\mu^* \circ Z = (Z \otimes \mathrm{id}) \circ \mu^*.$$

Remark that this condition can be split into the following two conditions:

$$\mu^* \circ Z_0 = (Z_0 \otimes \mathrm{id}) \circ \mu^* \text{ and } \mu^* \circ Z_1 = (Z_1 \otimes \mathrm{id}) \circ \mu^*.$$

Thus, if Z is a right-invariant vector field, the functions $\alpha, \beta, \gamma, \eta$ are characterized as follows:

$$\mu^* \circ Z_0(t) = \mu^*(\alpha) = \alpha(t_1 + t_2) + a\alpha'(t_1 + t_2)\tau_1\tau_2,$$

$$\begin{aligned}(Z_0 \otimes \mathrm{id}) \circ \mu^*(t) &= (Z_0 \otimes \mathrm{id})(t_1 \otimes 1 + 1 \otimes t_2 + a\tau_1 \otimes \tau_2) \\ &= Z_0(t_1) \otimes 1 + Z_0(1) \otimes t_2 + aZ_0(\tau_1) \otimes \tau_2 \\ &= \alpha(t_1) + a\beta(t_1)\tau_1\tau_2 \qquad .\end{aligned}$$

Consequently α is a constant and $a \cdot \beta = 0$. Similarly,

$$\begin{aligned}\mu^* \circ Z_0(\tau) &= \mu^*(\beta\tau) = \big(\beta(t_1 + t_2) + a\beta'(t_1 + t_2)\tau_1\tau_2\big) \cdot (\tau_1 + e^{bt_1}\tau_2) \\ &= \beta(t_1 + t_2)\tau_1 + \beta(t_1 + t_2)e^{bt_1}\tau_2,\end{aligned}$$

$$\begin{aligned}(Z_0 \otimes \mathrm{id}) \circ \mu^*(\tau) &= (Z_0 \otimes \mathrm{id})(\tau_1 \otimes 1 + e^{bt_1} \otimes \tau_2) \\ &= Z_0(\tau_1) \otimes 1 + Z_0(e^{bt_1}) \otimes \tau_2 \\ &= \beta(t_1)\tau_1 + \alpha \cdot be^{bt_1}\tau_2,\end{aligned}$$

implying that β is a constant and $b \cdot \alpha = \beta$. For the odd part we find

$$\mu^* \circ Z_1(t) = \mu^*(\eta\tau) = \eta(t_1 + t_2)\tau_1 + \eta(t_1 + t_2)e^{bt_1}\tau_2,$$

$$\begin{aligned}(Z_1 \otimes \mathrm{id}) \circ \mu^*(t) &= (Z_1 \otimes \mathrm{id})(t_1 \otimes 1 + 1 \otimes t_2 + a\tau_1 \otimes \tau_2) \\ &= \eta(t_1)\tau_1 + a\gamma(t_1)\tau_2.\end{aligned}$$

Thus η is a constant and $a \cdot e^{-bt_1} \cdot \gamma = \eta$. Finally,

$$\mu^* \circ Z_1(\tau) = \mu^*(\gamma) = \gamma(t_1 + t_2) + a\gamma'(t_1 + t_2)\tau_1\tau_2,$$

$$(Z_1 \otimes \mathrm{id}) \circ \mu^*(\tau) = \gamma(t_1) + \eta(t_1)be^{bt_1}\tau_1\tau_2,$$

thus γ is a constant and $b \cdot \eta = 0$.

Since $a \cdot b = 0$, it suffices to consider the cases $a = 0$ and $b = 0$. If $a = 0$, we have $Z_0 = \alpha\partial_t + b\alpha\tau\partial_\tau = \alpha D_0$ and $Z_1 = \gamma\partial_t = \gamma D_1$ where $\alpha, \gamma \in \mathbb{R}$. If $b = 0$, we have $Z_0 = \alpha\partial_t = \alpha D_0$ and $Z_1 = \gamma\partial_\tau + a\gamma\tau\partial_t = \gamma D_1$ where α, γ are real numbers. Reciprocally, in both cases, Z is a right-invariant vector field. □

Proof of Theorem 3.4.

Recall that F fulfills

$$\begin{aligned}(\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_t \circ F^* &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_0 \quad \text{and} \\ (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \circ F^* &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1.\end{aligned}$$

Denoting the projection from $\mathbb{R}^{1|1}$ to $\mathbb{R}$ by p, we have

$$\mathrm{id}^*_{\mathbb{R}^{1|1}} = p^* \circ (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* + \tau \cdot p^* \circ (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \tag{3.18}$$

which we will write more succinctly as

$$\mathrm{id}^*_{\mathbb{R}^{1|1}} = (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* + \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau. \tag{3.19}$$

Using relation (3.19) we get

$$\begin{aligned}F^* \circ X &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X + \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \circ F^* \circ X \\ &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* + \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1 \circ X \\ &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* + \tau \cdot (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \left([X_1, X_0] + X_0 \circ X_1 + \frac{1}{2}[X_1, X_1]\right).\end{aligned}$$

Since

$$\begin{aligned}(\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_0 \circ X_1 &= (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_t \circ F^* \circ X_1 \\ &= \partial_t \circ (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1 \\ &= \partial_t \circ (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ \partial_\tau \circ F^* \\ &= \partial_t \circ \partial_\tau \circ F^* \\ &= \partial_\tau \circ \partial_t \circ F^*,\end{aligned}$$

we arrive at

$$\begin{aligned}F^* \circ X = {} & (\mathrm{inj}\,{}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ (\partial_t + \partial_\tau) \circ F^* + \\ & \tau \cdot F^* \left([X_1, X_0] + \frac{1}{2}[X_1, X_1]\right) + \tau \cdot \partial_\tau \circ \partial_t \circ F^*.\end{aligned} \tag{3.20}$$

On the other hand, if a and b are real numbers, we have, again using (3.19)

$$\begin{aligned}(\partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)) \circ F^* &= \Big((\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* + \tau \cdot (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ \partial_\tau\Big) \circ (\partial_t + \partial_\tau) \circ F^* + \\ &\qquad \tau \cdot \Big(a \cdot (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ \partial_t + b \cdot (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ \partial_\tau\Big) \circ F^* \\ &= (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ (\partial_t + \partial_\tau) \circ F^* + \tau \cdot \partial_\tau \circ \partial_t \circ F^* + \\ &\qquad \tau \cdot F^* \circ (aX_0 + bX_1)\,.\end{aligned}$$

Thus we have

$$(\partial_t + \partial_\tau + \tau(a\partial_t + b\partial_\tau)) \circ F^* - F^* \circ X = \tau \cdot F^* \circ \left(aX_0 - \frac{1}{2}[X_1, X_1] + bX_1 - [X_1, X_0]\right). \tag{3.21}$$

Since F satisfies the initial condition $(\mathrm{inj}\,{}^{\mathcal{V}}_{\{0\}\times\mathcal{M}})^* \circ F^* = \mathrm{id}_{\mathcal{M}}$, $\tau \cdot F^*$ is injective and thus Equation (3.21) easily implies: $(i) \Longleftrightarrow (iii)$.

We remark that, in this case, we automatically have $a \cdot b = 0$ since the Jacobi identity implies that $[X_1, [X_1, X_1]] = [[X_1, X_1], X_1] + (-1)^{1\cdot 1}[X_1, [X_1, X_1]]$, i.e., $2a \cdot b \cdot X_1 = [X_1, [X_1, X_1]] = 0$.

Assume now that a and b be real numbers such that (i) satisfied, and let $\mu = \mu_{a,b}$ be as in Lemma 3.5. We have to show that F is a local action of $(\mathbb{R}^{1|1}, \mu)$.

Let us define

$$G := F \circ (\mathrm{id}_{\mathbb{R}^{1|1}} \times F) : \mathbb{R}^{1|1} \times (\mathbb{R}^{1|1} \times \mathcal{M}) \to \mathcal{M}$$

and

$$H := F \circ (\mu \times \mathrm{id}_{\mathcal{M}}) : (\mathbb{R}^{1|1} \times \mathbb{R}^{1|1}) \times \mathcal{M} \cong \mathbb{R}^{1|1} \times (\mathbb{R}^{1|1} \times \mathcal{M}) \to \mathcal{M}.$$

In order to prove that F is a $\mathbb{R}^{1|1}$-action on, we have to show that $G = H$. We observe that G is the integral curve of X subject to the initial condition $F \in \mathrm{Mor}(\mathbb{R}^{1|1} \times \mathcal{M}, \mathcal{M})$.

Let us prove that the morphism H satisfies the following conditions:

$$(\mathrm{inj}\,{}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ H^* = (\mathrm{inj}\,{}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})})^* \circ H^* \circ X \tag{3.22}$$

$$H \circ \mathrm{inj}\,{}^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\{0\}\times(\mathbb{R}^{1|1}\times\mathcal{M})} = F. \tag{3.23}$$

Then by the unicity of integral curves we have $H = G$.

Equation (3.23) holds true since $\mu \circ \mathrm{inj}\,{}^{\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}}_{\{0\}\times\mathbb{R}^{1|1}} = \mathrm{id}_{\mathbb{R}^{1|1}}$.

Defining $D := D_0 + D_1 = \partial_{t_1} + \partial_{\tau_1} + \tau_1(a\partial_{t_1} + b\partial_{\tau_1})$ and writing $\mathrm{inj}\,_{|t_1}$ for $\mathrm{inj}\,^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})}$ and using right-invariance of D, we arrive at equation (3.22) as follows

$$\begin{aligned}
(\mathrm{inj}\,^{\mathbb{R}^{1|1}\times(\mathbb{R}^{1|1}\times\mathcal{M})}_{\mathbb{R}\times(\mathbb{R}^{1|1}\times\mathcal{M})})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ H^* &= (\mathrm{inj}\,_{|t_1})^* \circ D \circ H^* \\
&= (\mathrm{inj}\,_{|t_1})^* \circ \Big(((D \otimes \mathrm{id}^*_{\mathbb{R}^{1|1}}) \circ \mu^*) \times \mathrm{id}^*_{\mathcal{M}}\Big) \circ F^* \\
&= (\mathrm{inj}\,_{|t_1})^* \circ ((\mu^* \circ D) \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \\
&= (\mathrm{inj}\,_{|t_1})^* \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \circ X \\
&= (\mathrm{inj}\,_{|t_1})^* \circ H^* \circ X.
\end{aligned}$$

Thus we obtain $(i) \Longrightarrow (ii)$.

Assume now that (ii) is satisfied, i.e., there exists a Lie supergroup structure on $\mathbb{R}^{1|1}$ with multiplication μ such that

$$F \circ (\mathrm{id}_{\mathbb{R}^{1|1}} \times F) = F \circ (\mu \times \mathrm{id}_{\mathcal{M}}). \tag{3.24}$$

Since F is a flow for X, with initial condition $\phi = \mathrm{id}_{\mathcal{M}}$, the LHS of the preceding equality is a flow for X with initial condition $\phi = F$, (3.24) implies

$$(\mathrm{inj}\,_{|t_1})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* = (\mathrm{inj}\,_{|t_1})^* \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \circ X. \tag{3.25}$$

By Equation 3.20, the RHS gives for $t_1 = 0$:

$$\begin{aligned}
(\mathrm{inj}\,_{|t_1=0})^* \circ (\mu^* \times \mathrm{id}^*_{\mathcal{M}}) \circ F^* \circ X &= F^* \circ X \\
&= (\mathrm{inj}\,^{\mathbb{R}^{1|1}}_{\mathbb{R}})^* \circ (\partial_t + \partial_\tau) \circ F^* \\
&\quad + \tau \cdot \left[F^* \left([X_1, X_0] + \frac{1}{2}[X_1, X_1]\right) + \partial_\tau \circ \partial_t \circ F^*\right].
\end{aligned}$$

Moreover, we have by direct comparison

$$(\partial_{t_1} + \partial_{\tau_1}) \circ \mu^* = (\partial_{t_1} + \partial_{\tau_1})(\mu^*(t)) \cdot (\mu^* \circ \partial_t) + (\partial_{t_1} + \partial_{\tau_1})(\mu^*(\tau)) \cdot (\mu^* \circ \partial_\tau).$$

Thus, if $\mu : \mathbb{R}^{1|1} \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}$ is given by

$$\begin{aligned}
\mu^*(t) &= \widetilde{\mu}(t_1, t_2) + \alpha(t_1, t_2)\tau_1\tau_2 \\
\mu^*(\tau) &= \beta(t_1, t_2)\tau_1 + \gamma(t_1, t_2)\tau_2,
\end{aligned}$$

and upon using $(\mathrm{inj}\,^{\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}}_{\{0\}\times\mathbb{R}^{1|1}})^* \circ \mu^* = \mathrm{id}^*_{\mathbb{R}^{1|1}}$, we have

$$\begin{aligned}
(\mathrm{inj}\,^{\mathbb{R}^{1|1}\times\mathbb{R}^{1|1}}_{\{0\}\times\mathbb{R}^{1|1}})^*(\partial_{t_1} + \partial_{\tau_1}) \circ \mu^* &= ((\partial_{t_1}\widetilde{\mu})(0, t) + \alpha(0, t)\tau) \cdot \partial_t \\
&\quad + (\beta(0, t) + (\partial_{t_1}\gamma)(0, t)\tau) \cdot \partial_\tau.
\end{aligned}$$

Using again (3.19), we have

$$\begin{aligned}\partial_t \circ F^* &= (\text{inj}\ {}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_0 + \tau \cdot \partial_\tau \circ \partial_t \circ F^* \quad \text{and}\\ \partial_\tau \circ F^* &= (\text{inj}\ {}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1.\end{aligned}$$

Then the LHS of (3.25) at $t_1 = 0$ is

$$\begin{aligned}(\text{inj}\ _{|t_1=0})^* \circ (\partial_{t_1} + \partial_{\tau_1}) \circ (\mu^* \times \text{id}^*_{\mathcal{M}}) \circ F^* &= (\partial_{t_1}\widetilde{\mu})(0,t) \cdot (\text{inj}\ {}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_0\\ &+\beta(0,t) \cdot (\text{inj}\ {}_{\mathbb{R}}^{\mathbb{R}^{1|1}})^* \circ F^* \circ X_1\\ &+\tau \cdot \big((\partial_{t_1}\widetilde{\mu})(0,t) \cdot \partial_\tau \circ \partial_t \circ F^*\\ &\qquad +\alpha(0,t) \cdot F^* \circ X_0\\ &\qquad\quad +(\partial_{t_1}\gamma)(0,t) \cdot F^* \circ X_1\big).\end{aligned}$$

Using the obtain identities for its LHS and RHS, the "τ-part" of Equation (3.25) at $t_1 = 0$ gives us:

$$\begin{aligned}\tau \cdot \left[F^*\left([X_1, X_0] + \frac{1}{2}[X_1, X_1]\right) + \partial_\tau \circ \partial_t \circ F^*\right] &= \tau \cdot \big[(\partial_{t_1}\widetilde{\mu})(0,t) \cdot \partial_\tau \circ \partial_t \circ F^* +\\ &\alpha(0,t) \cdot F^* \circ X_0 + (\partial_{t_1}\gamma)(0,t) \cdot F^* \circ X_1\big].\end{aligned}$$

Since $\widetilde{\mu}(t_1, 0) = t_1$, we have $(\partial_{t_1}\widetilde{\mu})(0,0) = 1$ and therefore the preceding equation evaluated at $t = 0$ yields

$$[X_1, X_0] + \frac{1}{2}[X_1, X_1] = (\partial_{t_1}\gamma)(0,0) \cdot X_1 + \alpha(0,0) \cdot X_0$$

finishing the proof that $(ii) \Longrightarrow (iii)$. □

Remarks. (1) If one of the equivalent condition of Theorem 3.4 is satisfied, $[X_1, X_0] = bX_1$ and $[X_1, X_1] = 2aX_1$. On the other hand, if $\mu = \mu_{a,b}$ is used as multiplication on $\mathbb{R}^{1|1}$, we have

$$[D_1, D_0] = bD_1 \text{ and } [D_1, D_1] = 2aD_0,$$

for $D_0 := \partial_t + b \cdot \tau\partial_\tau$ and $D_1 := \partial_\tau + a \cdot \tau\partial_t$. This relation shows that X_0 resp. X_1 behave as "the fundamental vector fields" of the $\mathbb{R}^{1|1}$-action on $\mathcal{M}$ associated to the right-invariant vector fields D_0 and D_1 on $(\mathbb{R}^{1|1}, \mu)$.

(2) All results and proofs given in this chapter hold true in the category of complex supermanifolds for holomorphic super vector fields. Of course, the initial conditions should be holomorphic morphisms defined on a complex supermanifold $\mathcal{S}$ and all flows and actions should be considered with $\mathbb{C}^{1|1}$ replacing $\mathbb{R}^{1|1}$. Since reference [15] uses a Batchelor model $E \to M$ (and a connection on E), it yields only information for a restricted class of complex supermanifolds. Since the results of [15] are used in [17], the results in [17] do not extend directly to the holomorphic case. By contrast, the strategy of proof, we adopt here, immediately applies in the holomorphic case.

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