On a new Itô-type formula in law

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Chapter 1

Introduction

1.1 Introduction à la thèse

Le but principal de cette thèse est de construire un calcul stochastique pour le mouvement brownien fractionnaire en temps brownien (mbftb en abrégé.) Dans cette introduction, nous introduisons le mbftb ainsi que certains résultats basiques le concernant. Nous donnons également un rapide aperçu des résultats principaux que nous avons obtenus dans notre thèse.

1.1.1 Définitions de base et quelques résultats connus à propos du mbftb

Soient X^H un mouvement brownien fractionnaire (mbf) sur $\mathbb{R}$ d'indice de Hurst $H \in (0,1)$, et Y un mouvement brownien standard sur $\mathbb{R}^+$ indépendant de X^H.

Définition 1.1.1 *Le processus $Z^H = X^H \circ Y$ est appelé mouvement brownien fractionnaire en temps brownien (mbftb).*

On peut vérifier immédiatement que Z^H est un processus auto-similaire d'ordre $H/2$ à accroissements stationnaires, et qu'il n'est pas gaussien (pour ce dernier point, voir la Proposition 1.1.3 ci-dessous).

Le cas $H = \frac{1}{2}$

Historiquement, l'étude du mbftb a commencé avec le cas $H = \frac{1}{2}$, c'est-à-dire lorsque $X^{\frac{1}{2}}$ est un mouvement brownien standard. Il s'agit alors du fameux *mouvement brownien itéré* (mbi). Cette dénomination a été introduite par Burdzy en 1993 (voir [5]). Mais l'idée d'introduire le mbi est plus ancienne que ça. En fait, Funaki [12] a découvert en 1979 que le mbi peut servir à représenter la solution de l'équation aux dérivées partielles parabolique suivante:

$$\frac{\partial u}{\partial t} = \frac{1}{8}\left(\frac{\partial u}{\partial x}\right)^4, \ (t,x) \in (0,\infty) \times \mathbb{R}.$$

Le lecteur intéressé peut consulter les travaux de Nane (voir, par exemple, [23] et ses références) pour d'autres résultats intéressants qui montrent le lien entre les processus itérés et les équations aux dérivées partielles.

En 1998, Burdzy et Khoshnevisan [7] ont montré que le mbi peut être considéré comme le mouvement canonique dans une fissure brownienne indépendante. Autrement dit, le mbi est un candidat convenable pour modéliser le phénomène de diffusion dans une fissure brownienne. Pour asseoir leur théorie, ils ont montré que les deux composantes d'un mouvement brownien bidimensionnel réfléchi dans une saucisse de Wiener convergent conjointement en loi vers un mouvement brownien et un mbi, respectivement, lorsque le rayon $\epsilon > 0$ de la saucisse tend vers zéro.

On peut démontrer que $Z^{\frac{1}{2}}$ n'est pas un processus de Markov dans sa filtration naturelle. De plus, d'après [6], les trajectoires de $Z^{\frac{1}{2}}$ sont à variation quartique finie presque sûrement. Une conséquence immédiate de ce dernier fait est que $Z^{\frac{1}{2}}$ admet une variation quadratique infinie sur un ensemble de probabilité 1. En particulier, $Z^{\frac{1}{2}}$ n'est ni une semi-martingale, ni un processus de Dirichlet.

Ainsi, un problème intéressant et loin d'être trivial* était de construire un processus d'intégrale stochastique de la forme $\int_0^{\cdot} f(Z_s^{\frac{1}{2}})dZ_s^{\frac{1}{2}}$, pour f appartenant à une certaine classe de fonctions suffisamment régulières. Ce problème a été abordé et résolu par Khoshnevisan et Lewis [20, 21] en 1999. Ils ont construit une intégrale stochastique par rapport au mbi, après avoir analysé ses variations puissance. Ils ont aussi trouvé une formule de changement de variable pour le mbi.

Le cas $H \neq \frac{1}{2}$

Une suite naturelle des travaux [20, 21] précédemment cités de Khoshnevisan et Lewis consiste à étudier la situation lorsque $H \neq \frac{1}{2}$. Même si la situation pourrait sembler similaire, ce changement de cadre nécessite une analyse radicalement différente.

Avant d'entrer dans le coeur de cette thèse, il est intéressant de collecter quelques propriétés basiques du mbftb qui ont déjà été obtenues dans la littérature. Durant l'écriture de notre manuscrit, nous avons découvert que le mbftb avait été le sujet principal de la thèse récente de Joseph A. Zadeh [39], rédigée sous la direction de Frederi Viens et soutenue à l'Université de Purdue en 2012. Nous allons décrire quelques-uns des résultats obtenus par Zadeh. Commençons par le résultat suivant, qui correspond au Théorème 3.2.4 de [39].

Théorème 1.1.2 *Z^H satisfait le module de continuité suivant: P-p.s.,*

$$\lim_{\eta\to 0} \sup_{0\leqslant|s-t|\leqslant\eta,\, 0\leqslant s,t\leqslant 1} \frac{|Z_t^H - Z_s^H|}{\eta^{\frac{H}{2}}|\log\eta|^{-\frac{1}{2}(1+\frac{3H}{2})}} < \infty.$$

On a aussi un résultat simple concernant la densité de Z_t^H.

Proposition 1.1.3 *Pour tout $t > 0$, la variable aléatoire Z_t^H admet pour densité $f_H(t,\cdot)$ donnée par*

$$f_H(t,x) = \int_{\mathbb{R}} \frac{e^{-\frac{x^2}{2|y|^{2H}}-\frac{y^2}{2t}}}{2\pi\sqrt{t|y|^{2H}}}\, dy.$$

*En effet, il existe un résultat de Bichteler et Dellacherie qui affirme que pour que $\int MdN$ existe pour une large classe de processus M, alors N doit être une semimartingale.

Un exemple d'application de la formule précédente est l'encadrement de la queue de distribution de Z_1^H. Nous renvoyons au Corollaire 4.1.6 de [39] pour un énoncé précis. Bien entendu, grâce à un argument d'autosimilarité on pourrait énoncer un résultat similaire pour Z_t^H.

Théorème 1.1.4 *Soit* $H^* = H^{\frac{1}{2H+2}}$ *et* $L(H^*) = \frac{1}{2}(H^{-\frac{H}{H+1}} + H^{\frac{1}{H+1}})$. *Alors, il existe des constantes* $k(H), K(H) > 0$ *dépendant seulement en* H *telles que, pour tout* $x > \frac{H^{H+1}}{\sqrt{H}}$,

$$k(H)x^{-\frac{1}{H+1}}\exp(-L(H^*)x^{\frac{2}{H+1}}) \leqslant P(Z_1^H > x) \leqslant K(H)\exp(-L(H^*)x^{\frac{2}{H+1}}).$$

Pour finir, observons qu'il est mentionné dans la thèse de Zadeh (Chapitre 5) qu'un étudiant d'Orsingher a montré en 2010 que la densité $f_H(t,\cdot)$ de Z_t^H est solution de l'équation aux dérivées partielles de second ordre suivante, montrant ainsi à nouveau comment ce processus est lié aux EDP.

Théorème 1.1.5 *On a*

$$(1+\frac{H}{2})\,t\,\frac{\partial f_H}{\partial t} + t^2\frac{\partial^2 f_H}{\partial t^2} = \frac{H^2}{4}\left(2x\frac{\partial f_H}{\partial x} + x^2\frac{\partial^2 f_H}{\partial x^2}\right).$$

1.1.2 Formules de type Itô: un très bref aperçu historique

Entrons maintenant dans le coeur de cette thèse qui, rappelons-le, a comme but principal de construire un calcul stochastique pour le mbftb. Nous commençons d'abord par une présentation succincte de l'état de l'art, ceci afin de positionner notre étude dans le contexte scientifique existant.

Calcul stochastique et formule d'Itô pour les semimartingales

Soit $(B_t)_{t\geqslant 0}$ un mouvement brownien défini sur un espace de probabilité $(\Omega, \mathcal{F}, P)$. Comme il est bien connu, Wiener fut le premier à donner un sens à l'intégrale

$$\int_0^t f(s)dB_s(w),$$

lorsque f est une fonction déterministe telle que $\int_0^t f^2(s)ds < \infty$. Cette intégrale est définie globalement sur Ω, mais ce n'est pas une intégrale trajectorielle. Ceci est dû au fait que B est à variation infinie presque sûrement. Itô a ensuite étendu en 1944 (voir [17]) la construction de Wiener, en considérant plus généralement des intégrales de la forme

$$\int_0^t f_s(w)dB_s(w) \tag{1.1.1}$$

où (f_t) est un processus adapté à la filtration naturelle de (B_t). Itô a non seulement donné un sens à (1.1.1), mais il a aussi mis en évidence une formule de changement de variable associée qui, dans sa forme la plus simple, prend la forme suivante.

Théorème 1.1.6 (Formule d'Itô) *Soit* $(B_t)_{t\geqslant 0}$ *un mouvement brownien et soit* f *une fonction de classe* C^2 *sur* $\mathbb{R}$. *Alors, pour tout* $t \geqslant 0$,

$$f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds.$$

En utilisant son intégrale, Itô fut capable de construire un processus de diffusion de la forme

$$X_t = x + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dB_s$$

(où, bien sûr, $a(s,x)$ et $b(s,x)$ doivent satisfaire certaines conditions). Avant Itô il est intéressant de se rappeler que, pour construire un processus de diffusion avec un générateur infinitésimal donné G, la stratégie consistait à résoudre l'équation de Kolmogorov

$$\frac{\partial}{\partial t} P_t f = G P_t f,$$

et ensuite d'utiliser le semi-groupe $(P_t)_{t\geqslant 0}$ et le théorème de Kolmogorov pour en déduire le processus de diffusion.

Après la découverte d'Itô, la communauté probabilistes s'est montrée de plus en plus intéressée par le calcul stochastique, et plusieurs autres résultats fondamentaux ont été découverts. Il est bien entendu impossible de les citer tous ici: nous renvoyons le lecteur à n'importe quel traîté de référence (par exemple, [22]) dédié à l'histoire fascinante du calcul stochastique.

Pour clore cette section, nous allons seulement rappeler la formule d'Itô pour les semimartingales. C'est d'ailleurs sous cette forme que la formule d'Itô est généralement enseignée aujourd'hui partout dans le monde.

Théorème 1.1.7 (Formule d'Itô pour les semimartingales continues) *Soit $(X_t^1, \ldots, X_t^n)_{t\geqslant 0}$ une semimartingale continue et soit f une fonction de classe C^2 sur $\mathbb{R}^n$. Pour tout $i, j \in \{1, \ldots, n\}$, soit $\langle X^i, X^j\rangle$ le crochet de X^i et X^j. Alors, pour tout $t \geqslant 0$:*

$$\begin{aligned} f(X_t^1, \ldots, X_t^n) &= f(X_0^1, \ldots, X_0^n) + \sum_{i=1}^n \int_0^t \frac{\partial f}{\partial x_i}(X_s^1, \ldots, X_s^n) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(X_s^1, \ldots, X_s^n) d\langle X^i, X^j\rangle_s. \end{aligned} \tag{1.1.2}$$

Généralisation du calcul stochastique et de la formule d'Itô au cas du mbf

Vu le grand succès rencontré par la théorie du calcul stochastique, de nombreux chercheurs se sont intéressés à étendre l'intégrale stochastique à une classe plus générale de processus et à sortir du cadre des semimartingales. Plusieurs approches ont été proposées. Citons, par exemple, les travaux de Skorokhod [38], Asch et Potthoff [1], Nualart et Pardoux [32], ou bien encore Russo et Vallois [35]. Arrêtons-nous un instant sur cette dernière référence. En 1993, Russo et Vallois ont défini ce qui est connu aujourd'hui sous le nom *d'intégrale de Russo-Vallois*. Cette intégrale généralise à la fois l'intégrale classique d'Itô et celle de Stratonovich. Une des conséquences les plus importantes de l'intégrale de Russo-Vallois est la possibilité de construire une intégrale stochastique par rapport à un processus qui n'est pas une semimartingale. L'exemple typique et populaire d'un tel processus est le mouvement brownien fractionnaire (mbf).

Rappelons les définitions basiques de la théorie de Russo-Vallois.

Définition 1.1.8 *Si $X = \{X_t, t \in \mathbb{R}_+\}$ et $Y = \{Y_t, t \in \mathbb{R}_+\}$ sont deux processus continus, on définit, lorsque la limite existe*

$$\begin{aligned}
\int_0^{\cdot} Y_s d^- X_s &= \lim_{\epsilon\to 0} ucp \int_0^{\cdot} Y_s \frac{X_{s+\epsilon}-X_s}{\epsilon} ds && \text{"intégrale progressive"},\\
\int_0^{\cdot} Y_s d^+ X_s &= \lim_{\epsilon\to 0} ucp \int_0^{\cdot} Y_{s+\epsilon} \frac{X_{s+\epsilon}-X_s}{\epsilon} ds && \text{"intégrale rétrograde"},\\
\int_0^{\cdot} Y_s d^\circ X_s &= \lim_{\epsilon\to 0} ucp \int_0^{\cdot} (Y_{s+\epsilon}+Y_s) \frac{X_{s+\epsilon}-X_s}{2\epsilon} ds && \text{"intégrale symétrique"},\\
[X,X] &= \lim_{\epsilon\to 0} ucp \int_0^{\cdot} \frac{(X_{s+\epsilon}-X_s)^2}{\epsilon} ds && \text{"variation quadratique"},\\
[X,X,X] &= \lim_{\epsilon\to 0} ucp \int_0^{\cdot} \frac{(X_{s+\epsilon}-X_s)^3}{\epsilon} ds && \text{"variation cubique"}.
\end{aligned}$$

Le terme “ucp” dans la définition précédente signifie la “convergence en probabilité uniforme sur les compacts”. Rappelons qu'une famille de processus $(Y^\epsilon)_{\epsilon>0}$ est dite converger en probabilité vers Y uniformément sur les compacts lorsque $\epsilon \to 0$ si

$$\forall T>0, \forall \delta>0: \quad \lim_{\epsilon\to 0} P\big(\sup_{t\in[0,T]} |X_t^\epsilon - X_t| > \delta\big) = 0.$$

On peut démontrer que l'intégrale progressive (resp. l'intégrale symétrique, la variation quadratique) généralise l'intégrale d'Itô (resp. l'intégrale de Stratonovich, le crochet) lorsque Y est adapté et X est une semimartingale (resp. X et Y sont deux semimartingales, X est une semimartingale). Le lecteur intéressé pourra consulter [35] pour plus de détails à propos de ces résultats.

Étant donné que nous sommes intéressés par une formule de changement de variable, il nous parait important de rappeler le résultat suivant, démontré par F. Russo et P. Vallois [36, 37].

Proposition 1.1.9 *Soit $X = \{X_t, t \in \mathbb{R}_+\}$ un processus continu. L'intégrale progressive $\int_0^{\cdot} f'(X_s) d^- X_s$ existe pour tout $f \in C^2(\mathbb{R})$ si et seulement si X possède une variation quadratique. Dans ce cas,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^- X_s + \frac{1}{2}\int_0^t f''(X_s) d[X,X]_s, \quad t \in \mathbb{R}_+.$$

La proposition précédente illustre un phénomène typique dans le cadre de la théorie de Russo-Vallois: l'existence de $\int_0^{\cdot} f'(X_s) d^- X_s$ est équivalente à l'existence de la formule de type Itô.

Considérons maintenant le cas où $X = X^H$ est un mouvement brownien fractionnaire d'indice de Hurst $H \in (0,1)$. D'après, par exemple, le Corollaire 2.1 de [24], on a

$$[X^H, X^H]_t = \begin{cases} 0 & \text{si } H > \frac{1}{2} \\ t & \text{si } H = \frac{1}{2} \\ +\infty & \text{si } H < \frac{1}{2} \end{cases}.$$

On déduit de la Proposition 1.1.9 que, pour toute fonction $f \in C^2(\mathbb{R})$ et tout $t \geqslant 0$,

$$\begin{aligned}
f(X_t^H) &= f(X_0^H) + \int_0^t f'(X_s^H) d^- X_s^H \quad \text{si } H > \tfrac{1}{2},\\
f(X_t^H) &= f(X_0^H) + \int_0^t f'(X_s^H) d^- X_s^H + \frac{1}{2}\int_0^t f''(X_s^H) ds \quad \text{si } H = \tfrac{1}{2}.
\end{aligned}$$

La dernière formule n'est autre que la formule d'Itô classique donnée dans le Théorème 1.1.6.

En 2002, M. Errami et F. Russo [11] ont découvert la formule de changement de variable suivante pour des processus à variation cubique forte finie.

Proposition 1.1.10 *Soit $X = \{X_t, t \in \mathbb{R}_+\}$ un processus continu à variation cubique forte finie, et soit $f \in C^3$. Alors,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) d^\circ X_s - \frac{1}{12} \int_0^t f'''(X_s) d[X, X, X]_s, \quad t \in \mathbb{R}_+.$$

Les mêmes autheurs ont démontré aussi une extension multidimensionnelle de la Proposition 1.1.10 qui est donnée par la proposition suivante. (Le lecteur intéressé peut consulter [11] pour n'importe quelle notion qui n'est pas expliquée dans (1.1.3).)

Proposition 1.1.11 *Soit $F \in C^3(\mathbb{R}^n)$ et $X = (X^1, \ldots, X^n)$ un vecteur de processus continus ayant tous leurs 3-covariances mutuelles fortes. Alors,*

$$F(X_t) = F(X_0) + \int_0^t \nabla F(X).d^\circ X - \frac{1}{12}[X, \mathrm{Hess} F(X), X^T](t), \quad t \in \mathbb{R}_+. \tag{1.1.3}$$

Revenons maintenant au cas spécifique du mouvement brownien fractionnaire. En considérant l'intégrale symétrique $\int_0^t f'(X_s^H) d^\circ X_s^H$, on s'attend à aller au-delà du seuil $H = \frac{1}{2}$. C'est effectivement ce qu'il se passe. Plus précisément, nous avons le résultat suivant, qui fut démontré simultanément et indépendamment d'une part par Cheredito et Nualart [9] et d'autre part par Gradinaru, Nourdin, Russo et Vallois [13] en 2004.

Théorème 1.1.12 (Une formule de type Itô pour le mbf avec $H > \frac{1}{6}$) *Soit $H > \frac{1}{6}$ et soit $f \in C^6(\mathbb{R})$. Alors $\int_0^t f'(X_s^H) d^\circ X_s^H$ existe et on a*

$$f(X_t^H) = f(X_0^H) + \int_0^t f'(X_s^H) d^\circ X_s^H,\ t \in \mathbb{R}_+.$$

Comme c'était déjà le cas dans la Proposition 1.1.9, on voit que l'existence de l'intégrale symétrique $\int_0^\cdot f'(X_s) d^\circ X_s$ est démontrée en même temps que la formule de changement de variable qui lui est associée.

Le Théorème 1.1.12 ne dit rien à propos du cas $H \leqslant \frac{1}{6}$. En fait, en faisant appel à un théorème central limite convenable on peut prouver que $\int_0^t (X_s^H)^2 d^\circ X_s^H$ n'existe pas lorsque $H < \frac{1}{6}$ et qu'elle existe, en loi seulement, dans le cas critique $H = \frac{1}{6}$. Comme conséquence, ce n'est pas possible de trouver une formule de changement de variable pour $f(x) = x^3$ lorsque $H < \frac{1}{6}$. Pour avoir une compréhension complète du phénomène, il reste à comprendre ce qui se passe dans le cas critique $H = \frac{1}{6}$. La réponse a été donnée en 2010 par Nourdin, Réveillac et Swanson [30]. En utilisant des techniques du calcul de Malliavin, ils ont découvert une formule de type Itô dans laquelle le terme additionnel prend la forme d'une intégrale de Wiener par rapport à un mouvement brownien indépendant.

Théorème 1.1.13 (Formule de type Itô pour le mbf avec $H = \frac{1}{6}$) *Pour tout $f \in C^\infty(\mathbb{R})$, on a*

$$f(X_t^{\frac{1}{6}}) \overset{\text{law}}{=} f(0) + \int_0^t f'(X_s^{\frac{1}{6}}) d^\circ X_s^{\frac{1}{6}} - \frac{\kappa_3}{12} \int_0^t f'''(X^{\frac{1}{6}}) dW_s, \quad t \geqslant 0.$$

Ici, W est un mouvement brownien standard indépendant de $X^{\frac{1}{6}}$, $\kappa_3 \simeq 2.322$, et $\int_0^t f'(X_s^{\frac{1}{6}})d^\circ X_s^{\frac{1}{6}}$ existe comme une limite stable en loi (lorsque $n \to \infty$) de la somme trapézoïdale de Riemann suivante:

$$\frac{1}{2}\sum_{k=0}^{\lfloor 2^n t\rfloor -1}\big(f'(X_{k2^{-n}}^{\frac{1}{6}})+f'(X_{(k+1)2^{-n}}^{\frac{1}{6}})\big)\big(X_{(k+1)2^{-n}}^{\frac{1}{6}}-X_{k2^{-n}}^{\frac{1}{6}}\big). \tag{1.1.4}$$

Autres formules de type Itô

En considérant la méthode du point médian à la place de la méthode trapézoïdale dans (1.1.4), on obtient une autre formule de type Itô avec une autre valeur critique de H (précisement, $H = \frac{1}{4}$). Ce fait, découvert par Nourdin et Réveillac dans [29], a historiquement précédé le Théorème 1.1.13. Un peu plus tard, Harnett et Nualart [15, 14, 16] ont étendu une partie des résultats précédents à une large classe de processus gaussiens. Tous ces résultats ont en commun l'utilisation du calcul de Malliavin pour montrer des convergences en loi.

Dans notre description historique, nous ne nous sommes focalisés que sur certains façons dont on peut construire un calcul stochastique lorsque l'intégrateur est le mouvement brownien fractionnaire. Ceci car notre but premier est de décrire les résultats qui ont inspiré notre travail. Mais il existe d'autres approches intéressantes pour construire une telle intégrale stochastique. Une d'elle est celle obtenue par Christian Bender [3] en 2003 en utilisant le produit de Wick et l'intégrale de Pettis dans l'espace des distributions d'Hida. À l'aide de ces techniques, Bender est capable de prouver une formule de changement de variable pour le mouvement brownien fractionnaire X^H, et ce quelle que soit la valeur du paramètre d'Hurst H entre 0 et 1:

$$f(X_t^H) = f(X_0^H) + \int_0^t f'(X_s^H)dX_s^H + H\int_0^t u^{2H-1}f''(X_s^H)ds, \quad t \geqslant 0.$$

1.1.3 Un résumé de nos résultats principaux

Nous allons maintenant présenter nos propres résultats.

Fluctuations des variations du mbftb

Notre première étude se concentre sur le comportement asymptotique des variations du mbftb. Elle correspond au contenu d'un article accepté pour publication dans le journal *Bernoulli*. Dans cette sous-section, nous présentons seulement les résultats. Pour les détails et les preuves, nous renvoyons au chapitre 2.

Soit Z un mbftb comme dans la Définition 1.1.1. (Dans la suite, on écrira Z au lieu de Z^H pour simplifier les notations.) Pour tout entier naturel p, on considère le processus de la variation d'ordre p associée à Z, défini par:

$$R_n^{(p)}(t) = \sum_{k=0}^{\lfloor 2^n t\rfloor -1}(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p, \quad n \in \mathbb{N}, \quad t \geqslant 0.$$

Il faut noter, d'une manière plus précise, qu'on est en train de travailler avec la variation d'ordre p de Z, au sens classique du mot, seulement lorsque p est pair. Lorsque p est impair on travaille plutôt avec ce qu'on appelle la variation *signée* d'ordre p de Z. Le lecteur intéressé peut consulter [2, 10] pour plus d'informations sur les variations d'ordre p.

La version renormalisée de $R_n^{(p)}$ est

$$S_n^{(p)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \big((Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p - E[(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p]\big), \quad n\in\mathbb{N}, \quad t\geqslant 0,$$

où $\kappa > 0$ est une certaine constante qu'il reste à déterminer. La valeur de cette dernière sera à choisir pour assurer la convergence au sens des distributions fini-dimensionnelles de $S_n^{(p)}$ vers une limite non-dégénérée, ce qui est exactement notre but dans cette section. Pour l'atteindre, nous allons utiliser une stratégie classique dans ce genre de problème, qui consiste à développer le monôme x^p en fonction des polynômes d'Hermite H_r ($H_1(x) = x$, $H_2(x) = x^2 - 1$, etc.). En suivant cette stratégie, notre problème sera réduit à l'analyse jointe des quantités suivantes:

$$U_n^{(r)}(t) = 2^{-n\widetilde{\kappa}} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} H_r(Z_{(k+1)2^{-n}} - Z_{k2^{-n}}), \quad n\in\mathbb{N}, \quad t\geqslant 0, \quad r\in\mathbb{N}^*, \tag{1.1.5}$$

où $\widetilde{\kappa} > 0$ est une certaine constante qui dépend à priori de r. Mais, vu qu'on ne peut pas séparer X de Y dans Z dans la définition de $U_n^{(r)}$, il est difficile de travailler directement avec (1.1.5) (voir aussi [21, Problem 5.1]). Pour cette raison, en suivant une idée introduite par Khosnevisan et Lewis [20] dans leur étude du cas $H = 1/2$ (c'est-à-dire lorsque Z est un mbi), on va plutôt analyser $U_n^{(r)}$ en utilisant une certaine suite de temps d'arrêts de Y. L'idée de Khoshnevisan et Lewis est la suivante: en arrêtant le mouvement brownien Y lorsqu'il atteint certains niveaux, et en évaluant Z en ces temps, on peut effectivement séparer X de Y. Pour être plus précis, introduisons la collection suivante de temps d'arrêts (par rapport à la filtration naturelle de Y), notée

$$\mathscr{T}_n = \{T_{k,n} : k\geqslant 0\}, \quad n\geqslant 0, \tag{1.1.6}$$

qui est à son tour exprimée en termes d'une suite de temps d'atteintes d'une partition dyadique de l'axe réel. Plus précisement, soit $\mathscr{D}_n = \{j2^{-n/2} : j\in\mathbb{Z}\}$, $n\geqslant 0$, la partition dyadique (de $\mathbb{R}$) d'ordre $n/2$. Pour tout $n\geqslant 0$, les temps d'arrêts $T_{k,n}$ qui apparaissent dans (1.1.6) sont donnés par la définition récursive suivante: $T_{0,n} = 0$, et

$$T_{k,n} = \inf\big\{s > T_{k-1,n} : \quad Y(s)\in\mathscr{D}_n\setminus\{Y(T_{k-1,n})\}\big\}, \quad k\geqslant 1.$$

Il est à noter que la définition de $T_{k,n}$, et par conséquent celle de $\mathscr{T}_n$, fait appel seulement au mouvement brownien Y. De plus, pour tout $n\geqslant 0$, le processus stochastique discret

$$\mathscr{Y}_n = \{Y(T_{k,n}) : k\geqslant 0\}$$

définit une marche aléatoire simple sur $\mathscr{D}_n$. Dans [20], les auteurs ont prouvé que, lorsque n tend vers l'infini, la collection $\{T_{k,n} : 1\leqslant k\leqslant 2^n t\}$ approxime la partition dyadique $\{k2^{-n} : 1\leqslant k\leqslant 2^n t\}$ d'ordre n de l'intervale $[0,t]$ (voir [20, Lemma 2.2] pour la formulation précise de ce résultat). En se basant sur ce fait, on peut introduire l'homologue de (1.1.5) basé sur $\mathscr{T}_n$, qui n'est autre que

$$V_n^{(r)}(t) = 2^{-n\widetilde{\kappa}} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_r(Z_{T_{k+1,n}} - Z_{T_{k,n}}), \quad n \in \mathbb{N}, \quad t \geqslant 0, \quad r \in \mathbb{N}^*.$$

Nous sommes maintenant prêts à énoncer le résultat principal de cette section.

Théorème 1.1.14 *Pour tout entier $N \geqslant 1$, les deux convergences en loi suivantes ont lieu au sens des distributions fini-dimensionnelles lorsque $n \to \infty$.*

(1) Supposons $H \leqslant \frac{1}{2}$. On a alors

$$\begin{aligned} & 2^{-n/4} \left\{ \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{2r-1}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}})) : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0} \\ \xrightarrow{\text{fdd}} \quad & \left\{ \sigma_{2r-1} E_t^{(r)} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}, \end{aligned}$$

où σ_{2r-1} est une constante (explicite) et $E = (B^{(1)} \circ Y, \dots, B^{(N)} \circ Y)$, avec $B = (B^{(1)}, \dots, B^{(N)})$ un mouvement brownien N-dimensionnel sur $\mathbb{R}$ indépendant de Y.

(2) Supposons $H < \frac{3}{4}$. On a alors

$$\begin{aligned} & 2^{-3n/4} \left\{ \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{2r}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}})) : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0} \\ \xrightarrow{\text{fdd}} \quad & \left\{ \sigma_{2r} \int_{-\infty}^{\infty} L_t^x(Y) dB_x^{(r)} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}, \end{aligned} \tag{1.1.7}$$

où σ_{2r} est une constante (explicite), $B = (B^{(1)}, \dots, B^{(N)})$ est un mouvement brownien N-dimensionnel sur $\mathbb{R}$ indépendant de Y et $L_t^x(Y)$ est le temps local passé par Y avant le temps t au niveau x.

Noter que le processus $\{\int_{\mathbb{R}} L_t^x(Y) dB_x^{(r)}\}_{t \geqslant 0}$ qui apparait dans (1.1.7) est le *mouvement brownien en scène aléatoire* introduit par Kesten et Spitzer dans [18].

Comme corollaire du Théorème 1.1.14, on peut déduire les fluctuations des variations d'ordre p de Z.

Corollaire 1.1.15 *Pour tout entier $N \geqslant 1$, les deux convergences en loi suivantes ont lieu au sens des distributions fini-dimensionnelles lorsque $n \to \infty$.*

(1) Supposons $H \leqslant \frac{1}{2}$. On a alors

$$\begin{aligned} & \left\{ 2^{\frac{-n}{4}(1-(4r-2)H)} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (Z_{T_{k+1,n}} - Z_{T_{k,n}})^{2r-1} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0} \\ \xrightarrow{\text{fdd}} \quad & \left\{ \sum_{k=1}^{r} a_{r,k} \sigma_{2k-1} E_t^{(k)} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}, \end{aligned} \tag{1.1.8}$$

où $a_{r,k} = \displaystyle\sum_{l=0}^{k-1} \frac{(-1)^l (2(r+k-l-1))!}{l!(2(k-l)-1)!(r+k-l-1)!2^{r+k-1}}.$

(2) Supposons $H < \frac{3}{4}$*. On a alors*

$$\left\{ 2^{\frac{-3n}{4}(1-\frac{4rH}{3})} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \big((Z_{T_{k+1,n}} - Z_{T_{k,n}})^{2r} - 2^{-nrH} b_{r,0} \big) \,:\, 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}$$

$$\xrightarrow{\text{fdd}} \left\{ \sum_{k=1}^{r} b_{r,k} \sigma_{2k} \int_{-\infty}^{\infty} L_t^x(Y) dB_x^{(k)} \,:\, 1 \leqslant r \leqslant N \right\}_{t \geqslant 0},$$

où $b_{r,k} = \displaystyle\sum_{l=0}^{k} \frac{(-1)^l (2(r+k-l))!}{l!(2(k-l))!(r+k-l)!2^{r+k}}.$

Observons que $b_{r,0} = E\big[(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r}\big] = E[N^{2r}]$, avec $N \sim \mathcal{N}(0,1)$.

Le Corollaire 1.1.15 contient, comme cas particuliers ($H = 1/2$ et $r = 2, 3, 4$), les Théorèmes 3.2, 4.4 et 4.5 de Khoshnevisan et Lewis [20].

Vers une formule de type Itô pour le mbftb en dimension 1

Le problème qu'on cherche à résoudre ici est de trouver une formule de type Itô satisfaite par le mbftb Z. Pour toute fonction suffisamment régulière f, on s'attend à une formule ayant la forme suivante:

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s) dZ_s + \text{reste}, \quad t \geqslant 0. \tag{1.1.9}$$

Bien sûr, dans (1.1.9) on doit d'abord définir $\int_0^t f'(Z_s)dZ_s$ et ensuite découvrir la forme exacte du reste.

Rappelons que Z n'est pas un processus gaussien et qu'il est très irrégulier (précisement: il est Höldérien d'ordre α avec α strictement plus petit que $H/2$). À cause de cela, il semble difficile de définir l'intégrale stochastique comme une limite d'une somme de Riemann par rapport à une partition *déterministe* de l'axe des temps. C'est pour cela que nous allons plutôt travailler avec la partition stochastique $T_{k,n}$ qu'on a introduite dans la sous-section précédente. À ce stade, on a besoin de définir d'autres objets. Pour tout entier $n \geqslant 0$, $k \in \mathbb{Z}$ et tout nombre réel $t \geqslant 0$, on note $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) le nombre de *montées* (resp. *descentes*) dans l'intervalle $[j2^{-n/2}, (j+1)2^{-n/2}]$ durant les $\lfloor 2^n t\rfloor$ premières étapes de la marche aléatoire $\{Y(T_{k,n})\}_{k \geqslant 1}$. Autrement dit,

$$\begin{aligned} U_{j,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^n t\rfloor - 1 : \\ &\qquad Y(T_{k,n}) = j2^{-n/2} \text{ et } Y(T_{k+1,n}) = (j+1)2^{-n/2}\big\}; \\ D_{j,n}(t) &= \sharp\big\{k = 0, \ldots, \lfloor 2^n t\rfloor - 1 : \\ &\qquad Y(T_{k,n}) = (j+1)2^{-n/2} \text{ et } Y(T_{k+1,n}) = j2^{-n/2}\big\}. \end{aligned}$$

On dit aussi que $\varphi : \mathbb{R}^2 \to \mathbb{R}$ est symétrique (resp. anti-symétrique) si $\varphi(x,y) = \varphi(y,x)$ (resp. $\varphi(x,y) = -\varphi(y,x)$) pour tout $x, y \in \mathbb{R}$. Le lemme suivant correspond à [20, Lemma 2.4].

Lemme 1.1.16 *Si φ est symétrique, alors*

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \varphi(Z_{T_{k+1,n}}, Z_{T_{k,n}}) = \sum_{j\in\mathbb{Z}} \varphi(X_{(j+1)2^{-\frac{n}{2}}}, X_{j2^{-\frac{n}{2}}})(U_{j,n}(t) + D_{j,n}(t)).$$

Si φ est anti-symétrique, alors

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \varphi(Z_{T_{k+1,n}}, Z_{T_{k,n}}) = \sum_{j\in\mathbb{Z}} \varphi(X_{(j+1)2^{-\frac{n}{2}}}, X_{j2^{-\frac{n}{2}}})(U_{j,n}(t) - D_{j,n}(t)).$$

L'intérêt principal du Lemme 1.1.16 est la séparation de X et Y dans Z. Nous allons notamment l'utiliser pour trouver une représentation plus facile à analyser de

$$V_n^{(r)}(f,t) = \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}\big(f(Z_{T_{k+1,n}} + f(Z_{T_{k,n}})\big)(Z_{T_{k+1,n}} - Z_{T_{k,n}})^r. \tag{1.1.10}$$

Nous choisissons d'appeler $V_n^r(f,t)$ la r-variation à poids (symétrique) de Z.

Fixons une fonction $f:\mathbb{R}\to\mathbb{R}$ appartenant à C_b^∞, l'ensemble des fonctions qui sont C^∞ et bornées avec des dérivées bornées. Dans le chapitre 3 nous prouvons que, pour tout $H\in\left[\frac{1}{6},\frac{1}{2}\right)$, tout entier $r\geqslant 3$ et tout $t>0$,

$$V_n^{(2r-1)}(f,t) \overset{\text{prob}}{\longrightarrow} 0 \quad \text{quand } n\to\infty. \tag{1.1.11}$$

Nous prouvons aussi que, pour tout $H\in\left(\frac{1}{6},\frac{1}{2}\right)$ et tout $t>0$,

$$V_n^{(3)}(f,t) \overset{\text{prob}}{\longrightarrow} 0 \quad \text{quand } n\to\infty, \tag{1.1.12}$$

alors que, lorsque $H=\frac{1}{6}$,

$$\left(X_t, Y_t, V_n^{(3)}(f,t)\right)_{t\geqslant 0} \overset{\text{fdd}}{\to} \left(X_t, Y_t, \int_0^t f(Z_s)d^{\circ 3}Z_s\right)_{t\geqslant 0} \quad \text{quand } n\to\infty. \tag{1.1.13}$$

Autrement dit, (1.1.13) signifie que $V_n^{(3)}(f,\cdot)$ converge stablement en loi (au sens de la convergence des distributions fini-dimensionnelles) vers un processus limite noté $\int_0^\cdot f(Z_s)d^{\circ 3}Z_s$.

La formule de type Taylor suivante se vérifie aisément: pour tous $a,b\in\mathbb{R}$, et pour certaines constantes c_r dont les valeurs explicites sont inutiles pour ce qu'on a à faire,

$$\begin{aligned} f(b)-f(a) &= \frac{1}{2}\big(f'(a)+f'(b)\big)(b-a) - \frac{1}{24}\big(f'''(a)+f'''(b)\big)(b-a)^3 \\ &\quad + \sum_{r=3}^{7} c_r\big(f^{(2r-1)}(a)+f^{(2r-1)}(b)\big)(b-a)^{2r-1} + O(|b-a|^{14}), \end{aligned} \tag{1.1.14}$$

où $|O(|b-a|^{14})|\leqslant C_f|b-a|^{14}$, avec C_f une constante qui dépend seulement f. En insérant (1.1.11), (1.1.12) et (1.1.13) dans (1.1.14), on en déduit (voir le chapitre 3 pour les détails) les deux premiers points du théorème suivant.

Théorème 1.1.17 *Soit $f\in C_b^\infty$ et soit $t>0$.*

1. *Si $H > \frac{1}{6}$ alors*

$$f(Z_t) - f(0) = \int_0^t f'(Z_s) d^\circ Z_s, \tag{1.1.15}$$

où $\int_0^t f'(Z_s)d^\circ Z_s$ est la limite en probabilité de $V_n^{(1)}(f', t)$ défini dans (1.1.10). (Son existence fait partie de la conclusion.)

2. *Si $H = \frac{1}{6}$ alors, avec $\kappa_3 \simeq 2.322$,*

$$f(Z_t) - f(0) + \frac{\kappa_3}{12}\int_0^t f'''(Z_s) d^{\circ 3} Z_s \overset{\text{(loi)}}{=} \int_0^t f'(Z_s) d^\circ Z_s. \tag{1.1.16}$$

où $\int_0^\cdot f'(Z_s)d^\circ Z_s$ désigne la limite en loi de $V_n^{(1)}(f', \cdot)$ défini dans (1.1.10). (Son existence fait partie de la conclusion.) De plus, pour tout $t \geqslant 0$,

$$\int_0^t f'''(Z_s) d^{\circ 3} Z_s := \int_0^{Y_t} f'''(X_s) dW_s,$$

où W est un mouvement brownien sur $\mathbb{R}$ indépendant du couple (X, Y) définissant Z, et où l'intégrale par rapport à W est au sens de Wiener-Itô.

3. *Si $H < \frac{1}{6}$ alors*

$$V_n^{(1)}(x \mapsto x^2, t) \text{ ne converge pas, même stablement en loi.} \tag{1.1.17}$$

Ceci signifie qu'on ne peut pas espérer de formule de changement de variable pour $f(x) = x^3$.

La preuve de (1.1.15) dans le cas $H = \frac{1}{2}$ a déjà été donnée dans [21]. Donnons une explication de la preuve de (1.1.17). En utilisant

$$b^3 - a^3 = \frac{3}{2}(a^2 + b^2)(b - a) - \frac{1}{2}(b - a)^3$$

(qui correspond à l'identité (1.1.14) avec $f(x) = x^3$), nous pouvons écrire, avec $\mathbf{1}$ désignant la fonction constante égale à 1,

$$\begin{aligned} V_n(x \mapsto x^2, t) - \frac{1}{3}Z_t^3 &= \frac{1}{6}V_n^{(3)}(\mathbf{1}, t) + \frac{1}{3}\sum_{k=0}^{\lfloor 2^n t \rfloor - 1}(Z_{T_{k+1,n}}^3 - Z_{T_{k,n}}^3) - \frac{1}{3}Z_t^3 \\ &= \frac{1}{6}V_n^{(3)}(\mathbf{1}, t) + \frac{1}{3}(Z_{T_{\lfloor 2^n t \rfloor, n}}^3 - Z_t^3). \end{aligned}$$

Comme $T_{\lfloor 2^n t \rfloor, n} \to t$ presque sûrement et vu que Z est à trajectoires continues, on en déduit que $Z_{T_{\lfloor 2^n t \rfloor, n}} \to Z_t$ lorsque $n \to \infty$ presque sûrement. Ainsi, la convergence stable de $V_n(x \mapsto x^2, t)$ est équivalente à la convergence stable de $V_n^{(3)}(\mathbf{1}, t)$. Mais nous avons démontré dans le Corollaire 1.1.15 (voir (1.1.8)) que $2^{-n(1-6H)/4}V_n^{(3)}(\mathbf{1}, t)$ converge en loi vers une limite non dégénérée. Comme ceci est clairement en contradiction avec la convergence de $V_n^{(3)}(\mathbf{1}, t)$, on en déduit la conclusion (1.1.17).

Vers une formule de type Itô pour le mbftb en dimension 2

Notre but dans cette sous-section est d'étendre la formule de changement de variable pour le mbftb en dimension 1, qui était donnée dans la sous-section précédente, au cas de la dimension 2. Considérons

$$Z_t := (Z_t^1, Z_t^2) = (X_{Y_t}^1, X_{Y_t}^2), \ \ t \geqslant 0,$$

où X^1, X^2 sont deux mouvements browniens fractionnaires sur $\mathbb{R}$ indépendants ayant le même paramètre d'Hurst $H \in (0,1)$, et Y un mouvement brownien standard indépendant de (X^1, X^2). La définition suivante est d'une grande importance dans la suite.

Définition 1.1.18 *Soit $f : \mathbb{R}^2 \to \mathbb{R}$ une fonction de classe C^1, et fixons un temps $t > 0$.*
1) On définit $\int_0^t \nabla f(X_s)dX_s$ comme étant la limite en probabilité (si elle existe), lorsque $n \to \infty$, de

$$\begin{aligned} & O_n(f,t) \qquad\qquad (1.1.18)\\ = & \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor -1} \frac{\partial f}{\partial x}\Big(\frac{X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}}{2}, \frac{X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}}}{2}\Big)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big)\\ & + \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor -1} \frac{\partial f}{\partial y}\Big(\frac{X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}}{2}, \frac{X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}}}{2}\Big)\big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big). \end{aligned}$$

2) Lorsque $O_n(f,t)$ défini par (1.1.18) ne converge pas en probabilité mais converge stablement en loi, on note sa limite $\int_0^t \nabla f(X_s)d^*X_s$.

On peut maintenant énoncer un premier résultat préliminaire, qui représente l'extension au cas bidimensionnel du Théorème 1.1.13 et qui est la première étape vers l'obtention de la formule de changement de variable pour le mbftb en dimension 2.

Théorème 1.1.19 *Soit $f : \mathbb{R}^2 \to \mathbb{R}$ une fonction appartenant à C_b^∞, et fixons un temps $t > 0$.*

1. *Si $H > 1/6$ alors $\int_0^t \nabla f(X_s)dX_s$ est bien défini, et on a*
$$f(X_t) = f(0) + \int_0^t \nabla f(X_s)dX_s.$$
2. *Si $H = 1/6$ alors $\int_0^t \nabla f(X_s)d^*X_s$ est bien défini, et on a*
$$f(X_t) - f(0) - \int_0^t D^3 f(X_s)d^3X_s \overset{law}{=} \int_0^t \nabla f(X_s)d^*X_s$$
où $\int_0^t D^3 f(X_s)d^3X_s$ désigne
$$\begin{aligned}\int_0^t D^3 f(X_s)d^3X_s &= \kappa_1 \int_0^t \frac{\partial^3 f}{\partial x^3}(X_s^1, X_s^2)dB_s^1 + \kappa_2 \int_0^t \frac{\partial^3 f}{\partial y^3}(X_s^1, X_s^2)dB_s^2\\ &+\kappa_3 \int_0^t \frac{\partial^3 f}{\partial x^2 \partial y}(X_s^1, X_s^2)dB_s^3 + \kappa_4 \int_0^t \frac{\partial^3 f}{\partial x \partial y^2}(X_s^1, X_s^2)dB_s^4\end{aligned}$$
avec $B = (B^1, \dots, B^4)$ un mouvement brownien 4-dimensionnel indépendant de X, $\kappa_1^2 = \kappa_2^2 = \frac{1}{96}\sum_{r\in\mathbb{Z}} \rho^3(r)$, $\kappa_3^2 = \kappa_4^2 = \frac{1}{32}\sum_{r\in\mathbb{Z}} \rho^3(r)$ et $\rho(r) = \frac{1}{2}(|r+1|^{\frac{1}{3}} + |r-1|^{\frac{1}{3}} - 2|r|^{\frac{1}{3}})$.

3. *Si $H < 1/6$, pour $f(x,y) = x^3$ l'intégrale $\int_0^t \nabla f(X_s) d^* X_s$ n'existe pas, même stablement en loi. Donc, c'est impossible d'avoir une formule de type Itô dans ce cas.*

Pour passer du mbf au mbftb, notre stratégie consiste à essayer d'utiliser les mêmes objets introduits dans l'étude précédente correspondant à la dimension 1. Ainsi, il est naturel de poser la définition suivante pour l'intégrale par rapport au mbftb en dimension 2. À nouveau, cette définition est basée sur la collection $\mathscr{T}_n$ de temps d'arrêts définie dans (1.1.6).

Définition 1.1.20 *Soit $f : \mathbb{R}^2 \to \mathbb{R}$ une fonction de classe C^1, et fixons un temps $t > 0$. On définit $\int_0^t \nabla f(Z_s) dZ_s$ comme étant la limite en probabilité (si elle existe), lorsque $n \to \infty$, de*

$$\begin{aligned} \tilde{O}_n(f,t) \quad := \quad & \sum_{j=0}^{\lfloor 2^{n/2} t \rfloor - 1} \frac{\partial f}{\partial x}\left(\frac{Z^1_{T_{j+1,n}} + Z^1_{T_{j,n}}}{2}, \frac{Z^2_{T_{j+1,n}} + Z^2_{T_{j,n}}}{2} \right) \left(Z^1_{T_{j+1,n}} - Z^1_{T_{j,n}} \right) \\ & + \sum_{j=0}^{\lfloor 2^{n/2} t \rfloor - 1} \frac{\partial f}{\partial y}\left(\frac{Z^1_{T_{j+1,n}} + Z^1_{T_{j,n}}}{2}, \frac{Z^2_{T_{j+1,n}} + Z^2_{T_{j,n}}}{2} \right) \left(Z^2_{T_{j+1,n}} - Z^2_{T_{j,n}} \right). \end{aligned} \tag{1.1.19}$$

Si $\tilde{O}_n(f,t)$ défini dans (1.1.19) ne converge pas en probabilité mais converge stablement en loi, on note sa limite $\int_0^t \nabla f(Z_s) d^* Z_s$.

À l'aide de la définition précédente et du Théorème 1.1.19 nous arrivons à la formule de changement de variable pour le mbftb en dimension 2 suivante, qui représente la découverte principale de cette sous-section et qui résoud le problème posé au début.

Théorème 1.1.21 *Soit $f : \mathbb{R}^2 \to \mathbb{R}$ une fonction appartenant à C_b^∞, et fixons un temps $t > 0$.*

1. *Si $H > 1/6$ alors $\int_0^t \nabla f(Z_s) dZ_s$ est bien défini, et on a*

$$f(Z_t) = f(0) + \int_0^t \nabla f(Z_s) dZ_s.$$

2. *Si $H = 1/6$ alors $\int_0^t \nabla f(Z_s) d^* Z_s$ est bien défini, et on a*

$$f(Z_t) - f(0) - \int_0^t D^3 f(Z_s) d^3 Z_s \overset{law}{=} \int_0^t \nabla f(Z_s) d^* Z_s$$

où $\int_0^t D^3 f(Z_s) d^3 Z_s$ désigne

$$\begin{aligned} \int_0^t D^3 f(Z_s) d^3 Z_s \quad = \quad & \kappa_1 \int_0^{Y_t} \frac{\partial^3 f}{\partial x^3}(X_s) dB^1_s + \kappa_2 \int_0^{Y_t} \frac{\partial^3 f}{\partial y^3}(X_s) dB^2_s \\ & + \kappa_3 \int_0^{Y_t} \frac{\partial^3 f}{\partial x^2 \partial y}(X_s) dB^3_s + \kappa_4 \int_0^{Y_t} \frac{\partial^3 f}{\partial x \partial y^2}(X_s) dB^4_s, \end{aligned}$$

avec $B = (B^1, \ldots, B^4)$ un mouvement brownien 4-dimensionnel sur $\mathbb{R}$ indépendant de X, et les constantes $\kappa_1, \ldots, \kappa_4$ sont les mêmes que dans le Théorème 1.1.19. (B est aussi indépendant de Y.)

3. *Si $H < 1/6$, pour $f(x,y) = x^3$ l'intégrale $\int_0^t \nabla f(Z_s) d^* Z_s$ n'existe pas, même stablement en loi. Donc, c'est impossible d'avoir une formule de type Itô dans ce cas.*

1.2 Introduction to the thesis

Our main goal in this PhD thesis is to build a stochastic calculus for the so-called fractional Brownian motion in Brownian time (fBmBt in short). We have organized our manuscript as follows. In this first chapter, we will introduce the fBmBt and provide some useful basic results about it; we will also give a flavour of the main results we have obtained during our PhD thesis. All the details will be then provided in the next three chapters. More precisely, in Chapter 2 we will carefully analyse the fluctuations of the power variations of the fBmBt. In Chapter 3, we will explain how to deduce from this an Itô's type formula for fBmBt in the one-dimensional case. Finally, we will devote Chapter 4 to the study of the two-dimensional case.

1.2.1 Basic definitions and some known results about fBmBt

Consider a two-sided fractional Brownian motion (fBm) X^H with Hurst parameter $H \in (0,1)$, as well as a standard Brownian motion Y on $\mathbb{R}^+$ independent of X^H.

Definition 1.2.1 *The process $Z^H = X^H \circ Y$ is called the fractional Brownian motion in Brownian time (fBmBt).*

At this stage, it is straightforward to check that Z^H is a self-similar process of order $H/2$ with stationary increments, which is not Gaussian (for this latter point, see Proposition 1.2.3 below).

The case $H = \frac{1}{2}$

Historically, the study of fBmBt started with the case $H = \frac{1}{2}$, that is, with the case where $X^{\frac{1}{2}}$ is a *standard* Brownian motion. One then recovers the celebrated *iterated Brownian motion* (iBm). This terminology was coined by Burdzy in 1993 (see [5]). But the idea of introducing iBm is actually older than that. Indeed, Funaki [12] discovered in 1979 that iBm may be used to represent the solution of the following parabolic partial differential equation:

$$\frac{\partial u}{\partial t} = \frac{1}{8}\left(\frac{\partial u}{\partial x}\right)^4, \ (t,x) \in (0,\infty) \times \mathbb{R}.$$

We refer the interested reader to the research works of Nane (see, e.g., [23] and the references therein) for many other interesting relationships between iterated processes and partial differential equations.

In 1998, Burdzy and Khoshnevisan [7] showed that iBm can be somehow considered as the canonical motion in an independent Brownian fissure. As such, iBm reveals to be a suitable candidate to model a diffusion in a Brownian crack. To support their claim, they have shown that the two components of a reflected two-dimensional Brownian motion in a Wiener sausage of width $\epsilon > 0$ converge to the usual Brownian motion and iterated Brownian motion, respectively, when ϵ tends to zero.

It can be shown that $Z^{\frac{1}{2}}$ is not a Markov process in its natural filtration. In addition, according to [6], the sample paths of $Z^{\frac{1}{2}}$ have finite quartic variation almost surely. As an immediate consequence of this last fact, it appears that $Z^{\frac{1}{2}}$ admits an infinite quadratic variation on a set of full measure. It means in particular that $Z^{\frac{1}{2}}$ is not a semi-martingale, nor a Dirichlet process.

Having said that, an interesting and admittedly non-trivial† problem was to construct a stochastic integral process of the form $\int_0^{\cdot} f(Z_s^{\frac{1}{2}})dZ_s^{\frac{1}{2}}$, where f is supposed to belong in a sufficiently rich class of smooth functions. This problem was tackled and solved by Khoshnevisan and Lewis [20, 21] in 1999. They were indeed able to construct a stochastic integral with respect to the iBm, after carefully analyzing its power variations. They were also able to associate a change of variable formula.

The case $H \neq \frac{1}{2}$

A natural follow-up of the aforementioned work [20, 21] by Khoshnevisan and Lewis consists in studying the situation where $H \neq \frac{1}{2}$. Even though it may look similar, this change of framework makes the analysis radically different.

Before going into the meat of this PhD thesis, let us review some basic properties of the fBmBt already existing in the literature. In the course of writing this manuscript, we discovered that fBmBt was the subject of a recent PhD thesis by Joseph A. Zadeh [39], written under the supervision of Frederi Viens and defended in Purdue University in 2012. Let us review here some of the results obtained in Joseph's thesis, after specializing them to the case of our fBmBt Z^H. The following result corresponds to Theorem 3.2.4 in the aforementioned PHD thesis.

Theorem 1.2.2 *We have the following uniform modulus of continuity for the fBmBt Z^H: P-a.s.,*

$$\lim_{\eta\to 0} \sup_{0\leqslant |s-t|\leqslant \eta,\, 0\leqslant s,t\leqslant 1} \frac{|Z_t^H - Z_s^H|}{\eta^{\frac{H}{2}} |\log \eta|^{-\frac{1}{2}(1+\frac{3H}{2})}} < \infty.$$

Also, as far as the problem of the density for Z_t^H is concerned, one has the following simple result.

Proposition 1.2.3 *For any $t > 0$, the random variable Z_t^H admits a density $f_H(t,\cdot)$ given by*

$$f_H(t,x) = \int_{\mathbb{R}} \frac{e^{-\frac{x^2}{2|y|^{2H}} - \frac{y^2}{2t}}}{2\pi\sqrt{t|y|^{2H}}}\, dy.$$

An example of application of the previous formula is a bound for the tail of the distribution of Z_1^H, see indeed Corollary 4.1.6 in the Joseph's thesis [39]. Note that a similar result could of course have be obtained for Z_t^H as well, by merely relying to a selfsimilarity argument.

Theorem 1.2.4 *Set $H^* = H^{\frac{1}{2H+2}}$ and $L(H^*) = \frac{1}{2}(H^{-\frac{H}{H+1}} + H^{\frac{1}{H+1}})$. Then, their exist $k(H), K(H) > 0$ depending only on H such that, for all $x > \frac{H^{H+1}}{\sqrt{H}}$,*

$$k(H)x^{-\frac{1}{H+1}} \exp(-L(H^*)x^{\frac{2}{H+1}}) \leqslant P(Z_1^H > x) \leqslant K(H)\exp(-L(H^*)x^{\frac{2}{H+1}}).$$

†Indeed, a folk result by Bichteler and Dellacherie asserts that for $\int MdN$ to exist for a large class of M's, N needs to be a semi-martingale.

Finally, we would like to emphasize that it is claimed in Zadeh's thesis (in his Chapter 5) that a student of Orsingher derived in 2010 the following second-order partial differential equation satisfied by the density $f_H(t,\cdot)$ of Z_t^H, thus demonstrating once again how this process is heavily related to PDEs.

Theorem 1.2.5 *One has*

$$\left(1+\frac{H}{2}\right)t\,\frac{\partial f_H}{\partial t}+t^2\frac{\partial^2 f_H}{\partial t^2}=\frac{H^2}{4}\left(2x\frac{\partial f_H}{\partial x}+x^2\frac{\partial^2 f_H}{\partial x^2}\right).$$

1.2.2 Itô's type formulae: a historical review

We now go into the heart of this thesis, whose main aim is to construct a stochastic calculus for the fBmBt. We start by giving a very brief state of the art, in order to settle our study in the existing context.

Stochastic calculus and Itô's formulae for semi-martingales

Let $(B_t)_{t\geqslant 0}$ be a Brownian motion defined on a probability space $(\Omega,\mathcal{F},P)$. Wiener was the first to give a sense to the integral

$$\int_0^t f(s)dB_s(w),$$

when f is a deterministic function such that $\int_0^t f^2(s)ds<\infty$. This integral is defined globally on Ω, but it is not a path-wise integral. As is well-known, this is because B has an infinite variation almost surely. Then, Itô extended in 1944 (see [17]) the construction of Wiener, and was able to properly define

$$\int_0^t f_s(w)dB_s(w).$$

This time, (f_t) stands for a process adapted to the natural filtration associated to (B_t). In addition, Itô discovered the associated change-of-variable formula which, in its simplest form, can be stated as follows.

Theorem 1.2.6 *(Itô's formula) Let $(B_t)_{t\geqslant 0}$ be a Brownian motion and let f be a C^2 function on $\mathbb{R}$. Then, for any $t\geqslant 0$,*

$$f(B_t)=f(B_0)+\int_0^t f'(B_s)dB_s+\frac{1}{2}\int_0^t f''(B_s)ds.$$

Using his integral, Itô managed to construct a diffusion process of the form

$$X_t=x+\int_0^t a(s,X_s)ds+\int_0^t b(s,X_s)dB_s$$

(where, of course, $a(s,x)$ and $b(s,x)$ are assumed to satisfy adequate conditions). Prior to Itô, it is interesting to remember that, in order to construct a diffusion process with a given infinitesimal generator G, the strategy rather consisted in solving the Kolmogorov equation

$$\frac{\partial}{\partial t}P_t f=GP_t f,$$

and then to use the semi-group $(P_t)_{t\geqslant 0}$ and Kolmogorov Theorem to deduce the diffusion process.

After Itô's discovery, there has been an increasing interest in stochastic calculus among the probabilistic community, and many other fundamental results were quickly released. Since it is of course impossible to list them here, we refer the reader to any textbook (e.g., [22]) dedicated to the fascinating history of stochastic calculus.

To end up this section, let us only recall the Itô's formula in the framework of semi-martingales. It is the way this formula is nowadays taught all around the world.

Theorem 1.2.7 *(Itô's formula for continuous semimartingales) Let $(X_t^1,\dots,X_t^n)_{t\geqslant 0}$ be a continuous semimartingale and let f be a C^2 function on $\mathbb{R}^n$. For any $i,j\in\{1,\dots,n\}$, let us indicate by $\langle X^i,X^j\rangle$ the cross-variation process of X^i and X^j. Then, for all $t\geqslant 0$:*

$$\begin{aligned} f(X_t^1,\dots,X_t^n) &= f(X_0^1,\dots,X_0^n)+\sum_{i=1}^n\int_0^t \frac{\partial f}{\partial x_i}(X_s^1,\dots,X_s^n)dX_s^i \\ &\quad +\frac{1}{2}\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i\partial x_j}(X_s^1,\dots,X_s^n)d\langle X^i,X^j\rangle_s. \end{aligned} \tag{1.2.20}$$

Generalization of stochastic calculus and Itô's type formulae for fBm

Taking into account the huge success of the theory of stochastic calculus for integrals for other classes of processes. Different approaches and frameworks have been proposed, such as the ones of Skorokhod [38], Asch and Potthoff [1], Nualart and Pardoux [32], or Russo and Vallois [35]. Let us pause a little bit to discuss this latter reference. In 1993, Russo and Vallois defined what is nowadays refered to as the *Russo-Vallois integral*. It generalises the classical Itô and Stratonovich integrals. One of the most interesting features of the Russo-Vallois' theory is the possibility to construct integrals with respect to processes that are not semimartingales. As we will see, a popular and typical example of such processes is the fractional Brownian motion (fBm).

Let us recall the basic definitions of the Russo-Vallois theory.

Definition 1.2.8 *If $X=\{X_t,t\in\mathbb{R}_+\}$ and $Y=\{Y_t,t\in\mathbb{R}_+\}$ are two continuous processes, we define, provided the limit exists*

$$\begin{aligned} \int_0^\cdot Y_s d^- X_s &= \lim_{\epsilon\to 0} ucp \int_0^\cdot Y_s \frac{X_{s+\epsilon}-X_s}{\epsilon}ds && \textit{"forward integral"}, \\ \int_0^\cdot Y_s d^+ X_s &= \lim_{\epsilon\to 0} ucp \int_0^\cdot Y_{s+\epsilon} \frac{X_{s+\epsilon}-X_s}{\epsilon}ds && \textit{"backward integral"}, \\ \int_0^\cdot Y_s d^\circ X_s &= \lim_{\epsilon\to 0} ucp \int_0^\cdot (Y_{s+\epsilon}+Y_s) \frac{X_{s+\epsilon}-X_s}{2\epsilon}ds && \textit{"symmetric integral"}, \\ [X,X] &= \lim_{\epsilon\to 0} ucp \int_0^\cdot \frac{(X_{s+\epsilon}-X_s)^2}{\epsilon}ds && \textit{"quadratic variation"}, \\ [X,X,X] &= \lim_{\epsilon\to 0} ucp \int_0^\cdot \frac{(X_{s+\epsilon}-X_s)^3}{\epsilon}ds && \textit{"cubic variation"}. \end{aligned}$$

The term "ucp" in the previous definitions means uniform convergence on compact sets in probability. Recall that a family of processes $(Y^\epsilon)_{\epsilon>0}$ is said to converge in probability to Y

uniformly on compact sets as $\epsilon \to 0$ if

$$\forall T > 0, \forall \delta > 0 : \quad \lim_{\epsilon \to 0} P\big(\sup_{t \in [0,T]} |X_t^\epsilon - X_t| > \delta \big) = 0.$$

One can prove that the forward integral (resp. the symmetric integral, the quadratic variation) generalizes the Itô's integral (resp. the Stratonovich integral, the cross-variation process) when Y is adapted and X is a semimartingale (resp. X and Y are two semimartingales, X is a semimartingale). The reader is referred to [35] for more details about this type of results.

As far as a change-of-variable is concerned, we have the following result proved by F. Russo and P. Vallois [36, 37].

Proposition 1.2.9 *Let $X = \{X_t, t \in \mathbb{R}_+\}$ be a continuous process. The forward integral $\int_0^\cdot f'(X_s)d^-X_s$ exists for all $f \in C^2(\mathbb{R})$ if and only if X has a quadratic variation. In this case,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^-X_s + \frac{1}{2}\int_0^t f''(X_s)d[X,X]_s, \quad t \in \mathbb{R}_+.$$

The previous proposition highlights a phenomenon which is typical in the Russo-Vallois framework: the existence of $\int_0^\cdot f'(X_s)d^-X_s$ turns to be equivalent to the existence of the Itô's type formula.

Let us now consider the case where $X = X^H$ is a fractional Brownian motion of Hurst parameter $H \in (0,1)$. Recall, for example, from Corollary 2.1 in the PHD thesis of Ivan Nourdin [24] that

$$[X^H, X^H]_t = \left\{ \begin{array}{cl} 0 & \text{if } H > \frac{1}{2} \\ t & \text{if } H = \frac{1}{2} \\ +\infty & \text{if } H < \frac{1}{2} \end{array} \right. .$$

We deduce from Proposition 1.2.9 that, for any $f \in C^2(\mathbb{R})$ and $t \geqslant 0$,

$$\begin{aligned} f(X_t^H) &= f(X_0^H) + \int_0^t f'(X_s^H)d^-X_s^H \quad \text{if } H > \tfrac{1}{2}, \\ f(X_t^H) &= f(X_0^H) + \int_0^t f'(X_s^H)d^-X_s^H + \frac{1}{2}\int_0^t f''(X_s^H)ds \quad \text{if } H = \tfrac{1}{2}. \end{aligned}$$

This latter identity is nothing but the classical Itô's formula, as given in Theorem 1.2.6.

In 2002, M. Errami and F. Russo [11] unveiled the following change-of-variable formula for processes with finite strong cubic variation.

Proposition 1.2.10 *Let $X = \{X_t, t \in \mathbb{R}_+\}$ be a continuous process with finite strong cubic variation, and let $f \in C^3$. Then,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^\circ X_s - \frac{1}{12}\int_0^t f'''(X_s)d[X,X,X]_s, \quad t \in \mathbb{R}_+.$$

The same authors also prove a multi-dimensional extension of Proposition 1.2.10 which reads as follows. (We refer to [11] for any unexplained quantity appearing in (1.2.21).)

Proposition 1.2.11 *Let $F \in C^3(\mathbb{R}^n)$ and $X = (X^1, \ldots, X^n)$ be a vector of continuous processes having all its mutual strong 3-covariations. Then,*

$$F(X_t) = F(X_0) + \int_0^t \nabla F(X).d^\circ X - \frac{1}{12}[X, \mathrm{Hess}F(X), X^T](t), \quad t \in \mathbb{R}_+. \tag{1.2.21}$$

Let us go back to the specific fractional Brownian motion case. By considering the symmetric integral $\int_0^t f'(X_s^H)d^\circ X_s^H$ and in view of Proposition 1.2.10, one may expect to go beyond the threshold $H = \frac{1}{2}$. This is indeed what happens, as evidenced by the following result, which was shown simultaneously and independently by Cheredito and Nualart [9] on one hand and Gradinaru, Nourdin, Russo and Vallois [13] on the other hand in 2004.

Theorem 1.2.12 *(Itô's formula for fractional Brownian motion with $H > \frac{1}{6}$)*
Let $H > \frac{1}{6}$ and $f \in C^6(\mathbb{R})$. Then $\int_0^t f'(X_s^H)d^\circ X_s^H$ exists and we have

$$f(X_t^H) = f(X_0^H) + \int_0^t f'(X_s^H)d^\circ X_s^H,\ t \in \mathbb{R}_+.$$

As it was already the case in Proposition 1.2.9, we see that the existence of the symmetric integral $\int_0^\cdot f'(X_s)d^\circ X_s$ is shown at the same time as its associated change-of-variable formula.

Theorem 1.2.12 says nothing about the case $H \leqslant \frac{1}{6}$. In fact, relying to a suitable central limit theorem one can prove that $\int_0^t (X_s^H)^2 d^\circ X_s^H$ does not exists when $H < \frac{1}{6}$, while it exists only in law in the critical case $H = \frac{1}{6}$. As a result, it is not possible to write a change-of-variable formula for $f(x) = x^3$ when $H < \frac{1}{6}$. To have a complete picture, it remained to understand the critical case $H = \frac{1}{6}$. The answer was given in 2010 by Nourdin, Réveillac and Swanson [30]. By using techniques from the Malliavin calculus, they discovered an Itô's type formula in which the bracket term takes the form of a Wiener integral with respect to an independent Brownian motion.

Theorem 1.2.13 *(Itô's formula for fractional Brownian motion with $H = \frac{1}{6}$) For any $f \in C^\infty(\mathbb{R})$, one has*

$$f(X_t^{\frac{1}{6}}) \overset{\text{law}}{=} f(0) + \int_0^t f'(X_s^{\frac{1}{6}})d^\circ X_s^{\frac{1}{6}} - \frac{\kappa_3}{12}\int_0^t f'''(X^{\frac{1}{6}})dW_s, \quad t \geqslant 0.$$

Here, W is a standard Brownian motion independent of $X^{\frac{1}{6}}$, $\kappa_3 \simeq 2.322$, and $\int_0^t f'(X_s^{\frac{1}{6}})d^\circ X_s^{\frac{1}{6}}$ exists as a stable limit in law (as $n \to \infty$) of the following trapezoidal Riemann sum:

$$\frac{1}{2}\sum_{k=0}^{\lfloor 2^n t\rfloor - 1}\big(f'(X_{k2^{-n}}^{\frac{1}{6}}) + f'(X_{(k+1)2^{-n}}^{\frac{1}{6}})\big)(X_{(k+1)2^{-n}}^{\frac{1}{6}} - X_{k2^{-n}}^{\frac{1}{6}}). \tag{1.2.22}$$

Other Itô's type formulae

By considering the midpoint rule instead of the trapezoidal rule in (1.2.22), one obtains yet another Itô's type formula together with a new critical case (namely, $H = \frac{1}{4}$). This fact was unveiled by Nourdin and Réveillac in [29] prior to Theorem 1.2.13. Shortly afterwards, Harnett and Nualart [15, 14, 16] have extended a part of the previous results to a large class of Gaussian processes. All these results have in common to be based on a thorough use of Malliavin calculus.

In our historical review, we have covered only a few examples of how one can obtain a stochastic calculus with a fractional Brownian motion as integrator. Indeed, our purpose is only to focus on the results that have inspired our own work. Nevertheless, we would like to mention another nice construction of stochastic integral with respect to fractional Brownian motion, obtained by Christian Bender [3] in 2003 by means of the Wick product and the Pettis integrals in the space of Hida's distributions. With these tools at hand, he was able to prove the following change-of-variable formula for the fractional Brownian motion X^H, and this for any value of the Hurst parameter H between 0 and 1:

$$f(X_t^H) = f(X_0^H) + \int_0^t f'(X_s^H)dX_s^H + H\int_0^t u^{2H-1}f''(X_s^H)ds, \quad t \geqslant 0.$$

1.2.3 A summary of our main results

It is now time to present our own results.

Fluctuations of fBmBt power variations

Our first study is about the asymptotic behavior of the power variations of fBmBt. It corresponds to the content of a paper accepted for publication in the *Bernoulli* journal. In this subsection, we only present the results. Details and proofs will be given in Chapter 2.

Let Z be a fBmBt as given in Definition 1.2.1. (In what follows, we write Z instead of Z^H for the sake of simplicity.) For a given integer p, we consider the pth variation process associated to Z, defined as:

$$R_n^{(p)}(t) = \sum_{k=0}^{\lfloor 2^n t\rfloor -1} (Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p, \quad n \in \mathbb{N}, \quad t \geqslant 0.$$

At this stage it is worthwhile noting that, strictly speaking, we are dealing with the pth variations of Z in the classical sense only when p is even. When p is odd we are rather dealing with what is customarily called *signed* pth variations of Z. The interested reader may consult [2, 10] for relevant information about power variations.

Normalizing $R_n^{(p)}$ leads to

$$S_n^{(p)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \left((2^{\frac{nH}{2}}(Z_{(k+1)2^{-n}} - Z_{k2^{-n}}))^p - E[(2^{\frac{nH}{2}}(Z_{(k+1)2^{-n}} - Z_{k2^{-n}}))^p]\right), \quad n \in \mathbb{N},$$

for $t \geqslant 0$ and some $\kappa > 0$ to be computed. This latter coefficient will be chosen to ensure the fdd convergence of $S_n^{(p)}$ to a non-degenerate limit, which is exactly the statement we are looking for in this section. To reach our goal, we will make use of a somehow classical strategy in this kind of problem, which consists in expanding the power function x^p in terms of the Hermite polynomials H_r ($H_1(x) = x$, $H_2(x) = x^2 - 1$, etc.). Doing so, our problem then reduced to the joint analysis of the following quantities:

$$U_n^{(r)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} H_r(2^{\frac{nH}{2}}(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})), \quad n \in \mathbb{N}, \quad t \geqslant 0, \quad r \in \mathbb{N}^*. \quad (1.2.23)$$

But due to the fact that one cannot separate X from Y inside Z in the definition of $U_n^{(r)}$, working directly with (1.2.23) seems to be a difficult task (see also [21, Problem 5.1]). This is why, following an idea introduced by Khosnevisan and Lewis [20] in their study of the case $H = 1/2$ (that is, precisely when Z is the iBm), we will rather analyze $U_n^{(r)}$ by means of certain stopping times for Y. Khoshnevisan and Lewis' idea is indeed quite simple: by stopping Y as it crosses certain levels, and by sampling Z at these times, one can effectively separate X from Y. To be more specific, let us introduce the following collection of stopping times (with respect to the natural filtration of Y), noted

$$\mathscr{T}_n = \{T_{k,n} : k \geqslant 0\}, \quad n \geqslant 0, \tag{1.2.24}$$

which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} : j \in \mathbb{Z}\}$, $n \geqslant 0$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geqslant 0$, the stopping times $T_{k,n}$ appearing in (1.2.24) are given by the following recursive definition: $T_{0,n} = 0$, and

$$T_{k,n} = \inf\left\{s > T_{k-1,n} : \quad Y(s) \in \mathscr{D}_n \setminus \{Y(T_{k-1,n})\}\right\}, \quad k \geqslant 1.$$

Note that the definition of $T_{k,n}$, and therefore of $\mathscr{T}_n$, only involves the one-sided Brownian motion Y. Moreover, for every $n \geqslant 0$ the discrete stochastic process

$$\mathscr{Y}_n = \{Y(T_{k,n}) : k \geqslant 0\}$$

defines a simple random walk over $\mathscr{D}_n$. As shown in [20], as n tends to infinity the collection $\{T_{k,n} : 1 \leqslant k \leqslant 2^n t\}$ approximates the common dyadic partition $\{k2^{-n} : 1 \leqslant k \leqslant 2^n t\}$ of order n of the time interval $[0, t]$ (see [20, Lemma 2.2] for a precise statement). Based on this fact, one can introduce the counterpart of (1.2.23) based on $\mathscr{T}_n$, namely,

$$V_n^{(r)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} H_r(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}})), \quad n \in \mathbb{N}, \quad t \geqslant 0, \quad r \in \mathbb{N}^*.$$

We are now in a position to state the main result of this section.

Theorem 1.2.14 *The following two fdd convergences take place as $n \to \infty$ for any integer $N \geqslant 1$.*

(1) Assume that $H \leqslant \frac{1}{2}$. One has

$$2^{-n/4}\left\{\sum_{k=0}^{\lfloor 2^n t\rfloor - 1} H_{2r-1}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}})) : 1 \leqslant r \leqslant N\right\}_{t\geqslant 0}$$
$$\xrightarrow{\text{fdd}} \left\{\sigma_{2r-1} E_t^{(r)} : 1 \leqslant r \leqslant N\right\}_{t \geqslant 0},$$

where σ_{2r-1} is some (explicit) constant and $E = (B^{(1)} \circ Y, \dots, B^{(N)} \circ Y)$, with $B = (B^{(1)}, \dots, B^{(N)})$ a N-dimensional two-sided Brownian motion independent from Y.

(2) Assume that $H < \frac{3}{4}$. One has

$$2^{-3n/4}\left\{\sum_{k=0}^{\lfloor 2^n t\rfloor -1} H_{2r}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))\,:\, 1\leqslant r\leqslant N\right\}_{t\geqslant 0}$$

$$\overset{\text{fdd}}{\longrightarrow} \left\{\sigma_{2r}\int_{-\infty}^{\infty} L_t^x(Y)dB_x^{(r)}\,:\, 1\leqslant r\leqslant N\right\}_{t\geqslant 0}, \tag{1.2.25}$$

where σ_{2r} is some (explicit) constant, $B = (B^{(1)}, \ldots, B^{(N)})$ is a N-dimensional two-sided Brownian motion independent from Y and $L_t^x(Y)$ stands for the local time of Y before time t at level x.

Let us emphasize that the process $\{\int_{\mathbb{R}} L_t^x(Y)dB_x^{(r)}\}_{t\geqslant 0}$ appearing in (1.2.25) is nothing but the *Brownian motion in Random Scenery* introduced by Kesten and Spitzer in [18].

As a corollary of Theorem 1.2.14, one can deduce the fluctuations of the power variations of Z.

Corollary 1.2.15 *The following two fdd convergences in law take place as $n\to\infty$ for any integer $N\geqslant 1$.*

(1) Assume that $H\leqslant \frac{1}{2}$. One has

$$\left\{2^{\frac{-n}{4}(1-(4r-2)H)}\sum_{k=0}^{\lfloor 2^n t\rfloor -1} (Z_{T_{k+1,n}} - Z_{T_{k,n}})^{2r-1}\,:\, 1\leqslant r\leqslant N\right\}_{t\geqslant 0}$$

$$\overset{\text{fdd}}{\longrightarrow} \left\{\sum_{k=1}^{r} a_{r,k}\sigma_{2k-1}E_t^{(k)}\,:\, 1\leqslant r\leqslant N\right\}_{t\geqslant 0}, \tag{1.2.26}$$

where $a_{r,k} = \displaystyle\sum_{l=0}^{k-1}\frac{(-1)^l(2(r+k-l-1))!}{l!(2(k-l)-1)!(r+k-l-1)!2^{r+k-1}}$.

(2) Assume that $H < \frac{3}{4}$. One has

$$\left\{2^{\frac{-3n}{4}(1-\frac{4rH}{3})}\sum_{k=0}^{\lfloor 2^n t\rfloor -1} \big((Z_{T_{k+1,n}} - Z_{T_{k,n}})^{2r} - 2^{-nrH}b_{r,0}\big)\,:\, 1\leqslant r\leqslant N\right\}_{t\geqslant 0}$$

$$\overset{\text{fdd}}{\longrightarrow} \left\{\sum_{k=1}^{r} b_{r,k}\sigma_{2k}\int_{-\infty}^{\infty} L_t^x(Y)dB_x^{(k)}\,:\, 1\leqslant r\leqslant N\right\}_{t\geqslant 0},$$

where $b_{r,k} = \displaystyle\sum_{l=0}^{k}\frac{(-1)^l(2(r+k-l))!}{l!(2(k-l))!(r+k-l)!2^{r+k}}$.

Note that $b_{r,0} = E\big[(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r}\big] = E[N^{2r}]$, with $N\sim\mathcal{N}(0,1)$.

Corollary 1.2.15 contains Theorems 3.2, 4.4 and 4.5 from Khoshnevisan and Lewis [20] as a particular example ($H = 1/2$ and $r = 2, 3, 4$).

Towards an Itô's type formula for fBmBt in dimension 1

The problem we aim to solve here is to find an Itô's type formula satisfied by the fBmBt Z. For any smooth enough function f, one can legitimately expect the formula we are looking for to have the form

$$f(Z_t) - f(Z_0) = \int_0^t f'(Z_s)dZ_s + \text{bracket term}, \quad t \geqslant 0. \tag{1.2.27}$$

Of course, in (1.2.27) we have first to suitably define $\int_0^t f'(Z_s)dZ_s$ and then to discover the exact form of the remainder.

Recall that Z is not a Gaussian process and that it is quite irregular (precisely: it is Hölder continuous of order α for all α strictly less than $H/2$). Because of this, it would be actually too difficult to try to define our stochastic integral as a limit of Riemann sums with respect to a *deterministic* partition of the time axis. This is why we will rather work with the stochastic partition $T_{k,n}$ we already introduced in the previous subsection. At this stage, we need to define further objects. For each integer $n \geqslant 0$, $k \in \mathbb{Z}$ and real number $t \geqslant 0$, we let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2}, (j+1)2^{-n/2}]$ within the first $\lfloor 2^n t \rfloor$ steps of the random walk $\{Y(T_{k,n})\}_{k\geqslant 0}$, that is,

$$\begin{aligned} U_{j,n}(t) = \sharp\big\{k = 0, \ldots, \lfloor 2^n t\rfloor - 1 :& \\ & Y(T_{k,n}) = j2^{-n/2} \text{ and } Y(T_{k+1,n}) = (j+1)2^{-n/2}\big\}; \\ D_{j,n}(t) = \sharp\big\{k = 0, \ldots, \lfloor 2^n t\rfloor - 1 :& \\ & Y(T_{k,n}) = (j+1)2^{-n/2} \text{ and } Y(T_{k+1,n}) = j2^{-n/2}\big\}. \end{aligned}$$

Also, we will say that $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is symmetric (resp. skew symmetric) if $\varphi(x,y) = \varphi(y,x)$ (resp. $\varphi(x,y) = -\varphi(y,x)$) for all $x, y \in \mathbb{R}$. The following lemma corresponds to [20, Lemma 2.4].

Lemma 1.2.16 *If φ is symmetric, then*

$$\sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \varphi(Z_{T_{k+1,n}}, Z_{T_{k,n}}) = \sum_{j\in\mathbb{Z}} \varphi(X_{(j+1)2^{-\frac{n}{2}}}, X_{j2^{-\frac{n}{2}}})(U_{j,n}(t) + D_{j,n}(t)).$$

If φ is skew-symmetric, then

$$\sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \varphi(Z_{T_{k+1,n}}, Z_{T_{k,n}}) = \sum_{j\in\mathbb{Z}} \varphi(X_{(j+1)2^{-\frac{n}{2}}}, X_{j2^{-\frac{n}{2}}})(U_{j,n}(t) - D_{j,n}(t)).$$

The main feature of Lemma 1.2.16 is to separate X from Y in Z. We will indeed use it to provide a amenable-to-analysis representation of

$$V_n^{(r)}(f,t) = \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \frac{1}{2}\big(f(Z_{T_{k+1,n}} + f(Z_{T_{k,n}})\big)(Z_{T_{k+1,n}} - Z_{T_{k,n}})^r. \tag{1.2.28}$$

We call $V_n^r(f,t)$ the (symmetric) weighted r-variation of Z.

Let us fix a function $f : \mathbb{R} \to \mathbb{R}$ belonging to C_b^∞, the set of those functions that are C^∞ and bounded together with their derivatives. In Chapter 3 we will prove that, for any $H \in \left[\frac{1}{6}, \frac{1}{2}\right)$, any integer $r \geqslant 3$ and any $t > 0$,

$$V_n^{(2r-1)}(f,t) \overset{\text{prob}}{\longrightarrow} 0 \quad \text{as } n \to \infty. \tag{1.2.29}$$

We will also prove that, for any $H \in \left(\frac{1}{6}, \frac{1}{2}\right)$ and any $t > 0$,

$$V_n^{(3)}(f,t) \overset{\text{prob}}{\longrightarrow} 0 \quad \text{as } n \to \infty, \tag{1.2.30}$$

whereas, for $H = \frac{1}{6}$,

$$\left(X_t, Y_t, V_n^{(3)}(f,t)\right)_{t\geqslant 0} \overset{\text{fdd}}{\to} \left(X_t, Y_t, \int_0^t f(Z_s)d^{\circ 3}Z_s\right)_{t\geqslant 0} \quad \text{as } n \to \infty. \tag{1.2.31}$$

Otherwise stated, (1.2.31) means that $V_n^{(3)}(f,\cdot)$ converges stably in law (in the sense of the fdd convergence) to a limit process written $\int_0^\cdot f(Z_s)d^{\circ 3}Z_s$.

Now, observe that one has the following Taylor's type formula: for any a, $b \in \mathbb{R}$, and for some constants c_r whose explicit values are useless for our purpose,

$$\begin{aligned} f(b) - f(a) &= \frac{1}{2}\big(f'(a) + f'(b)\big)(b-a) - \frac{1}{24}\big(f'''(a) + f'''(b)\big)(b-a)^3 \\ &\quad + \sum_{r=3}^{7} c_r\big(f^{(2r-1)}(a) + f^{(2r-1)}(b)\big)(b-a)^{2r-1} + O(|b-a|^{14}), \end{aligned} \tag{1.2.32}$$

where $|O(|b-a|^{14})| \leqslant C_f |b-a|^{14}$, C_f being a constant depending only on f. By plugging (1.2.29), (1.2.30) and (1.2.31) into (1.2.32), one can deduce (see Chapter 3 for the details) the first two items of the following theorem.

Theorem 1.2.17 *Let $f \in C_b^\infty$ and $t > 0$.*

1. *If $H > \frac{1}{6}$ then*

$$f(Z_t) - f(0) = \int_0^t f'(Z_s)d^\circ Z_s, \tag{1.2.33}$$

where $\int_0^t f'(Z_s)d^\circ Z_s$ is the limit in probability of $V_n^{(1)}(f',t)$ defined in (1.2.28). (Its existence is part of the conclusion.)

2. *If $H = \frac{1}{6}$ then, with $\kappa_3 \simeq 2.322$,*

$$f(Z_t) - f(0) + \frac{\kappa_3}{12}\int_0^t f'''(Z_s)d^{\circ 3}Z_s \overset{\text{(law)}}{=} \int_0^t f'(Z_s)d^\circ Z_s. \tag{1.2.34}$$

Here, $\int_0^\cdot f'(Z_s)d^\circ Z_s$ denotes the limit in law of $V_n^{(1)}(f',\cdot)$ defined in (1.2.28). (Its existence is part of the conclusion.) Moreover, we have, for all $t \geqslant 0$,

$$\int_0^t f'''(Z_s)d^{\circ 3}Z_s := \int_0^{Y_t} f'''(X_s)dW_s,$$

where W is a two-sided Brownian motion independent of the pair (X,Y) defining Z, and where the integral with respect to W is understood in the Wiener-Itô sense.

3. *If $H < \frac{1}{6}$ then*

$$V_n^{(1)}(x \mapsto x^2, t) \text{ does not converge, even stably in law.} \tag{1.2.35}$$

This means that there is no way to get a change-of-variable formula for $f(x) = x^3$.

The proof of (1.2.33) in the case $H = \frac{1}{2}$ was already done in [21] by Khoshnevisan and Lewis. Let us also briefly explain how to prove (1.2.35). Using

$$b^3 - a^3 = \frac{3}{2}(a^2 + b^2)(b - a) - \frac{1}{2}(b - a)^3$$

(which corresponds to the identity (1.2.32) with $f(x) = x^3$), one can write, with $\mathbf{1}$ denoting the function constantly equal to 1,

$$\begin{aligned} V_n(x \mapsto x^2, t) - \frac{1}{3}Z_t^3 &= \frac{1}{6}V_n^{(3)}(\mathbf{1}, t) + \frac{1}{3}\sum_{k=0}^{\lfloor 2^n t\rfloor - 1}(Z_{T_{k+1,n}}^3 - Z_{T_{k,n}}^3) - \frac{1}{3}Z_t^3 \\ &= \frac{1}{6}V_n^{(3)}(\mathbf{1}, t) + \frac{1}{3}(Z_{T_{\lfloor 2^n t\rfloor,n}}^3 - Z_t^3). \end{aligned}$$

Since $T_{\lfloor 2^n t\rfloor,n} \to t$ almost surely and Z has continuous paths, we deduce that $Z_{T_{\lfloor 2^n t\rfloor,n}} \to Z_t$ as $n \to \infty$ almost surely. Thus, the stable convergence of $V_n(x \mapsto x^2, t)$ is equivalent to the stable convergence of $V_n^{(3)}(\mathbf{1}, t)$. But it is shown in Corollary 1.2.15 (see (1.2.26) therein) that $2^{-n(1-6H)/4}V_n^{(3)}(\mathbf{1}, t)$ converges in law to a non degenerate limit. This fact being clearly in contradiction with the convergence of $V_n^{(3)}(\mathbf{1}, t)$, we deduce that (1.2.35) holds.

Towards an Itô's type formula for the fBmBt in dimension 2

Our aim here is to extend the previous one-dimensional change-of-variable formula for the fBmBt in the case of dimension 2. Set

$$Z_t := (Z_t^1, Z_t^2) = (X_{Y_t}^1, X_{Y_t}^2), \ \ t \geqslant 0,$$

where X^1, X^2 are two independent two-sided fractional Brownian motions sharing the same Hurst parameter $H \in (0, 1)$, and Y is a standard (one-sided) Brownian motion independent of $X := (X^1, X^2)$. The following definition is of pivotal importance.

Definition 1.2.18 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and fix a time $t > 0$.*
1) Provided it exists, define $\int_0^t \nabla f(X_s) dX_s$ to be the limit in probability, as $n \to \infty$, of

$$\begin{aligned} &O_n(f, t) \qquad (1.2.36) \\ = &\sum_{j=0}^{\lfloor 2^{n/2}t\rfloor - 1}\frac{\partial f}{\partial x}\left(\frac{X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}}{2}, \frac{X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}}}{2}\right)\left(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\right) \\ &+ \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor - 1}\frac{\partial f}{\partial y}\left(\frac{X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}}{2}, \frac{X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}}}{2}\right)\left(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\right). \end{aligned}$$

2) When $O_n(f, t)$ defined by (1.2.36) does not converge in probability but converges stably instead, we denote the limit by $\int_0^t \nabla f(X_s) d^*X_s$.

One can now state a first preliminary result, which represents the bi-dimensional extension of Theorem 1.2.13 and which is the first step for obtaining a change-of-variable formula for the 2d fBmBt.

Theorem 1.2.19 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function belonging to C_b^∞, and fix a time $t > 0$.*

1. *If $H > 1/6$ then $\int_0^t \nabla f(X_s) dX_s$ is well-defined, and we have*
$$f(X_t) = f(0) + \int_0^t \nabla f(X_s) dX_s.$$
2. *If $H = 1/6$ then $\int_0^t \nabla f(X_s) d^* X_s$ is well-defined, and we have*
$$f(X_t) - f(0) - \int_0^t D^3 f(X_s) d^3 X_s \overset{law}{=} \int_0^t \nabla f(X_s) d^* X_s$$
where $\int_0^t D^3 f(X_s) d^3 X_s$ is short-hand for
$$\begin{aligned}\int_0^t D^3 f(X_s) d^3 X_s &= \kappa_1 \int_0^t \frac{\partial^3 f}{\partial x^3}(X_s^1, X_s^2) dB_s^1 + \kappa_2 \int_0^t \frac{\partial^3 f}{\partial y^3}(X_s^1, X_s^2) dB_s^2 \\ &\quad + \kappa_3 \int_0^t \frac{\partial^3 f}{\partial x^2 \partial y}(X_s^1, X_s^2) dB_s^3 + \kappa_4 \int_0^t \frac{\partial^3 f}{\partial x \partial y^2}(X_s^1, X_s^2) dB_s^4\end{aligned}$$
with $B = (B^1, \ldots, B^4)$ a 4-dimensional Brownian motion independent of X, $\kappa_1^2 = \kappa_2^2 = \frac{1}{96}\sum_{r\in\mathbb{Z}} \rho^3(r)$, $\kappa_3^2 = \kappa_4^2 = \frac{1}{32}\sum_{r\in\mathbb{Z}} \rho^3(r)$ and $\rho(r) = \frac{1}{2}(|r+1|^{\frac{1}{3}} + |r-1|^{\frac{1}{3}} - 2|r|^{\frac{1}{3}})$.
3. *If $H < 1/6$, for $f(x,y) = x^3$ then the integral $\int_0^t \nabla f(X_s) d^* X_s$ does not exist, even stably in law. So, it is impossible to write an Itô's type formula.*

To pass from the fBm to the fBmBt, our strategy is to make use of similar objects than the one we dealt with in dimension 1. That being said, it is natural to introduce the following definition for the integral with respect to the 2d fBmBt. Again, it is based on the collection $\mathscr{T}_n$ of stopping times defined in (1.2.24).

Definition 1.2.20 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and fix a time $t > 0$. Provided it exists, we define $\int_0^t \nabla f(Z_s) dZ_s$ to be the limit in probability, as $n \to \infty$, of*
$$\begin{aligned}\tilde{O}_n(f,t) &:= \sum_{j=0}^{\lfloor 2^{n/2} t \rfloor - 1} \frac{\partial f}{\partial x}\left(\frac{Z^1_{T_{j+1,n}} + Z^1_{T_{j,n}}}{2}, \frac{Z^2_{T_{j+1,n}} + Z^2_{T_{j,n}}}{2}\right)(Z^1_{T_{j+1,n}} - Z^1_{T_{j,n}}) \\ &\quad + \sum_{j=0}^{\lfloor 2^{n/2} t \rfloor - 1} \frac{\partial f}{\partial y}\left(\frac{Z^1_{T_{j+1,n}} + Z^1_{T_{j,n}}}{2}, \frac{Z^2_{T_{j+1,n}} + Z^2_{T_{j,n}}}{2}\right)(Z^2_{T_{j+1,n}} - Z^2_{T_{j,n}}).\end{aligned} \tag{1.2.37}$$

If $\tilde{O}_n(f,t)$ defined by (1.2.37) does not converge in probability but converges stably, we denote the limit by $\int_0^t \nabla f(Z_s) d^* Z_s$.

The previous definition together with Theorem 1.2.19 lead to the following change-of-variable formula for 2d fBmBt, which represents the main finding of this section and solves the problem we pose at the beginning.

Theorem 1.2.21 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function belonging to C_b^∞, and fix a time $t > 0$.*

1. *If $H > 1/6$ then $\int_0^t \nabla f(Z_s) dZ_s$ is well defined, and we have*
$$f(Z_t) = f(0) + \int_0^t \nabla f(Z_s) dZ_s.$$
2. *If $H = 1/6$ then $\int_0^t \nabla f(Z_s) d^* Z_s$ is well defined, and we have*
$$f(Z_t) - f(0) - \int_0^t D^3 f(Z_s) d^3 Z_s \overset{law}{=} \int_0^t \nabla f(Z_s) d^* Z_s$$
where $\int_0^t D^3 f(Z_s) d^3 Z_s$ is short-hand for
$$\begin{aligned}\int_0^t D^3 f(Z_s) d^3 Z_s &= \kappa_1 \int_0^{Y_t} \frac{\partial^3 f}{\partial x^3}(X_s) dB_s^1 + \kappa_2 \int_0^{Y_t} \frac{\partial^3 f}{\partial y^3}(X_s) dB_s^2 \\ &\quad + \kappa_3 \int_0^{Y_t} \frac{\partial^3 f}{\partial x^2 \partial y}(X_s) dB_s^3 + \kappa_4 \int_0^{Y_t} \frac{\partial^3 f}{\partial x \partial y^2}(X_s) dB_s^4,\end{aligned}$$
with $B = (B^1, \ldots, B^4)$ is a 4-dimensional two-sided Brownian motion independent of X, and $\kappa_1, \ldots, \kappa_4$ as in Theorem 1.2.19. (B is also independent from Y.)
3. *If $H < 1/6$, for $f(x, y) = x^3$ then the integral $\int_0^t \nabla f(Z_s) d^* Z_s$ does not exist, even stably in law. So, it is impossible to write an Itô's type formula.*

Chapter 2

Fluctuations of the power variations of fBmBt

In this chapter we present the results obtained in the paper: R. Zeineddine, "Fluctuations of the power variations of fractional Brownian motion in Brownian time", to appear in *Bernoulli*.

2.1 Introduction

Studying the variations of a stochastic process is of fundamental importance in probability theory. In this chapter, we are interested in the fractional Brownian motion in Brownian time, which is defined as follows. Consider a fractional Brownian motion X on $\mathbb{R}$ with Hurst parameter $H \in (0,1)$, as well as a standard Brownian motion Y on $\mathbb{R}_+$ independent from X. The process $Z = X \circ Y$ is the so-called *fractional Brownian motion in Brownian time* (F.B.M.B.T. in short). It is a self-similar process (of order $H/2$) with stationary increments, which is not Gaussian. When $H = 1/2$, one recovers the celebrated iterated Brownian motion.

In recent years, starting with the articles of Burdzy [5, 6], there has been an increased interest in iterated processes in which one changes the time parameter with one-dimensional Brownian motion, see, e.g., [7, 19, 20, 21, 27] to cite but a few. In the present chapter, we are concerned with the study of the fluctuations of the pth variation of Z for any integer p, defined as

$$R_n^{(p)}(t) = \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (Z_{(k+1)2^{-n}} - Z_{k2^{-n}})^p, \quad n \in \mathbb{N}, \quad t \geqslant 0.$$

At this stage, it is worthwhile noting that we are dealing with the pth variations of Z in the classical sense when p is even whereas, when p is odd, we are rather dealing with the *signed* pth variations of Z. The interested reader may read [2, 10] in order to find relevant information about power variations.

After proper normalization we may expect the f.d.d. convergence to a non-degenerate limit (to be determined) of

$$S_n^{(p)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \big((2^{\frac{nH}{2}}(Z_{(k+1)2^{-n}} - Z_{k2^{-n}}))^p - E[(2^{\frac{nH}{2}}(Z_{(k+1)2^{-n}} - Z_{k2^{-n}}))^p]\big), \quad n \in \mathbb{N},$$

for $t \geqslant 0$ and some $\kappa > 0$ to be discovered. To reach this goal, a classical strategy consists in expanding the power function x^p in terms of Hermite polynomials. Doing so, our problem is reduced to the joint analysis of the following quantities:

$$U_n^{(r)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t \rfloor -1} H_r(2^{\frac{nH}{2}}(Z_{(k+1)2^{-n}} - Z_{k2^{-n}})), \quad n \in \mathbb{N}, \quad t \geqslant 0, \quad r \in \mathbb{N}^*. \tag{2.1.1}$$

Here, H_r denotes the rth Hermite polynomial ($H_1(x) = x$, $H_2(x) = x^2 - 1$, etc.). Due to the fact that one cannot separate X from Y inside Z in the definition of $U_n^{(r)}$, working directly with (2.1.1) seems to be a difficult task (see also [21, Problem 5.1]). This is why, following an idea introduced by Khosnevisan and Lewis [20] in the study of the case $H = 1/2$, we will rather analyze $U_n^{(r)}$ by means of certain stopping times for Y. The idea is quite simple: by stopping Y as it crosses certain levels, and by sampling Z at these times, one can effectively separate X from Y. To be more specific, let us introduce the following collection of stopping times (with respect to the natural filtration of Y), noted

$$\mathscr{T}_n = \{T_{k,n} : k \geqslant 0\}, \quad n \geqslant 0, \tag{2.1.2}$$

which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} : j \in \mathbb{Z}\}$, $n \geqslant 0$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geqslant 0$, the stopping times $T_{k,n}$, appearing in (2.1.2), are given by the following recursive definition: $T_{0,n} = 0$, and

$$T_{k,n} = \inf \left\{ s > T_{k-1,n} : \quad Y(s) \in \mathscr{D}_n \setminus \{Y(T_{k-1,n})\} \right\}, \quad k \geqslant 1.$$

Note that the definition of $T_{k,n}$, and therefore of $\mathscr{T}_n$, only involves the one-sided Brownian motion Y, and that, for every $n \geqslant 0$, the discrete stochastic process

$$\mathscr{Y}_n = \{Y(T_{k,n}) : k \geqslant 0\}$$

defines a simple random walk over $\mathscr{D}_n$. As shown in [20], as n tends to infinity the collection $\{T_{k,n} : 1 \leqslant k \leqslant 2^n t\}$ approximates the common dyadic partition $\{k2^{-n} : 1 \leqslant k \leqslant 2^n t\}$ of order n of the time interval $[0, t]$ (see [20, Lemma 2.2] for a precise statement). Based on this fact, one can introduce the counterpart of (2.1.1) based on $\mathscr{T}_n$, namely,

$$V_n^{(r)}(t) = 2^{-n\kappa} \sum_{k=0}^{\lfloor 2^n t \rfloor -1} H_r(2^{\frac{nH}{2}}(Z_{T_{k+1,n}} - Z_{T_{k,n}})), \quad n \in \mathbb{N}, \quad t \geqslant 0, \quad r \in \mathbb{N}^*. \tag{2.1.3}$$

We are now in a position to state the main result of the present chapter.

Theorem 2.1.1 *The following two f.d.d. convergences in law take place as $n \to \infty$ for any integer $N \geqslant 1$.*

(1) Assume that $H \leqslant \frac{1}{2}$. One has

$$\begin{aligned} & 2^{-n/4} \left\{ \sum_{k=0}^{\lfloor 2^n t \rfloor -1} H_{2r-1}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}})) : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0} \\ \xrightarrow{\text{f.d.d.}} \; & \left\{ \sigma_{2r-1} E_t^{(r)} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}, \end{aligned} \tag{2.1.4}$$

where σ_{2r-1} is some (explicit) constant and $E = (B^{(1)} \circ Y, \ldots, B^{(N)} \circ Y)$, with $B = (B^{(1)}, \ldots, B^{(N)})$ a N-dimensional two-sided Brownian motion independent from Y.

(2) Assume that $H < \frac{3}{4}$. One has

$$2^{-3n/4} \left\{ \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} H_{2r}(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}})) : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}$$

$$\xrightarrow{\text{f.d.d.}} \left\{ \sigma_{2r} \int_{-\infty}^{\infty} L_t^x(Y) dB_x^{(r)} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}, \tag{2.1.5}$$

where σ_{2r} is some (explicit) constant, $B = (B^{(1)}, \ldots, B^{(N)})$ is a N-dimensional two-sided Brownian motion independent from Y and $L_t^x(Y)$ stands for the local time of Y before time t at level x.

The process $\{\int_{\mathbb{R}} L_t^x(Y) dB_x^{(r)}\}_{t \geqslant 0}$ appearing in (2.1.5) is nothing but the *Brownian motion in Random Scenery* introduced by Kesten and Spitzer (see [18]).

As a corollary of this theorem, we deduce the fluctuations of the power variation of Z:

Corollary 2.1.2 *The following two f.d.d. convergences in law take place as $n \to \infty$ for any integer $N \geqslant 1$.*

(1) Assume that $H \leqslant \frac{1}{2}$. One has

$$\left\{ 2^{\frac{-n}{4}(1-(4r-2)H)} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (Z_{T_{k+1,n}} - Z_{T_{k,n}})^{2r-1} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}$$

$$\xrightarrow{\text{f.d.d.}} \left\{ \sum_{k=1}^{r} a_{r,k} \sigma_{2k-1} E_t^{(k)} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}, \tag{2.1.6}$$

where $a_{r,k}$ is some constant given by: $a_{r,k} = \displaystyle\sum_{l=0}^{k-1} \frac{(-1)^l (2(r+k-l-1))!}{l!(2(k-l)-1)!(r+k-l-1)! 2^{r+k-1}}$.

(2) Assume that $H < \frac{3}{4}$. One has

$$\left\{ 2^{\frac{-3n}{4}(1-\frac{4rH}{3})} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \left((Z_{T_{k+1,n}} - Z_{T_{k,n}})^{2r} - 2^{-nrH} b_{r,0} \right) : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}$$

$$\xrightarrow{\text{f.d.d.}} \left\{ \sum_{k=1}^{r} b_{r,k} \sigma_{2k} \int_{-\infty}^{\infty} L_t^x(Y) dB_x^{(k)} : 1 \leqslant r \leqslant N \right\}_{t \geqslant 0}, \tag{2.1.7}$$

where $b_{r,k}$ is some constant given by: $b_{r,k} = \displaystyle\sum_{l=0}^{k} \frac{(-1)^l (2(r+k-l))!}{l!(2(k-l))!(r+k-l)! 2^{r+k}}$.

Note that $b_{r,0} = E\big[(2^{nH/2}(Z_{T_{k+1,n}} - Z_{T_{k,n}}))^{2r}\big] = E[N^{2r}]$, with $N \sim \mathcal{N}(0,1)$.

In the particular case where $H = 1/2$ (that is, when Z is an iterated Brownian motion) and $r = 2,3,4$, we emphasize that Corollary 2.1.2 allows one to recovers Theorems 3.2, 4.4 and 4.5 from Khoshnevisan and Lewis [20].

The organisation of the chapter is as follows. In Section 2, we provide some needed preliminaries. Theorem 2.1.1 and Corollary 2.1.2 are then shown in Section 3.

2.2 Preliminaries

In this section, we collect several results that are useful for the proof of Theorem 2.1.1 .

2.2.1 An algebraic lemma and some local time estimates

For each integer $n \geqslant 0$, $k \in \mathbb{Z}$ and real number $t \geqslant 0$, let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2}, (j+1)2^{-n/2}]$ within the first $\lfloor 2^n t \rfloor$ steps of the random walk $\{Y(T_{k,n})\}_{k\geqslant 0}$, that is,

$$U_{j,n}(t) = \sharp\big\{k = 0, \dots, \lfloor 2^n t\rfloor - 1 : \\ Y(T_{k,n}) = j2^{-n/2} \text{ and } Y(T_{k+1,n}) = (j+1)2^{-n/2}\big\}; \tag{2.2.8}$$

$$D_{j,n}(t) = \sharp\big\{k = 0, \dots, \lfloor 2^n t\rfloor - 1 : \\ Y(T_{k,n}) = (j+1)2^{-n/2} \text{ and } Y(T_{k+1,n}) = j2^{-n/2}\big\}. \tag{2.2.9}$$

The following lemma will play a crucial role in our study of the asymptotic behavior of $V_n^{(r)}$. Its main feature is to separate X from Y, thus providing a representation of $V_n^{(r)}$ which is amenable to analysis.

Lemma 2.2.1 *(See [20, Lemma 2.4]) Fix $t \geqslant 0$ and $r \in \mathbb{N}^*$. Then*

$$V_n^{(r)}(t) \;=\; 2^{-n\widetilde{\kappa}} \sum_{j\in\mathbb{Z}} H_r\left(2^{nH/2}\big(X_{(j+1)2^{-\frac{n}{2}}} - X_{j2^{-\frac{n}{2}}}\big)\right)\big(U_{j,n}(t) + (-1)^r D_{j,n}(t)\big) \tag{2.2.10}$$

Also, in order to prove the second point of Theorem 2.1.1 we will need estimates on the local time of Y taken from [20], that we collect in the following statement.

Proposition 2.2.2 *1. For every $x \in \mathbb{R}$, $p \in \mathbb{N}^*$ and $t > 0$, we have*

$$E\big[(L_t^x(Y))^p\big] \leqslant 2\,E\big[(L_1^0(Y))^p\big]\, t^{p/2} \exp\left(-\frac{x^2}{2t}\right).$$

2. There exists a positive constant μ such that, for every $a, b \in \mathbb{R}$ with $ab \geqslant 0$ and $t > 0$,

$$E\big[|L_t^b(Y) - L_t^a(Y)|^2\big]^{1/2} \leqslant \mu\,\sqrt{|b-a|}\,t^{1/4} \exp\left(-\frac{a^2}{4t}\right).$$

3. There exists a positive random variable $K \in L^8$ such that, for every $j \in \mathbb{Z}$, every $n \geqslant 0$ and every $t > 0$, one has that

$$|\mathcal{L}_{j,n}(t) - L_t^{j2^{-n/2}}(Y)| \leqslant 2Kn2^{-n/4}\sqrt{L_t^{j2^{-n/2}}(Y)},$$

where $\mathcal{L}_{j,n}(t) = 2^{-n/2}(U_{j,n}(t) + D_{j,n}(t))$.

2.2.2 Breuer-Major Theorem

Let $\{G_k\}_{k\geqslant 1}$ be a centered stationary Gaussian sequence. In this Gaussian context, stationary just means that there exist $\rho : \mathbb{Z} \to \mathbb{R}$ such that $E[G_k G_l] = \rho(k-l)$, $k, l \geqslant 1$. Assume further that $\rho(0) = 1$, that is, each G_k is $\mathscr{N}(0,1)$ distributed. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$E[\varphi^2(G_1)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \varphi^2(x) e^{-x^2/2} dx < +\infty. \tag{2.2.11}$$

The function φ may be expanded in $L^2(\mathbb{R}, \frac{e^{-x^2/2}}{\sqrt{2\pi}} dx)$ (in a unique way) in terms of Hermite polynomials as follows:

$$\varphi(x) = \sum_{q=0}^{+\infty} a_q H_q(x). \tag{2.2.12}$$

Let $d \geqslant 0$ be the *Hermite rank* of φ, that is, the first integer $q \geqslant 0$ such that $a_q \neq 0$ in (2.2.12). We then have the celebrated Breuer-Major Theorem (see [4], see also [28] for a modern proof).

Theorem 2.2.3 (Breuer-Major) *Let $\{G_k\}_{k\geqslant 1}$ (with covariance ρ) and $\varphi : \mathbb{R} \to \mathbb{R}$ (with Hermite index d) be as above. Assume further that $\sum_{k\in\mathbb{Z}} |\rho(k)|^d < +\infty$. Then, as $n \to +\infty$,*

$$2^{-n/2} \left\{ \sum_{k=1}^{\lfloor 2^n t \rfloor} \big(\varphi(G_k) - E[\varphi(G_k)] \big) \right\}_{t\geqslant 0} \overset{\text{f.d.d.}}{\to} \{\sigma B_t\}_{t\geqslant 0}, \tag{2.2.13}$$

with B a standard Brownian motion and $\sigma > 0$ given by

$$\sigma^2 = \sum_{q=d}^{+\infty} q! a_q^2 \sum_{k\in\mathbb{Z}} \rho(k)^q \in [0, +\infty[. \tag{2.2.14}$$

2.2.3 Peccati-Tudor Theorem

In a seminal paper of 2005, Nualart and Peccati [33] discovered a surprising central limit theorem (called the *Fourth Moment Theorem* nowadays) for sequences of multiple stochastic integrals of a fixed order: in this context, convergence in distribution to the standard normal law is actually equivalent to convergence of just the fourth moment. Shortly afterwards, Peccati and Tudor gave a multidimensional version of this characterization, making use of tools belonging to the Malliavin calculus. Since we will rely on this result in the present chapter, let us give more details.

Let $d \geqslant 2$ and $q_1, \ldots, q_d \geqslant 1$ be some fixed integers. Consider a sequence of random vectors $F_n = (F_{1,n}, \ldots, F_{d,n})$ of the following form. Each $F_{i,n}$ can be written as

$$F_{i,n} = \sum_{j=0}^{N_n} a_{j,n} H_{q_i}(Y_{j,n}),$$

where N_n is an integer, $a_{j,n}$ are real numbers and $\{Y_{j,n}\}_{j\geqslant 0}$ is a centered stationary Gaussian family with unit variance. We then have the following result, shown in [34].

Theorem 2.2.4 (Peccati-Tudor) *Let (F_n) be a sequence as above. Let $C \in \mathscr{M}_d(\mathbb{R})$ be a symmetric and positive matrix, and let N be a centered Gaussian vector with covariance C. Assume that*

$$\lim_{n\to+\infty} E[F_{i,n}F_{j,n}] = C(i,j),\ 1 \leqslant i,j \leqslant d. \tag{2.2.15}$$

Then, as $n \to +\infty$, the following two conditions are equivalent:
(a) F_n converges in law to N;
(b) for every $1 \leqslant i \leqslant d$, $F_{i,n}$ converges in law to $\mathscr{N}(0, C(i,i))$.

2.3 Proof of Theorem 2.1.1

2.3.1 Proof of (2.1.4)

Recall the definition (2.1.3) of $V_n^{(r)}(t)$ and let us fix $\widetilde{\kappa} = 1/4$. First of all, let us apply Lemma 2.2.1. Because $2r-1$ is an odd number, we obtain that

$$V_n^{(2r-1)}(t) = 2^{-n/4} \sum_{j\in\mathbb{Z}} H_{2r-1}(2^{nH/2}(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}))(U_{j,n}(t) - D_{j,n}(t)). \tag{2.3.16}$$

Now, let us observe (see also [20, Lemma 2.5]) that

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} \mathbf{1}(0 \leqslant j < j^*(n,t)) & \text{if } j^*(n,t) > 0 \\ 0 & \text{if } j^* = 0 \\ -\mathbf{1}(j^*(n,t) \leqslant j < 0) & \text{if } j^*(n,t) < 0 \end{cases},$$

where

$$j^*(n,t) = 2^{n/2} Y_{T_{\lfloor 2^n t\rfloor,n}}.$$

As a consequence,

$$V_n^{(2r-1)}(t) = \begin{cases} 2^{-n/4} \sum_{j=1}^{j^*(n,t)} H_{2r-1}(2^{nH/2}(X^+_{j2^{-n/2}} - X^+_{(j-1)2^{-n/2}})) & \text{if } j^*(n,t) > 0 \\ 0 & \text{if } j^* = 0 \\ 2^{-n/4} \sum_{j=1}^{|j^*(n,t)|} H_{2r-1}(2^{nH/2}(X^-_{j2^{-n/2}} - X^-_{(j-1)2^{-n/2}})) & \text{if } j^*(n,t) < 0 \end{cases},$$

where $X_t^+ = X_t$ for $t \geqslant 0$ and $X_{-t}^- = X_t$ for $t < 0$. Our analysis of $V_n^{(2r-1)}$ will become easier if one introduces the following sequence of processes $W_{\pm,n}^{(2r-1)}$, in which we have replaced $\sum_{j=1}^{\pm j^*(n,t)}$ by $\sum_{j=1}^{\lfloor 2^{n/2}t\rfloor}$, namely:

$$\begin{aligned} W_{+,n}^{(2r-1)}(t) &= 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2}t\rfloor} H_{2r-1}(2^{nH/2}(X^+_{j2^{-n/2}} - X^+_{(j-1)2^{-n/2}})), \quad t \geqslant 0 \\ W_{-,n}^{(2r-1)}(t) &= 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2}t\rfloor} H_{2r-1}(2^{nH/2}(X^-_{j2^{-n/2}} - X^-_{(j-1)2^{-n/2}})), \quad t \geqslant 0, \\ W_n^{(2r-1)}(t) &= \begin{cases} W_{+,n}^{(2r-1)}(t) & \text{if } t \geqslant 0 \\ W_{-,n}^{(2r-1)}(-t) & \text{if } t < 0 \end{cases}. \end{aligned}$$

It is clear, using the self-similarity property of X, that the f.d.d. convergence in law of the vector $(W_{+,n}^{(2r-1)}, W_{-,n}^{(2r-1)}, 1 \leqslant r \leqslant N)$ is equivalent to the f.d.d. convergence in law of the vector $(\overline{W}_{+,n}^{(2r-1)}, \overline{W}_{-,n}^{(2r-1)}, 1 \leqslant r \leqslant N)$ defined as:

$$\begin{aligned}
\overline{W}_{+,n}^{(2r-1)}(t) &= 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} H_{2r-1}(X_j^+ - X_{j-1}^+), \quad t \geqslant 0 \\
\overline{W}_{-,n}^{(2r-1)}(t) &= 2^{-n/4} \sum_{j=1}^{\lfloor 2^{n/2} t \rfloor} H_{2r-1}(X_j^- - X_{j-1}^-), \quad t \geqslant 0.
\end{aligned}$$

Let $G_j = X_j^+ - X_{j-1}^+$. The family $\{G_j\}$ is Gaussian, stationary, centered, with variance 1; moreover its covariance ρ is given by

$$\rho(k) = E[G_j G_{j+k}] = \frac{1}{2}\big(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}\big), \tag{2.3.17}$$

so that $\sum |\rho(k)| < \infty$ because $H \leqslant \frac{1}{2}$. Hence, Breuer-Major Theorem 2.2.3 applies and yields that, as $n \to \infty$ and for any fixed r,

$$\{\overline{W}_{+,n}^{(2r-1)}(t) : t \geqslant 0\} \overset{\text{f.d.d.}}{\longrightarrow} \sigma_{2r-1}\{B^{+,r}(t) : t \geqslant 0\},$$

with $B^{+,r}$ a standard Brownian motion and $\sigma_{2r-1} = \sqrt{(2r-1)! \sum_{a\in\mathbb{Z}} \rho(a)^{2r-1}}$. Note that $\sum_{a\in\mathbb{Z}} |\rho(a)|^{2r-1} < \infty$ if and only if $H < 1 - 1/(2(2r-1))$, which is satisfied for all $r \geqslant 1$ since we have supposed that $H \leqslant 1/2$ (the case $H = 1/2$ may be treated separately). Similarly,

$$\{\overline{W}_{-,n}^{(2r-1)}(t) : t \geqslant 0\} \overset{\text{f.d.d.}}{\longrightarrow} \sigma_{2r-1}\{B^{-,r}(t) : t \geqslant 0\},$$

with $B^{-,r}$ a standard Brownian motion and σ_{2r-1} as above. In order to deduce the joint convergence in law of $(\overline{W}_{+,n}^{(2r-1)}, \overline{W}_{-,n}^{(2r-1)}, 1 \leqslant r \leqslant N)$, from Peccati-Tudor Theorem 2.2.4 and taking into account that $E[\overline{W}_{\pm,n}^{(2r-1)}(t)\overline{W}_{\pm,n}^{(2l-1)}(t)] = 0$ for $l \neq r$ (since Hermite polynomials of different orders are orthogonal), it remains to check that, for any integer r and any real numbers $t, s \geqslant 0$,

$$\lim_{n\to\infty} E[\overline{W}_{+,n}^{(2r-1)}(t)\overline{W}_{-,n}^{(2r-1)}(s)] = 0. \tag{2.3.18}$$

Let us do it. One can write,

$$
\begin{aligned}
&E[\overline{W}^{(2r-1)}_{+,n}(t)\overline{W}^{(2r-1)}_{-,n}(s)]\\
&= 2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{l=1}^{\lfloor 2^{n/2}s\rfloor} E[H_{2r-1}(X^+_k - X^+_{k-1})H_{2r-1}(X^-_l - X^-_{l-1})]\\
&= (2r-1)!\,2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{l=1}^{\lfloor 2^{n/2}s\rfloor}\big(E[(X^+_k - X^+_{k-1})(X^-_l - X^-_{l-1})]\big)^{2r-1}\\
&= (2r-1)!\,2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{l=1}^{\lfloor 2^{n/2}s\rfloor}\big(E[(X_k - X_{k-1})(X_{-l} - X_{-l+1})]\big)^{2r-1}\\
&= (2r-1)!\,2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{l=1}^{\lfloor 2^{n/2}s\rfloor}\big(\tfrac{1}{2}[\,2|k+l-1|^{2H} - |k+l|^{2H} - |k+l-2|^{2H}]\big)^{2r-1}
\end{aligned}
$$

Setting $a = k+l$, we deduce that

$$
\begin{aligned}
&E[\overline{W}^{(2r-1)}_{+,n}(t)\overline{W}^{(2r-1)}_{-,n}(s)]\\
=\;& 2^{-(2r-1)}(2r-1)!\,2^{-n/2}\sum_{k=1}^{\lfloor 2^{n/2}t\rfloor}\sum_{a=k+1}^{k+\lfloor 2^{n/2}s\rfloor}\big(2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H}\big)^{2r-1},\\
=\;& 2^{-(2r-1)}(2r-1)!\,2^{-n/2}\sum_{a=2}^{\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor}\;\sum_{k=(a-\lfloor 2^{n/2}s\rfloor)\vee 1}^{(a-1)\wedge\lfloor 2^{n/2}t\rfloor}\big(2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H}\big)^{2r-1}\\
=\;& 2^{-(2r-1)}(2r-1)!\,2^{-n/2}\sum_{a\in\mathbb{N}} f_n(a),
\end{aligned}
$$

where

$$
\begin{aligned}
f_n(a) \;:=\;& \big(2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H}\big)^{2r-1}\big((a-1)\wedge\lfloor 2^{n/2}t\rfloor - (a-\lfloor 2^{n/2}s\rfloor)\vee 1 + 1\big)\\
&\times\mathbf{1}_{\{2\leqslant a\leqslant\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}.
\end{aligned}
$$

For any $a\in\{2,\ldots,\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}$, observe that $\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\geqslant 1$. Also, we have

$$
2^{-\frac{n}{2}}\big|(a-1)\wedge\lfloor 2^{n/2}t\rfloor\big|\mathbf{1}_{\{2\leqslant a\leqslant\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}\leqslant 2^{-\frac{n}{2}}\lfloor 2^{n/2}t\rfloor\leqslant t,
$$

as well as

$$
\begin{aligned}
&2^{-\frac{n}{2}}\big|(a-\lfloor 2^{n/2}s\rfloor)\vee 1\big|\mathbf{1}_{\{2\leqslant a\leqslant\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}\\
\leqslant\;& 2^{-\frac{n}{2}}(|a|+\lfloor 2^{n/2}s\rfloor+1)\mathbf{1}_{\{2\leqslant a\leqslant\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}\\
\leqslant\;& 2^{-\frac{n}{2}}(\lfloor 2^{n/2}t\rfloor+2\lfloor 2^{n/2}s\rfloor)+2^{-\frac{n}{2}}\mathbf{1}_{\{2\leqslant a\leqslant\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}\\
\leqslant\;& 2^{-\frac{n}{2}}(\lfloor 2^{n/2}t\rfloor+2\lfloor 2^{n/2}s\rfloor)+2^{-\frac{n}{2}}(\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor)\leqslant 2t+3s,
\end{aligned}
$$

and

$$
2^{-\frac{n}{2}}\mathbf{1}_{\{2\leqslant a\leqslant\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor\}}\leqslant 2^{-\frac{n}{2}}(\lfloor 2^{n/2}t\rfloor+\lfloor 2^{n/2}s\rfloor)\leqslant t+s.
$$

Plugging all these inequalities together leads to

$$2^{-n/2}|f_n(a)| \leqslant (4t+4s)\big|2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H}\big|^{2r-1}$$

for all n, with $\sum_{a\in\mathbb{N}} \big|2|a-1|^{2H} - |a|^{2H} - |a-2|^{2H}\big|^{2r-1} < \infty$ (recall that $H \leqslant 1/2$). Moreover, $2^{-n/2} f_n(a) \underset{n\to\infty}{\longrightarrow} 0$ for any fixed a because

$$\frac{(a-1)}{2^{n/2}} \wedge \frac{\lfloor 2^{n/2} t\rfloor}{2^{n/2}} - \frac{(a-\lfloor 2^{n/2} s\rfloor)}{2^{n/2}} \vee 2^{-n/2} + 2^{-n/2} \underset{n\to\infty}{\longrightarrow} 0 \wedge t - (-s) \vee 0 = 0$$

since $t, s \geqslant 0$. Hence, the dominated convergence theorem applies and yields

$$2^{-n/2} \sum_{a\in\mathbb{N}} f_n(a) \underset{n\to\infty}{\longrightarrow} 0,$$

that is, (2.3.18) holds true. As we said, using Peccati-Tudor Theorem 2.2.4 one thus obtains that

$$(W_{+,n}^{(2r-1)}, W_{-,n}^{(2r-1)},\, 1 \leqslant r \leqslant N) \overset{\text{f.d.d.}}{\longrightarrow} (\sigma_{2r-1} B^{+,r}, \sigma_{2r-1} B^{-,r},\, 1 \leqslant r \leqslant N), \tag{2.3.19}$$

with $(B^{+,r}, B^{-,r},\, 1 \leqslant r \leqslant N)$ a $2N$-dimensional standard Brownian motion. As a consequence, we have

$$(W_n^{(2r-1)}(t),\, 1 \leqslant r \leqslant N)_{t\in\mathbb{R}} \overset{\text{f.d.d.}}{\longrightarrow} (\sigma_{2r-1} B^{(r)}(t),\, 1 \leqslant r \leqslant N)_{t\in\mathbb{R}}, \tag{2.3.20}$$

with $(B^{(r)},\, 1 \leqslant r \leqslant N)$ a N-dimensional two-sided Brownian motion.

On the other hand, let us prove for any $r \in \mathbb{N}^*$ the existence of $C_r > 0$ such that, for any n and any $s, t \in \mathbb{R}$,

$$E[(W_n^{(2r-1)}(t) - W_n^{(2r-1)}(s))^2] \leqslant 8C_r(2^{-n/2} + |t-s|). \tag{2.3.21}$$

To do so, we distinguish three cases, according to the sign of $s, t \in \mathbb{R}$ (and reducing the problem by symmetry) :

(1) if $0 \leqslant s \leqslant t$:

$$
\begin{aligned}
&E[(W_n^{(2r-1)}(t) - W_n^{(2r-1)}(s))^2] \\
&= E[(\overline{W}_{+,n}^{(2r-1)}(t) - \overline{W}_{+,n}^{(2r-1)}(s))^2] = \Big|2^{-n/2}E[\Big(\sum_{j=\lfloor 2^{n/2}s\rfloor+1}^{\lfloor 2^{n/2}t\rfloor} H_{2r-1}(X_j^+ - X_{j-1}^+)\Big)^2]\Big| \\
&= \Big|(2r-1)!\, 2^{-n/2} \sum_{j,k=\lfloor 2^n s\rfloor+1}^{\lfloor 2^n t\rfloor} (\rho(j-k))^{2r-1}\Big| \\
&= \Big|(2r-1)!\, 2^{-n/2} \sum_{j=\lfloor 2^{n/2}s\rfloor+1}^{\lfloor 2^{n/2}t\rfloor} \sum_{a=j-\lfloor 2^{n/2}t\rfloor}^{j-\lfloor 2^{n/2}s\rfloor-1} (\rho(a))^{2r-1}\Big| \\
&\leqslant (2r-1)!\, 2^{-n/2} \sum_{a=\lfloor 2^{n/2}s\rfloor-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor-1} |\rho(a)|^{2r-1} \\
&\qquad \times \big|(a+\lfloor 2^n t\rfloor) \wedge \lfloor 2^n t\rfloor - (a+\lfloor 2^n s\rfloor) \vee (\lfloor 2^n s\rfloor)\big| \\
&\leqslant (2r-1)!\, 2^{-n/2} \sum_{a\in\mathbb{Z}} |\rho(a)|^{2r-1} \big|\lfloor 2^{n/2}t\rfloor - \lfloor 2^{n/2}s\rfloor\big| = C_r 2^{-n/2}\big|\lfloor 2^{n/2}t\rfloor - \lfloor 2^{n/2}s\rfloor\big| \\
&\leqslant C_r(|2^{-n/2}\lfloor 2^{n/2}t\rfloor - t| + |t-s| + |2^{-n/2}\lfloor 2^{n/2}s\rfloor - s|) \leqslant C_r(22^{-n/2} + |t-s|) \\
&\leqslant 2C_r(2^{-n/2} + |t-s|),
\end{aligned}
$$

with $C_r = (2r-1)! \sum_{a\in\mathbb{Z}} |\rho(a)|^{2r-1} < \infty$, hence (2.3.21) holds true.

(2) if $s \leqslant t \leqslant 0$: by the same argument as above

$$
\begin{aligned}
&E[(W_n^{(2r-1)}(t) - W_n^{(2r-1)}(s))^2] \\
&= E[(\overline{W}_{-,n}^{(2r-1)}(-s) - \overline{W}_{-,n}^{(2r-1)}(-t))^2] \leqslant C_r(2^{-n/2}\big|\lfloor 2^{n/2}|t|\rfloor - \lfloor 2^{n/2}|s|\rfloor\big|) \\
&\leqslant C_r(\big|2^{-n/2}\lfloor 2^{n/2}|t|\rfloor - |t|\big| + \big||t|-|s|\big| + \big|2^{-n/2}\lfloor 2^{n/2}|s|\rfloor - |s|\big|) \\
&\leqslant C_r(22^{-n/2} + \big||t|-|s|\big|) \\
&\leqslant 2C_r(2^{-n/2} + \big||t|-|s|\big|) \leqslant 2C_r(2^{-n/2} + |t-s|),
\end{aligned}
$$

so that (2.3.21) holds true as well.

(3) if $s < 0 < t$: using the two previous inequality (point (1) and point (2)) one has

$$
\begin{aligned}
E[(W_n^{(2r-1)}(t) - W_n^{(2r-1)}(s))^2] &\leqslant 2E[(W_n^{(2r-1)}(t) - W_n^{(2r-1)}(0))^2] \\
&\quad + 2E[(W_n^{(2r-1)}(s) - W_n^{(2r-1)}(0))^2] \\
&\leqslant 4C_r(2^{-n/2} + t) + 4C_r(2^{-n/2} + |s|) \\
&= 8C_r 2^{-n/2} + 4C_r(t + |s|) \\
&= 8C_r 2^{-n/2} + 4C_r|t-s| \\
&\leqslant 8C_r(2^{-n/2} + |t-s|).
\end{aligned}
$$

This proves (2.3.21).

Now, let us go back to $V_n^{(2r-1)}$. Observe that

$$V_n^{(2r-1)}(t) = W_n^{(2r-1)}(Y_{T_{\lfloor 2^n t \rfloor, n}})$$

and recall from [20, Lemma 2.3] that $E[|Y_{T_{\lfloor 2^n t\rfloor,n}} - Y(t)|] \to 0$ as $n \to \infty$ for any $t > 0$. We deduce, combining these two latter facts with (2.3.21), that

$$V_n^{(2r-1)}(t) - W_n^{(2r-1)}(Y_t) \xrightarrow{L^2} 0 \quad \text{as } n \to \infty.$$

But Y is independent from $W_n^{(2r-1)}$ so, from (2.3.20), it comes that

$$(W_n^{(2r-1)} \circ Y,\, 1 \leqslant r \leqslant N) \xrightarrow{\text{f.d.d.}} (\sigma_{2r-1} B^{(r)} \circ Y,\, 1 \leqslant r \leqslant N), \tag{2.3.22}$$

from which the desired conclusion (2.1.4) follows.

2.3.2 Proof of (2.1.5)

Recall the definition (2.1.3) and let us fix $\widetilde{\kappa} = 3/4$. First of all, let us apply Lemma 2.2.1. Because $2r$ is an even number, we obtain that

$$V_n^{(2r)}(t) = 2^{-3n/4} \sum_{j \in \mathbb{Z}} H_{2r}(2^{nH/2}(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}))(U_{j,n}(t) + D_{j,n}(t)). \tag{2.3.23}$$

Set $\mathcal{L}_{j,n}(t) = 2^{-n/2}(U_{j,n}(t) + D_{j,n}(t))$, so that

$$V_n^{(2r)}(t) \quad = \quad 2^{-n/4} \sum_{j \in \mathbb{Z}} H_{2r}(2^{nH/2}(X_{(j+1)2^{-n/2}} - X_{j2^{-n/2}}))\mathcal{L}_{j,n}(t).$$

At this stage, to simplify the exposition, let us introduce the short-hand notation

$$X_j^{(n)} = 2^{nH/2} X_{j2^{-n/2}}.$$

Fix $t \geqslant 0$. In order to study the convergence in law of $V_n^{(2r)}(t)$ as n tends to infinity, we shall consider (separately) the cases when n is even and when n is odd.

When n is even, for any even integers $n \geqslant m \geqslant 0$ and any integer $p \geqslant 0$, by following Nourdin and Peccati (see [27]) one can decompose $V_n^{(2r)}(t)$ as

$$V_n^{(2r)}(t) = A_{m,n,p}^{(2r)}(t) + B_{m,n,p}^{(2r)}(t) + C_{m,n,p}^{(2r)}(t) + D_{m,n,p}^{(2r)}(t),$$

where

$$
\begin{aligned}
A^{(2r)}_{m,n,p}(t) &= 2^{-n/4} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)(\mathcal{L}_{i,n}(t)-L^{i2^{-n/2}}_t(Y))\\
B^{(2r)}_{m,n,p}(t) &= 2^{-n/4} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)\\
&\qquad\qquad \times(L^{i2^{-n/2}}_t(Y)-L^{j2^{-m/2}}_t(Y))\\
C^{(2r)}_{m,n,p}(t) &= 2^{-n/4} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} L^{j2^{-m/2}}_t(Y) \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)\\
D^{(2r)}_{m,n,p}(t) &= 2^{-n/4} \sum_{i\geqslant p2^{n/2}} H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)\mathcal{L}_{i,n}(t)+\sum_{i<-p2^{n/2}} H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)\mathcal{L}_{i,n}(t).
\end{aligned}
$$

We can see that since we have taken even integers $n\geqslant m\geqslant 0$ then $2^{m/2}$, $2^{\frac{n-m}{2}}$ and $2^{n/2}$ are integers as well. This justifies the validity of the previous decomposition.

When n is odd, for any odd integers $n\geqslant m\geqslant 0$ we can work with the same decomposition for $V^{(2r)}_n(t)$. The only difference is that we have to replace the sum $\sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}}$ in $A^{(2r)}_{m,n,p}(t)$, $B^{(2r)}_{m,n,p}(t)$ and $C^{(2r)}_{m,n,p}(t)$ by $\sum_{-p2^{\frac{m+1}{2}}+1\leqslant j\leqslant p2^{\frac{m+1}{2}}}$. And instead of $\sum_{i\geqslant p2^{n/2}}$ and $\sum_{i<-p2^{n/2}}$ in $D^{(2r)}_{m,n,p}(t)$, we must consider $\sum_{i\geqslant p2^{\frac{n+1}{2}}}$ and $\sum_{i<-p2^{\frac{n+1}{2}}}$ respectively. The analysis can then be done *mutatis mutandis*.

Let us go back to our proof. Firstly, we will prove that $A^{(2r)}_{m,n,p}(t)$, $B^{(2r)}_{m,n,p}(t)$ and $D^{(2r)}_{m,n,p}(t)$ converge to 0 in L^2 by letting n, then m, then p tend to infinity. Secondly, we will study the convergence in law (in the sense f.d.d) of

$$\big(C^{(2r)}_{m,n,p},\,1\leqslant r\leqslant N\big), \tag{2.3.24}$$

which will then be equivalent to the convergence in law (in the sense f.d.d) of

$$\big(V^{(2r)}_n,\,1\leqslant r\leqslant N\big).$$

We will prove that $E[(A^{(2r)}_{m,n,p}(t))^2]\to 0$ as $n\to\infty$. We have, with ρ given by (2.3.17) (note that $\sum_{a\in\mathbb{Z}}|\rho(a)|^{2r}<\infty$ if and only if $H<1-1/(4r)$, which is satisfied for any $r\geqslant 1$ because

$H < 3/4$),

$$
\begin{aligned}
& E[(A^{(2r)}_{m,n,p}(t))^2] \\
= & \Big| 2^{-\frac{n}{2}} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} E\big[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_{i}) \\
& \times H_{2r}(X^{(n)}_{i'+1}-X^{(n)}_{i'})\big] E\big[(\mathcal{L}_{i,n}(t)-L_t^{i2^{-n/2}}(Y))(\mathcal{L}_{i',n}(t)-L_t^{i'2^{-n/2}}(Y))\big]\Big| \\
\leqslant & (2r)!\, 2^{-\frac{n}{2}} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r} \\
& \|\mathcal{L}_{i,n}(t)-L_t^{i2^{-n/2}}(Y)\|_2 \times \|\mathcal{L}_{i',n}(t)-L_t^{i'2^{-n/2}}(Y)\|_2,
\end{aligned}
$$

where, in the first equality, we used the independence between X and Y. By the point 3 of Proposition 2.2.2, we have

$$\|\mathcal{L}_{i,n}(t)-L_t^{i2^{-n/2}}(Y)\|_2 \leqslant 2n2^{-n/4}\|K\|_4\|L_t^{i2^{-n/2}}(Y)\|_2^{1/2}. \tag{2.3.25}$$

On the other hand

$$\|L_t^{i2^{-n/2}}(Y)\|_2 \leqslant \|L_t^{i2^{-n/2}}(Y)-L_t^0(Y)\|_2+\|L_t^0(Y)\|_2. \tag{2.3.26}$$

By the point 2 of Proposition 2.2.2, we have

$$\|L_t^{i2^{-n/2}}(Y)-L_t^0(Y)\|_2 \leqslant \mu\sqrt{|i|2^{-n/2}}\, t^{\frac{1}{4}}. \tag{2.3.27}$$

By combining (2.3.26) and (2.3.27), we get that $\|L_t^{i2^{-n/2}}(Y)\|_2 \leqslant \mu\sqrt{|i|}\,2^{-n/4}t^{\frac{1}{4}}+\|L_t^0(Y)\|_2$. Since $\sqrt{a+b}\leqslant\sqrt{a}+\sqrt{b}$ for all $a,b\geqslant 0$, we deduce that

$$\|L_t^{i2^{-n/2}}(Y)\|_2^{1/2} \leqslant \sqrt{\mu}|i|^{1/4}2^{-n/8}t^{1/8}+\|L_t^0(Y)\|_2^{1/2}. \tag{2.3.28}$$

Finally, (2.3.28) together with (2.3.25) show that

$$\|\mathcal{L}_{(i,n)}(t)-L_t^{i2^{-n/2}}(Y)\|_2 \leqslant 2\sqrt{\mu}\|K\|_4\, t^{1/8}n2^{-n/4}\,2^{-n/8}|i|^{1/4}+2\|K\|_4\|L_t^0(Y)\|_2^{1/2}n2^{-n/4}. \tag{2.3.29}$$

As a result,

$$
\begin{aligned}
& E[(A^{(2r)}_{m,n,p}(t))^2] \\
\leqslant\ & 4(2r)!\,\mu\, t^{1/8}t^{1/8}\|K\|_4^2\, 2^{-n}2^{-n/4}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}}\ \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} && (2.3.30)\\
& \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\ \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r}|ii'|^{1/4} \\
+\ & 4(2r)!\,\sqrt{\mu}t^{1/8}\|K\|_4^2\|L_t^0(Y)\|_2^{1/2}2^{-n}2^{-n/8}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} && (2.3.31)\\
& \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}}\ \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\ \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r}|i|^{1/4} \\
+\ & 4(2r)!\,\sqrt{\mu}t^{1/8}\|K\|_4^2\|L_t^0(Y)\|_2^{1/2}2^{-n}2^{-n/8}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} && (2.3.32)\\
& \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}}\ \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\ \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r}|i'|^{1/4} \\
+\ & 4(2r)!\,\|K\|_4^2\|L_t^0(Y)\|_2^{1/2}\|L_t^0(Y)\|_2^{1/2}2^{-n}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} && (2.3.33)\\
& \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}}\ \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\ \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r},
\end{aligned}
$$

and we are thus left to prove the convergence to 0 of (2.3.30)-(2.3.33) as $n\to\infty$. Let us do it.

(a) We have

$$
\begin{aligned}
& 2^{-n}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}}\ \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}}\ \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1}\ \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r} \\
&= 2^{-n}n^2 \sum_{i=-p2^{n/2}}^{p2^{n/2}-1}\ \sum_{i'=-p2^{n/2}}^{p2^{n/2}-1} \rho(i-i')^{2r} \\
&\leqslant 2^{-n}n^2 \sum_{i=-p2^{n/2}}^{p2^{n/2}-1}\ \sum_{i'\in\mathbb{Z}} \rho(i')^{2r} = \sum_{i'\in\mathbb{Z}} \rho(i')^{2r} n^2 2^{-n}(2p2^{n/2}).
\end{aligned}
$$

Since it is clear that the last quantity converges to 0 as $n\to\infty$, one deduces that (2.3.33) tends to zero.

(b) Since $-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}$ and $(j'-1)2^{\frac{n-m}{2}}\leqslant i'\leqslant j'2^{\frac{n-m}{2}}-1$, we deduce that $-p2^{n/2}\leqslant i'\leqslant p2^{n/2}-1$. So, $|i'|\leqslant p2^{n/2}$. Consequently we have that $|i'|^{1/4}\leqslant p^{1/4}2^{n/8}$, which shows that

$$2^{-n}2^{-n/8}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r}|i'|^{1/4}$$
$$\leqslant p^{1/4}2^{-n}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r}$$

and this last quantity converges to 0 by the same argument as above. This shows that (2.3.32) tends to zero.

(c) Following the same strategy as in point (b), one deduces that (2.3.31) tends to zero. Details are left to the reader.

(d) By the same arguments as above, one can see that $|ii'|^{1/4} \leqslant p^{1/2}2^{n/4}$. It follows that

$$2^{-n}2^{-n/4}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r}|ii'|^{1/4}$$
$$\leqslant p^{1/2}2^{-n}n^2 \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1} \rho(i-i')^{2r},$$

which converges to 0 by the same arguments as above. Hence, (2.3.30) tends to zero. The proof of $E[(A^{(2r)}_{m,n,p}(t))^2] \to 0$ as $n \to \infty$ is complete.

Now, let us prove the convergence of $B^{(2r)}_{m,n,p}(t)$ to 0 in L^2 as $m \to \infty$, uniformly in n. We have

$$E[(B^{(2r)}_{m,n,p}(t))^2]$$
$$= 2^{-n/2} \sum_{-p2^{m/2}+1\leqslant j\leqslant p2^{m/2}} \sum_{-p2^{m/2}+1\leqslant j'\leqslant p2^{m/2}} \sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} \sum_{i'=(j'-1)2^{\frac{n-m}{2}}}^{j'2^{\frac{n-m}{2}}-1}$$
$$(2r)!\rho(i-i')^{2r} \times E[(L_t^{i2^{-n/2}}(Y) - L_t^{j2^{-m/2}}(Y))(L_t^{i'2^{-n/2}}(Y) - L_t^{j'2^{-m/2}}(Y))].$$

By Proposition 2.2.2 (point 2) and Cauchy-Schwarz, there is a universal constant μ such that

$$\left|E[(L_t^{i2^{-n/2}}(Y) - L_t^{j2^{-m/2}}(Y))(L_t^{i'2^{-n/2}}(Y) - L_t^{j'2^{-m/2}}(Y))]\right|$$
$$\leqslant \mu^2\sqrt{t}\sqrt{|i2^{-n/2} - j2^{-m/2}||i'2^{-n/2} - j'2^{-m/2}|} \leqslant \mu^2\sqrt{t}2^{-m/2}.$$

This yields

$$\begin{aligned}\sup_n E[(B^{(2r)}_{m,n,p}(t))^2] &\leqslant \mu^2(2r)!\,2^{-m/2}\sqrt{t}\\ &\qquad\times\sup_n\left\{2^{-n/2}\sum_{i=-p2^{n/2}}^{p2^{n/2}-1}\sum_{i'=-p2^{n/2}}^{p2^{n/2}-1}\rho(i-i')^{2r}\right\}\\ &\leqslant \mu^2(2r)!\,2^{-m/2}\sqrt{t}\,2p\sum_{i\in\mathbb{Z}}\rho(i)^{2r},\end{aligned}$$

which converges to 0 as $m\to\infty$.

Finally, let us prove that $D^{(2r)}_{m,n,p}(t)$ converges to 0 in L^2 as $p\to\infty$, uniformly in m and n. We have

$$\begin{aligned}&E\big[(D^{(2r)}_{m,n,p}(t))^2\big]\\ &= 2^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\sum_{j\geqslant p2^{n/2}}E\big[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)H_{2r}(X^{(n)}_{j+1}-X^{(n)}_j)\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big] && (2.3.34)\\ &\quad+22^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\sum_{j<-p2^{n/2}}E\big[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)H_{2r}(X^{(n)}_{j+1}-X^{(n)}_j) && (2.3.35)\\ &\qquad\qquad\times\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big]\\ &\quad+2^{-\frac{n}{2}}\sum_{i<-p2^{n/2}}\sum_{j<-p2^{n/2}}E\big[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)H_{2r}(X^{(n)}_{j+1}-X^{(n)}_j) && (2.3.36)\\ &\qquad\qquad\times\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big],\end{aligned}$$

and we are thus left to prove the convergence to 0 of (2.3.34)-(2.3.36) as $p\to\infty$, uniformly in m and n. Let us do it.

(a) We have

$$\begin{aligned}&\left|2^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\sum_{j\geqslant p2^{n/2}}E\big[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)H_{2r}(X^{(n)}_{j+1}-X^{(n)}_j)\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big]\right|\\ &= \left|2^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\sum_{j\geqslant p2^{n/2}}E\big[H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i)H_{2r}(X^{(n)}_{j+1}-X^{(n)}_j)\big]\right.\\ &\qquad\qquad\left.\times E\big[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big]\right|\\ &= (2r)!\,2^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\sum_{j\geqslant p2^{n/2}}\rho(i-j)^{2r}E\big[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big]\end{aligned}$$

where, in the second equality, we used the independence between X and Y. It is enough to prove that, uniformly in n and m, and as $p\to\infty$:

$$2^{-\frac{n}{2}}\sum_{i\geqslant p2^{n/2}}\sum_{j\geqslant p2^{n/2}}\rho(i-j)^{2r}E\big[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big]\to 0. \qquad (2.3.37)$$

We can write

$$
\begin{aligned}
&2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} \sum_{j \geqslant p2^{n/2}} \rho(i-j)^{2r} E\big[\mathcal{L}_{i,n}(t)\mathcal{L}_{j,n}(t)\big] \\
&\leqslant 2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} \sum_{j \geqslant p2^{n/2}} \rho(i-j)^{2r} E\big[\frac{1}{2}\big(\mathcal{L}_{i,n}(t)^2 + \mathcal{L}_{j,n}(t)^2\big)\big] \\
&= 2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} E\big[\mathcal{L}_{i,n}(t)^2\big] \sum_{j \geqslant p2^{n/2}} \rho(i-j)^{2r} \\
&\leqslant 2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} E\big[\mathcal{L}_{i,n}(t)^2\big] \sum_{j \in \mathbb{Z}} \rho(j)^{2r} = C_r 2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} E\big[\mathcal{L}_{i,n}(t)^2\big]
\end{aligned}
$$

where $C_r := \sum_{j \in \mathbb{Z}} \rho(j)^{2r} < \infty$. By the third point of Proposition 2.2.2, we have

$$|\mathcal{L}_{i,n}(t)| \leqslant L_t^{i2^{-n/2}}(Y) + 2Kn2^{-n/4}\sqrt{L_t^{i2^{-n/2}}(Y)}$$

so that

$$E\big[\mathcal{L}_{i,n}(t)^2\big] \leqslant 2E\big[L_t^{i2^{-n/2}}(Y)^2\big] + 8n^2 2^{-n/2}\|K^2\|_2\|L_t^{i2^{-n/2}}(Y)\|_2. \tag{2.3.38}$$

On the other hand, thanks to the point 1 of Proposition 2.2.2, we have

$$E\big[L_t^{i2^{-n/2}}(Y)^2\big] \leqslant Ct\exp\big(-\frac{(i2^{-n/2})^2}{2t}\big). \tag{2.3.39}$$

Consequently, we get

$$\|L_t^{i2^{-n/2}}(Y)\|_2 \leqslant C^{1/2}t^{1/2}\exp\big(-\frac{(i2^{-n/2})^2}{4t}\big). \tag{2.3.40}$$

By combining (2.3.38) with (2.3.39) and (2.3.40), we deduce that

$$
\begin{aligned}
2^{-\frac{n}{2}} \sum_{i \geqslant p2^{n/2}} E\big[(\mathcal{L}_{i,n}(t)^2\big] \leqslant 2Ct2^{-n/2} &\sum_{i \geqslant p2^{n/2}} \exp\big(-\frac{(i2^{-n/2})^2}{2t}\big) \\
&+ 8C^{1/2}t^{1/2}\|K^2\|_2 n^2 2^{-n/2} \\
&\quad \times 2^{-n/2} \sum_{i \geqslant p2^{n/2}} \exp\big(-\frac{(i2^{-n/2})^2}{4t}\big).
\end{aligned}
$$

But, for $a \in \{2,4\}$,

$$2^{-n/2} \sum_{i \geqslant p2^{n/2}} \exp\big(-\frac{(i2^{-n/2})^2}{at}\big) \leqslant \int_{p-1}^{\infty} \exp\big(\frac{-x^2}{at}\big)dx \underset{p\to\infty}{\longrightarrow} 0.$$

This proves (2.3.37). Hence, we deduce that (2.3.34) converges to 0 as $p \to \infty$ uniformly in n and m.

(b) Following the same strategy as in point (a), one deduces that (2.3.35) and (2.3.36) converge to 0 as $p \to \infty$ uniformly in n and m. Details are left to the reader.

This shows that $D^{(2r)}_{m,n,p}(t)$ converges to 0 in L^2 as $p \to \infty$, uniformly in m and n.

To finish our proof of (2.1.5), it remains to prove that, by letting n, then m, then p tend to infinity, we get

$$\left\{C^{(2r)}_{m,n,p}(t),\, 1 \leqslant r \leqslant N\right\}_{t\geqslant 0} \overset{\text{f.d.d.}}{\longrightarrow} \left\{\sigma_{2r}\int_{-\infty}^{\infty} L^x_t(Y)dB^{(r)}_x \,:\, 1 \leqslant r \leqslant N\right\}_{t\geqslant 0}. \tag{2.3.41}$$

Since $H < 3/4$, we claim that, as $n \to \infty$,

$$\left(2^{-n/4}\sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i),\, 1 \leqslant r \leqslant N : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}\right) \overset{\text{law}}{\longrightarrow} \tag{2.3.42}$$

$$\left(\sigma_{2r}(B^{(r)}_{(j+1)2^{-m/2}} - B^{(r)}_{j2^{-m/2}}),\, 1 \leqslant r \leqslant N : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}\right)$$

where $(B^{(1)}, \ldots, B^{(N)})$ is a N-dimensional two-sided Brownian motion.

Indeed, it is clear, using the self-similarity property of X, that the convergence in law of

$$\left(2^{-n/4}\sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2r}(X^{(n)}_{i+1}-X^{(n)}_i),\, 1 \leqslant r \leqslant N : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}\right)$$

is equivalent to the convergence in law of

$$\left(2^{-n/4}\sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2r}(X_{i+1}-X_i),\, 1 \leqslant r \leqslant N : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}\right).$$

Then, Breuer-Major Theorem 2.2.3 applies and yields that, as $n \to \infty$ and for any fixed $1 \leqslant r \leqslant N$,

$$\left(2^{-n/4}\sum_{i=(j-1)2^{\frac{n-m}{2}}}^{j2^{\frac{n-m}{2}}-1} H_{2r}(X_{i+1}-X_i) : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}\right) \overset{\text{law}}{\longrightarrow}$$
$$\left(\sigma_{2r}(B^{(r)}_{(j+1)2^{-m/2}} - B^{(r)}_{j2^{-m/2}}) : -p2^{m/2}+1 \leqslant j \leqslant p2^{m/2}\right).$$

In addition, from Peccati-Tudor Theorem 2.2.4 and taking into account the orthogonality of Hermite polynomial with different orders, we deduce (2.3.42). (The detailed proof of this result is similar to the proof of (2.3.20)).

As a consequence of (2.3.42), and thanks to the independence of X and Y, we have that as $n \to \infty$,

$$\left\{C^{(2r)}_{m,n,p}(t),\, 1 \leqslant r \leqslant N\right\}_{t\geqslant 0} \overset{\text{f.d.d.}}{\longrightarrow}$$
$$\left\{\sigma_{2r} \sum_{j=-p2^{m/2}+1}^{p2^{m/2}} L_t^{j2^{-m/2}}(Y)(B^{(r)}_{(j+1)2^{-m/2}} - B^{(r)}_{j2^{-m/2}}),\, 1 \leqslant r \leqslant N\right\}_{t\geqslant 0}.$$

Since, for any fixed $t \geqslant 0$ and $1 \leqslant r \leqslant N$ and as $m \to \infty$,

$$\sum_{j=-p2^{m/2}+1}^{p2^{m/2}} L_t^{j2^{-m/2}}(Y)(B^{(r)}_{(j+1)2^{-m/2}} - B^{(r)}_{j2^{-m/2}}) \overset{P}{\longrightarrow} \int_{-p}^{p} L_t^x(Y) dB_x^{(r)},$$

and since $\int_{-p}^{p} L_t^x(Y) dB_x^{(r)} \overset{P}{\longrightarrow} \int_{\mathbb{R}} L_t^x(Y) dB_x^{(r)}$ as $p \to \infty$, we deduce finally that by letting m, then p tend to infinity, we get

$$\left\{\sigma_{2r} \sum_{j=-p2^{m/2}+1}^{p2^{m/2}} L_t^{j2^{-m/2}}(Y)(B^{(r)}_{(j+1)2^{-m/2}} - B^{(r)}_{j2^{-m/2}}),\, 1 \leqslant r \leqslant N\right\}_{t\geqslant 0} \overset{\text{f.d.d.}}{\longrightarrow}$$
$$\left\{\sigma_{2r} \int_{-\infty}^{\infty} L_t^x(Y) dB_x^{(r)} \,:\, 1 \leqslant r \leqslant N\right\}_{t\geqslant 0}.$$

This proves (2.3.41), and consequently (2.1.5).

2.3.3 Proof of Corollary 2.1.2

Let us decompose x^p in terms of Hermite polynomials. We have $x^p = \sum_{k=0}^{p} a_{p,k} H_k(x)$, where $a_{p,k}$ is some (explicit) integer. To calculate $a_{p,k}$, let N be a centred Gaussian variable with variance one. We have

$$N^p = \sum_{k=0}^{p} a_{p,k} H_k(N). \tag{2.3.43}$$

Thanks to the orthogonality property of Hermite polynomials with different orders and to the well known fact that $E[H_k(N)^2] = k!$, we get

$$a_{p,k} = \frac{1}{k!} E[N^p H_k(N)]. \tag{2.3.44}$$

On the other hand (see, eg., [28] page 19) we have, for all $k \geqslant 1$,

$$H_k(x) = \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{k!(-1)^l}{l!(k-2l)!2^l} x^{k-2l}. \tag{2.3.45}$$

By combining (2.3.44) with (2.3.45), we deduce that

$$a_{p,k} = \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(-1)^l}{l!(k-2l)!2^l} E(N^{p+k-2l}). \tag{2.3.46}$$

Thus,

$$a_{p,k} = \begin{cases} \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(-1)^l (p+k-2l)!}{l!(k-2l)!2^l 2^{\frac{p+k-2l}{2}} (\frac{p+k-2l}{2})!} & \text{if } p \text{ and } k \text{ are odd} \\ \sum_{l=0}^{\lfloor k/2 \rfloor} \frac{(-1)^l (p+k-2l)!}{l!(k-2l)!2^l 2^{\frac{p+k-2l}{2}} (\frac{p+k-2l}{2})!} & \text{if } p \text{ and } k \text{ are even} \\ 0 & \text{otherwise} \end{cases} \cdot$$

As a result, we deduce that if p is odd, then

$$x^p = \sum_{k=1}^{\lfloor p/2 \rfloor + 1} a_{p,2k-1} H_{2k-1}(x), \tag{2.3.47}$$

whereas if p is even, then

$$x^p = \sum_{k=0}^{p/2} a_{p,2k} H_{2k}(x), \tag{2.3.48}$$

Finally, thanks to (2.3.47), (2.3.48), Theorem 2.1.1 and the Continuous Mapping Theorem, we deduce the content of Corollary 2.1.2.

Chapter 3

An Itô's type formula for the fBmBt in dimension 1

In this chapter we present the results obtained in: I. Nourdin and R. Zeineddine, "An Itô-type formula for the fractional Brownian motion in Brownian time", *Electron. J. Probab.* 19 (2014), no. 99, 1-15.

3.1 Introduction

If $f : \mathbb{R}_+ \to \mathbb{R}$ is C^1 then $f(t) = f(0) + \int_0^t f'(s)ds$ for all $t \geqslant 0$ whereas, if W is a standard Brownian motion and if $f : \mathbb{R} \to \mathbb{R}$ is C^2 then, by the Itô's formula,

$$f(W_t) = f(0) + \int_0^t f'(W_s)d^-W_s + \frac{1}{2}\int_0^t f''(W_s)ds, \quad t \geqslant 0. \tag{3.1.1}$$

In (3.1.1) the Itô integral, namely

$$\int_0^t X_s d^-Y_s := \lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} X_{k2^{-n}}(Y_{(k+1)2^{-n}} - Y_{k2^{-n}}), \tag{3.1.2}$$

is of forward type. It is well-known that the additional bracket term $\frac{1}{2}\int_0^t f''(W_s)ds$ appearing in (3.1.1) comes from the non-negligibility of the quadratic variation of W in the large limit; more precisely,

$$\sum_{k=0}^{\lfloor 2^n t\rfloor -1} (W_{(k+1)2^{-n}} - W_{k2^{-n}})^2 \overset{\text{a.s.}}{\longrightarrow} t \quad \text{as } n \to \infty. \tag{3.1.3}$$

Introducing a family $\{B^H\}_{H\in(0,1)}$ of fractional Brownian motions parametrized by the Hurst parameter H may help to reinterpret (3.1.1) in a more dynamical way. Let us elaborate this point of view further. Recall that $B^{\frac{1}{2}}$ is nothing but the standard Brownian motion, whereas B^1 is the process $B^1_t = tG$, $t \geqslant 0$, $G \sim N(0,1)$. The extension of (3.1.3) to any $H \in (0,1)$ is well-known: one has

$$2^{n(2H-1)} \sum_{k=0}^{\lfloor 2^n t\rfloor -1} (B^H_{(k+1)2^{-n}} - B^H_{k2^{-n}})^2 \overset{\text{a.s.}}{\longrightarrow} t \quad \text{as } n \to \infty. \tag{3.1.4}$$

Based on (3.1.4), it is then not difficult to prove the following two facts:

1. If $H > \frac{1}{2}$ and $f : \mathbb{R} \to \mathbb{R}$ is C^2 (actually, C^1 is enough), then $\int_0^t f'(B_s^H) d^- B_s^H$ exists as a limit in probability and we have
$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^- B_s^H, \quad t \geqslant 0.$$

2. If $H < \frac{1}{2}$, then
$$\int_0^t B_s^H d^- B_s^H = -\infty \quad \text{a.s.},$$
meaning that there is no possible change-of-variable formula for $f(x) = x^2$.

Thus, $H = \frac{1}{2}$ appears to be a critical value for the change-of-variable formula involving the forward integral (3.1.2). This is because it is precisely the value from which the sign of $2H - 1$ changes in (3.1.4). The chain rule being (3.1.1) in the critical case $H = \frac{1}{2}$, one has a complete picture for the forward integral (3.1.2).

To go one step further, one may wonder what kind of change-of-variable formula one would obtain after replacing the definition (3.1.2) by its symmetric counterpart, namely
$$\int_0^t X_s d^\circ Y_s := \lim_{n\to\infty} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} \frac{1}{2}\big(X_{k2^{-n}} + X_{(k+1)2^{-n}}\big)\big(Y_{(k+1)2^{-n}} - Y_{k2^{-n}}\big) \tag{3.1.5}$$
(provided the limit exists in some sense). As it turns out, it is arguably a much more difficult problem, which has been solved only recently. In this context, the crucial quantity is now the cubic variation. And this latter is known to satisfy, for any $H < \frac{1}{2}$,
$$2^{n(3H-\frac{1}{2})} \sum_{k=0}^{\lfloor 2^n t \rfloor - 1} (B^H_{(k+1)2^{-n}} - B^H_{k2^{-n}})^3 \overset{\text{law}}{\to} N(0, \sigma_H^2) \quad \text{as } n \to \infty. \tag{3.1.6}$$
With a lot of efforts, one can prove (see [9, 13] when $H \neq \frac{1}{6}$ and [30] when $H = \frac{1}{6}$) the following three facts, which hold for any smooth enough real function $f : \mathbb{R} \to \mathbb{R}$:

1. If $H > \frac{1}{6}$ then $\int_0^t f'(B_s^H) d^\circ B_s^H$ exists as a limit in probability and one has
$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\circ B_s^H, \quad t \geqslant 0. \tag{3.1.7}$$

2. If $H = \frac{1}{6}$ then $\int_0^t f'(B_s^{\frac{1}{6}}) d^\circ B_s^{\frac{1}{6}}$ exists as a stable limit in law and one has, with W a standard Brownian motion independent of $B^{\frac{1}{6}}$ and $\kappa_3 \simeq 2.322$,
$$f(B_t^{\frac{1}{6}}) = f(0) + \int_0^t f'(B_s^{\frac{1}{6}}) d^\circ B_s^{\frac{1}{6}} - \frac{\kappa_3}{12} \int_0^t f'''(B^{\frac{1}{6}}) dW_s, \quad t \geqslant 0. \tag{3.1.8}$$

3. If $H < \frac{1}{6}$ then

$$\int_0^t (B_s^H)^2 d^\circ B_s^H \text{ does not exist in law.} \tag{3.1.9}$$

Thus, as we see, the critical value for the symmetric integral is now $H = \frac{1}{6}$; it is exactly the value of H from which the sign of $3H - \frac{1}{2}$ changes in (3.1.6).

In [5, 6] (see also [7]), Burdzy has introduced the so-called iterated Brownian motion. This process, which can be regarded as the realization of a Brownian motion on a random fractal, is defined as

$$Z_t = X(Y_t), \quad t \geqslant 0,$$

where X is a two-sided Brownian motion and Y is a standard (one-sided) Brownian motion independent of X. Note that Z is self-similar of order $\frac{1}{4}$ and has stationary increments; hence, in some sense, Z is close to the fractional Brownian motion $B^{\frac{1}{4}}$ of index $H = \frac{1}{4}$. As is the case for $B^{\frac{1}{4}}$, Z is neither a Dirichlet process nor a semimartingale or a Markov process in its own filtration. A crucial question is therefore how to define a stochastic calculus with respect to it. This issue has been tackled by Khoshnevisan and Lewis in [20, 21], where the authors develop a Stratonovich-type stochastic calculus with respect to Z, by extensively using techniques based on the properties of some special arrays of Brownian stopping times, as well as on excursion-theoretic arguments. See also the paper [27] which may be seen as a follow-up of [20]. The formula obtained in [20, 21] reads, unsurprisingly (due to (3.1.7) and the similarities between Z and $B^{\frac{1}{4}}$) and losely speaking, as follows:

$$f(Z_t) = f(0) + \int_0^t f'(Z_s) d^\circ Z_s, \quad t \geqslant 0. \tag{3.1.10}$$

The change-of-variable formula (3.1.10) is of the same kind than (3.1.7). In view of what has been done so far for the fractional Brownian motion B^H, aiming to provide an answer to the following problem is somehow natural: can we also reinterpret (3.1.10) in a dynamical way, in the spirit of (3.1.7), (3.1.8) and (3.1.9)? To this end, we first need to introduce a family of processes that contains the iterated Brownian motion Z as a particular element. The family consisting in the so-called fractional Brownian motions in Brownian time, introduced in chapter 1, does the job. More specifically, it is the family $\{Z^H\}_{H\in(0,1)}$ defined as follows:

$$Z_t^H = X^H(Y_t), \quad t \geqslant 0,$$

where X^H is a two-sided fractional Brownian motion of index H and Y is a standard (one-sided) Brownian motion independent of X. Roughly speaking, in the present chapter we are going to show the following three assertions (see Theorem 3.2.1 for a precise statement): for any smooth real function $f : \mathbb{R} \to \mathbb{R}$,

1. If $H > \frac{1}{6}$ then

$$f(Z_t^H) = f(0) + \int_0^t f'(Z_s^H) d^\circ Z_s^H, \quad t \geqslant 0.$$

2. If $H = \frac{1}{6}$ then, with $\kappa_3 \simeq 2.322$,

$$f(Z_t^{\frac{1}{6}}) - f(0) + \frac{\kappa_3}{12} \int_0^t f'''(Z_s^{\frac{1}{6}}) d^{\circ 3} Z_s^{\frac{1}{6}} = \int_0^t f'(Z_s^{\frac{1}{6}}) d^\circ Z_s^{\frac{1}{6}}, \quad t \geqslant 0. \tag{3.1.11}$$

Where, we have, for all $t \geqslant 0$,

$$\int_0^t f'''(Z_s^{\frac{1}{6}}) d^{\circ 3} Z_s^{\frac{1}{6}} := \int_0^{Y_t} f'''(X_s^{\frac{1}{6}}) dW_s,$$

where W is a two-sided Brownian motion independent of the pair $(X^{\frac{1}{6}}, Y)$ defining $Z^{\frac{1}{6}}$.

3. If $H < \frac{1}{6}$, then

$$\int_0^t (Z_s^H)^2 d^\circ Z_s^H \text{ does not exist.}$$

The formula (3.1.11) is related to a recent line of research in which, by means of Malliavin calculus, one aims to exhibit change-of-variable formulas in law with a correction term which is an Itô integral with respect to martingale independent of the underlying Gaussian processes. Papers dealing with this problem and which are prior to our work include [8, 15, 14, 16, 25, 29, 30]; however, it is worthwhile noting that all these mentioned references only deal with Gaussian processes, not with iterated processes (which are arguably more difficult to handle).

A brief outline of the chapter is as follows. In Section 2, we introduce the framework in which our study takes place and we provide an exact statement of our result, namely Theorem 3.2.1. Finally, Section 3 contains the proof of Theorem 3.2.1, which is divided into several steps.

3.2 Framework and exact statement of our results

For simplicity, throughout the chapter we remove the superscript H, that is, we write Z (resp. X) instead of Z^H (resp. X^H).

Let Z be a fractional Brownian motion in Brownian time of Hurst parameter $H \in (0, 1)$, defined as

$$Z_t = X(Y_t), \quad t \geqslant 0, \tag{3.2.12}$$

where X is a two-sided fractional Brownian motion of parameter H and Y is a standard (one-sided) Brownian motion independent of X.

The paths of Z being very irregular (precisely: Hölder continuous of order α if and only if α is strictly less than $H/2$), we will not be able to define a stochastic integral with respect to it as the limit of Riemann sums with respect to a *deterministic* partition of the time axis. However, a winning idea borrowed from Khoshnevisan and Lewis [20, 21] is to approach deterministic partitions by means of random partitions defined in terms of hitting times of the underlying Brownian motion Y. As such, one can bypass the random "time-deformation" forced by (3.2.12), and perform asymptotic procedures by separating the roles of X and Y in the overall definition of Z.

Following Khoshnevisan and Lewis [20, 21], we start by introducing the so-called intrinsic skeletal structure of Z. This structure is defined through a sequence of collections of stopping times (with respect to the natural filtration of Y), noted

$$\mathscr{T}_n = \{T_{k,n} : k \geqslant 0\}, \quad n \geqslant 1, \tag{3.2.13}$$

which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} : j \in \mathbb{Z}\}$, $n \geqslant 1$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geqslant 1$, the stopping times $T_{k,n}$, appearing in (3.2.13), are given by the following recursive definition: $T_{0,n} = 0$, and

$$T_{k,n} = \inf\big\{s > T_{k-1,n} : \quad Y(s) \in \mathscr{D}_n \setminus \{Y(T_{k-1,n})\}\big\}, \quad k \geqslant 1.$$

Note that the definition of $T_{k,n}$, and therefore of $\mathscr{T}_n$, only involves the one-sided Brownian motion Y, and that, for every $n \geqslant 1$, the discrete stochastic process

$$\mathscr{Y}_n = \{Y(T_{k,n}) : k \geqslant 0\}$$

defines a simple random walk over $\mathscr{D}_n$. As shown in [20, Lemma 2.2], as n tends to infinity the collection $\{T_{k,n} : 1 \leqslant k \leqslant 2^n t\}$ approximates the common dyadic partition $\{k2^{-n} : 1 \leqslant k \leqslant 2^n t\}$ of order n of the time interval $[0, t]$. More precisely,

$$\sup_{0\leqslant s\leqslant t} \left|T_{\lfloor 2^n s\rfloor,n} - s\right| \to 0 \quad \text{almost surely and in } L^2(\Omega). \tag{3.2.14}$$

Based on this fact, one may introduce the counterpart of (3.1.5) based on $\mathscr{T}_n$, namely,

$$V_n(f,t) = \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}\big(f(Z_{T_{k,n}}) + f(Z_{T_{k+1,n}})\big)(Z_{T_{k+1,n}} - Z_{T_{k,n}}). \tag{3.2.15}$$

Let C_b^∞ denote the class of those functions $f : \mathbb{R} \to \mathbb{R}$ that are C^∞ and bounded together with their derivatives. We then have the following result.

Theorem 3.2.1 *Let $f \in C_b^\infty$ and $t > 0$.*

1. *If $H > \frac{1}{6}$ then*

$$f(Z_t) - f(0) = \int_0^t f'(Z_s)d^\circ Z_s, \tag{3.2.16}$$

where $\int_0^t f'(Z_s)d^\circ Z_s$ is the limit in probability of $V_n(f', t)$ defined in (3.2.15) as $n \to \infty$.

2. *If $H = \frac{1}{6}$ then, with $\kappa_3 \simeq 2.322$,*

$$f(Z_t) - f(0) + \frac{\kappa_3}{12}\int_0^t f'''(Z_s)d^{\circ 3}Z_s \overset{\text{(law)}}{=} \int_0^t f'(Z_s)d^\circ Z_s. \tag{3.2.17}$$

Here, $\int_0^\cdot f'(Z_s)d^\circ Z_s$ denotes the limit in law of $V_n(f',\cdot)$ defined in (3.2.15) as $n\to\infty$ (its existence is part of the conclusion). Moreover, we have, for all $t\geqslant 0$,

$$\int_0^t f'''(Z_s)d^{\circ 3}Z_s := \int_0^{Y_t} f'''(X_s)dW_s,$$

where W is a two-sided Brownian motion independent of the pair (X,Y) defining Z, and where the integral with respect to W is understood in the Wiener-Itô sense.

3. *If $H<\frac{1}{6}$ then*

$$V_n(x\mapsto x^2,t) \text{ does not converge, even stably in law.} \tag{3.2.18}$$

This means that there is no way to get a change-of-variable formula for $f(x)=x^3$.

3.3 Proof of Theorem 3.2.1

3.3.1 Elements of Malliavin calculus

In this section, we gather some elements of Malliavin calculus we shall need thoughout the proof of Theorem 3.2.1. The reader already familiar with this topic may skip this section.

We continue to denote by $X=(X_t)_{t\in\mathbb{R}}$ a two-sided fractional Brownian motion with Hurst parameter $H\in(0,1)$. That is, X is a zero mean Gaussian process, defined on a complete probability space $(\Omega,\mathscr{A},P)$, with covariance function,

$$C_H(t,s)=E(X_tX_s)=\frac{1}{2}(|s|^{2H}+|t|^{2H}-|t-s|^{2H}),\quad s,t\in\mathbb{R}.$$

We suppose that $\mathscr{A}$ is the σ-field generated by X. For all $n\in\mathbb{N}^*$, we let $\mathscr{E}_n$ be the set of step functions on $[-n,n]$, and $\mathscr{E}:=\cup_n\mathscr{E}_n$. Set $\xi_t=\mathbf{1}_{[0,t]}$ (resp. $\mathbf{1}_{[t,0]}$) if $t\geqslant 0$ (resp. $t<0$). Let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the inner product

$$\langle\xi_t,\xi_s\rangle_{\mathscr{H}}=C_H(t,s),\quad s,t\in\mathbb{R}.$$

The mapping $\xi_t\mapsto X_t$ can be extended to an isometry between $\mathscr{H}$ and the Gaussian space $\mathbb{H}_1$ associated with X. We will denote this isometry by $\varphi\mapsto X(\varphi)$.

Let $\mathscr{F}$ be the set of all smooth cylindrical random variables, i.e. of the form

$$F=\phi(X_{t_1},...,X_{t_l}),$$

where $l\in\mathbb{N}^*$, $\phi:\mathbb{R}^l\to\mathbb{R}$ is C_b^∞ and $t_1<...<t_l$ are some real numbers. The derivative of F with respect to X is the element of $L^2(\Omega,\mathscr{H})$ defined by

$$D_sF=\sum_{i=1}^l\frac{\partial\phi}{\partial x_i}(X_{t_1},...,X_{t_l})\xi_{t_i}(s),\qquad s\in\mathbb{R}.$$

In particular $D_sX_t=\xi_t(s)$. For any integer $k\geqslant 1$, we denote by $\mathbb{D}^{k,2}$ the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{k,2}^2=E(F^2)+\sum_{j=1}^k E[\|D^jF\|^2_{\mathscr{H}^{\otimes j}}].$$

The Malliavin derivative D satisfies the chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C_b^1 and if $F_1, \dots, F_n$ are in $\mathbb{D}^{1,2}$, then $\varphi(F_1, ..., F_n) \in \mathbb{D}^{1,2}$ and we have

$$D\varphi(F_1, ..., F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, ..., F_n) DF_i.$$

We have the following Leibniz formula, whose proof is straightforward by induction on q. Let $\varphi, \psi \in C_b^q$ ($q \geqslant 1$), and fix $0 \leqslant u < v$ and $0 \leqslant s < t$. Then $\varphi(X_t - X_s)\psi(X_v - X_u) \in \mathbb{D}^{q,2}$ and

$$D^q\big(\varphi(X_t - X_s)\psi(X_v - X_u)\big) = \sum_{a=0}^{q} \binom{q}{a} \varphi^{(a)}(X_t - X_s)\psi^{(q-a)}(X_v - X_u)\mathbf{1}_{[s,t]}^{\otimes a} \tilde{\otimes} \mathbf{1}_{[u,v]}^{\otimes (q-a)}, \tag{3.3.19}$$

where $\tilde{\otimes}$ stands for the symmetric tensor product. A similar statement holds for $u < v \leqslant 0$ and $s < t \leqslant 0$.

If a random element $u \in L^2(\Omega, \mathscr{H})$ belongs to the domain of the divergence operator, that is, if it satisfies

$$|E\langle DF, u\rangle_{\mathscr{H}}| \leqslant c_u \sqrt{E(F^2)} \text{ for any } F \in \mathscr{F},$$

then $I(u)$ is defined by the duality relationship

$$E\big(FI(u)\big) = E\big(\langle DF, u\rangle_{\mathscr{H}}\big),$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geqslant 1$, let $\mathbb{H}_n$ be the nth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathscr{A}, P)$ generated by the random variables $\{H_n(B(h)), h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, where H_n is the nth Hermite polynomial. The mapping $I_n(h^{\otimes n}) = H_n(B(h))$ provides a linear isometry between the symmetric tensor product $\mathscr{H}^{\odot n}$ and $\mathbb{H}_n$. For $H = \frac{1}{2}$, I_n coincides with the multiple Wiener-Itô integral of order n. The following duality formula holds

$$E\big(FI_n(h)\big) = E\big(\langle D^n F, h\rangle_{\mathscr{H}^{\otimes n}}\big), \tag{3.3.20}$$

for any element $h \in \mathscr{H}^{\odot n}$ and any random variable $F \in \mathbb{D}^{n,2}$.

Let $\{e_k, k \geqslant 1\}$ be a complete orthonormal system in $\mathscr{H}$. Given $f \in \mathscr{H}^{\odot n}$ and $g \in \mathscr{H}^{\odot m}$, for every $r = 0, ..., n \wedge m$, the contraction of f and g of order r is the element of $\mathscr{H}^{\otimes(n+m-2r)}$ defined by

$$f \otimes_r g = \sum_{k_1, \dots, k_r = 1}^{\infty} \langle f, e_{k_1} \otimes ... \otimes e_{k_r}\rangle_{\mathscr{H}^{\otimes r}} \otimes \langle g, e_{k_1} \otimes ... \otimes e_{k_r}\rangle_{\mathscr{H}^{\otimes r}}.$$

Note that $f \otimes_r g$ is not necessarily symmetric: we denote its symmetrization by $f \tilde{\otimes}_r g \in \mathscr{H}^{\odot(n+m-2r)}$. Finally, we recall the following product formula: if $f \in \mathscr{H}^{\odot n}$ and $g \in \mathscr{H}^{\odot m}$ then

$$I_n(f)I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f \tilde{\otimes}_r g). \tag{3.3.21}$$

3.3.2 Notation and reduction of the problem

Throughout all the proof, we shall use the following notation. For all $k,n\in\mathbb{N}$ we write

$$\begin{aligned}\xi_{k2^{-n/2}} &= \mathbf{1}_{[0,k2^{-n/2}]}, \quad \xi^-_{k2^{-n/2}} = \mathbf{1}_{[-k2^{-n/2},0]},\\ \delta_{k2^{-n/2}} &= \mathbf{1}_{[(k-1)2^{-n/2},k2^{-n/2}]}, \quad \delta^-_{k2^{-n/2}} = \mathbf{1}_{[-k2^{-n/2},(-k+1)2^{-n/2}]}.\end{aligned}$$

Also, $\langle\cdot,\cdot\rangle$ ($\|\cdot\|$, respectively) will always stand for inner product (the norm, respectively) in an appropriate tensor product $\mathscr{H}^{\otimes s}$.

In the sequel, we only consider the case $H<\frac12$. The proof of (3.2.16) in the case $H>\frac12$ is easier and left to the reader, whereas the proof when $H=\frac12$ was already done in [20, 21] by Khoshnevisan and Lewis.

That said, we now divide the proof of Theorem 3.2.1 in several steps.

3.3.3 Step 1: A key algebraic lemma

For each integer $n\geqslant 1$, $k\in\mathbb{Z}$ and real number $t\geqslant 0$, let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2},(j+1)2^{-n/2}]$ within the first $\lfloor 2^n t\rfloor$ steps of the random walk $\{Y(T_{k,n})\}_{k\geqslant 0}$, that is,

$$\begin{aligned}U_{j,n}(t) &= \sharp\big\{k=0,\ldots,\lfloor 2^n t\rfloor-1:\\ &\qquad Y(T_{k,n})=j2^{-n/2} \text{ and } Y(T_{k+1,n})=(j+1)2^{-n/2}\big\};\\ D_{j,n}(t) &= \sharp\big\{k=0,\ldots,\lfloor 2^n t\rfloor-1:\\ &\qquad Y(T_{k,n})=(j+1)2^{-n/2} \text{ and } Y(T_{k+1,n})=j2^{-n/2}\big\}.\end{aligned}$$

While easy, the following lemma taken from [20, Lemma 2.4] is going to be the key when studying the asymptotic behavior of the weighted power variation $V_n^{(2r-1)}(f,t)$ of *odd* order $2r-1$, defined as:

$$V_n^{(2r-1)}(f,t)=\sum_{k=0}^{\lfloor 2^n t\rfloor-1}\frac12\big(f(Z_{T_{k,n}})+f(Z_{T_{k+1,n}})\big)(Z_{T_{k+1,n}}-Z_{T_{k,n}})^{2r-1},\quad t\geqslant 0.$$

Its main feature is to separate X from Y, thus providing a representation of $V_n^{(2r-1)}(f,t)$ which is amenable to analysis.

Lemma 3.3.1 *Fix $f\in C_b^\infty$, $t\geqslant 0$ and $r\in\mathbb{N}^*$. Then*

$$\begin{aligned}&V_n^{(2r-1)}(f,t)\\ =&\sum_{j\in\mathbb{Z}}\frac12\left(f(X_{j2^{-\frac n2}})+f(X_{(j+1)2^{-\frac n2}})\right)\left(X_{(j+1)2^{-\frac n2}}-X_{j2^{-\frac n2}}\right)^{2r-1}\big(U_{j,n}(t)-D_{j,n}(t)\big).\end{aligned}$$

Observe that $V_n^{(1)}(f,t)=V_n(f,t)$, see (3.2.15).

3.3.4 Step 2: Transforming the weighted power variations of odd order

By [20, Lemma 2.5], one has

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} 1_{\{0\leqslant j<j^*(n,t)\}} & \text{if } j^*(n,t)>0 \\ 0 & \text{if } j^*=0 \\ -1_{\{j^*(n,t)\leqslant j<0\}} & \text{if } j^*(n,t)<0 \end{cases},$$

where $j^*(n,t) = 2^{n/2} Y_{T_{\lfloor 2^n t\rfloor,n}}$. As a consequence, $V_n^{(2r-1)}(f,t)$ is equal to

$$\begin{cases} 2^{-nH(r-\frac{1}{2})}\sum_{j=0}^{j^*(n,t)-1}\frac{1}{2}\big(f(X^+_{j2^{-n/2}})+f(X^+_{(j+1)2^{-n/2}})\big)\big(X^{n,+}_{j+1}-X^{n,+}_{j}\big)^{2r-1} & \text{if } j^*(n,t)>0 \\ 0 & \text{if } j^*=0 \\ 2^{-nH(r-\frac{1}{2})}\sum_{j=0}^{|j^*(n,t)|-1}\frac{1}{2}\big(f(X^-_{j2^{-n/2}})+f(X^-_{(j+1)2^{-n/2}})\big)\big(X^{n,-}_{j+1}-X^{n,-}_{j}\big)^{2r-1} & \text{if } j^*(n,t)<0 \end{cases},$$

where $X_t^+ := X_t$ for $t \geqslant 0$, $X_{-t}^- := X_t$ for $t<0$, $X_t^{n,+} := 2^{nH/2}X^+_{2^{-n/2}t}$ for $t\geqslant 0$ and $X_{-t}^{n,-} := 2^{nH/2}X^-_{2^{-n/2}(-t)}$ for $t<0$.

Let us now introduce the following sequence of processes $W^{(2r-1)}_{\pm,n}$, in which H_p stands for the pth Hermite polynomial:

$$\begin{aligned} W^{(2r-1)}_{\pm,n}(f,t) &= \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor-1}\frac{1}{2}\big(f(X^\pm_{j2^{-\frac{n}{2}}})+f(X^\pm_{(j+1)2^{-\frac{n}{2}}})\big)H_{2r-1}(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}), \quad t\geqslant 0 \\ W^{(2r-1)}_{n}(f,t) &= \begin{cases} W^{(2r-1)}_{+,n}(f,t) & \text{if } t\geqslant 0 \\ W^{(2r-1)}_{-,n}(f,-t) & \text{if } t<0 \end{cases}. \end{aligned}$$

We then have, using the decomposition $x^{2r-1} = \sum_{l=1}^r a_{r,l}H_{2l-1}(x)$ (with $a_{r,r}=1$, which is the only explicit value of $a_{l,r}$ we will need in the sequel),

$$V_n^{(2r-1)}(f,t) = 2^{-nH(r-\frac{1}{2})}\sum_{l=1}^r a_{r,l}W_n^{(2l-1)}(f, Y_{T_{\lfloor 2^n t\rfloor,n}}). \tag{3.3.22}$$

3.3.5 Step 3: Asymptotic behaviour of $W^{(2r-1)}_{\pm,n}(f,\cdot)$

We have the following proposition :

Proposition 3.3.2 *If $H\in(\frac{1}{6},\frac{1}{2})$, if $r\geqslant 2$ then, for any $f\in C_b^\infty$ and as $n\to\infty$,*

$$\Big(X_t, 2^{-n/4}W^{(2r-1)}_{\pm,n}(f,t)\Big)_{t\geqslant 0} \xrightarrow{\text{fdd}} \Big(X_t, \kappa_{2r-1}\int_0^t f(X_s^\pm)dW_s^\pm\Big)_{t\geqslant 0}, \tag{3.3.23}$$

where κ_{2r-1} is some positive constant depending on the order of the Hermite polynomial H_{2r-1}, $W_t^+ = W_t$ if $t>0$ and $W_t^- = W_{-t}$ if $t<0$, with W a two-sided Brownian motion independent of X, and where $\int_0^t f(X_s^\pm)dW_s^\pm$ must be understood in the Wiener-Itô sense.

Note that in the boundary case $r=2$ and $H=\frac{1}{6}$, (3.3.23) continues to hold, as was shown in [30, Theorem 3.1].

In the case $r=1$, it was shown in [26, Theorem 4] (case $H>\frac{1}{6}$) and [30, Theorem 2.13] (case $H=\frac{1}{6}$) that, for any fixed $t>0$, the sequence $W^{(1)}_{\pm,n}(f,t)$ converges in probability (when $H>\frac{1}{6}$) or only in law (when $H=\frac{1}{6}$) to a non degenerate limit as $n\to\infty$.

Proof.

In all the proof C shall denote a positive, finite constant that may change value from line to line. In what follows we may study separately the finite dimensional convergence in law of $(X, 2^{-n/4}W_{+,n}^{(2r-1)}(f,\cdot), 2^{-n/4}W_{-,n}^{(2r-1)}(f,\cdot))$ when n is even and when n is odd. For the sake of simplicity, we will only consider the even case, the analysis when n is odd being *mutatis mutandis* the same. So, assume that n is even and let m be another even integer such that $n \geqslant m \geqslant 0$. We shall apply a coarse graining argument. We have

$$\begin{aligned} W_{\pm,n}^{(2r-1)}(f,t) &= 2^{-n/4}\sum_{i=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}\frac{1}{2}\big(f(X^{\pm}_{j2^{-n/2}})+f(X^{\pm}_{(j+1)2^{-n/2}})\big) && (3.3.24)\\ &\qquad\qquad\times H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big) \\ &+ 2^{-n/4}\sum_{j=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1}\frac{1}{2}\big(f(X^{\pm}_{j2^{-n/2}})+f(X^{\pm}_{(j+1)2^{-n/2}})\big) && (3.3.25)\\ &\qquad\qquad\times H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big). \end{aligned}$$

Observe that $2^{\frac{n-m}{2}}$ is an integer precisely because we have assumed that n and m are even numbers. We have

$$(3.3.24) = A^{\pm}_{n,m}(t)+B^{\pm}_{n,m}(t)+C^{\pm}_{n,m}(t)$$

where

$$\begin{aligned} A^{\pm}_{n,m}(t) &= 2^{-n/4}\sum_{i=1}^{\lfloor 2^{m/2}t\rfloor}\frac{1}{2}\big(f(X^{\pm}_{i2^{-m/2}})+f(X^{\pm}_{(i+1)2^{-m/2}})\big)\\ &\qquad\qquad\times\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big)\\ B^{\pm}_{n,m}(t) &= 2^{-n/4}\sum_{i=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}\frac{1}{2}\big(f(X^{\pm}_{(j+1)2^{-n/2}})-f(X^{\pm}_{(i+1)2^{-m/2}})\big)\\ &\qquad\qquad\times H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big)\\ C^{\pm}_{n,m}(t) &= 2^{-n/4}\sum_{i=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}\frac{1}{2}\big(f(X^{\pm}_{j2^{-n/2}})-f(X^{\pm}_{i2^{-m/2}})\big)\\ &\qquad\qquad\times H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big). \end{aligned}$$

Here is a sketch of what remains to be done in order to complete the proof of (3.3.23). Firstly, we will prove **(a)** the finite dimensional convergence in law of $(X, A^{+}_{n,m}, A^{-}_{n,m})$ to $(X, \int_0^{\cdot} f(X_s^+)dW_s^+, \int_0^{\cdot} f(X_s^-)dW_s^-)$ as $n\to\infty$ and then $m\to\infty$. Secondly, we will show that **(b)** $B^{\pm}_{n,m}(t)$ converges to 0 in $L^2(\Omega)$ as $n\to\infty$ and then $m\to\infty$. By applying the same techniques, we would also obtain that the same holds with $C^{\pm}_{n,m}(t)$. Thirdly, we will prove that **(c)** (3.3.25) converges to 0 in $L^2(\Omega)$ as $n \to \infty$ and then $m \to \infty$. Once this has been done, one can easily deduce the fdd convergence of $(X, 2^{-n/4} W^{(2r-1)}_{+,n}(f,\cdot),$ $2^{-n/4} W^{(2r-1)}_{-,n}(f,\cdot))$ to $(X, \int_0^\cdot f(X_s^+) dW_s^+, \int_0^\cdot f(X_s^-) dW_s^-)$ as $n \to \infty$, which is equivalent to (3.3.23).

(a) Finite dimensional convergence in law of $(X, A^+_{n,m}, A^-_{n,m})$. Fix m. Showing the fdd convergence of $(X, A^+_{n,m}, A^-_{n,m})$ as $n \to \infty$ can be easily reduced to checking the fdd convergence of the following random-vector valued process:

$$\Bigg(X_x : x \geqslant 0,\ 2^{-n/4} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2r-1}\big(X^{n,+}_{j+1} - X^{n,+}_j\big)\,,$$
$$2^{-n/4} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2r-1}\big(X^{n,-}_{j+1} - X^{n,-}_j\big) : 1 \leqslant i \leqslant \lfloor 2^{m/2} t \rfloor \Bigg).$$

Using (2.3.42) in chapter 2 and Peccati-Tudor Theorem (see, e.g., [28, Theorem 6.2.3]), we have

$$\Bigg(2^{-n/4} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2r-1}(X^{n,+}_{j+1} - X^{n,+}_j), 2^{-n/4} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2r-1}(X^{n,-}_{j+1} - X^{n,-}_j) :$$
$$1 \leqslant i \leqslant \lfloor 2^{m/2} t \rfloor \Bigg) \xrightarrow{\text{law}}$$

$$\Bigg(\kappa_{2r-1}(W^+_{(i+1)2^{-m/2}} - W^+_{i2^{-m/2}}), \kappa_{2r-1}(W^-_{(i+1)2^{-m/2}} - W^-_{i2^{-m/2}}) \ : 1 \leqslant i \leqslant \lfloor 2^{m/2} t \rfloor \Bigg)$$

where W is a two-sided Brownian motion, and we have set $W_t^+ := W_t$ and $W_t^- := W_{-t}$ for any $t \geqslant 0$. Moreover, $\kappa_{2r-1} := \sqrt{(2r-1)! \sum_{a \in \mathbb{Z}} \rho(a)^{2r-1}}$ with $\rho(a) = \frac{1}{2}\big(|a+1|^{2H} + |a-1|^{2H} - 2|a|^{2H}\big)$. Since $E[X_x H_{2r-1}(X^{n,\pm}_{j+1} - X^{n,\pm}_j)] = 0$ when $r \geqslant 2$ (Hermite polynomials of different orders are orthogonal), Peccati-Tudor Theorem applies and yields

$$\Bigg(X_x : x \geqslant 0,\ 2^{-n/4} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2r-1}\big(X^{n,+}_{j+1} - X^{n,+}_j\big),$$
$$2^{-n/4} \sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1} H_{2r-1}\big(X^{n,+}_{j+1} - X^{n,+}_j\big) : 1 \leqslant i \leqslant \lfloor 2^{m/2} t \rfloor \Bigg) \xrightarrow{\text{f.d.d}}$$

$$\Bigg(X_x : x \geqslant 0, \kappa_{2r-1}(W^+_{(i+1)2^{-m/2}} - W^+_{i2^{-m/2}}), \kappa_{2r-1}(W^-_{(i+1)2^{-m/2}} - W^-_{i2^{-m/2}}) :$$
$$1 \leqslant i \leqslant \lfloor 2^{m/2} t \rfloor \Bigg),$$

with W is independent of X (and independent of Y as well).

We then have, as $n \to \infty$ and m is fixed,

$$
\begin{aligned}
&(X, A^+_{n,m}, A^-_{n,m}) \xrightarrow{\text{f.d.d}} \\
&\Big(X, \kappa_{2r-1} \sum_{i=1}^{\lfloor 2^{m/2} \cdot \rfloor} \Big(\frac{f(X^+_{i2^{-m/2}}) + f(X^+_{(i+1)2^{-m/2}})}{2}\Big)(W^+_{(i+1)2^{-m/2}} - W^+_{i2^{-m/2}}), \\
&\qquad \kappa_{2r-1} \sum_{i=1}^{\lfloor 2^{m/2} \cdot \rfloor} \Big(\frac{f(X^-_{i2^{-m/2}}) + f(X^-_{(i+1)2^{-m/2}})}{2}\Big)(W^-_{(i+1)2^{-m/2}} - W^-_{i2^{-m/2}})\Big).
\end{aligned}
$$

One can write

$$
\sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} \left(\frac{f(X^\pm_{i2^{-m/2}}) + f(X^\pm_{(i+1)2^{-m/2}})}{2}\right)(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}}) = K^\pm_m(t) + L^\pm_m(t),
$$

with

$$
\begin{aligned}
K^\pm_m(t) &= \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} f(X^\pm_{i2^{-m/2}})(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}}), \\
L^\pm_m(t) &= \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} \left(\frac{f(X^\pm_{(i+1)2^{-m/2}}) - f(X^\pm_{i2^{-m/2}})}{2}\right)(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}}).
\end{aligned}
$$

It is clear that $K^\pm_m(t) \xrightarrow{\text{prob}} \int_0^t f(X^\pm_s) dW^\pm_s$, as $m \to \infty$. On the other hand $L^\pm_m(t)$ converges to 0 in $L^2(\Omega)$ as $m \to \infty$. Indeed, by independence,

$$
\begin{aligned}
&E[L^\pm_m(t)^2] \qquad (3.3.26)\\
&= \sum_{i,j=1}^{\lfloor 2^{m/2} t \rfloor} E\Big[\Big(\frac{f(X^\pm_{(i+1)2^{-m/2}}) - f(X^\pm_{i2^{-m/2}})}{2}\Big)\Big(\frac{f(X^\pm_{(j+1)2^{-m/2}}) - f(X^\pm_{j2^{-m/2}})}{2}\Big)\Big] \\
&\qquad \times E\Big[(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}})(W^\pm_{(j+1)2^{-m/2}} - W^\pm_{j2^{-m/2}})\Big] \\
&= \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} E\Big[\Big(\frac{f(X^\pm_{(i+1)2^{-m/2}}) - f(X^\pm_{i2^{-m/2}})}{2}\Big)^2\Big] \times E\Big[\big(W^\pm_{(i+1)2^{-m/2}} - W^\pm_{i2^{-m/2}}\big)^2\Big] \\
&= 2^{-m/2} \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} E\Big[f'(X_{\theta_i})^2 \big(X^\pm_{(i+1)2^{-m/2}}) - X^\pm_{i2^{-m/2}}\big)^2\Big], \qquad (3.3.27)
\end{aligned}
$$

where θ_i denotes a random real number satisfying $i2^{-m/2} < \theta_i < (i+1)2^{-m/2}$. Since $f \in C^\infty_b$ and by Cauchy-Schwarz inequality, we deduce that

$$
\begin{aligned}
(3.3.27) &\leqslant C_f 2^{-m/2} \sum_{i=1}^{\lfloor 2^{m/2} t \rfloor} E\Big[\big(X^\pm_{(i+1)2^{-m/2}}) - X^\pm_{i2^{-m/2}}\big)^4\Big]^{1/2} \\
&= C_f 2^{-m/2} \lfloor 2^{m/2} t \rfloor 2^{-mH} \sqrt{3} \leqslant C_f 2^{-mH} t,
\end{aligned}
$$

from which the claim follows. Summarizing, we just showed that

$$(X, A^+_{n,m}, A^-_{n,m}) \xrightarrow{\text{fdd}} (X, \kappa_{2r-1}\int_0^\cdot f(X_s^+)dW_s^+, \kappa_{2r-1}\int_0^\cdot f(X_s^-)dW_s^-)$$

as $n\to\infty$ then $m\to\infty$.

(b) $B^{\pm}_{n,m}(t)$ converges to 0 in $L^2(\Omega)$ as $n\to\infty$ and then $m\to\infty$.

With obvious notation, we have that

$$B^{\pm}_{n,m}(t) = 2^{-n/4}\sum_{i=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}\Delta^{n,m}_{i,j}f(X^{\pm})H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_j\big).$$

We will only prove the convergence to 0 of $B^+_{n,m}(t)$, the proof for $B^-_{n,m}(t)$ being exactly the same. Using the relationship between Hermite polynomials and multiple stochastic integrals, namely $H_r\big(2^{nH/2}(X^+_{(j+1)2^{-n/2}}-X^+_{j2^{-n/2}})\big)=2^{nrH/2}I_r(\delta^{\otimes r}_{(j+1)2^{-n/2}})$, we obtain, using (3.3.21) as well,

$$\begin{aligned}
&E\big[(B^+_{n,m}(t))^2\big] && (3.3.28)\\
&= \Bigg|a^2_{2r-1}2^{-n/2}2^{nH(2r-1)}\sum_{i=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}\sum_{i'=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j'=(i'-1)2^{\frac{n-m}{2}}}^{i'2^{\frac{n-m}{2}}-1}\\
&\quad\sum_{l=0}^{2r-1} l!\binom{2r-1}{l}^2 E\big[\Delta^{n,m}_{i,j}f(X^+)\Delta^{n,m}_{i',j'}f(X^+)I_{2(2r-1)-2l}(\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}})\big]\\
&\qquad\times\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle^l\Bigg|\\
&\leqslant a^2_{2r-1}2^{-n/2}2^{nH(2r-1)}\sum_{i=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j=(i-1)2^{\frac{n-m}{2}}}^{i2^{\frac{n-m}{2}}-1}\sum_{i'=1}^{\lfloor 2^{m/2}t\rfloor}\sum_{j'=(i'-1)2^{\frac{n-m}{2}}}^{i'2^{\frac{n-m}{2}}-1}\sum_{l=0}^{2r-1} l!\\
&\quad\binom{2r-1}{l}^2\Big|E\big[\Delta^{n,m}_{i,j}f(X^+)\Delta^{n,m}_{i',j'}f(X^+)I_{2(2r-1)-2l}(\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}})\big]\Big|\\
&\qquad\times\big|\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\big|^l\\
&= a^2_{2r-1}2^{-n/2}2^{nH(2r-1)}\sum_{l=0}^{2r-1} l!\binom{2r-1}{l}^2 O^{+,l}_{n,m}(t), && (3.3.29)
\end{aligned}$$

with obvious notation. Thanks both to the duality formula (3.3.20) and (3.3.19), we have

$$\begin{aligned}
& b_{n,m}^{(+,l)}(i,i',j,j') \\
:= \; & E\big[\Delta_{i,j}^{n,m} f(X^+)\Delta_{i',j'}^{n,m} f(X^+) I_{2(2r-1)-2l}(\delta_{(j+1)2^{-n/2}}^{\otimes(2r-1-l)} \tilde{\otimes} \delta_{(j'+1)2^{-n/2}}^{\otimes(2r-1-l)})\big] \\
= \; & E\big[\big\langle D^{2(2r-1-l)}(\Delta_{i,j}^{n,m} f(X^+)\Delta_{i',j'}^{n,m} f(X^+))\,;\, \delta_{(j+1)2^{-n/2}}^{\otimes(2r-1-l)} \tilde{\otimes} \delta_{(j'+1)2^{-n/2}}^{\otimes(2r-1-l)}\big\rangle\big] \\
= \; & \frac{1}{4}\sum_{a=0}^{2(2r-1-l)} C_{2(2r-1-l)}^{a} E\Big[\Big\langle \big(f^{(a)}(X_{(j+1)2^{-n/2}}^{+})\xi_{(j+1)2^{-n/2}}^{\otimes a} - f^{(a)}(X_{(i+1)2^{-m/2}}^{+})\xi_{(i+1)2^{-m/2}}^{\otimes a}\big) \\
& \tilde{\otimes}\big(f^{(2(2r-1-l)-a)}(X_{(j'+1)2^{-n/2}}^{+})\xi_{(j'+1)2^{-n/2}}^{\otimes(2(2r-1-l)-a)} - f^{(2(2r-1-l)-a)}(X_{(i'+1)2^{-m/2}}^{+}) \\
& \qquad\qquad \times\xi_{(i'+1)2^{-m/2}}^{\otimes(2(2r-1-l)-a)}\big)\,;\, \delta_{(j+1)2^{-n/2}}^{\otimes(2r-1-l)} \tilde{\otimes} \delta_{(j'+1)2^{-n/2}}^{\otimes(2r-1-l)}\Big\rangle\Big].
\end{aligned}$$

The term $O_{n,m}^{+,2r-1}(t)$ in (3.3.29) can be bounded by

$$\frac{1}{4}\sup_{|x-y|\leqslant 2^{-m/2}} E(|f(X_x)-f(X_y)|^2)\sum_{j,j'=0}^{\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-1} |\langle \delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle|^{2r-1}. \tag{3.3.30}$$

Observe that

$$\begin{aligned}
& \sum_{j,j'=0}^{\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-1} \big|\langle \delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\big|^{2r-1} && (3.3.31) \\
= \; & 2^{-(2r-1)nH}\sum_{j,j'=0}^{\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-1} \big|\frac{1}{2}(|j-j'+1|^{2H}+|j-j'-1|^{2H}-2|j-j'|^{2H})\big|^{2r-1} \\
\leqslant \; & 2^{-(2r-1)nH}\sum_{j=0}^{\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-1}\sum_{q\in\mathbb{Z}}\big|\rho(q)\big|^{2r-1} \\
\leqslant \; & C_H 2^{n(\frac{1}{2}-(2r-1)H)}t,
\end{aligned}$$

with $\rho(q) := \frac{1}{2}(|q+1|^{2H}+|q-1|^{2H}-2|q|^{2H})$. Note that $\sum_{q\in\mathbb{Z}}|\rho(q)|^{2r-1}\leqslant C_H$ since $H<\frac{1}{2}\leqslant 1-\frac{1}{4r-2}$. So, by plugging (3.3.31) in (3.3.30) we deduce that

$$O_{n,m}^{+,2r-1}(t) \leqslant \frac{1}{4}C_H 2^{n(\frac{1}{2}-(2r-1)H)}t\sup_{|x-y|\leqslant 2^{-m/2}} E(|f(X_x)-f(X_y)|^2). \tag{3.3.32}$$

In order to handle the terms $O_{n,m}^{+,l}(t)$ in (3.3.29) when $0\leqslant l\leqslant 2r-2$, we rely to the decomposition

$$
\begin{aligned}
&|b^{(+,l)}_{n,m}(i,i',j,j')| \\
&\leqslant \frac{1}{4}\Big(\sum_{a=0}^{2(2r-1-l)} \binom{2(2r-1-l)}{a} \big|E[f^{(a)}(X^+_{(j+1)2^{-n/2}})f^{(2(2r-1-l)-a)}(X^+_{(j'+1)2^{-n/2}})]\big| \\
&\qquad\qquad \times \big|\langle \xi^{\otimes a}_{(j+1)2^{-n/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{(j'+1)2^{-n/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}} \rangle\big| \\
&\quad + \sum_{a=0}^{2(2r-1-l)} \binom{2(2r-1-l)}{a} \big|E[f^{(a)}(X^+_{(i+1)2^{-m/2}})f^{(2(2r-1-l)-a)}(X^+_{(j'+1)2^{-n/2}})]\big| \\
&\qquad\qquad \times \big|\langle \xi^{\otimes a}_{(i+1)2^{-m/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{(j'+1)2^{-n/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}} \rangle\big| \\
&\quad + \sum_{a=0}^{2(2r-1-l)} \binom{2(2r-1-l)}{a} \big|E[f^{(a)}(X^+_{(j+1)2^{-n/2}})f^{(2(2r-1-l)-a)}(X^+_{(i'+1)2^{-m/2}})]\big| \\
&\qquad\qquad \times \big|\langle \xi^{\otimes a}_{(j+1)2^{-n/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{(i'+1)2^{-m/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}} \rangle\big| \\
&\quad + \sum_{a=0}^{2(2r-1-l)} \binom{2(2r-1-l)}{a} \big|E[f^{(a)}(X^+_{(i+1)2^{-m/2}})f^{(2(2r-1-l)-a)}(X^+_{(i'+1)2^{-m/2}})]\big| \\
&\qquad\qquad \times \big|\langle \xi^{\otimes a}_{(i+1)2^{-m/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{(i'+1)2^{-m/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}} \rangle\big|\Big) \\
&\quad \leqslant C_f (2^{-nH})^{2(2r-1-l)},
\end{aligned}
$$

where C_f is a constant only depending on f (recall that $f \in C_b^\infty$). The last inequality holds since, by (3.3.45), $|\langle \xi_t ; \delta_{(j+1)2^{-n/2}} \rangle| \leqslant 2^{-nH}$ for all $t \geqslant 0$ and all $j \in \mathbb{N}$. At this stage, the proof of the claim (**b**) is going to be different according to the value of l:

- If $1 \leqslant l \leqslant 2r-2$ then

$$
\begin{aligned}
O^{+,l}_{n,m}(t) &\leqslant C_f (2^{-nH})^{2(2r-1-l)} \sum_{j,j'=0}^{\lfloor 2^{m/2} t \rfloor 2^{\frac{n-m}{2}} - 1} \big|\langle \delta_{(j+1)2^{-n/2}} ; \delta_{(j'+1)2^{-n/2}} \rangle\big|^l \\
&\leqslant C_f C_H (2^{-nH})^{2(2r-1-l)}\, 2^{n(\frac{1}{2} - lH)} t, \qquad (3.3.33)
\end{aligned}
$$

  where, we have the last inequality by (3.3.31).

- If $l = 0$ then

$$
O^{+,0}_{n,m}(t) \leqslant C_f (2^{-nH})^{2(2r-1)} (\lfloor 2^{m/2} t \rfloor 2^{\frac{n-m}{2}})^2 \leqslant C_f (2^{-nH})^{2(2r-1)} 2^n t^2. \qquad (3.3.34)
$$

  By combining (3.3.29), (3.3.32), (3.3.33) and (3.3.34) we obtain

$$
\begin{aligned}
E\big[(B^+_{n,m}(t))^2\big] \leqslant Ct\Big[&\sup_{|x-y| \leqslant 2^{-m/2}} E(|f(X_x) - f(X_y)|^2) + \sum_{l=1}^{2r-2} 2^{-nH(2r-1-l)} \\
&+ 2^{-n(H(2r-1) - \frac{1}{2})} t \Big]. \qquad (3.3.35)
\end{aligned}
$$

  Since $H > \frac{1}{6}$, we deduce that $B^\pm_{n,m}(t)$ converges to 0 in $L^2(\Omega)$ as $n \to \infty$ and then $m \to \infty$.

(c) (3.3.25) converges to 0 **in** $L^2(\Omega)$ **as** $n\to\infty$ **and then** $m\to\infty$. Let $J^{\pm}_{n,m}(t)$ denote the quantity given in (3.3.25). That is,

$$\begin{aligned} J^{\pm}_{n,m}(t) &= 2^{-n/4}\sum_{j=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1}\left(\frac{f(X^{\pm}_{j2^{-n/2}})+f(X^{\pm}_{(j+1)2^{-n/2}})}{2}\right)H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big)\\ &= 2^{-n/4}\sum_{j=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1}\Theta^n_j f(X^{\pm})H_{2r-1}\big(X^{n,\pm}_{j+1}-X^{n,\pm}_{j}\big), \end{aligned}$$

with obvious notation. We will only prove the convergence to 0 of $J^{+}_{n,m}(t)$, the proof for $J^{-}_{n,m}(t)$ being exactly the same. Using the relationship between Hermite polynomials and multiple stochastic integrals, namely $H_r\big(2^{nH/2}(X^+_{(j+1)2^{-n/2}}-X^+_{j2^{-n/2}})\big) = 2^{nrH/2}I_r(\delta^{\otimes r}_{(j+1)2^{-n/2}})$, we obtain, using (3.3.21) as well,

$$\begin{aligned} E\big[(J^{+}_{n,m}(t))^2\big] &= \Bigg|2^{-n/2}2^{nH(2r-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1}\sum_{l=0}^{2r-1} l!\binom{2r-1}{l}^2\\ &\qquad\times E\big[\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+)I_{2(2r-1)-2l}(\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}})\big]\\ &\qquad\times\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle^l\Bigg|\\ &\leqslant 2^{-n/2}2^{nH(2r-1)}\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1}\sum_{l=0}^{2r-1} l!\binom{2r-1}{l}^2\\ &\qquad\times\Big|E\big[\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+)I_{2(2r-1)-2l}(\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}})\big]\Big|\\ &\qquad\times\big|\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\big|^l\\ &= \sum_{l=0}^{2r-1} l!\binom{2r-1}{l}^2 Q^{+,l}_{n,m}(t), \end{aligned}\tag{3.3.36}$$

with obvious notation. Thanks both to the duality formula (3.3.20) and to (3.3.19), we have

$$\begin{aligned} d^{(+,l)}_n(j,j') &:= E\big[\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+)I_{2(2r-1)-2l}(\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}})\big]\\ &= E\big[\big\langle D^{2(2r-1-l)}(\Theta^n_j f(X^+)\Theta^n_{j'}f(X^+))\,;\,\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}}\big\rangle\big]\\ &= \frac{1}{4}\sum_{a=0}^{2(2r-1-l)} C^a_{2(2r-1-l)}E\Bigg[\Bigg\langle\big(f^{(a)}(X^+_{j2^{-n/2}})\xi^{\otimes a}_{j2^{-n/2}}+f^{(a)}(X^+_{(j+1)2^{-n/2}})\xi^{\otimes a}_{(j+1)2^{-n/2}}\big)\\ &\quad\tilde{\otimes}\big(f^{(2(2r-1-l)-a)}(X^+_{j'2^{-n/2}})\xi^{\otimes(2(2r-1-l)-a)}_{j'2^{-n/2}}+f^{(2(2r-1-l)-a)}(X^+_{(j'+1)2^{-n/2}})\\ &\qquad\times\xi^{\otimes(2(2r-1-l)-a)}_{(j'+1)2^{-n/2}}\big)\,;\,\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}}\Bigg\rangle\Bigg]. \end{aligned}$$

At this stage, the proof of the claim (**c**) is going to be different according to the value of l:

- If $l = 2r-1$ in (3.3.36) then

$$
\begin{aligned}
Q_{n,m}^{+,2r-1}(t) &= 2^{-n/2}2^{nH(2r-1)} \sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1} \Big| E\big[\Theta_j^n f(X^+)\Theta_{j'}^n f(X^+)\big]\Big| \\
&\qquad\qquad \times \big|\langle \delta_{(j+1)2^{-n/2}}; \delta_{(j'+1)2^{-n/2}}\rangle\big|^{2r-1} \\
&\leqslant C_f 2^{-n/2}2^{nH(2r-1)} \sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1} \big|\langle \delta_{(j+1)2^{-n/2}}; \delta_{(j'+1)2^{-n/2}}\rangle\big|^{2r-1} \\
&= C_f 2^{-n/2} \sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1} \big|\frac{1}{2}(|j-j'+1|^{2H}+|j-j'-1|^{2H}-2|j-j'|^{2H})\big|^{2r-1} \\
&= C_f 2^{-n/2} \sum_{j=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1} \sum_{p=j-\lfloor 2^{n/2}t\rfloor+1}^{j-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}} \big|\frac{1}{2}(|p+1|^{2H}+|p-1|^{2H}-2|p|^{2H})\big|^{2r-1}
\end{aligned}
\tag{3.3.37}
$$

where we have the first inequality since f belongs to C_b^∞ and the last one follows by the change of variable $p = j-j'$. Using the notation $\rho(p) := \frac{1}{2}(|p+1|^{2H}+|p-1|^{2H}-2|p|^{2H})$, and by a Fubini argument, we get that the quantity given in (3.3.37) is equal to

$$
\begin{aligned}
C_f 2^{-n/2} \sum_{p=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}-1} |\rho(p)|^{2r-1}\Big((p+\lfloor 2^{n/2}t\rfloor)\wedge\lfloor 2^{n/2}t\rfloor-(p+ \\
\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}})\vee\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\Big).
\end{aligned}
\tag{3.3.38}
$$

By separating the cases when $0 \leqslant p \leqslant \lfloor 2^{n/2}t\rfloor - \lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}} - 1$ or when $\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}} - \lfloor 2^{n/2}t\rfloor + 1 \leqslant p < 0$ we deduce that

$$
\begin{aligned}
0 &\leqslant \Big(\frac{(p+\lfloor 2^{n/2}t\rfloor)}{2^{n/2}}\wedge\frac{(\lfloor 2^{n/2}t\rfloor)}{2^{n/2}} - \frac{(p+\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}})}{2^{n/2}}\vee\lfloor 2^{m/2}t\rfloor 2^{-m/2}\Big) \\
&\leqslant \lfloor 2^{n/2}t\rfloor 2^{-n/2} - \lfloor 2^{m/2}t\rfloor 2^{-m/2} = \big|\lfloor 2^{n/2}t\rfloor 2^{-n/2} - \lfloor 2^{m/2}t\rfloor 2^{-m/2}\big| \\
&\leqslant \big|\lfloor 2^{n/2}t\rfloor 2^{-n/2} - t\big| + \big|t - \lfloor 2^{m/2}t\rfloor 2^{-m/2}\big| \leqslant 2^{-n/2}+2^{-m/2}.
\end{aligned}
$$

As a result, the quantity given in (3.3.38) is bounded by

$$
C_f \sum_{p\in\mathbb{Z}} |\rho(p)|^{2r-1}(2^{-n/2}+2^{-m/2}),
$$

with $\sum_{p\in\mathbb{Z}}|\rho(p)|^{2r-1} < \infty$ (because $H < 1/2 \leqslant 1-\frac{1}{4r-2}$). Finally, we have

$$
Q_{n,m}^{+,2r-1}(t) \leqslant C(2^{-n/2}+2^{-m/2}). \tag{3.3.39}
$$

- Preparation to the cases $0 \leqslant l \leqslant 2r-2$ In order to handle the terms $Q^{+,l}_{n,m}(t)$ whenever $0 \leqslant l \leqslant 2r-2$, we will make use of the following decomposition:

$$|d_n^{(+,l)}(j,j')| \leqslant \frac{1}{4}\big(\Omega_n^{(1,l)}(j,j') + \Omega_n^{(2,l)}(j,j') + \Omega_n^{(3,l)}(j,j') + \Omega_n^{(4,l)}(j,j')\big), \qquad (3.3.40)$$

where

$$\begin{aligned}
\Omega_n^{(1,l)}(j,j') = & \sum_{a=0}^{2(2r-1-l)} C^a_{2(2r-1-l)} \big|E[f^{(a)}(X^+_{j2^{-n/2}}) f^{(2(2r-1-l)-a)}(X^+_{j'2^{-n/2}})]\big| \\
& \qquad \times \big|\langle \xi^{\otimes a}_{j2^{-n/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{j'2^{-n/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}}\rangle\big| \\
\Omega_n^{(2,l)}(j,j') = & \sum_{a=0}^{2(2r-1-l)} C^a_{2(2r-1-l)} \big|E[f^{(a)}(X^+_{j2^{-n/2}}) f^{(2(2r-1-l)-a)}(X^+_{(j'+1)2^{-n/2}})]\big| \\
& \qquad \times \big|\langle \xi^{\otimes a}_{j2^{-n/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{(j'+1)2^{-n/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}}\rangle\big| \\
\Omega_n^{(3,l)}(j,j') = & \sum_{a=0}^{2(2r-1-l)} C^a_{2(2r-1-l)} \big|E[f^{(a)}(X^+_{(j+1)2^{-n/2}}) f^{(2(2r-1-l)-a)}(X^+_{j'2^{-n/2}})]\big| \\
& \qquad \times \big|\langle \xi^{\otimes a}_{(j+1)2^{-n/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{j'2^{-n/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}}\rangle\big| \\
\Omega_n^{(4,l)}(j,j') = & \sum_{a=0}^{2(2r-1-l)} C^a_{2(2r-1-l)} \big|E[f^{(a)}(X^+_{(j+1)2^{-n/2}}) f^{(2(2r-1-l)-a)}(X^+_{(j'+1)2^{-n/2}})]\big| \\
& \qquad \times \big|\langle \xi^{\otimes a}_{(j+1)2^{-n/2}} \tilde{\otimes} \xi^{\otimes(2(2r-1-l)-a)}_{(j'+1)2^{-n/2}} ; \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}}\rangle\big|.
\end{aligned}$$

- For $1 \leqslant l \leqslant 2r-2$: Since f belongs to C_b^∞ and since, by (3.3.45), $|\langle \xi_t ; \delta_{(j+1)2^{-n/2}}\rangle| \leqslant 2^{-nH}$ for all $t \geqslant 0$ and all $j \in \mathbb{N}$, we deduce that

$$d_n^{(+,l)}(j,j') \leqslant C(2^{-nH})^{2(2r-1-l)}.$$

As a consequence of this previous inequality we have

$$\begin{aligned}
& Q^{+,l}_{n,m}(t) \\
& \leqslant C(2^{-nH})^{2(2r-2)}\, 2^{-n/2}\, 2^{nH(2r-1)} \sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor -1} \big|\langle \delta_{(j+1)2^{-n/2}} ; \delta_{(j'+1)2^{-n/2}}\rangle\big|^l \\
& \leqslant C(2^{-nH})^{2(2r-2)}\, 2^{nH(2r-1)} 2^{-nHl} \Big(\sum_{p\in\mathbb{Z}} |\rho(p)|^l\Big)(2^{-n/2} + 2^{-m/2}) \\
& \leqslant C\, 2^{-nH(2r-2)}(2^{-n/2} + 2^{-m/2}),
\end{aligned} \qquad (3.3.41)$$

where we have the second inequality by the same arguments that have been used previously in the case $l = 2r-1$.

- <u>For $l=0$:</u> Thanks to the decomposition (3.3.40) we get

$$Q_{n,m}^{+,0}(t) \leqslant \frac{1}{4} 2^{-n/2} 2^{nH(2r-1)} \sum_{k=1}^{4} \sum_{j,j'=\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2} t\rfloor -1} \Omega_n^{(k,0)}(j,j') \tag{3.3.42}$$

We will study only the term corresponding to $\Omega_n^{(2,0)}(j,j')$ in (3.3.42), which is representative to the difficulty. It is given by

$$\begin{aligned}
& \frac{1}{4} 2^{-n/2} 2^{nH(2r-1)} \sum_{j,j'=\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2} t\rfloor -1} \sum_{a=0}^{2(2r-1)} C_{2(2r-1)}^{a} \big| E[f^{(a)}(X_{j2^{-n/2}}^{+}) \\
& \times f^{(2(2r-1)-a)}(X_{(j'+1)2^{-n/2}}^{+})] \big| \big| \langle \xi_{j2^{-n/2}}^{\otimes a} \tilde{\otimes} \xi_{(j'+1)2^{-n/2}}^{\otimes (2(2r-1)-a)} ; \delta_{(j+1)2^{-n/2}}^{\otimes (2r-1)} \tilde{\otimes} \delta_{(j'+1)2^{-n/2}}^{\otimes (2r-1)} \rangle \big| \\
\leqslant \; & C 2^{-n/2} 2^{nH(2r-1)} \sum_{j,j'=\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2} t\rfloor -1} \sum_{a=0}^{2(2r-1)} \big| \langle \xi_{j2^{-n/2}}^{\otimes a} \tilde{\otimes} \xi_{(j'+1)2^{-n/2}}^{\otimes (2(2r-1)-a)} ; \\
& \qquad \delta_{(j+1)2^{-n/2}}^{\otimes (2r-1)} \tilde{\otimes} \delta_{(j'+1)2^{-n/2}}^{\otimes (2r-1)} \rangle \big| .
\end{aligned}$$

We define $E_n^{(a,r)}(j,j') := \big| \langle \xi_{j2^{-n/2}}^{\otimes a} \tilde{\otimes} \xi_{(j'+1)2^{-n/2}}^{\otimes (2(2r-1)-a)} ; \delta_{(j+1)2^{-n/2}}^{\otimes (2r-1)} \tilde{\otimes} \delta_{(j'+1)2^{-n/2}}^{\otimes (2r-1)} \rangle \big|$. By (3.3.45), recall that $|\langle \xi_t ; \delta_{(j+1)2^{-n/2}} \rangle| \leqslant 2^{-nH}$ for all $t \geqslant 0$ and all $j \in \mathbb{N}$. We thus get, with $\tilde{c}_a$ some combinatorial constants,

$$\begin{aligned}
E_n^{(a,r)}(j,j') \;\; \leqslant \;\; & \tilde{c}_a \, 2^{-nH(4r-3)} \big(|\langle \xi_{j2^{-n/2}} ; \delta_{(j+1)2^{-n/2}} \rangle| + |\langle \xi_{j2^{-n/2}} ; \delta_{(j'+1)2^{-n/2}} \rangle| \\
& + |\langle \xi_{(j'+1)2^{-n/2}} ; \delta_{(j+1)2^{-n/2}} \rangle| + |\langle \xi_{(j'+1)2^{-n/2}} ; \delta_{(j'+1)2^{-n/2}} \rangle| \big).
\end{aligned}$$

For instance, we can write

$$\begin{aligned}
& \sum_{j,j'=\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2} t\rfloor -1} |\langle \xi_{(j'+1)2^{-n/2}} ; \delta_{(j+1)2^{-n/2}} \rangle| \\
= \;\; & 2^{-nH-1} \sum_{j,j'=\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2} t\rfloor -1} \big| (j+1)^{2H} - j^{2H} + |j'-j+1|^{2H} - |j'-j|^{2H} \big| \\
\leqslant \;\; & 2^{-nH-1} \sum_{j,j'=\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2} t\rfloor -1} \big((j+1)^{2H} - j^{2H} \big) \\
& + 2^{-nH-1} \sum_{\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}} \leqslant j \leqslant j' \leqslant \lfloor 2^{n/2} t\rfloor -1} \big((j'-j+1)^{2H} - (j'-j)^{2H} \big) \\
& + 2^{-nH-1} \sum_{\lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}} \leqslant j' < j \leqslant \lfloor 2^{n/2} t\rfloor -1} \big((j-j')^{2H} - (j-j'-1)^{2H} \big) \\
\leqslant \;\; & \frac{3}{2} 2^{-nH} \big(\lfloor 2^{n/2} t\rfloor - \lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}} \big) \lfloor 2^{n/2} t\rfloor^{2H} \leqslant \frac{3t^{2H}}{2} \big(2^{n/2} t - \lfloor 2^{m/2} t\rfloor 2^{\frac{n-m}{2}} \big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} |\langle \xi_{j2^{-n/2}};\delta_{(j+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{2H}}{2}\big(2^{n/2}t-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\big);\\
\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} |\langle \xi_{j2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{2H}}{2}\big(2^{n/2}t-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\big);\\
\sum_{j,j'=\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}}^{\lfloor 2^{n/2}t\rfloor-1} |\langle \xi_{(j'+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{2H}}{2}\big(2^{n/2}t-\lfloor 2^{m/2}t\rfloor 2^{\frac{n-m}{2}}\big).
\end{aligned}$$

As a consequence, we deduce

$$Q^{(+,0)}_{n,m}(t) \leqslant C\,2^{-nH(2r-2)}\big(t-\lfloor 2^{m/2}t\rfloor 2^{\frac{-m}{2}}\big) \leqslant C\,2^{-nH(2r-2)}2^{-m/2}. \tag{3.3.43}$$

Combining (3.3.39), (3.3.41) and (3.3.43) finally shows

$$\begin{aligned}
E\big[\big(J^+_{n,m}(t)\big)^2\big] \leqslant C\Big(&2^{-n/2}+2^{-m/2}+2^{-nH(2r-2)}(2^{-n/2}+2^{-m/2})\\
&+2^{-nH(2r-2)}2^{-m/2}\Big).
\end{aligned}$$

So, we deduce that $J^+_{n,m}(t)$ converges to 0 in $L^2(\Omega)$ as $n\to\infty$ and then $m\to\infty$. Finally, thanks to (a), (b) and (c), (3.3.23) holds true. ■

3.3.6 Step 4: Moment bounds for $W^{(2r-1)}_n(f,\cdot)$

Fix an integer $r\geqslant 1$ as well as a function $f\in C^\infty_b$. We claim the existence of $c>0$ such that, for all real numbers $s<t$ and all $n\in\mathbb{N}$,

$$E\big[\big(W^{(2r-1)}_n(f,t)-W^{(2r-1)}_n(f,s)\big)^2\big] \leqslant c\max(|s|^{2H},|t|^{2H})\big(|t-s|2^{n/2}+1\big). \tag{3.3.44}$$

In order to prove (3.3.44), we will need the following lemma.

Lemma 3.3.3 *If $s,t,u>0$ or if $s,t,u<0$ then*

$$|E\big(X_u(X_t-X_s)\big)| \leqslant |t-s|^{2H}. \tag{3.3.45}$$

Proof. When $s,t,u>0$ we have

$$E\big(X_u(X_t-X_s)\big)=\frac{1}{2}\big(t^{2H}-s^{2H}\big)+\frac{1}{2}\big(|s-u|^{2H}-|t-u|^{2H}\big).$$

Since $|b^{2H}-a^{2H}|\leqslant|b-a|^{2H}$ for any $a,b\in\mathbb{R}_+$, we immediately deduce (3.3.45). The proof when $s,t,u<0$ is similar.

■

We are now ready to show (3.3.44). We distinguish two cases according to the signs of $s,t\in\mathbb{R}$ (and reducing the problem by symmetry):

(1) if $0 \leqslant s < t$ (the case $s < t \leqslant 0$ being similar), then

$$
\begin{aligned}
& E[(W_n^{(2r-1)}(f,t) - W_n^{(2r-1)}(f,s))^2] = E[(W_{+,n}^{(2r-1)}(f,t) - W_{+,n}^{(2r-1)}(f,s))^2] \\
= \; & \frac{1}{4} \sum_{j,j'=\lfloor 2^{n/2}s \rfloor}^{\lfloor 2^{n/2}t \rfloor -1} \Big| E\big[\big(f(X^+_{j2^{-\frac{n}{2}}}) + f(X^+_{(j+1)2^{-\frac{n}{2}}})\big) \\
& \times \big(f(X^+_{j'2^{-\frac{n}{2}}}) + f(X^+_{(j'+1)2^{-\frac{n}{2}}})\big) H_{2r-1}(X^{n,+}_{j+1} - X^{n,+}_{j}) H_{2r-1}(X^{n,+}_{j'+1} - X^{n,+}_{j'})\big]\Big| \\
= \; & \frac{1}{4} 2^{nH(2r-1)} \sum_{j,j'=\lfloor 2^{n/2}s \rfloor}^{\lfloor 2^{n/2}t \rfloor -1} \Big| E\big[\Theta^n_j f(X^+) \Theta^n_{j'} f(X^+) I_{2r-1}(\delta^{\otimes(2r-1)}_{(j+1)2^{-n/2}}) I_{2r-1}(\delta^{\otimes(2r-1)}_{(j'+1)2^{-n/2}})\big]\Big|,
\end{aligned}
$$

with obvious notation. Relying to the product formula (3.3.21), we deduce that this latter quantity is less than or equal to

$$
\begin{aligned}
& \frac{1}{4} 2^{nH(2r-1)} \sum_{j,j'=\lfloor 2^{n/2}s \rfloor}^{\lfloor 2^{n/2}t \rfloor -1} \sum_{l=0}^{2r-1} l! \binom{2r-1}{l}^2 \big|\langle \delta_{(j+1)2^{-n/2}}; \delta_{(j'+1)2^{-n/2}} \rangle\big|^l \\
& \qquad \times \Big| E\big[\Theta^n_j f(X^+) \Theta^n_{j'} f(X^+) I_{4r-2-2l}(\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}})\big]\Big| \\
=: \; & \frac{1}{4} \sum_{l=0}^{2r-1} l! \binom{2r-1}{l}^2 Q_n^{(+,r,l)}(s,t). \qquad (3.3.46)
\end{aligned}
$$

By the duality formula (3.3.20) and the Leibniz rule (3.3.19), one has that

$$
\begin{aligned}
& d_n^{(+,r,l)}(j,j') := E\big[\Theta^n_j f(X^+) \Theta^n_{j'} f(X^+) I_{4r-2-2l}(\delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}})\big] \\
= \; & E\big[\big\langle D^{4r-2-2l}(\Theta^n_j f(X^+) \Theta^n_{j'} f(X^+))\,;\, \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}} \big\rangle\big] \\
= \; & \sum_{a=0}^{4r-2-2l} \binom{4r-2-2l}{a} E\Big[\Big\langle \big(f^{(a)}(X^+_{j2^{-n/2}}) \xi^{\otimes a}_{j2^{-n/2}} + f^{(a)}(X^+_{(j+1)2^{-n/2}}) \xi^{\otimes a}_{(j+1)2^{-n/2}}\big) \\
& \tilde{\otimes} \big(f^{(4r-2-2l-a)}(X^+_{j'2^{-n/2}}) \xi^{\otimes(4r-2-2l-a)}_{j'2^{-n/2}} + f^{(4r-2-2l-a)}(X^+_{(j'+1)2^{-n/2}}) \xi^{\otimes(4r-2-2l-a)}_{(j'+1)2^{-n/2}}\big)\,; \\
& \qquad \delta^{\otimes(2r-1-l)}_{(j+1)2^{-n/2}} \tilde{\otimes} \delta^{\otimes(2r-1-l)}_{(j'+1)2^{-n/2}} \Big\rangle\Big].
\end{aligned}
$$

Let now c denote a generic constant that may differ from one line to another and recall that $f \in C_b^\infty$. We then have the following estimates.

- Case $l = 2r - 1$

$$
\begin{aligned}
&Q_n^{(+,r,2r-1)}(s,t)\\
&\leqslant\; c\,2^{nH(2r-1)}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\big|\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\big|^{2r-1}\\
&=\; c\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\big|\frac{1}{2}(|j-j'+1|^{2H}+|j-j'-1|^{2H}-2|j-j'|^{2H})\big|^{2r-1}\\
&=\; c\sum_{j=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\;\sum_{q=j-\lfloor 2^{n/2}t\rfloor+1}^{j-\lfloor 2^{n/2}s\rfloor}\big|\rho(q)\big|^{2r-1},
\end{aligned}
$$

with $\rho(q):=\frac{1}{2}(|q+1|^{2H}+|q-1|^{2H}-2|q|^{2H})$. By a Fubini argument, it comes

$$
\begin{aligned}
&Q_n^{(+,r,2r-1)}(s,t)\\
&\leqslant\; c\sum_{q=\lfloor 2^{n/2}s\rfloor-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor-1}|\rho(q)|^{2r-1}\Big((q+\lfloor 2^{n/2}t\rfloor)\wedge\lfloor 2^{n/2}t\rfloor-(q+\lfloor 2^{n/2}s\rfloor)\vee\lfloor 2^{n/2}s\rfloor\Big)\\
&\leqslant\; c\sum_{q=\lfloor 2^{n/2}s\rfloor-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor-1}|\rho(q)|^{2r-1}\big(\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big)\\
&\leqslant\; c\sum_{q\in\mathbb{Z}}|\rho(q)|^{2r-1}\big|\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big| = c\big|\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big|\\
&\leqslant\; c\big(\big|\lfloor 2^{n/2}t\rfloor-2^{n/2}t\big|+2^{n/2}\big|t-s\big|+\big|\lfloor 2^{n/2}s\rfloor-2^{n/2}s\big|\big)\\
&\leqslant\; c(1+2^{n/2}|t-s|).
\end{aligned}
\tag{3.3.47}
$$

Note that $\sum_{q\in\mathbb{Z}}|\rho(q)|^{2r-1}<\infty$ since $H<\frac{1}{2}\leqslant 1-\frac{1}{4r-2}$.

- Preparation to the cases where $0\leqslant l\leqslant 2r-2$

In order to handle the terms $Q_n^{(+,r,l)}(s,t)$ whenever $0\leqslant l\leqslant 2r-2$, we will make use of the following decomposition:

$$
|d_n^{(+,r,l)}(j,j')|\leqslant\sum_{u,v=0}^{1}\Omega_n^{(u,v,r,l)}(j,j'),
\tag{3.3.48}
$$

where

$$
\begin{aligned}
\Omega_n^{(u,v,r,l)}(j,j') \;=\; &\sum_{a=0}^{4r-2-2l}\binom{4r-2-2l}{a}\big|E[f^{(a)}(X^+_{(j+u)2^{-n/2}})f^{(4r-2-2l-a)}(X^+_{(j'+v)2^{-n/2}})]\big|\\
&\times\big|\big\langle\xi_{(j+u)2^{-n/2}}^{\otimes a}\tilde{\otimes}\xi_{(j'+v)2^{-n/2}}^{\otimes(4r-2-2l-a)};\delta_{(j+1)2^{-n/2}}^{\otimes(2r-1-l)}\tilde{\otimes}\delta_{(j'+1)2^{-n/2}}^{\otimes(2r-1-l)}\big\rangle\big|.
\end{aligned}
$$

- Case $1\leqslant l\leqslant 2r-2$ (only when $r\geqslant 2$)

Since f belongs to C_b^∞ and since, by (3.3.45), we have $|\langle\xi_t;\delta_{(j+1)2^{-n/2}}\rangle|\leqslant 2^{-nH}$ for all $t\geqslant 0$ and all $j\in\mathbb{N}$, we deduce that

$$
|d_n^{(+,r,l)}(j,j')|\leqslant c\,2^{-nH(4r-2-2l)}.
$$

As a consequence, and relying to the same arguments that have been used previously in the case $l=2r-1$, we get

$$\begin{aligned} Q_n^{(+,r,l)}(s,t) &\leqslant c\,2^{-nH(4r-2-2l)}\,2^{nH(2r-1)}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\left|\langle\delta_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle\right|^l \\ &\leqslant c\,2^{-nH(4r-2-2l)}\,2^{nH(2r-1)}2^{-nHl}\sum_{q\in\mathbb{Z}}|\rho(q)|^l(1+2^{n/2}|t-s|) \\ &= c\,2^{-nH(2r-1-l)}(1+2^{n/2}|t-s|)\leqslant c\,(1+2^{n/2}|t-s|). \end{aligned} \tag{3.3.49}$$

- Case $l=0$

Relying to the decomposition (3.3.48), we get

$$Q_n^{(+,r,0)}(s,t) \leqslant 2^{nH(2r-1)}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{u,v=0}^{1}\Omega_n^{(u,v,r,0)}(j,j'). \tag{3.3.50}$$

We will study only the term corresponding to $\Omega_n^{(0,1,r,0)}(j,j')$ in (3.3.50), which is representative of the difficulty. It is given by

$$\begin{aligned} & 2^{nH(2r-1)}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{a=0}^{4r-2}\binom{4r-2}{a}\left|E[f^{(a)}(X^+_{j2^{-n/2}})f^{(4r-2-a)}(X^+_{(j'+1)2^{-n/2}})]\right| \\ &\qquad\qquad\times\left|\langle\xi^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\xi^{\otimes(4r-2-a)}_{(j'+1)2^{-n/2}};\delta^{\otimes(2r-1)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1)}_{(j'+1)2^{-n/2}}\rangle\right| \\ \leqslant\ & c\,2^{nH(2r-1)}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\sum_{a=0}^{4r-2}\left|\langle\xi^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\xi^{\otimes(4r-2-a)}_{(j'+1)2^{-n/2}};\delta^{\otimes(2r-1)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1)}_{(j'+1)2^{-n/2}}\rangle\right|. \end{aligned}$$

We define $E_n^{(a,r)}(j,j') := \left|\langle\xi^{\otimes a}_{j2^{-n/2}}\tilde{\otimes}\xi^{\otimes(4r-2-a)}_{(j'+1)2^{-n/2}};\delta^{\otimes(2r-1)}_{(j+1)2^{-n/2}}\tilde{\otimes}\delta^{\otimes(2r-1)}_{(j'+1)2^{-n/2}}\rangle\right|$. By (3.3.45), recall that $|\langle\xi_t;\delta_{(j+1)2^{-n/2}}\rangle|\leqslant 2^{-nH}$ for all $t\geqslant 0$ and all $j\in\mathbb{N}$. We thus get, with $\tilde{c}_a$ some combinatorial constants,

$$\begin{aligned} E_n^{(a,r)}(j,j') &\leqslant \tilde{c}_a\,2^{-nH(4r-3)}\big(|\langle\xi_{j2^{-n/2}};\delta_{(j+1)2^{-n/2}}\rangle|+|\langle\xi_{j2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| \\ &\qquad +|\langle\xi_{(j'+1)2^{-n/2}};\delta_{(j+1)2^{-n/2}}\rangle|+|\langle\xi_{(j'+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle|\big). \end{aligned}$$

For instance, we can write

$$\begin{aligned}
&\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} |\langle \xi_{(j'+1)2^{-n/2}};\delta_{(j+1)2^{-n/2}}\rangle| \\
&= 2^{-nH-1}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \big|(j+1)^{2H}-j^{2H}+|j'-j+1|^{2H}-|j'-j|^{2H}\big| \\
&\leqslant 2^{-nH-1}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \big((j+1)^{2H}-j^{2H}\big) \\
&\quad +2^{-nH-1}\sum_{\lfloor 2^{n/2}s\rfloor\leqslant j\leqslant j'\leqslant\lfloor 2^{n/2}t\rfloor-1} \big((j'-j+1)^{2H}-(j'-j)^{2H}\big) \\
&\quad +2^{-nH-1}\sum_{\lfloor 2^{n/2}s\rfloor\leqslant j'< j\leqslant\lfloor 2^{n/2}t\rfloor-1} \big((j-j')^{2H}-(j-j'-1)^{2H}\big) \\
&\leqslant \frac{3}{2}\,2^{-nH}\big(\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big)\lfloor 2^{n/2}t\rfloor^{2H}\leqslant\frac{3t^{2H}}{2}\big(1+2^{n/2}|t-s|\big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \langle \xi_{j2^{-n/2}};\delta_{(j+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{2H}}{2}\big(1+2^{n/2}|t-s|\big); \\
\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \langle \xi_{j2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{2H}}{2}\big(1+2^{n/2}|t-s|\big); \\
\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \langle \xi_{(j'+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{2H}}{2}\big(1+2^{n/2}|t-s|\big).
\end{aligned}$$

As a consequence, we deduce

$$Q_n^{(+,r,0)}(s,t)\leqslant c\,2^{-nH(2r-2)}t^{2H}\big(2^{n/2}|t-s|+1\big)\leqslant c\,t^{2H}\big(2^{n/2}|t-s|+1\big). \tag{3.3.51}$$

Combining (3.3.46), (3.3.47), (3.3.49) and (3.3.51) finally shows our claim (3.3.44).

(2) if $s<0\leqslant t$, then

$$\begin{aligned}
&E[(W_n^{(2r-1)}(f,t)-W_n^{(2r-1)}(f,s))^2]=E[(W_{+,n}^{(2r-1)}(f,t)-W_{-,n}^{(2r-1)}(f,-s))^2] \\
&\leqslant 2E[(W_{+,n}^{(2r-1)}(f,t))^2]+2E[(W_{-,n}^{(2r-1)}(f,-s))^2].
\end{aligned}$$

By (1) with $s=0$, one can write

$$E[(W_{+,n}^{(2r-1)}(f,t))^2]\leqslant c\,t^{2H}(t2^{n/2}+1).$$

Similarly

$$E[(W_{-,n}^{(2r-1)}(f,-s))^2]\leqslant c(-s)^{2H}\big((-s)2^{n/2}+1\big)$$

We deduce that

$$E[(W_n^{(2r-1)}(f,t) - W_n^{(2r-1)}(f,s))^2] \leqslant c \max(t^{2H}, (-s)^{2H})\big((t-s)2^{n/2}+1\big).$$

That is, (3.3.44) also holds true in this case.

3.3.7 Step 5: Limits of the weighted power variations of odd order

Fix $f \in C_b^\infty$ and $t \geqslant 0$. We claim that, if $H \in \left[\frac{1}{6}, \frac{1}{2}\right)$ and $r \geqslant 3$ then, as $n \to \infty$,

$$V_n^{(2r-1)}(f,t) \overset{\text{prob}}{\longrightarrow} 0. \tag{3.3.52}$$

Moreover, if $H \in \left(\frac{1}{6}, \frac{1}{2}\right)$ then, as $n \to \infty$,

$$V_n^{(3)}(f,t) \overset{\text{prob}}{\longrightarrow} 0, \tag{3.3.53}$$

whereas, if $H = \frac{1}{6}$ then, as $n \to \infty$,

$$\Big(X_t, Y_t, V_n^{(3)}(f,t)\Big)_{t\geqslant 0} \overset{\text{fdd}}{\to} \left(X_t, Y_t, \kappa_3 \int_0^{Y_t} f(X_s)dW_s\right)_{t\geqslant 0}, \tag{3.3.54}$$

with $W = (W_t)_{t\in\mathbb{R}}$ a two-sided Brownian motion independent of the pair (X, Y).

Indeed, using the decomposition (3.3.22), the conclusion of Step 4 (to pass from $Y_{T_{\lfloor 2^n t\rfloor,n}}$ to Y_t) and since by [20, Lemma 2.3], we have $Y_{T_{\lfloor 2^n t\rfloor,n}} \overset{L^2}{\longrightarrow} Y_t$ as $n \to \infty$, we deduce that the limit of $V_n^{(2r-1)}(f,t)$ is the same as that of

$$2^{-nH(r-\frac{1}{2})} \sum_{l=1}^{r} a_{r,l} W_n^{(2l-1)}(f, Y_t).$$

Thus, the proofs of (3.3.52), (3.3.53) and (3.3.54) then follow directly from (3.3.23) and the other results recalled in Step 3, as well as the fact that X and Y are independent.

3.3.8 Step 6: Proving (3.2.16) and (3.2.17)

We assume $H \in [\frac{1}{6}, \frac{1}{2})$. We will make use of the following Taylor's type formula. Fix $f \in C_b^\infty$. For any a, $b \in \mathbb{R}$ and for some constants c_r whose explicit values are immaterial here,

$$\begin{aligned} f(b) - f(a) &= \frac{1}{2}\big(f'(a) + f'(b)\big)(b-a) - \frac{1}{24}\big(f'''(a) + f'''(b)\big)(b-a)^3 \\ &\quad + \sum_{r=3}^{7} c_r \big(f^{(2r-1)}(a) + f^{(2r-1)}(b)\big)(b-a)^{2r-1} + O(|b-a|^{14}), \end{aligned}$$

where $|O(|b-a|^{14})| \leqslant C_f|b-a|^{14}$, C_f being a constant depending only on f. One can thus write

$$\begin{aligned} f(Z_{T_{\lfloor 2^n t\rfloor,n}}) - f(0) &= \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \big(f(Z_{T_{k+1,n}}) - f(Z_{T_{k,n}})\big) \\ &= \sum_{k=0}^{\lfloor 2^n t\rfloor -1} \frac{1}{2}\big(f'(Z_{T_{k,n}}) + f'(Z_{T_{k+1,n}})\big)(Z_{T_{k+1,n}} - Z_{T_{k,n}}) \\ &\quad - \frac{1}{12}V_n^{(3)}(f''',t) + \sum_{r=3}^{7} 2c_r V_n^{(2r-1)}(f^{(2r-1)},t) \\ &\quad + \sum_{k=0}^{\lfloor 2^n t\rfloor -1} O((Z_{T_{k+1,n}} - Z_{T_{k,n}})^{14}). \end{aligned} \tag{3.3.55}$$

As far as the big O in (3.3.55) is concerned, we have, with $G \sim N(0,1)$,

$$\begin{aligned} E\Big[\Big|\sum_{k=0}^{\lfloor 2^n t\rfloor -1} O((Z_{T_{k+1,n}} - Z_{T_{k,n}})^{14})\Big|\Big] &\leqslant C_f \sum_{k=0}^{\lfloor 2^n t\rfloor -1} E\big[(Z_{T_{k+1,n}} - Z_{T_{k,n}})^{14}\big] \\ = \; C_f \sum_{k=0}^{\lfloor 2^n t\rfloor -1} 2^{-7nH} E[G^{14}] &\leqslant C_f E[G^{14}]t\, 2^{n(1-7H)} \to_{n\to\infty} 0 \quad \text{since } H \geqslant \tfrac{1}{6}. \end{aligned} \tag{3.3.56}$$

On the other hand, by continuity of $f \circ Z$ and due to (3.2.14), one has, almost surely and as $n \to \infty$,

$$f(Z_{T_{\lfloor 2^n t\rfloor,n}}) - f(0) \to f(Z_t) - f(0). \tag{3.3.57}$$

Finally, when $H > \frac{1}{6}$ the desired conclusion (3.2.16) follows from (3.3.56), (3.3.57), (3.3.52) and (3.3.53) plugged into (3.3.55). The proof of (3.2.17) when $H = \frac{1}{6}$ is similar, the only difference being that one has (3.3.54) instead of (3.3.53), thus leading to the bracket term $\frac{\kappa_3}{12}\int_0^{Y_t} f'''(X_s)dW_s =: \frac{\kappa_3}{12}\int_0^t f'''(Z_s)d^{\circ 3}Z_s$ in (3.2.17).

3.3.9 Step 7: Proving (3.2.18)

Using $b^3 - a^3 = \frac{3}{2}(a^2+b^2)(b-a) - \frac{1}{2}(b-a)^3$, one can write, with $\mathbf{1}$ denoting the function constantly equal to 1,

$$\begin{aligned} V_n(x \mapsto x^2, t) - \frac{1}{3}Z_t^3 &= \frac{1}{6}V_n^{(3)}(\mathbf{1},t) + \frac{1}{3}\sum_{k=0}^{\lfloor 2^n t\rfloor -1} (Z^3_{T_{k+1,n}} - Z^3_{T_{k,n}}) - \frac{1}{3}Z_t^3 \\ &= \frac{1}{6}V_n^{(3)}(\mathbf{1},t) + \frac{1}{3}(Z^3_{T_{\lfloor 2^n t\rfloor,n}} - Z_t^3). \end{aligned}$$

As a result, and thank to (3.2.14), one deduces that if $V_n(x \mapsto x^2, t)$ converges stably in law, then $V_n^{(3)}(\mathbf{1},t)$ must converge as well. But it is shown in Corollary 2.1.2 in chapter 2 that $2^{-n(1-6H)/4}V_n^{(3)}(\mathbf{1},t)$ converges in law to a non degenerate limit. This being clearly in contradiction with the convergence of $V_n^{(3)}(\mathbf{1},t)$, we deduce that (3.2.18) holds.

Chapter 4

An Itô's type formula for the fBmBt in dimension 2

In this chapter we present the results obtained in: R. Zeineddine, "Change-of-variable formula for the bi-dimensional fractional Brownian motion in Brownian time", submitted to *ALEA*.

4.1 Introduction

Our aim in the present chapter is to provide a change-of-variable formula for the fractional Brownian motion in Brownian time (fBmBt) in multi-dimension. For simplicity of the exposition and because the computations are rather involved, we will stick on dimension 2, which is representative of the difficulty. In dimension 1, the mathematical definition of fBmBt (together with its terminology) were introduced in chapter 1. Let us give an analogue definition in dimension 2. Set

$$Z_t := (Z_t^1, Z_t^2) = (X_{Y_t}^1, X_{Y_t}^2), \;\; t \geqslant 0, \tag{4.1.1}$$

where X^1, X^2 are two independent (two-sided) fractional Brownian motions having the same Hurst parameter $H \in (0,1)$, and Y is a standard (one-sided) Brownian motion independent of (X^1, X^2).

The present work may be seen a natural follow-up of chapter 3, in which we proved a change-of-variable for fBmBt in dimension one, that is, for Z^1. Recall that in chapter 3 we have proved the following theorem.

Theorem 4.1.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth and bounded enough function.*

1. *If $H > \frac{1}{6}$ then*

$$f(Z_t^1) = f(0) + \int_0^t f'(Z_s^1) d^\circ Z_s^1, \quad t \geqslant 0.$$

2. *If $H = \frac{1}{6}$ then, with $\kappa_3 \simeq 2.322$,*

$$f(Z_t^1) - f(0) + \frac{\kappa_3}{12} \int_0^t f'''(Z_s^1) d^{\circ 3} Z_s^1 \overset{law}{=} \int_0^t f'(Z_s^1) d^\circ Z_s^1, \quad t \geqslant 0,$$

where $\int_0^t f'''(Z_s^1) d^{\circ 3} Z_s^1$ is a random variable equal in law to $\int_0^{Y_t} f'''(X_s^1) dW_s$, for W a two-sided Brownian motion independent of the pair (X^1, Y).

3. *If $H < \frac{1}{6}$, then*

$$\int_0^t (Z_s^1)^2 d^\circ Z_s^1 \text{ does not exist (even stably in law).}$$

In the present chapter, our main aim is to extend Theorem 4.1.1 to the bi-dimensional case. To reach this goal, we follow and use a strategy introduced in [25] and [30]. We continue to let X^1, X^2, Y, Z be as in (4.1.1), and we set $X = (X^1, X^2)$. The following definition will play a pivotal role in the sequel.

Definition 4.1.2 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and fix a time $t > 0$.*
1) Provided it exists, we define $\int_0^t \nabla f(X_s)dX_s$ to be the limit in probability, as $n \to \infty$, of

$$\begin{aligned} & O_n(f,t) \qquad (4.1.2) \\ = & \sum_{j=0}^{\lfloor 2^{n/2}t \rfloor -1} \frac{\partial f}{\partial x}\left(\frac{X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}}{2}, \frac{X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}}}{2}\right)\left(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\right) \\ + & \sum_{j=0}^{\lfloor 2^{n/2}t \rfloor -1} \frac{\partial f}{\partial y}\left(\frac{X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}}{2}, \frac{X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}}}{2}\right)\left(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\right). \end{aligned}$$

2) When $O_n(f,t)$ defined by (4.1.2) does not converge in probability but converges stably instead, we denote the limit by $\int_0^t \nabla f(X_s)d^*X_s$.

A first preliminary result, which concerns the bi-dimensional fractional Brownian motion X, can now be stated. An analogue result for the fBmBt Z will be the object of the forthcoming Theorem 4.1.4.

Theorem 4.1.3 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function belonging to C_b^∞, and fix a time $t > 0$.*

1. *If $H > 1/6$ then $\int_0^t \nabla f(X_s)dX_s$ is well-defined, and we have*

$$f(X_t) = f(0) + \int_0^t \nabla f(X_s)dX_s. \qquad (4.1.3)$$

2. *If $H = 1/6$ then $\int_0^t \nabla f(X_s)d^*X_s$ is well-defined, and we have*

$$f(X_t) - f(0) - \int_0^t D^3 f(X_s)d^3 X_s \overset{law}{=} \int_0^t \nabla f(X_s)d^*X_s \qquad (4.1.4)$$

where $\int_0^t D^3 f(X_s)d^3X_s$ is short-hand for

$$\begin{aligned} \int_0^t D^3 f(X_s)d^3X_s &= \kappa_1 \int_0^t \frac{\partial^3 f}{\partial x^3}(X_s^1, X_s^2)dB_s^1 + \kappa_2 \int_0^t \frac{\partial^3 f}{\partial y^3}(X_s^1, X_s^2)dB_s^2 \quad (4.1.5) \\ &+ \kappa_3 \int_0^t \frac{\partial^3 f}{\partial x^2 \partial y}(X_s^1, X_s^2)dB_s^3 + \kappa_4 \int_0^t \frac{\partial^3 f}{\partial x \partial y^2}(X_s^1, X_s^2)dB_s^4 \end{aligned}$$

with $B = (B^1, \dots, B^4)$ a 4-dimensional Brownian motion independent of X, $\kappa_1^2 = \kappa_2^2 = \frac{1}{96}\sum_{r\in\mathbb{Z}} \rho^3(r)$ and $\kappa_3^2 = \kappa_4^2 = \frac{1}{32}\sum_{r\in\mathbb{Z}} \rho^3(r)$ with ρ defined in (4.2.23).

3. *If $H < 1/6$, for $f(x,y) = x^3$ then*

$$\int_0^t \nabla f(X_s) d^* X_s \text{ does not exist, even stably in law.} \tag{4.1.6}$$

So, it is impossible to write an Itô's type formula.

Theorem 4.1.3 together with a suitable extension of the Khoshnevisan-Lewis definition for the Stratonovich integral (see the next section for a precise statement) with respect to Z then lead to the following change-of-variable formula for 2D fBmBt, which represents the main finding of this chapter.

Theorem 4.1.4 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function belonging to C_b^∞, and fix a time $t > 0$.*

1. *If $H > 1/6$ then $\int_0^t \nabla f(Z_s) dZ_s$ is well defined, and we have*

$$f(Z_t) = f(0) + \int_0^t \nabla f(Z_s) dZ_s. \tag{4.1.7}$$

2. *If $H = 1/6$ then $\int_0^t \nabla f(Z_s) d^* Z_s$ is well defined, and we have*

$$f(Z_t) - f(0) - \int_0^t D^3 f(Z_s) d^3 Z_s \overset{law}{=} \int_0^t \nabla f(Z_s) d^* Z_s \tag{4.1.8}$$

where $\int_0^t D^3 f(Z_s) d^3 Z_s$ is short-hand for

$$\begin{aligned} \int_0^t D^3 f(Z_s) d^3 Z_s &= \kappa_1 \int_0^{Y_t} \frac{\partial^3 f}{\partial x^3}(X_s^1, X_s^2) dB_s^1 + \kappa_2 \int_0^{Y_t} \frac{\partial^3 f}{\partial y^3}(X_s^1, X_s^2) dB_s^2 \\ &\quad + \kappa_3 \int_0^{Y_t} \frac{\partial^3 f}{\partial x^2 \partial y}(X_s^1, X_s^2) dB_s^3 + \kappa_4 \int_0^{Y_t} \frac{\partial^3 f}{\partial x \partial y^2}(X_s^1, X_s^2) dB_s^4, \end{aligned}$$

with $B = (B^1, \ldots, B^4)$ is a 4-dimensional two-sided Brownian motion independent of X, and $\kappa_1, \ldots, \kappa_4$ as in Theorem 4.1.3. (B is also independent from Y.)

3. *If $H < 1/6$, for $f(x,y) = x^3$ then*

$$\int_0^t \nabla f(Z_s) d^* Z_s \quad \text{does not exist, even stably in law.} \tag{4.1.9}$$

So, it is impossible to write an Itô's type formula.

A brief outline of the chapter is as follows. In section 2, we introduce the framework and the preliminaries to prove our results, as well as the notation and some technical lemmas. In section 3, we prove Theorem 4.1.3 and finally in section 4, we prove Theorem 4.1.4.

4.2 Framework, preliminaries, notation and technical lemmas

4.2.1 The framework of Theorem 4.1.4

In this section, we explain and introduce the missing definitions of the mathematical objects appearing in Theorem 4.1.4.

1. **Khoshnevisan-Lewis' definition of the Stratonovich-integral with respect to the 1D fBmBt**

   Recall from the second section of chapter 3 that Khoshnevisan and Lewis [20, 21], have introduced the so-called intrinsic skeletal structure of Z^1. This structure is defined through a sequence of collections of stopping times (with respect to the natural filtration of Y), noted

   $$\mathscr{T}_n = \{T_{k,n} : k \geqslant 0\}, \quad n \geqslant 1, \tag{4.2.10}$$

   which are in turn expressed in terms of the subsequent hitting times of a dyadic grid cast on the real axis. More precisely, let $\mathscr{D}_n = \{j2^{-n/2} : j \in \mathbb{Z}\}$, $n \geqslant 1$, be the dyadic partition (of $\mathbb{R}$) of order $n/2$. For every $n \geqslant 1$, the stopping times $T_{k,n}$, appearing in (4.2.10), are given by the following recursive definition: $T_{0,n} = 0$, and

   $$T_{k,n} = \inf\big\{s > T_{k-1,n} : \quad Y(s) \in \mathscr{D}_n \setminus \{Y(T_{k-1,n})\}\big\}, \quad k \geqslant 1.$$

   Note that the definition of $T_{k,n}$, and therefore of $\mathscr{T}_n$, only involves the one-sided Brownian motion Y. Also, for every $n \geqslant 1$, the discrete stochastic process

   $$\mathscr{Y}_n = \{Y(T_{k,n}) : k \geqslant 0\}$$

   defines a simple random walk over $\mathscr{D}_n$. As shown in [20, Lemma 2.2], as n tends to infinity the collection $\{T_{k,n} : 1 \leqslant k \leqslant 2^n t\}$ approximates the common dyadic partition $\{k2^{-n} : 1 \leqslant k \leqslant 2^n t\}$ of order n of the time interval $[0,t]$. More precisely,

   $$\sup_{0 \leqslant s \leqslant t} \big|T_{\lfloor 2^n s\rfloor,n} - s\big| \to 0 \quad \text{almost surely and in } L^2(\Omega). \tag{4.2.11}$$

   Based on this fact, the integral of $f(Z^1)$ with respect to Z^1 is defined as

   $$\int_0^t f(Z^1_s)dZ^1_s := \lim_{n\to\infty} V_n(f,t), \tag{4.2.12}$$

   provided the limit exists in some sense, with $V_n(f,t)$ defined as follows,

   $$V_n(f,t) = \sum_{k=0}^{\lfloor 2^n t\rfloor - 1} \frac{1}{2}\big(f(Z^1_{T_{k,n}}) + f(Z^1_{T_{k+1,n}})\big)(Z^1_{T_{k+1,n}} - Z^1_{T_{k,n}}).$$

2. **A suitable definition for the Stratonovich-integral with respect to the 2D fBmBt**

   In the light of the previous definition of the integral with respect to 1D fBmBt and of Definition 4.1.2, it might seem natural to introduce the following definition for the integral with respect to the 2D fBmBt based on $\mathscr{T}_n$.

Definition 4.2.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and fix a time $t > 0$. Provided it exists, we define $\int_0^t \nabla f(Z_s)dZ_s$ to be the limit in probability, as $n \to \infty$, of*

$$\begin{aligned} \tilde{O}_n(f,t) \quad := & \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor -1} \frac{\partial f}{\partial x}\left(\frac{Z^1_{T_{j+1,n}} + Z^1_{T_{j,n}}}{2}, \frac{Z^2_{T_{j+1,n}} + Z^2_{T_{j,n}}}{2}\right)\left(Z^1_{T_{j+1,n}} - Z^1_{T_{j,n}}\right) \\ & + \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor -1} \frac{\partial f}{\partial y}\left(\frac{Z^1_{T_{j+1,n}} + Z^1_{T_{j,n}}}{2}, \frac{Z^2_{T_{j+1,n}} + Z^2_{T_{j,n}}}{2}\right)\left(Z^2_{T_{j+1,n}} - Z^2_{T_{j,n}}\right). \end{aligned} \tag{4.2.13}$$

If $\tilde{O}_n(f,t)$ defined by (4.2.13) does not converge in probability but converges stably, we denote the limit by $\int_0^t \nabla f(Z_s)d^*Z_s$.

4.2.2 Some preliminary results

We provide now a description of the tools of Malliavin calculus that we need in this chapter. We follow in this section the idea introduced in [25]. The reader in referred to [28] for details and any unexplained result.

Let $X = (X^1_t, X^2_t)_{t\in\mathbb{R}}$ be a 2D fBm with Hurst parameter belonging to $(0,1)$. For all $n \in \mathbb{N}^*$, we let $\mathscr{E}_n$ be the set of step $\mathbb{R}^2$-valued functions on $[-n,n]$, and $\mathscr{E} := \cup_n \mathscr{E}_n$. Set $\varepsilon_t = \mathbf{1}_{[0,t]}$ (resp. $\mathbf{1}_{[t,0]}$) if $t \geqslant 0$ (resp. $t < 0$). Let $\mathscr{H}$ be the Hilbert space defined as the closure of $\mathscr{E}$ with respect to the inner product

$$\langle (\varepsilon_{t_1}, \varepsilon_{t_2}), (\varepsilon_{s_1}, \varepsilon_{s_2})\rangle_{\mathscr{H}} = C_H(t_1, s_1) + C_H(t_2, s_2), \quad s_1, s_2, t_1, t_2 \in \mathbb{R},$$

where $C_H(t,s) = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H}) = E\big(X^i_s X^i_t\big)$ (i equals 1 or 2). The mapping $(\varepsilon_{t_1}, \varepsilon_{t_2}) \mapsto X^1_{t_1} + X^2_{t_2}$ can be extended to an isometry between $\mathscr{H}$ and the Gaussian space associated with X. Also, let $\mathscr{F}_n$ denote the set of step $\mathbb{R}$-valued functions on $[-n,n]$, $\mathscr{F} := \cup_n \mathscr{F}_n$ and $\mathcal{G}$ denote the Hilbert space defined as the closure of $\mathscr{F}$ with respect to the scaler product induced by

$$\langle \varepsilon_t, \varepsilon_s\rangle_{\mathscr{G}} = C_H(t,s), \quad s,t \in \mathbb{R}. \tag{4.2.14}$$

The mapping $\varepsilon_t \mapsto X^i_t$ (i equals 1 or 2) can be extended to an isometry between $\mathcal{G}$ and the Gaussian space associated with X^i.

We consider the set of smooth cylindrical random variables, i.e. of the form

$$F = f\big(X(\rho_1), \dots, X(\rho_m)\big), \quad \rho_i \in \mathscr{H},\ i = 1, \dots, m,$$

where $f \in C^\infty_b$ is bounded with bounded derivatives. The derivative operator D of a smooth random variable of the above form is defined as the $\mathscr{H}$-valued random variable

$$DF = \sum_{i=1}^m \frac{\partial f}{\partial x_i}\big(X(\rho_1), \dots, X(\rho_m)\big)\rho_i =: \big(D_{X^1}F, D_{X^2}F\big). \tag{4.2.15}$$

For example, if $F = f(X^1_t, X^2_s)$ with $f \in C^\infty_b(\mathbb{R}^2)$, then

$$DF = \frac{\partial f}{\partial x}(X^1_t, X^2_s)(\varepsilon_t, 0) + \frac{\partial f}{\partial y}(X^1_t, X^2_s)(0, \varepsilon_s).$$

So, we deduce from (4.2.15) that

$$D_{X^1}F = \frac{\partial f}{\partial x}(X_t^1, X_s^2)\varepsilon_t \quad \text{and} \quad D_{X^2}F = \frac{\partial f}{\partial y}(X_t^1, X_s^2)\varepsilon_s.$$

In particular, we have

$$D_{X^j}X_t^k = \delta_{jk}\varepsilon_t \quad \text{for } j,k \in \{1,2\}, \text{and } \delta_{jk} \text{ the Kronecker symbol.}$$

For any integer $k \geqslant 2$, one can define, by iteration, the k-th derivative $D^k F$ (which is a symmetric element of $L^2(\Omega, \mathscr{H}^{\otimes k})$). As usual, for any $k \geqslant 1$, the space $\mathbb{D}^{k,2}$ denotes the closure of the set of smooth random variables with respect to the norm $\|.\|_{k,2}$ defined by

$$\|F\|_{k,2}^2 = E(F^2) + \sum_{j=1}^{k} E[\|D^j F\|_{\mathscr{H}^{\otimes j}}^2].$$

The Malliavin derivative D satisfies the chain rule. If $\varphi : \mathbb{R}^n \to \mathbb{R}$ is C_b^1 and if $F_1, \dots, F_n$ are in $\mathbb{D}^{1,2}$, then $\varphi(F_1, ..., F_n) \in \mathbb{D}^{1,2}$ and we have

$$D\varphi(F_1, ..., F_n) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(F_1, ..., F_n) DF_i.$$

We have the following Leibniz formula. For any $F, G \in \mathbb{D}^{q,2}$ $(q \geqslant 1)$ such that $FG \in \mathbb{D}^{q,2}$, for $i \in \{1,2\}$, we have

$$D_{X^i}^q(FG) = \sum_{l=0}^{q} \binom{q}{l} (D_{X^i}^l(F)) \tilde{\otimes} (D_{X^i}^{q-l}G), \tag{4.2.16}$$

where $\tilde{\otimes}$ stands for the symmetric tensor product. In particular, we have the following formula. Let $\varphi, \psi \in C_b^q(\mathbb{R}^2)$ $(q \geqslant 1)$, and fix $0 \leqslant u < v$ and $0 \leqslant s < t$. Then $\varphi\big(\frac{X_t^1+X_s^1}{2}, \frac{X_t^2+X_s^2}{2}\big) \times\psi\big(\frac{X_v^1+X_u^1}{2}, \frac{X_v^2+X_u^2}{2}\big) \in \mathbb{D}^{q,2}$ and for $i \in \{1,2\}$ we have

$$\begin{aligned} & D_{X^i}^q\left(\varphi\left(\frac{X_t^1+X_s^1}{2}, \frac{X_t^2+X_s^2}{2}\right)\psi\left(\frac{X_v^1+X_u^1}{2}, \frac{X_v^2+X_u^2}{2}\right)\right) \\ = & \sum_{l=0}^{q} \binom{q}{l} \frac{\partial^l \varphi}{\partial x_i^l}\left(\frac{X_t^1+X_s^1}{2}, \frac{X_t^2+X_s^2}{2}\right) \frac{\partial^{q-l}\psi}{\partial x_i^{q-l}}\left(\frac{X_v^1+X_u^1}{2}, \frac{X_v^2+X_u^2}{2}\right) \\ & \times \left(\frac{\varepsilon_s+\varepsilon_t}{2}\right)^{\otimes l} \tilde{\otimes} \left(\frac{\varepsilon_u+\varepsilon_v}{2}\right)^{\otimes(q-l)}. \end{aligned} \tag{4.2.17}$$

A similar statement holds for $u < v \leqslant 0$ and $s < t \leqslant 0$.

If a random element $u \in L^2(\Omega, \mathscr{H})$ belongs to the domain of the divergence operator, that is, if it satisfies

$$|E\langle DF, u\rangle_{\mathscr{H}}| \leqslant c_u \sqrt{E(F^2)} \quad \text{for any } F \in \mathscr{F},$$

then $I(u)$ is defined by the duality relationship

$$E\big(FI(u)\big) = E\big(\langle DF, u\rangle_{\mathscr{H}}\big),$$

for every $F \in \mathbb{D}^{1,2}$.

For every $n \geqslant 1$, let $\mathbb{H}_n$ be the nth Wiener chaos of X, that is, the closed linear subspace of $L^2(\Omega, \mathscr{A}, P)$ generated by the random variables $\{H_n(X(h)), h \in \mathscr{H}, \|h\|_{\mathscr{H}} = 1\}$, where H_n is the nth Hermite polynomial. The mapping

$$I_n(h^{\otimes n}) = H_n(X(h)), \tag{4.2.18}$$

provides a linear isometry between the symmetric tensor product $\mathscr{H}^{\odot n}$ and $\mathbb{H}_n$. The following duality formula holds

$$E\big(FI_n(h)\big) = E\big(\langle D^n F, h\rangle_{\mathscr{H}^{\otimes n}}\big),$$

for any element $h \in \mathscr{H}^{\odot n}$ and any random variable $F \in \mathbb{D}^{n,2}$. In particular, we have

$$E\big(FI_n^{(i)}(h)\big) = E\big(\langle D^n_{X^i} F, h\rangle_{\mathcal{G}^{\otimes n}}\big), \;\; i = 1, 2, \tag{4.2.19}$$

for any $h \in \mathcal{G}^{\odot n}$ and $F \in \mathbb{D}^{n,2}$, where we write $I_n^{(i)}(h)$ whenever the corresponding n-th multiple integral is only with respect to X^i.

Finally, we mention the following particular cases (the only one we will need in the sequel): if $f, g \in \mathcal{G}$, $n, m \geqslant 1$ and $i \in \{1, 2\}$, then we have the classical multiplication formula

$$I_n^{(i)}(f^{\otimes n}) I_m^{(i)}(g^{\otimes m}) = \sum_{r=0}^{n\wedge m} r! \binom{n}{r}\binom{m}{r} I^{(i)}_{n+m-2r}(f^{\otimes n+m-r} \otimes g^{\otimes n+m-r})\langle f, g\rangle^r_{\mathcal{G}}. \tag{4.2.20}$$

We have also the following isometric property,

$$E[I_n^{(i)}(f^{\otimes n})^2] = n!\langle f, f\rangle^n_{\mathcal{G}}, \tag{4.2.21}$$

and, for $j \in \{1, 2\}$,

$$D_{X^j}\big(I_n^{(i)}(f^{\otimes n})\big) = \delta_{ji} n I^{(i)}_{n-1}(f^{\otimes n-1}). \tag{4.2.22}$$

4.2.3 Notation

Throughout all the forthcoming proofs, we shall use the following notation. For all $k, n \in \mathbb{N}$ we write

$$\varepsilon_{k2^{-n/2}} = \mathbf{1}_{[0, k2^{-n/2}]}, \quad \delta_{k2^{-n/2}} = \mathbf{1}_{[(k-1)2^{-n/2}, k2^{-n/2}]}.$$

For all $k \in \mathbb{Z}$, $H \in (0, 1)$, we write

$$\rho(k) = \frac{1}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}). \tag{4.2.23}$$

For any sufficiently smooth function $f : \mathbb{R}^2 \to \mathbb{R}$, the notation $\partial^{k,l}_{1\ldots12\ldots2} f$ (where the index 1 is repeated k times and the index 2 is repeated l times) means that f is differentiated k times with respect to the first component and l times with respect to the second one.

We denote for any $j \in \mathbb{Z}$, $\Delta_{j,n} f(X^1, X^2) := f\left(\frac{X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}}{2}, \frac{X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}}}{2}\right)$.

For $i \in \{1, 2\}$, $H \in (0, 1)$, $X_j^{i,n} := 2^{\frac{nH}{2}} X^i_{j2^{-\frac{n}{2}}}$.

Definition 4.2.2 *For any $t \in \mathbb{R}^+$ and any $n \in \mathbb{N}$, we define :*

$$\begin{aligned}
K_n^{(1)}(f,t) &:= \frac{1}{24}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Delta_{j,n}\partial_{111} f(X^1,X^2) I_3^{(1)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 3}\big) \\
K_n^{(2)}(f,t) &:= \frac{1}{24}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Delta_{j,n}\partial_{222} f(X^1,X^2) I_3^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 3}\big) \\
K_n^{(3)}(f,t) &:= \frac{1}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Delta_{j,n}\partial_{122} f(X^1,X^2) I_1^{(1)}\big(\delta_{(j+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2}\big) \\
K_n^{(4)}(f,t) &:= \frac{1}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Delta_{j,n}\partial_{112} f(X^1,X^2) I_2^{(1)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2}\big) I_1^{(2)}\big(\delta_{(j+1)2^{-n/2}}\big).
\end{aligned}$$

For any $r \in \mathbb{N}^*$ and $\psi \in C_b^\infty(\mathbb{R}^{2r},\mathbb{R})$, we define ξ as follows :

$$\xi = \psi\big(X_{s_1}^1, X_{s_1}^2, \ldots, X_{s_r}^1, X_{s_r}^2\big), \tag{4.2.24}$$

where $s_1, \ldots, s_r \in \mathbb{R}$. In the proofs contained in this paper, C shall denote a positive, finite constant that may change value from line to line.

4.2.4 Some technical lemmas

A key tool in our analysis will be the next lemma, which can be deduced from the following Taylor's theorem with remainder.

Theorem 4.2.3 *Let n be a nonnegative integer. If $g \in C^n(\mathbb{R}^2)$, then*

$$g(k) = \sum_{|\alpha|\leqslant n} \partial^\alpha g(l)\frac{(k-l)^\alpha}{\alpha!} + R_n(l,k), \tag{4.2.25}$$

where

$$R_n(l,k) = n\sum_{|\alpha|=n} \frac{(k-l)^\alpha}{\alpha!}\int_0^1 (1-u)^n[\partial^\alpha g(l+u(k-l)) - \partial^\alpha g(l)]du$$

if $n \geqslant 1$, and $R_0(l,k) = g(k) - g(l)$. In particular, $R_n(l,k) = \sum_{|\alpha|=n} h_\alpha(l,k)(k-l)^\alpha$, where h_α is a continuous function with $h_\alpha(l,l) = 0$ for all l. Moreover,

$$\big|R_n(l,k)\big| \leqslant (n\vee 1)\sum_{|\alpha|=n} M_\alpha\big|(k-l)^\alpha\big|,$$

where $M_\alpha = \sup\{|\partial^\alpha g(l+u(k-l)) - \partial^\alpha g(l)| : 0 \leqslant u \leqslant 1\}$.

Thanks to the previous theorem we deduce the following lemma.

Lemma 4.2.4 *Let $f \in C_b^{13}(\mathbb{R}^2)$, then*

$$\begin{aligned} f(b,d) = f(a,c) \quad &+ \sum_{i=1}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, \partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)(b-a)^{\alpha_1}(d-c)^{\alpha_2} \\ &+ R_{13}\big((b,d),(a,c)\big), \end{aligned}$$

where $\alpha_1, \alpha_2 \in \mathbb{N}$, and

$$|R_{13}\big((b,d),(a,c)\big)| \leqslant C_f \sum_{\alpha_1+\alpha_2=13} |b-a|^{\alpha_1}|d-c|^{\alpha_2},$$

with C_f is a constant depending only on f. On the other hand, we have $C(1,0) = C(0,1) = 1$, $C(3,0) = C(0,3) = \frac{1}{24}$ and $C(2,1) = C(1,2) = \frac{1}{8}$. The other constants: $C(\alpha_1,\alpha_2)$ could also be determined explicitly, but won't need their explicit values.

Proof. By applying (4.2.25) to f, we get

$$f(b,d) = f(a,c) + \sum_{\alpha_1+\alpha_2 \leqslant 13} \partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f(a,c) \frac{(b-a)^{\alpha_1}}{\alpha_1!}\frac{(d-c)^{\alpha_2}}{\alpha_2!} + R_{13}\big((b,d),(a,c)\big).$$

Since $f \in C_b^{13}$,observe that $\exists C_f > 0$ such that

$$\Big| \sum_{\alpha_1+\alpha_2=13} \partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f(a,c) \frac{(b-a)^{\alpha_1}}{\alpha_1!}\frac{(d-c)^{\alpha_2}}{\alpha_2!} \Big| \;\leqslant\; C_f \sum_{\alpha_1+\alpha_2=13} |b-a|^{\alpha_1}|d-c|^{\alpha_2},$$

and such that

$$\begin{aligned} |R_{13}\big((b,d),(a,c)\big)| &\leqslant 13 \sum_{\alpha_1+\alpha_2=13} M_{(\alpha_1,\alpha_2)} |b-a|^{\alpha_1}|d-c|^{\alpha_2} \\ &\leqslant 13 \sup_{\alpha_1+\alpha_2=13} M_{(\alpha_1,\alpha_2)} \sum_{\alpha_1+\alpha_2=13} |b-a|^{\alpha_1}|d-c|^{\alpha_2} \\ &\leqslant C_f \sum_{\alpha_1+\alpha_2=13} |b-a|^{\alpha_1}|d-c|^{\alpha_2}. \end{aligned}$$

So, we deduce that

$$f(b,d) = f(a,c) + \sum_{\alpha_1+\alpha_2 \leqslant 12} \partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f(a,c) \frac{(b-a)^{\alpha_1}}{\alpha_1!}\frac{(d-c)^{\alpha_2}}{\alpha_2!} + \tilde{R}_{13}\big((b,d),(a,c)\big), \quad (4.2.26)$$

where

$$|\tilde{R}_{13}\big((b,d),(a,c)\big)| \leqslant C_f \sum_{\alpha_1+\alpha_2=13} |b-a|^{\alpha_1}|d-c|^{\alpha_2}.$$

For each $i \in \{1,\ldots,12\}$ and each $\alpha_1, \alpha_2 \in \mathbb{N}$ such that $\alpha_1 + \alpha_2 = i$, we define $g_{\alpha_1,\alpha_2,i}$ as $g_{\alpha_1,\alpha_2,i} := \partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f$ and we set $k := (a,c)$, $l := \left(\frac{a+b}{2}, \frac{c+d}{2}\right)$. So, by applying (4.2.25) to $g_{\alpha_1,\alpha_2,i}$

with $k := (a, c)$, $l := \big(\frac{a+b}{2}, \frac{c+d}{2}\big)$ and $n = 13 - i$, we get

$$\begin{aligned} g_{\alpha_1,\alpha_2,i}(a,c) &= g_{\alpha_1,\alpha_2,i}\big(\frac{a+b}{2}, \frac{c+d}{2}\big) + \sum_{\beta_1+\beta_2 \leqslant 13-i} \partial^{\beta_1,\beta_2}_{1\ldots 12\ldots 2} g_{\alpha_1,\alpha_2,i}\big(\frac{a+b}{2}, \frac{c+d}{2}\big) \\ &\quad \times \frac{(a-b)^{\beta_1}}{2^{\beta_1}\beta_1!} \frac{(c-d)^{\beta_2}}{2^{\beta_2}\beta_2!} + \bar{R}_{13-i}\big((a,c), \big(\frac{a+b}{2}, \frac{c+d}{2}\big)\big) \\ &= g_{\alpha_1,\alpha_2,i}\big(\frac{a+b}{2}, \frac{c+d}{2}\big) + \sum_{\beta_1+\beta_2 \leqslant 13-i} C(\beta_1,\beta_2) \partial^{\beta_1,\beta_2}_{1\ldots 12\ldots 2} g_{\alpha_1,\alpha_2,i}\big(\frac{a+b}{2}, \frac{c+d}{2}\big) \\ &\quad \times (b-a)^{\beta_1}(d-c)^{\beta_2} + \bar{R}_{13-i}\big((a,c), \big(\frac{a+b}{2}, \frac{c+d}{2}\big)\big). \end{aligned}$$

By replacing $g_{\alpha_1,\alpha_2,i}(a,c)$ in (4.2.26) we get

$$\begin{aligned} f(b,d) = f(a,c) \quad &+ \sum_{\alpha_1+\alpha_2 \leqslant 13} \tilde{C}(\alpha_1,\alpha_2) \partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} f\big(\frac{a+b}{2}, \frac{c+d}{2}\big)(b-a)^{\alpha_1}(d-c)^{\alpha_2} \\ &+ R_{13}\big((a,c), \big(\frac{a+b}{2}, \frac{c+d}{2}\big)\big), \end{aligned} \tag{4.2.27}$$

where

$$\big|R_{13}\big((a,c), \big(\frac{a+b}{2}, \frac{c+d}{2}\big)\big)\big| \leqslant C_f \sum_{\alpha_1+\alpha_2=13} |b-a|^{\alpha_1}|d-c|^{\alpha_2}.$$

Let us prove that $\forall i \in \{1, \ldots, 6\}$ and $\forall \alpha_1, \alpha_2 \in \mathbb{N} \,/\, \alpha_1 + \alpha_2 = 2i$, we have

$$\tilde{C}(\alpha_1, \alpha_2) = 0. \tag{4.2.28}$$

Indeed, let $f(x,y) = x^{\alpha_1} y^{\alpha_2}$. Thanks to (4.2.27), we get

$$\begin{aligned} b^{\alpha_1} d^{\alpha_2} = a^{\alpha_1} c^{\alpha_2} \quad &+ \sum_{\beta_1+\beta_2 \leqslant 2i-1} \tilde{C}(\beta_1,\beta_2) \partial^{\beta_1,\beta_2}_{1\ldots 12\ldots 2} f\big(\frac{a+b}{2}, \frac{c+d}{2}\big)(b-a)^{\beta_1}(d-c)^{\beta_2} \\ &+ \alpha_1!\alpha_2!\tilde{C}(\alpha_1,\alpha_2)(b-a)^{\alpha_1}(d-c)^{\alpha_2}. \end{aligned} \tag{4.2.29}$$

Let us now change a into b and c into d in the previous formula, so to get

$$\begin{aligned} a^{\alpha_1} c^{\alpha_2} = b^{\alpha_1} d^{\alpha_2} \quad &+ \sum_{\beta_1+\beta_2 \leqslant 2i-1} \tilde{C}(\beta_1,\beta_2) \partial^{\beta_1,\beta_2}_{1\ldots 12\ldots 2} f\big(\frac{b+a}{2}, \frac{d+c}{2}\big)(a-b)^{\beta_1}(c-d)^{\beta_2} \\ &+ \alpha_1!\alpha_2!\tilde{C}(\alpha_1,\alpha_2)(a-b)^{\alpha_1}(c-d)^{\alpha_2}. \end{aligned} \tag{4.2.30}$$

Observe that if in (4.2.30) $\beta_1+\beta_2$ is odd (resp. is even) then $(a-b)^{\beta_1}(c-d)^{\beta_2} = (-1)^{\beta_1+\beta_2}(b-a)^{\beta_1}(d-c)^{\beta_2} = -(b-a)^{\beta_1}(d-c)^{\beta_2}$ (resp. $(b-a)^{\beta_1}(d-c)^{\beta_2}$). So, by taking the sum of (4.2.29) and (4.2.30) we get

$$\begin{aligned} b^{\alpha_1} d^{\alpha_2} + a^{\alpha_1} c^{\alpha_2} &= a^{\alpha_1} c^{\alpha_2} + b^{\alpha_1} d^{\alpha_2} + \sum_{k=1}^{i-1} \sum_{\beta_1+\beta_2=2k} 2\tilde{C}(\beta_1,\beta_2) \partial^{\beta_1,\beta_2}_{1\ldots 12\ldots 2} f\big(\frac{b+a}{2}, \frac{d+c}{2}\big)(b-a)^{\beta_1} \\ &\quad \times (d-c)^{\beta_2} + 2\alpha_1!\alpha_2!\tilde{C}(\alpha_1,\alpha_2)(b-a)^{\alpha_1}(d-c)^{\alpha_2}, \end{aligned}$$

leading to

$$\begin{aligned} 0 &= \sum_{k=1}^{i-1} \sum_{\beta_1+\beta_2=2k} 2\tilde{C}(\beta_1,\beta_2)\partial^{\beta_1,\beta_2}_{1\ldots 12\ldots 2} f\big(\frac{b+a}{2},\frac{d+c}{2}\big)(b-a)^{\beta_1}(d-c)^{\beta_2} \\ &\quad +2\alpha_1!\alpha_2!\tilde{C}(\alpha_1,\alpha_2)(b-a)^{\alpha_1}(d-c)^{\alpha_2}. \end{aligned} \tag{4.2.31}$$

We deduce thanks to (4.2.31) that, for $i=1$ and each $\alpha_1,\alpha_2\in\mathbb{N}$ satisfying $\alpha_1+\alpha_2=2$, we have $\forall a,b,c,d\in\mathbb{R}$,

$$2\alpha_1!\alpha_2!\tilde{C}(\alpha_1,\alpha_2)(b-a)^{\alpha_1}(d-c)^{\alpha_2}=0,$$

implying in turn $\tilde{C}(\alpha_1,\alpha_2)=0$. Then, a simple recursive argument shows that for all $\forall i\in\{1,\ldots,6\}$ and $\forall\alpha_1,\alpha_2\in\mathbb{N}\,/\,\alpha_1+\alpha_2=2i$ we have $\tilde{C}(\alpha_1,\alpha_2)=0$. As a result, (4.2.28) holds true.

It remains to prove that $\tilde{C}(1,0)=\tilde{C}(0,1)=1$, $\tilde{C}(3,0)=\tilde{C}(0,3)=\frac{1}{24}$ and $\tilde{C}(2,1)=\tilde{C}(1,2)=\frac{1}{8}$. Thanks to (4.2.27) and (4.2.28), by taking $f(x,y)=x$ (resp. $f(x,y)=y$) we deduce immediately that $\tilde{C}(1,0)$ (resp. $\tilde{C}(0,1)$) equals 1. By taking $f(x,y)=x^3$ (resp. $f(x,y)=y^3$) we deduce that $\tilde{C}(3,0)$ (resp. $\tilde{C}(0,3)$) equals $\frac{1}{24}$. Finally, by taking $f(x,y)=x^2y$ (resp. $f(x,y)=xy^2$) we deduce that $\tilde{C}(2,1)$ (resp. $\tilde{C}(1,2)$) equals $\frac{1}{8}$. The proof of Lemma 4.2.4 is complete. ■

The following lemma gathers several estimates that will be needed while completing the proof of our theorems .

Lemma 4.2.5 *Suppose that $H<1/2$. Then*

1. *For all $j,k\in\mathbb{N}$ and $u\in\mathbb{R}$,*

$$|\langle\varepsilon_u,\delta_{(j+1)2^{-n/2}}\rangle_{\mathcal{G}}| \;\leqslant\; 2^{-nH}, \tag{4.2.32}$$

2. *For all integers $r,n\geqslant 1$ and all $t\in\mathbb{R}_+$, and with $C_{H,r}$ a constant depending only on H and r (but independent of t and n),*

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1} |\langle\delta_{(k+1)2^{-n/2}};\delta_{(l+1)2^{-n/2}}\rangle_{\mathcal{G}}|^r \;\leqslant\; C_{H,r}\,t\,2^{n(\frac{1}{2}-rH)}, \tag{4.2.33}$$

3. *For all integer $n\geqslant 1$ and all $t\in\mathbb{R}_+$,*

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1} |\langle\varepsilon_{k2^{-n/2}};\delta_{(l+1)2^{-n/2}}\rangle_{\mathcal{G}}| \;\leqslant\; 2^{-nH-1}+2^{1+n/2}t^{2H+1}, \tag{4.2.34}$$

$$\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor-1} |\langle\varepsilon_{(k+1)2^{-n/2}};\delta_{(l+1)2^{-n/2}}\rangle_{\mathcal{G}}| \;\leqslant\; 2^{-nH-1}+2^{1+n/2}t^{2H+1}, \tag{4.2.35}$$

Proof.

1) We have, for all $0 \leqslant s \leqslant t$ and $i \in \{1,2\}$,

$$E\big(X_u^i(X_t^i - X_s^i)\big) = \frac{1}{2}\big(t^{2H} - s^{2H}\big) + \frac{1}{2}\big(|s-u|^{2H} - |t-u|^{2H}\big).$$

Thanks to (4.2.14), we have

$$\langle \varepsilon_u, \delta_{(j+1)2^{-n/2}} \rangle_{\mathcal{G}} = E\big(X_u^i(X^i_{(j+1)2^{-n/2}} - X^i_{j2^{-n/2}})\big).$$

Since for $H < 1/2$ one has $|b^{2H} - a^{2H}| \leqslant |b-a|^{2H}$ for any $a, b \in \mathbb{R}_+$, we immediately deduce (4.2.32).

2) Thanks to (4.2.14), for $i \in \{1,2\}$, we have that

$$\langle \delta_{(k+1)2^{-n/2}}; \delta_{(l+1)2^{-n/2}} \rangle^r_{\mathcal{G}} = (E[(X^i_{(k+1)2^{-n/2}} - X^i_{k2^{-n/2}})(X^i_{(l+1)2^{-n/2}} - X^i_{l2^{-n/2}})])^r.$$

Thus,

$$\begin{aligned}
&\sum_{k,l=0}^{\lfloor 2^{n/2}t \rfloor - 1} |\langle \delta_{(k+1)2^{-n/2}}; \delta_{(l+1)2^{-n/2}} \rangle_{\mathcal{G}}|^r \\
= \;& 2^{-nrH-r} \sum_{k,l=0}^{\lfloor 2^{n/2}t \rfloor - 1} \big||k-l+1|^{2H} + |k-l-1|^{2H} - 2|k-l|^{2H}\big|^r.
\end{aligned}$$

By first setting $p = k - l$ and then applying Fubini, we get that the latter quantity is equal to:

$$\begin{aligned}
& 2^{-nrH-r} \sum_{p=1-\lfloor 2^{n/2}t \rfloor}^{\lfloor 2^{n/2}t \rfloor - 1} \big||p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H}\big|^r \big((p + \lfloor 2^{n/2}t \rfloor) \wedge \lfloor 2^{n/2}t \rfloor - p \vee 0\big) \\
\leqslant \;& 2^{-nrH-r} \lfloor 2^{n/2}t \rfloor \sum_{p=1-\lfloor 2^{n/2}t \rfloor}^{\lfloor 2^{n/2}t \rfloor - 1} \big||p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H}\big|^r \\
\leqslant \;& C_{H,r}\, t\, 2^{\frac{n}{2} - nrH},
\end{aligned}$$

where $C_{H,r} := \frac{1}{2^r} \sum_{p=-\infty}^{+\infty} \big||p+1|^{2H} + |p-1|^{2H} - 2|p|^{2H}\big|^r$. Observe that $C_{H,r}$ is finite because $H < \frac{1}{2}$. This shows (4.2.33).

3) Thanks to (4.2.14), for $i \in \{1,2\}$, we have that

$$\langle \varepsilon_{k2^{-n/2}}; \delta_{(l+1)2^{-n/2}} \rangle_{\mathcal{G}} = E[X^i_{k2^{-n/2}}(X^i_{(l+1)2^{-n/2}} - X^i_{l2^{-n/2}})].$$

Thus,

$$\begin{aligned}
&\sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} |\langle \varepsilon_{k2^{-n/2}}; \delta_{(l+1)2^{-n/2}}\rangle_{\mathcal{G}}| \\
= \quad & 2^{-nH-1} \sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} |(l+1)^{2H} - l^{2H} + |k-l|^{2H} - |k-l-1|^{2H}| \\
\leqslant \quad & 2^{-nH-1} \sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} |(l+1)^{2H} - l^{2H}| + 2^{-nH-1} \sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} \big||k-l|^{2H} - |k-l-1|^{2H}\big|.
\end{aligned}$$

We have

$$\begin{aligned}
& 2^{-nH-1} \sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} |(l+1)^{2H} - l^{2H}| = 2^{-nH-1}\lfloor 2^{n/2}t\rfloor \sum_{l=0}^{\lfloor 2^{n/2}t\rfloor -1} ((l+1)^{2H} - l^{2H}) \\
= \quad & 2^{-nH-1}\lfloor 2^{n/2}t\rfloor (\lfloor 2^{n/2}t\rfloor)^{2H} \leqslant \frac{1}{2} 2^{n/2} t^{2H+1}. \qquad (4.2.36)
\end{aligned}$$

On the other hand, a telescoping sum argument leads to

$$2^{-nH-1} \sum_{k,l=0}^{\lfloor 2^{n/2}t\rfloor -1} \big||k-l|^{2H} - |k-l-1|^{2H}\big| \leqslant 2^{-nH-1} + 2^{n/2} t^{2H+1}. \qquad (4.2.37)$$

By combining (4.2.36) and (4.2.37) we deduce (4.2.34). The proof of (4.2.35) may be done similarly.

■

Lemma 4.2.6 *Suppose that $H < 1/2$. Then*

1. *For $t \geqslant 0$*

$$\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\left\langle \frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}}\right| \leqslant \frac{1}{2} t^{2H}, \qquad (4.2.38)$$

2. *For $s \in \mathbb{R}$ and $t \geqslant 0$*

$$\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\langle \varepsilon_s, \delta_{(j+1)2^{-n/2}}\rangle_{\mathcal{G}}\right| \leqslant 2t^{2H}, \qquad (4.2.39)$$

Proof.

1) Thanks to (4.2.14) we have

$$\begin{aligned}\left\langle \varepsilon_{j2^{-n/2}}, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}} &= \frac{1}{2} 2^{-nH} (|j+1|^{2H} - |j|^{2H} - 1), \\ \left\langle \varepsilon_{(j+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}} &= \frac{1}{2} 2^{-nH} (|j+1|^{2H} - |j|^{2H} + 1).\end{aligned}$$

Hence, we get that

$$\left\langle \frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}} = \frac{1}{2} 2^{-nH} (|j+1|^{2H} - |j|^{2H}),$$

and, by a telescoping argument, it yields

$$\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t \rfloor - 1} \left| \left\langle \frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}} \right| \leqslant \frac{1}{2} t^{2H},$$

which proves (4.2.38).

2) Thanks to (4.2.14), for $i \in \{1, 2\}$, we have

$$\begin{aligned}&\left\langle \varepsilon_s, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}} = E[X_s^i (X^i_{(j+1)2^{-n/2}} - X^i_{j2^{-n/2}})] \\ = \ & \frac{1}{2} 2^{-nH} \big(|j+1|^{2H} - |j|^{2H} \big) + \frac{1}{2} 2^{-nH} \big(|s2^{-n/2} - j|^{2H} - |s2^{-n/2} - j - 1|^{2H} \big).\end{aligned}$$

We deduce that

1. If $s \geqslant 0$:

   (a) if $s \leqslant t$:

$$\begin{aligned}&\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t \rfloor - 1} \left| \left\langle \varepsilon_s, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}} \right| \\ \leqslant \ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} s \rfloor - 1} \big((s2^{n/2} - j)^{2H} - (s2^{n/2} - j - 1)^{2H} \big) \\ & + \frac{1}{2} 2^{-nH} \big| |s2^{n/2} - \lfloor s2^{\frac{n}{2}} \rfloor|^{2H} - |s2^{n/2} - \lfloor s2^{\frac{n}{2}} \rfloor - 1|^{2H} \big| \\ & + \frac{1}{2} 2^{-nH} \sum_{j=\lfloor 2^{\frac{n}{2}} s \rfloor + 1}^{\lfloor 2^{\frac{n}{2}} t \rfloor - 1} \big((j + 1 - s2^{n/2})^{2H} - (j - s2^{n/2})^{2H} \big) \\ = \ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH} \big((s2^{n/2})^{2H} - (s2^{n/2} - \lfloor s2^{\frac{n}{2}} \rfloor)^{2H} \big) \\ & + \frac{1}{2} 2^{-nH} \big| |s2^{n/2} - \lfloor s2^{\frac{n}{2}} \rfloor|^{2H} - |s2^{n/2} - \lfloor s2^{\frac{n}{2}} \rfloor - 1|^{2H} \big| \\ & + \frac{1}{2} 2^{-nH} \big((\lfloor 2^{\frac{n}{2}} t \rfloor - s2^{n/2})^{2H} - (\lfloor s2^{\frac{n}{2}} \rfloor - s2^{n/2} + 1)^{2H} \big),\end{aligned}$$

where we have obtained the first equality by a telescoping argument. As a consequence of the previous calculation, and since $|b^{2H} - a^{2H}| \leqslant |b-a|^{2H}$ for $H < 1/2$ and $a, b \in \mathbb{R}_+$, we deduce that

$$
\begin{aligned}
&\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\left\langle \varepsilon_s, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}}\right| \\
\leqslant\ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH}(s^{2H} 2^{nH}) + \frac{1}{2} 2^{-nH} 1^{2H} + \frac{1}{2} 2^{-nH}(\lfloor 2^{\frac{n}{2}} t\rfloor - \lfloor 2^{\frac{n}{2}} s\rfloor - 1)^{2H} \\
\leqslant\ & \frac{1}{2} t^{2H} + \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH}(\lfloor 2^{\frac{n}{2}} t\rfloor)^{2H} + \frac{1}{2} 2^{-nH}(\lfloor 2^{\frac{n}{2}} t\rfloor)^{2H} \\
\leqslant\ & 2t^{2H},
\end{aligned}
$$

meaning that (4.2.39) holds true.

(b) if $s > t$: by the same argument that was used in the previous case, we have

$$
\begin{aligned}
&\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\left\langle \varepsilon_s, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}}\right| \\
\leqslant\ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left((s 2^{n/2} - j)^{2H} - (s 2^{n/2} - j - 1)^{2H} \right) \\
=\ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH} \left((s 2^{n/2})^{2H} - (s 2^{n/2} - \lfloor 2^{\frac{n}{2}} t\rfloor)^{2H} \right) \\
\leqslant\ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH}(\lfloor 2^{\frac{n}{2}} t\rfloor)^{2H} \leqslant t^{2H},
\end{aligned}
$$

and (4.2.39) holds true.

2. If $s < 0$:

$$
\begin{aligned}
&\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\left\langle \varepsilon_s, \delta_{(j+1)2^{-n/2}} \right\rangle_{\mathcal{G}}\right| \\
\leqslant\ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left| |s 2^{n/2} - j|^{2H} - |s 2^{n/2} - j - 1|^{2H} \right| \\
=\ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left((j + 1 + (-s) 2^{n/2})^{2H} - (j + (-s) 2^{n/2})^{2H} \right) \\
=\ & \frac{1}{2} t^{2H} + \frac{1}{2} 2^{-nH} \left((\lfloor 2^{\frac{n}{2}} t\rfloor + (-s) 2^{n/2})^{2H} - (-s 2^{n/2})^{2H} \right) \\
\leqslant\ & \frac{1}{2} t^{2H} + \frac{1}{2} \left((t + (-s))^{2H} - (-s)^{2H} \right) \\
\leqslant\ & \frac{1}{2} t^{2H} + \frac{1}{2} t^{2H},
\end{aligned}
$$

where the second equality follows from a telescoping argument and the last inequality follows from the relation $|b^{2H} - a^{2H}| \leqslant |b-a|^{2H}$ for any $a, b \in \mathbb{R}_+$. The last inequality shows that (4.2.39) holds true.

■

Lemma 4.2.7 *Suppose that $f_1, f_2, f_3, f_4 \in C_b^\infty(\mathbb{R}^2)$ and set $H = 1/6$. Fix $t \geqslant 0$. Then*

$$\begin{aligned}
&\sup_{n\geqslant 1} \sup_{i_1,i_2\in\{0,\ldots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}} \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \bigg| E\bigg(\Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2) \\
&\times \Delta_{i_4,n}f_4(X^1,X^2) I_1^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big) I_1^{(1)}\big(\delta_{(i_4+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\bigg)\bigg| \\
\leqslant\ & C(t+t^2),
\end{aligned}$$

where C is a positive constant depending only on f (but independent of n and t).

Proof. By the product formula (4.2.20), we have

$$\begin{aligned}
I_1^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\big) I_1^{(1)}\big(\delta_{(i_4+1)2^{-n/2}}\big) &= I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big) \\
&\quad + \langle \delta_{(i_3+1)2^{-n/2}}, \delta_{(i_4+1)2^{-n/2}}\rangle && (4.2.40)\\
I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big) &= I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big) && (4.2.41)\\
&\quad + 4 I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\langle \delta_{(i_3+1)2^{-n/2}}, \delta_{(i_4+1)2^{-n/2}}\rangle \\
&\quad + 2\langle \delta_{(i_3+1)2^{-n/2}}, \delta_{(i_4+1)2^{-n/2}}\rangle^2.
\end{aligned}$$

Thanks to (4.2.40) we deduce that, for all $i_1, i_2 \in \{0, \ldots, \lfloor 2^{\frac{n}{2}}t\rfloor - 1\}$,

$$\begin{aligned}
&\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \bigg| E\bigg(\Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2) \\
&\times \Delta_{i_4,n}f_4(X^1,X^2) I_1^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big) I_1^{(1)}\big(\delta_{(i_4+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\bigg)\bigg| \\
\leqslant\ & \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \bigg| E\bigg(\Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2) \\
&\times \Delta_{i_4,n}f_4(X^1,X^2) I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\bigg)\bigg| \\
&+ \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \bigg| E\bigg(\Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2) \\
&\times \Delta_{i_4,n}f_4(X^{(1)},X^{(2)}) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\bigg)\bigg| \big|\langle \delta_{(i_3+1)2^{-n/2}}, \delta_{(i_4+1)2^{-n/2}}\rangle\big| \\
=\ & M_{n,1}(i_1,i_2,t) + M_{n,2}(i_1,i_2,t),
\end{aligned}$$

with obvious notation at the last line. Set

$$\phi(i_1,i_2,i_3,i_4) := \Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2)\Delta_{i_4,n}f_4(X^1,X^2).$$

Let us prove that, for $i \in \{1, 2\}$, $\exists C > 0$ such that :

$$\sup_{n\geqslant 0} \sup_{i_1,i_2\in\{0,\ldots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}} M_{n,i}(i_1,i_2,t) \leqslant C(t+t^2). \tag{4.2.42}$$

1. for $i = 1$: thanks to the duality formula (4.2.19), we have

$$M_{n,1}(i_1,i_2,t) = \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\Big(\Big\langle D_{X^1}\big(\phi(i_1,i_2,i_3,i_4)\big), \delta_{(i_3+1)2^{-n/2}} \otimes \delta_{(i_4+1)2^{-n/2}}\Big\rangle \times I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\Big)\Big|.$$

Observe that, thanks to (4.2.16) and (4.2.17), we have

$$D_{X^1}\big(\phi(i_1,i_2,i_3,i_4)\big) = \sum_{j=1}^{4} \phi_j(i_1,i_2,i_3,i_4)\left(\frac{\varepsilon_{i_j 2^{-n/2}} + \varepsilon_{(i_j+1)2^{-n/2}}}{2}\right),$$

where $\phi_j(i_1,i_2,i_3,i_4)$ is a quantity having a similar form as $\phi(i_1,i_2,i_3,i_4)$ and arising when one differentiates $\Delta_{i_j,n} f_j(X^1,X^2)$ in $\phi(i_1,i_2,i_3,i_4)$ with respect to X^1. By combining this fact with (4.2.32), we get

$$\begin{aligned} M_{n,1}(i_1,i_2,t) &\leqslant (2^{-n/6})^2 \sum_{j=1}^{4} \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\Big(\phi_j(i_1,i_2,i_3,i_4) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\Big)\Big| \\ &= \sum_{j=1}^{4} M_{n,1}^{(j)}(i_1,i_2,t) \end{aligned}$$

with obvious notation at the last line. We have to prove that, for all $j \in \{1,\dots,4\}$, one has $\sup_{n\geqslant 0} \sup_{i_1,i_2\in\{0,\dots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}} M_{n,1}^{(j)}(i_1,i_2,t) \leqslant C(t+t^2)$. Let us do it. Thanks to (4.2.41), we have

$$\begin{aligned} M_{n,1}^{(j)}(i_1,i_2,t) &= 2^{-n/3} \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\Big(\phi_j(i_1,i_2,i_3,i_4) I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}} \otimes \delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\Big)\Big| \\ &\quad + 4 2^{-n/3} \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\Big(\phi_j(i_1,i_2,i_3,i_4) I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}} \otimes \delta_{(i_4+1)2^{-n/2}}\big)\Big)\Big| \\ &\qquad \times |\langle \delta_{(i_3+1)2^{-n/2}}, \delta_{(i_4+1)2^{-n/2}}\rangle| \\ &\quad + 2 2^{-n/3} \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\Big(\phi_j(i_1,i_2,i_3,i_4)\Big)\Big| \langle \delta_{(i_3+1)2^{-n/2}}, \delta_{(i_4+1)2^{-n/2}}\rangle^2. \end{aligned}$$

Thanks to the duality formula (4.2.19), to (4.2.17) and to (4.2.32) and since ϕ_j is bounded, we deduce that

$$
\begin{aligned}
& \left|E\left(\phi_j(i_1,i_2,i_3,i_4)I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\right)\right| \\
= & \left|E\left(\langle D^4_{X^2}\big(\phi_j(i_1,i_2,i_3,i_4)\big),\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\rangle\right)\right| \\
& \leqslant C_j(2^{-n/6})^4, \\
& \left|E\left(\phi_j(i_1,i_2,i_3,i_4)I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\right)\right| \\
= & \left|E\left(\langle D^2_{X^2}\big(\phi_j(i_1,i_2,i_3,i_4)\big),\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\rangle\right)\right| \\
& \leqslant C_j(2^{-n/6})^2.
\end{aligned}
$$

By combining these inequalities with (4.2.33), we get

$$
\begin{aligned}
M^{(j)}_{n,1}(i_1,i_2,t) &\leqslant C_j 2^{-n}2^n t^2 + C_j t 2^{-2n/3}2^{n/3} + C_j t 2^{-n/3}2^{n/6} \\
&\leqslant C_j(t+t^2).
\end{aligned}
$$

Hence $\exists C>0$ such that for all $j\in\{1,\dots,4\}$, $\sup_{n\geqslant 0}\sup_{i_1,i_2\in\{0,\dots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}} M^{(j)}_{n,1}(i_1,i_2,t)\leqslant C(t+t^2)$. So, we have the desired conclusion (4.2.42) for $i=1$.

2. for $i=2$: Thanks to (4.2.41), we have

$$
\begin{aligned}
M_{n,2}(i_1,i_2,t) = & \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\left(\phi(i_1,i_2,i_3,i_4)I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\right)\right| \\
& \qquad \times\left|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle\right| \\
& +4\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\left(\phi(i_1,i_2,i_3,i_4)I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\right)\right| \\
& \qquad \times\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2 \\
& +2\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\big(\phi(i_1,i_2,i_3,i_4)\big)\right|\left|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle\right|^3.
\end{aligned}
$$

By the same arguments as used in the previous case, we deduce that

$$
M_{n,2}(i_1,i_2,t)\leqslant C(2^{-n/6})^4 t2^{n/3}+C(2^{-n/6})^2 t2^{n/6}+Ct\leqslant Ct.
$$

Hence, $\exists C>0$ such that $\sup_{n\geqslant 0}\sup_{i_1,i_2\in\{0,\dots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}} M_{n,2}(i_1,i_2,t)\leqslant C(t+t^2)$. So, we have the desired conclusion (4.2.42) for $i=2$. This ends the proof of Lemma 4.2.7.

∎

Lemma 4.2.8 *Suppose that $f_1, f_2, f_3, f_4 \in C_b^\infty(\mathbb{R}^2)$ and set $H = 1/6$. Fix $t \geqslant 0$. Then*

$$\begin{aligned}
&\sup_{n\geqslant 1} \sup_{i_1,i_2\in\{0,\ldots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}} \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \bigg| E\bigg(\Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2)\\
&\times\Delta_{i_4,n}f_4(X^1,X^2) I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\big) I_1^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\big) I_2^{(1)}\big(\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\big) I_1^{(2)}\big(\delta_{(i_4+1)2^{-n/2}}\big)\bigg)\bigg|\\
\leqslant\ & C(t+t^2),
\end{aligned}$$

where C is a positive constant depending only on f (but independent of n and t).

Proof. By symmetry, the proof is very similar to the proof of Lemma 4.2.7 and is left to the reader. ■

Lemma 4.2.9 *Suppose that $f_1, f_2, f_3, f_4 \in C_b^\infty(\mathbb{R}^2)$ and set $H = 1/6$. Fix $t \geqslant 0$. For $i \in \{1,2\}$, we have*

$$\begin{aligned}
&\sup_{n\geqslant 1} \sup_{i_1,i_2\in\{0,\ldots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}} \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \bigg| E\bigg(\Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2)\\
&\qquad\qquad\times\Delta_{i_4,n}f_4(X^1,X^2) I_3^{(i)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 3}\big) I_3^{(i)}\big(\delta_{(i_4+1)2^{-n/2}}^{\otimes 3}\big)\bigg)\bigg|\\
\leqslant\ & C(t+t^2),
\end{aligned}$$

where C is a positive constant depending only on f (but independent of n and t).

Proof. We will consider only the case $i = 2$ (by symmetry, the proof is very similar for $i = 1$). Set

$$\phi(i_1,i_2,i_3,i_4) := \Delta_{i_1,n}f_1(X^1,X^2)\Delta_{i_2,n}f_2(X^1,X^2)\Delta_{i_3,n}f_3(X^1,X^2) \quad \Delta_{i_4,n}f_4(X^1,X^2).$$

Using the product formula (4.2.20), we have that $I_3^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 3}\big) I_3^{(2)}\big(\delta_{(i_4+1)2^{-n/2}}^{\otimes 3}\big)$ equals

$$\begin{aligned}
&I_6^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 3}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 3}\big) + 9 I_4^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\big)\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle\\
&+18 I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2 + 6\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^3.
\end{aligned}$$

So, we get

$$
\begin{aligned}
&\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_3+1)2^{-n/2}}\big)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_4+1)2^{-n/2}}\big)\Big)\right|\\
=&\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)I_6^{(2)}\big(\delta^{\otimes 3}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 3}_{(i_4+1)2^{-n/2}}\big)\Big)\right|\\
&+9\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\Big)\right||\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|\\
&+18\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\Big)\right|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2\\
&+6\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)\Big)\right||\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|^3.
\end{aligned}
$$

Thanks to the duality formula (4.2.19), we get

$$
\begin{aligned}
&\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_3+1)2^{-n/2}}\big)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_4+1)2^{-n/2}}\big)\Big)\right|\\
=&\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\big\langle D^6_{X^2}\big(\phi(i_1,i_2,i_3,i_4)\big),\delta^{\otimes 3}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 3}_{(i_4+1)2^{-n/2}}\big\rangle\Big)\right|\\
&+9\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\big\langle D^4_{X^2}\big(\phi(i_1,i_2,i_3,i_4)\big),\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big\rangle\Big)\right|\\
&\qquad\times|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|\\
&+18\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\langle D^2_{X^2}\big(\phi(i_1,i_2,i_3,i_4)\big),\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\rangle\Big)\right|\\
&\qquad\times\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2\\
&+6\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)\Big)\right||\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|^3.
\end{aligned}
$$

Observe that, thanks to (4.2.16) and (4.2.17), for any $k\in\{1,2,3\}$, we have

$$
\begin{aligned}
&D^{2k}_{X^2}\big(\phi(i_1,i_2,i_3,i_4)\big) \qquad (4.2.43)\\
=&\sum_{a_1+a_2+a_3+a_4=2k}\phi_{(a_1,a_2,a_3,a_4)}(i_1,i_2,i_3,i_4)\left(\frac{\varepsilon_{i_1 2^{-n/2}}+\varepsilon_{(i_1+1)2^{-n/2}}}{2}\right)^{\otimes a_1}\\
&\tilde{\otimes}\left(\frac{\varepsilon_{i_2 2^{-n/2}}+\varepsilon_{(i_2+1)2^{-n/2}}}{2}\right)^{\otimes a_2}\tilde{\otimes}\left(\frac{\varepsilon_{i_3 2^{-n/2}}+\varepsilon_{(i_3+1)2^{-n/2}}}{2}\right)^{\otimes a_3}\tilde{\otimes}\left(\frac{\varepsilon_{i_4 2^{-n/2}}+\varepsilon_{(i_4+1)2^{-n/2}}}{2}\right)^{\otimes a_4}
\end{aligned}
$$

where $(a_1,a_2,a_3,a_4)\in\mathbb{N}^4$ and $\phi_{(a_1,a_2,a_3,a_4)}(i_1,i_2,i_3,i_4)$ is a quantity having a similar form as $\phi(i_1,i_2,i_3,i_4)$ and arising when one differentiates $\Delta_{i_j,n}f_j(X^1,X^2)$ in $\phi(i_1,i_2,i_3,i_4)$ a_j-times with respect to X^2. Thanks to (4.2.43), (4.2.32), (4.2.33) and since $\phi_{(a_1,a_2,a_3,a_4)}(i_1,i_2,i_3,i_4)$ is bounded, we deduce that

$$
\begin{aligned}
&\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_3+1)2^{-n/2}}\big)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_4+1)2^{-n/2}}\big)\Big)\right|\\
\leqslant\ & C(2^{-n/6})^6\,2^n\,t^2+C(2^{-n/6})^4\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|\\
&+C(2^{-n/6})^2\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\langle\delta_{(i_3+1)2^{-n/2}}\delta_{(i_4+1)2^{-n/2}}\rangle^2+C\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\langle\delta_{(i_3+1)2^{-n/2}}\delta_{(i_4+1)2^{-n/2}}\rangle|^3\\
\leqslant\ & Ct^2+C2^{-n/3}t+C2^{-n/6}t+Ct.
\end{aligned}
$$

Hence, we deduce immediately that

$$
\sup_{n\geqslant 1}\ \sup_{i_1,i_2\in\{0,\dots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}}\ \sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_3+1)2^{-n/2}}\big)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_4+1)2^{-n/2}}\big)\Big)\right|\leqslant C(t+t^2),
$$

which end the proof of Lemma 4.2.9. ∎

Lemma 4.2.10 *Suppose that $f_1,f_2,f_3,f_4\in C_b^\infty(\mathbb{R}^2)$ and set $H=1/6$. Fix $t\geqslant 0$. Then*

$$
\begin{aligned}
&\sup_{n\geqslant 1}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\prod_{a=1}^4\Delta_{i_a,n}f_a(X^1,X^2)I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big)\right|\\
\leqslant\ & C(t+t^2+t^3+t^4),
\end{aligned}
$$

where C is a positive constant depending only on f (but independent of n and t).

Proof. The proof, which is quite long and rather technical, is postponed in Section 5. ∎

Lemma 4.2.11 *Suppose that $f_1,f_2,f_3,f_4\in C_b^\infty(\mathbb{R}^2)$ and set $H=1/6$. Fix $t\geqslant 0$. Then*

$$
\begin{aligned}
&\sup_{n\geqslant 1}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\prod_{a=1}^4\Delta_{i_a,n}f_a(X^1,X^2)I_2^{(1)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\Big)\right|\\
\leqslant\ & C(t+t^2+t^3+t^4),
\end{aligned}
$$

where C is a positive constant depending only on f (but independent of n and t).

Proof. By symmetry, the proof is very similar to the proof of Lemma 4.2.10 and is left to the reader. ∎

Lemma 4.2.12 *Suppose that $f_1, f_2, f_3, f_4 \in C_b^\infty(\mathbb{R}^2)$ and set $H = 1/6$. Fix $t \geqslant 0$. For $i \in \{1,2\}$, we have*

$$\begin{aligned}&\sup_{n\geqslant 1} \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \left| E\Big(\prod_{a=1}^4 \Delta_{i_a,n} f_a(X^1,X^2) I_3^{(i)}\big(\delta^{\otimes 3}_{(i_a+1)2^{-n/2}}\big)\Big)\right| \\ \leqslant\ & C(t+t^2+t^3+t^4),\end{aligned}$$

where C is a positive constant depending only on f (but independent of n and t).

Proof. The proof is similar to the proof of Lemma 4.2.10 and is left to the reader. See also [30, Lemma 3.5] for a very similar result. ■

Lemma 4.2.13 *Suppose $H = 1/6$. Then*

1. *For all $j \in \mathbb{N}$*

$$\begin{aligned} \left|\left\langle D_{X^2}\xi, \delta_{(j+1)2^{-n/2}}\right\rangle\right| &\leqslant C2^{-n/6}, && (4.2.44)\\ \left|\left\langle D^2_{X^2}\xi, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\right| &\leqslant C2^{-n/3}, && (4.2.45)\end{aligned}$$

   where C in a positive constant depending only on ψ introduced in (4.2.24).

2. *For all $j \in \mathbb{N}$, for $i \in \{1,2,3,4\}$*

$$E[\langle D_{X^2} K_n^{(i)}(f,t), \delta_{(j+1)2^{-n/2}}\rangle^2] \leqslant C2^{-n/3}(t^2+t+1), \qquad (4.2.46)$$

   where C in a positive constant depending only on f.

3. *For all $j \in \mathbb{N}$, for $i \in \{1,2,3,4\}$*

$$E[\langle D^2_{X^2}\big(K_n^{(i)}(f,t)\big), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\rangle^2] \leqslant C2^{-2n/3}(t^2+t+1), \qquad (4.2.47)$$

   where C in a positive constant depending only on f.

4. *For all $j \in \mathbb{N}$, for $i \in \{1,2,3,4\}$*

$$E[\langle D_{X^2}\big(K_n^{(i)}(f,t)\big), \delta_{(j+1)2^{-n/2}}\rangle^4] \leqslant C2^{-2n/3}(t^3+t^2+t+1), \qquad (4.2.48)$$

   where C in a positive constant depending only on f.

Proof.

1) Observe that, thanks to (4.2.15) (see also the example given after (4.2.15)), we have

$$D_{X^2}\xi = \sum_{k=1}^r \frac{\partial \psi}{\partial x_{2k}}\big(X^1_{s_1}, X^2_{s_1}, \dots, X^1_{s_r}, X^2_{s_r}\big)\varepsilon_{s_k},$$

As a consequence, since ψ is bounded and thanks to (4.2.32), we deduce that

$$\begin{aligned}\left|\left\langle D_{X^2}\xi, \delta_{(j+1)2^{-n/2}}\right\rangle\right| &\leqslant C\sum_{k=1}^r |\langle \varepsilon_{s_k}, \delta_{(j+1)2^{-n/2}}\rangle| \\ &\leqslant C2^{-n/6},\end{aligned}$$

which proves (4.2.44). On the other hand, we have

$$D^2_{X^2}\xi = \sum_{k,k'=1}^{r} \frac{\partial^2\psi}{\partial x_{2k}\partial x_{2k'}}\big(X^1_{s_1}, X^2_{s_1}, \ldots, X^1_{s_r}, X^2_{s_r}\big)\varepsilon_{s_k}\otimes\varepsilon_{s_{k'}}.$$

So, as previously, we deduce that

$$\big|\big\langle D^2_{X^2}\xi, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big| \leqslant C(2^{-n/6})^2 = C2^{-n/3},$$

which proves (4.2.45).

2) We will prove (4.2.46) for $i = 2, 3$. The proof is similar for the other values of i.

(a) For $i = 2$: Thanks to (4.2.16) and (4.2.22), we have

$$\begin{aligned}
&D_{X^2}K_n^{(2)}(f,t)\\
= &\frac{1}{24}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{2222}f(X^1,X^2)I_3^{(2)}\big(\delta^{\otimes 3}_{(l+1)2^{-n/2}}\big)\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)\\
&+\frac{1}{8}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)\delta_{(l+1)2^{-n/2}}.
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
&E\big[\big\langle D_{X^2}K_n^{(2)}(f,t), \delta_{(j+1)2^{-n/2}}\big\rangle^2\big]\\
\leqslant\ &2(\frac{1}{24})^2\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|E[\Delta_{l,n}\partial_{2222}f(X^1,X^2)\Delta_{k,n}\partial_{2222}f(X^1,X^2)\\
&\qquad\times I_3^{(2)}\big(\delta^{\otimes 3}_{(l+1)2^{-n/2}}\big)I_3^{(2)}\big(\delta^{\otimes 3}_{(k+1)2^{-n/2}}\big)]\big|\\
&\qquad\times\left|\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right), \delta_{(j+1)2^{-n/2}}\right\rangle\right|\\
&\qquad\times\left|\left\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right), \delta_{(j+1)2^{-n/2}}\right\rangle\right|\\
&+2(\frac{1}{8})^2\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|E[\Delta_{l,n}\partial_{222}f(X^1,X^2)\Delta_{k,n}\partial_{222}f(X^1,X^2)\\
&\qquad\times I_2^{(2)}\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(k+1)2^{-n/2}}\big)]\big|\\
&\qquad\times\big|\big\langle\delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big|\big|\big\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big|.
\end{aligned}$$

By Lemma 4.2.9, (4.2.32), (4.2.14), (4.2.23) and since

$$\big|E[\Delta_{l,n}\partial_{222}f(X^1,X^2)\Delta_{k,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(k+1)2^{-n/2}}\big)]\big|$$

is uniformly bounded in n $\big($because $f\in C_b^\infty(\mathbb{R}^2)$ and thanks to the Cauchy-Schwarz inequality and to (4.2.21) $\big)$, we deduce that

$$\begin{aligned}
E\big[\big\langle D_{X^2}K_n^{(2)}(f,t), \delta_{(j+1)2^{-n/2}}\big\rangle^2\big] &\leqslant C(2^{-n/6})^2(t+t^2) + C(2^{-n/6})^2(\sum_{n\in\mathbb{Z}}|\rho(n)|)^2\\
&\leqslant C2^{-n/3}(t^2+t+1),
\end{aligned}$$

which proves (4.2.46) for $i=2$.

(b) For $i=3$: Thanks to (4.2.16) and (4.2.22), we have

$$\begin{aligned}
&D_{X^2}K_n^{(3)}(f,t)\\
&= \frac{1}{8}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{1222}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big)\\
&\qquad\times\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)\\
&\quad+\frac{1}{4}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{122}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)\delta_{(l+1)2^{-n/2}}.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
&E[\langle D_{X^2}K_n^{(3)}(f,t),\delta_{(j+1)2^{-n/2}}\rangle^2]\\
&\leqslant 2(\frac{1}{8})^2\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|E[\Delta_{l,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)\\
&\qquad\times I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(k+1)2^{-n/2}}^{\otimes 2}\big)]\big|\\
&\qquad\times\left|\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\right|\\
&\qquad\times\left|\left\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\right|\\
&\quad+2(\frac{1}{4})^2\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|E[\Delta_{l,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
&\qquad\times I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(k+1)2^{-n/2}}\big)]\big|\\
&\qquad\times\big|\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|.
\end{aligned}$$

Since $f\in C_b^\infty(\mathbb{R}^2)$, since X^1 and X^2 are independent, and thanks to the Cauchy-Schwarz inequality and to (4.2.21), we have

$$\begin{aligned}
&\big|E[\Delta_{l,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)\\
&\qquad\times I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(k+1)2^{-n/2}}\big)]\big|\\
&\leqslant CE\big(\big|I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)\big|\big|I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)\big|\big)E\big(\big|I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)\big|\big|I_1^{(2)}\big(\delta_{(k+1)2^{-n/2}}\big)\big|\big)\\
&\leqslant C\|I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)\|_2\|I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)\|_2\|I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)\|_2\|I_1^{(2)}\big(\delta_{(k+1)2^{-n/2}}\big)\|_2\\
&\leqslant C(2^{-n/12})^4\leqslant C.
\end{aligned}$$

Thanks to the previous estimation, to Lemma 4.2.7 and to (4.2.32) (see also (4.2.23) for the definition of ρ), we deduce that

$$\begin{aligned}
E[\langle D_{X^2}K_n^{(3)}(f,t),\delta_{(j+1)2^{-n/2}}\rangle^2] &\leqslant C(2^{-n/6})^2(t+t^2)+C(2^{-n/6})^2(\sum_{n\in\mathbb{Z}}|\rho(n)|)^2\\
&\leqslant C2^{-n/3}(t^2+t+1),
\end{aligned}$$

which proves (4.2.46) for $i=3$.

Finally, we have proved (4.2.46).

3) We will prove (4.2.47) for $i=2,3$. The proof is similar for the other values of i.

(a) For $i=2$: Thanks to (4.2.16) and (4.2.22), we have

$$
\begin{aligned}
& D^2_{X^2}\big(K_n^{(2)}(f,t)\big)\\
= \;& \frac{1}{24}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{22222}f(X^1,X^2)I_3^{(2)}\big(\delta^{\otimes 3}_{(l+1)2^{-n/2}}\big)\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)^{\otimes 2}\\
& +\frac{1}{4}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{2222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(l+1)2^{-n/2}}\\
& +\frac{1}{4}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{222}f(X^1,X^2)I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)\delta^{\otimes 2}_{(l+1)2^{-n/2}}.
\end{aligned}
$$

So, we have

$$
\begin{aligned}
& \big\langle D^2_{X^2}\big(K_n^{(2)}(f,t)\big),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle^2\\
\leqslant \;& C\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{22222}f(X^1,X^2)\Delta_{k,n}\partial_{22222}f(X^1,X^2)I_3^{(2)}\big(\delta^{\otimes 3}_{(l+1)2^{-n/2}}\big)\\
& \times I_3^{(2)}\big(\delta^{\otimes 3}_{(k+1)2^{-n/2}}\big)\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle^2\\
& \times\left\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle^2\\
& +C\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{2222}f(X^1,X^2)\Delta_{k,n}\partial_{2222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)\\
& \times I_2^{(2)}\big(\delta^{\otimes 2}_{(k+1)2^{-n/2}}\big)\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\\
& \times\left\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\\
& +C\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{222}f(X^1,X^2)\Delta_{k,n}\partial_{222}f(X^1,X^2)I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)\\
& \times I_1^{(2)}\big(\delta_{(k+1)2^{-n/2}}\big)\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle^2\big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle^2.
\end{aligned}
$$

Thanks to (4.2.32) (see also (4.2.23) for the definition of ρ), we get

$$\begin{aligned}
& E\big(\big\langle D^2_{X^2}\big(K^{(2)}_n(f,t)\big),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle^2\big) \\
\leqslant\ & C(2^{-n/6})^4\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|E\big(\Delta_{l,n}\partial_{22222}f(X^1,X^2)\Delta_{k,n}\partial_{22222}f(X^1,X^2)I^{(2)}_3\big(\delta^{\otimes 3}_{(l+1)2^{-n/2}}\big) \\
& \times I^{(2)}_3\big(\delta^{\otimes 3}_{(k+1)2^{-n/2}}\big)\big)\big| \\
& +C(2^{-n/6})^4\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|E\big(\Delta_{l,n}\partial_{2222}f(X^1,X^2)\Delta_{k,n}\partial_{2222}f(X^1,X^2)I^{(2)}_2\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big) \\
& \times I^{(2)}_2\big(\delta^{\otimes 2}_{(k+1)2^{-n/2}}\big)\big)\big||\rho(l-j)||\rho(k-j)| \\
& +C(2^{-n/6})^4\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|E\big(\Delta_{l,n}\partial_{222}f(X^1,X^2)\Delta_{k,n}\partial_{222}f(X^1,X^2)I^{(2)}_1\big(\delta_{(l+1)2^{-n/2}}\big) \\
& \times I^{(2)}_1\big(\delta_{(k+1)2^{-n/2}}\big)\big)\big||\rho(l-j)|^2|\rho(k-j)|^2 \\
\leqslant\ & C2^{-2n/3}(t+t^2)+C2^{-2n/3}\big(\sum_{r\in\mathbb{Z}}|\rho(r)|\big)^2+C2^{-2n/3}\big(\sum_{r\in\mathbb{Z}}|\rho(r)|^2\big)^2 \\
\leqslant\ & C2^{-2n/3}(1+t+t^2),
\end{aligned}$$

where we the second inequality follows by Lemma 4.2.9 and by the uniformly boundedness on n of :
$\big|E\big(\Delta_{l,n}\partial_{2222}f(X^1,X^2)\Delta_{k,n}\partial_{2222}f(X^1,X^2)I^{(2)}_2\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)I^{(2)}_2\big(\delta^{\otimes 2}_{(k+1)2^{-n/2}}\big)\big)\big|$
and $\big|E\big(\Delta_{l,n}\partial_{222}f(X^1,X^2)\Delta_{k,n}\partial_{222}f(X^1,X^2)I^{(2)}_1\big(\delta_{(l+1)2^{-n/2}}\big)I^{(2)}_1\big(\delta_{(k+1)2^{-n/2}}\big)\big)\big|$.
We deduce from the last inequality that (4.2.47) holds true for $i=2$.

(b) For $i=3$: Thanks to (4.2.16) and (4.2.22), we have

$$\begin{aligned}
& D^2_{X^2}\big(K^{(3)}_n(f,t)\big) \\
=\ & \frac{1}{8}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{12222}f(X^1,X^2)I^{(1)}_1\big(\delta_{(l+1)2^{-n/2}}\big)I^{(2)}_2\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big) \\
& \qquad\times\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)^{\otimes 2} \\
& +\frac{1}{2}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{1222}f(X^1,X^2)I^{(1)}_1\big(\delta_{(l+1)2^{-n/2}}\big)I^{(2)}_1\big(\delta_{(l+1)2^{-n/2}}\big) \\
& \qquad\times\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(l+1)2^{-n/2}} \\
& +\frac{1}{4}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{122}f(X^1,X^2)I^{(1)}_1\big(\delta_{(l+1)2^{-n/2}}\big)\delta^{\otimes 2}_{(l+1)2^{-n/2}}.
\end{aligned}$$

So, we deduce that

$$
\begin{aligned}
&\big\langle D^2_{X^2}\big(K_n^{(3)}(f,t)\big), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & C\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Delta_{l,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{12222}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big) \\
&\qquad \times I_2^{(2)}\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(k+1)2^{-n/2}}\big) \\
&\qquad \times\Big\langle\Big(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\Big), \delta_{(j+1)2^{-n/2}}\Big\rangle^2 \\
&\qquad \times\Big\langle\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big), \delta_{(j+1)2^{-n/2}}\Big\rangle^2 \\
& +C\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Delta_{l,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big) \\
&\qquad \times I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(k+1)2^{-n/2}}\big) \\
&\qquad \times\Big\langle\Big(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\Big), \delta_{(j+1)2^{-n/2}}\Big\rangle \\
&\qquad \times\Big\langle\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big), \delta_{(j+1)2^{-n/2}}\Big\rangle \\
&\qquad \times\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle \\
& +C\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Delta_{l,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big) \\
& \times I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle^2\big\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle^2.
\end{aligned}
$$

Thanks to (4.2.32), we get

$$
\begin{aligned}
&E\Big(\big\langle D^2_{X^2}\big(K_n^{(3)}(f,t)\big), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle^2\Big) \\
\leqslant\ & C(2^{-n/6})^4\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\big(\Delta_{l,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{12222}f(X^1,X^2) \\
& \times I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(k+1)2^{-n/2}}\big)\big)\Big| \\
& +C(2^{-n/6})^4\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\big(\Delta_{l,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2) \\
& \times I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(k+1)2^{-n/2}}\big)\big)\Big| \\
& \times|\rho(l-j)||\rho(k-j)| \\
& +C(2^{-n/6})^4\sum_{l,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Big|E\big(\Delta_{l,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2) \\
& \times I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(k+1)2^{-n/2}}\big)\big)\Big||\rho(l-j)|^2|\rho(k-j)|^2.
\end{aligned}
$$

By Lemma 4.2.7 and similar arguments as in the case $i=2$, we get

$$\begin{aligned}
& E\big(\langle D^2_{X^2}\big(K_n^{(3)}(f,t)\big),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\rangle^2\big)\\
\leqslant\ & C2^{-2n/3}(t+t^2)+C2^{-2n/3}\big(\sum_{r\in\mathbb{Z}}|\rho(r)|\big)^2+C2^{-2n/3}\big(\sum_{r\in\mathbb{Z}}|\rho(r)|^2\big)^2\\
\leqslant\ & C2^{-2n/3}(1+t+t^2).
\end{aligned}$$

Thus, (4.2.47) holds true for $i=3$.

4) We will prove (4.2.48) for $i=2,3$. The proof is similar for the other values of i.

(a) For $i=2$:

$$\begin{aligned}
& \langle D_{X^2}(K_n^{(2)}(f,t)),\delta_{(j+1)2^{-n/2}}\rangle^4\\
\leqslant\ & C\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{2222}f(X^1,X^2)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_a+1)2^{-n/2}}\big)\\
& \times\prod_{a=1}^{4}\Big\langle\frac{\varepsilon_{i_a2^{-n/2}}+\varepsilon_{(i_a+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\\
& +C\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\\
& \times\prod_{a=1}^{4}\big\langle\delta_{(i_a+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& E\big(\langle D_{X^2}(K_n^{(2)}(f,t)),\delta_{(j+1)2^{-n/2}}\rangle^4\big)\\
\leqslant\ & C\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{2222}f(X^1,X^2)I_3^{(2)}\big(\delta^{\otimes 3}_{(i_a+1)2^{-n/2}}\big)\bigg)\bigg|\\
& \times\prod_{a=1}^{4}\bigg|\Big\langle\frac{\varepsilon_{i_a2^{-n/2}}+\varepsilon_{(i_a+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\bigg|\\
& +C\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\bigg)\bigg|\\
& \times\prod_{a=1}^{4}\big|\big\langle\delta_{(i_a+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
=\ & Z_{n,1}^{(2)}(t)+Z_{n,2}^{(2)}(t).
\end{aligned}$$

Thanks to (4.2.32) and to Lemma 4.2.12, we have

$$Z_{n,1}^{(2)}(t)\leqslant C(2^{-n/6})^4(t+t^2+t^3)=C2^{-2n/3}(t+t^2+t^3).$$

On the other hand,

$$\Big|E\Big(\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big)\Big|$$

is uniformly bounded in n. In fact, since $f\in C_b^\infty$ and by applying the Cauchy-Schwarz inequality two times, we get

$$\Big|E\Big(\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big)\Big|\leqslant\prod_{a=1}^{4}\|I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\|_{L^4}.$$

Thanks to the Hypercontractivity property of the pth multiple integral with $p\geqslant 1$ (see for example Theorem 2.7.2 in [28]), we have for $a\in\{1,\dots,4\}$,

$$\|I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\|_{L^4}\leqslant C\|I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\|_{L^2}\leqslant C2^{-n/6},$$

where C is some positive and finite constant, and we have the last inequality thanks to (4.2.21). We deduce immediately from the last inequality that $\exists C>0$ such that for all $n\in\mathbb{N}$

$$\Big|E\Big(\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big)\Big|\leqslant C. \tag{4.2.49}$$

So, we get

$$\begin{aligned}
Z_{n,2}^{(2)}(t) &\leqslant C(2^{-n/6})^4\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\rho(i_1-j)||\rho(i_2-j)||\rho(i_3-j)||\rho(i_4-j)|\\
&\leqslant C2^{-2n/3}\big(\sum_{r\in\mathbb{Z}}|\rho(r)|\big)^4=C2^{-2n/3}.
\end{aligned}$$

Consequently, we deduce that $E\big(\langle D_{X^2}(K_n^{(2)}(f,t)),\delta_{(j+1)2^{-n/2}}\rangle^4\big)\leqslant C2^{-2n/3}(1+t+t^2+t^3)$. Hence, we have proved (4.2.48) for $i=2$.

(b) For $i=3$:

$$\begin{aligned}
&\langle D_{X^2}(K_n^{(3)}(f,t)),\delta_{(j+1)2^{-n/2}}\rangle^4\\
\leqslant\;& C\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{1222}f(X^1,X^2)I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\\
&\times\prod_{a=1}^{4}\Big\langle\frac{\varepsilon_{i_a2^{-n/2}}+\varepsilon_{(i_a+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\\
&+C\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\prod_{a=1}^{4}\Delta_{i_a,n}\partial_{122}f(X^1,X^2)I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\\
&\times\prod_{a=1}^{4}\big\langle\delta_{(i_a+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle.
\end{aligned}$$

Thus, we have

$$
\begin{aligned}
& E\big(\langle D_{X^2}(K_n^{(3)}(f,t)), \delta_{(j+1)2^{-n/2}}\rangle^4\big) \\
\leqslant\ & C \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left| E\Big(\prod_{a=1}^{4} \Delta_{i_a,n}\partial_{1222} f(X^1,X^2) I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big) I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\Big)\right| \\
& \times \prod_{a=1}^{4} \left| \left\langle \frac{\varepsilon_{i_a 2^{-n/2}} + \varepsilon_{(i_a+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \right\rangle \right| \\
& + C \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left| E\Big(\prod_{a=1}^{4} \Delta_{i_a,n}\partial_{122} f(X^1,X^2) I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big) I_1^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\Big)\right| \\
& \times \prod_{a=1}^{4} \left|\langle \delta_{(i_a+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle\right| \\
=\ & Z_{n,1}^{(3)}(t) + Z_{n,2}^{(3)}(t),
\end{aligned}
$$

with obvious notation at the last line. Thanks to (4.2.32) and to Lemma 4.2.10, we have

$$
Z_{n,1}^{(3)}(t) \leqslant C(2^{-n/6})^4 (t+t^2+t^3) = C2^{-2n/3}(t+t^2+t^3).
$$

On the other hand, we have that

$$
\left| E\Big(\prod_{a=1}^{4} \Delta_{i_a,n}\partial_{122} f(X^1,X^2) I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big) I_1^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\Big)\right|
$$

is uniformly bounded in n (by the same arguments that has been used to prove (4.2.49)). Consequently, we get

$$
\begin{aligned}
Z_{n,2}^{(3)}(t) &\leqslant C(2^{-n/6})^4 \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} |\rho(i_1-j)||\rho(i_2-j)||\rho(i_3-j)||\rho(i_4-j)| \\
&\leqslant C2^{-2n/3} \Big(\sum_{r\in\mathbb{Z}} |\rho(r)|\Big)^4 = C2^{-2n/3}.
\end{aligned}
$$

We deduce that $E\big(\langle D_{X^2}(K_n^{(3)}(f,t)), \delta_{(j+1)2^{-n/2}}\rangle^4\big) \leqslant C2^{-2n/3}(1+t+t^2+t^3)$. Hence (4.2.48) holds true for $i=3$.

Finally, we have proved (4.2.48), which ends the proof of Lemma 4.2.13.

■

4.3 Proof of Theorem 4.1.3

We suppose $H < 1/2$. The proof in the case $H \geqslant 1/2$ is immediate and, consequently, is left to the reader.

Definition 4.3.1 *For all $p, q \in \mathbb{N}$ such that $p+q$ is odd, we define $V_n^{p,q}(f,t)$ as follows:*

$$V_n^{p,q}(f,t) := \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n} f(X^1, X^2)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big)^p \big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big)^q. \tag{4.3.50}$$

We have the following proposition which will play a pivotal role in the sequel :

Proposition 4.3.2 *If ($H > 1/6$ and $p+q \geqslant 3$) or if ($H = 1/6$ and $p+q \geqslant 5$), then*

$$V_n^{p,q}(f,t) \xrightarrow{L^2} 0, \quad \textit{as } n \to \infty.$$

Proof. We suppose that p is even and q is odd (the proof when p is odd and q is even is exactly the same). We have, for all $k \in \mathbb{N}^*$, $x^{2k} = \sum_{i=1}^{k} b_{2k,i} H_{2i}(x) + b_{2k,0}$ and $x^{2k-1} = \sum_{i=1}^{k} a_{2k-1,i} H_{2i-1}(x)$, where H_n is the nth Hermite polynomial, $b_{2k,i}$ and $a_{2k-1,i}$ are some explicit constants (if interested, the reader can find these explicit constants, in Corollary 2.1.2 in chapter 2). Set

$$\phi(j,j') := \Delta_{j,n} f(X^1, X^2) \Delta_{j',n} f(X^1, X^2).$$

Recall that for $i \in \{1,2\}$ we denote $X_j^{i,n} := 2^{\frac{nH}{2}} X^i_{j2^{-\frac{n}{2}}}$. We distinguish two cases: if $p \neq 0$ and if $p = 0$,

1. <u>If $p \neq 0$:</u> Then, we have

$$\begin{aligned} V_n^{p,q}(f,t) &= b_{p,0} 2^{-\frac{nH(p+q)}{2}} \sum_{k'=1}^{\frac{q+1}{2}} a_{q,k'} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n} f(X^1, X^2) H_{2k'-1}\big(X^{2,n}_{j+1} - X^{2,n}_j\big) \\ &\quad + 2^{-\frac{nH(p+q)}{2}} \sum_{k=1}^{\frac{p}{2}} \sum_{k'=1}^{\frac{q+1}{2}} a_{q,k'} b_{p,k} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n} f(X^1, X^2) H_{2k}\big(X^{1,n}_{j+1} - X^{1,n}_j\big) \\ &\qquad \times H_{2k'-1}\big(X^{2,n}_{j+1} - X^{2,n}_j\big) \\ &= b_{p,0} \sum_{k'=1}^{\frac{q+1}{2}} a_{q,k'} V_{n,1}(f,k',t) + \sum_{k=1}^{\frac{p}{2}} \sum_{k'=1}^{\frac{q+1}{2}} a_{q,k'} b_{p,k} V_{n,2}(f,k,k',t), \end{aligned} \tag{4.3.51}$$

with obvious notation at the last equality. Let us now prove the convergence to 0 as n tends to infinity, in the L^2 sense, of $V_{n,1}(f,k',t)$ and $V_{n,2}(f,k,k',t)$ for all $k \in \{1,\dots,\frac{p}{2}\}$ and $k' \in \{1,\dots,\frac{q+1}{2}\}$,

(a) Convergence to 0, in L^2, of $V_{n,1}(f,k',t)$: For all $k' \in \{1,\ldots,\frac{q+1}{2}\}$,

$$
\begin{aligned}
& E\big[\big(V_{n,1}(f,k',t)\big)^2\big] \\
= \;& 2^{-nH(p+q)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\phi(j,j')H_{2k'-1}\big(X^{2,n}_{j+1}-X^{2,n}_{j}\big)H_{2k'-1}\big(X^{2,n}_{j'+1}-X^{2,n}_{j'}\big)\Big) \\
= \;& 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\phi(j,j')I^{(2)}_{2k'-1}\big(\delta^{\otimes 2k'-1}_{(j+1)2^{-n/2}}\big)I^{(2)}_{2k'-1}\big(\delta^{\otimes 2k'-1}_{(j'+1)2^{-n/2}}\big)\Big) \\
= \;& 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{a=0}^{2k'-1} a!\binom{2k'-1}{a}^2 \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\phi(j,j') \\
& \times I^{(2)}_{4k'-2-2a}\big(\delta^{\otimes 2k'-1-a}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes 2k'-1-a}_{(j'+1)2^{-n/2}}\big)\Big)\langle\delta_{(j+1)2^{-n/2}},\delta_{(j'+1)2^{-n/2}}\rangle^a \\
= \;& 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{a=0}^{2k'-1} a!\binom{2k'-1}{a}^2 \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Big\langle D^{4k'-2-2a}_{X^2}\big(\phi(j,j')\big), \\
& \delta^{\otimes 2k'-1-a}_{(j+1)2^{-n/2}}\otimes\delta^{\otimes 2k'-1-a}_{(j'+1)2^{-n/2}}\Big\rangle\Big)\langle\delta_{(j+1)2^{-n/2}},\delta_{(j'+1)2^{-n/2}}\rangle^a \\
= \;& \sum_{a=0}^{2k'-1} a!\binom{2k'-1}{a}^2 Q_n^{(k',a)}(t),
\end{aligned}
$$

with obvious notation at the last equality and with the second equality following from (4.2.18), the third one from (4.2.20) and the fourth one from (4.2.19). Recall that $f\in C_b^\infty$. We have the following estimates.

- Case $a=2k'-1$

$$
\begin{aligned}
|Q_n^{(k',2k'-1)}(t)| \;&\leqslant\; 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\big(|\phi(j,j')|\big) \\
&\qquad \times\big|\langle\delta_{(j+1)2^{-n/2}},\delta_{(j'+1)2^{-n/2}}\rangle\big|^{2k'-1} \\
&\leqslant\; C2^{-nH\left(p+q-(2k'-1)\right)}t2^{n(\frac12-(2k'-1)H)} = Ct2^{-n[H(p+q)-\frac12]},
\end{aligned}
$$

where we have the second inequality by (4.2.33).

- Preparation to the cases where $0\leqslant a\leqslant 2k'-2$

Thanks to (4.2.17) we have

$$
\begin{aligned}
D^{4k'-2-2a}_{X^2}\big(\phi(j,j')\big) = & \sum_{l=0}^{4k'-2-2a}\phi_l(j,j')\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)^l \\
\tilde{\otimes}\left(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\right)^{4k'-2-2a-l}, & \qquad\qquad (4.3.52)
\end{aligned}
$$

where $\phi_l(j,j')$ is a quantity having a similar form as $\phi(j,j')$. So, we have

- <u>Case $1 \leqslant a \leqslant 2k'-2$</u>

$$\begin{aligned}
&|Q_n^{(k',a)}(t)| \\
\leqslant\ & 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{l=0}^{4k'-2-2a} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\big|\phi_l(j,j')\big|\big) \times \\
& \bigg|\bigg\langle \left(\frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{l} \tilde{\otimes} \left(\frac{\varepsilon_{j'2^{-n/2}} + \varepsilon_{(j'+1)2^{-n/2}}}{2}\right)^{4k'-2-2a-l}, \\
& \delta_{(j+1)2^{-n/2}}^{\otimes 2k'-1-a} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2k'-1-a} \bigg\rangle\bigg| \big|\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle\big|^{a} \\
\leqslant\ & C 2^{-nH\left(p+q-(2k'-1)\right)} (2^{-nH})^{4k'-2-2a} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big|\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle\big|^{a} \\
\leqslant\ & C t 2^{-nH\left(p+q-(2k'-1)\right)} (2^{-nH})^{4k'-2-2a} 2^{n(\frac{1}{2}-aH)} \\
=\ & C t 2^{-n[H(p+q+2k'-1-a)-\frac{1}{2}]},
\end{aligned}$$

where we have the second inequality thanks to (4.2.32) and the third one thanks to (4.2.33).

- <u>Case $a=0$</u>

$$\begin{aligned}
&|Q_n^{(k',0)}(t)| \qquad (4.3.53) \\
\leqslant\ & 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{l=0}^{4k'-2} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\big|\phi_l(j,j')\big|\big) \times \\
& \bigg|\bigg\langle \left(\frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{l} \tilde{\otimes} \left(\frac{\varepsilon_{j'2^{-n/2}} + \varepsilon_{(j'+1)2^{-n/2}}}{2}\right)^{4k'-2-l}, \\
& \qquad \delta_{(j+1)2^{-n/2}}^{\otimes 2k'-1} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2k'-1} \bigg\rangle\bigg| \\
\leqslant\ & C 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{l=0}^{4k'-2} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \bigg|\bigg\langle \left(\frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{l} \\
& \tilde{\otimes} \left(\frac{\varepsilon_{j'2^{-n/2}} + \varepsilon_{(j'+1)2^{-n/2}}}{2}\right)^{4k'-2-l}, \delta_{(j+1)2^{-n/2}}^{\otimes 2k'-1} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2k'-1} \bigg\rangle\bigg| \qquad (4.3.54)
\end{aligned}$$

We define

$$\begin{aligned}
E_n^{(k',l)}(j,j') \ :=\ & \bigg|\bigg\langle \left(\frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{l} \tilde{\otimes} \left(\frac{\varepsilon_{j'2^{-n/2}} + \varepsilon_{(j'+1)2^{-n/2}}}{2}\right)^{4k'-2-l}, \\
& \qquad \delta_{(j+1)2^{-n/2}}^{\otimes 2k'-1} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2k'-1} \bigg\rangle\bigg|,
\end{aligned}$$

If $l=0$, observe by (4.2.32) that

$$E_n^{(k',0)}(j,j') \leqslant (2^{-nH})^{4k'-3} \bigg|\bigg\langle \left(\frac{\varepsilon_{j'2^{-n/2}} + \varepsilon_{(j'+1)2^{-n/2}}}{2}\right), \delta_{(j+1)2^{-n/2}} \bigg\rangle\bigg|.$$

If $l \neq 0$ then

$$E_n^{(k',l)}(j,j') \leqslant (2^{-nH})^{4k'-3} \left| \left\langle \left(\frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2} \right), \delta_{(j'+1)2^{-n/2}} \right\rangle \right|.$$

By combining these previous estimates with (4.3.54), (4.2.34) and (4.2.35) we deduce that

$$\begin{aligned} |Q_n^{(k',0)}(t)| &\leqslant C2^{-nH\big(p+q-(2k'-1)\big)}(2^{-nH})^{4k'-3}(2^{-nH-1} + 2^{1+n/2}t^{2H+1}) \\ &\leqslant C2^{-nH[p+q+2k'-1]} + C2^{-n[H(p+q+2k'-2)-\frac{1}{2}]}t^{2H+1}. \end{aligned}$$

Finally, we deduce that for all $k' \in \{1, \ldots, \frac{q+1}{2}\}$,

$$\begin{aligned} E\big[\big(V_{n,1}(f,k',t)\big)^2\big] &\leqslant Ct\Big(\sum_{a=1}^{2k'-1} 2^{-n[H(p+q+2k'-1-a)-\frac{1}{2}]}\Big) \\ &\quad + C2^{-n[H(p+q+2k'-2)-\frac{1}{2}]}t^{2H+1} + C2^{-nH[p+q+2k'-1]}. \end{aligned} \tag{4.3.55}$$

It is clear that either for $H > 1/6$ and $p+q \geqslant 3$ or for $H = 1/6$ and $p+q \geqslant 5$, we have $V_{n,1}(f,k',t) \xrightarrow{L^2} 0$ as n tends to infinity.

(b) Convergence to 0, in L^2, of $V_{n,2}(f,k,k',t)$: For all $k \in \{1, \ldots, \frac{p}{2}\}$ and $k' \in \{1, \ldots, \frac{q+1}{2}\}$, thanks to (4.2.18) we have

$$\begin{aligned} V_{n,2}(f,k,k',t) &= 2^{-\frac{nH(p+q)}{2}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Delta_{j,n}f(X^1,X^2)H_{2k}\big(X^{1,n}_{j+1} - X^{1,n}_j\big) \\ &\qquad \times H_{2k'-1}\big(X^{2,n}_{j+1} - X^{2,n}_j\big) \\ &= 2^{-\frac{nH(p+q)}{2}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Delta_{j,n}f(X^1,X^2)I^{(1)}_{2k}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big) \\ &\qquad \times I^{(2)}_{2k'-1}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k'-1}\big). \end{aligned}$$

We thus get

$$
\begin{aligned}
& E\left[\left(V_{n,2}(f,k,k',t)\right)^2\right] \\
= \; & 2^{-nH[p+q-(2k'-1)]} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\bigg(\phi(j,j') I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big) \\
& \qquad \times I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big) I_{2k'-1}^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2k'-1}\big) I_{2k'-1}^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2k'-1}\big)\bigg) \\
= \; & 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{a=0}^{2k'-1} a! \binom{2k'-1}{a}^2 \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\bigg(\phi(j,j') I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big) \\
& \qquad \times I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big) I_{4k'-2-2a}^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2k'-1-a} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2k'-1-a}\big)\bigg) \\
& \qquad \times \langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle^a \\
= \; & 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{a=0}^{2k'-1} a! \binom{2k'-1}{a}^2 \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\bigg(\Big\langle D_{X^2}^{4k'-2-2a}\big(\phi(j,j')\big), \\
& \delta_{(j+1)2^{-n/2}}^{\otimes 2k'-1-a} \otimes \delta_{(j'+1)2^{-n/2}}^{\otimes 2k'-1-a}\Big\rangle I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big) I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big)\bigg) \\
& \qquad \times \langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle^a \\
= \; & \sum_{a=0}^{2k'-1} a! \binom{2k'-1}{a}^2 \tilde{Q}_n^{(k',a)}(t),
\end{aligned}
$$

with obvious notation at the last equality. Recall that $f \in C_b^\infty$. We have the following estimates.

- <u>Case $a = 2k'-1$</u>

$$
\begin{aligned}
& \big|\tilde{Q}_n^{(k',2k'-1)}(t)\big| \\
\leqslant \; & 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\bigg(\Big|\phi(j,j') I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big) \\
& \qquad \times I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big)\bigg)\Big|\Big) \big|\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle\big|^{2k'-1} \\
\leqslant \; & C 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \|I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big)\|_2 \\
& \qquad \times \|I_{2k}^{(1)}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big)\|_2 \big|\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle\big|^{2k'-1} \\
\leqslant \; & C 2^{-nH\left(p+q-(2k'-1)\right)} \sum_{j,j'=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big|\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle\big|^{2k'-1} \\
\leqslant \; & C t 2^{-nH\left(p+q-(2k'-1)\right)} 2^{n(\frac{1}{2}-(2k'-1)H)} = Ct 2^{-n[H(p+q)-\frac{1}{2}]},
\end{aligned}
$$

  where the third inequality is a consequence of (4.2.21) and the fourth one a consequence of (4.2.33).

- Preparation to the cases where $0 \leqslant a \leqslant 2k'-2$
  Thanks to (4.3.52), we have

$$
\begin{aligned}
&\Big|E\Big(\Big\langle D^{4k'-2-2a}_{X^2}\big(\phi(j,j')\big), \delta^{\otimes 2k'-1-a}_{(j+1)2^{-n/2}} \otimes \delta^{\otimes 2k'-1-a}_{(j'+1)2^{-n/2}}\Big\rangle \\
&\qquad I^{(1)}_{2k}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big) I^{(1)}_{2k}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big)\Big)\Big| \\
\leqslant\ & \sum_{l=0}^{4k'-2-2a} E\Big(\big|\phi_l(j,j') I^{(1)}_{2k}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big) I^{(1)}_{2k}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big)\big|\Big) \\
&\Big|\Big\langle \Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{l} \tilde{\otimes} \Big(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\Big)^{4k'-2-2a-l}, \\
&\qquad \delta^{\otimes 2k'-1-a}_{(j+1)2^{-n/2}} \otimes \delta^{\otimes 2k'-1-a}_{(j'+1)2^{-n/2}}\Big\rangle\Big| \\
\leqslant\ & C \sum_{l=0}^{4k'-2-2a} \|I^{(1)}_{2k}\big((2^{\frac{nH}{2}}\delta_{(j+1)2^{-n/2}})^{\otimes 2k}\big)\|_2 \|I^{(1)}_{2k}\big((2^{\frac{nH}{2}}\delta_{(j'+1)2^{-n/2}})^{\otimes 2k}\big)\|_2 \\
&\Big|\Big\langle \Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{l} \tilde{\otimes} \Big(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\Big)^{4k'-2-2a-l}, \\
&\qquad \delta^{\otimes 2k'-1-a}_{(j+1)2^{-n/2}} \otimes \delta^{\otimes 2k'-1-a}_{(j'+1)2^{-n/2}}\Big\rangle\Big| \\
\leqslant\ & C \sum_{l=0}^{4k'-2-2a} \Big|\Big\langle \Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{l} \tilde{\otimes} \Big(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\Big)^{4k'-2-2a-l}, \\
&\qquad \delta^{\otimes 2k'-1-a}_{(j+1)2^{-n/2}} \otimes \delta^{\otimes 2k'-1-a}_{(j'+1)2^{-n/2}}\Big\rangle\Big|,
\end{aligned}
$$

  By the same arguments that was used in the study of $V_{n,1}(f,k',t)$, we deduce that

- Case $1 \leqslant a \leqslant 2k'-2$

$$|\tilde{Q}^{(k',a)}_n(t)| \leqslant Ct2^{-n[H(p+q+2k'-1-a)-\frac{1}{2}]}.$$

- Case $a=0$

$$|\tilde{Q}^{(k',0)}_n(t)| \leqslant C2^{-nH[p+q+2k'-1]} + C2^{-n[H(p+q+2k'-2)-\frac{1}{2}]}t^{2H+1}.$$

Finally, we deduce that, for all $k \in \{1,\ldots,\frac{p}{2}\}$ and $k' \in \{1,\ldots,\frac{q+1}{2}\}$,

$$
\begin{aligned}
E\big[\big(V_{n,2}(f,k,k',t)\big)^2\big] \ &\leqslant\ Ct\Big(\sum_{a=1}^{2k'-1} 2^{-n[H(p+q+2k'-1-a)-\frac{1}{2}]}\Big) \\
&\quad + C2^{-n[H(p+q+2k'-2)-\frac{1}{2}]}t^{2H+1} + C2^{-nH[p+q+2k'-1]}.
\end{aligned}
\tag{4.3.56}
$$

It is clear that either for $H > 1/6$ and $p+q \geqslant 3$ or for $H = 1/6$ and $p+q \geqslant 5$, we have $V_{n,2}(f,k,k',t) \xrightarrow{L^2} 0$ as n tends to infinity.

Combining (4.3.51), (4.3.55) and (4.3.56), we deduce that

$$E\big[\big(V_n^{p,q}(f,t)\big)^2\big] \leqslant C\sum_{k'=1}^{\frac{q+1}{2}}\Bigg(\Big(\sum_{a=1}^{2k'-1} 2^{-n[H(p+q+2k'-1-a)-\frac{1}{2}]}\Big)t + 2^{-n[H(p+q+2k'-2)-\frac{1}{2}]}t^{2H+1} + 2^{-nH[p+q+2k'-1]}\Bigg). \tag{4.3.57}$$

Hence, the desired conclusion of Proposition 4.3.2 holds true when $p \neq 0$.

2. <u>If $p = 0$:</u> We have

$$V_n^{0,q}(f,t) = 2^{-\frac{nHq}{2}}\sum_{k'=1}^{\frac{q+1}{2}} a_{q,k'}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{j,n}f(X^1,X^2)H_{2k'-1}\big(X_{j+1}^{2,n}-X_j^{2,n}\big).$$

By the same calculation as in the previous case, we deduce that

$$E\big[\big(V_n^{0,q}(f,t)\big)^2\big] \leqslant C\sum_{k'=1}^{\frac{q+1}{2}}\Bigg(\Big(\sum_{a=1}^{2k'-1} 2^{-n[H(q+2k'-1-a)-\frac{1}{2}]}\Big)t + 2^{-n[H(q+2k'-2)-\frac{1}{2}]}t^{2H+1} + 2^{-nH[q+2k'-1]}\Bigg). \tag{4.3.58}$$

Hence, the desired conclusion of Proposition 4.3.2 holds true when $p = 0$ as well, which ends up the proof of Proposition 4.3.2.

■

4.3.1 Proof of (4.1.3)

Thanks to Lemma 4.2.4, we have

$$\begin{aligned}
& f(X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}) - f(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}}) \\
= {} & \Delta_{j,n}\frac{\partial f}{\partial x}(X^1,X^2)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big) + \Delta_{j,n}\frac{\partial f}{\partial y}(X^1,X^2)\big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big) \\
& + \sum_{i=2}^{7}\sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\,\Delta_{j,n}\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2}f(X^1,X^2)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big)^{\alpha_1} \\
& \quad \times\big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big)^{\alpha_2} + R_{13}\big(\big(X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}\big), \big(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}}\big)\big).
\end{aligned}$$

By Definition 4.1.2 and (4.3.50), we can write

$$\begin{aligned}
& f(X^1_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}}) - f(0,0) \\
= {} & O_n(f,t) + \sum_{i=2}^{7}\sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\,V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2}f,t) \\
& + \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} R_{13}\big(\big(X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}\big), \big(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}}\big)\big).
\end{aligned} \tag{4.3.59}$$

By Lemma 4.2.4, we have

$$\begin{aligned}
&\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|R_{13}\big((X^1_{(j+1)2^{-n/2}},X^2_{(j+1)2^{-n/2}})\big),\big(X^1_{j2^{-n/2}},X^2_{j2^{-n/2}})\big)\big)\right|\\
\leqslant\;& C_f\sum_{\alpha_1+\alpha_2=13}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|X^1_{(j+1)2^{-n/2}}-X^1_{j2^{-n/2}}|^{\alpha_1}|X^2_{(j+1)2^{-n/2}}-X^2_{j2^{-n/2}})|^{\alpha_2}.
\end{aligned}$$

So, we deduce by the independence of X^1 and X^2 that

$$\begin{aligned}
&\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Big|R_{13}\big((X^1_{(j+1)2^{-n/2}},X^2_{(j+1)2^{-n/2}})\big),\big(X^1_{j2^{-n/2}},X^2_{j2^{-n/2}})\big)\big)\Big|\Big) \qquad (4.3.60)\\
\leqslant\;& C_f\sum_{\alpha_1+\alpha_2=13}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\big(|X^1_{(j+1)2^{-n/2}}-X^1_{j2^{-n/2}}|^{\alpha_1}\big)E\big(|X^2_{(j+1)2^{-n/2}}-X^2_{j2^{-n/2}})|^{\alpha_2}\big)\\
=\;& C_f\sum_{\alpha_1+\alpha_2=13}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}2^{\frac{-nH(\alpha_1+\alpha_2)}{2}}E\big[|G|^{\alpha_1}\big]E\big[|G|^{\alpha_2}\big]\\
\leqslant\;& C_f\sum_{\alpha_1+\alpha_2=13}E\big[|G|^{\alpha_1}\big]E\big[|G|^{\alpha_2}\big]2^{\frac{-n[H(\alpha_1+\alpha_2)-1]}{2}}t\to_{n\to\infty}0,
\end{aligned}$$

with $G\sim N(0,1)$. On the other hand, by the almost sure continuity of $f\big(X^1,X^2\big)$, one has, almost surely and as $n\to\infty$,

$$f(X^1_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}},X^2_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}})-f(0,0)\to f\big(X^1_t,X^2_t\big)-f(0,0). \qquad (4.3.61)$$

Finally, the desired conclusion (4.1.3) follows from (4.3.60), (4.3.61) and the conclusion of Proposition 4.3.2, plugged into (4.3.59).

4.3.2 Proof of (4.1.4)

We define $W_n(f,t)$ by

$$W_n(f,t)\;:=\;\Big(K_n^{(1)}(f,t),K_n^{(2)}(f,t),K_n^{(3)}(f,t),K_n^{(4)}(f,t)\Big),$$

where, for all $i\in\{1,\dots,4\}$, $K_n^{(i)}(f,t)$ is given in Definition 4.2.2. Let also define $W(f,t)$ as follows,

$$\begin{aligned}
W(f,t)\;:=\;&\Big(\kappa_1\int_0^t\partial_{111}f(X^1_s,X^2_s)dB^1_s,\kappa_2\int_0^t\partial_{222}f(X^1_s,X^2_s)dB^2_s,\\
&\kappa_3\int_0^t\partial_{122}f(X^1_s,X^2_s)dB^3_s,\kappa_4\int_0^t\partial_{112}f(X^1_s,X^2_s)dB^4_s\Big),
\end{aligned}$$

where $(B^1,\dots,B^4)$ is a 4-dimensional Brownian motion independent of (X^1,X^2), $\kappa_1^2=\kappa_2^2=\frac{1}{96}\sum_{r\in\mathbb{Z}}\rho^3(r)$ and $\kappa_3^2=\kappa_4^2=\frac{1}{32}\sum_{r\in\mathbb{Z}}\rho^3(r)$ with ρ defined in (4.2.23).

The following theorem will play a crucial role in the proof of (4.1.4).

Theorem 4.3.3 *Suppose $H = 1/6$ and fix $t \geqslant 0$. Then, as $n \to \infty$,*

$$\big(X^1, X^2, W_n(f,t)\big) \to \big(X^1, X^2, W(f,t)\big),$$

in $D_{\mathbb{R}^2}[0,\infty) \times \mathbb{R}^4$.

Proof. The proof of Theorem 4.3.3 will be done in several steps.

Step 1: Tightness of $\big(X^1, X^2, W_n(f,t)\big)$ in $D_{\mathbb{R}^2}[0,\infty) \times \mathbb{R}^4$.

It suffices to prove that $\big(W_n(f,t)\big)_n$ is bounded in $L^2(\mathbb{R}^4)$. We have

$$E\big(\|W_n(f,t)\|^2_{\mathbb{R}^4}\big) = E\big(\|(K_n^{(1)}(f,t), K_n^{(2)}(f,t), K_n^{(3)}(f,t), K_n^{(4)}(f,t))\|^2_{\mathbb{R}^4}\big) \quad (4.3.62)$$
$$= \sum_{p=1}^{4} E\big((K_n^{(p)}(f,t))^2\big).$$

On the other hand:

1. For $p = 1, 2$, thanks to Lemma 4.2.9, we have
$$\sup_{n\in\mathbb{N}} E\big((K_n^{(p)}(f,t))^2\big) \leqslant C(t+t^2).$$
2. Thanks to Lemma 4.2.7, we have
$$\sup_{n\in\mathbb{N}} E\big((K_n^{(3)}(f,t))^2\big) \leqslant C(t+t^2).$$
3. Thanks to Lemma 4.2.8, we have also
$$\sup_{n\in\mathbb{N}} E\big((K_n^{(4)}(f,t))^2\big) \leqslant C(t+t^2).$$

By combining these previous estimates with (4.3.62), we deduce that $\exists C > 0$ independent of n and t such that

$$E\big(\|W_n(f,t)\|^2_{\mathbb{R}^4}\big) \leqslant C(t+t^2),$$

which proves the boundedness, in $L^2(\mathbb{R}^4)$, of $(W_n(f,t))_n$ and consequently its tightness.

Step 2.

By Step 1, the sequence $\big(X^1, X^2, W_n(f,t)\big)$ is tight in $D_{\mathbb{R}^2}[0,\infty) \times \mathbb{R}^4$. Consider a subsequence converging in law to some limit denoted by

$$\big(X^1, X^2, W_\infty(f,t)\big)$$

(for convenience, we keep the same notation for this subsequence and for the sequence itself).

We have to show in this step that, conditioned on (X^1, X^2), the laws of $W_\infty(f,t)$ and $W(f,t)$ are the same: Let $\lambda = (\lambda_1, \ldots, \lambda_4)$ denote a generic element of $\mathbb{R}^4$ and, for $\lambda, \beta \in \mathbb{R}^4$, write $\langle \lambda, \beta \rangle$ for $\sum_{i=1}^{4} \lambda_i \beta_i$. Let us define $g_1 := \partial_{111} f$, $g_2 := \partial_{222} f$, $g_3 := \partial_{122} f$ and $g_4 := \partial_{112} f$. We consider the conditional characteristic function of $W(f,t)$ given (X^1, X^2):

$$\phi(\lambda) := E\big(e^{i\langle \lambda, W(f,t)\rangle} | X^1, X^2\big).$$

Observe that $\phi(\lambda) = e^{-\frac{1}{2}\sum_{p=1}^{4} \lambda_p^2 q_p}$ where, for $p \in \{1, \dots, 4\}$,

$$q_p = \kappa_p^2 \Big(\int_0^t g_p^2(X_s^1, X_s^2) ds \Big).$$

Observe that ϕ is the unique solution of the following system of PDEs:

$$\frac{\partial \phi}{\partial \lambda_p}(\lambda) = \phi(\lambda)\big(-\lambda_p q_p\big), \quad p = 1, \dots, 4, \tag{4.3.63}$$

where the unknown function $\phi : \mathbb{R}^4 \to \mathbb{C}$ satisfies the initial condition $\phi(0) = 1$.

Recall that our purpose is to show that

$$E\big(e^{i\langle \lambda, W_\infty(f,t)\rangle} | X^1, X^2\big) = E\big(e^{i\langle \lambda, W(f,t)\rangle} | X^1, X^2\big).$$

Thanks to (4.3.63) this is equivalent to show that

$$\frac{\partial}{\partial \lambda_p} E\big(e^{i\langle \lambda, W_\infty(f,t)\rangle} | X^1, X^2\big) = E\big(e^{i\langle \lambda, W_\infty(f,t)\rangle} | X^1, X^2\big)\big(-\lambda_p \kappa_p^2 \int_0^t g_p^2(X_s^1, X_s^2) ds\big),.$$

Hence we have to show that, for all $p \in \{1, \dots, 4\}$,

$$\begin{aligned} E\big(iK_\infty^{(p)}(f,t)\, e^{i\langle \lambda, W_\infty(f,t)\rangle} | X^1, X^2\big) &= E\big(e^{i\langle \lambda, W_\infty(f,t)\rangle} | X^1, X^2\big) \\ &\quad \times \big(-\lambda_p \kappa_p^2 \int_0^t g_p^2(X_s^1, X_s^2) ds\big). \end{aligned}$$

Equivalently, for every random variable ξ of the form $\psi\big(X_{s_1}^1, X_{s_1}^2, \dots, X_{s_r}^1, X_{s_r}^2\big)$, with $r \in \mathbb{N}^*$, $\psi : \mathbb{R}^{2r} \to \mathbb{R}$ belonging to $C_b^\infty(\mathbb{R}^{2r})$ and $s_1, \dots, s_r \in \mathbb{R}$, for all $p \in \{1, \dots, 4\}$, we have to prove that

$$E\big(iK_\infty^{(p)}(f,t)\, e^{i\langle \lambda, W_\infty(f,t)\rangle} \xi\big) = E\big(\xi e^{i\langle \lambda, W_\infty(f,t)\rangle} \times \big(-\lambda_p \kappa_p^2 \int_0^t g_p^2(X_s^1, X_s^2) ds\big)\big).$$

Since $\big(X^1, X^2, W_\infty(f,t)\big)$ is defined as the limit in law of $\big(X^1, X^2, W_n(f,t)\big)$ and since $W_n(f,t)$ is bounded in L^2, we have

$$E\big(iK_\infty^{(p)}(f,t)\, e^{i\langle \lambda, W_\infty(f,t)\rangle} \xi\big) = \lim_{n\to\infty} E\big(iK_n^{(p)}(f,t)\, e^{i\langle \lambda, W_n(f,t)\rangle} \xi\big).$$

Thus, we have to prove that, for all $p \in \{1, \dots, 4\}$,

$$\lim_{n\to\infty} E\big(iK_n^{(p)}(f,t)\, e^{i\langle \lambda, W_n(f,t)\rangle} \xi\big) = E\big(\xi e^{i\langle \lambda, W_\infty(f,t)\rangle} \times \big(-\lambda_p \kappa_p^2 \int_0^t g_p^2(X_s^1, X_s^2) ds\big)\big). \tag{4.3.64}$$

Let us compute $E\big(iK_n^{(p)}(f,t)\,e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big)$ for $p=3$ (the calculations are very similar for the other values of p). We have

$$
\begin{aligned}
& E\big(iK_n^{(3)}(f,t)\,e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big)\\
= {} & \frac{i}{8}2^{-\frac{n}{4}}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)H_1\big(X^{1,n}_{j+1}-X^{1,n}_j\big)H_2\big(X^{2,n}_{j+1}-X^{2,n}_j\big)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\
= {} & \frac{i}{8}2^{-\frac{n}{4}}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)I_1^{(1)}(2^{\frac{n}{12}}\delta_{(j+1)2^{-n/2}})I_2^{(2)}(2^{\frac{n}{6}}\delta^{\otimes 2}_{(j+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\
= {} & \frac{i}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)I_1^{(1)}(\delta_{(j+1)2^{-n/2}})I_2^{(2)}(\delta^{\otimes 2}_{(j+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\
= {} & \frac{i}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\langle D_{X^1}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big),\delta_{(j+1)2^{-n/2}}\rangle I_2^{(2)}(\delta^{\otimes 2}_{(j+1)2^{-n/2}})\Big),
\end{aligned}
\tag{4.3.65}
$$

where the second equality follows from (4.2.18) and the last one follows from (4.2.19). Thanks to (4.2.16), the first Malliavin derivative with respect to X^1 of $\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi$ is given by

$$
\begin{aligned}
& D_{X^1}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big)\\
= {} & \Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\\
& +i\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi D_{X^1}\langle\lambda,W_n(f,t)\rangle+\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^1}\xi.
\end{aligned}
$$

Thus, by (4.3.65), we have

$$
\begin{aligned}
& E\big(iK_n^{(3)}(f,t)\,e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big)\\
= {} & \frac{i}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\; I_2^{(2)}(\delta^{\otimes 2}_{(j+1)2^{-n/2}})\Big)\\
& \qquad\qquad\times\left\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle\\
- {} & \frac{1}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\;\big\langle D_{X^1}\langle\lambda,W_n(f,t)\rangle,\delta_{(j+1)2^{-n/2}}\big\rangle I_2^{(2)}(\delta^{\otimes 2}_{(j+1)2^{-n/2}})\Big)\\
+ {} & \frac{i}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\;\big\langle D_{X^1}\xi,\delta_{(j+1)2^{-n/2}}\big\rangle I_2^{(2)}(\delta^{\otimes 2}_{(j+1)2^{-n/2}})\Big)\\
= {} & A_n(t)+B_n(t)+C_n(t),
\end{aligned}
$$

with obvious notation at the last equality.

In the next steps we will prove firstly that $A_n(t)\to 0$ as $n\to\infty$, then that $B_n(t)\to -\lambda_3\kappa_3^2E\big(\xi e^{i\langle\lambda,W_\infty(f,t)\rangle}\times\int_0^t g_3^2(X_s^1,X_s^2)ds\big)$ and finally that $C_n(t)\to 0$.

Step 3: Proof of the convergence to 0 of $A_n(t)$.

Thanks to the duality formula (4.2.19), we have

$$A_n(t) = \frac{i}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Big\langle D^2_{X^2}\big(\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big),\delta^{\otimes 2}_{(j+1)2^{-n/2})}\Big\rangle\Big)$$
$$\times\Big\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle. \tag{4.3.66}$$

Thanks to (4.2.16), the second Malliavin derivative of $\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi$ with respect to X^2 is given by

$$\begin{aligned}
& D^2_{X^2}\big(\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big)\\
= \; & \Delta_{j,n}\partial_{112222}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{\otimes 2}\\
& +2\Delta_{j,n}\partial_{11222}f(X^1,X^2)\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}D_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi)\\
& +\Delta_{j,n}\partial_{1122}f(X^1,X^2)D^2_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi).
\end{aligned}$$

On the other hand, we have

$$D_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi) = ie^{i\langle\lambda,W_n(f,t)\rangle}\xi D_{X^2}\langle\lambda,W_n(f,t)\rangle + e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}\xi,$$

and

$$\begin{aligned}
& D^2_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi)\\
= \; & -e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big)^{\otimes 2} + ie^{i\langle\lambda,W_n(f,t)\rangle}\xi D^2_{X^2}\langle\lambda,W_n(f,t)\rangle\\
& +2ie^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}\langle\lambda,W_n(f,t)\rangle\tilde{\otimes}D_{X^2}\xi + e^{i\langle\lambda,W_n(f,t)\rangle}D^2_{X^2}\xi.
\end{aligned}$$

Thus, we finally get

$$\begin{aligned}
& D^2_{X^2}\big(\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big) \qquad (4.3.67)\\
= \; & \Delta_{j,n}\partial_{112222}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{\otimes 2}\\
& +2i\Delta_{j,n}\partial_{11222}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}D_{X^2}\langle\lambda,W_n(f,t)\rangle\\
& +2\Delta_{j,n}\partial_{11222}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}D_{X^2}\xi\\
& -\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big)^{\otimes 2}\\
& +i\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi D^2_{X^2}\langle\lambda,W_n(f,t)\rangle\\
& +2i\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}\langle\lambda,W_n(f,t)\rangle\tilde{\otimes}D_{X^2}\xi\\
& +\Delta_{j,n}\partial_{1122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}D^2_{X^2}\xi.
\end{aligned}$$

By replacing the quantity (4.3.67) in (4.3.66), we get

$$
\begin{aligned}
A_n(t) &= \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{112222} f(X^1,X^{(2)}) e^{i\langle\lambda, W_n(f,t)\rangle}\xi\big)\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle^3 \\
&\quad +2i \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{11222} f(X^1,X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\xi\big\langle D_{X^2}\langle\lambda, W_n(f,t)\rangle, \delta_{(j+1)2^{-n/2}}\big\rangle\big) \\
&\qquad\qquad \times\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle^2 \\
&\quad +2 \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{11222} f(X^1,X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\big\langle D_{X^2}\xi, \delta_{(j+1)2^{-n/2}}\big\rangle\big) \\
&\qquad\qquad \times\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle^2 \\
&\quad - \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{1122} f(X^1,X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\xi\big\langle D_{X^2}\langle\lambda, W_n(f,t)\rangle, \delta_{(j+1)2^{-n/2}}\big\rangle^2\big) \\
&\qquad\qquad \times\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle \\
&\quad +i \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{1122} f(X^1,X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\xi\big\langle D^2_{X^2}\langle\lambda, W_n(f,t)\rangle, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big) \\
&\qquad\qquad \times\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle \\
&\quad +2i \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{1122} f(X^1,X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\big\langle D_{X^2}\langle\lambda, W_n(f,t)\rangle\tilde{\otimes} D_{X^2}\xi, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big) \\
&\qquad\qquad \times\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle \\
&\quad + \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{1122} f(X^1,X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\big\langle D^2_{X^2}\xi, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big) \\
&\qquad\qquad \times\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle \\
&= \sum_{i=1}^{7} A_{n,i}(t),
\end{aligned}
$$

with obvious notation at the last equality. Let us prove the convergence to 0 of $A_{n,i}(t)$ for $i \in \{1,\dots,7\}$.

convergence of $A_{n,1}(t)$ to 0

Thanks to the boundedness of $e^{i\langle\lambda, W_n(f,t)\rangle}$ and ξ, and since $f \in C_b^\infty(\mathbb{R}^2)$, we get

$$\begin{aligned}|A_{n,1}(t)| &\leqslant C(2^{-n/6})^2 \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \left|\left\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\right\rangle\right| \\ &\leqslant C2^{-n/3}t^{1/3},\end{aligned}$$

where the last inequality follows from (4.2.38). Hence, $A_{n,1}(t)\to 0$ as $n\to\infty$.

Convergence of $A_{n,2}(t)$ to 0

Thanks to the boundedness of $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ, and since $f\in C_b^\infty(\mathbb{R}^2)$, we get

$$\begin{aligned}|A_{n,2}(t)| &= \Bigg|2i\sum_{p=1}^{4}\lambda_p \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{11222}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi \\ &\quad \big\langle D_{X^2}K_n^{(p)}(t),\delta_{(j+1)2^{-n/2}}\big\rangle\Big)\times\left\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle^2\Bigg| \\ &\leqslant C\sum_{p=1}^{4}|\lambda_p| \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|\big\langle D_{X^2}K_n^{(p)}(t),\delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big) \\ &\qquad \times\left\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle^2 \\ &= C\sum_{p=1}^{4}|\lambda_p|A_{n,2}^{(p)}(t),\end{aligned}$$

with obvious notation at the last equality. We will prove that

$$A_{n,2}^{(p)}(t)\leqslant C2^{-n/12}(t^{4/3}+1)+C2^{-n/6}t. \tag{4.3.68}$$

In fact, it suffices to consider the cases $p=2,3$. The previous inequality could be proved similarly for the other values of p.

1. For $p=2$: By Definition 4.2.2

$$K_n^{(2)}(t)=\frac{1}{24}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{222}f(X^1,X^2)I_3^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 3}\big).$$

So, thanks to (4.2.16) and (4.2.22), we have

$$\begin{aligned}&D_{X^2}K_n^{(2)}(t)\\ &= \frac{1}{24}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{2222}f(X^1,X^2)I_3^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 3}\big)\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right) \\ &\quad +\frac{1}{8}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{222}f(X^1,X^2)I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big)\delta_{(l+1)2^{-n/2}},\end{aligned}$$

and we get

$$
\begin{aligned}
A_{n,2}^{(2)}(t) &\leqslant C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_3^{(2)}(\delta_{(l+1)2^{-n/2}}^{\otimes 3})\big|\Big)\Big|\Big\langle\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\qquad\qquad\times\Big\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle^2\\
&\quad +C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(2)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big|\Big)\Big|\Big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\qquad\qquad\times\Big\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle^2\\
&\leqslant C(2^{-n/6})^2\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I_3^{(2)}(\delta_{(l+1)2^{-n/2}}^{\otimes 3})\|_2\Big|\Big\langle\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\quad +C(2^{-n/6})^2\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I_2^{(2)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\|_2\Big|\Big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\leqslant C2^{-n/3}\,2^{-n/4}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|\Big\langle\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\quad +C2^{-n/3}\,2^{-n/6}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|\Big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\leqslant C2^{-7n/12}(2^{-n/6}+2^{n/2}t^{4/3})+C2^{-n/6}t\\
&\leqslant C2^{-n/12}(t^{4/3}+1)+C2^{-n/6}t.
\end{aligned}
$$

The second inequality follows from Cauchy-Schwarz inequality and (4.2.32). The third inequality follows from (4.2.21) , whereas the fourth one follows from (4.2.33), (4.2.34) and (4.2.35). As a result, (4.3.68) holds true for $p=2$.

2. For $p=3$: By definition 4.2.2

$$K_n^{(3)}(t)=\frac{1}{8}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{122}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big).$$

Thus, by (4.2.16) and (4.2.22), we have

$$
\begin{aligned}
&D_{X^2}K_n^{(3)}(t)\\
=\;&\frac{1}{8}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{1222}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big)\Big(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\Big)\\
&+\frac{1}{4}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{l,n}\partial_{122}f(X^1,X^2)I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)\delta_{(l+1)2^{-n/2}}.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
& A^{(3)}_{n,2}(t) \\
\leqslant\ & C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I^{(1)}_1\big(\delta_{(l+1)2^{-n/2}}\big)\big|\big|I^{(2)}_2\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)\big|\Big) \\
& \times\Big|\Big\langle\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\Big\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle^2 \\
& +C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I^{(1)}_1\big(\delta_{(l+1)2^{-n/2}}\big)\big|\big|I^{(2)}_1\big(\delta_{(l+1)2^{-n/2}}\big)\big|\Big)\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
& \qquad\times\Big\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle^2 \\
\leqslant\ & C(2^{-n/6})^2\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I^{(1)}_1\big(\delta_{(l+1)2^{-n/2}}\big)\|_2\|I^{(2)}_2\big(\delta^{\otimes 2}_{(l+1)2^{-n/2}}\big)\|_2 \\
& \qquad\times\Big|\Big\langle\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
& +C(2^{-n/6})^2\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I^{(1)}_1\big(\delta_{(l+1)2^{-n/2}}\big)\|_2\|I^{(2)}_1\big(\delta_{(l+1)2^{-n/2}}\big)\|_2\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C2^{-n/3}\,2^{-n/12}\,2^{-n/6}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|\Big\langle\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
& +C2^{-n/3}(2^{-n/12})^2\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C2^{-n/12}(t^{4/3}+1)+C2^{-n/6}t,
\end{aligned}
$$

where the previous inequalities are shown by using the same arguments as in the case of $p=2$. So, (4.3.68) holds true for $p=3$.

Finally, we deduce that $|A_{n,2}(t)|\leqslant C2^{-n/12}(t^{4/3}+1)+C2^{-n/6}t$.

Convergence of $A_{n,3}(t)$ to 0

Thanks to the boundedness of $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ, and since $f\in C^\infty_b(\mathbb{R}^2)$, we get

$$
\begin{aligned}
|A_{n,3}(t)| &\leqslant C\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\big(\big|\big\langle D_{X^2}\xi,\delta_{(j+1)2^{-n/2}}\big\rangle\big|\big)\times\Big\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle^2 \\
&\leqslant C(2^{-n/6})^2\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|\Big\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
&\leqslant C2^{-n/3}t^{1/3},
\end{aligned}
$$

where the second inequality comes from (4.2.44) and (4.2.32), and the last one from (4.2.38). We deduce that $A_{n,3}(t) \to 0$ as $n \to \infty$.

Convergence of $A_{n,4}(t)$ to 0

Thanks to the boundedness of $e^{i\langle\lambda, W_n(f,t)\rangle}$ and ξ, and since $f \in C_b^\infty(\mathbb{R}^2)$, we get

$$\begin{aligned}
|A_{n,4}(t)| &\leqslant C \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\big\langle D_{X^2}\langle\lambda, W_n(f,t)\rangle, \delta_{(j+1)2^{-n/2}}\big\rangle^2\big) \\
&\qquad\qquad \times \left|\left\langle \frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
&\leqslant C \sum_{p=1}^{4} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\big\langle D_{X^2} K_n^{(p)}(t), \delta_{(j+1)2^{-n/2}}\big\rangle^2\big) \\
&\qquad\qquad \times \left|\left\langle \frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
&= C \sum_{p=1}^{4} A_{n,4}^{(p)}(t),
\end{aligned}$$

with obvious notation at the last equality. Thanks to (4.2.46) and to (4.2.38), we have

$$\begin{aligned}
A_{n,4}^{(p)}(t) &\leqslant C 2^{-n/3}(t^2+t+1) \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left|\left\langle \left(\frac{\varepsilon_{j2^{-n/2}} + \varepsilon_{(j+1)2^{-n/2}}}{2}\right), \delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
&\leqslant C 2^{-n/3}(t^2+t+1) t^{1/3}.
\end{aligned}$$

We deduce that

$$|A_{n,4}(t)| \leqslant C 2^{-n/3}(t^2+t+1) t^{1/3}.$$

As a consequence, the convergence of $A_{n,4}(t)$ to 0 is shown.

Convergence of $A_{n,5}(t)$ to 0

Thanks to the boundedness of $e^{i\langle\lambda, W_n(f,t)\rangle}$ and ξ, and since $f \in C_b^\infty(\mathbb{R}^2)$, we deduce that

$$\begin{aligned}
|A_{n,5}(t)| &= \Big| i \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\Delta_{j,n}\partial_{1122} f(X^1,X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\xi \big\langle D^2_{X^2}\langle\lambda, W_n(f,t)\rangle, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big) \\
&\qquad \times \Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle \Big| \\
&\leqslant C\sum_{p=1}^{4}|\lambda_p| \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\big|\big\langle D^2_{X^2}\big(K_n^{(p)}(t)\big), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big|\big) \\
&\qquad \times \Big|\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
&\leqslant C\sum_{p=1}^{4}|\lambda_p| \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \|\big\langle D^2_{X^2}\big(K_n^{(p)}(t)\big), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\|_2 \\
&\qquad \times \Big|\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
&\leqslant C2^{-n/3}(t^2+t+1)^{\frac{1}{2}} t^{1/3},
\end{aligned}$$

where the last inequality comes from (4.2.47) and (4.2.38). As a consequence, the convergence to 0 of $A_{n,5}(t)$ is shown.

Convergence of $A_{n,6}(t)$ to 0

Thanks to the boundedness of $e^{i\langle\lambda, W_n(f,t)\rangle}$ and ξ, and since $f\in C_b^\infty(\mathbb{R}^2)$, we deduce that

$$\begin{aligned}
&|A_{n,6}(t)| \\
&\leqslant C \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\big|\big\langle D_{X^2}\langle\lambda, W_n(f,t)\rangle, \delta_{(j+1)2^{-n/2}}\big\rangle\big|\big|\big\langle D_{X^2}\xi, \delta_{(j+1)2^{-n/2}}\big\rangle\big|\big) \\
&\qquad \times \Big|\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
&\leqslant C2^{-n/6}\sum_{p=1}^{4}|\lambda_p| \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\big(\big|\big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\big) \\
&\qquad \times \Big|\Big\langle \frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
&= C\sum_{p=1}^{4}|\lambda_p| A_{n,6}^{(p)}(t),
\end{aligned}$$

with obvious notation at the last equality, and where the second inequality comes from (4.2.44).

We will prove that

$$A_{n,6}^{(p)}(t) \leqslant C2^{-n/12}(t^{4/3}+t+1). \tag{4.3.69}$$

Actually, it suffices to prove this inequality for $p=2,3$, the proof for the other values of p being entirely similar.

1. For $p=2$:

$$
\begin{aligned}
&\big|\big\langle D_{X^2}(K_n^{(2)}(t)),\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|I_3^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 3}\big)\big|\Big|\Big\langle\Big(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&+C\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big)\big|\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|.
\end{aligned}
$$

We deduce that

$$
\begin{aligned}
&A_{n,6}^{(2)}(t)\\
\leqslant\;& C2^{-n/6}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\big(\big|I_3^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 3}\big)\big|\big)\Big|\Big\langle\Big(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\qquad\qquad\times\Big|\Big\langle\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&+C2^{-n/6}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\big(\big|I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big)\big|\big)\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
&\qquad\qquad\times\Big|\Big\langle\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
\leqslant\;& C2^{-7n/12}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|\Big\langle\Big(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&+C2^{-n/2}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C2^{-7n/12}(2^{-n/6}+2^{n/2}t^{4/3})+C2^{-n/6}t\leqslant C2^{-n/12}(t^{4/3}+t+1),
\end{aligned}
$$

where the second inequality comes from (4.2.32), the Cauchy-Schwarz inequality and (4.2.21). The third inequality is a consequence of (4.2.33), (4.2.34) and (4.2.35). As a result, (4.3.69) holds true for $p=2$.

2. For $p=3$:

$$
\begin{aligned}
&\big|\big\langle D_{X^2}(K_n^{(3)}(t)),\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)\big|\big|I_2^{(2)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 2}\big)\big|\Big|\Big\langle\Big(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&+C\sum_{l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|I_1^{(1)}\big(\delta_{(l+1)2^{-n/2}}\big)\big|\big|I_1^{(2)}\big(\delta_{(l+1)2^{-n/2}}\big)\big|\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|.
\end{aligned}
$$

Using the same arguments as in the previous case, we deduce that

$$\begin{aligned}
& A^{(3)}_{n,6}(t) \\
\leqslant\ & C2^{-n/6}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\big(\big|I^{(1)}_1(\delta_{(l+1)2^{-n/2}})\big|\big|I^{(2)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\big|\big) \\
& \times\left|\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\right|\left|\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
& +C2^{-n/6}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\big(\big|I^{(1)}_1(\delta_{(l+1)2^{-n/2}})\big|\big|I^{(2)}_1(\delta_{(l+1)2^{-n/2}})\big|\big)\big|\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big| \\
& \qquad\qquad \times\left|\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
\leqslant\ & C2^{-n/3}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I^{(1)}_1(\delta_{(l+1)2^{-n/2}})\|_2\|I^{(2)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\|_2 \\
& \qquad\qquad \times\left|\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
& +C2^{-n/3}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I^{(1)}_1(\delta_{(l+1)2^{-n/2}})\|_2\|I^{(2)}_1(\delta_{(l+1)2^{-n/2}})\|_2\big|\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big| \\
\leqslant\ & C2^{-7n/12}(2^{-n/6}+2^{n/2}t^{4/3})+C2^{-n/6}t\leqslant C2^{-n/12}(t^{4/3}+t+1).
\end{aligned}$$

Hence (4.3.69) holds true for $p=3$.

Finally, we have shown that $|A_{n,6}(t)|\leqslant C2^{-n/12}(t^{4/3}+t+1)$. As a result, it converges to 0.

Convergence of $A_{n,7}(t)$ to 0

Thanks to the boundedness of $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ, and since $f\in C^\infty_b(\mathbb{R}^2)$, we deduce that

$$\begin{aligned}
|A_{n,7}(t)| &\leqslant C\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\big(\big|\langle D^2_{X^2}\xi,\delta^{\otimes 2}_{(j+1)2^{-n/2}}\rangle\big|\big) \\
&\qquad\qquad \times\left|\left\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
&\leqslant C2^{-n/3}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|\left\langle\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle\right| \\
&\leqslant C2^{-n/3}t^{1/3},
\end{aligned}$$

where the second inequality comes from (4.2.45) and the last one from (4.2.38). It is now clear that $A_{n,7}(t)$ converges to 0.

Step 4: Study of the convergence of $B_n(t)$.

$$
\begin{aligned}
& B_n(t) \\
= & -\frac{1}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\,\big\langle D_{X^1}\langle\lambda,W_n(f,t)\rangle,\delta_{(j+1)2^{-n/2}}\big\rangle \\
& \qquad \times I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\Big) \\
= & -\frac{1}{8}\sum_{p=1}^{4}\lambda_p\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\,\big\langle D_{X^1}\big(K_n^{(p)}(t)\big),\delta_{(j+1)2^{-n/2}}\big\rangle \\
& \qquad \times I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\Big) \\
= & \sum_{p=1}^{4}\lambda_p B_n^{(p)}(t),
\end{aligned}
\tag{4.3.70}
$$

with obvious notation at the last equality. We will prove that

$$
B_n(t) \underset{n\to\infty}{\longrightarrow} -\kappa_3^2 E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times\int_0^t g_3^2(X_s^1,X_s^2)ds\Big), \tag{4.3.71}
$$

where $g_3 := \partial_{122}f$, see the beginning of Step 2. The proof of (4.3.71) will be done in several steps. Firstly, we will prove the convergence of $B_n^{(3)}(t)$ to $-\kappa_3^2 E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times \int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2 ds\Big)$. Then, we will prove the convergence to 0 of $B_n^{(p)}(t)$ for the other values of p.

Study of the convergence of $B_n^{(3)}(t)$.

$$
\begin{aligned}
& B_n^{(3)}(t) \\
= & -\frac{1}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\,\big\langle D_{X^1}\big(K_n^{(3)}(t)\big),\delta_{(j+1)2^{-n/2}}\big\rangle I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\Big).
\end{aligned}
\tag{4.3.72}
$$

Our purpose in this subsection is to prove the following result :

$$
B_n^{(3)}(t) \underset{n\to\infty}{\longrightarrow} -\kappa_3^2 E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times\int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2 ds\Big). \tag{4.3.73}
$$

Proof of (4.3.73). Thanks to (4.2.16) and (4.2.22), we have

$$\begin{aligned}
&\big\langle D_{X^1}\big(K_n^{(3)}(t)\big), \delta_{(j+1)2^{-n/2}}\big\rangle \\
= &\ \frac{1}{8}\sum_{k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2}) \\
&\qquad\qquad\qquad\times\Big\langle\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle \\
&+\frac{1}{8}\sum_{k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle.
\end{aligned}$$

Using the duality formula (4.2.19), we deduce that

$$\begin{aligned}
&B_n^{(3)}(t) \\
= &-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2}) \\
&\qquad\times I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\Big\langle\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle \\
&-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2}) \\
&\qquad\times I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle \\
= &-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Big\langle D_{X^2}^2\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}}) \\
&\qquad\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Big)\Big\langle\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle \\
&-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Big\langle D_{X^2}^2\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2}) \\
&\qquad\times e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Big)\big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle \\
= &\ B_n^{(3,1)}(t)+B_n^{(3,2)}(t), \qquad (4.3.74)
\end{aligned}$$

with obvious notation at the last equality. In the following two steps, we will prove firstly that $B_n^{(3,2)}(t)\to-\kappa_3^2E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times\int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2ds\Big)$, then that $B_n^{(3,1)}(t)\to 0$ as $n\to\infty$.

1. Convergence of $B_n^{(3,2)}(t)$ as $n\to\infty$: Let us prove that

$$B_n^{(3,2)}(t)\underset{n\to\infty}{\longrightarrow}-\kappa_3^2E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times\int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2ds\Big). \qquad (4.3.75)$$

Proof of (4.3.75). Thanks to (4.2.16) and (4.2.22) , we have

$$\begin{aligned} & D^2_{X^2}\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^{(1)},X^{(2)})I^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big) \quad (4.3.76)\\ = \;& D^2_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\big)I^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\ & +2D_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\big)\tilde{\otimes}D_{X^2}\big(I^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big)\\ & +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)D^2_{X^2}\big(I^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big). \end{aligned}$$

We also have

$$\begin{aligned} & D_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\big) \quad (4.3.77)\\ = \;& \Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\\ & +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big), \end{aligned}$$

and

$$\begin{aligned} & D^2_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\big) \quad (4.3.78)\\ = \;& \Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{\otimes 2}\\ & +2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}\\ & \Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big)\\ & +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{12222}f(X^1,X^2)\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big)^{\otimes 2}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & D_{X^2}\big(I^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big) \quad (4.3.79)\\ = \;& 2I^{(2)}_1(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta_{(k+1)2^{-n/2}}+I^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})D_{X^2}\big(e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big)\\ = \;& 2I^{(2)}_1(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta_{(k+1)2^{-n/2}}+iI^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\ & \times D_{X^2}(\langle\lambda,W_n(f,t)\rangle)+I^{(2)}_2(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}(\xi). \end{aligned}$$

Also

$$
\begin{aligned}
& D^2_{X^2}\big(I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big) \qquad (4.3.80)\\
= \;& 2e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta^{\otimes 2}_{(k+1)2^{-n/2}} + 4I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi)\\
& +I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})D^2_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi)\\
= \;& 2e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta^{\otimes 2}_{(k+1)2^{-n/2}} + 4iI_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta_{(k+1)2^{-n/2}}\tilde{\otimes}\\
& D_{X^2}(\langle\lambda,W_n(f,t)\rangle) + 4I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\xi)\\
& +iI_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi D^2_{X^2}(\langle\lambda,W_n(f,t)\rangle)\\
& -I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big(D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\big)^{\otimes 2}\\
& +2iI_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\tilde{\otimes}D_{X^2}(\xi)\\
& +I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}D^2_{X^2}(\xi).
\end{aligned}
$$

From (4.3.77), (4.3.78), (4.3.79) and (4.3.80) plugged into (4.3.76), we deduce that

$$
\begin{aligned}
& D^2_{X^2}\Bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Bigg) \qquad (4.3.81)\\
= \;& \Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
& \times\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{\otimes 2} + 2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)\\
& \times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\\
& +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{12222}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
& \times\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)^{\otimes 2} + 4\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
& I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(k+1)2^{-n/2}}\\
& +2i\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
& \times\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle) + 2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\\
& \times\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\xi)\\
& +4\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
& \times\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(k+1)2^{-n/2}} + 2i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)\\
& \times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)
\end{aligned}
$$

$$
\begin{aligned}
&+2\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\\
&\times\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\xi)+2\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
&\times e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta^{\otimes 2}_{(k+1)2^{-n/2}}+4i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
&\times I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)+4\Delta_{j,n}\partial_{122}f(X^1,X^2)\\
&\times\Delta_{k,n}\partial_{122}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\xi)\\
&+i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
&\times D^2_{X^2}(\langle\lambda,W_n(f,t)\rangle)-\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\\
&\times e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big(D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\big)^{\otimes 2}+2i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
&\times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\tilde{\otimes}D_{X^2}(\xi)+\Delta_{j,n}\partial_{122}f(X^1,X^2)\\
&\times\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}D^2_{X^2}(\xi).
\end{aligned}
$$

By plugging (4.3.81) in $B_n^{(3,2)}(t)$, we get

$$
\begin{aligned}
&B_n^{(3,2)}(t)\\
=\;&-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\
&\times\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{\otimes 2},\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\\
&-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\
&\times\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\\
&\times\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\\
&-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{12222}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\
&\left\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)^{\otimes 2},\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\\
&-\frac{4}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\
&\times\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(k+1)2^{-n/2}},\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle
\end{aligned}
$$

$$-\frac{2i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi$$
$$\times\bigg\langle\bigg(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\bigg)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)$$
$$\times\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}$$
$$\times\bigg\langle\bigg(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\bigg)\tilde{\otimes}D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{4}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\bigg)$$
$$\times\bigg\langle\bigg(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\bigg)\tilde{\otimes}\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{2i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi$$
$$\times\bigg\langle\bigg(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\bigg)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1222}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}$$
$$\times\bigg\langle\bigg(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\bigg)\tilde{\otimes}D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\bigg)$$
$$\times\langle\delta_{(k+1)2^{-n/2}}^{\otimes 2},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\rangle\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{4i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi$$
$$\times\bigg\langle\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{4}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}$$
$$\times\bigg\langle\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi$$
$$\times\bigg\langle D_{X^2}^2(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$

$$+\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi$$
$$\times\Big\langle\big(D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\big)^{\otimes 2},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{2i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}$$
$$\times\Big\langle D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\tilde{\otimes}D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}$$
$$\times\Big\langle D^2_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Bigg)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle$$
$$=\sum_{i=1}^{16}B_{n,i}^{(3,2)}(t).$$

Now, we will prove firstly that $B_{n,10}^{(3,2)}(t)\to-\kappa_3^2E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times\int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2ds\Big)$, then we will prove the convergence to 0 of $B_{n,i}^{(3,2)}(t)$ for $i\in\{1,\dots,16\}\backslash\{10\}$.

- Convergence of $B_{n,10}^{(3,2)}(t)$:

$$\begin{aligned}B_{n,10}^{(3,2)}(t) &= -\frac{1}{32}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\ &\quad\times\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle^3.\end{aligned}$$

Let us show that

$$B_{n,10}^{(3,2)}(t)\underset{n\to\infty}{\longrightarrow}-\kappa_3^2E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times\int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2ds\Big).\qquad(4.3.82)$$

Let us prove firstly that, a.s.,

$$\frac{1}{32}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle^3$$
$$\underset{n\to\infty}{\longrightarrow}\kappa_3^2\int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2ds.\qquad(4.3.83)$$

We have, see (4.2.23) for the definition of ρ :

$$
\begin{aligned}
&\frac{1}{32}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle^3\\
=\;&\frac{2^{-n/2}}{32}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\rho(j-k)^3\\
=\;&\frac{2^{-n/2}}{32}\sum_{r=1-\lfloor 2^{\frac{n}{2}}t\rfloor}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\rho(r)^3\sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)}\Delta_{r+k,n}\partial_{122}f(X^1,X^2) \qquad (4.3.84)\\
&\times\Delta_{k,n}\partial_{122}f(X^1,X^2),
\end{aligned}
$$

where the last equality comes from the simple change of variable $r=j-k$, together with a Fubini argument. Observe that

$$
\begin{aligned}
&\Bigg|2^{-n/2}\sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)}\Delta_{r+k,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
&-\int_0^t\big(\partial_{122}f(X^1_s,X^2_s)\big)^2ds\Bigg|\\
\leqslant\;&\Bigg|2^{-n/2}\sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)}\Delta_{r+k,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
&-2^{-n/2}\sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)}\partial_{122}f(X^1_{k2^{-n/2}},X^2_{k2^{-n/2}})\Delta_{k,n}\partial_{122}f(X^1,X^2)\Bigg|\\
&+\Bigg|2^{-n/2}\sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)}\partial_{122}f(X^1_{k2^{-n/2}},X^2_{k2^{-n/2}})\Delta_{k,n}\partial_{122}f(X^1,X^2)\\
&-2^{-n/2}\sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)}\big(\partial_{122}f(X^1_{k2^{-n/2}},X^2_{k2^{-n/2}})\big)^2\Bigg|\\
&+\Bigg|2^{-n/2}\sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)}\big(\partial_{122}f(X^1_{k2^{-n/2}},X^2_{k2^{-n/2}})\big)^2\\
&-\int_0^t\big(\partial_{122}f(X^1_s,X^2_s)\big)^2ds\Bigg|\\
=\;&r_{1,n}+r_{2,n}+r_{3,n},
\end{aligned}
$$

with obvious notation at the last equality. Let us prove the convergence to 0 of $r_{1,n}$, $r_{2,n}$ and $r_{3,n}$.

For any fixed integer $r>0$ (the case $r\leqslant 0$ being similar), by Theorem 4.2.3 and since $f \in C_b^\infty$, we have

$$\begin{aligned}
r_{1,n} &\leqslant \|\partial_{122}f\|_\infty 2^{-n/2} \sum_{k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \left|\Delta_{r+k,n}\partial_{122}f(X^1,X^2) - \partial_{122}f(X^1_{k2^{-n/2}},X^2_{k2^{-n/2}})\right| \\
&\leqslant C2^{-n/2} \sum_{k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \left|\partial_{1122}f(X^1_{k2^{-n/2}},X^2_{k2^{-n/2}})\right| \\
&\qquad\qquad \times\left|X^1_{(r+k+1)2^{-n/2}} + X^1_{(r+k)2^{-n/2}} - 2X^1_{k2^{-n/2}}\right| \\
&\quad +C2^{-n/2} \sum_{k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \left|\partial_{1222}f(X^1_{k2^{-n/2}},X^2_{k2^{-n/2}})\right| \\
&\qquad\qquad \times\left|X^2_{(r+k+1)2^{-n/2}} + X^2_{(r+k)2^{-n/2}} - 2X^2_{k2^{-n/2}}\right| \\
&\quad +C2^{-n/2} \sum_{k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Bigg|R_1\Bigg((X^1_{k2^{-n/2}},X^2_{k2^{-n/2}}), \\
&\qquad \left(\frac{X^1_{(r+k+1)2^{-n/2}} + X^1_{(r+k)2^{-n/2}}}{2}, \frac{X^2_{(r+k+1)2^{-n/2}} + X^2_{(r+k)2^{-n/2}}}{2}\right)\Bigg)\Bigg| \\
&\leqslant Ct \sup_{0\leqslant k\leqslant \lfloor 2^{\frac{n}{2}}t\rfloor -1} \left|X^1_{(r+k+1)2^{-n/2}} + X^1_{(r+k)2^{-n/2}} - 2X^1_{k2^{-n/2}}\right| \\
&\quad +Ct \sup_{0\leqslant k\leqslant \lfloor 2^{\frac{n}{2}}t\rfloor -1} \left|X^2_{(r+k+1)2^{-n/2}} + X^2_{(r+k)2^{-n/2}} - 2X^2_{k2^{-n/2}}\right| \\
&\quad +C2^{-n/2} \sum_{k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Bigg|R_1\Bigg((X^1_{k2^{-n/2}},X^2_{k2^{-n/2}}), \\
&\qquad \left(\frac{X^1_{(r+k+1)2^{-n/2}} + X^1_{(r+k)2^{-n/2}}}{2}, \frac{X^2_{(r+k+1)2^{-n/2}} + X^2_{(r+k)2^{-n/2}}}{2}\right)\Bigg)\Bigg|.
\end{aligned}$$

By Theorem 4.2.3 and since $f \in C_b^\infty$, we have

$$\begin{aligned}
&\Bigg|R_1\Bigg((X^1_{k2^{-n/2}},X^2_{k2^{-n/2}}), \Bigg(\frac{X^1_{(r+k+1)2^{-n/2}} + X^1_{(r+k)2^{-n/2}}}{2}, \\
&\qquad\qquad \frac{X^2_{(r+k+1)2^{-n/2}} + X^2_{(r+k)2^{-n/2}}}{2}\Bigg)\Bigg)\Bigg| \\
&\leqslant C\Bigg(\left|X^1_{(r+k+1)2^{-n/2}} + X^1_{(r+k)2^{-n/2}} - 2X^1_{k2^{-n/2}}\right| \\
&\qquad\qquad +\left|X^2_{(r+k+1)2^{-n/2}} + X^2_{(r+k)2^{-n/2}} - 2X^2_{k2^{-n/2}}\right|\Bigg).
\end{aligned}$$

We finally get

$$\begin{aligned}
r_{1,n} &\leqslant C \sup_{0\leqslant k\leqslant \lfloor 2^{\frac{n}{2}}t\rfloor -1} \left|X^1_{(r+k+1)2^{-n/2}} + X^1_{(r+k)2^{-n/2}} - 2X^1_{k2^{-n/2}}\right| \\
&\quad +C \sup_{0\leqslant k\leqslant \lfloor 2^{\frac{n}{2}}t\rfloor -1} \left|X^2_{(r+k+1)2^{-n/2}} + X^2_{(r+k)2^{-n/2}} - 2X^2_{k2^{-n/2}}\right|.
\end{aligned}$$

By Heine's theorem, the last quantities converge to 0 almost surely. Thus, $r_{1,n} \to 0$ as $n \to \infty$. We can prove similarly that $r_{2,n} \to 0$ as $n \to \infty$. Finally, it is clear that $r_{3,n} \to 0$ as $n \to \infty$. Hence, we have proved that, for all fixed $r \in \mathbb{Z}$, a.s.,

$$2^{-n/2} \sum_{k=0\vee(-r)}^{(\lfloor 2^{\frac{n}{2}}t\rfloor-1-r)\wedge(\lfloor 2^{\frac{n}{2}}t\rfloor-1)} \Delta_{r+k,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2) \underset{n\to\infty}{\longrightarrow} \int_0^t \big(\partial_{122}f(X_s^1,X_s^2)\big)^2 ds.$$

By combining a bounded convergence argument with (4.3.84) (observe that $\kappa_3^2 := \frac{1}{32}\sum_{r\in\mathbb{Z}}\rho^3(r) < \infty$), we deduce that, a.s.,

$$\frac{1}{32}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle^3 \underset{n\to\infty}{\longrightarrow} \kappa_3^2\int_0^t \big(\partial_{122}f(X_s^1,X_s^2)\big)^2 ds.$$

Thus (4.3.83) holds true.

Since $\big(X^1,X^2,W_n(f,t)\big) \to \big(X^1,X^2,W_\infty(f,t)\big)$ in $D_{\mathbb{R}^2}[0,\infty)\times\mathbb{R}^4$, we deduce the following convergence in law in $\mathbb{R}^3$,

$$\Bigg(e^{i\langle\lambda,W_n(f,t)\rangle},\ \xi,\ \frac{1}{32}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2) \times\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle^3\Bigg) \overset{Law}{\longrightarrow} \Bigg(e^{i\langle\lambda,W_\infty(f,t)\rangle},\ \xi,\ \kappa_3^2\int_0^t\big(\partial_{122}f(X_s^1,X_s^2)\big)^2ds\Bigg).$$

By boundedness of $e^{i\langle\lambda,W_n(f,t)\rangle}$, ξ and $\partial_{122}f$, we deduce that (4.3.82) follows.

- <u>Convergence to 0 of $B_{n,1}^{(3,2)}(t)$, $B_{n,2}^{(3,2)}(t)$ and $B_{n,3}^{(3,2)}(t)$.</u> Let us first focus on $B_{n,1}^{(3,2)}(t)$. Since $f\in C_b^\infty$, $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ are bounded and by Cauchy-Schwarz inequality and (4.2.21), we have

$$\Big|E\Big(\Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\Big| \leqslant C\|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2 \leqslant C2^{-n/6}.$$

Combining this fact with (4.2.32) and (4.2.33), we deduce that

$$\begin{aligned} |B_{n,1}^{(3,2)}(t)| &\leqslant C2^{-n/6}(2^{-n/6})^2\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big| \\ &\leqslant C2^{-n/6}2^{-n/3}t2^{n/3} = Ct2^{-n/6}. \end{aligned}$$

Let us now turn to $B_{n,2}^{(3,2)}(t)$ and $B_{n,3}^{(3,2)}(t)$. Relying to the same arguments, we get

$$\begin{aligned} B_{n,2}^{(3,2)}(t) &\leqslant Ct2^{-n/6} \\ B_{n,3}^{(3,2)}(t) &\leqslant Ct2^{-n/6}. \end{aligned}$$

It is now clear that $B_{n,1}^{(3,2)}(t)$, $B_{n,2}^{(3,2)}(t)$ and $B_{n,3}^{(3,2)}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B_{n,4}^{(3,2)}(t)$ and $B_{n,7}^{(3,2)}(t)$. Let us first focus on $B_{n,4}^{(3,2)}(t)$. Since $f\in C_b^\infty$, $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ are bounded and by Cauchy-Schwarz inequality and (4.2.21) we have

$$\begin{aligned}
&\left|E\bigg(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{122}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\bigg)\right|\\
\leqslant\ & C\|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\|_2\leqslant C2^{-n/12}.
\end{aligned}$$

Combining this fact with (4.2.32) and (4.2.33), we get

$$\begin{aligned}
|B_{n,4}^{(3,2)}(t)| &\leqslant C2^{-n/12}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\right|\\
&\quad\times\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle^2\\
&\leqslant C2^{-n/12}2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle^2\\
&\leqslant C2^{-n/12}2^{-n/6}t2^{n/6}\leqslant Ct2^{-n/12}.
\end{aligned}$$

Let us now turn to $B_{n,7}^{(3,2)}(t)$. By the same arguments, we get

$$|B_{n,7}^{(3,2)}(t)|\leqslant Ct2^{-n/12}.$$

It is now clear that $B_{n,4}^{(3,2)}(t)$ and $B_{n,7}^{(3,2)}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B_{n,5}^{(3,2)}(t)$ and $B_{n,8}^{(3,2)}(t)$. Let us first focus on $B_{n,5}^{(3,2)}(t)$. Since $f\in C_b^\infty$, $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ are bounded and thanks to (4.2.32) and to the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|B_{n,5}^{(3,2)}(t)|\\
\leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\left|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\right|\left|\langle D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}\rangle\right|\bigg)\\
&\times\left|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\right|\\
\leqslant\ & C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\left|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\right|\left|\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\rangle\right|\bigg)\\
&\times\left|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\right|\\
\leqslant\ & C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2\|\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\rangle\|_2\\
&\times\left|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\right|.
\end{aligned}$$

Due to (4.2.21), (4.2.46) and (4.2.33), we have

$$
\begin{aligned}
|B_{n,5}^{(3,2)}(t)| &\leqslant C2^{-n/6}2^{-n/6}2^{-n/6}\big(t^2+t+1\big)^{\frac{1}{2}}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant Ct\big(t^2+t+1\big)^{\frac{1}{2}}2^{-n/6}.
\end{aligned}
$$

Let us now turn to $B_{n,8}^{(3,2)}(t)$. The same arguments shows that

$$
|B_{n,8}^{(3,2)}(t)| \leqslant Ct\big(t^2+t+1\big)^{\frac{1}{2}}2^{-n/6}.
$$

It is now clear that $B_{n,5}^{(3,2)}(t)$ and $B_{n,8}^{(3,2)}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B_{n,6}^{(3,2)}(t)$ and $B_{n,9}^{(3,2)}(t)$. Let us first focus on $B_{n,6}^{(3,2)}(t)$. Since $f\in C_b^\infty$, $e^{i\langle\lambda,W_n(f,t)\rangle}$ is bounded and due to (4.2.32), (4.2.44), (4.2.21) and (4.2.33), we have

$$
\begin{aligned}
|B_{n,6}^{(3,2)}(t)| &\leqslant C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big|\big|\langle D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}\rangle\big|\Big) \\
&\quad\times\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C2^{-n/6}2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C2^{-n/2}t2^{n/3}=Ct2^{-n/6}.
\end{aligned}
$$

Let us now turn to $B_{n,9}^{(3,2)}(t)$. The same arguments shows that

$$
|B_{n,9}^{(3,2)}(t)| \leqslant Ct2^{-n/6}.
$$

We deduce that $B_{n,6}^{(3,2)}(t)$ and $B_{n,9}^{(3,2)}(t)$ both converge to 0 as $n\to\infty$.

- Convergence to 0 of $B_{n,11}^{(3,2)}(t)$: Since $f\in C_b^\infty$, $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ are bounded and due to (4.2.46), (4.2.21) and (4.2.33), we have

$$\begin{aligned}
&|B_{n,11}^{(3,2)}(t)| \\
\leqslant\ & C\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\big|\big|\big\langle D_{X^2}(\langle\lambda, W_n(f,t)\rangle), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big) \\
&\times\big\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & C\sum_{p=1}^{4}|\lambda_p| \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\big|\big|\big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big) \\
&\times\big\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & C\sum_{p=1}^{4}|\lambda_p| \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\|_2\|\big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle\|_2 \\
&\times\big\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & C2^{-n/12}2^{-n/6}(t^2+t+1)^{\frac{1}{2}} \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \big\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & Ct(t^2+t+1)^{\frac{1}{2}}2^{-n/12}.
\end{aligned}$$

Hence, $B_{n,11}^{(3,2)}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B_{n,12}^{(3,2)}(t)$: Since $f\in C_b^\infty$, $e^{i\langle\lambda, W_n(f,t)\rangle}$ is bounded and due to (4.2.44), (4.2.21) and (4.2.33), we have

$$\begin{aligned}
|B_{n,12}^{(3,2)}(t)| \leqslant\ & C\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\big|\big|\big\langle D_{X^2}(\xi), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big) \\
&\times\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle^2 \\
\leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\|_2\langle \delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle^2 \\
\leqslant\ & C2^{-n/12}t.
\end{aligned}$$

Thus, $B_{n,12}^{(3,2)}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B^{(3,2)}_{n,13}(t)$:

$$
\begin{aligned}
|B^{(3,2)}_{n,13}(t)| &\leqslant C\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\big|\langle D^2_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\rangle\big|\Big)\\
&\quad\times\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\\
&\leqslant C\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\big|\langle D^2_{X^2}(K_n^{(p)}),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\rangle\big|\Big)\\
&\quad\times\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\\
&\leqslant C\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2\|\langle D^2_{X^2}(K_n^{(p)}),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\rangle\|_2\\
&\quad\times\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\\
&\leqslant C2^{-n/6}2^{-n/3}(t^2+t+1)^{\frac{1}{2}}2^{n/3}t = C2^{-n/6}(t^2+t+1)^{\frac{1}{2}}t,
\end{aligned}
$$

where the fourth inequality is due to (4.2.21), (4.2.47) and (4.2.33). Hence, $B^{(3,2)}_{n,13}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B^{(3,2)}_{n,14}(t)$: Since $f\in C_b^\infty$ and thanks to (4.2.21), (4.2.48) and (4.2.33), we have

$$
\begin{aligned}
|B^{(3,2)}_{n,14}(t)| &\leqslant C\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\big\langle D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}\big\rangle^2\Big)\\
&\quad\times\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\\
&\leqslant C\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2\|\langle D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}\big\rangle^2\|_2\\
&\quad\times\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\\
&\leqslant C\sum_{p=1}^{4}(\lambda_p)^2\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2\|\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\big\rangle^2\|_2\\
&\quad\times\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\\
&\leqslant C2^{-n/6}2^{-n/3}(1+t+t^2+t^3)^{\frac{1}{2}}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle\big|\\
&\leqslant C2^{-n/6}(1+t+t^2+t^3)^{\frac{1}{2}}t.
\end{aligned}
$$

It is now clear that $B^{(3,2)}_{n,14}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B^{(3,2)}_{n,15}(t)$: Since $f\in C_b^\infty$ and thanks to (4.2.44), (4.2.46), (4.2.21) and (4.2.33), we obtain

$$
\begin{aligned}
|B_{n,15}^{(3,2)}(t)| &\leqslant C \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big|\big|\langle D_{X^2}(\langle\lambda, W_n(f,t)\rangle), \delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\quad \times\big|\langle D_{X^2}(\xi), \delta_{(j+1)2^{-n/2}}\rangle\big|\Big)\big|\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C2^{-n/6}\sum_{p=1}^{4}|\lambda_p| \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big|\big|\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\rangle\big|\Big) \\
&\quad \times\big|\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C2^{-n/6}\sum_{p=1}^{4}|\lambda_p| \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2\|\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\rangle\|_2 \\
&\quad \times\big|\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C(2^{-n/6})^3(t^2+t+1)^{\frac{1}{2}} \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \big|\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C2^{-n/6}(t^2+t+1)^{\frac{1}{2}}t.
\end{aligned}
$$

Hence, $B_{n,15}^{(3,2)}(t) \to 0$ as $n \to \infty$.

- Convergence to 0 of $B_{n,16}^{(3,2)}(t)$: Since $f \in C_b^\infty$ and thanks to (4.2.45), (4.2.21) and (4.2.33), we deduce that

$$
\begin{aligned}
|B_{n,16}^{(3,2)}(t)| &\leqslant C \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big|\big|\langle D_{X^2}^2(\xi), \delta_{(j+1)2^{-n/2}}^{\otimes 2}\rangle\big|\Big) \\
&\quad \times\big|\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C2^{-n/3} \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2\big|\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\rangle\big| \\
&\leqslant C2^{-n/6}t.
\end{aligned}
$$

Thus, $B_{n,16}^{(3,2)}(t) \to 0$ as $n \to \infty$.

Finally, we have shown that

$$
B_n^{(3,2)}(t) \underset{n\to\infty}{\longrightarrow} -\kappa_3^2 E\Big(e^{i\langle\lambda, W_\infty(f,t)\rangle}\xi \times \int_0^t \big(\partial_{122}f(X_s^1, X_s^2)\big)^2 ds\Big),
$$

and (4.3.75) holds true. ■

Recall that we are proving (4.3.73). Moreover, by (4.3.74), $B_n^{(3)}(t) = B_n^{(3,1)}(t) + B_n^{(3,2)}(t)$. So, it remains to prove the convergence to 0 of $B_n^{(3,1)}(t)$.

2. Convergence to 0 of $B_n^{(3,1)}(t)$ as $n\to\infty$.

$$\begin{aligned}B_n^{(3,1)}(t) &= -\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Big\langle D^2_{X^2}\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)\\ &\quad \times I_1^{(1)}(\delta_{(k+1)2^{-n/2}})I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\Big\rangle\Big)\\ &\quad \times\Big\langle\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big),\delta_{(j+1)2^{-n/2}}\Big\rangle.\end{aligned}$$

Observe that

$$\begin{aligned}&D^2_{X^2}\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\\ &\qquad \times e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\ &= I_1^{(1)}(\delta_{(k+1)2^{-n/2}})D^2_{X^2}\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\\ &\qquad \times e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big).\end{aligned}$$

As in the case of (4.3.81), the same calculations show that

$$\begin{aligned}&D^2_{X^2}\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)\\ &= \Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\ &\quad\times\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{\otimes 2}+2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)\\ &\quad\times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big)\\ &\quad+\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{112222}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\ &\quad\times\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big)^{\otimes 2}+4\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)\\ &\quad I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}\delta_{(k+1)2^{-n/2}}\\ &\quad+2i\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\ &\quad\times\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)+2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\\ &\quad\times\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}D_{X^2}(\xi)\end{aligned}$$

$$
\begin{aligned}
&+4\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
&\times\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(k+1)2^{-n/2}}+2i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)\\
&\times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\\
&+2\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\\
&\times\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\xi)+2\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)\\
&\times e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta^{\otimes 2}_{(k+1)2^{-n/2}}+4i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)\\
&\times I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)+4\Delta_{j,n}\partial_{122}f(X^1,X^2)\\
&\times\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\xi)\\
&+i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
&\times D^2_{X^2}(\langle\lambda,W_n(f,t)\rangle)-\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\\
&\times e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big(D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\big)^{\otimes 2}+2i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)\\
&\times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\tilde{\otimes}D_{X^2}(\xi)+\Delta_{j,n}\partial_{122}f(X^1,X^2)\\
&\times\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}D^2_{X^2}(\xi).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
&B_n^{(3,1)}(t)\\
=\;&-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\bigg)\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{\otimes 2},\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\\
&\times\left\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle\\
&-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\bigg(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\bigg)\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right),\right.\\
&\qquad\qquad\left.\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\left\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{112222}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Bigg)\Bigg\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)^{\otimes 2},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Bigg\rangle\\
&\times\Bigg\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Bigg\rangle\\
&-\frac{4}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Bigg)\Bigg\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Bigg\rangle\\
&\times\Bigg\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Bigg\rangle\\
&-\frac{2i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\\
&\times\Bigg\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Bigg\rangle\Bigg)\\
&\times\Bigg\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Bigg\rangle\\
&-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\Bigg\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Bigg\rangle\Bigg)\\
&\times\Bigg\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Bigg\rangle\\
&-\frac{4}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Bigg)\Bigg\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Bigg\rangle\\
&\times\Bigg\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Bigg\rangle\\
&-\frac{2i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Bigg(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Bigg\langle\left(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\\
&\qquad\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Bigg\rangle\Bigg)\times\Bigg\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Bigg\rangle
\end{aligned}
$$

$$-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{11222}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})$$

$$\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\Big\langle\Big(\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Big)$$

$$\times\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle$$

$$-\frac{2}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big)$$

$$\times\langle\delta_{(k+1)2^{-n/2}}^{\otimes 2},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\rangle\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle$$

$$-\frac{4i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})$$

$$\times I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big\langle\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Big)$$

$$\times\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle$$

$$-\frac{4}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})$$

$$\times I_1^{(2)}(\delta_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\Big\langle\delta_{(k+1)2^{-n/2}}\tilde{\otimes}D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Big)$$

$$\times\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle$$

$$-\frac{i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})$$

$$\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big\langle D_{X^2}^2(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Big)$$

$$\times\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle$$

$$+\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})$$

$$\times I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})e^{i\langle\lambda,W_n(f,t)\rangle}\xi\Big\langle\big(D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\big)^{\otimes 2},\delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle\Big)$$

$$\times\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle$$

$$-\frac{2i}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\times$$

$$
\begin{aligned}
&I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\Big\langle D_{X^2}(\langle\lambda,W_n(f,t)\rangle)\tilde{\otimes}D_{X^2}(\xi),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\Big\rangle\Big)\\
&\times\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\\
&-\frac{1}{64}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{k,n}\partial_{1122}f(X^1,X^2)I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\\
&\times I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})e^{i\langle\lambda,W_n(f,t)\rangle}\Big\langle D^2_{X^2}(\xi),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\Big\rangle\Big)\\
&\times\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\\
&=\sum_{i=1}^{16}B_{n,i}^{(3,1)}(t).
\end{aligned}
$$

Let us prove the convergence to 0, as $n\to\infty$, of $B_{n,i}^{(3,1)}(t)$ for $i\in\{1,\dots,16\}$.

- Convergence to 0 of $B_{n,1}^{(3,1)}(t)$, $B_{n,2}^{(3,1)}(t)$ and $B_{n,3}^{(3,1)}(t)$. Since $f\in C_b^\infty$ and thanks to (4.2.32), (4.2.21), (4.2.34) and (4.2.35) we have

$$
\begin{aligned}
|B_{n,1}^{(3,1)}(t)| &\leqslant C(2^{-n/6})^2\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\Big)\\
&\quad\times\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\leqslant C(2^{-n/6})^2\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2\\
&\quad\times\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\leqslant C2^{-n/2}2^{-n/12}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
&\leqslant C2^{-n/2}2^{-n/12}(2^{-n/6}+2^{n/2}t^{4/3})=C2^{-3n/4}+C2^{-n/12}t^{4/3}.
\end{aligned}
$$

Hence, $B_{n,1}^{(3,1)}(t)\to 0$ as $n\to\infty$.
The same arguments show that

$$
\begin{aligned}
|B_{n,2}^{(3,1)}(t)| &\leqslant C2^{-3n/4}+C2^{-n/12}t^{4/3},\\
|B_{n,3}^{(3,1)}(t)| &\leqslant C2^{-3n/4}+C2^{-n/12}t^{4/3}.
\end{aligned}
$$

Thus, $B_{n,2}^{(3,1)}(t)$ and $B_{n,3}^{(3,1)}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B_{n,4}^{(3,1)}(t)$ and $B_{n,7}^{(3,1)}(t)$. By the same arguments that was used previously, we have

$$\begin{aligned}|B_{n,3}^{(3,1)}(t)| &\leqslant C(2^{-n/6})^2\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\|_2\\ &\qquad\qquad\times\left|\left\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle\right|\\ &\leqslant C2^{-n/6}t,\end{aligned}$$

where the last inequality follows from (4.2.21) and (4.2.33).
We can prove similarly that $|B_{n,7}^{(3,1)}(t)|\leqslant C2^{-n/6}t$. Thus, $B_{n,4}^{(3,1)}(t)$ and $B_{n,7}^{(3,1)}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B_{n,5}^{(3,1)}(t)$ and $B_{n,8}^{(3,1)}(t)$.

$$\begin{aligned}&|B_{n,5}^{(3,1)}(t)|\\ \leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\left|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\right|\left|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\right|\\ &\qquad\qquad\times\left|\left\langle D_{X^2}(\langle\lambda,W_n(f,t)\rangle),\delta_{(j+1)2^{-n/2}}\right\rangle\right|\Big)\\ &\times\left|\left\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle\right|\\ \leqslant\ & C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\left|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\right|\left|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\right|\\ &\qquad\qquad\times\left|\left\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\right\rangle\right|\Big)\\ &\times\left|\left\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle\right|.\end{aligned}$$

Observe that by the Cauchy-Schwarz inequality, the independence of $X^{(1)}$ and $X^{(2)}$, (4.2.21) and (4.2.46), we get, for all $p\in\{1,\ldots,4\}$,

$$\begin{aligned}&E\Big(\left|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\right|\left|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\right|\left|\left\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\right\rangle\right|\Big)\\ \leqslant\ & \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2\|\left\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\right\rangle\|_2\\ =\ & \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2\|\left\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\right\rangle\|_2\\ \leqslant\ & C2^{-n/12}2^{-n/6}2^{-n/6}(t^2+t+1)^{\frac{1}{2}}.\end{aligned}$$

We deduce that

$$\begin{aligned}&|B_{n,5}^{(3,1)}(t)|\\ \leqslant\ & C2^{-n/12}2^{-n/2}(t^2+t+1)^{\frac{1}{2}}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|\left\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\right\rangle\right|\\ \leqslant\ & C2^{-3n/4}(t^2+t+1)^{\frac{1}{2}}+C2^{-n/12}(t^2+t+1)^{\frac{1}{2}}t^{4/3}.\end{aligned}$$

By the same arguments, we get

$$|B_{n,8}^{(3,1)}(t)| \leqslant C2^{-3n/4}(t^2+t+1)^{\frac{1}{2}} + C2^{-n/12}(t^2+t+1)^{\frac{1}{2}}t^{4/3}.$$

Hence, we deduce that $B_{n,5}^{(3,1)}(t)$ and $B_{n,8}^{(3,1)}(t)$ converge to 0 as $n \to \infty$.

- Convergence to 0 of $B_{n,6}^{(3,1)}(t)$ and $B_{n,9}^{(3,1)}(t)$. We can write, using (4.2.44) among other things and following the same route as previously,

$$\begin{aligned}
&|B_{n,6}^{(3,1)}(t)| \\
\leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big|\big|\langle D_{X^2}(\xi),\delta_{(j+1)2^{-n/2}}\rangle\big|\Big) \\
&\times\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
\leqslant\ & C2^{-n/3}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2 \\
&\times\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
\leqslant\ & C2^{-n/12}2^{-n/2}(2^{-n/6}+2^{n/2}t^{4/3}) = C2^{-n/4}+C2^{-n/12}t^{4/3}.
\end{aligned}$$

We can prove similarly that

$$|B_{n,9}^{(3,1)}(t)| \leqslant C2^{-n/4} + C2^{-n/12}t^{4/3}.$$

Hence, it is now clear that $B_{n,6}^{(3,1)}(t)$ and $B_{n,9}^{(3,1)}(t)$ converge to 0 as $n \to \infty$.

- Convergence to 0 of $B_{n,10}^{(3,1)}(t)$. We can write, using (4.2.33) and proceeding as previously,

$$\begin{aligned}
&|B_{n,10}^{(3,1)}(t)| \\
\leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\Big)\big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & C2^{-n/6}2^{-n/12}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \big\langle\delta_{(k+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle^2 \\
\leqslant\ & C2^{-n/12}t.
\end{aligned}$$

Hence, $B_{n,10}^{(3,1)}(t) \to 0$ as $n \to \infty$.

- Convergence to 0 of $B_{n,11}^{(3,1)}(t)$. By similar arguments, we have

$$\begin{aligned}
&|B_{n,11}^{(3,1)}(t)|\\
\leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\big|\\
&\times\big|\big\langle D_{X^2}(\langle\lambda, W_n(f,t)\rangle), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\big|\big\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\ & C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\big|\\
&\times\big|\big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\big|\big\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big|.
\end{aligned}$$

Observe that, using the Cauchy-Schwarz inequality, (4.2.21) and (4.2.46), we get

$$\begin{aligned}
&E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\big|\big|\big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\\
\leqslant\ & \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\|_2\|\big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle\|_2\\
=\ & \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\|_2\|\big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle\|_2\\
\leqslant\ & C(2^{-n/12})^2 2^{-n/6}(t^2+t+1)^{\frac{1}{2}}.
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
|B_{n,11}^{(3,1)}(t)| &\leqslant C2^{-n/2}(t^2+t+1)^{\frac{1}{2}}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\big\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
&\leqslant C2^{-n/6}(t^2+t+1)^{\frac{1}{2}}t.
\end{aligned}$$

Thus, it is now clear that $B_{n,11}^{(3,1)}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B_{n,12}^{(3,1)}(t)$. Since $f\in C_b^\infty$ and thanks to (4.2.32), (4.2.21), (4.2.44) and (4.2.33), we have

$$\begin{aligned}
&|B_{n,12}^{(3,1)}(t)|\\
\leqslant\ & C2^{-n/6}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\big|\big|\big\langle D_{X^2}(\xi), \delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\\
&\times\big|\big\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\ & C2^{-n/3}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_1^{(2)}(\delta_{(k+1)2^{-n/2}})\|_2\big|\big\langle\delta_{(k+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\ & C2^{-n/3}(2^{-n/12})^2 2^{n/3}t = C2^{-n/6}t.
\end{aligned}$$

Hence, $B_{n,12}^{(3,1)}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B^{(3,1)}_{n,13}(t)$. By similar arguments as before, we have

$$
\begin{aligned}
&|B^{(3,1)}_{n,13}(t)|\\
\leqslant\ & \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\big|\big\langle D^2_{X^2}(\langle\lambda, W_n(f,t)\rangle), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\\
&\times\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
\leqslant\ & \sum_{p=1}^{4}|\lambda_p| \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\\
&\times\big|\big\langle D^2_{X^2}(K_n^{(p)}(t)), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle\Big|.
\end{aligned}
$$

Observe that, thanks to Cauchy-Schwarz inequality, the independence of X^1 and X^2, (4.2.21) and (4.2.47), we have

$$
\begin{aligned}
& E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\big|\big\langle D^2_{X^2}(K_n^{(p)}(t)), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\\
\leqslant\ & \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2\|\big\langle D^2_{X^2}(K_n^{(p)}(t)), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\|_2\\
=\ & \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2\|\big\langle D^2_{X^2}(K_n^{(p)}(t)), \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\|_2\\
\leqslant\ & C2^{-n/12}2^{-n/6}2^{-n/3}(t^2+t+1)^{\frac{1}{2}}.
\end{aligned}
$$

So, we deduce that

$$
\begin{aligned}
&|B^{(3,1)}_{n,13}(t)|\\
\leqslant\ & C2^{-n/12}2^{-n/2}(t^2+t+1)^{\frac{1}{2}} \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|\Big\langle\frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle\Big|\\
\leqslant\ & C2^{-n/12}2^{-n/2}(t^2+t+1)^{\frac{1}{2}}(2^{-n/6}+2^{n/2}t^{4/3})\\
=\ & C2^{-n/4}(t^2+t+1)^{\frac{1}{2}}+C2^{-n/12}(t^2+t+1)^{\frac{1}{2}}t^{4/3}.
\end{aligned}
$$

We deduce that $B^{(3,1)}_{n,13}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B_{n,14}^{(3,1)}(t)$. Using (4.2.48) among other things, we obtain

$$
\begin{aligned}
& |B_{n,14}^{(3,1)}(t)| \\
\leqslant\ & \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big| \big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big| \big\langle D_{X^2}^2(\langle \lambda, W_n(f,t)\rangle), \delta_{(j+1)2^{-n/2}}^{\otimes 2}\big\rangle^2 \Big) \\
& \times \Big| \Big\langle \frac{\varepsilon_{k2^{-n/2}} + \varepsilon_{(k+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \Big\rangle \Big| \\
\leqslant\ & C \sum_{p=1}^{4} (\lambda_p)^2 \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big| \big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big| \\
& \times \big\langle D_{X^2}^2(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}^{\otimes 2}\big\rangle^2 \Big) \Big| \Big\langle \frac{\varepsilon_{k2^{-n/2}} + \varepsilon_{(k+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \Big\rangle \Big| \\
\leqslant\ & C \sum_{p=1}^{4} (\lambda_p)^2 \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2 \|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\|_2 \\
& \times \|\big\langle D_{X^2}^2(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}^{\otimes 2}\big\rangle^2\|_2 \Big| \Big\langle \frac{\varepsilon_{k2^{-n/2}} + \varepsilon_{(k+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \Big\rangle \Big| \\
\leqslant\ & C 2^{-n/12} 2^{-n/6} 2^{-n/3} (t^3+t^2+t+1)^{\frac{1}{2}} (2^{-n/6} + 2^{n/2} t^{4/3}) \\
=\ & C 2^{-n/4} (t^3+t^2+t+1)^{\frac{1}{2}} + C 2^{-n/12} (t^3+t^2+t+1)^{\frac{1}{2}} t^{4/3}.
\end{aligned}
$$

Hence, $B_{n,14}^{(3,1)}(t) \to 0$ as $n \to \infty$.

- Convergence to 0 of $B_{n,15}^{(3,1)}(t)$. Arguing as previously, we obtain

$$
\begin{aligned}
& |B_{n,15}^{(3,1)}(t)| \\
\leqslant\ & C \sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big| \big|I_2^{(2)}(\delta_{(k+1)2^{-n/2}}^{\otimes 2})\big| \\
& \times \big|\big\langle D_{X^2}(\langle \lambda, W_n(f,t)\rangle), \delta_{(j+1)2^{-n/2}}\big\rangle\big| \big|\big\langle D_{X^2}(\xi), \delta_{(j+1)2^{-n/2}}\big\rangle\big| \Big) \\
& \times \Big| \Big\langle \frac{\varepsilon_{k2^{-n/2}} + \varepsilon_{(k+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}} \Big\rangle \Big|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big| \\
&\quad \times\big|\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\rangle\big|\Big) \\
&\quad \times\Big|\Big\langle \frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
&\leqslant C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2 \\
&\quad \times\|\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\rangle\|_2 \\
&\quad \times\Big|\Big\langle \frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle\Big| \\
&\leqslant C2^{-n/6}2^{-n/12}2^{-n/6}2^{-n/6}(t^2+t+1)^{\frac{1}{2}}(2^{-n/6}+2^{n/2}t^{4/3}) \\
&= C2^{-n/4}(t^2+t+1)^{\frac{1}{2}}+C2^{-n/12}(t^2+t+1)^{\frac{1}{2}}t^{4/3}.
\end{aligned}
$$

So, we deduce that $B^{(3,1)}_{n,15}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B^{(3,1)}_{n,16}(t)$. Thanks to (4.2.45) among other things, we obtain

$$
\begin{aligned}
&|B^{(3,1)}_{n,16}(t)| \\
&\leqslant C\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\big|\big|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\big|\big|\langle D^2_{X^2}(\xi),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\rangle\big|\Big) \\
&\quad \times\Big\langle \frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle \\
&\leqslant C2^{-n/3}\sum_{j,k=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_1^{(1)}(\delta_{(k+1)2^{-n/2}})\|_2\|I_2^{(2)}(\delta^{\otimes 2}_{(k+1)2^{-n/2}})\|_2 \\
&\quad \times\Big\langle \frac{\varepsilon_{k2^{-n/2}}+\varepsilon_{(k+1)2^{-n/2}}}{2},\delta_{(j+1)2^{-n/2}}\Big\rangle \\
&\leqslant C2^{-n/3}2^{-n/12}2^{-n/6}(2^{-n/6}+2^{n/2}t^{4/3}) \\
&= C2^{-n/4}+C2^{-n/12}t^{4/3}.
\end{aligned}
$$

Hence, we get that $B^{(3,1)}_{n,16}(t)\to 0$ as $n\to\infty$.

Finally, we have shown that $B^{(3,1)}_{n}(t)\to 0$ as $n\to\infty$.

We have proved that

$$B^{(3,2)}_{n}(t)\underset{n\to\infty}{\longrightarrow}-\kappa_3^2 E\Big(e^{i\langle\lambda,W_\infty(f,t)\rangle}\xi\times\int_0^t\big(\partial_{122}f(X^1_s,X^2_s)\big)^2ds\Big),$$

and

$$B^{(3,1)}_{n}(t)\underset{n\to\infty}{\longrightarrow}0.$$

Taking into account that $B_n^{(3)}(t) = B_n^{(3,1)}(t) + B_n^{(3,2)}(t)$ by (4.3.74), we deduce that

$$B_n^{(3)}(t) \underset{n\to\infty}{\longrightarrow} -\kappa_3^2 E\Big(e^{i\langle\lambda, W_\infty(f,t)\rangle}\xi \times \int_0^t \big(\partial_{122} f(X_s^1, X_s^2)\big)^2 ds\Big).$$

Consequently, (4.3.73) holds true. ■

In order to prove (4.3.71), it remains to prove the convergence to 0 of $B_n^{(p)}(t)$, defined in (4.3.70), for $p = 1, 2, 4$.

Proof of the convergence to 0 of $B_n^{(1)}(t)$.

$$B_n^{(1)}(t) = -\frac{1}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Delta_{j,n}\partial_{122} f(X^1, X^2) e^{i\langle\lambda, W_n(f,t)\rangle}\xi \,\langle D_{X^1}\big(K_n^{(1)}(t)\big), \delta_{(j+1)2^{-n/2}}\rangle \times I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\Big).$$

Recall that, by Definition 4.2.2,

$$K_n^{(1)}(t) = \frac{1}{24}\sum_{l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{l,n}\partial_{111} f(X^1, X^2) I_3^{(1)}\big(\delta_{(l+1)2^{-n/2}}^{\otimes 3}\big).$$

We deduce that

$$\begin{aligned} & B_n^{(1)}(t) \\ = \ & -\frac{1}{64}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Delta_{j,n}\partial_{122} f(X^1, X^2)\Delta_{l,n}\partial_{111} f(X^1, X^2) I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2}) \\ & \times I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2}) e^{i\langle\lambda, W_n(f,t)\rangle}\xi\Big)\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle \\ & -\frac{1}{8\times 24}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Delta_{j,n}\partial_{122} f(X^1, X^2)\Delta_{l,n}\partial_{1111} f(X^1, X^2) I_3^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 3}) \\ & \times I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2}) e^{i\langle\lambda, W_n(f,t)\rangle}\xi\Big)\Big\langle \frac{\varepsilon_{l2^{-n/2}} + \varepsilon_{(l+1)2^{-n/2}}}{2}, \delta_{(j+1)2^{-n/2}}\Big\rangle \\ = \ & B_n^{(1,1)}(t) + B_n^{(1,2)}(t). \end{aligned}$$

Let us prove the convergence to 0 of $B_n^{(1,1)}(t)$ and $B_n^{(1,2)}(t)$.

1. Convergence to 0 of $B_n^{(1,1)}(t)$. Observe that, thanks to (4.2.19), we have

$$\begin{aligned} & B_n^{(1,1)}(t) \qquad\qquad (4.3.85) \\ = \ & -\frac{1}{64}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Big\langle D_{X^2}^2\big(\Delta_{j,n}\partial_{122} f(X^1, X^2)\Delta_{l,n}\partial_{111} f(X^1, X^2) \\ & e^{i\langle\lambda, W_n(f,t)\rangle}\xi\big), \delta_{(j+1)2^{-n/2}}^{\otimes 2}\Big\rangle I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\Big)\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle. \end{aligned}$$

We have shown before that

$$\begin{aligned}
& D^2_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big) \\
= \;& D^2_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)\big)e^{i\langle\lambda,W_n(f,t)\rangle}\xi \\
& +2D_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)\big)\tilde{\otimes}D_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi) \\
& +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)D^2_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi).
\end{aligned}$$

that

$$\begin{aligned}
& D_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)\big) \\
= \;& \Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right) \\
& +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),
\end{aligned}$$

that

$$\begin{aligned}
& D^2_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)\big) \\
= \;& \Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{\otimes 2} \\
& +2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes} \\
& \qquad\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right) \\
& +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{11122}f(X^1,X^2)\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)^{\otimes 2},
\end{aligned}$$

that

$$D_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi) = ie^{i\langle\lambda,W_n(f,t)\rangle}\xi D_{X^2}\langle\lambda,W_n(f,t)\rangle + e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}\xi,$$

and that

$$\begin{aligned}
& D^2_{X^2}(e^{i\langle\lambda,W_n(f,t)\rangle}\xi) \\
= \;& -e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big)^{\otimes 2} + ie^{i\langle\lambda,W_n(f,t)\rangle}\xi D^2_{X^2}\langle\lambda,W_n(f,t)\rangle \\
& +2ie^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}\langle\lambda,W_n(f,t)\rangle\tilde{\otimes}D_{X^2}\xi + e^{i\langle\lambda,W_n(f,t)\rangle}D^2_{X^2}\xi.
\end{aligned}$$

We deduce that

$$
\begin{aligned}
& D^2_{X^2}\big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big) \qquad (4.3.86)\\
= \quad & \Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)^{\otimes 2}\\
& +2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\\
& \qquad\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)\\
& +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{11122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)^{\otimes 2}\\
& +2i\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\\
& \qquad D_{X^2}\langle\lambda,W_n(f,t)\rangle\\
& +2i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)\tilde{\otimes}\\
& \qquad D_{X^2}\langle\lambda,W_n(f,t)\rangle\\
& +2\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}\xi\\
& +2\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right)\tilde{\otimes}D_{X^2}\xi\\
& -\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi\big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big)^{\otimes 2}\\
& +i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi D^2_{X^2}\langle\lambda,W_n(f,t)\rangle\\
& +2i\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}D_{X^2}\langle\lambda,W_n(f,t)\rangle\tilde{\otimes}D_{X^2}\xi\\
& +\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}D^2_{X^2}\xi.
\end{aligned}
$$

By plugging (4.3.86) into (4.3.85), we deduce that

$$
\begin{aligned}
& B_n^{(1,1)}(t)\\
= \quad & -\frac{1}{64}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{12222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\Big)\\
& \times\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle^2\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle\\
& -\frac{1}{32}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\Big)\times
\end{aligned}
$$

$$\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$\times\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$-\frac{1}{64}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{11122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\Big)$$
$$\times\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle^2\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$-\frac{i}{32}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})$$
$$\left\langle D_{X^2}\langle\lambda,W_n(f,t)\rangle,\delta_{(j+1)2^{-n/2}}\right\rangle\Big)\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$\times\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$-\frac{i}{32}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})$$
$$\left\langle D_{X^2}\langle\lambda,W_n(f,t)\rangle,\delta_{(j+1)2^{-n/2}}\right\rangle\Big)\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$\times\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$-\frac{1}{32}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{1222}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})$$
$$\left\langle D_{X^2}\xi,\delta_{(j+1)2^{-n/2}}\right\rangle\Big)\left\langle\left(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$\times\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$-\frac{1}{32}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{1112}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})$$
$$\left\langle D_{X^2}\xi,\delta_{(j+1)2^{-n/2}}\right\rangle\Big)\left\langle\left(\frac{\varepsilon_{l2^{-n/2}}+\varepsilon_{(l+1)2^{-n/2}}}{2}\right),\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$\times\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$+\frac{1}{64}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})$$
$$\left\langle\big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big)^{\otimes 2},\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\Big)\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$
$$-\frac{i}{64}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)\Delta_{l,n}\partial_{111}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\xi I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})$$
$$\left\langle D^2_{X^2}\langle\lambda,W_n(f,t)\rangle,\delta^{\otimes 2}_{(j+1)2^{-n/2}}\right\rangle\Big)\left\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\right\rangle$$

$$
\begin{aligned}
& -\frac{i}{32} \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Delta_{j,n}\partial_{122} f(X^1,X^2)\Delta_{l,n}\partial_{111} f(X^1,X^2) e^{i\langle\lambda,W_n(f,t)\rangle} I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}}) \\
& \big\langle \big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big)\tilde{\otimes} D_{X^2}\xi, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\Big)\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle \\
& -\frac{1}{64} \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Delta_{j,n}\partial_{122} f(X^1,X^2)\Delta_{l,n}\partial_{111} f(X^1,X^2) e^{i\langle\lambda,W_n(f,t)\rangle} I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}}) \\
& \big\langle D^2_{X^2}\xi, \delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\Big)\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle, \\
= \; & \sum_{i=1}^{11} B^{(1,1)}_{n,i}(t).
\end{aligned}
$$

Let us prove the convergence to 0 of $B^{(1,1)}_{n,i}(t)$ for all $i \in \{1,\dots,11\}$.

- Convergence to 0 of $B^{(1,1)}_{n,1}(t)$, $B^{(1,1)}_{n,2}(t)$ and $B^{(1,1)}_{n,3}(t)$. Since $f \in C^\infty_b$, $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ are bounded and thanks to (4.2.32), the Cauchy-Schwarz inequality, (4.2.21) and (4.2.33), we deduce that

$$
\begin{aligned}
& |B^{(1,1)}_{n,1}(t)| \\
\leqslant\; & C(2^{-n/6})^2 \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\big|I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\big|\Big)\big|\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\; & C2^{-n/3} \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \|I_2^{(1)}(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\|_2\big|\big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\; & C2^{-n/3}2^{-n/6}2^{n/3}t = C2^{-n/6}t.
\end{aligned}
$$

  We can prove by the same arguments that

$$
\begin{aligned}
|B^{(1,1)}_{n,2}(t)| &\leqslant C2^{-n/6}t, \\
|B^{(1,1)}_{n,3}(t)| &\leqslant C2^{-n/6}t.
\end{aligned}
$$

  Hence, $B^{(1,1)}_{n,1}(t)$, $B^{(1,1)}_{n,2}(t)$ and $B^{(1,1)}_{n,3}(t)$ converge to 0 as $n\to\infty$.

- Convergence to 0 of $B^{(1,1)}_{n,4}(t)$ and $B^{(1,1)}_{n,5}(t)$. Since $f \in C^\infty_b$, $e^{i\langle\lambda,W_n(f,t)\rangle}$ and ξ are bounded and thanks to (4.2.32), the Cauchy-Schwarz inequality, (4.2.21), (4.2.46) and (4.2.33), we have

$$
\begin{aligned}
&|B_{n,4}^{(1,1)}(t)| \\
\leqslant\ & C2^{-n/6} \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\big| I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big| \big| \big\langle D_{X^2}\langle \lambda, W_n(f,t)\rangle, \delta_{(j+1)2^{-n/2}}\big\rangle \big| \Big) \\
&\times \big| \big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}} \big\rangle \big| \\
\leqslant\ & C2^{-n/6} \sum_{p=1}^{4} |\lambda_p| \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\big| I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big| \big| \big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle \big| \Big) \\
&\times \big| \big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}} \big\rangle \big| \\
\leqslant\ & C2^{-n/6} \sum_{p=1}^{4} |\lambda_p| \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \| I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\|_2 \| \big\langle D_{X^2}(K_n^{(p)}(t)), \delta_{(j+1)2^{-n/2}}\big\rangle \|_2 \\
&\times \big| \big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}} \big\rangle \big| \\
\leqslant\ & C(2^{-n/6})^2 2^{-n/6} (t^2+t+1)^{\frac{1}{2}} 2^{n/3} t = C2^{-n/6}(t^2+t+1)^{\frac{1}{2}} t.
\end{aligned}
$$

We can prove similarly that $|B_{n,5}^{(1,1)}(t)| \leqslant C2^{-n/6}(t^2+t+1)^{\frac{1}{2}} t$. We deduce that $B_{n,4}^{(1,1)}(t)$ and $B_{n,5}^{(1,1)}(t)$ converge to 0 as $n \to \infty$.

- Convergence to 0 of $B_{n,6}^{(1,1)}(t)$ and $B_{n,7}^{(1,1)}(t)$. Since $f \in C_b^\infty$, $e^{i\langle \lambda, W_n(f,t)\rangle}$ and ξ are bounded and thanks to (4.2.32), (4.2.44), the Cauchy-Schwarz inequality, (4.2.21) and (4.2.33), we get

$$
\begin{aligned}
&|B_{n,6}^{(1,1)}(t)| \\
\leqslant\ & C2^{-n/6} \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\big| I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big| \big| \big\langle D_{X^2}(\xi), \delta_{(j+1)2^{-n/2}}\big\rangle \big| \Big) \\
&\times \big| \big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}} \big\rangle \big| \\
\leqslant\ & C2^{-n/3} \sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \| I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\|_2 \big| \big\langle \delta_{(l+1)2^{-n/2}}, \delta_{(j+1)2^{-n/2}} \big\rangle \big| \\
\leqslant\ & C2^{-n/3} 2^{-n/6} 2^{n/3} t = C2^{-n/6} t.
\end{aligned}
$$

We can prove similarly that $|B_{n,7}^{(1,1)}(t)| \leqslant C2^{-n/6}t$. Thus, we get that $B_{n,6}^{(1,1)}(t)$ and $B_{n,7}^{(1,1)}(t)$ converge to 0 as $n \to \infty$.

- Convergence to 0 of $B_{n,8}^{(1,1)}(t)$. Thanks to (4.2.48) among other things, we deduce that

$$
\begin{aligned}
&|B^{(1,1)}_{n,8}(t)| \\
\leqslant\;& C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I^{(1)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\big|\big\langle D_{X^2}\langle\lambda,W_n(f,t)\rangle,\delta_{(j+1)2^{-n/2}}\big\rangle^2\Big)\\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C\sum_{p=1}^{4}\lambda_p^2\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I^{(1)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\big|\big\langle D_{X^2}(K^{(p)}_n(t)),\delta_{(j+1)2^{-n/2}}\big\rangle^2\Big)\\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C\sum_{p=1}^{4}\lambda_p^2\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I^{(1)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\|_2\|\big\langle D_{X^2}(K^{(p)}_n(t)),\delta_{(j+1)2^{-n/2}}\big\rangle^2\|_2\\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C2^{-n/6}2^{-n/3}(t^3+t^2+t+1)^{\frac{1}{2}}2^{n/3}t = C2^{-n/6}(t^3+t^2+t+1)^{\frac{1}{2}}t.
\end{aligned}
$$

It is now clear that $B^{(1,1)}_{n,8}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B^{(1,1)}_{n,9}(t)$. Thanks to (4.2.47) among other things, we deduce that

$$
\begin{aligned}
&|B^{(1,1)}_{n,9}(t)| \\
\leqslant\;& C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I^{(1)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\big|\big|\big\langle D^2_{X^2}\langle\lambda,W_n(f,t)\rangle,\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C\sum_{p=1}^{4}|\lambda_p|\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I^{(1)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\big|\big|\big\langle D^2_{X^2}(K^{(p)}_n(t)),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C\sum_{p=1}^{4}|\lambda_p|\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I^{(1)}_2(\delta^{\otimes 2}_{(l+1)2^{-n/2}})\|_2\|\big\langle D^2_{X^2}(K^{(p)}_n(t)),\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big\rangle\|_2\\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\\
\leqslant\;& C2^{-n/6}2^{-n/3}(t^2+t+1)^{\frac{1}{2}}2^{n/3}t = C2^{-n/6}(t^2+t+1)^{\frac{1}{2}}t.
\end{aligned}
$$

We deduce that $B^{(1,1)}_{n,9}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B^{(1,1)}_{n,10}(t)$. Thanks to (4.2.44) and (4.2.46) among other things, we have

$$
\begin{aligned}
&|B_{n,10}^{(1,1)}(t)| \\
\leqslant\ & C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big|\big|\big\langle\big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big),\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
&\times\big|\big\langle D_{X^2}\xi,\delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big)\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C2^{-n/6}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big|\big|\big\langle\big(D_{X^2}\langle\lambda,W_n(f,t)\rangle\big),\delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big) \\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big|\big|\big\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\big\rangle\big|\Big) \\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C2^{-n/6}\sum_{p=1}^{4}|\lambda_p|\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\|_2\|\big\langle D_{X^2}(K_n^{(p)}(t)),\delta_{(j+1)2^{-n/2}}\big\rangle\|_2 \\
&\times\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C(2^{-n/6})^2 2^{-n/6}(t^2+t+1)^{\frac{1}{2}}2^{n/3}t = C2^{-n/6}(t^2+t+1)^{\frac{1}{2}}t.
\end{aligned}
$$

Thus, $B_{n,10}^{(1,1)}(t)\to 0$ as $n\to\infty$.

- Convergence to 0 of $B_{n,11}^{(1,1)}(t)$: Thanks to (4.2.45) among other things, we have

$$
\begin{aligned}
&|B_{n,11}^{(1,1)}(t)| \\
\leqslant\ & C\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\big|I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\big|\big|\big\langle D_{X^2}^2\xi,\delta_{(j+1)2^{-n/2}}^{\otimes 2}\big\rangle\big|\Big)\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C2^{-n/3}\sum_{j,l=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} \|I_2^{(1)}(\delta_{(l+1)2^{-n/2}}^{\otimes 2})\|_2\big|\big\langle\delta_{(l+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \\
\leqslant\ & C2^{-n/3}2^{-n/6}2^{n/3}t = C2^{-n/6}t.
\end{aligned}
$$

It is now clear that $B_{n,11}^{(1,1)}(t)\to 0$ as $n\to\infty$.

Finally we have shown that $B_n^{(1,1)}(t)\to 0$ as $n\to\infty$.

2. Convergence to 0 of $B_n^{(1,2)}(t)$. The proof is very similar to the previous one and is left to the reader.

Proof of the convergence to 0 of $B_n^{(2)}(t)$ and $B_n^{(4)}(t)$.

The motivated reader may check that there is no additional difficulties to prove the convergence to 0 of $B_n^{(p)}(t)$ for $p\in\{2,4\}$. Indeed, all the arguments and techniques which are needed to this proof, were already introduced and used along the analysis of the asymptotic behaviour of $B_n^{(p)}(t)$ for $p \in \{1,3\}$.

Hence, we may consider that the proof of (4.3.71) is done.

Step 5: Convergence of $C_n(t)$ to 0.

Since $e^{i\langle\lambda,W_n(f,t)\rangle}$ is bounded and $f \in C_b^\infty(\mathbb{R}^2)$, we deduce that

$$\begin{aligned} |C_n(t)| &= \Bigg|\frac{i}{8}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\Big(\Delta_{j,n}\partial_{122}f(X^1,X^2)e^{i\langle\lambda,W_n(f,t)\rangle}\,\big\langle D_{X^1}\xi,\delta_{(j+1)2^{-n/2}}\big\rangle I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\Big)\Bigg| \\ &\leqslant C\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} E\big(\big|\big\langle D_{X^1}\xi,\delta_{(j+1)2^{-n/2}}\big\rangle\big|\big|I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\big|\big). \end{aligned}$$

Observe that

$$D_{X^1}\xi = \sum_{k=1}^{r}\frac{\partial\psi}{\partial x_{2k-1}}(X_{s_1}^1,X_{s_1}^2,\ldots,X_{s_r}^1,X_{s_r}^2)\varepsilon_{s_k}.$$

Since $\psi \in C_b^\infty(\mathbb{R}^{2r})$, we get

$$\big|\big\langle D_{X^1}\xi,\delta_{(j+1)2^{-n/2}}\big\rangle\big| \leqslant C\sum_{k=1}^{r}\big|\big\langle\varepsilon_{s_k},\delta_{(j+1)2^{-n/2}}\big\rangle\big|.$$

We deduce that

$$\begin{aligned} |C_n(t)| &= C\sum_{k=1}^{r}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\big\langle\varepsilon_{s_k},\delta_{(j+1)2^{-n/2}}\big\rangle\big|E\big(\big|I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\big|\big) \\ &\leqslant C\sum_{k=1}^{r}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\big\langle\varepsilon_{s_k},\delta_{(j+1)2^{-n/2}}\big\rangle\big|\|I_2^{(2)}(\delta_{(j+1)2^{-n/2}}^{\otimes 2})\|_2 \\ &\leqslant C2^{-n/6}\sum_{k=1}^{r}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\big|\big\langle\varepsilon_{s_k},\delta_{(j+1)2^{-n/2}}\big\rangle\big| \leqslant C2^{-n/6}t^{1/3}, \end{aligned}$$

where the third inequality follows from (4.2.21) and the last one by (4.2.39). It is now clear that $C_n(t)$ converges to 0.

Finally, putting together the respective conclusions of Steps 1 to 5 lead to the end of the proof ofTheorem 4.3.3. ■

Definition 4.3.4 *For $f \in C_b^\infty(\mathbb{R}^2)$, for all $t \geqslant 0$, we define $V_n^3(f,t)$ as follows:*

$$\begin{aligned}
V_n^3(f,t) &:= \frac{1}{24}2^{-\frac{3nH}{2}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{111} f(X^1,X^2)\big(X^{1,n}_{j+1}-X^{1,n}_j\big)^3 \\
&\quad + \frac{1}{24}2^{-\frac{3nH}{2}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{222} f(X^1,X^2)\big(X^{2,n}_{j+1}-X^{2,n}_j\big)^3 \\
&\quad + \frac{1}{8}2^{-\frac{3nH}{2}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{122} f(X^1,X^2)\big(X^{1,n}_{j+1}-X^{1,n}_j\big)\big(X^{2,n}_{j+1}-X^{2,n}_j\big)^2 \\
&\quad + \frac{1}{8}2^{-\frac{3nH}{2}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{112} f(X^1,X^2)\big(X^{1,n}_{j+1}-X^{1,n}_j\big)^2\big(X^{2,n}_{j+1}-X^{2,n}_j\big),
\end{aligned}$$

where, for $i \in \{1,2\}$, $X^{i,n}_j := 2^{\frac{nH}{2}} X^i_{j2^{-\frac{n}{2}}}$.

Since $x^3 = H_3(x) + 3x$, $x^2 = H_2(x) + 1$ and $x = H_1(x)$. We get, for $H = 1/6$,

$$\begin{aligned}
V_n^3(f,t) &= \frac{1}{24}2^{-\frac{n}{4}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{111} f(X^1,X^2) H_3\big(X^{1,n}_{j+1}-X^{1,n}_j\big) \\
&\quad + \frac{1}{24}2^{-\frac{n}{4}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{222} f(X^1,X^2) H_3\big(X^{2,n}_{j+1}-X^{2,n}_j\big) \\
&\quad + \frac{1}{8}2^{-\frac{n}{4}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{122} f(X^1,X^2) H_1\big(X^{1,n}_{j+1}-X^{1,n}_j\big) H_2\big(X^{2,n}_{j+1}-X^{2,n}_j\big) \\
&\quad + \frac{1}{8}2^{-\frac{n}{4}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \Delta_{j,n}\partial_{112} f(X^1,X^2) H_2\big(X^{1,n}_{j+1}-X^{1,n}_j\big) H_1\big(X^{2,n}_{j+1}-X^{2,n}_j\big) \\
&\quad + \frac{1}{8}2^{-\frac{n}{6}} \Bigg(\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big(\Delta_{j,n}\partial_{111} f(X^1,X^2) + \Delta_{j,n}\partial_{122} f(X^1,X^2)\big)(X^1_{j+1}-X^1_j) \\
&\qquad\qquad + \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big(\Delta_{j,n}\partial_{222} f(X^1,X^2) + \Delta_{j,n}\partial_{112} f(X^1,X^2)\big)(X^2_{j+1}-X^2_j)\Bigg)
\end{aligned}$$

Let us define $P_n(f,t)$ as follows:

$$\begin{aligned}
P_n(f,t) := \frac{1}{8}2^{-\frac{n}{6}} \Bigg(&\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big(\Delta_{j,n}\partial_{111} f(X^1,X^2) + \Delta_{j,n}\partial_{122} f(X^1,X^2)\big)(X^1_{j+1}-X^1_j) \\
&+ \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big(\Delta_{j,n}\partial_{222} f(X^1,X^2) + \Delta_{j,n}\partial_{112} f(X^1,X^2)\big)(X^2_{j+1}-X^2_j)\Bigg).
\end{aligned}$$

Then, thanks to (4.2.18) and to the Definition 4.2.2, we deduce that, for $H = 1/6$,

$$V_n^3(f,t) = K_n^{(1)}(f,t) + K_n^{(2)}(f,t) + K_n^{(3)}(f,t) + K_n^{(4)}(f,t) + P_n(f,t). \tag{4.3.87}$$

We have the following corollary of Theorem 4.3.3.

Corollary 4.3.5 *Suppose $H = 1/6$. Fix $t \geqslant 0$. Then*

$$\big(X^1, X^2, V_n^3(f,t)\big) \xrightarrow{f.d.d.} \big(X^1, X^2, \int_0^t D^3 f(X_s) d^3 X_s\big), \tag{4.3.88}$$

where $\int_0^t D^3 f(X_s) d^3 X_s$ is short-hand for

$$\begin{aligned} \int_0^t D^3 f(X_s) d^3 X_s &= \kappa_1 \int_0^t \frac{\partial^3 f}{\partial x^3}(X_s^1, X_s^2) dB_s^1 + \kappa_2 \int_0^t \frac{\partial^3 f}{\partial y^3}(X_s^1, X_s^2) dB_s^2 \\ &\quad + \kappa_3 \int_0^t \frac{\partial^3 f}{\partial x^2 \partial y}(X_s^1, X_s^2) dB_s^3 + \kappa_4 \int_0^t \frac{\partial^3 f}{\partial x \partial y^2}(X_s^1, X_s^2) dB_s^4 \end{aligned}$$

with $B = (B^1, \ldots, B^4)$ a 4-dimensional Brownian motion independent of X, $\kappa_1^2 = \kappa_2^2 = \frac{1}{96}\sum_{r\in\mathbb{Z}} \rho^3(r)$ and $\kappa_3^2 = \kappa_4^2 = \frac{1}{32}\sum_{r\in\mathbb{Z}} \rho^3(r)$ with ρ defined in (4.2.23). Otherwise stated, (4.3.88) means that $V_n^{(3)}(f,t)$ converges stably in law to the random variable $\int_0^t D^3 f(X_s) d^3 X_s$.

Proof. Thanks to (4.3.87), if we prove that

$$P_n(f,t) \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty, \tag{4.3.89}$$

then we can deduce (4.3.88) immediately from Theorem 4.3.3. So, let us prove (4.3.89).
We define $g := \partial_{11} f + \partial_{22} f$. Thanks to Lemma 4.2.4, we have

$$\begin{aligned} & g(X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}) - g(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}}) \\ = \; & \Delta_{j,n}\partial_1 g\big(X^1, X^2\big)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big) + \Delta_{j,n}\partial_2 g\big(X^1, X^2\big)\big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big) \\ & + \sum_{i=2}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, \Delta_{j,n}\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} g\big(X^1, X^2\big)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big)^{\alpha_1} \\ & \qquad \times \big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big)^{\alpha_2} + R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}})\big), \big(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big)\big). \end{aligned}$$

Since $f \in C^\infty$, we have in particular that $\partial_1 g = \partial_{111} f + \partial_{122} f$ and $\partial_2 g = \partial_{112} f + \partial_{222} f$. So, by combining this fact with the definition of $V_n^{\alpha_1,\alpha_2}(\cdot,t)$ given in (4.3.50) and a telescoping argument, we get

$$\begin{aligned} & g(X^1_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}) - g(0,0) \\ = \; & 82^{n/6} P_n(f,t) + \sum_{i=2}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} g, t) \\ & + \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}})\big), \big(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big)\big). \end{aligned}$$

This way, we deduce that

$$\begin{aligned} P_n(f,t) &= \frac{1}{8}2^{-n/6}\big(g(X^1_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}}) - g(0,0)\big) \\ &\quad -\frac{1}{8}2^{-n/6}\sum_{i=2}^{7}\sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2}g,t) \\ &\quad -\frac{1}{8}2^{-n/6}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big) \\ &= -\frac{1}{8}2^{-n/6}\sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2}g,t) \\ &\quad + r_{n,1}(t), \end{aligned}$$

with obvious notation at the last equality. Thanks to Proposition 4.3.2, (4.3.60) and since, by continuity of $g(X^1,X^2)$, we have a.s. $g(X^1_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}}t\rfloor 2^{-n/2}}) \to g(X^1_t, X^2_t)$, we deduce that

$$r_{n,1}(t) \overset{P}{\to} 0 \quad \text{as} \quad n\to\infty. \tag{4.3.90}$$

So, it remains to prove that

$$2^{-n/6}\sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2}g,t) \xrightarrow{P} 0. \tag{4.3.91}$$

By Lemma 4.2.4, we have $C(3,0)=C(0,3)=\frac{1}{24}$ and $C(2,1)=C(1,2)=\frac{1}{8}$. As a result,

$$\sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2}g,t) = V_n^3(g,t), \tag{4.3.92}$$

with $V_n^3(g,t)$ given in Definition 4.3.4. Thanks to (4.3.87), we have

$$2^{-n/6}V_n^3(g,t) = 2^{-n/6}\big(K_n^{(1)}(g,t)+K_n^{(2)}(g,t)+K_n^{(3)}(g,t)+K_n^{(4)}(g,t)\big) + 2^{-n/6}P_n(g,t).$$

By Theorem 4.3.3, we have that $2^{-n/6}\big(K_n^{(1)}(g,t)+K_n^{(2)}(g,t)+K_n^{(3)}(g,t)+K_n^{(4)}(g,t)\big) \xrightarrow{P} 0$. So, in order to prove (4.3.91), we have to show that, as $n\to\infty$

$$2^{-n/6}P_n(g,t) \xrightarrow{P} 0. \tag{4.3.93}$$

Set $h := \partial_{11}g + \partial_{22}g$. Thanks to Lemma 4.2.4, we have

$$\begin{aligned} & h(X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}) - h(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}}) \\ = \;& \Delta_{j,n}\partial_1 h(X^1,X^2)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big) + \Delta_{j,n}\partial_2 h(X^1,X^2)\big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big) \\ & + \sum_{i=2}^{7}\sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\,\Delta_{j,n}\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2}h(X^1,X^2)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big)^{\alpha_1} \\ & \times\big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big)^{\alpha_2} + R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big). \end{aligned}$$

Observe that $\partial_1 h = \partial_{111} g + \partial_{122} g$ and $\partial_2 h = \partial_{112} g + \partial_{222} g$. By the same arguments that has been used previously, we get

$$\begin{aligned}
& h(X^1_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}) - h(0,0) \\
= \ & 8 2^{n/6} P_n(g,t) + \sum_{i=2}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} h, t) \\
& + \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
P_n(g,t) &= \frac{1}{8} 2^{-n/6}\big(h(X^1_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}) - h(0,0)\big) \\
& \quad -\frac{1}{8} 2^{-n/6} \sum_{i=2}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} h, t) \\
& \quad -\frac{1}{8} 2^{-n/6} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big) \\
&= -\frac{1}{8} 2^{-n/6} \sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} h, t) \\
& \quad + r_{n,2}(t),
\end{aligned}$$

with obvious notation at the last equality. Thus, we finally have

$$\begin{aligned}
2^{-n/6} P_n(g,t) &= -\frac{1}{8} 2^{-n/3} \sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} h, t) \\
& \quad + 2^{-n/6} r_{n,2}(t).
\end{aligned}$$

By the same arguments that has been used to prove (4.3.90), we deduce that $r_{n,2}(t) \overset{P}{\to} 0$ as $n \to \infty$. Hence, to prove (4.3.93) it remains to show that, as $n \to \infty$

$$2^{-n/3} \sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} h, t) \overset{P}{\longrightarrow} 0. \tag{4.3.94}$$

In fact, since $h \in C_b^\infty$ and by the definition of $V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} h, t)$ given in (4.3.50), we deduce that

$$\begin{aligned}
& 2^{-n/3} \Big| \sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} h, t) \Big| \\
\leqslant \ & C 2^{-n/3} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big|X^1_{j+1} - X^1_j\big|^3 + C 2^{-n/3} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big|X^2_{j+1} - X^2_j\big|^3 \qquad (4.3.95) \\
& + C 2^{-n/3} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big|X^1_{j+1} - X^1_j\big| \big|X^2_{j+1} - X^2_j\big|^2 + C 2^{-n/3} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big|X^1_{j+1} - X^1_j\big|^2 \big|X^2_{j+1} - X^2_j\big|.
\end{aligned}$$

Recall the following notation: for $i \in \{1,2\}$, $X_j^{i,n} := 2^{\frac{n}{12}} X^i_{j2^{-\frac{n}{2}}}$. We deduce that, for all $p \in \mathbb{N}^*$,

$$E[|X^i_{j+1} - X^i_j|^p] = 2^{-\frac{np}{12}} E[|X^{i,n}_{j+1} - X^{i,n}_j|^p] = 2^{-\frac{np}{12}} E[|G|^p],$$

where $G \sim N(0,1)$. Thanks to this identity , to the independence of X^1, X^2 and to (4.3.95), we deduce that

$$\begin{aligned}
& 2^{-n/3} E\Big(\big| \sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} h, t) \big| \Big) \\
\leqslant\ & C 2^{-n/3} 2^{-n/4} E[|G|^3] \lfloor 2^{\frac{n}{2}} t \rfloor + C 2^{-n/3} 2^{-n/4} E[|G|] E[|G|^2] \lfloor 2^{\frac{n}{2}} t \rfloor \\
\leqslant\ & C 2^{-n/12} t \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Convergence (4.3.94) follows immediately. Consequently, we have proved (4.3.93), (4.3.91) and (4.3.89). It finishes the proof of Corollary 4.3.5. ■

<u>We are now ready to prove (4.1.4).</u> Thanks to Lemma 4.2.4, we have

$$\begin{aligned}
& f(X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}) - f(X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}}) \\
=\ & \Delta_{j,n} \frac{\partial f}{\partial x}(X^1, X^2)\big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big) + \Delta_{j,n} \frac{\partial f}{\partial y}(X^1, X^2)\big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big) \\
& + \sum_{i=2}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, \Delta_{j,n} \partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f(X^1,X^2) \big(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}}\big)^{\alpha_1} \\
& \qquad \times \big(X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}}\big)^{\alpha_2} + R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big).
\end{aligned}$$

Then, by Definition 4.1.2 and (4.3.50), we can write

$$\begin{aligned}
& f(X^1_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}) - f(0,0) \\
=\ & O_n(f,t) + \sum_{i=2}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f, t) \\
& + \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big).
\end{aligned}$$

By the same arguments that has been used to show (4.3.92), we get

$$\sum_{\alpha_1+\alpha_2=3} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f, t) = V_n^3(f,t).$$

Combining this fact with our Taylor's expansion, we deduce that

$$\begin{aligned}
& O_n(f,t) \qquad\qquad (4.3.96)\\
=\ & f(X^1_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}) - f(0,0) - V_n^3(f,t) \\
& - \sum_{i=3}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f, t) \\
& - \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big).
\end{aligned}$$

Thanks to Proposition 4.3.2, we have

$$\sum_{i=3}^{7} \sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, V_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots12\ldots2} f, t) \xrightarrow{P} 0. \tag{4.3.97}$$

On the other hand, by (4.3.60) we have

$$\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} R_{13}\big((X^1_{(j+1)2^{-n/2}}, X^2_{(j+1)2^{-n/2}}), (X^1_{j2^{-n/2}}, X^2_{j2^{-n/2}})\big) \xrightarrow{P} 0 \tag{4.3.98}$$

Observe also that, by the almost sure continuity of $f\big(X^1, X^2\big)$, one has, almost surely and as $n \to \infty$,

$$f(X^1_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}, X^2_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}}) - f(0,0) \to f\big(X^1_t, X^2_t\big) - f(0,0). \tag{4.3.99}$$

Finally, the desired conclusion (4.1.4) follows from (4.3.97), (4.3.98), (4.3.99) and the conclusion of Corollary 4.3.5, plugged into (4.3.96).

4.3.3 Proof of (4.1.6)

Using $b^3 - a^3 = 3\big(\frac{a+b}{2}\big)^2(b-a) + \frac{1}{4}(b-a)^3$, one can write

$$O_n(x\mapsto x^3,t) - (X^1_t)^3 \;\;=\;\; -V_n^3(x\mapsto x^3,t) + \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \big((X^1_{(j+1)2^{-n/2}})^3 - (X^1_{j2^{-n/2}})^3\big) - (X^1_t)^3,$$

where $O_n(\cdot,t)$ is introduced in Definition 4.1.2 and $V_n^3(\cdot,t)$ is given in Definition 4.3.4. As a result, and since $(X^1_{\lfloor 2^{\frac{n}{2}} t\rfloor 2^{-n/2}})^3 \to (X^1_t)^3$ a.s. as $n\to\infty$, one deduces that if $O_n(x\mapsto x^3,t)$ converges stably in law, then $V_n^3(x\mapsto x^3,t)$ must converge as well. But it is known (see for example (1.8) in [26]) that $2^{-n(\frac{1}{2}-3H)} V_n^3(x\mapsto x^3,t)$ converges in law to a non degenerate limit. This fact being in contradiction with the convergence of $V_n^3(x\mapsto x^3,t)$, we deduce that (4.1.6) holds.

4.4 Proof of Theorem 4.1.4

We divide the proof of Theorem 4.1.4 in several steps.

4.4.1 Step 1: A key algebraic lemma

For each integer $n\geqslant 1$, $k\in\mathbb{Z}$ and real number $t\geqslant 0$, let $U_{j,n}(t)$ (resp. $D_{j,n}(t)$) denote the number of *upcrossings* (resp. *downcrossings*) of the interval $[j2^{-n/2},(j+1)2^{-n/2}]$ within the first $\lfloor 2^n t\rfloor$ steps of the random walk $\{Y(T_{k,n})\}_{k\geqslant 0}$, where $(T_{k,n})_{k\geqslant 0}$ is introduced in (4.2.10). That is,

$$\begin{aligned} U_{j,n}(t) &= \sharp\big\{k=0,\ldots,\lfloor 2^n t\rfloor -1 : \\ &\qquad\qquad Y(T_{k,n}) = j2^{-n/2} \text{ and } Y(T_{k+1,n}) = (j+1)2^{-n/2}\big\};\\ D_{j,n}(t) &= \sharp\big\{k=0,\ldots,\lfloor 2^n t\rfloor -1 : \\ &\qquad\qquad Y(T_{k,n}) = (j+1)2^{-n/2} \text{ and } Y(T_{k+1,n}) = j2^{-n/2}\big\}. \end{aligned}$$

Definition 4.4.1 *For $f \in C_b^\infty$ and $t \geqslant 0$, for all $p, q \in \mathbb{N}$ such that $p+q$ is odd, we define $\tilde{V}_n^{p,q}(f,t)$ as follows:*

$$\tilde{V}_n^{p,q}(f,t) = \sum_{k=0}^{\lfloor 2^n t\rfloor -1} f\big(\frac{1}{2}(Z^1_{T_{k,n}} + Z^1_{T_{k+1,n}}), \frac{1}{2}(Z^2_{T_{k,n}} + Z^2_{T_{k+1,n}})\big)(Z^1_{T_{k+1,n}} - Z^1_{T_{k,n}})^p \times (Z^2_{T_{k+1,n}} - Z^2_{T_{k,n}})^q. \quad (4.4.100)$$

While easy, the following lemma taken from [20, Lemma 2.4] is going to be the key when studying the asymptotic behavior $\tilde{V}_n^{p,q}(f,t)$. Its main feature is to separate (X^1, X^2) from Y, thus providing a representation of $\tilde{V}_n^{p,q}(f,t)$ which is amenable to analysis.

Lemma 4.4.2 *Fix $f \in C_b^\infty$, $t \geqslant 0$ and for all $p, q \in \mathbb{N}$ such that $p+q$ is odd, we have then*

$$\begin{aligned}&\tilde{V}_n^{p,q}(f,t)\\ &= \sum_{j\in\mathbb{Z}} f\big(\frac{1}{2}(X^1_{(j+1)2^{-n/2}} + X^1_{j2^{-n/2}}), \frac{1}{2}(X^2_{(j+1)2^{-n/2}} + X^2_{j2^{-n/2}})\big)(X^1_{(j+1)2^{-n/2}} - X^1_{j2^{-n/2}})^p\\ &\qquad \times (X^2_{(j+1)2^{-n/2}} - X^2_{j2^{-n/2}})^q \big(U_{j,n}(t) - D_{j,n}(t)\big).\end{aligned}$$

4.4.2 Step 2: Transforming the 2D weighted power variations of odd order

By [20, Lemma 2.5], one has

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} 1_{\{0\leqslant j < j^*(n,t)\}} & \text{if } j^*(n,t) > 0 \\ 0 & \text{if } j^* = 0 \\ -1_{\{j^*(n,t)\leqslant j<0\}} & \text{if } j^*(n,t) < 0 \end{cases},$$

where $j^*(n,t) = 2^{n/2} Y_{T_{\lfloor 2^n t\rfloor ,n}}$. As a consequence, we have

1. If $j^*(n,t) > 0$:

$$\begin{aligned}&\tilde{V}_n^{p,q}(f,t)\\ &= \sum_{j=0}^{j^*(n,t)-1} f\big(\frac{1}{2}(X^{1,+}_{(j+1)2^{-n/2}} + X^{1,+}_{j2^{-n/2}}), \frac{1}{2}(X^{2,+}_{(j+1)2^{-n/2}} + X^{2,+}_{j2^{-n/2}})\big)\\ &\qquad \times \big(X^{1,+}_{(j+1)2^{-n/2}} - X^{1,+}_{j2^{-n/2}}\big)^p \big(X^{2,+}_{(j+1)2^{-n/2}} - X^{2,+}_{j2^{-n/2}}\big)^q.\end{aligned}$$

2. If $j^* = 0$: $\tilde{V}_n^{p,q}(f,t) = 0$.

3. If $j^*(n,t) < 0$:

$$\begin{aligned}&\tilde{V}_n^{p,q}(f,t)\\ &= \sum_{j=0}^{|j^*(n,t)|-1} f\big(\frac{1}{2}(X^{1,-}_{(j+1)2^{-n/2}} + X^{1,-}_{j2^{-n/2}}), \frac{1}{2}(X^{2,-}_{(j+1)2^{-n/2}} + X^{2,-}_{j2^{-n/2}})\big)\\ &\qquad \times \big(X^{1,-}_{(j+1)2^{-n/2}} - X^{1,-}_{j2^{-n/2}}\big)^p \big(X^{2,-}_{(j+1)2^{-n/2}} - X^{2,-}_{j2^{-n/2}}\big)^q,\end{aligned}$$

where, for $i \in \{1,2\}$, $X^{i,+}_t := X^i_t$ for $t \geqslant 0$, $X^{i,-}_{-t} := X^i_t$ for $t < 0$.

Let us introduce the following sequence of processes $W^{p,q}_{\pm,n}$:

$$
\begin{aligned}
W^{p,q}_{\pm,n}(f,t) &= \sum_{j=0}^{\lfloor 2^{n/2}t\rfloor-1} f\Big(\frac{1}{2}(X^{1,\pm}_{(j+1)2^{-n/2}}+X^{1,\pm}_{j2^{-n/2}}),\frac{1}{2}(X^{2,\pm}_{(j+1)2^{-n/2}}+X^{2,\pm}_{j2^{-n/2}})\Big)\\
&\qquad\times\big(X^{1,\pm}_{(j+1)2^{-n/2}}-X^{1,\pm}_{j2^{-n/2}}\big)^p\big(X^{2,\pm}_{(j+1)2^{-n/2}}-X^{2,\pm}_{j2^{-n/2}}\big)^q,\quad t\geqslant 0\\
W^{p,q}_{n}(f,t) &= \begin{cases} W^{p,q}_{+,n}(f,t) & \text{if } t\geqslant 0\\ W^{p,q}_{-,n}(f,-t) & \text{if } t<0\end{cases}.
\end{aligned}
$$

We then have that

$$
\tilde{V}^{p,q}_n(f,t)=W^{p,q}_n(f,Y_{T_{\lfloor 2^n t\rfloor,n}}). \tag{4.4.101}
$$

4.4.3 Step 3: Known results for the 2D fractional Brownian motion

- If $H>1/6$, $p+q\geqslant 3$ and if $H=1/6$, $p+q\geqslant 5$, then, thanks to (4.3.57) and (4.3.58), we have for all $t\geqslant 0$

$$
\begin{aligned}
E\big[\big(W^{p,q}_{\pm,n}(f,t)\big)^2\big] &\leqslant C\sum_{k'=1}^{\frac{q+1}{2}}\Bigg(\Big(\sum_{a=1}^{2k'-1}2^{-n[H(p+q+2k'-1-a)-\frac{1}{2}]}\Big)t+2^{-n[H(p+q+2k'-2)-\frac{1}{2}]}\\
&\qquad\times t^{2H+1}+2^{-nH[p+q+2k'-1]}\Bigg).
\end{aligned} \tag{4.4.102}
$$

- If $H=1/6$, for all $t\in\mathbb{R}$, we define $W^{(3)}_n(f,t)$ as follows:

$$
W^{(3)}_n(f,t)=\sum_{p+q=3}C(p,q)W^{p,q}_n(\partial^{p,q}_{1\ldots12\ldots2}f,t), \tag{4.4.103}
$$

where $C(3,0)=C(0,3)=\frac{1}{24}$ and $C(2,1)=C(1,2)=\frac{1}{8}$. Then, thanks to Corollary 4.3.3 we have, for $H=1/6$, for any fixed $t\in\mathbb{R}$ and as $n\to\infty$

$$
(X^1,X^2,W^{(3)}_n(f,t))\xrightarrow{fdd}(X^1,X^2,\int_0^t D^3f(X_s)d^3X_s) \tag{4.4.104}
$$

where $\int_0^t D^3f(X_s)d^3X_s$ is short-hand for

$$
\begin{aligned}
\int_0^t D^3f(X_s)d^3X_s &= \kappa_1\int_0^t\frac{\partial^3 f}{\partial x^3}(X^1_s,X^2_s)dB^1_s+\kappa_2\int_0^t\frac{\partial^3 f}{\partial y^3}(X^1_s,X^2_s)dB^2_s\\
&\quad+\kappa_3\int_0^t\frac{\partial^3 f}{\partial x^2\partial y}(X^1_s,X^2_s)dB^3_s+\kappa_4\int_0^t\frac{\partial^3 f}{\partial x\partial y^2}(X^1_s,X^2_s)dB^4_s
\end{aligned}
$$

with $B=(B^1,\ldots,B^4)$ a 4-dimensional two-sided Brownian motion independent of (X^1,X^2), $\kappa_1^2=\kappa_2^2=\frac{1}{96}\sum_{r\in\mathbb{Z}}\rho^3(r)$ and $\kappa_3^2=\kappa_4^2=\frac{1}{32}\sum_{r\in\mathbb{Z}}\rho^3(r)$ with ρ defined in (4.2.23).

4.4.4 Step 4: Moment bounds for $W_n^{(3)}(f,\cdot)$

Fix $f \in C_b^\infty$ and set $H = 1/6$. We claim the existence of $C > 0$ such that, for all real numbers $s < t$ and all $n \in \mathbb{N}$,

$$E[(W_n^{(3)}(f,t) - W_n^{(3)}(f,s))^2] \leqslant C \max\big(|s|^{1/3}, |t|^{1/3}\big)\big(2^{-n/2} + |t-s|\big). \tag{4.4.105}$$

Proof. By the definition of $W_n^{(3)}(f,t)$ in (4.4.103), we deduce that

$$E[(W_n^{(3)}(f,t) - W_n^{(3)}(f,s))^2] \leqslant C \sum_{p+q=3} E[(W_n^{p,q}(\partial^{p,q}_{1\dots12\dots2}f,t) - W_n^{p,q}(\partial^{p,q}_{1\dots12\dots2}f,s))^2].$$

So, if we prove that for all $f \in C_b^\infty$ and for all $p,q \in \mathbb{N}$ such that $p+q=3$,

$$E[(W_n^{p,q}(f,t) - W_n^{p,q}(f,s))^2] \leqslant C \max\big(|s|^{1/3}, |t|^{1/3}\big)\big(2^{-n/2} + |t-s|\big),$$

then the conclusion (4.4.105) follows immediately. In fact, we will prove the last inequality only for $p=1$ and $q=2$, the proof being similar for the other values* of p and q.

For $p=1$, $q=2$, bearing the notation of Step 2 in mind, we have

$$\begin{aligned}
W^{1,2}_{\pm,n}(f,t) &= \frac{1}{8}2^{-\frac{n}{4}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Delta_{j,n} f(X^{1,\pm}, X^{2,\pm})\big(2^{\frac{n}{12}}(X^{1,\pm}_{(j+1)2^{-n/2}} - X^{1,\pm}_{j2^{-n/2}})\big)\\
&\quad \times \big(2^{\frac{n}{12}}(X^{2,\pm}_{(j+1)2^{-n/2}} - X^{2,\pm}_{j2^{-n/2}})\big)^2\\
&= \frac{1}{8}2^{-\frac{n}{4}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Delta_{j,n} f(X^{1,\pm}, X^{2,\pm}) H_1\big(2^{\frac{n}{12}}(X^{1,\pm}_{(j+1)2^{-n/2}} - X^{1,\pm}_{j2^{-n/2}})\big)\\
&\quad + \frac{1}{8}2^{-\frac{n}{4}} \sum_{j=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor -1} \Delta_{j,n} f(X^{1,\pm}, X^{2,\pm}) H_1\big(2^{\frac{n}{12}}(X^{1,\pm}_{(j+1)2^{-n/2}} - X^{1,\pm}_{j2^{-n/2}})\big)\\
&\quad \times H_2\big(2^{\frac{n}{12}}(X^{2,\pm}_{(j+1)2^{-n/2}} - X^{2,\pm}_{j2^{-n/2}})\big)\\
&=: \widetilde{W}^{1,2}_{\pm,n}(f,t) + \overline{W}^{1,2}_{\pm,n}(f,t).
\end{aligned}$$

We claim that:

$$E[(\widetilde{W}_n^{1,2}(f,t) - \widetilde{W}_n^{1,2}(f,s))^2] \leqslant C \max\big(|s|^{1/3}, |t|^{1/3}\big)\big(2^{-n/2} + |t-s|\big); \tag{4.4.106}$$

$$E[(\overline{W}_n^{1,2}(f,t) - \overline{W}_n^{1,2}(f,s))^2] \leqslant C \max\big(|s|^{1/3}, |t|^{1/3}\big)\big(2^{-n/2} + |t-s|\big). \tag{4.4.107}$$

It suffices to prove (4.4.107) which is representative of the difficulty. To do so, we distinguish two cases according to the signs of $s,t \in \mathbb{R}$ (and reducing the problem by symmetry):

*When $p=3$, $q=0$ or $p=0$, $q=3$ the reader will find a very similar result in Step 4 of chapter 3

(1) If $0 \leqslant s < t$ (the case $s < t \leqslant 0$ being similar), then

$$
\begin{aligned}
& E[(\overline{W}_n^{1,2}(f,t) - \overline{W}_n^{1,2}(f,s))^2] = E[(\overline{W}_{+,n}^{1,2}(f,t) - \overline{W}_{+,n}^{1,2}(f,s))^2] \\
= \; & \frac{1}{64} 2^{-n/2} \sum_{j,j'=\lfloor 2^{n/2} s \rfloor}^{\lfloor 2^{\frac{n}{2}} t \rfloor - 1} E\Big(\Delta_{j,n} f(X^{1,+}, X^{2,+}) \Delta_{j',n} f(X^{1,+}, X^{2,+}) I_1^{(1)}\big((2^{\frac{n}{12}} \delta_{(j+1)2^{-n/2}}) \big) \\
& \times I_2^{(2)}\big((2^{\frac{n}{12}} \delta_{(j+1)2^{-n/2}})^{\otimes 2} \big) I_1^{(1)}\big((2^{\frac{n}{12}} \delta_{(j'+1)2^{-n/2}}) \big) I_2^{(2)}\big((2^{\frac{n}{12}} \delta_{(j'+1)2^{-n/2}})^{\otimes 2} \big) \Big) \\
= \; & \frac{1}{64} \sum_{j,j'=\lfloor 2^{n/2} s \rfloor}^{\lfloor 2^{\frac{n}{2}} t \rfloor - 1} E\Big(\Delta_{j,n} f(X^{1,+}, X^{2,+}) \Delta_{j',n} f(X^{1,+}, X^{2,+}) I_1^{(1)}\big(\delta_{(j+1)2^{-n/2}} \big) \\
& \times I_1^{(1)}\big(\delta_{(j'+1)2^{-n/2}} \big) I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2} \big) I_2^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2} \big) \Big),
\end{aligned}
$$

where we have the first equality by (4.2.18). Relying to the product formula (4.2.20), we deduce that this latter quantity is less than or equal to

$$
\begin{aligned}
& \frac{1}{64} \sum_{j,j'=\lfloor 2^{n/2} s \rfloor}^{\lfloor 2^{\frac{n}{2}} t \rfloor - 1} \Big| E\Big(\Delta_{j,n} f(X^{1,+}, X^{2,+}) \Delta_{j',n} f(X^{1,+}, X^{2,+}) I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2} \big) \\
& \qquad \times I_2^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2} \big) \Big) \Big| |\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}} \rangle| \\
& + \frac{1}{64} \sum_{j,j'=\lfloor 2^{n/2} s \rfloor}^{\lfloor 2^{\frac{n}{2}} t \rfloor - 1} \Big| E\Big(\Delta_{j,n} f(X^{1,+}, X^{2,+}) \Delta_{j',n} f(X^{1,+}, X^{2,+}) I_2^{(1)}\big(\delta_{(j+1)2^{-n/2}} \otimes \\
& \qquad \delta_{(j'+1)2^{-n/2}} \big) I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2} \big) I_2^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2} \big) \Big) \Big| \\
& =: \sum_{i=1}^{2} Q_n^{+,i}(s,t).
\end{aligned}
\tag{4.4.108}
$$

We have then the following estimates.

- Case $i = 1$. Since $f \in C_b^\infty$, and thanks to the Cauchy-Schwarz inequality and to (4.2.21), we have

$$
\begin{aligned}
& \Big| E\Big(\Delta_{j,n} f(X^{1,+}, X^{2,+}) \Delta_{j',n} f(X^{1,+}, X^{2,+}) I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2} \big) \\
& \qquad \times I_2^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2} \big) \Big) \Big| \\
\leqslant \; & C \| I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2} \big) \|_2 \| I_2^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2} \big) \|_2 \\
\leqslant \; & C (2^{-n/6})^2.
\end{aligned}
\tag{4.4.109}
$$

We deduce that

$$
\begin{aligned}
& Q_n^{+,1}(s,t) \\
\leqslant\ & C2^{-n/3} \sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1} |\langle \delta_{(j+1)2^{-n/2}}, \delta_{(j'+1)2^{-n/2}}\rangle| \\
\leqslant\ & C2^{-n/2} \sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \left|\frac{1}{2}(|j-j'+1|^{1/3}+|j-j'-1|^{1/3}-2|j-j'|^{1/3})\right| \\
=\ & C2^{-n/2} \sum_{j=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1} \sum_{q=j-\lfloor 2^{n/2}t\rfloor+1}^{j-\lfloor 2^{n/2}s\rfloor} \big|\rho(q)\big|,
\end{aligned}
$$

with $\rho(q)$ defined in (4.2.23). By a Fubini argument, it comes

$$
\begin{aligned}
& Q_n^{+,1}(s,t) \\
\leqslant\ & C2^{-n/2} \sum_{q=\lfloor 2^{n/2}s\rfloor-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor-1} |\rho(q)|\Big((q+\lfloor 2^{n/2}t\rfloor)\wedge\lfloor 2^{n/2}t\rfloor-(q+\lfloor 2^{n/2}s\rfloor)\vee\lfloor 2^{n/2}s\rfloor\Big) \\
\leqslant\ & C2^{-n/2} \sum_{q=\lfloor 2^{n/2}s\rfloor-\lfloor 2^{n/2}t\rfloor+1}^{\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor-1} |\rho(q)|\big(\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big) \\
\leqslant\ & C2^{-n/2}\sum_{q\in\mathbb{Z}}|\rho(q)|\big|\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big| = C2^{-n/2}\big|\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big| \\
\leqslant\ & C2^{-n/2}\big(\big|\lfloor 2^{n/2}t\rfloor-2^{n/2}t\big|+2^{n/2}\big|t-s\big|+\big|\lfloor 2^{n/2}s\rfloor-2^{n/2}s\big|\big) \\
\leqslant\ & C(2^{-n/2}+|t-s|). \qquad (4.4.110)
\end{aligned}
$$

Note that $\sum_{q\in\mathbb{Z}}|\rho(q)|<\infty$ since $H<\frac{1}{2}$.

- Case $i=2$. Thanks to the duality formula (4.2.19) and to the Leibniz rule (4.2.17), one has that

$$
\begin{aligned}
& \Bigg|E\Bigg(\Delta_{j,n}f(X^{1,+},X^{2,+})\Delta_{j',n}f(X^{1,+},X^{2,+})I_2^{(1)}\big(\delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\big) \\
& \qquad \times I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2}\big)I_2^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2}\big)\Bigg)\Bigg| \\
=\ & \Bigg|E\Bigg(\Big\langle D_{X^1}^2\big(\Delta_{j,n}f(X^{1,+},X^{2,+})\Delta_{j',n}f(X^{1,+},X^{2,+})\big),\delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\Big\rangle \\
& \qquad \times I_2^{(2)}\big(\delta_{(j+1)2^{-n/2}}^{\otimes 2}\big)I_2^{(2)}\big(\delta_{(j'+1)2^{-n/2}}^{\otimes 2}\big)\Bigg)\Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leqslant\ & E\Big(\Big|\Delta_{j,n}\partial_{11}f(X^{1,+},X^{2,+})\Delta_{j',n}f(X^{1,+},X^{2,+})\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big)\\
& \times I_2^{(2)}\big(\delta^{\otimes 2}_{(j'+1)2^{-n/2}}\big)\Big|\Big)\Big|\Big\langle\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{\otimes 2},\delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\Big\rangle\Big|\\
& +2E\Big(\Big|\Delta_{j,n}\partial_1 f(X^{1,+},X^{2,+})\Delta_{j',n}\partial_1 f(X^{1,+},X^{2,+})\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big)\\
& \times I_2^{(2)}\big(\delta^{\otimes 2}_{(j'+1)2^{-n/2}}\big)\Big|\Big)\Big|\Big\langle\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}\Big(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\Big),\\
& \qquad\qquad \delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\Big\rangle\Big|\\
& +E\Big(\Big|\Delta_{j,n}f(X^{1,+},X^{2,+})\Delta_{j',n}\partial_{11}f(X^{1,+},X^{2,+})\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(j+1)2^{-n/2}}\big)\\
& \times I_2^{(2)}\big(\delta^{\otimes 2}_{(j'+1)2^{-n/2}}\big)\Big|\Big)\Big|\Big\langle\Big(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\Big)^{\otimes 2},\delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\Big\rangle\Big|\\
& =: d_n^1+d_n^2+d_n^3.
\end{aligned}
$$

Observe that, thanks to (4.4.109), we get

$$
\begin{aligned}
d_n^1 &\leqslant C2^{-n/3}\Big|\Big\langle\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)^{\otimes 2},\delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\Big\rangle\Big|,\\
d_n^2 &\leqslant C2^{-n/3}\Big|\Big\langle\Big(\frac{\varepsilon_{j2^{-n/2}}+\varepsilon_{(j+1)2^{-n/2}}}{2}\Big)\tilde{\otimes}\Big(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\Big),\\
&\qquad\qquad \delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\Big\rangle\Big|,\\
d_n^3 &\leqslant C2^{-n/3}\Big|\Big\langle\Big(\frac{\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}}}{2}\Big)^{\otimes 2},\delta_{(j+1)2^{-n/2}}\otimes\delta_{(j'+1)2^{-n/2}}\Big\rangle\Big|.
\end{aligned}
$$

By (4.2.32), recall that $|\langle\varepsilon_u,\delta_{(j+1)2^{-n/2}}\rangle|\leqslant 2^{-n/6}$ for all $u\geqslant 0$ and all $j\in\mathbb{N}$. We thus get,

$$
\begin{aligned}
d_n^1+d_n^2+d_n^3 \leqslant\ & C2^{-n/2}\big(|\langle\varepsilon_{j2^{-n/2}},\delta_{(j'+1)2^{-n/2}}\rangle|+|\langle\varepsilon_{(j+1)2^{-n/2}},\delta_{(j'+1)2^{-n/2}}\rangle|\\
& +|\langle\varepsilon_{j'2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle|+|\langle\varepsilon_{(j'+1)2^{-n/2}},\delta_{(j+1)2^{-n/2}}\rangle|\\
& +|\langle\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}},\delta_{(j'+1)2^{-n/2}}\rangle|\big).
\end{aligned}
$$

For instance, we can write

$$\begin{aligned}
&2^{-n/2}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}|\langle\varepsilon_{(j'+1)2^{-n/2}};\delta_{(j+1)2^{-n/2}}\rangle|\\
=\;&\frac{1}{2}2^{-2n/3}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\big|(j+1)^{1/3}-j^{1/3}+|j'-j+1|^{1/3}-|j'-j|^{1/3}\big|\\
\leqslant\;&\frac{1}{2}2^{-2n/3}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}\big((j+1)^{1/3}-j^{1/3}\big)\\
&+\frac{1}{2}2^{-2n/3}\sum_{\lfloor 2^{n/2}s\rfloor\leqslant j\leqslant j'\leqslant\lfloor 2^{n/2}t\rfloor-1}\big((j'-j+1)^{1/3}-(j'-j)^{1/3}\big)\\
&+\frac{1}{2}2^{-2n/3}\sum_{\lfloor 2^{n/2}s\rfloor\leqslant j'<j\leqslant\lfloor 2^{n/2}t\rfloor-1}\big((j-j')^{1/3}-(j-j'-1)^{1/3}\big)\\
\leqslant\;&\frac{3}{2}\,2^{-2n/3}\big(\lfloor 2^{n/2}t\rfloor-\lfloor 2^{n/2}s\rfloor\big)\lfloor 2^{n/2}t\rfloor^{1/3}\leqslant\frac{3t^{1/3}}{2}\,\big(2^{-n/2}+|t-s|\big).
\end{aligned}$$

Similarly,

$$\begin{aligned}
2^{-n/2}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}|\langle\varepsilon_{j2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{1/3}}{2}\,\big(2^{-n/2}+|t-s|\big);\\
2^{-n/2}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}|\langle\varepsilon_{(j+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{1/3}}{2}\,\big(2^{-n/2}+|t-s|\big);\\
2^{-n/2}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}|\langle\varepsilon_{j'2^{-n/2}};\delta_{(j+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{1/3}}{2}\,\big(2^{-n/2}+|t-s|\big);\\
2^{-n/2}\sum_{j,j'=\lfloor 2^{n/2}s\rfloor}^{\lfloor 2^{n/2}t\rfloor-1}|\langle\varepsilon_{j'2^{-n/2}}+\varepsilon_{(j'+1)2^{-n/2}};\delta_{(j'+1)2^{-n/2}}\rangle| &\leqslant \frac{3t^{1/3}}{2}\,\big(2^{-n/2}+|t-s|\big).
\end{aligned}$$

As a consequence, we deduce

$$Q_n^{+,2}(s,t)\leqslant Ct^{1/3}\big(2^{-n/2}+|t-s|\big). \tag{4.4.111}$$

Combining (4.4.108), (4.4.110) and (4.4.111) finally shows our claim (4.4.107).

(2) If $s<0\leqslant t$, then

$$\begin{aligned}
&E[(\overline{W}_n^{1,2}(f,t)-\overline{W}_n^{1,2}(f,s))^2]=E[(\overline{W}_{+,n}^{1,2}(f,t)-\overline{W}_{-,n}^{1,2}(f,-s))^2]\\
\leqslant\;&2E[(\overline{W}_{+,n}^{1,2}(f,t))^2]+2E[(\overline{W}_{-,n}^{1,2}(f,-s))^2].
\end{aligned}$$

By (1) with $s=0$, one can write

$$E[(\overline{W}_{+,n}^{1,2}(f,t))^2]\leqslant Ct^{1/3}\big(2^{-n/2}+t\big).$$

Similarly

$$E[(\overline{W}^{1,2}_{-,n}(f,-s))^2] \leqslant C(-s)^{1/3}\big(2^{-n/2}+(-s)\big).$$

We deduce that

$$E[(\overline{W}^{1,2}_{n}(f,t)-\overline{W}^{1,2}_{n}(f,s))^2] \leqslant C\max\big(t^{1/3},(-s)^{1/3}\big)\big(2^{-n/2}+(t-s)\big).$$

That is, (4.4.107) holds true in this case.

■

4.4.5 Step 5: Limits of the 2D weighted power variations of odd order

Fix $f \in C_b^\infty$ and $t \geqslant 0$. We claim that, if $H \in \left[\frac{1}{6},\frac{1}{2}\right)$ and $p+q \geqslant 5$ then, as $n \to \infty$,

$$\tilde{V}_n^{p,q}(f,t) \xrightarrow{\text{prob}} 0. \tag{4.4.112}$$

Moreover, if $H \in \left(\frac{1}{6},\frac{1}{2}\right)$ and $p+q=3$ then, as $n \to \infty$,

$$\tilde{V}_n^{p,q}(f,t) \xrightarrow{\text{prob}} 0. \tag{4.4.113}$$

For all $t \geqslant 0$, we define $\tilde{V}_n^{(3)}(f,t)$ as follows

$$\tilde{V}_n^{(3)}(f,t) = \sum_{p+q=3} C(p,q)\tilde{V}_n^{p,q}(\partial^{p,q}_{1\ldots 12\ldots 2}f,t), \tag{4.4.114}$$

with $C(3,0)=C(0,3)=\frac{1}{24}$ and $C(2,1)=C(1,2)=\frac{1}{8}$. Observe that thanks to (4.4.101) and (4.4.103), we have

$$\tilde{V}_n^{(3)}(f,t) = W_n^{(3)}(f,Y_{T_{\lfloor 2^n t\rfloor,n}}). \tag{4.4.115}$$

Then, we claim that, for $H=1/6$, for any fixed $t \geqslant 0$, as $n \to \infty$

$$(X^1,X^2,Y,\tilde{V}_n^{(3)}(f,t)) \xrightarrow{fdd} (X^1,X^2,Y,\int_0^{Y_t} D^3f(X_s)d^3X_s), \tag{4.4.116}$$

where $\int_0^t D^3f(X_s)d^3X_s$ is short-hand for

$$\begin{aligned}\int_0^t D^3f(X_s)d^3X_s &= \kappa_1\int_0^t \frac{\partial^3 f}{\partial x^3}(X_s^1,X_s^2)dB_s^1 + \kappa_2\int_0^t \frac{\partial^3 f}{\partial y^3}(X_s^1,X_s^2)dB_s^2 \\ &\quad + \kappa_3\int_0^t \frac{\partial^3 f}{\partial x^2\partial y}(X_s^1,X_s^2)dB_s^3 + \kappa_4\int_0^t \frac{\partial^3 f}{\partial x\partial y^2}(X_s^1,X_s^2)dB_s^4\end{aligned}$$

with $B=(B^1,\ldots,B^4)$ a 4-dimensional two-sided Brownian motion independent of (X^1,X^2) and also independent of Y. The constants $\kappa_1,\ldots,\kappa_4$ are the same as in (4.4.104). Otherwise stated, (4.4.116) means that $\tilde{V}_n^{(3)}(f,t)$ converges stably in law to the random variable $\int_0^{Y_t} D^3f(X_s)d^3X_s$.

Indeed, combining (4.4.101), (4.4.102) together with the independence of Y and (X^1, X^2) (by the definition of Z in (4.1.1)), we deduce that

$$\begin{aligned} E\big[\big(\tilde{V}_n^{p,q}(f,t)\big)^2\big] &\leqslant C\sum_{k'=1}^{\frac{q+1}{2}}\Bigg(\Big(\sum_{a=1}^{2k'-1} 2^{-n[H(p+q+2k'-1-a)-\frac{1}{2}]}\Big)E(|Y_{T_{\lfloor 2^n t\rfloor,n}}|) \\ &\quad +2^{-n[H(p+q+2k'-2)-\frac{1}{2}]}E\big(|Y_{T_{\lfloor 2^n t\rfloor,n}}|^{2H+1}\big)+2^{-nH[p+q+2k'-1]}\Bigg). \end{aligned}$$

On the other hand, recall from [20, Lemma 2.3] that $Y_{T_{\lfloor 2^n t\rfloor,n}} \xrightarrow{L^2} Y_t$ as $n\to\infty$. So, combining this fact with the last inequality, we deduce that (4.4.112) and (4.4.113) hold true.

Now, using the decomposition (4.4.115), the conclusion of Step 4 (to pass from $Y_{T_{\lfloor 2^n t\rfloor,n}}$ to Y_t) and the convergence: $Y_{T_{\lfloor 2^n t\rfloor,n}} \xrightarrow{L^2} Y_t$, we deduce that the limit of $\tilde{V}_n^{(3)}(f,t)$ is the same as that of

$$W_n^{(3)}(f, Y_t).$$

Thus, the proof of (4.4.116) then follows directly from (4.4.104) and the fact that Y is independent of (X^1, X^2) and independent of $(B^1,\dots,B^4)$.

4.4.6 Step 6: Proving (4.1.7) and (4.1.8)

Let us introduce the following notation: for $f\in C_b^\infty$, for $j\in\mathbb{N}$, $\Delta_{j,n}f(Z^1,Z^2) := f\big(\frac{1}{2}(Z^1_{T_{j,n}} + Z^1_{T_{j+1,n}}), \frac{1}{2}(Z^2_{T_{j,n}} + Z^2_{T_{j+1,n}})\big)$. Then, thanks to Lemma 4.2.4, we have

$$\begin{aligned} &f(Z^1_{T_{j+1,n}}, Z^2_{T_{j+1,n}}) - f(Z^1_{T_{j,n}}, Z^2_{T_{j,n}}) \\ =\ & \Delta_{j,n}\frac{\partial f}{\partial x}(Z^1,Z^2)\big(Z^1_{T_{j+1,n}} - Z^1_{T_{j,n}}\big) + \Delta_{j,n}\frac{\partial f}{\partial y}(Z^1,Z^2)\big(Z^2_{T_{j+1,n}} - Z^2_{T_{j,n}}\big) \\ &+\sum_{i=2}^{7}\sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\,\Delta_{j,n}\partial^{\alpha_1,\alpha_2}_{1\dots 12\dots 2}f(Z^1,Z^2)\big(Z^1_{T_{j+1,n}} - Z^1_{T_{j,n}}\big)^{\alpha_1} \\ &\qquad\times\big(Z^2_{T_{j+1,n}} - Z^2_{T_{j,n}}\big)^{\alpha_2} + R_{13}\big((Z^1_{T_{j+1,n}}, Z^2_{T_{j+1,n}})\big),\big(Z^1_{T_{j,n}}, Z^2_{T_{j,n}}\big)\big)\big). \end{aligned}$$

Then, by the Definition 4.2.1 and (4.4.100), we can write

$$\begin{aligned} &f(Z^1_{T_{\lfloor 2^n t\rfloor,n}}, Z^2_{T_{\lfloor 2^n t\rfloor,n}}) - f(0,0) \\ =\ & \tilde{O}_n(f,t) + \sum_{i=2}^{7}\sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\,\tilde{V}_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\dots 12\dots 2}f,t) \\ &+\sum_{j=0}^{\lfloor 2^n t\rfloor-1} R_{13}\big((Z^1_{T_{j+1,n}}, Z^2_{T_{j+1,n}})\big),\big(Z^1_{T_{j,n}}, Z^2_{T_{j,n}}\big)\big)\big). \end{aligned}$$

Thanks to (4.4.114), we can write

$$\begin{aligned}
& \tilde{O}_n(f,t) \qquad (4.4.117)\\
= \;& f(Z^1_{T_{\lfloor 2^n t\rfloor,n}}, Z^2_{T_{\lfloor 2^n t\rfloor,n}}) - f(0,0) - \tilde{V}_n^{(3)}(f,t)\\
& -\sum_{i=3}^{7}\sum_{\alpha_1+\alpha_2=2i-1} C(\alpha_1,\alpha_2)\, \tilde{V}_n^{\alpha_1,\alpha_2}(\partial^{\alpha_1,\alpha_2}_{1\ldots 12\ldots 2} f, t)\\
& - \sum_{j=0}^{\lfloor 2^n t\rfloor -1} R_{13}\big((Z^1_{T_{j+1,n}}, Z^2_{T_{j+1,n}})), (Z^1_{T_{j,n}}, Z^2_{T_{j,n}}))\big).
\end{aligned}$$

By Lemma 4.2.4, we have, with $G \sim N(0,1)$,

$$\begin{aligned}
& \sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} E\Big(\Big| R_{13}\big((Z^1_{T_{j+1,n}}, Z^2_{T_{j+1,n}})), (Z^1_{T_{j,n}}, Z^2_{T_{j,n}}))\big)\Big|\Big)\\
\leqslant\; & C_f \sum_{\alpha_1+\alpha_2=13}\sum_{j=0}^{\lfloor 2^n t\rfloor -1} E\Big(\big|Z^1_{T_{j+1,n}} - Z^1_{T_{j,n}}\big|^{\alpha_1}\big|Z^2_{T_{j+1,n}} - Z^2_{T_{j,n}}\big|^{\alpha_2}\Big)\\
\leqslant\; & C_f \sum_{\alpha_1+\alpha_2=13}\sum_{j=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \||Z^1_{T_{j+1,n}} - Z^1_{T_{j,n}}|^{\alpha_1}\|_2 \||Z^2_{T_{j+1,n}} - Z^2_{T_{j,n}}|^{\alpha_2}\|_2\\
=\; & C_f 2^{-\frac{13nH}{2}} \sum_{\alpha_1+\alpha_2=13}\sum_{j=0}^{\lfloor 2^n t\rfloor -1} \big(E[G^{2\alpha_1}]E[G^{2\alpha_2}]\big)^{1/2} \leqslant Ct2^{-n(\frac{13H}{2}-1)}. \qquad (4.4.118)
\end{aligned}$$

On the other hand, by continuity of $f\circ Z$ and due to (4.2.11), one has, almost surely and as $n\to\infty$,

$$f(Z^1_{T_{\lfloor 2^n t\rfloor,n}}, Z^2_{T_{\lfloor 2^n t\rfloor,n}}) - f(0,0) \to f(Z^1_t, Z^2_t) - f(0,0). \qquad (4.4.119)$$

Finally, when $H > \frac{1}{6}$ the desired conclusion (4.1.7) follows from (4.4.112), (4.4.113), (4.4.118) and (4.4.119) plugged into (4.4.117). The proof of (4.1.8) when $H = \frac{1}{6}$ is similar, the only difference being that one has (4.4.116) instead of (4.4.113), thus leading to the bracket term $\int_0^t D^3 f(Z_s) d^3 Z_s$ in (4.1.8).

4.4.7 Step 7: Proving (4.1.9)

Using $b^3 - a^3 = 3\big(\frac{a+b}{2}\big)^2(b-a) + \frac{1}{4}(b-a)^3$, one can write,

$$\tilde{O}_n(x\mapsto x^3,t) - (Z^1_t)^3 \;=\; -\tilde{V}^3_n(x\mapsto x^3,t) + \sum_{j=0}^{\lfloor 2^n t\rfloor -1}\big((Z^1_{T_{j+1,n}})^3 - (Z^1_{T_{j,n}})^3\big) - (Z^1_t)^3.$$

As a result and since, by (4.2.11), $(Z^1_{T_{\lfloor 2^n t\rfloor,n}})^3 \to (Z^1_t)^3$ a.s. as $n\to\infty$, one deduces that if $\tilde{O}_n(x\mapsto x^3,t)$ converges stably in law, then $\tilde{V}^3_n(x\mapsto x^3,t)$ must converge as well. But it is shown in Corollary 2.1.2 in chapter 2 that $2^{-n(1-6H)/4}\tilde{V}^{(3)}_n(x\mapsto x^3,t)$ converges in law to a non degenerate limit. This fact being in contradiction with the convergence of $\tilde{V}^3_n(x\mapsto x^3,t)$, we deduce that (4.1.9) holds.

4.5 Proof of Lemma 4.2.10

Thanks to (4.2.20), we have

$$\begin{aligned}
& I_1^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(i_4+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big) \\
= \; & I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big) \\
& +\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle I_2^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big) \\
= \; & I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big) \\
& +4I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle \\
& +2I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2 \\
& +I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle \\
& +4I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2 \\
& +2\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^3.
\end{aligned}$$

For $i_1,i_2,i_3,i_4\in\mathbb{N}$, set $\phi(i_1,i_2,i_3,i_4):=\prod_{a=1}^4\Delta_{i_a,n}f_a(X^1,X^2)$. Then, thanks to the previous estimate, we get

$$\begin{aligned}
& \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\left(\prod_{a=1}^4\Delta_{i_a,n}f_a(X^1,X^2)I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\right)\right| \\
\leqslant \; & \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\left(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^2 I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\right.\right. \\
& \qquad\left.\left.\times I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\right)\right| \\
& +4\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\left(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^2 I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\right.\right. \\
& \qquad\left.\left.\times I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\right)\right| \\
& \qquad\times|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|
\end{aligned}$$

$$
\begin{aligned}
&+2\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\\
&\qquad\qquad\times I_2^{(1)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\bigg)\bigg|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2\\
&+\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\\
&\qquad\qquad\times I_4^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\big)\bigg)\bigg||\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|\\
&+4\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\\
&\qquad\qquad\times I_2^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\big)\bigg)\bigg|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle^2\\
&+2\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\bigg)\bigg|\\
&\qquad\qquad\times|\langle\delta_{(i_3+1)2^{-n/2}},\delta_{(i_4+1)2^{-n/2}}\rangle|^3\\
&=:\sum_{i=1}^{6}L_{n,i}(t).
\end{aligned}
$$

Let us prove that, for any $i\in\{1,\ldots,6\}$ there exists $C>0$ (depending only on f) such that

$$L_{n,i}(t)\leqslant C(t+t^2+t^3+t^4). \tag{4.5.120}$$

Then the desired conclusion of Lemma 4.2.10 will follow immediately.

Thanks to the duality formula (4.2.19), we have

$$
\begin{aligned}
L_{n,1}(t) &= \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\bigg\langle D^2_{X^{(1)}}\bigg(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\bigg),\\
&\qquad\qquad \delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\bigg\rangle I_4^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\big)\bigg)\bigg|\\
&= \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\bigg\langle D^2_{X^{(1)}}\bigg(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\bigg),\\
&\qquad \delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\bigg\rangle\prod_{a=1}^{2}I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)I_4^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\big)\bigg)\bigg|
\end{aligned}
$$

When computing the second Malliavin derivative

$$D^2_{X^{(1)}}\bigg(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\bigg),$$

there are three types of terms:

(1) The first type consists in terms arising when one only differentiates $\phi(i_1,i_2,i_3,i_4)$. By (4.2.32), these terms are all bounded by

$$C2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\widetilde{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\\ \times I_4^{(2)}\big(\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\big)\bigg)\bigg|,$$

where $\widetilde{\phi}(i_1,i_2,i_3,i_4)$ is a quantity having a similar form as $\phi(i_1,i_2,i_3,i_4)$. By the duality formula (4.2.19), we deduce that the last quantity is equal to

$$\begin{aligned}
&C2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\bigg\langle D_{X^{(2)}}^4\bigg(\widetilde{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\bigg),\\
&\qquad\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\bigg)\bigg|\\
= {}&C2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\bigg\langle D_{X^{(2)}}^4\bigg(\widetilde{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\bigg),\\
&\qquad\delta_{(i_3+1)2^{-n/2}}^{\otimes 2}\otimes\delta_{(i_4+1)2^{-n/2}}^{\otimes 2}\bigg\rangle\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\bigg)\bigg|.
\end{aligned}$$

When computing the fourth Malliavin derivative

$$D_{X^{(2)}}^4\bigg(\widetilde{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\bigg),$$

there are three types of terms:

(a) The first type consists in terms arising when one only differentiates $\widetilde{\phi}(i_1,i_2,i_3,i_4)$. Thanks to (4.2.32), these terms are all bounded by

$$C2^{-n}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\bar{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\bigg)\bigg|,$$

where $\bar{\phi}(i_1,i_2,i_3,i_4)$ is a quantity having a similar form as $\widetilde{\phi}(i_1,i_2,i_3,i_4)$. Observe that the last quantity is less than

$$\begin{aligned}
&Ct^2\sup_{i_3,i_4\in\{0,\ldots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}}\sum_{i_1,i_2=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\bigg(\bar{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\\
&\qquad\times I_2^{(2)}\big(\delta_{(i_a+1)2^{-n/2}}^{\otimes 2}\big)\bigg)\bigg|\\
\leqslant{}& C(t^3+t^4),
\end{aligned}\tag{4.5.121}$$

where the last inequality is a consequence of Lemma 4.2.7.

(b) The second type consists in terms arising when one differentiates $\widetilde{\phi}(i_1, i_2, i_3, i_4)$ and $I_2^{(2)}(\delta^{\otimes 2}_{(i_1+1)2^{-n/2}})$ but not $I_2^{(2)}(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}})$ (the case when one differentiates $\widetilde{\phi}(i_1, i_2, i_3, i_4)$ and $I_2^{(2)}(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}})$ but not $I_2^{(2)}(\delta^{\otimes 2}_{(i_1+1)2^{-n/2}})$ is completely similar). In this case, with ρ defined in (4.2.23) and $\alpha \in \{0, 1\}$, the corresponding terms are bounded either by

$$C2^{-n} \sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} \left| E\Big(\bar{\phi}(i_1, i_2, i_3, i_4) I_\alpha^{(2)}(\delta^{\otimes \alpha}_{(i_1+1)2^{-n/2}}) I_2^{(2)}(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}) \times \prod_{a=1}^{2} I_1^{(1)}(\delta_{(i_a+1)2^{-n/2}}) \Big) \right| |\rho(i_1 - i_3)| \qquad (4.5.122)$$

or by the same quantity with $|\rho(i_1 - i_4)|$ instead of $|\rho(i_1 - i_3)|$. We have obtained the previous estimate by using (4.2.22) and (4.2.32). Observe that, by the duality formula (4.2.19), we have

$$\begin{aligned}
&\left| E\Big(\bar{\phi}(i_1, i_2, i_3, i_4) I_\alpha^{(2)}(\delta^{\otimes \alpha}_{(i_1+1)2^{-n/2}}) I_2^{(2)}(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}) \prod_{a=1}^{2} I_1^{(1)}(\delta_{(i_a+1)2^{-n/2}}) \Big) \right| \\
= & \left| E\Big(\Big\langle D^2_{X^{(2)}}\Big(\bar{\phi}(i_1, i_2, i_3, i_4) I_\alpha^{(2)}(\delta^{\otimes \alpha}_{(i_1+1)2^{-n/2}}) \Big), \delta^{\otimes 2}_{(i_2+1)2^{-n/2}} \Big\rangle \times \prod_{a=1}^{2} I_1^{(1)}(\delta_{(i_a+1)2^{-n/2}}) \Big) \right| \\
=: & \; F(i_1, i_2, i_3, i_4, \alpha).
\end{aligned}$$

We have

- For $\alpha = 0$:

$$\begin{aligned}
& F(i_1, i_2, i_3, i_4, 0) \\
= & \left| E\Big(\Big\langle D^2_{X^{(2)}}(\bar{\phi}(i_1, i_2, i_3, i_4)), \delta^{\otimes 2}_{(i_2+1)2^{-n/2}} \Big\rangle \prod_{a=1}^{2} I_1^{(1)}(\delta_{(i_a+1)2^{-n/2}}) \Big) \right| \\
\leqslant & \; C(2^{-n/6})^2 \|I_1^{(1)}(\delta_{(i_1+1)2^{-n/2}})\|_2 \|I_1^{(1)}(\delta_{(i_2+1)2^{-n/2}})\|_2 \\
\leqslant & \; C2^{-n/2},
\end{aligned}$$

  where we have the first inequality since $f \in C_b^\infty$ and thanks to (4.2.32) and to the Cauchy-Schwarz inequality. The second inequality follows from (4.2.21).

- For $\alpha = 1$: Thanks to (4.2.16),(4.2.17), (4.2.22),(4.2.32) and (4.2.21), we have

$$\begin{aligned}
& F(i_1, i_2, i_3, i_4, 1) \\
\leqslant & \; C2^{-n/3} E\big(|I_1^{(1)}(\delta_{(i_1+1)2^{-n/2}})| |I_1^{(1)}(\delta_{(i_2+1)2^{-n/2}})| \big) \\
\leqslant & \; C2^{-n/3} \|I_1^{(1)}(\delta_{(i_1+1)2^{-n/2}})\|_2 \|I_1^{(1)}(\delta_{(i_2+1)2^{-n/2}})\|_2 \leqslant C2^{-n/2}.
\end{aligned}$$

For $\alpha \in \{0, 1\}$, by plugging $F(i_1, i_2, i_3, i_4, \alpha)$ into (4.5.122) we deduce that the quantity given in (4.5.122) is bounded by

$$Ct^2 2^{-n/2} \sum_{i_1,i_3=0}^{\lfloor 2^{\frac{n}{2}} t\rfloor -1} |\rho(i_1 - i_3)| \leqslant Ct^3 (\sum_{r \in \mathbb{Z}} |\rho(r)|) \leqslant Ct^3. \qquad (4.5.123)$$

Note that $\sum_{r\in\mathbb{Z}}|\rho(r)|<\infty$ because $H=1/6<1/2$.

(c) The third type consists in terms arising when one differentiates $\widetilde{\phi}(i_1,i_2,i_3,i_4)$, $I_2^{(2)}(\delta^{\otimes 2}_{(i_1+1)2^{-n/2}})$ and $I_2^{(2)}(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}})$. In this case, thanks to (4.2.22) and (4.2.32), for $\alpha,\beta\in\{0,1\}$ the corresponding terms can be bounded either by

$$C2^{-n}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\bar{\phi}(i_1,i_2,i_3,i_4)I_\alpha^{(2)}(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}})I_\beta^{(2)}(\delta^{\otimes\beta}_{(i_2+1)2^{-n/2}})\times\prod_{a=1}^{2}I_1^{(1)}(\delta_{(i_a+1)2^{-n/2}})\Big)\Big||\rho(i_1-i_3)||\rho(i_2-i_3)|, \qquad (4.5.124)$$

or by the same quantity with $|\rho(i_1-i_4)||\rho(i_2-i_4)|$ or $|\rho(i_1-i_3)||\rho(i_2-i_4)|$ or $|\rho(i_2-i_3)||\rho(i_1-i_4)|$ instead of $|\rho(i_1-i_3)||\rho(i_2-i_3)|$. Observe that

$$\Big|E\Big(\bar{\phi}(i_1,i_2,i_3,i_4)I_\alpha^{(2)}(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}})I_\beta^{(2)}(\delta^{\otimes\beta}_{(i_2+1)2^{-n/2}})\prod_{a=1}^{2}I_1^{(1)}(\delta_{(i_a+1)2^{-n/2}})\Big)\Big|$$

is uniformly bounded in n. So, the quantity given in (4.5.124) is bounded by

$$Ct2^{-n/2}\sum_{i_1,i_2,i_3=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\rho(i_1-i_3)||\rho(i_2-i_3)|\leqslant Ct^2(\sum_{r\in\mathbb{Z}}|\rho(r)|)^2\leqslant Ct^2. \qquad (4.5.125)$$

Thanks to (4.5.125), (4.5.123) and (4.5.121), we deduce that the terms of the first type in $L_{n,1}(t)$ agree with the desired conclusion (4.5.120).

(2) The second type consists in terms arising when one differentiates $\phi(i_1,i_2,i_3,i_4)$ and $I_1^{(1)}(\delta_{(i_1+1)2^{-n/2}})$, but not $I_1^{(1)}(\delta_{(i_2+1)2^{-n/2}})$ (the case where one differentiates $\phi(i_1,i_2,i_3,i_4)$ and $I_1^{(1)}(\delta_{(i_2+1)2^{-n/2}})$, but not $I_1^{(1)}(\delta_{(i_1+1)2^{-n/2}})$ is completely similar). In this case, thanks to (4.2.32), the corresponding terms are all bounded either by

$$C2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\widetilde{\phi}(i_1,i_2,i_3,i_4)I_1^{(1)}(\delta_{(i_2+1)2^{-n/2}})\prod_{a=1}^{2}I_2^{(2)}(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}})\times I_4^{(2)}(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}})\Big)\Big||\rho(i_1-i_3)|,$$

or by the same quantity with $|\rho(i_1-i_4)|$ instead of $|\rho(i_1-i_3)|$. By the duality formula (4.2.19), the previous quantity is equal to

$$C2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\Big\langle D^4_{X^{(2)}}\Big(\widetilde{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}})\Big),\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\Big\rangle I_1^{(1)}(\delta_{(i_2+1)2^{-n/2}})\Big)\Big||\rho(i_1-i_3)|.$$

When computing the fourth Malliavin derivative

$$D^4_{X^{(2)}}\Big(\widetilde{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2} I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big),$$

there are three types of terms, exactly as it has been proved previously:

(a) The first type consists in terms arising when one only differentiates $\widetilde{\phi}(i_1,i_2,i_3,i_4)$. Thanks to (4.2.32), these terms are all bounded by

$$C2^{-n}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\bar{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2} I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\\ \times I_1^{(1)}\big(\delta_{(i_2+1)2^{-n/2}}\big)\Big)\Big||\rho(i_1-i_3)|. \tag{4.5.126}$$

Observe that since $f\in C_b^\infty$ and thanks to (4.2.19), (4.2.32) and (4.2.21), we have

$$\begin{aligned}
&\Big|E\Big(\bar{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2} I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(i_2+1)2^{-n/2}}\big)\Big)\Big|\\
= &\Big|E\Big(\big\langle D_{X^{(1)}}\big(\bar{\phi}(i_1,i_2,i_3,i_4)\prod_{a=1}^{2} I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\big),\delta_{(i_2+1)2^{-n/2}}\big\rangle\Big)\Big|\\
\leqslant &\ C2^{-n/6}E\Big(\prod_{a=1}^{2}\big|I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\big|\Big)\\
\leqslant &\ C2^{-n/6}\|I_2^{(2)}\big(\delta^{\otimes 2}_{(i_1+1)2^{-n/2}}\big)\|_2\|I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)\|_2\leqslant C2^{-n/2}.
\end{aligned}$$

Hence, we deduce that the quantity given in (4.5.126) is bounded by

$$Ct^2 2^{-n/2}\sum_{i_1,i_3=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\rho(i_1-i_3)|\leqslant Ct^3\big(\sum_{r\in\mathbb{Z}}|\rho(r)|\big)\leqslant Ct^3. \tag{4.5.127}$$

(b) The second type consists in terms arising when one differentiates $\widetilde{\phi}(i_1,i_2,i_3,i_4)$ and $I_2^{(2)}\big(\delta^{\otimes 2}_{(i_1+1)2^{-n/2}}\big)$ but not $I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)$ (the case when one differentiates $\widetilde{\phi}(i_1,i_2,i_3,i_4)$ and $I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)$ but not $I_2^{(2)}\big(\delta^{\otimes 2}_{(i_1+1)2^{-n/2}}\big)$ is completely similar). In this case, thanks to (4.2.32) and for $\alpha\in\{0,1\}$, the corresponding terms are all bounded either by

$$C2^{-n}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\bar{\phi}(i_1,i_2,i_3,i_4)I_\alpha^{(2)}\big(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)\\ \times I_1^{(1)}\big(\delta_{(i_2+1)2^{-n/2}}\big)\Big)\Big||\rho(i_1-i_3)||\rho(i_1-i_4)| \tag{4.5.128}$$

or by the same quantity with $|\rho(i_1-i_3)|$ instead of $|\rho(i_1-i_4)|$. Observe that, by

(4.2.19) and (4.2.32) among other things and since $f \in C_b^\infty$, we have

$$\begin{aligned}
&\Big|E\Big(\bar\phi(i_1,i_2,i_3,i_4)I_\alpha^{(2)}\big(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(i_2+1)2^{-n/2}}\big)\Big)\Big| \\
= &\Big|E\Big(\langle D_{X^{(1)}}\big(\bar\phi(i_1,i_2,i_3,i_4)\big),\delta_{(i_2+1)2^{-n/2}}\rangle I_\alpha^{(2)}\big(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)\Big)\Big| \\
\leqslant & C2^{-n/6}\Big|E\Big(\chi(i_1,i_2,i_3,i_4)I_\alpha^{(2)}\big(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)\Big)\Big| \\
= & C2^{-n/6}\Big|E\Big(\langle D^2_{X^{(2)}}\big(\chi(i_1,i_2,i_3,i_4)I_\alpha^{(2)}\big(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}}\big)\big),\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\rangle\Big)\Big| \\
\leqslant & C2^{-n/2},
\end{aligned}$$

where $\chi(i_1,i_2,i_3,i_4)$ is a quantity having a similar form as $\bar\phi(i_1,i_2,i_3,i_4)$. Thus, we get that the quantity given by (4.5.128) is bounded by

$$\begin{aligned}
C2^{-n}t\sum_{i_1,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\rho(i_1-i_3)||\rho(i_1-i_4)| &\leqslant Ct^2 2^{-n/2}\big(\sum_{r\in\mathbb{Z}}|\rho(r)|\big)^2 \\
&\leqslant C2^{-n/2}t^2. \qquad (4.5.129)
\end{aligned}$$

(c) The third type consists in terms arising when one differentiates $\widetilde\phi(i_1,i_2,i_3,i_4)$, $I_2^{(2)}\big(\delta^{\otimes 2}_{(i_1+1)2^{-n/2}}\big)$ and $I_2^{(2)}\big(\delta^{\otimes 2}_{(i_2+1)2^{-n/2}}\big)$. In this case, thanks to (4.2.32), for $\alpha,\beta\in\{0,1\}$ the corresponding terms can be bounded either by

$$\begin{aligned}
C2^{-n}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}&\Big|E\Big(\bar\phi(i_1,i_2,i_3,i_4)I_\alpha^{(2)}\big(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}}\big)I_\beta^{(2)}\big(\delta^{\otimes\beta}_{(i_2+1)2^{-n/2}}\big) \\
&\times I_1^{(1)}\big(\delta_{(i_2+1)2^{-n/2}}\big)\Big)\Big||\rho(i_1-i_3)||\rho(i_1-i_4)||\rho(i_2-i_4)|, \qquad (4.5.130)
\end{aligned}$$

or by the same quantity with $|\rho(i_1-i_3)||\rho(i_2-i_3)|$ or $|\rho(i_1-i_3)||\rho(i_2-i_4)|$ or $|\rho(i_2-i_3)||\rho(i_1-i_4)|$ instead of $|\rho(i_1-i_4)||\rho(i_2-i_4)|$. Observe that

$$\Big|E\Big(\bar\phi(i_1,i_2,i_3,i_4)I_\alpha^{(2)}\big(\delta^{\otimes\alpha}_{(i_1+1)2^{-n/2}}\big)I_\beta^{(2)}\big(\delta^{\otimes\beta}_{(i_2+1)2^{-n/2}}\big)I_1^{(1)}\big(\delta_{(i_2+1)2^{-n/2}}\big)\Big)\Big|$$

is uniformly bounded in n. So, we deduce that the quantity given by (4.5.130) is bounded by

$$\begin{aligned}
C2^{-n}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\rho(i_1-i_3)||\rho(i_1-i_4)||\rho(i_2-i_4)| &\leqslant C2^{-n/2}t\big(\sum_{r\in\mathbb{Z}}|\rho(r)|\big)^3 \\
&\leqslant C2^{-n/2}t. \qquad (4.5.131)
\end{aligned}$$

Combining (4.5.131), (4.5.129) and (4.5.127), we deduce that the terms of the second type in $L_{n,1}(t)$ agree with the desired conclusion (4.5.120).

(3) The third type consists in terms arising when one only differentiates $\prod_{a=1}^2 I_1^{(1)}\big(\delta_{(i_a+1)2^{-\frac{n}{2}}}\big)$.

In this case, thanks to (4.2.16) and (4.2.22), the corresponding term is equal to:

$$
\begin{aligned}
&\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\Big)\Big|\\
&\qquad\qquad\times\big|\langle\delta_{(i_1+1)2^{-n/2}}\tilde{\otimes}\delta_{(i_2+1)2^{-n/2}},\delta_{(i_3+1)2^{-n/2}}\otimes\delta_{(i_4+1)2^{-n/2}}\rangle\big|\\
\leqslant\;& 2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\\
&\qquad\qquad\times I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\Big)\Big||\rho(i_1-i_3)||\rho(i_2-i_4)|\\
&+2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\\
&\qquad\qquad\times I_4^{(2)}\big(\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\big)\Big)\Big||\rho(i_2-i_3)||\rho(i_1-i_4)|.
\end{aligned}
$$

It suffices to prove that the second quantity agree with the desired conclusion (4.5.120) (similarly, the first quantity agree as well with (4.5.120)). Thanks to the duality formula (4.2.19), we have that the last quantity is equal to

$$
\begin{aligned}
2^{-n/3}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Big|E\Big(\Big\langle D^4_{X^{(2)}}\Big(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big),\qquad (4.5.132)\\
\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\Big\rangle\Big)\Big||\rho(i_2-i_3)||\rho(i_1-i_4)|.
\end{aligned}
$$

Observe that one can prove, thanks to (4.2.32) among other things (and following the approach already used several times previously) that

$$
\begin{aligned}
&\Big|E\Big(\Big\langle D^4_{X^{(2)}}\Big(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big),\delta^{\otimes 2}_{(i_3+1)2^{-n/2}}\otimes\delta^{\otimes 2}_{(i_4+1)2^{-n/2}}\Big\rangle\Big)\Big|\\
\leqslant\;& C2^{-2n/3}.
\end{aligned}
$$

Hence, we get that the quantity given in (4.5.132) is bounded by

$$
C2^{-n}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\rho(i_2-i_3)||\rho(i_1-i_4)|\leqslant Ct^2\big(\sum_{r\in\mathbb{Z}}|\rho(r)|\big)^2\leqslant Ct^2,
$$

which agrees with the desired conclusion (4.5.120).

Finally, we have proved that $L_{n,1}(t)$ agrees with the desired conclusion (4.5.120).

The motivated reader may check that there is no additional difficulties to prove that for all $i\in\{2,\ldots,5\}$, $L_{n,i}(t)$ agrees with the desired conclusion (4.5.120). Indeed, all the arguments and techniques which are needed to prove this claim, were already introduced and used along the previous proof.

It remains to prove that $L_{n,6}(t)$ agrees with the desired conclusion (4.5.120). Observe that

$$\begin{aligned}
&L_{n,6}(t)\\
&= 22^{-n/2}\sum_{i_1,i_2,i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\left|E\Big(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big)\right|\\
&\qquad\qquad\times|\rho(i_3-i_4)|^3\\
&\leqslant 22^{-n/2}\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\Bigg(\sup_{i_3,i_4\in\{0,\dots,\lfloor 2^{\frac{n}{2}}t\rfloor-1\}}\sum_{i_1,i_2=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}\bigg|E\Big(\phi(i_1,i_2,i_3,i_4)\prod_{a=1}^{2}I_1^{(1)}\big(\delta_{(i_a+1)2^{-n/2}}\big)\\
&\qquad\qquad\times I_2^{(2)}\big(\delta^{\otimes 2}_{(i_a+1)2^{-n/2}}\big)\Big)\bigg|\Bigg)|\rho(i_3-i_4)|^3\\
&\leqslant C(t+t^2)2^{-n/2}\sum_{i_3,i_4=0}^{\lfloor 2^{\frac{n}{2}}t\rfloor-1}|\rho(i_3-i_4)|^3\leqslant C(t+t^2)t\big(\sum_{r\in\mathbb{Z}}|\rho(r)|^3\big)\leqslant C(t^2+t^3),
\end{aligned}$$

where the second inequality is a consequence of Lemma 4.2.7. Thanks to the previous estimate, it is clear that $L_{n,6}(t)$ agrees with the desired conclusion (4.5.120). The proof of Lemma 4.2.10 is now complete.

Bibliography

[1] J. Asch and J. Potthoff (1991): Itô's lemma without non-anticipatory conditions. *Probab. Rel. Fields* (1991), volume 88, issue 1, pp 17-46.

[2] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2009): Power variation for Gaussian processes with stationary increments. Stoch. Proc. Appl. **119**, pp. 1845-1865.

[3] C. Bender (2003): An Itô Formula for Generalised Functionals of a Fractional Brownian Motion With Arbitrary Hurst Parameter. Stoch. Process. Appl. **104**, pp. 81-106.

[4] P. Breuer and P. Major (1983): Central limit theorems for non-linear functionals of Gaussian fields. J. Multiv. Anal **13**, pp. 425-441.

[5] K. Burdzy (1993): Some path properties of iterated Brownian motion. In *Seminar on Stochastic Processes* (E. Cinlar, K. L. Chung and M. J. Sharpe, eds.) pp. 67-87. Birkhaüser, Boston.

[6] K. Burdzy (1994): Variation of iterated Brownian motion. In *Workshop and Conference on Measure-Valued Processes, Stochastic Partial Differential Equations and Interacting Particle Systems* (D. A. Dawson, ed.), pp. 35-53. Amer. Math. Soc., Providence, RI.

[7] K. Burdzy and D. Khoshnevisan (1998): Brownian motion in a Brownian crack. *Ann. Appl. Probab.* **8**, pp. 708-748.

[8] K. Burdzy and J. Swanson (2010): A change of variable formula with Itô correction term. *Ann. Probab.* **38**, no. 5, pp. 1817-1869.

[9] P. Cheridito and D. Nualart (2005): Stochastic integral of divergence type with respect to fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2})$. *Ann. Inst. H. Poincaré Probab. Statist.* **41**, no. 6, pp. 1049-1081.

[10] J.M. Corcuera, D. Nualart and J.H.C. Woerner (2006): Power variation for some integral fractional processes. *Bernoulli* **12**, no. 4, pp. 713-735.

[11] M. Errami and F. Russo (2003): n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stoch. Proc. Appl.* 104 (2003) 259-299.

[12] T. Funaki (1979): Probabilistic Construction of the Solution of Some Higher Order Parabolic Differential Equation. *Proc. Japan Acad.* **55**, Ser. A.

[13] M. Gradinaru, I. Nourdin, F. Russo and P. Vallois (2005): m-order integrals and generalized Itô's formula: the case of a fractional Brownian motion with any Hurst index. *Ann. Inst. H. Poincaré Probab. Statist.* **41**, no. 4, pp. 781-806.

[14] D. Harnett and D. Nualart (2012): Weak convergence of the Stratonovich integral with respect to a class of Gaussian processes. *Stoch. Proc. Appl.* **122**, pp. 3460-3505.

[15] D. Harnett and D. Nualart (2013): Central limit theorem for a Stratonovich integral with Malliavin calculus. *Ann. Probab.* **41**, no. 4, pp. 2820-2879.

[16] D. Harnett and D. Nualart (2013): On Simpson's rule and fractional Brownian motion with $H = 1/10$. *J. Theoret. Probab.* To appear.

[17] K. Itô (1944): Stochastic integral. Proc. Imp. Acad. Tokyo. **20**.

[18] H. Kesten and F. Spitzer (1979): A limit theorem related to a new class of self-similar process. *Z. Wahrsch. Verw. Gebiete* **50**, pp. 5-25.

[19] D. Khoshnevisan and T. M. Lewis (1996): The uniform modulus of continuity for iterated Brownian motion. *J. Theoret. Probab.* **9**, pp. 317-333.

[20] D. Khoshnevisan and T. M. Lewis (1999): Stochastic calculus for Brownian motion on a Brownian fracture. *Ann. Appl. Probab.* **9**, no. 3, pp. 629-667.

[21] D. Khoshnevisan and T. M. Lewis (1999): Iterated Brownian motion and its intrinsic skeletal structure. *Progress in Probability* **45**, pp. 201-210, Birkhaüser Verlag, Basel.

[22] P-A. Meyer (1976): Un cours sur les intégrales stochastiques (exposés 1 à 6). *Séminaire de probabilités* (Strasbourg) **X**, pp. 245-400.

[23] E. Nane (2008): Higher order PDE's and iterated processes. *Trans. Amer. Math. Soc.* **360**, pp. 2681-2692.

[24] I. Nourdin (2004): Calcul stochastique généralisé et applications au mouvement brownien fractionnaire; Estimation non-paramétrique de la volatilité et test d'adéquation. *PhD thesis, Université Henri Poincaré*, 2004.

[25] I. Nourdin (2009): A change of variable formula for the 2D fractional Brownian motion of Hurst index bigger or equal to 1/4. *J. Funct. Anal.* **256**, pp. 2303-2320.

[26] I. Nourdin, D. Nualart and C. Tudor (2009): Central and non-central limit theorems for weighted power variations of the fractional Brownian motion. *Ann. Inst. H. Poincaré Probab. Statist.* **46**, no. 4, pp. 1055-1079.

[27] I. Nourdin and G. Peccati (2008): Weighted power variations of iterated Brownian motion. *Electron. J. Probab.* **13**, no. 43, pp. 1229-1256.

[28] I. Nourdin and G. Peccati (2012): *Normal Approximations using Malliavin Calculus: from Stein's Method to the Universality.* Cambridge University Press.

[29] I. Nourdin and A. Réveillac (2009): Asymptotic behavior of weighted quadratic variations of fractional Brownian motion: the critical case $H = 1/4$. *Ann. Probab.* **37**, pp. 2200-2230.

[30] I. Nourdin, A. Réveillac and J. Swanson (2010): The weak Stratonovich integral with respect to fractional Brownian motion with Hurst parameter 1/6. *Electron. J. Probab.* **15**, no. 70, pp. 2117-2162.

[31] I. Nourdin and R. Zeineddine (2013): An Itô-type formula for the fractional Brownian motion in Brownian time. *Electron. J. Probab.* 19 (2014), no. 99, 1-15.

[32] D. Nualart and E. Pardoux (1988): Stochastic calculus with anticipating integrands. *Probab. Rel. Fields*, 78, 1988, p. 80-129.

[33] D. Nualart and G. Peccati (2005): Central limit theorems for sequences of multiple stochastic integrals. *Ann. Probab.* **33**, no. 1, pp. 177-193.

[34] G. Peccati and C.A. Tudor (2004): Gaussian limits for vector-valued multiple stochastic integrals. *Séminaire de Probabilités* **XXXVIII**, pp. 247-262.

[35] F. Russo and P. Vallois (1993): Forward, backward and symmetric stochastic integration. *Probab. Rel. Fields* **97**, pp. 403-421.

[36] F. Russo and P. Vallois (1995): The generalized covariation process and Itô formula. *Stoch. Proc. Appl.* **59**, 81-104.

[37] F. Russo and P. Vallois (2000): Stochastic calculus with respect to a finite quadratic variation process. *Stochastics and Stochastics Reports* **70**, 1-40.

[38] A. V. Skorokhod (1975): On a generalization of a stochastic integral. *Theory Prob. Applications* (1975), pp. 219-233.

[39] J. A. Zadeh (2012): Iterated fractional Brownian motion: Tail estimates and large deviations. *PhD thesis, Purdue University*, (2012).

[40] R. Zeineddine (2013): Fluctuations of the power variation of fractional Brownian motion in Brownian time. *Bernoulli*, to appear.

Résumé de la thèse en français:

Le mouvement brownien fractionnaire en temps brownien Z est un processus qui sert de modèle à la diffusion d'un gaz le long d'une fissure. Dans cette thèse, réalisée sous la direction d'Ivan Nourdin, nous prouvons des formules de type Itô pour Z. Nos principaux outils sont le calcul de Malliavin, le calcul stochastique et l'utilisation de théorèmes limites. Une des spécificités des formules de changement de variables que nous avons obtenues est qu'elles ont lieu en loi, avec création d'un nouvel aléa.

Ce mémoire est constitué d'un chapitre introductif, suivi de trois autres chapitres qui correspondent chacun à différents résultats obtenus lors de la préparation de cette thèse et rédigés sous forme d'articles de recherche. Plus précisément:

1) Dans un premier article, nous introduisons le processus central de cette thèse, à savoir le mouvement brownien fractionnaire en temps brownien Z. Nous étudions ensuite les fluctuations de ses variations d'ordre p, où p est n'importe quel entier supérieur ou égal à 1.

2) Dans un deuxième article, avec mon encadrant Ivan Nourdin nous avons utilisé les résultats du premier article pour construire une formule de type Itô pour Z. Pour ce faire, nous avons étendu à notre cadre une idée due originellement à Khoshnevisan et Lewis, consistant à travailler avec une partition aléatoire du temps au lieu de la partition déterministe classique.

3) Enfin, dans un troisième et dernier article, nous avons prolongé la formule unidimensionnelle décrite en 2) au cadre multidimensionnel.

Résumé de la thèse en anglais :

Fractional Brownian motion in Brownian time Z may serve as a model for the motion of a single gas particle constrained to evolve inside a crack. In this PhD thesis, written under the supervision of Ivan Nourdin, we prove Itô's type formulas for Z. To achieve this goal, our main tools are the Malliavin calculus, the stochastic calculus and the use of limit theorems. One of the specificity of the formula we have obtained is that they hold in law, with creation of a new alea. This manuscript consists in an introductory chapter, followed by three other chapters, each one corresponding to different results obtained along the preparation of this thesis and written is the form of research papers. More precisely:

1) In a first paper, we introduce the central process of this thesis, namely the fractional Brownian motion in Brownian time Z. Then, we study the fluctuations of its power variations of order p, for any integer p greater than or equal to 1.

2) In a second paper, written jointly with my supervisor Ivan Nourdin, we use the results obtained in 1) to build an Itô's type formula for Z. To do so, we need to extend to our setting an approach originally due to Khoshnevisan and Lewis, consisting in rather working with a random partition of time, instead of the classical uniform deterministic partition.

3) Finally, in a third and last paper, we extend to multidimension the one-dimensional formula obtained in 2)

Mots-clés : Calcul stochastique, calcul de Malliavin, théorèmes limites, formules de type Itô, mouvement brownien fractionnaire et mouvement brownien fractionnaire en temps brownien.