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## Extension theorems and Kahler-Einstein matrices

Li Yi

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Thèse présentée par

Li Yi

pour l'obtention du

Doctorat de l'Université de Lorraine  
Spécialité : Mathématiques

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**Théorèmes d'extension et métriques de  
Kähler-Einstein Généralisées**

---

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---



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# Introduction

## Résumé

Dans cette thèse nous allons traiter deux problèmes importantes en géométrie complexe : une version kählérienne du célèbre théorème d'extension d'Ohsawa-Takegoshi, et l'unicité des solutions des équations de type Monge-Ampère généralisées.

Pour ce qui est du premier thème que nous avons abordé, notre motivation provient du problème suivant, formulé par Y.-T. Siu dans [Siu02] :

**Conjecture 1.** *Soit  $\mathcal{X} \rightarrow \mathbb{D}$  une famille kählérienne de variétés sur le disque unité  $\mathbb{D}$ . Soit*

$$u \in H^0(\mathcal{X}_0, m_0 K_{\mathcal{X}_0})$$

*une section pluricanonique définie sur le fibré central. Alors  $u$  admet une extension à  $\mathcal{X}$ .*

Il s'agit donc d'établir l'invariance des plurigenres dans le cas d'une famille kählérienne. Dans le cas projectif, une preuve de ce résultat (cruciale pour la théorie des modèles minimaux en géométrie algébrique) a été obtenue par Y.-T. Siu dans une suite d'articles [Siu98], [Siu02]. Nous remarquons que dans le cadre projectif, le problème précédent est de nature purement algébrique ; malgré cela, c'est la seule preuve connue actuellement utilisée de manière essentielle un résultat central de la théorie  $L^2$  : le théorème d'extension d'Ohsawa-Takegoshi (dans sa version "singulière").

Il est naturel de penser que la preuve de la conjecture précédente fera également intervenir un théorème d'extension de type Ohsawa-Takegoshi, cette fois-ci dans le cas d'une famille de variétés kählériennes compactes. C'est le sujet que nous allons étudier dans le premier chapitre de cette thèse.

Étant donnée une variété kählérienne  $\mathcal{X}$ , un fibré en droites hermitien  $(L, h_L)$  sur  $\mathcal{X}$  et une sous-variété  $\mathcal{X}_0 \subset \mathcal{X}$ , la question qui nous intéresse est de caractériser les sections du fibré  $K_{\mathcal{X}_0} + L|_{\mathcal{X}_0}$  qui admettent un prolongement à  $\mathcal{X}$ . Par exemple, si le fibré  $L$  admet une métrique non-singulière  $h_L$ , dont la forme de courbure vérifie certaines conditions de positivité naturelles, alors le morphisme de restriction :

$$H^0(\mathcal{X}, K_{\mathcal{X}} + L) \rightarrow H^0(\mathcal{X}_0, K_{\mathcal{X}_0} + L)$$

est surjectif.

Si  $h_L$  est autorisée à avoir des singularités (on suppose que les conditions de courbure sont vérifiées au sens de courants), alors il est raisonnable de prédire que les sections de  $K_{\mathcal{X}_0} + L$  intégrables par rapport à  $h_L$  vont se prolonger à  $\mathcal{X}$  (compte tenu du cas projectif). Les résultats que nous obtenons vont dans cette direction. Suite aux difficultés techniques que

nous avons rencontrés (qui proviennent de la régularisation des fonctions quasi-psh sur les variétés kählériennes compactes), nous obtenons seulement deux cas particuliers du résultat espéré. Plus précisément, on est amené à considérer les hypothèses suivantes concernant la métrique singulière  $h_L$  :

- (I) La métrique  $h_L$  a des pôles logarithmiques ;
- (II) Le faisceau d'idéaux multiplicateurs associé à la métrique  $h_{L|\mathcal{X}_0}$  est trivial.

On remarque que la seconde hypothèse considérée ici apparaît naturellement en géométrie algébrique. Quant à la première hypothèse, elle est également naturelle, compte tenu d'un résultat attendu dans le cadre de la désingularisation des fonctions psh : modulo une modification de la variété, toute fonction psh se décompose en une partie avec des pôles logarithmiques, et une partie dont les singularités sont "petites". Dans la première partie de cette thèse, nous montrons que le résultat d'extension de type Ohsawa-Takegoshi est vérifié si la métrique de  $L$  vérifie une des conditions ci-dessus.

Les résultats que nous présentons dans la seconde partie de notre travail gravitent également autour de la conjecture de Siu : on s'intéresse aux prolongements des courants positifs fermés

$$\Theta_0 \in c_1(K_{\mathcal{X}}) + \{\alpha\}|_{\mathcal{X}_0}.$$

Notre théorème principal est un cas particulier de la conjecture qui prédit que tout courant positif fermé  $\Theta_0 \in c_1(K_{\mathcal{X}}) + \{\alpha\}|_{\mathcal{X}_0}$  admet un "prolongement"

$$\Theta \in c_1(K_{\mathcal{X}}) + \{\alpha\}$$

dont la restriction à  $\mathcal{X}_0$  est bien définie et moins singulière que  $\Theta_0$ , et qui vérifie l'estimée

$$\int_{\mathcal{X}} e^{\varphi_{\Theta} - \varphi_{\alpha}} dV_{\omega} \leq C \int_{\mathcal{X}_0} e^{\varphi_{\Theta_0} - \varphi_{\alpha}} dV_{\omega_0}. \quad (1)$$

où  $C$  est une constante numérique. Dans l'énoncé précédent on désigne par  $\alpha \geq 0$  une forme de type  $(1, 1)$  sur la famille kählérienne  $\mathcal{X}$  et  $\varphi_{\alpha}$  est un potentiel local de  $\alpha$ .

Nous avons vérifié cet énoncé pour une famille  $\mathcal{X}$  projective, et dans le cas où  $\alpha$  est la forme de courbure d'un  $\mathbb{Q}$ -fibré pseudo-effectif, qui vérifie certaines conditions (modèles d'après la condition "klt" en géométrie algébrique). Il serait intéressant de traiter le cas d'une classe transcendante  $\{\alpha\}$ , même si on maintient l'hypothèse sur la projectivité de la famille  $\mathcal{X}$  ; malheureusement, les techniques dont on dispose actuellement ne nous permettent pas de conclure.

Dans un autre registre on obtient une généralisation d'un théorème de Bando-Mabuchi concernant les métriques de Kähler-Einstein sur les variétés de Fano.

Etant donnée une variété  $X$  telle que  $c_1(X)$  a un signe défini (i.e. positif, négatif ou zéro), on peut se poser la question de l'existence et l'unicité d'une métrique de Kähler-Einstein. Autrement dit, on cherche une métrique kählérienne  $\omega$  sur  $X$ , telle que

$$\text{Ricci } \omega = \lambda \omega,$$

où  $\lambda \in \{-1, 0, 1\}$ . Dans le cas où  $\lambda = -1$  ( $K_X$  est ample en géométrie algébrique), l'existence et l'unicité d'une telle métrique a été établi par Aubin et Yau (cf. [Aub76], [Aub78], [Yau77], [Yau78]). Si  $\lambda = 0$ , alors on sait que pour toute classe kählérienne  $\{\alpha\}$ , on peut

trouver une métrique  $\omega \in \{\alpha\}$ , telle que  $\text{Ricci } \omega = 0$ ; cette métrique est également unique (cf. [Yau78]). Pour ce qui est de l'existence d'une métrique de Kähler-Einstein dans le cas  $\lambda = 1$  (i.e.  $c_1(X) > 0$ ), la situation est beaucoup moins bien comprise-on sait qu'il existe des obstructions à l'existence des métriques de Kähler-Einstein, mais on ne dispose pas encore d'un critère algébrique nécessaire et suffisant. En revanche, on sait que deux métriques de Kähler-Einstein sont identiques, modulo un automorphisme de  $X$ , grâce aux travaux de Bando-Mabuchi en 1987 (cf. [BM87]).

Récemment, une nouvelle preuve de ce résultat a été trouvée (cf. [Ber10], [Ber11]). Cela a permis d'obtenir un résultat un peu plus général traitant de l'unicité de métrique  $\omega$  dont les potentiels sont bornés, solutions de l'équation

$$\text{Ricci } \omega = \omega + [D].$$

Ici  $(X, D)$  est une paire klt- $D$  est effectif et  $\frac{d\lambda}{\prod |f_j|^{2\nu_j}} \in L^1_{\text{loc}}$  (on désigne ici  $D = \sum \nu_j Y_j$ , et  $f_j = 0$  sont les équations locales des  $Y_j$ ). Dans le troisième chapitre, nous avons généralisé ce type de résultats en travaillant avec un courant positif fermé  $T$  à la place de  $D$ . On s'intéresse donc à l'équation

$$\text{Ricci } \omega = \omega + T;$$

comme dans le cas d'un diviseur, on doit supposer que les singularités algébriques de  $T$  ne soient pas trop grandes.

Bien que ces deux thèmes soient relativement indépendants, leur outil commun est la technique de régularisation des courants positifs fermés. La théorie  $L^2$  ainsi que les propriétés de convexité des métriques de Bergman jouent des rôles importants dans notre travail.

## Structure de la thèse

Comme nous l'avons indiqué dans le résumé, cette thèse est composée de trois chapitres. Au début de chaque chapitre, nous essayons de présenter quelques aspects de l'origine du problème que nous allons traiter. Ensuite, nous présentons les outils techniques dont nous avons besoin, puis la preuve détaillée des théorèmes. Pour finir, nous faisons quelques commentaires sur les difficultés que nous avons rencontrés et sur les problèmes futures que nous avons en vue. Dans la suite, nous allons faire une présentation plus détaillée des résultats obtenus.

### Chapitre 1

Ce chapitre commence par quelques rappels au sujet du théorème d'Ohsawa-Takegoshi. Ce théorème joue un rôle important dans l'évaluation au bord du noyau de Bergman. C'était la motivation initiale pour les recherches dans cette direction. Rappelons que le noyau de Bergman est un noyau reproduisant pour l'espace de Hilbert des fonctions holomorphes de carré sommable dans un domaine de  $\mathbb{C}^n$ . Plus précisément, soient  $\mathcal{D}$  un domaine dans  $\mathbb{C}^n$  et  $\mathbb{A}^2(\mathcal{D})$  l'espace de Hilbert des fonctions holomorphes de carré sommable dans  $\mathcal{D}$ . Pour une fonction holomorphe  $f \in \mathbb{A}^2(\mathcal{D})$ , l'évaluation

$$\text{ev}_z f \rightarrow f(z)$$

est une forme linéaire continue sur  $\mathbb{A}^2(\mathcal{D})$ . Par le théorème de représentation de Riesz, cette forme peut être représentée comme un produit scalaire de la fonction holomorphe  $f$  avec un élément de  $\mathbb{A}^2(\mathcal{D})$ , autrement dit,

$$\text{ev}_z f = \int_{\mathcal{D}} f(\xi) \overline{\eta_z(\xi)} d\mu(\xi).$$

Le noyau de Bergman  $K$  est défini par

$$K(z, \xi) = \overline{\eta_z(\xi)}.$$

Il est holomorphe en  $z$  et antiholomorphe en  $\xi$ . De plus, on a

$$f(z) = \int_{\mathcal{D}} K(z, \xi) f(\xi) d\mu(\xi).$$

On dit qu'une fonction  $\psi$  définie sur un ensemble ouvert  $\Omega \subset \mathbb{C}^n$  est *plurisousharmonique* (psh en abrégé) si elle est semi-continue sur  $\Omega$  et sous-harmonique en restriction aux droites complexes  $L \subset \mathbb{C}^n$ ; autrement dit, si elle satisfait

$$\psi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(a + e^{i\theta} \xi) d\theta$$

pour tout  $a \in \Omega$ ,  $\xi \in \mathbb{C}^n$  et  $|\xi| < d(a, \mathbb{C}\Omega)$ . Un domaine  $\Omega$  est dit *pseudoconvexe* s'il existe une fonction psh continue  $\varphi(z)$  en  $\Omega$  telle que l'ensemble

$$\{z \in \Omega \mid \varphi(z) < c\}$$

est relativement compact dans  $\Omega$  pour tous  $c \in \mathbb{R}$ . Dans ce cas, on appelle la fonction  $\varphi(z)$  une *exhaustion*.

En 1987, Ohsawa et Takegoshi ont démontré un théorème d'extension  $L^2$  pour des fonctions holomorphes avec contrôle de leur croissance.

**Théorème 1.** ([OT87]). Soient  $\Omega$  un domaine pseudoconvexe borné dans  $\mathbb{C}^n$ ,  $\psi : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  une fonction psh et  $H \subset \mathbb{C}^n$  un hyperplan complexe. Alors il existe une constante  $C$  qui ne dépend que du diamètre de  $\Omega$  telle que : pour toute fonction holomorphe  $f$  sur  $\Omega \cap H$  satisfaisant

$$\int_{\Omega \cap H} |f|^2 e^{-\psi} dV_{n-1} < +\infty$$

où  $dV_{n-1}$  est la mesure de Lebesgue sur  $\mathbb{R}^{2n-2}$ , il existe une fonction holomorphe  $F$  sur  $\Omega$  telle que  $F|_{\Omega \cap H} = f$  et que

$$\int_{\Omega} |F|^2 e^{-\psi} dV_n \leq C \int_{\Omega \cap H} |f|^2 e^{-\psi} dV_{n-1}.$$

Une conséquence presque immédiate de cet énoncé est l'estimée suivante :

**Théorème 2.** Soit  $\Omega \subset \mathbb{C}^n$  un domaine pseudoconvexe borné; alors

$$K(z) \geq \frac{1}{C \cdot \delta_{\Omega}^2},$$

où  $\delta_{\Omega}$  est la fonction de distance de  $z$  au  $\partial\Omega$  et  $C > 0$  est une constante. En particulier,

$$\lim_{z \rightarrow \partial\Omega} K(z) = +\infty.$$

Si on veut transposer le résultat précédent (Théorème 1) dans le contexte des variétés, on fait les remarques suivantes. Le facteur  $e^{-\psi}$  sous le signe de la somme peut être vu comme le poids de la métrique d'un fibré  $L$ , et la fonction  $f$  correspond à une  $(n, 0)$ -forme à valeurs dans  $L$ .

L'interprétation géométrique suivante a été trouvée par L. Manivel dans [Man93] :

**Théorème 3.** ([Man93]). Soient  $X$  une variété complexe faiblement pseudoconvexe de dimension  $n$  munie d'une métrique kählérienne  $\omega$ ,  $(L, h_L)$  (resp  $E$ ) un fibré en droites hermitien holomorphe muni d'une métrique lisse  $h_L$  (resp un fibré vectoriel hermitien holomorphe de rang  $r$  sur  $X$ ), et  $s$  une section holomorphe globale de  $E$ . Supposons que  $s$  soit généralement transverse à la section zéro. On définit

$$Y = \{x \in X; s(x) = 0, \wedge^r ds(x) \neq 0\}, \quad p = \dim Y = n - r.$$

De plus, on suppose que la forme  $\sqrt{-1}\Theta(L) + r\sqrt{-1}\partial\bar{\partial} \log |s|^2$  soit semi-positive et qu'il existe une fonction continue  $\alpha \geq 1$  telle que les deux inégalités suivantes soient vérifiées

- (a)  $\sqrt{-1}\Theta(L) + r\sqrt{-1}\partial\bar{\partial} \log |s|^2 \geq \frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)_{s,s}\}}{|s|^2}$ ;  
 (b)  $|s| \leq e^{-\alpha}$ .

Alors pour chaque section holomorphe  $f$  de  $\wedge^n T_X^* \otimes L$  sur  $Y$  satisfaisant

$$\int_Y \frac{|f|^2}{|\wedge^r(ds)|^2} dV_\omega < +\infty,$$

il existe une section holomorphe  $F$  sur  $X$  telles que  $F|_Y = f$  et que

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log |s|)^2} dV_{X,\omega} \leq C_r \int_Y \frac{|f|^2}{|\wedge^r(ds)|^2} dV_{Y,\omega},$$

où  $C_r$  est une constante numérique qui ne dépend que de  $r$ .

Par définition, une variété est *faiblement pseudoconvexe* si elle possède une exhaustion psh de classe  $C^\infty$ . Dans cette classe très générale de variétés, on trouve deux exemples de natures assez différentes : les variétés de Stein et les variétés complexes compactes. On remarque que dans l'énoncé précédent, la métrique de  $L$  (i.e. l'analogue de la fonction  $e^{-\psi}$  dans le théorème d'Ohsawa-Takegoshi) est supposée non-singulière. Les applications suivantes montrent d'une part l'importance du théorème 1, et en même temps, elles indiquent qu'il serait souhaitable d'avoir une version du théorème de Manivel pour les métriques singulières.

**Théorème 4.** ([Dem94]). Soit  $\varphi$  une fonction psh sur un domaine pseudoconvexe borné  $\Omega \subset \mathbb{C}^n$ . Pour chaque  $m > 0$ , soient  $\mathcal{H}_\Omega(m\varphi)$  l'espace de Hilbert des fonctions holomorphes  $f$  sur  $\Omega$  telles que  $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$  et  $\varphi_m = \frac{1}{2m} \log \sum |\sigma_l|^2$  où  $(\sigma_l)$  est une base orthogonale de  $\mathcal{H}_\Omega(m\varphi)$ . Alors il existe des constantes  $C_1, C_2 > 0$ , indépendantes de  $m$  telles que

(a)

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\xi-z|<r} \varphi(\xi) + \frac{1}{m} \log \frac{C_2}{r^n}$$

pour chaque  $z \in \Omega$  et  $r < d(z, \partial\Omega)$ . En particulier,  $\varphi_m$  converge vers  $\varphi$  ponctuellement dans la topologie de  $L_{loc}^1$  sur  $\Omega$  lors que  $m \rightarrow +\infty$  et

(b)

$$\nu(\varphi, z) - \frac{n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z)$$

pour chaque  $z \in \Omega$ .

Nous finissons la première section par mentionner une autre application du théorème 3, c'est un théorème de type Briançon-Skoda pour les faisceaux d'idéaux multiplicateurs de Nadel :

**Théorème 5.** ([Dem00]). Soient  $X$  une variété complexe de dimension  $n$ , et  $\varphi, \psi$  des fonctions psh sur  $X$ . Alors pour chaque entier  $k \geq n$  on a

$$\mathcal{I}(k\varphi + \psi) \subset \mathcal{I}(\varphi)^{k-n} \mathcal{I}(\psi).$$

Dans la deuxième section, nous rappelons les définitions de quelques notions telles que les métriques singulières d'un fibré en droites, les notions de positivité associées et aussi les faisceaux d'idéaux multiplicateurs.

Soient  $X$  une variété complexe munie d'une métrique hermitienne  $\omega$  et  $L$  un fibré en droites holomorphe sur  $X$ . Soient  $U_i$  un ensemble ouvert dans  $X$  sur lequel  $L$  est trivial, et  $s$  une section de  $L$  représentée localement par  $f_i$  sur  $U_i$ . Une *métrique singulière*  $h_L$  sur  $L$  est définie par une collection de fonctions  $\phi_i$  localement intégrables telles que la norme de  $s$  satisfait

$$\|s\|^2 = |f_i|^2 e^{-\phi_i}.$$

$\phi_i$  est le *poids local* associé à la trivialisaton  $U_i$ . Si on note  $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$  la fonction de transition, on a  $f_i = g_{ij} f_j$  sur l'intersection  $U_i \cap U_j \neq \emptyset$ . Les représentantes locales de la métrique singulière  $h_L$  satisfont la condition de compatibilité

$$\phi_i - \phi_j = \log |g_{ij}|^2.$$

Grâce à l'intégrabilité locale de poids  $\phi_j$ , nous pouvons définir un courant de courbure correspondant à la métrique  $h_L$ . En effet, nous pouvons calculer

$$\sqrt{-1} \partial \bar{\partial} \phi_i \quad \text{et} \quad \sqrt{-1} \partial \bar{\partial} \phi_j$$

au sens des distributions : ce sont des courants de type  $(1, 1)$  sur  $U_i$  et  $U_j$  respectivement. Puisque

$$\log |g_{ij}|^2$$

est pluriharmonique,  $\sqrt{-1} \partial \bar{\partial} \phi_i$  est égale à  $\sqrt{-1} \partial \bar{\partial} \phi_j$  sur l'intersection  $U_i \cap U_j$ .  $\sqrt{-1} \partial \bar{\partial} \phi_i$  est ainsi globalement défini. Nous désignons la *courbure*  $\sqrt{-1} \Theta_{h_L}(L)$  associée à la métrique  $h_L$  par

$$\sqrt{-1} \Theta_{h_L}(L) = \sqrt{-1} \partial \bar{\partial} \phi.$$

Une fois la définition de courant de courbure donnée, nous rappelons les notions de positivité associée à un fibré en droites. Avant cela, rappelons qu'un courant  $T$  de type  $(1, 1)$  s'écrit localement comme suit :

$$\sqrt{-1} \sum_{j,k} T_{j,k} dz_j \wedge \bar{d}z_k$$

avec des coefficients distributions.  $T$  est dit *positif* si et seulement si pour tout  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ ,

$$\sum_{1 \leq j, k \leq n} \lambda_j \bar{\lambda}_k T_{j,k}$$

est une mesure positive.

Nous allons rappeler par la suite quelques notions de positivité et leurs contreparties en géométrie algébrique. Ce dictionnaire a été établi par J-P. Demailly dans [Dem92].

En substance, on veut établir une connection entre les propriétés de positivité des métriques d'un fibré en droites et l'existence des sections holomorphes de ses multiples.

Par exemple, si  $u_1, \dots, u_N \in H^0(X, mL)$  sont des sections holomorphes de  $mL$ , alors la métrique définie localement en  $U_i \subset X$  par

$$\phi_i := \frac{1}{m} \log \left( \sum_{j_i=1}^N |f_{j_i}|^2 \right)$$

sera singulière, ici  $f_{j_i}$  est la fonction holomorphe qui correspond à  $u_j$  en  $U_i$ , mais le courant de courbure associé est positif. On rappelle maintenant quelques résultats dans cette direction.

Un fibré en droites  $L \rightarrow X$  est dit *positif* s'il admet une métrique non-singulière  $h_L$  telle que

$$\Theta_{h_L}(L) \geq \varepsilon_0 \omega,$$

où  $\varepsilon_0 > 0$  et  $\omega$  est une métrique kählérienne sur  $X$ . Le résultat suivant est la base des beaucoup de développements en géométrie algébrique/analytique.

**Théorème 6.** ([Kodaira])  $L$  est positif si et seulement si  $L$  est *ample*. C'est-à-dire, l'application induite par les sections globales de  $mL$  est un plongement pour tout  $m \gg 0$ .

Si  $\sigma_0, \dots, \sigma_N \in H^0(X, mL)$  sont des sections holomorphes de  $mL$ , l'application induite par cette famille de sections est définie comme suit :

$$\begin{aligned} \phi : X &\dashrightarrow \mathbb{P}^N \\ x &\rightarrow [f_{0_i}(x) : \dots : f_{N_i}(x)] \end{aligned}$$

si  $x \in U_i$ , et  $f_{j_i}$  est une fonction holomorphe qui correspond à  $\sigma_{j_i}$ . Etant donné le fait que  $\sigma_i$  soient des sections globales, la fonction  $\phi$  est bien définie globalement. Une notion beaucoup plus flexible que l'amplitude est celle de fibré en droites numériquement effectif (nef en abrégé). De point de vue métrique,  $L$  est dit *nef* si  $\forall \varepsilon > 0$ , il existe une métrique non-singulière  $h_\varepsilon$  telle que

$$\Theta_{h_\varepsilon}(L) \geq -\varepsilon \omega.$$

La définition correspondante en géométrie algébrique est :  $L$  est nef si et seulement si  $L \cdot C \geq 0$  pour toute courbe  $C \subset X$ . L'équivalence entre ces deux notions a été établi dans [DPS93].

Si on autorise des métriques singulières, les notions précédentes changent comme suit. Un fibré  $L \rightarrow X$  est dit *gros* si  $h^0(X, mL) \geq Cm^n$ , où  $C$  est une constante positive, et  $n = \dim X$ . Via la théorie  $L^2$ , on dit que  $L$  est gros si et seulement s'il admet une métrique  $h_L$  (éventuellement singulière) dont le courant de courbure est un courant kählérien. C'est-à-dire,  $\Theta_{h_L}(L) \geq \varepsilon_0 \omega$ . Ainsi, on a une version "singulière" du théorème de Kodaira (cf. [Dem93]).

Finalement, un fibré  $L$  est *pseudo-effectif* si  $kL + A$  ( $A$  est un fibré ample) admet des  $\mathbb{Q}$ -sections (non-nulles),  $\forall k \geq 1$ . La contrepartie analytique de cette notion est la suivante :  $L$  est

pseudo-effectif si et seulement s'il admet une métrique dont le courant de courbure est positif.

Dans le contexte de notre thèse, une autre notion importante est celle de faisceaux d'idéaux multiplicateurs, introduit par Nadel [Nad90] pour l'étude des obstructions à l'existence d'une métrique de Kähler-Einstein sur une variété de Fano. Nous le rappelons maintenant.

Soit  $\varphi$  une fonction psh sur un ensemble ouvert  $\Omega \subset X$ ; Le faisceau d'idéaux multiplicateurs associé à  $\varphi$ ,  $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$  est défini par des germes de fonctions holomorphes  $f \in \mathcal{O}_{\Omega, x}$  telles que  $|f|^2 e^{-\varphi}$  est intégrable dans un voisinage local de  $X$ . La variété des zéros  $V(\mathcal{I}(\varphi))$  est l'ensemble des points dans le voisinage duquel  $e^{-\varphi}$  est non intégrable. De tels points n'apparaissent que lorsque  $\varphi$  a des pôles logarithmiques "de grande taille". Plus précisément, on définit le *nombre de Lelong* de  $\varphi$  en  $x$  par

$$\nu(\varphi, x) := \liminf_{z \rightarrow x} \frac{\varphi(z)}{\log |z - x|}.$$

Si  $\nu(\varphi, x) > 0$ , on dit que  $\varphi$  admet un pôle logarithmique d'ordre  $\gamma = \nu(\varphi, x)$  en  $x$ . Par exemple, le nombre de Lelong  $\nu([A], x)$  d'un courant d'intégration sur un ensemble analytique  $A \subset X$  est égale à la multiplicité de cet ensemble à  $x$  (cf. [Thi67]). Un résultat de Skoda [Sko72a] établit une relation entre l'intégrabilité de  $e^{-\varphi}$  et les nombres de Lelong  $\nu(\varphi, x)$ .

**Lemme 1.** *Soit  $\varphi$  une fonction psh sur un ensemble ouvert  $\Omega$ ,  $x \in \Omega$ .*

- (a) *Si  $\nu(\varphi, x) < 1$ , alors  $e^{-\varphi}$  est intégrable dans un voisinage de  $x$ , en particulier,  $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega, x}$ .*
- (b) *Si  $\nu(\varphi, x) \geq n + s$  pour certain  $s \geq 0$ , alors  $e^{-\varphi} \geq C|z - x|^{-n-s}$  dans un voisinage de  $x$  et  $\mathcal{I}(\varphi)_x = \mathcal{M}_{\Omega, x}^{s+1}$ , où  $\mathcal{M}_{\Omega, x}$  est l'idéal maximal de  $\mathcal{O}_{\Omega, x}$ .*
- (c) *La variété des zéros  $V(\mathcal{I}(\varphi))$  de  $\mathcal{I}(\varphi)$  satisfait*

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_1(\varphi)$$

où  $E_c(\varphi) = \{x \in X : \nu(\varphi, x) \geq c\}$  est l'ensemble de niveau supérieur de  $c$  de nombres de Lelong de  $\varphi$ .

Un résultat crucial affirme que l'ensemble au voisinage duquel  $e^{-\varphi}$  est non intégrable est un sous-ensemble analytique de la variété  $X$  : c'est une application du théorème d'Hörmander-Bombieri-Skoda (cf. [Bom70] [Sko72a] [Sko77]). Quant aux ensembles  $E_c(\varphi)$ , un résultat analogue a été établi par Siu [Siu74].

**Théorème 7.** *Si  $T$  est un courant positif fermé de type  $(p, p)$  sur une variété  $X$ , alors l'ensemble de niveau supérieur  $E_c(T) = \{x \in X : \nu(T, x) \geq c\}$  est analytique de dimension  $\leq p$ .*

Nadel [Nad90] a démontré que  $\mathcal{I}(\varphi)$  est un faisceau cohérent sur  $\Omega$ . Si de plus  $\Omega$  est un ensemble ouvert borné de Stein,  $\mathcal{I}(\varphi)$  est engendré par une base de l'espace de Hilbert  $L^2 \mathcal{H}^2(\Omega, \varphi)$  de fonctions holomorphes  $f$  sur  $\Omega$  telles que  $\int_\Omega |f|^2 e^{-\varphi} d\lambda < +\infty$ .

Revenons au contexte de fibré pseudo-effectif  $(L, h_L)$ , les poids locaux  $\varphi_j$  sont des fonctions psh qui définissent le faisceau cohérent  $\mathcal{I}(\varphi_j)$ . Compte tenu de la relation de transition,  $\mathcal{I}(\varphi_i)|_{U_i \cap U_j} = \mathcal{I}(\varphi_j)|_{U_i \cap U_j}$ . Il en résulte que  $\mathcal{I}(h_L)$  est un faisceau cohérent bien défini. Une propriété remarquable de faisceau  $\mathcal{I}(\varphi)$  est qu'il permet d'établir le théorème d'annulation suivant (également dû à Nadel, [Nad90]), qui est une généralisation extrêmement utile du théorème correspondant de Kodaira.



**Théorème 8.** Soient  $(X, \omega)$  une variété kählérienne faiblement pseudoconvexe,  $L$  un fibré en droites au-dessus de  $X$  muni d'une métrique hermitienne singulière  $h_L$ . Si  $\sqrt{-1}\Theta_{h_L} \geq \varepsilon\omega$  pour une certaine fonction positive continue  $\varepsilon$  sur  $X$ , alors

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(h_L)) = 0$$

pour tout  $q \geq 1$ .

Il est parfois utile de considérer la version suivante du faisceau d'idéaux multiplicateurs

$$\mathcal{I}_+(\varphi) = \lim_{\varepsilon \rightarrow 0_+} \mathcal{I}((1 + \varepsilon)\varphi),$$

qui a été introduit dans [DP03]. Conjecturalement, on a  $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ .

Dans le troisième paragraphe, nous énonçons nos résultats principaux. Nous allons démontrer deux théorèmes d'extension qui généralisent le théorème de Manivel sous deux hypothèses différentes.

Soient  $X$  une variété compacte kählérienne et  $Z \subset X$  l'ensemble des zéros d'une section holomorphe  $s \in H^0(X, E)$  d'un fibré en droites  $(E, h_E)$ . L'hypersurface  $Z$  est non singulière. Soit  $L$  un fibré en droites holomorphe, muni d'une métrique éventuellement singulière  $h_L$  telle que :

- (1)  $\sqrt{-1}\Theta_{h_L}(L) \geq 0$  en tant qu'un courant sur  $X$ ;
- (2)  $\sqrt{-1}\Theta_{h_L}(L) \geq \frac{1}{\alpha}\sqrt{-1}\Theta(E)$  pour un certain  $\alpha \geq 1$ ,  $\alpha \in C^\infty(X)$ ;
- (3)  $|s|_{h_E}^2 \leq \exp(-\alpha)$  sur  $X$ , et la restriction de la métrique  $h_L$  en  $Z$  est bien définie.

Les types de singularités de la métrique  $h_L$  qu'on peut traiter actuellement sont les suivants :

- (I) La métrique  $h_L$  a des pôles logarithmiques et non identiquement égale à  $-\infty$  sur  $Z$ ;
- (II) La restriction du faisceau d'idéaux multiplicateurs associé à la métrique  $h_L|_Z$  est égale au faisceau structural  $\mathcal{O}_Z$ .

Les résultats que nous obtenons sont les suivants :

**Théorème 9.** Soient  $X$  une variété compacte kählérienne et  $(L, h_L)$  un fibré en droites holomorphe tel que les propriétés (1) – (3) soient satisfaites. De plus, supposons que la métrique  $h_L$  satisfasse hypothèse (I). Etant donné une section

$$u \in H^0(Z, (K_Z + L|_Z) \otimes \mathcal{I}(\varphi_L|_Z)),$$

il existe une section holomorphe  $U$  de  $K_X + Z + L$  telle que  $U|_Z = u \wedge ds$  et que

$$\int_X \frac{|U|^2 e^{-\varphi_L - \varphi_Z}}{|s|^2 \log^2(|s|^2)} \leq C_0 \int_Z |u|^2 e^{-\varphi_L}$$

où  $C_0$  est une constant numérique.

**Théorème 10.** Soient  $X$  une variété compacte kählérienne et  $(L, h_L)$  un fibré en droites holomorphe tel que les propriétés (1) – (3) soient satisfaites. De plus, supposons que la métrique  $h_L$  satisfasse hypothèse (II). C'est-à-dire,

$$\int_Z e^{-\varphi_L} d\lambda_Z < \infty$$

avec  $d\lambda_Z$  la mesure de Lebesgue de  $Z$ . Etant donné une section

$$u \in H^0(Z, (K_Z + L|_Z)),$$

il existe une section holomorphe  $U$  de  $K_X + Z + L$  telle que  $U|_Z = u \wedge ds$  et que

$$\int_X \frac{|U|^2 e^{-\varphi_L - \varphi_E}}{|s|^2 \log^2(|s|^2)} \leq C_X \sup_X |u|^2 \int_Z e^{-\varphi_L} d\lambda_Z$$

Ici  $C_X$  est une constante qui dépend de  $X$ .

Avant d'entrer dans les détails de notre démonstration, nous discutons la méthode employée dans le cas projectif afin d'illustrer les difficultés que nous avons rencontré pour le cas générale. En effet, si  $X$  est projective, il existe un hyperplan  $H$  qui ne contient pas  $Z$ , et tel que

$$X \setminus H = \bigcup_k \Omega_k$$

avec  $(\Omega_k)$  une suite croissante de sous-domaines de Stein de  $X$ . La restriction de  $h_L$  à chaque  $\Omega_k$  peut s'écrire comme la limite décroissante des métriques non-singulière  $h_{L,\rho}$ , telle que les hypothèses sur les courbures soient préservées. C'est ici que l'on utilise la propriété de Stein de  $\Omega_k$  : on peut supposer que  $h_L|_{\Omega_k}$  est non-singulière, à condition que les estimées pour la norme d'extension que l'on obtient restent indépendantes de  $k$ . En conclusion, on peut appliquer sur chaque  $\Omega_k$  le théorème d'Ohsawa-Takegoshi et passer ensuite à la limite. On remarque qu'il y a 3 paramètres qui interviennent dans la démonstration :  $k$ ,  $\varepsilon$  (pour concentrer la masse sur  $Z \cap \Omega_k$  (voir [Ber96], [Siu02]), et  $\rho$  : ils sont complètement indépendants. Par conséquent, on peut passer à la limite dans l'ordre qui est le plus convenable.

En revanche, si  $X$  est seulement kählérienne compacte, nous n'avons plus la suite  $(\Omega_k)$  à notre disposition. Cependant, nous pouvons toujours régulariser la métrique  $h_L$  et obtenir une suite  $h_{L,\rho}$  en utilisant la technique de régularisation des courants positifs fermés [Dem94]. La conséquence est que les hypothèses sur la positivité de la courbure (1) et (2) ne sont vérifiées qu'à un petit facteur négatif  $-\delta_\rho \omega$  près, où  $\delta_\rho \rightarrow 0$  lorsque  $\rho \rightarrow 0$ . C'est ce petit facteur négatif qui nous cause des problèmes : nous ne pouvons plus supposer que  $\varepsilon$  et  $\rho$  se varient indépendamment, notamment à cause des estimées que l'on obtiennent à l'aide d'une application de l'identité de Bochner. En fait, au cours de la preuve, la forme de courbure est multipliée par la fonction  $\log \frac{1}{\varepsilon^2 + |s|^2}$ ; la perte de positivité est, par conséquent, de l'ordre  $\delta_\rho \log \frac{1}{\varepsilon^2 + |s|^2}$ . Les hypothèses (I) et (II) sont ainsi suffisantes pour assurer que quelques procédures de limite soient justifiés.

Nous allons présenter par la suite les étapes principales de notre démonstration :

- Dans un premier temps, on fait la démarche de l'approximation des fonctions quasi-psh par une suite des fonctions presque partout lisses sur  $X$  en utilisant un théorème de Demailly ([Dem94]) :

**Théorème 11.** *Soit  $T := \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$  un courant fermé de type  $(1,1)$  sur une variété complexe compacte  $X$  ; ici  $\alpha$  est une forme de type  $(1,1)$  et  $\varphi$  est quasi-psh. Soit  $\gamma$  une forme continue de type  $(1,1)$ , telle que  $T \geq \gamma$ . On suppose que le fibré tangent  $T_X$  est muni d'une métrique hermitienne lisse  $\omega$ . Alors pour chaque  $\rho > 0$ , il existe une fonction  $\varphi_\rho \in L^1(X)$  telle que :*

- (i) *La fonction  $\varphi_\rho$  est lisse sur  $X \setminus E_\rho(T)$ , où  $E_\rho(T)$  est l'ensemble de niveau supérieur de  $\rho$  de nombres de Lelong de  $T$ .*

- (ii) On a  $T_\rho := \alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\rho \geq \gamma - \delta_\rho\omega$ , où  $\delta_\rho \rightarrow 0$  lorsque  $\rho \rightarrow 0$ .  
 (iii) On a  $\varphi \leq \varphi_\rho$ , pour chaque  $\rho > 0$ .

Un fait important de ce théorème est que la construction de  $\varphi_\rho$  est indépendante de  $\gamma$ . On obtient ainsi une suite de métriques  $h_{L,\rho} = e^{-\varphi_{L,\rho}}$  sur  $L$  en utilisant un noyaux régularisé suivie de la transformée de Legendre.

**Lemme 2.** Pour tout  $\rho > 0$ , il existe une métrique  $h_{L,\rho} = e^{-\varphi_{L,\rho}}$  sur  $L$  telle que

$$\sqrt{-1}\Theta_{h_{L,\rho}}(L) \geq -\delta_\rho\omega, \quad \sqrt{-1}\Theta_{h_{L,\rho}}(L) \geq \frac{1}{\alpha}\sqrt{-1}\Theta(E) - \delta_\rho\omega.$$

De plus, pour chaque  $\rho > 0$ , il existe un ensemble analytique  $Y_\rho \subset X$  tel que  $\varphi_{L,\rho}$  est lisse sur  $X \setminus Y_\rho$ .

Cette étape est nécessaire seulement pour la preuve du théorème 10. Dans le cas où la métrique  $h_L$  a des pôles logarithmiques, on peut travailler directement sur la variété

$$M : X \setminus \{h_l = \infty\}.$$

C'est une variété complète et on peut ainsi utiliser la théorie  $L^2$  standard (cf. [Dem82b]).

- On construit ensuite une extension  $U_\infty$  de  $u$  de classe  $\mathcal{C}^\infty$ , et on étudie les propriétés  $L^2$  de  $\bar{\partial}U_\infty$ . On voudrait que l'intégrale suivante

$$I_\varepsilon := \int_{\{|s|^2 < \varepsilon^2\}} \frac{|\bar{\partial}U_\infty|_\omega^2 e^{-\varphi_L}}{|s|^2} dV_\omega$$

soit bornée par une petite constante dans les deux cas (I) et (II). La construction de  $U_\infty$  diffère selon les deux hypothèses concernant les singularités de  $h_L$ . En effet, dans le cas (I), on a besoin d'une extension spécifique afin d'avoir une estimée de  $I_\varepsilon$ . Pour cela, on utilise la version locale du théorème du premier cas et obtient  $U_\infty$  par la partition de unité. Plus précisément,  $U_\infty$  est construite par

$$U_\infty := \sum_\alpha \theta_\alpha \tilde{f}_\alpha$$

où  $\tilde{f}_\alpha$  satisfait  $\tilde{f}_\alpha|_{Z \cap U_\alpha} = f_\alpha$ , et

$$\int_{U_\alpha} \frac{|\tilde{f}_\alpha|^2 e^{-(1+\varepsilon_0)\varphi_L}}{|s|^2 \log^2 \frac{1}{|s|^2}} d\lambda \leq C_0 \int_{Z \cap U_\alpha} |f_\alpha \otimes e_\alpha|^2 e^{-(1+\varepsilon_0)\varphi_L} d\lambda$$

avec  $C_0$  une constante numérique. On obtient dans ce cas le lemme suivant :

**Lemme 3.** Il existe une extension  $\mathcal{C}^\infty$   $U_\infty$  de  $u$  telle que

$$\int_{\{|s|^2 < \varepsilon^2\}} \frac{|\bar{\partial}U_\infty|_\omega^2 e^{-\varphi_L}}{|s|^2} dV_\omega \leq C \left( \varepsilon^2 \log \frac{2}{\varepsilon^2} \frac{1}{\varepsilon^2} \right)^{\frac{\varepsilon_0}{1+\varepsilon_0}}.$$

La construction de  $U_\infty$  dans le cas (II) est arbitraire localement. On définit

$$U_\infty := \sum_\alpha \theta_\alpha \tilde{f}_\alpha$$

avec  $\tilde{f}_\alpha = (\rho_\alpha^* f_\alpha) \otimes e_\alpha$  où  $\rho_\alpha : U_\alpha \rightarrow Z \cap U_\alpha$  est une retraction holomorphe. Pour démontrer que la quantité  $I_\varepsilon$  soit petite dans ce cas, nous avons besoin du lemme suivant.

Désignons  $\Omega := \mathbb{D} \times \Delta \subset \mathbb{C}^n$  un domaine qui est le produit du disque unité  $\mathbb{D}$  dans  $\mathbb{C}$  et du disque unité  $\Delta$  dans  $\mathbb{C}^{n-1}$ . Considérons  $\varphi \in \text{Psh}(\Omega)$  telle que  $\sup_\Omega \varphi = 0$ .

**Lemme 4.** *Supposons que l'intégrale suivante est convergente :*

$$\int_{\Delta} e^{-\varphi(0,z)} d\lambda(z) < \infty.$$

Alors pour tout  $\delta > 0$ , on a :

$$\int_{\Omega'_\delta} e^{-\varphi(t,z)} \frac{d\lambda(t,z)}{\delta^2} \leq C_1 \left( \int_{\Delta} e^{-\varphi(0,z)} d\lambda(z) \right)^4$$

où  $\Omega'_\delta := (|t| < \delta) \times \Delta$  s'intersecte avec le polydisque de rayon  $\frac{1}{2}$  et  $C_1$  est une constante indépendante de  $\varphi$ .

A l'aide de ce lemme, on obtient dans ce cas le résultat suivant sur  $U_\infty$  :

**Lemme 5.** *L'extension  $U_\infty$  de  $u$  satisfait*

$$\int_{\{|s|^2 < \varepsilon^2\}} \frac{|\bar{\partial}U_\infty|_\omega^2 e^{-\varphi_L}}{|s|^2} dV_\omega \leq \varepsilon^2 C_X \int_Z e^{-\varphi_L} d\lambda_Z$$

où  $C_X$  dépend de  $X$  mais est indépendante de  $\varepsilon$ .

- La troisième étape consiste à rappeler la formule de Bochner. Commençons par l'identité de Bochner-Kodaira-Nakano classique. Etant donné une variété kählérienne faiblement pseudoconvexe et un fibré en droites holomorphe  $(L, h_L)$ , pour chaque forme de type  $(n, 1)$  à support compact  $u \in \mathcal{D}(X, \Lambda^{n,1} T_X^* \otimes L)$ , on a

$$\int_X |\bar{\partial}u|_h^2 dV_\omega + \int_X |\bar{\partial}^* u|_h^2 dV_\omega = \int_X |\partial_h^* u|_h^2 dV_\omega + \int_X \langle [\sqrt{-1}\Theta_h(L), \Lambda_\omega] u, u \rangle_h dV_\omega.$$

Observons que si on change de métrique, l'égalité précédente implique l'inégalité suivante :

**Lemme 6.** *Soient  $\eta, \lambda$  des fonctions lisses sur  $X$ , et  $u \in \mathcal{D}(X, \Lambda^{n,1} T_X^* \otimes L)$ . Alors on a l'inégalité suivante :*

$$\begin{aligned} \int_X |\bar{\partial}u|_h^2 \eta dV_\omega + \int_X (\eta + \lambda) |\bar{\partial}^* u|_h^2 dV_\omega &\geq \\ &\geq \int_X \left\langle \left[ \eta \sqrt{-1} \Theta_h(L) - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} \frac{\partial \eta \wedge \bar{\partial} \eta}{\lambda}, \Lambda_\omega \right] u, u \right\rangle_h dV_\omega \end{aligned}$$

- La quatrième étape est d'utiliser les arguments d'analyse fonctionnelle standard afin de résoudre une équation  $\bar{\partial}$ . Plus précisément, on définit d'abord deux opérateurs de type  $\bar{\partial}$  suivants :

$$Tu := \bar{\partial} \left( \sqrt{\eta + \lambda} u \right) \text{ et } Su := \sqrt{\eta} (\bar{\partial} u)$$

avec  $\eta, \lambda$  bien choisies. Le lemme précédent implique l'inégalité

$$\|T^*u\|^2 + \|Su\|^2 + \delta \|u\|^2 \geq \|\sqrt{\tau} (\bar{\partial} \mu)^* u\|^2$$

où  $\delta \geq 0$  est un nombre réel,  $\tau$  une fonction positive et  $\mu$  une fonction arbitraire.

L'étape suivante est de résoudre l'équation

$$Tv = g$$

où  $g$  est une forme de type  $(n, 1)$  fermée à valeurs dans  $F := Z + L$  telle que  $Sg = 0$ . En particulier, on prend  $g = \bar{\partial} \tau \wedge g_0 + g_1$  avec  $g_0$  une forme de type  $(n, 0)$  et  $g_1$  une forme de

type  $(n, 1)$ .

Ensuite on fait les calculs selon les deux hypothèses concernant  $h_L$ . Dans le premier cas, on a  $\delta = 0$  et

$$\frac{1}{2} \left| \int_X \langle g, u \rangle_h dV_\omega \right|^2 \leq C(g_0, \tau) \left( \|T^*u\|^2 + \frac{\|g_1\|^2}{C(g_0, \tau)} \|u\|^2 \right)$$

avec

$$C(g_0, \tau) = \int_X 1/\tau |g_0|^2 e^{-\varphi_F} dV_\omega.$$

Désignons

$$C(g_0, g_1, \tau) = \frac{\|g_1\|}{\sqrt{C(g_0, \tau)}},$$

alors l'application

$$(T^*u, C(g_0, g_1, \tau)u) \rightarrow \langle g, u \rangle$$

est continue, et on a

$$Tv + C(g_0, g_1, \tau)w = \bar{\partial}\tau \wedge g_0 + g_1$$

avec l'estimée

$$\int_X |v|^2 e^{-\varphi_F} + \int_X |w|_\omega^2 e^{-\varphi_F} dV_\omega \leq C(g_0, \tau).$$

De façon similaire, dans le deuxième cas, on a  $\delta > 0$ , alors on obtient

$$\frac{1}{2} \left| \int_X \langle g, u \rangle_h dV_\omega \right|^2 \leq C(g, \tau, \delta) (\|T^*u\|^2 + \delta \|u\|^2)$$

avec

$$C(g, \tau, \delta) := \int_X 1/\tau |g_0|^2 e^{-\varphi_F} + \frac{1}{\delta} \int_X |g_1|_\omega^2 e^{-\varphi_F} dV_\omega.$$

En conclusion, l'application

$$(T^*u, \sqrt{\delta}u) \rightarrow \langle g, u \rangle$$

est continue. En appliquant le théorème de Riesz, on obtient

$$Tv + \sqrt{\delta}w = \bar{\partial}\tau \wedge g_0 + g_1$$

avec l'estimée

$$\int_X |v|^2 e^{-\varphi_F} + \int_X |w|_\omega^2 e^{-\varphi_F} dV_\omega \leq C(g, \tau, \delta).$$

- Dans la dernière partie de ce chapitre, on construit une extension de  $u$  avec les propriétés  $L^2$  dans théorème 9 et théorème 10 respectivement. Cette extension est obtenue par la somme de la section  $C^\infty U_\infty$  et de la solution de l'équation dans l'étape précédente plus un terme qui tend vers zéro lorsque  $\varepsilon$  tend vers zéro. La vérification de la propriété  $L^2$  est faite dans les étapes précédentes. Par exemple, pour le premier cas, on obtient d'abord l'extension locale

$$U_{j,\varepsilon} := \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_\infty - \sqrt{\eta_\varepsilon + \lambda_\varepsilon} v_\varepsilon - C_\varepsilon f_{j,\varepsilon}$$

sur  $Z \cap \Omega_j$ , où  $\theta$  est une fonction fréquante,  $f_{j,\varepsilon}$  est la solution de l'équation

$$\bar{\partial}f_{j,\varepsilon} = w_\varepsilon,$$

$v_\varepsilon$  et  $w_\varepsilon$  sont respectivement la solution et le terme supplémentaire de l'équation de type  $\bar{\partial}$  que l'on vient de résoudre dans l'étape précédente. On montre que  $f_{j,\varepsilon} \rightarrow 0$  lorsque on fait tendre  $\varepsilon$  vers zéro,  $U_{j,\varepsilon}$  coïncide ainsi avec  $U_{l,\varepsilon}$  sur  $\Omega_j \cap \Omega_l$ . Ces extensions locales se "recolent", et elles définissent l'extension finale que l'on cherche.

A la fin de ce chapitre, nous indiquons un contexte plus général dans lequel nos arguments fonctionnent.

## Chapitre 2

Parmi les nombreuses conséquences des théorèmes d'Ohsawa-Takegoshi, la plus spectaculaire est l'invariance des plurigenres établie par Siu en 1998. Rappelons que le  $m$ -ème plurigendre d'une variété complexe compacte  $X$  est la dimension de  $H^0(X, mK_X)$ , le  $m$ -ème composante de l'anneau pluricanonique gradué de  $X$ . C'est un outil puissant pour étudier la géométrie de la variété sous-jacente. Le théorème de Siu est le suivant :

**Théorème 12.** ([Siu98]). Soit  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  une famille projective de variétés compactes paramétrisées par le disque unité ouvert  $\mathbb{D}$ . Supposons que la famille  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  soit de type général. Alors pour chaque entier positif  $m$ , le plurigendre  $\dim_{\mathbb{C}} \Gamma(\mathcal{X}_t, mK_{\mathcal{X}_t})$  est indépendant de  $t \in \mathbb{D}$ , où  $\mathcal{X}_t = \pi^{-1}(t)$  et  $K_{\mathcal{X}_t}$  est le fibré canonique de  $\mathcal{X}_t$ .

Quelques années plus tard, Siu a montré le cas général du théorème précédent :

**Théorème 13.** ([Siu02]). Soient  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  une famille projective,  $(L, h)$  un fibré pseudo-effectif sur  $\mathcal{X}$  avec  $h_{\mathcal{X}}$  bien définie et  $\mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0}) = \mathcal{O}_{\mathcal{X}_0}$ . Soit  $m \geq 1$  un entier. Alors chaque section  $s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0} + L)$  satisfaisant  $\|s\|_{\infty, h} < +\infty$  se prolonge à une section  $\tilde{s} \in H^0(\mathcal{X}, mK_{\mathcal{X}} + L)$ .

Dans ses grandes lignes, la preuve de Siu est la suivante. On voudrait prolonger la section  $s^k \otimes s_A \in H^0(\mathcal{X}_0, mk(K_{\mathcal{X}_0} + L) + A|_{\mathcal{X}_0})$  où  $s$  est la section de  $m(K_{\mathcal{X}} + L)$  donnée et  $s_A$  est une section d'un fibré  $A$  suffisamment ample. Pour cela, on "contracte" d'abord cette section avec des sections locales de  $-K_{\mathcal{X}}$  jusqu'à ce que l'on obtienne des sections locales de  $K_{\mathcal{X}_0} + L + A$  : en effet, si  $\alpha$  est une section locale de  $-K_{\mathcal{X}}$ ,  $\alpha \otimes s^k \otimes s_A$  est alors une section locale de  $(km - 1)(K_{\mathcal{X}_0} + L) + A$ . On applique ensuite un théorème de type Ohsawa-Takegoshi, afin d'obtenir des extensions des sections locales en  $K_{\mathcal{X}} + L + A$ . Avec ces nouvelles extensions, on peut remonter pas à pas jusqu'on arrive à  $km(K_{\mathcal{X}} + L) + A$  et ainsi obtenir une extension de  $s^k \otimes s_A$ .

Dans ce procédé, le théorème de division de Skoda joue un rôle cruciale ; on mentionne aussi le fait que toutes les extensions intermédiaire sont effectives-ce qui nous permet des passages à la limite.

La preuve de ce théorème a été considérablement simplifiée par Păun ; le résultat dans [Pău07] est le suivant :

**Théorème 14.** ([Pău07]). Soient  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  une famille de variétés projectives et  $(L, h_L)$  un fibré en droites pseudo-effectif sur  $\mathcal{X}$  avec  $h_{L|_{\mathcal{X}_0}}$  bien définie. Soit  $m \geq 1$  un entier. Alors chaque section  $s \in H^0((\mathcal{X}_0, mK_{\mathcal{X}_0} + L) \otimes \mathcal{I}(\mathcal{X}_0, h_{L|_{\mathcal{X}_0}}))$  se prolonge à une section  $\tilde{s} \in H^0(\mathcal{X}, mK_{\mathcal{X}} + L)$ .

Sa méthode est la suivante : soit  $\mathcal{X}$  une variété complexe, et  $(A, h_A)$  un fibré en droites à courbure strictement positive (dans le cas où  $\mathcal{X}$  est projective,  $(A, h_A)$  correspond à un fibré ample). Soient également  $\mathcal{X}_0$  le fibré central et  $s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$  une section à prolonger. Pour un fibré suffisamment ample, on impose deux conditions :

- (A) Chaque section de  $mK_{\mathcal{X}} + A$  se prolonge dans  $\mathcal{X}$  ;
- (B) Pour chaque  $p = 0, \dots, m-1$ , le fibré  $pK_{\mathcal{X}} + A$  est engendré par une famille de sections  $(s_j^{(p)})_{j=1, \dots, N_p}$  dans un voisinage de  $\mathcal{X}_0$ .

Un tel fibré  $A$  peut toujours se trouver puisqu'on impose un nombre fini des conditions. On peut alors lancer le processus inductif, qui consiste en deux étapes principales :

- Tout d'abord, l'extensions des sections  $s^k \otimes s_j^{(p)}$  pas à pas : selon un théorème de type Ohsawa-Takegoshi et deux conditions imposées, on produit une extension  $s_j^{(km+p)}$  de  $s^k \otimes s_j^{(p)}$  : elle s'y prend par récurrence en utilisant à chaque fois la famille d'extensions  $(s_j^{(km+p-1)})_{j=1, \dots, N_{p-1}}$  construite dans l'étape précédente pour concentrer les singularités le long des zéros de  $s$ . Pour  $p = m - 1$ , on utilise la famille  $(s_j^{(k-1)m+m-1})_{j=1, \dots, N_{m-1}}$  à la place.
- Ensuite, la construction de la métrique : la famille de sections  $(s_j^{(km)})_{j=1, \dots, N_0}$  construite dans la première étape induit une métrique sur le fibré  $(km)K_{\mathcal{X}} + A$ , sa  $k$ -ème racine est alors une métrique sur le  $\mathbb{Q}$ -fibré  $mK_{\mathcal{X}} + \frac{1}{k}A$ . Lorsque  $k$  tend vers l'infini, on voudrait produire une métrique sur  $(m - 1)K_{\mathcal{X}}$  par rapport à laquelle le module au carré  $|s|^2$  est intégrable. La démonstration l'achève en appliquant une dernière fois le théorème d'extensions en écrivant  $mK_{\mathcal{X}} = K_{\mathcal{X}} + (m - 1)K_{\mathcal{X}}$ .

Cette méthode est ensuite appliquée dans plusieurs cas encore plus généraux. Par exemple, dans [Cla07] :

**Théorème 15.** ([Cla07]). Soient  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  une famille projective et  $(L, h_L)$  un fibré pseudo-effectif sur  $\mathcal{X}$  avec  $h_{\mathcal{X}_0}$  bien définie. Supposons que  $\mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0}) = \mathcal{O}_{\mathcal{X}_0}$ . Alors pour chaque entier  $m \geq 1$ , l'application

$$H^0(\mathcal{X}, m(K_{\mathcal{X}} + L)) \rightarrow H^0(\mathcal{X}_0, m(K_{\mathcal{X}_0} + L))$$

est surjective.

Elle est également utilisée dans [Dem06] :

**Théorème 16.** ([Dem06]). Soient  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  une famille projective et  $(L_j, h_j)$  ( $0 \leq j \leq m - 1$ ) des fibrés pseudo-effectifs sur  $\mathcal{X}$ . Supposons que

- (i) Pour tout  $0 \leq j \leq m - 1$ , la restriction de  $h_j$  à  $\mathcal{X}_0$  soit bien définie.
- (ii) Pour tout  $1 \leq j \leq m - 1$ , le faisceau d'idéaux multiplicateurs  $\mathcal{I}(h_j|_{\mathcal{X}_0})$  soit trivial.

Alors chaque section

$$s \in H^0 \left( \mathcal{X}_0, (mK_{\mathcal{X}_0} + \sum_{j=0}^{m-1} L_j) \otimes \mathcal{I}(h_0|_{\mathcal{X}_0}) \right)$$

se prolonge à  $\mathcal{X}$ .

On mentionne également [EP12], [FH11], [FH11b], [Hac08], [HM06], [HM07], [Kaw99], [Kaw99], [Kim08], [Laz04], [Tak05], [Tak07], [Tsu05], [Tsu11] et [Var08] pour d'autres résultats obtenus dans ce contexte.

Notre théorème principal dans ce chapitre est d'établir un résultat d'extensions des courants positifs fermés dans la classe  $c_1(K_{\mathcal{X}} + L)|_{\mathcal{X}_0}$ . Ce théorème peut se voir comme une généralisation naturelle de "l'invariance des plurigenres" présentée dans [Siu98]. Nous nous plaçons dans le contexte suivant, qui est inspiré par les hypothèses usuelles en géométrie algébrique :

Soit  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  une famille projective lisse au-dessus du disque unité  $\mathbb{D}$ .

- (i)  $L$  est un  $\mathbb{Q}$ -fibré pseudo-effectif : il existe une constante  $m_0 \in \mathbb{Z}_+$  telle que  $m_0L$  est un fibré en droites qui peut se décomposer comme

$$m_0L = L_1 + L_2 + \cdots + L_{m_0-1},$$

avec  $(L_j, h_j)$  des fibrés en droites hermitiens. Supposons que  $h_j|_{\mathcal{X}_0}$  soit bien définie, c'est-à-dire que les potentiels locaux  $\varphi_{L_j}$  de  $h_j$  ne sont pas identiquement égaux à  $-\infty$  lorsqu'ils sont restreints à  $\mathcal{X}_0$ . On suppose que pour chaque  $j$ ,  $\sqrt{-1}\Theta_{h_j}(L_j) \geq 0$  au sens des courants.

- (ii) On suppose qu'il existe un courant positif fermé  $T \in c_1(K_{\mathcal{X}} + L)$  tel que  $T|_{\mathcal{X}_0}$  est bien défini.  
 (iii) Il existe un  $\varepsilon > 0$  telle que pour tout  $\delta$  positif et  $j = 1, \dots, m_0 - 1$ , on a

$$(\mathcal{I}_j) \quad \mathcal{I}((1 - \delta)\varphi_{L_j} + \varepsilon\varphi_{T|_{\mathcal{X}_0}}) = \mathcal{O}_{\mathcal{X}_0}.$$

Notons pour  $0 < \eta \ll 1$ ,  $\pi_\eta : \mathcal{X}_\eta \rightarrow \mathbb{D}_\eta$  l'application voisine autour de  $\pi_0$  où  $\mathbb{D}_\eta$  a les coordonnées locales  $|z| < \eta$ . Le théorème qu'on va démontrer est le suivant :

**Théorème 17.** *On considère une famille projective, non-singulière  $\mathcal{X} \rightarrow \mathbb{D}$  sur un disque unité, avec un fibré  $L \rightarrow \mathcal{X}$  satisfaisant les propriétés (i) – (iii) de l'hypothèse. Soit  $\Theta_0$  un courant positif fermé sur  $\mathcal{X}_0$  tel que*

$$\Theta_0 \in c_1(K_{\mathcal{X}} + L|_{\mathcal{X}_0}),$$

la première classe de Chern du fibré adjoint  $K_{\mathcal{X}} + L$  restreint à  $\mathcal{X}_0$ . On suppose qu'il existe une constante  $\delta_0 > 0$  telle que

$$\int_{\mathcal{X}_0} e^{(1+\delta_0)(\varphi_{\Theta_0} - \varphi_L)} dV_{\omega_0} < \infty.$$

Alors il existe un courant positif fermé  $\Theta$  sur  $\mathcal{X}_\eta$  dans la classe de cohomologie du  $\mathbb{Q}$ -fibré  $K_{\mathcal{X}} + L$  tel que

$$\int_{\mathcal{X}_\eta} e^{\varphi_\Theta - \varphi_L} dV_{\omega_\eta} \leq C \int_{\mathcal{X}_0} e^{\varphi_{\Theta_0} - \varphi_L} dV_{\omega_0}.$$

Dans ce théorème, on désigne par  $\varphi_\Theta$  le poids local du courant  $\Theta$ ,  $dV_{\omega_0}$  et  $dV_{\omega_\eta}$  sont des formes de volume de  $\mathcal{X}_0$  et de  $\mathcal{X}_\eta$  respectivement. Ce théorème implique que la classe  $K_{\mathcal{X}} + L$  est pseudo-effective, à condition que sa restriction à un fibré central ait cette propriété. Nous espérons que la version transcendante de ce théorème reste valable.

**Conjecture 2.** *Soient  $\alpha \geq 0$  une forme de type (1,1) semi-positives sur une famille kählérienne  $\mathcal{X}$ ,  $\Theta_0 \in \{K_{\mathcal{X}} + \alpha\}|_{\mathcal{X}_0}$  un courant positif fermé. Alors il existe un courant  $\Theta \in \{K_{\mathcal{X}} + \alpha\}$  tels que*

1. Le courant  $\Theta|_{\mathcal{X}_0}$  est bien défini et moins singulier que  $\Theta_0$ .
2. Il existe une constante numérique  $C > 0$  telle que

$$\int_{\mathcal{X}} e^{\varphi_\Theta - \varphi_\alpha} dV_\omega \leq C \int_{\mathcal{X}_0} e^{\varphi_{\Theta_0} - \varphi_\alpha} dV_{\omega_0}.$$

Nous remarquons que cette conjecture implique l'invariance des plurigenres de Siu dans le cas d'une famille kählérienne. Si  $\mathcal{X} \rightarrow \mathbb{D}$  est une famille kählérienne au-dessus du disque unité, étant donné une section  $u$  du fibré  $m_0K_{\mathcal{X}}|_{\mathcal{X}_0}$ , la conjecture implique l'existence d'une



métrique  $h$  sur le fibré  $K_{\mathcal{X}}$  à courbure positive telle que la section  $u$  est bornée par rapport à  $h^{m_0}$ . Alors  $u$  se prolonge à toute la famille par le théorème d'Ohsawa-Takegoshi. On remarque que cette conjecture est la motivation principale de notre théorème.

La partie qualitative du théorème 17 peut se résumer comme suit :

**Théorème 18.** *Soient  $\mathcal{X} \rightarrow \mathbb{D}$  une famille projective lisse,  $L$  un  $\mathbb{Q}$ -fibré pseudo-effectif et  $T$  un courant positif fermé satisfaisant les propriétés (i)-(iii). Alors chaque section de  $m_0(K_{\mathcal{X}} + L)|_{\mathcal{X}_0}$  admet une extension à  $\mathcal{X}_\eta$ .*

Nous allons donc d'abord démontrer ce théorème. L'outil principal est un théorème de type Ohsawa-Takegoshi.

**Théorème 19.** *Soit  $\mathcal{X} \rightarrow \mathbb{D}$  une famille projective lisse et  $L \rightarrow \mathcal{X}$  un fibré en droites muni d'une métrique éventuellement singulière  $h$  à courbure semi-positive. Alors il existe une constante  $C_0$  telle que pour chaque section  $\sigma \in H^0(\mathcal{X}_0, K_{\mathcal{X}_0} + L)$  satisfaisant :*

$$\int_{\mathcal{X}_0} \|\sigma\|_h^2 < +\infty,$$

il existe  $\tilde{\sigma} \in H^0(\mathcal{X}, K_{\mathcal{X}} + L)$  avec  $\tilde{\sigma}|_{\mathcal{X}_0} = \sigma \wedge dt$  et de plus :

$$\int_{\mathcal{X}} \|\tilde{\sigma}\|_h^2 \leq C_0 \int_{\mathcal{X}_0} \|\sigma\|_h^2.$$

Notons que

$$m_0(K_{\mathcal{X}} + L) = K_{\mathcal{X}} + (m_0 - 1)(K_{\mathcal{X}} + L) + L,$$

donc il faut trouver une métrique du fibré

$$(m_0 - 1)(K_{\mathcal{X}} + L) + L$$

telle que

- sa courbure est semi-positive;
- la section  $\sigma$  appartient au faisceau d'idéaux multiplicateurs défini par sa restriction au fibré central  $\mathcal{X}_0$ , si on envisage l'extension de  $\sigma$  via le théorème d'Ohsawa-Takegoshi.

La construction de la métrique consiste en deux étapes principales :

- Construire une suite de métriques des fibrés

$$km_0(K_{\mathcal{X}} + L) + A$$

où  $A$  est un fibré suffisamment ample. Notons pour chaque entier  $p \in [0, m_0 - 1]$ ,

$$L^{(p)} := pK_{\mathcal{X}} + L_1 + \dots + L_p.$$

(1) Pour chaque  $0 \leq p \leq m_0 - 1$ ,  $L^{(p)} + A$  est engendré par des sections globales

$$(s_j^{(p)})_{j=1, \dots, N_p}.$$

(2) Chaque section de  $m_0(K_{\mathcal{X}} + L) + A$  se prolonge à  $\mathcal{X}$ .

Ensuite, nous procédons par récurrence, en formulant l'affirmation suivante :

$\mathcal{P}_{k,p}$  : Pour  $k \in \mathbb{N}$  et  $0 \leq p \leq m_0 - 1$ , la section

$$\sigma^k \otimes s_j^{(p)} \in H^0(\mathcal{X}_0, L^{(p)} + km_0(K_{\mathcal{X}} + L) + A)$$

admet une extension à  $\mathcal{X}$ , pour chaque  $j = 1, \dots, N_p$ .

On montre que  $\mathcal{P}_{k,p}$  est vraie de la manière suivante :

- si  $p < m_0 - 1$ , on définit une famille de sections

$$\sigma^k \otimes s_j^{(p+1)} \in H^0(\mathcal{X}_0, L^{(p+1)} + km_0(K_{\mathcal{X}} + L) + A|_{\mathcal{X}_0})$$

où

$$L^{(p+1)} = K_{\mathcal{X}} + L_{p+1} + L^{(p)}.$$

Pour pouvoir appliquer le théorème de type Ohsawa-Takegoshi, nous munissons le fibré

$$F := L_{p+1} + L^{(p)} + km_0(K_{\mathcal{X}} + L) + A$$

une métrique comme suit :

$$\varphi_{\delta, \varepsilon}^{(km_0+p+1)} := (1 - \delta)\varphi_{L_{p+1}} + \delta\tilde{\varphi}_{L_{p+1}} + (1 - \varepsilon)\varphi^{(km_0+p)} + \varepsilon(\tilde{\varphi}_{L^{(p)}} + m_0k\varphi_T + \varphi_A)$$

où les paramètres  $\varepsilon, \delta$  sont tels que  $m_0k\varepsilon \ll 1$  et  $\delta \ll \varepsilon$ .

- Si  $p = m_0 - 1$ , le fibré  $(k + 1)m_0(K_{\mathcal{X}} + L) + A$  peut s'écrire comme

$$(k + 1)m_0(K_{\mathcal{X}} + L) + A = K_{\mathcal{X}} + L^{(m_0-1)} + km_0(K_{\mathcal{X}} + L) + A,$$

et si on note  $F$  le fibré suivant :

$$F := L^{(m_0-1)} + km_0(K_{\mathcal{X}} + L) + A,$$

on peut le munir d'une métrique  $e^{-\varphi^{(km_0+m_0-1)}}$  induite par  $(U_j^{(km_0+m_0-1)})$ , l'extension de  $\sigma^k \otimes s_j^{(m_0-1)}$  et appliquer ensuite le théorème d'extension.

En conclusion, nous disposons d'une suite de métriques  $e^{-\varphi^{(km_0)}}$  à courbure semi-positives des fibrés

$$km_0(K_{\mathcal{X}} + L) + A,$$

les singularités desquels sont équivalentes, lorsqu'elles sont restreintes au fibré central, à l'ensemble des zéros de la section  $\sigma$ .

- Ensuite, on voudrait construire une métrique sur  $K_{\mathcal{X}} + L$  comme la limite de la suite  $e^{-\frac{\varphi^{(km_0)}}{km_0}}$ . Afin de justifier le passage à la limite, les potentiels  $\frac{1}{km_0}\varphi^{(km_0)}$  doivent être bornés supérieurement. Donc, nous allons rendre "effective" la construction des métriques.

La reste de la preuve du théorème principal consiste à appliquer la technique de la régularisation de courants, le courant  $\Theta_0$  peut être approximé par des courants qui sont associés à certain espace de sections du fibré

$$k(K_{\mathcal{X}} + L)|_{\mathcal{X}_0} + A.$$

L'extension  $\Theta$  de  $\Theta_0$  est construite en considérant le courant de courbure associée à la métrique trouvée par extension.

## Chapitre 3

Dans ce dernier chapitre de notre thèse, nous obtenons une version du théorème de Bando-Mabuchi. Il s'agit d'un théorème d'unicité des solutions pour certaines équations de type Monge-Ampère.

Etant donné une variété complexe  $X$  munie d'une métrique kählérienne  $\omega$ , la courbure de Ricci de  $\omega$  est donné localement par

$$\text{Ricci}(\omega) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\det\omega).$$

La classe de cohomologie de la forme Ricci( $\omega$ ) ne dépend pas de  $\omega$  : c'est la première classe de Chern de  $X$  (la classe de Chern du fibré anticanonique).

**Question.** Soit  $X$  une variété kählérienne lisse, existe-t-il une métrique  $g$  sur  $X$  telle que pour  $\lambda \in \mathbb{R}$ ,

$$\text{Ricci}(g) = \lambda\omega_g, \quad (2)$$

où  $\omega_g$  est la forme kählérienne associée à  $g$ ?

L'équation (2) est l'équation de Kähler-Einstein, les solutions de cette équation sont des métriques de Kähler-Einstein. Par un changement d'échelle, on peut supposer que  $\lambda \in \{-1, 0, 1\}$ . On remarque que l'existence d'une métrique de Kähler-Einstein impose des restrictions considérables sur  $c_1(X)$  : par exemple, si on a une métrique  $\omega$  qui vérifie (2) avec  $\lambda = -1$ , cela implique que  $K_X$  est ample. Dans ce contexte, un résultat fondamental a été établi par Aubin et Yau ([Aub76], [Aub78], [Yau77], [Yau78]) :

**Théorème 20.** (i) Supposons que  $c_1(X) = 0$ . Alors pour  $\{\omega_0\}$  une classe de Kähler, il existe  $\omega \in \{\omega_0\}$  une métrique kählérienne telle que Ricci( $\omega$ ) = 0.

(ii) Si  $c_1(X) < 0$ , alors il existe  $\omega \in c_1(X)$  telle que

$$\text{Ricci}(\omega) = -\omega.$$

Le cas  $c_1(X) > 0$  (autrement dit, si  $X$  est une variété de Fano) n'est pas encore bien compris. On connaît des exemples de variétés de Fano qui n'admettent pas des métriques de Kähler-Einstein, mais on ne dispose pas d'un critère algébro-géométrique pour l'existence de ce type de métriques.

En revanche, l'unicité des métriques de Kähler-Einstein a été établie par Bando-Mabuchi :

**Théorème 21.** ([BM87]) Soient  $\omega_1, \omega_2 \in c_1(X)$  deux métriques telles que

$$\text{Ricci}(\omega_j) = \omega_j, \quad \forall j = 0, 1.$$

Alors il existe un automorphisme  $\varphi : X \rightarrow X$  tel que

$$\varphi^*\omega_1 = \omega_0.$$

Soit  $D = \sum d^j Y_j$  un diviseur effectif sur  $X$ , on suppose que  $(Y_j)$  sont non-singulières, et qu'elles s'intersectent transversalement. Si de plus on a  $d^j < 1$ , alors  $(X, D)$  est un exemple de paire klt.

Dans ce contexte, on dit qu'une paire  $(X, D)$  est de Fano si  $-(K_X + D)$  contient une métrique kählérienne. L'équation de Kähler-Einstein devient

$$\text{Ricci}(\omega) = \omega + [D]; \quad (3)$$

et plus précisément, comme dans le cas "classique",  $D = 0$ , on peut se poser la question de l'unicité de la métrique, solution de (3) à biholomorphisme près.

Une réponse complète à cette question a été apporté par Berndtsson dans [Ber11]. Son résultat s'énonce comme suit :

**Théorème 22.** *On suppose que  $-(K_X + D)$  est ample, et soient  $\omega_{\varphi_1}, \omega_{\varphi_2} \in -c_1(K_X + D)$  deux métriques dont les potentiels sont bornés, et tels que*

$$\text{Ricci}(\omega_{\varphi_j}) = \omega_{\varphi_j} + [D], \quad \forall j = 0, 1.$$

*Alors, il existe un automorphisme  $F : X \rightarrow X$ , tel que*

$$F^*\omega_{\varphi_1} = \omega_{\varphi_0}$$

*et que*

$$F^*(D) = D.$$

Le résultat principal obtenu dans le troisième chapitre de cette thèse est une généralisation de ce résultat.

Soit  $\{\alpha\}$  une classe  $(1, 1)$  sur  $X$ ; ici on désigne  $\alpha$  un représentant non-singulier. On suppose que les conditions suivantes soient satisfaites :

- (i) la classe  $c_1(X) - \{\alpha\}$  est de Kähler, autrement dit, elle contient une métrique kählérienne  $\omega$ ;
- (ii) la classe  $\{\alpha\}$  est pseudo-effective; soit  $\Theta \in \{\alpha\}$  un courant positif fermé;
- (iii) si on désigne  $\text{Ricci}(\omega)$  la courbure de Ricci associée à la métrique  $\omega$ , alors on a

$$\text{Ricci}(\omega) = \omega + \Theta - \sqrt{-1}\partial\bar{\partial}f_\Theta$$

pour certaine fonction  $f_\Theta$  qui est unique à une constante additive près. On suppose qu'il existe un nombre réel  $\varepsilon_0 > 0$  such that

$$\int_X e^{-(1+\varepsilon_0)\frac{\alpha}{2}} f_\Theta dV < \infty.$$

Autrement dit, les singularités du courant  $\Theta$  ne sont pas trop larges.

D'après Bedford et Taylor (cf. [BT76]), il est logique de considérer l'équation

$$(\star) \quad (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi - f_\Theta} \omega^n$$

où la solution  $\varphi$  est supposée bornée, telle que

$$\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi$$

est un courant kählérien (autrement dit, il est plus grand qu'une métrique kählérienne).

On remarque que si  $\Theta$  est non-singulier, l'équation  $(\star)$  est équivalente à l'identité suivante :

$$\text{Ricci}(\omega_\varphi) = \omega_\varphi + \Theta.$$

Dans ce chapitre nous démontrons qu'on a la généralisation suivante du théorème 22.

**Théorème 23.** *Soient  $\varphi_0$  et  $\varphi_1$  deux solutions bornées de l'équation  $(\star)$ . Alors il existe un biholomorphisme  $F : X \rightarrow X$  tels que*

$$F^*(\omega_{\varphi_1}) = \omega_{\varphi_0} \quad \text{et que} \quad F^*(\Theta) = \Theta.$$

Les arguments que nous allons invoquer afin de démontrer ce résultat généralisent l'approche par Berndtsson dans [Ber11]. Ils sont basés sur les propriétés de convexité de noyau de Bergman, ce que nous rappellent ensuite.

Soit  $L \rightarrow X$  un fibré en droites, muni d'une famille des métriques non-singulières  $h_t = e^{-\varphi_t}$ ,  $t \in \mathbb{D}$ , le disque d'unité. On suppose que la fonction

$$\varphi(t, x) = \varphi_t(x)$$

est non-singulière et psh. On peut regarder  $L$  comme un fibré sur la famille  $X \times \mathbb{D}$ ; la structure fibrale est indépendante de  $t \in \mathbb{D}$ , mais la structure hermitienne se varie avec  $t$ . Soit

$$E := p_*(K_{X \times \mathbb{D}/\mathbb{D}} + L),$$

l'image directe du fibré adjoint relatif de  $L$ , et soit  $(u_t)_{t \in \mathbb{D}}$  une section holomorphe de  $E$ . Bien entendu, en générale  $(u_t)$  ne correspond pas à une  $n$ -forme holomorphe en  $X \times \mathbb{D}$  : on ne peut que représenter  $(u_t)_{t \in \mathbb{D}}$  par une  $n$ -forme  $U$  dans  $X \times \mathbb{D}$  telles que  $\bar{\partial}U$  est égal à un multiple de  $dt$ , et que  $U|_{X \times \{t\}} = u_t$ . Afin d'évaluer la courbure  $\Theta_E$  de  $E$  dans la direction de la section  $(u_t)$ , on introduit une famille  $(v_t)$  de  $(n-1)$ -formes à valeurs dans  $L$  telles que

- \*  $\bar{\partial}v_t \wedge \omega = 0$ ,
- \*  $\partial^{\varphi_t} v_t = P_{\perp}(\dot{\varphi}_t u_t)$

où  $P$  est la projection sur le complément orthogonal des  $n$ -formes holomorphes sur chaque fibré, alors on a la formule suivante (cf. [Ber11]) :

$$\langle \Theta_E u_t, u_t \rangle = p_*(c_n \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \hat{u} \wedge \bar{\hat{u}} e^{-\varphi_t}) + \int_X \|\bar{\partial} v_t\|^2 e^{-\varphi_t} \sqrt{-1} dt \wedge d\bar{t}$$

où on désigne  $\hat{u} = u_t - dt \wedge v_t$ , une autre représentant de la section initiale  $(u_t)$ .

Nous supposons maintenant que pour des raisons géométriques, la courbure de  $E$  s'annule. En conséquence,  $v_t$  sera holomorphe, ainsi que le premier terme du côté droit de la formule ci-dessus. Mais alors on peut définir un champ de vecteurs par la formule

$$-v_t = V_t \lrcorner u_t;$$

il sera holomorphe en dehors de  $u_t = 0$ . En fait, sous la condition  $\Theta_E = 0$ , on peut voir que  $V_t$  est le gradient complexe de  $\sqrt{-1} \partial \varphi_t$ ; enfin, un simple calcul indique que le flot associé à  $V_t$ ,  $F$ , est l'application holomorphe que l'on cherche.

Tout en essayant de mettre en œuvre ce programme dans notre milieu, nous devons faire face à deux difficultés importantes. Dans notre cas, le fibré  $L$  sera juste  $-K_X$ , muni d'une famille des métriques en ci-dessous. On connecte  $\varphi_0$  et  $\varphi_1$  par une géodésie, (l'existence de cette dernière est connue), elle nous donne une métrique  $\varphi_t$  dont la courbure

$$\omega_t = \omega + \sqrt{-1} \partial \bar{\partial}_X \varphi_t$$

est dans la classe  $c_1(X) - \{\alpha\}$ . Alors le courant

$$\omega_t + \Theta$$

avec  $\Theta \in \{\alpha\}$  positif fermé, est dans la classe  $c_1(X)$  pour chaque  $t$ . Nos principales considérations sont :

\*  $\Theta$  n'est pas nécessairement lisse ;

\* le courant  $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}_X\varphi_t$  n'est pas nécessairement kälérien, il est seulement positif.

Notre schéma est le suivant : on approxime  $\omega_t$  et  $\Theta$  par des objets non-singuliers (soub-section 3.4.1), ensuite on construit  $v_t'$ , l'analogue de la  $(n-1)$ -forme  $v_t$  mentionnée en haut. Grâce au fait que  $\varphi_0$  et  $\varphi_1$  sont deux solutions de  $(\star)$ , on sait que la courbure de  $E$ , équipée d'une métrique régularisée, converge vers zéro. Un argument d'uniformité (ce qui est assez subtile, compte tenu que l'ensemble de Lelong de  $\Theta$  pourrait être assez compliqué) montre que  $v_t'$  converge à une  $(n-1)$ -forme holomorphe. Ensuite, le champ de vecteurs sera défini comme ci-dessus.

Première partie  
Extension theorems.





# Chapitre 1

## An Ohsawa-Takegoshi theorem on compact Kähler manifolds

In the present chapter, we establish an extension theorem of Ohsawa-Takegoshi type for twisted canonical forms on compact Kähler manifolds. We start with a few preliminaries and notations. Then, we will highlight some of the main steps of the proof in the case of projective manifolds. This will allow us to point out the main difficulties encountered in the Kähler case. We explain afterwards the details of our arguments and we provide a few comments concerning some open problems.

### 1.1 Origin of the problem and a few related results

#### 1.1.1 The Ohsawa-Takegoshi theorem

Consider a Stein manifold  $X$  and a closed regular submanifold  $Y \subset X$ , it is well known that every holomorphic function defined on  $Y$  has holomorphic extensions on  $X$ . What is considerably more difficult is to construct an extension whose  $L^2$  norm is bounded in terms of the function defined on  $Y$ . Several researches have been done on this subject (see [Dem82], [Yos82], [Nis80], [Nak86] and [FO87]). In 1987, Ohsawa and Takegoshi published a remarkable article where they proved an extension theorem of holomorphic functions defined on a bounded domain in  $\mathbb{C}^n$  with growth control (cf. [OT87]). This theorem, known as the Ohsawa-Takegoshi theorem, plays an important role in the evaluation of the Bergman kernel. This was the initial motivation for the researches in this direction. Recall that the Bergman kernel is a reproducing kernel for the Hilbert space of all square integrable holomorphic functions on a domain in  $\mathbb{C}^n$ . More precisely, let  $\mathcal{D}$  be a domain in  $\mathbb{C}^n$  and let  $\mathbb{A}^2(\mathcal{D})$  the Hilbert space of all square integrable holomorphic functions on  $\mathcal{D}$ . For a holomorphic function  $f \in \mathbb{A}^2(\mathcal{D})$ , the evaluation

$$\text{ev}_z f \rightarrow f(z)$$

is a continuous linear functional on  $\mathbb{A}^2(\mathcal{D})$ . By the Riesz representation theorem, this functional can be represented as an inner product  $f$  with an element of  $\mathbb{A}^2(\mathcal{D})$ , which is to say that,

$$\text{ev}_z f = \int_{\mathcal{D}} f(\xi) \overline{\eta_z(\xi)} d\mu(\xi).$$

The *Bergman Kernel*  $K$  is defined by

$$K(z, \xi) = \overline{\eta_z(\xi)}.$$

It is holomorphic in  $z$  and antiholomorphic in  $\xi$ . Moreover, we have the "reproducing property" :

$$f(z) = \int_{\mathcal{D}} K(z, \xi) f(\xi) d\mu(\xi).$$

We say that a function  $\psi$  defined over an open set  $\Omega \subset \mathbb{C}^n$  is *plurisubharmonic* (psh in abbreviation) if it is semicontinuous and satisfies the mean value inequality

$$\psi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(a + e^{i\theta} \xi) d\theta$$

for all  $a \in \Omega$ ,  $\xi \in \mathbb{C}^n$  and  $|\xi| < d(a, \mathbb{C}\Omega)$ . i.e. it is subharmonic when restricted to a complex line  $L \cap \Omega$ . The domain  $\Omega$  is *pseudoconvex* if there exists a continuous psh function  $\varphi(z)$  over  $\Omega$  such that the set

$$\{z \in \Omega \mid \varphi(z) < c\}$$

is relatively compact in  $\Omega$  for all  $c \in \mathbb{R}$ . The function  $\varphi(z)$  is called an *exhaustion*.

The result in [OT87] is as follows :

**Theorem 1.1.** *Let  $\Omega$  be a pseudoconvex domain in  $\mathbb{C}^n$ ,  $\psi : \Omega \rightarrow \mathbb{R} \cup \{-\infty\}$  a psh function and  $H \subset \mathbb{C}^n$  a complex hyperplane. There exists a constant  $C$  which depends only on the diameter of  $\Omega$  such that : for every holomorphic function  $f$  on  $\Omega \cap H$  which satisfies*

$$\int_{\Omega \cap H} |f|^2 e^{-\psi} dV_{n-1} < +\infty$$

where  $dV_{n-1}$  is the Lebesgue measure in  $\mathbb{R}^{(2n-2)}$ , there exists a holomorphic function  $F$  on  $\Omega$  such that  $F|_{\Omega \cap H} = f$  and

$$\int_{\Omega} |F|^2 e^{-\psi} dV_n \leq C \int_{\Omega \cap H} |f|^2 e^{-\psi} dV_{n-1}.$$

The proof of Theorem 1.1 is based on the  $\bar{\partial}$ -equation theory on complete manifolds. Using techniques developed by S. Bochner, K. Kodaira and S. Nakano (cf. [Kod54], [Gri69], [Nak55] and [Nak73]), Ohsawa and Takegoshi established an  $L^2$  estimate for the operator  $\bar{\partial} \circ \eta$  with  $\eta$  a positive function. This was inspired by the works of H. Donnelly, C. Fefferman and F. Xavier (cf. [DF83], [DX84]). The key point of the Ohsawa-Takegoshi theorem is the uniform estimate of the extension. An almost immediate consequence and the original motivation of Theorem 1.1 is the following estimate (cf. [Ohs84]) :

**Theorem 1.2.** *Let  $\Omega \subset \mathbb{C}^n$  be a bounded pseudoconvex domain, then we have*

$$K(z) \geq \frac{1}{C \cdot \delta_{\Omega}^2},$$

where  $\delta_{\Omega}$  is the distance function from  $z$  to  $\mathbb{C}\Omega$  and  $C > 0$  is a constant depending on  $\Omega$ . In particular, we have

$$\lim_{z \rightarrow \mathbb{C}\Omega} K(z) = +\infty.$$

Here  $K(z)$  is the restriction of the Bergman kernel on the diagonal.

### 1.1.2 Manivel's interpretation

Since we intend to use extension statements of Ohsawa-Takegoshi type in order to analyze problems in algebraic/Kähler geometry, one should establish a similar result in the context of manifolds. This was started by Manivel in [Man93]. Remark that the factor  $e^{-\psi}$  in integrals in Theorem 1.1 could be seen as the weight of a metric of a line bundle  $L$  over the manifold, and the function  $f$  corresponds to an  $(n,0)$ -form with values in  $L$ . The next geometric interpretation of the Ohsawa-Takegoshi theorem was established by Manivel [Man93]. We quote here the version appeared in [Dem00] :

**Theorem 1.3.** *Let  $X$  be a weakly pseudoconvex  $n$ -dimensional complex manifold equipped with a Kähler metric  $\omega$ , let  $L$  (resp.  $E$ ) be a hermitian holomorphic line bundle (resp. a hermitian holomorphic vector bundle of rank  $r$  over  $X$ ) equipped with a smooth metric  $h_L$  (resp. a smooth metric  $h_E$  for  $E$ ), and  $s$  a global holomorphic section of  $E$ . Assume that  $s$  is generically transverse to the zero section, and let*

$$Y = \{x \in X; s(x) = 0, \wedge^r ds(x) \neq 0\}, \quad p = \dim Y = n - r.$$

Moreover, assume that the  $(1,1)$ -form  $\sqrt{-1}\Theta(L) + r\sqrt{-1}\partial\bar{\partial}\log|s|^2$  is semipositive and that there is a continuous function  $\alpha \geq 1$  such that the following two inequalities hold everywhere on  $X$  :

$$(a) \quad \sqrt{-1}\Theta(L) + r\sqrt{-1}\partial\bar{\partial}\log|s|^2 \geq \frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)s, s\}}{|s|^2},$$

$$(b) \quad |s| \leq e^{-\alpha}.$$

Therefore for every holomorphic section  $f$  of  $\wedge^n T_X^* \otimes L$  over  $Y$ , such that

$$\int_Y \frac{|f|^2}{|\wedge^r(ds)|^2} dV_\omega < +\infty,$$

there exists a holomorphic extension  $F$  to  $X$  such that  $F|_Y = f$  and

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log|s|)^2} dV_{X,\omega} \leq C_r \int_Y \frac{|f|^2}{|\wedge^r(ds)|^2} dV_{Y,\omega}$$

where  $C_r$  is a numerical constant depending only on  $r$ .

In particular, the result above does apply to the Stein manifolds as well as to the compact Kähler manifolds. However, there is a drawback : the metric  $h_L$  is assumed to be non singular. The fact that one would need an analogue result where  $h_L$  is allowed to have singularities is illustrated by the following two examples. The first one is contained in the work of Demailly, concerning the approximation of psh functions (compared with [Bre54] and [Lel72]) :

**Theorem 1.4.** ([Dem94]) Let  $\varphi$  be a psh function on a bounded pseudoconvex open set  $\Omega \subset \mathbb{C}^n$ . For every  $m > 0$ , let  $\mathcal{H}_\Omega(m\varphi)$  be the Hilbert space of holomorphic functions  $f$  on  $\Omega$  such that  $\int_\Omega |f|^2 e^{-2m\varphi} d\lambda < +\infty$  and let  $\varphi_m = \frac{1}{2m} \log \sum |\sigma_l|^2$  where  $(\sigma_l)$  is an orthonormal basis of  $\mathcal{H}_\Omega(m\varphi)$ . Then there are constants  $C_1, C_2 > 0$  independent of  $m$  such that

(a)

$$\varphi(z) - \frac{C_1}{m} \leq \varphi_m(z) \leq \sup_{|\xi-z|<r} \varphi(\xi) + \frac{1}{m} \log \frac{C_2}{r^n}$$

for every  $z \in \Omega$  and  $r < d(z, \mathbb{C}\Omega)$ . In particular,  $\varphi_m$  converges to  $\varphi$  pointwise and in  $L^1_{loc}$  topology on  $\Omega$  when  $m \rightarrow +\infty$  and

(b)

$$\nu(\varphi, z) - \frac{n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z)$$

for every  $z \in \Omega$ .

This result contains in particular the result in [Siu74], that is, the analyticity of the upperlevel set of Lelong numbers. We remind that the proof of the above theorem uses only the zero dimension case of the  $L^2$  extension theorem. We finish this section by giving another application of Theorem 1.3. This is a Briançon-Skoda type theorem for multiplier ideal sheafs of Nadel ([Nad90]) :

**Theorem 1.5.** ([Dem00]) Let  $X$  be an  $n$ -dimensional complex manifold,  $\varphi$  and  $\psi$  be psh functions on  $X$ . Then for every integer  $k \geq n$  we have

$$\mathcal{I}(k\varphi + \psi) \subset \mathcal{I}(\varphi)^{k-n} \mathcal{I}(\psi).$$

## 1.2 Preliminaires

In this section we first recall the notion of singular metrics on a line bundle as well as some notions of positivity associated to that line bundle. Then we give the notions of multiplier ideal sheaf and upper regularized multiplier ideal sheaf in this set-up respectively.

### 1.2.1 Singular metrics and their associated curvature

Let  $X$  be a complex manifold equipped with a metric  $\omega$  and  $L$  a holomorphic line bundle over  $X$ . Let  $U_i$  be an open set in  $X$  over which  $L$  is trivial,  $s$  be a section of  $L$  locally presented by  $s_i$  on  $U_i$ , a *singular metric*  $h_L$  on  $L$  is defined by a collection of locally integrable functions  $\phi_i$  such that the norm of  $s$  satisfies

$$\|s\|^2 = |s_i|^2 e^{-\phi_i}.$$

The function  $\phi_i$  is the *local weight* of the metric associated to the trivialisation  $L|_{U_i}$ . For example, if  $\sigma_1, \dots, \sigma_N \in H^0(X, mL)$  are holomorphic sections of  $mL$ , then the metric locally defined by

$$\phi_i := \frac{1}{m} \log \left( \sum_{j=1}^N |f_{j_i}|^2 \right)$$

is singular, here  $f_{j_i}$  are local representatives of  $\sigma_j$  on  $U_i$ . Note  $g_{ij} \in \mathcal{O}^*(U_i \cap U_j)$  the transition function, we have  $s_i = g_{ij} s_j$  on the intersection  $U_i \cap U_j \neq \emptyset$ . Since we want the quantity  $\|s\|^2$  to have a global meaning, the local representatives of the singular metric  $h_L$  satisfy the compatibility condition

$$\phi_i - \phi_j = \log |g_{ij}|^2.$$

Thanks to the local integrability of local weights, we can associate a global curvature current to the metric  $h_L$ . In fact, we can calculate

$$\sqrt{-1} \partial \bar{\partial} \phi_i \quad \text{and} \quad \sqrt{-1} \partial \bar{\partial} \phi_j$$

in the sense of distributions : they are (1,1)-currents on  $U_i$  and  $U_j$  respectively. Since

$$\log |g_{ij}|^2$$

is pluriharmonic,  $\sqrt{-1} \partial \bar{\partial} \phi_i$  is equal to  $\sqrt{-1} \partial \bar{\partial} \phi_j$  over the intersection  $U_i \cap U_j$ . Therefore  $\sqrt{-1} \partial \bar{\partial} \phi_i$  is globally defined. We define the *curvature*  $\sqrt{-1} \Theta_{h_L}(L)$  associate to the metric  $h_L$  by

$$\sqrt{-1} \Theta_{h_L}(L) = \sqrt{-1} \partial \bar{\partial} \phi.$$

### 1.2.2 Positivity concepts for line bundles

Once the curvature current is defined, we recall next a few notions of positivity associated to a hermitian line bundle. We first recall here the notion of closed positive  $(1,1)$ -current. For more details on the positivity of the current in general, we refer to the book of J. P. Demailly [Dem09b]).

In a local setting, a current  $T$  of type  $(1,1)$  can be seen as a  $(1,1)$ -form

$$\sqrt{-1} \sum_{j,k} T_{j,k} dz_j \wedge d\bar{z}_k$$

with distributional coefficients.  $T$  is said *positive* if and only if for all  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ ,

$$\sum_{1 \leq j,k \leq n} \lambda_j \bar{\lambda}_k T_{j,k}$$

is a positive measure.

With this definition, we present in the following some positivity notions in geometric ways as well as their algebraic counterparts. This dictionary was established by J-P. Demailly in [Dem92].

The general idea is to establish the link between the existence of sections of multiples of a line bundle and the existence of (possibly singular) metrics on  $L$  whose curvature has some positivity properties. We have seen in the last paragraph that a family of sections induces a metric in  $L$ , whose curvature is semi-positive definite. The hope is that the  $L^2$  theory will allow us to reverse this process. We recall next some results in this direction.

A line bundle  $L \rightarrow X$  is said *positive* if it admits a nonsingular metric  $h_L$  such that

$$\Theta_{h_L}(L) \geq \varepsilon_0 \omega,$$

where  $\varepsilon_0 > 0$  and  $\omega$  is a Kähler metric on  $X$ . The next result is the foundation of many developments in algebraic and analytic geometry.

**Theorem 1.6.** ([Kodaira]).  $L$  is positive if and only if  $L$  is ample, that is, the map induced by the global sections of  $mL$  is an embedding, for  $m \gg 0$ .

A more flexible positive notion is that of numerically effective (nef in abbreviation). The metric version of this notion is that  $L$  is a *nef* line bundle if for all  $\varepsilon > 0$ , there exists a nonsingular metric  $h_\varepsilon$  such that

$$\Theta_{h_\varepsilon}(L) \geq -\varepsilon \omega.$$

The corresponding definition in algebraic geometry says that  $L$  is nef if and only if  $L \cdot C \geq 0$  for all curves  $C \subset X$ . The equivalence of these two definitions was established in [DPS93].

If metrics are allowed to be singular, the previous notions change as follows : a line bundle  $L \rightarrow X$  is *big* if  $h^0(X, mL) \geq Cm^n$ , where  $C$  is a positive constant and  $n = \dim X$ . Through the  $L^2$  theory, we see that  $L$  is big if and only if it admits a metric  $h_L$  (possibly singular) whose curvature current is a Kähler current. i.e.  $\Theta_{h_L}(L) \geq \varepsilon_0 \omega$ . Therefore, we have a singular version of the Kodaira's theorem.

Finally, a line bundle  $L$  is *pseudo-effective* if for any  $k \geq 1$ ,  $kL + A$  admits  $\mathbb{Q}$ -sections (non-zero). The analytic definition of this notion is as follows :  $L$  is pseudo-effective if and only if it admits a metric whose curvature current is positive.

### 1.2.3 Multiplier ideal sheaf and upper regularized multiplier ideal sheaf

Given a psh function on an open set  $\Omega \subset \mathbb{C}^n$ , its associated multiplier ideal sheaf was introduced by Nadel [Nad90], his main motivation was the study of obstructions to the existence of a Kähler-Einstein metric on Fano manifolds. The original idea of this notion goes back to the fundamental works of Bombieri [Bom70] and Skoda [Sko72b].

Let  $\varphi$  be a psh function over an open set  $\Omega \subset X$ ; The *multiplier ideal sheaf* associated to  $\varphi$ ,  $\mathcal{I}(\varphi) \subset \mathcal{O}_\Omega$  is defined by the germs of holomorphic functions  $f \in \mathcal{O}_{\Omega, X}$  such that  $|f|^2 e^{-\varphi}$  is integrable in the local neighborhood coordinates of  $X$ . The zero variety  $V(\mathcal{I}(\varphi))$  is therefore the set of points in a neighborhood of which  $e^{-\varphi}$  is non integrable. Such points appear only when  $\varphi$  has logarithmic poles. Concretely speaking, we say that a psh function  $\varphi$  has a logarithmic pole of coefficient  $\gamma$  at a point  $x \in X$  if the *Lelong number*

$$\nu(\varphi, x) := \liminf_{z \rightarrow x} \frac{\varphi(z)}{\log |z - x|}$$

is non zero and if  $\nu(\varphi, x) = \gamma$ . For example, the Lelong number  $\nu([A], x)$  of a current of integration over an analytic set  $A \subset X$  is equal to the multiplicity of this set at  $x$  (cf. [Thi67]). A lemma of Skoda [Sko 72a] establishes the relationship between the integrability of  $e^{-\varphi}$  near  $x$  and the Lelong number  $\nu(\varphi, x)$ .

**Lemma 1.7.** *Let  $\varphi$  be a psh function defined on an open set  $\Omega$  and let  $x \in \Omega$ .*

- (a) *If  $\nu(\varphi, x) < 1$ , then  $e^{-\varphi}$  is integrable in a neighborhood of  $x$ , in particular,  $\mathcal{I}(\varphi)_x = \mathcal{O}_{\Omega, x}$ .*
- (b) *If  $\nu(\varphi, x) \geq n + s$  for some integer  $s \geq 0$ , then  $e^{-\varphi} \geq C|z - x|^{-n-s}$  in a neighborhood of  $x$  and  $\mathcal{I}(\varphi)_x = \mathcal{M}_{\Omega, x}^{s+1}$ , where  $\mathcal{M}_{\Omega, x}$  is the maximal ideal of  $\mathcal{O}_{\Omega, x}$ .*
- (c) *The zero variety  $V(\mathcal{I}(\varphi))$  of  $\mathcal{I}(\varphi)$  satisfies*

$$E_n(\varphi) \subset V(\mathcal{I}(\varphi)) \subset E_1(\varphi)$$

where  $E_c(\varphi) = \{x \in X : \nu(\varphi, x) \geq c\}$  is the  $c$ -upperlevel set of Lelong numbers of  $\varphi$ .

An application of the Hörmander-Bombieri-Skoda theorem (cf. [Bom 70] [Sko 72a] [Sko 77]) yields a crucial result, namely that the set in a neighborhood of which  $e^{-\varphi}$  is not integrable is an analytic subset of the manifold  $X$ . In the seminal work [Siu 74], Siu proved an analogue result for the sets  $E_c(\varphi)$  (actually, his result is more general, it concerns  $(p, p)$ -currents)

**Corollary 1.8.** *If  $T$  is a closed positive current of type  $(p, p)$  over a manifold  $X$ , then the upperlevel sets  $E_c(T) = \{x \in X : \nu(T, x) \geq c\}$  are analytic subsets of dimension  $\leq p$ .*

Thus we see that the multiplier ideal sheaf measures in a very subtle way the size of the algebraic singularities of a metric. By a result equally due to Nadel (cf. [Nad90]), the sheaf  $\mathcal{I}(\varphi)$  is coherent. If moreover  $\Omega$  is a bounded Stein open set,  $\mathcal{I}(\varphi)$  is generated by a base of  $L^2$  Hilbert space  $\mathcal{H}^2(\Omega, \varphi)$  of holomorphic functions  $f$  over  $\Omega$  such that  $\int_\Omega |f|^2 e^{-\varphi} d\lambda < +\infty$ .

Let  $(L, h_L)$  be a hermitian line bundle, the metric  $h_L$  is allowed to be singular, but we assume that  $\Theta_{h_L}(L) \geq 0$  (so  $L$  is pseudo-effective). The local weights  $\varphi_j$  are psh functions which define the coherent sheaf  $\mathcal{I}(\varphi_j)$ , considering the transition relation, we have  $\mathcal{I}(\varphi_i)|_{U_i \cap U_j} = \mathcal{I}(\varphi_j)|_{U_i \cap U_j}$ . Then  $\mathcal{I}(\varphi_L)$  is a well defined coherent sheaf. The ideal sheaf  $\mathcal{I}(\varphi)$  allows us to formulate the following singular version of the Kodaira vanishing theorem, established by Nadel in [Nad90] (see also Kawamata-Viehweg vanishing theorem in the algebro-geometric context).

**Theorem 1.9.** *Let  $(X, \omega)$  be a weakly pseudoconvex Kähler manifold,  $L$  a line bundle over  $X$  equipped with a singular hermitian metric  $h_L$  with weights  $\varphi$ . If  $\sqrt{-1}\Theta_{h_L} \geq \varepsilon\omega$  holds for some continuous positive function  $\varepsilon$  on  $X$ , then we have*

$$H^q(X, \mathcal{O}(K_X + L) \otimes \mathcal{I}(\varphi)) = 0$$

for all  $q \geq 1$ .

The following results can be seen as consequences of the statement 1.9; they show in particular the strength of this result (cf. [Dem09]).

**Corollary 1.10.** *Let  $(X, \omega)$ ,  $L$  and  $\varphi$  be like in the vanishing theorem and  $x_1, \dots, x_N$  the isolated points in the zero variety  $V(\mathcal{I}(\varphi))$ . Then there exists a surjective map*

$$H^0(X, K_X + L) \rightarrow \bigoplus_{1 \leq j \leq N} \mathcal{O}(K_X + L)_{x_j} \otimes (\mathcal{O}_X / \mathcal{I}(\varphi))_{x_j}.$$

**Corollary 1.11.** *Let  $(X, \omega)$ ,  $L$  and  $\varphi$  be like in the vanishing theorem and the weight  $\varphi$  satisfies  $\nu(\varphi, x) \geq n + s$  at certain point  $x \in X$ , which is an isolated point of  $E_1(\varphi)$ . Then  $H^0(X, K_X + L)$  generates all  $s$ -jets at  $x$ .*

Sometimes it is useful to consider the following version of multiplier ideal sheafs :

$$\mathcal{I}_+(\varphi) = \lim_{\varepsilon \rightarrow 0_+} \mathcal{I}((1 + \varepsilon)\varphi).$$

This notion was established in [DP03]. Conjecturally, we have  $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$ .

## 1.3 Principal theorems

### 1.3.1 Statement of the main results

Before the announcement of our main theorems of this chapter, we begin by fixing some notations and presenting the general context in which we will work.

Let  $X$  be a compact Kähler manifold and let  $Z \subset X$  be the zero set of a holomorphic section  $s \in H^0(X, E)$  of a holomorphic line bundle  $(E, h_E)$  with  $h_E$  smooth. The hypersurface  $Z$  is assumed to be non-singular. Let  $L$  be a holomorphic line bundle, endowed with a possibly singular metric  $h_L$  such that :

- (1)  $\sqrt{-1}\Theta_{h_L}(L) \geq 0$  as a current on  $X$ ;
- (2)  $\sqrt{-1}\Theta_{h_L}(L) \geq \frac{1}{\alpha}\sqrt{-1}\Theta(E)$  for some  $\alpha \geq 1$ ,  $\alpha \in \mathcal{C}^\infty(X)$ ;
- (3)  $|s|_{h_E}^2 \leq \exp(-\alpha)$  on  $X$ , and the restriction of the metric  $h_L$  to  $Z$  is well defined.

As far as the singularities of the metric  $h_L$  are concerned, in algebraic geometry one usually deals with the following two special cases :

- (I) The metric  $h_L$  has logarithmic poles and is not identically equal to  $-\infty$  on  $Z$ . In the standard terminology, it means that  $h_L$  has local representatives that can be written as

$$\varphi_i = c \log \sum_j |f_j|^2 + \text{smooth term}$$

where  $f_j$  are holomorphic functions and  $c$  a positive constant. Moreover, we assume that the poles of  $h_L$  does not contain any component of  $Z$ ;

- (II) The multiplier ideal sheaf associated with the metric  $h_{L|Z}$  equals the structure sheaf  $\mathcal{O}_Z$ . This is the analytic version of the "klt" hypothesis in algebraic geometry.

In this context, our results are as follows :

**Theorem 1.12.** *Let  $X$  be a compact Kähler manifold, and let  $(L, h_L)$  be a holomorphic line bundle, such that the properties (1)-(3) above are satisfied. In addition, we assume that the metric  $h_L$  is under the hypothesis (I). Given a section*

$$u \in H^0(Z, (K_Z + L|_Z) \otimes \mathcal{I}(\varphi_{L|Z})),$$

there exists a holomorphic section  $U$  of  $K_X + Z + L$  such that  $U|_Z = u \wedge ds$  and

$$\int_X \frac{|U|^2 e^{-\varphi_L - \varphi_Z}}{|s|^2 \log^2(|s|^2)} \leq C_0 \int_Z |u|^2 e^{-\varphi_L}$$

where  $C_0$  is a numerical constant.

Concerning the second hypothesis, our result states as follows :

**Theorem 1.13.** *Let  $X$  and  $(L, h_L)$  be defined as in Theorem 1.12. In addition, we assume that the metric  $h_L$  is under the hypothesis (II), i.e.*

$$\int_Z e^{-\varphi_L} d\lambda_Z < \infty$$

with  $d\lambda_Z$  the Lebesgue measure on  $Z$ . Given a section

$$u \in H^0(Z, (K_Z + L|_Z)),$$

there exists a holomorphic section  $U$  of  $K_X + Z + L$  such that  $U|_Z = u \wedge ds$  and

$$\int_X \frac{|U|^2 e^{-\varphi_L - \varphi_E}}{|s|^2 \log^2(|s|^2)} \leq C_X \sup_X |u|^2 \int_Z e^{-\varphi_L} d\lambda_Z.$$

Here  $C_X$  is a constant depending on  $X$ .

**Remark 1.14.** *Theorem 1.12 (resp. Theorem 1.13) is still valid if we replace the holomorphic line bundle  $(E, h_E)$  in the setting by a holomorphic vector bundle  $(E, h_E)$  of rank  $r$ . In this case, the curvature assumptions (1) and (2) should be modified like in Theorem 1.3. The proof of Theorem 1.12 (resp. Theorem 1.13) is almost the same as in the line bundle case. To avoid falling into technically complicated calculations, without loss of generality, we treat only the case that  $E$  is a line bundle in the proof of our main theorems and discuss the general case in the appendix.*



**Remark 1.15.** *If the hypersurface  $Z$  is assumed to be singular, given a section*

$$u \in H^0(Z, (K_X + L + Z)|_Z \otimes \mathcal{I}(\varphi_L|_Z)),$$

*we assume that the integral*

$$\int_{Z_{\text{reg}}} \frac{|u|^2 e^{-\varphi_L}}{|ds|^2} < \infty$$

*is finite. Then the extension  $U \in K_X + L + Z$  of  $u$  satisfies, in case (I), the following estimate :*

$$\int_X \frac{|U|^2 e^{-\varphi_L - \varphi_E}}{|s|^2 \log^2(|s|^2)} \leq C_0 \int_{Z_{\text{reg}}} \frac{|u|^2 e^{-\varphi_L}}{|ds|^2}.$$

*Here,  $Z_{\text{reg}}$  denotes the regular part of  $Z$ .*

**Remark 1.16.** *In fact, our arguments go through under more general assumptions on the singularities of the metric  $h_L$ . We will discuss this at the end of the proof.*

### 1.3.2 Projective case

If  $X$  is projective, theorem 1.12 can be proved regardless of the type of singularities of  $h_L$ . What makes the difference is as follows : there exists a hyperplane section  $H$  which does not contain  $Z$ , and such that

$$X \setminus H = \bigcup_k \Omega_k$$

where  $(\Omega_k)$  is an increasing sequence of Stein sub-domains of  $X$ . The restriction of  $h_L$  to each  $\Omega_k$  can be written as the (decreasing) limit of  $C^\infty$  metrics  $h_{L,\rho}$ , in such a way that the curvature/normalization hypotheses in the main theorem above are preserved. This is precisely the reason for which the Stein property of  $\Omega_k$  is important : we can assume that  $h_L|_{\Omega_k}$  is non-singular, provided that the estimates we obtain for the norm of the extension are independent of  $k$ . In conclusion, we use the well-known version of Ohsawa-Takegoshi theorem on each  $\Omega_k$ , and then let  $k \rightarrow \infty$ . We remark here that there are 3 parameters involved in the proof :  $k$ ,  $\varepsilon$  (used to concentrate the mass on  $Z \cap \Omega_k$  (see [Ber96], [Siu02]), and  $\rho$  : they are completely independent, i.e. for each  $k$ , we are allowed to let first  $\varepsilon \rightarrow 0$ , and then  $\delta \rightarrow 0$ .

By contrast, if  $X$  is only assumed to be Kähler compact, then we do not have at our disposal the sequence  $(\Omega_k)$ . However, we can still regularize the metric  $h_L$  and obtain a sequence  $h_{L,\rho}$  by using the main result in [Dem94] (in this way we can compensate for the absence of the "large" coordinate systems used in the projective case). The important difference is that in the actual context, the curvature hypotheses (1) and (2) above are only verified up to a negative factor  $-\delta_\rho \omega$ , where  $\delta_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ . The occurrence of this small negative factor has a huge consequence : we cannot allow  $\varepsilon$  and  $\rho$  to be independent anymore, mainly because of the estimates we obtained as a consequence of the twisted Bochner identity.

Actually, during the proof, the curvature form gets multiplied with the function  $\log \frac{1}{\varepsilon^2 + |s|^2}$ ; as a conclusion, the loss of positivity is of order  $\delta_\rho \log \frac{1}{\varepsilon^2 + |s|^2}$ . The hypotheses (I) and (II) are needed to insure that a few limiting processes involved in our proof are justified.

## 1.4 Proof of the main theorem

We divide our arguments in several steps. In the first part of the proof, we will use an approximation theorem for quasi-psh functions, due to J.-P. Demailly (cf. [Dem94]). Then we

construct a smooth extension of the given section  $u$ , and we analyze its properties (norm of its  $\bar{\partial}$ , etc). Finally, we convert this extension to a holomorphic one, by solving a  $\bar{\partial}$  equation, using a twisted Bochner inequality. The two cases correspond to hypotheses (I) and (II) will be treated in parallel in each section.

### 1.4.1 Approximation of quasi-psh functions

The regularization procedure which will be realized in this paragraph is only needed for Theorem 1.13. We remark that if we assume that  $h_L$  has logarithmic poles ( for example, if  $h_L$  verifies (I)), then we can work directly on the manifold

$$M := X \setminus \{h_L = \infty\}.$$

Indeed, the manifold  $M$  is Kähler complete, hence we can use the standard  $L^2$  theory (cf. [Dem00]).

For the second case, the regularization result we need is the following (cf. [Dem94], page 18).

**Theorem 1.17.** *Let  $T := \alpha + \sqrt{-1}\partial\bar{\partial}\varphi$  be a closed  $(1,1)$ -current on a compact complex manifold  $X$ ; here  $\alpha$  is a closed  $(1,1)$ -form. We assume that  $\varphi$  is quasi-psh, and let  $\gamma$  be a continuous form of  $(1,1)$ -type, such that  $T \geq \gamma$ . Suppose that  $T_X$  is equipped with a smooth hermitian metric  $\omega$ . Then for each  $\rho > 0$ , there exists a function  $\varphi_\rho \in L^1(X)$  such that :*

- (i) *The function  $\varphi_\rho$  is smooth on  $X \setminus E_\rho(T)$ , where  $E_\rho(T)$  is the  $\rho$ -upper level set of Lelong numbers of  $T$ .*
- (ii) *We have  $T_\rho := \alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\rho \geq \gamma - \delta_\rho\omega$ , where  $\delta_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ .*
- (iii) *We have  $\varphi \leq \varphi_\rho$ , for each  $\rho > 0$ .*

One of the important aspects of the previous result is that the construction of  $\varphi_\rho$  is independent of  $\gamma$ , in the following sense. Given a quasi-psh function  $\varphi$  on  $X$ , the family of functions  $(\varphi_\rho)_{\rho>0}$  is obtained by using a regularizing kernel, followed by a Legendre transform. As we can see in the following(cf. [Dem94]), these operations are independent of the form  $\gamma$ ; in conclusion, given two forms  $\gamma_1$  and  $\gamma_2$  such that

$$T \geq \gamma_1, \quad T \geq \gamma_2.$$

We have

$$T_\rho \geq \gamma_j - \delta_\rho\omega$$

for each  $j = 1, 2$ . Concretely speaking, we obtain the current  $T_\rho$  as follows : we first construct a modified exponential map

$$T_X \rightarrow X \quad (x, \zeta) \rightarrow \text{exp}_x(\zeta), \quad \zeta \in T_{X,x}$$

from the tangent bundle of  $X$  to  $X$  with the property that for every  $x \in X$ , the map  $\zeta \rightarrow \text{exp}_x(\zeta)$  has a holomorphic Taylor expansion at  $\zeta = 0$ . Then we set  $\tilde{\varphi}_\rho(z)$  to be the convolution of  $\varphi(\text{exp}_x(\zeta))$  with a cut-off function  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  of class  $\mathcal{C}^\infty$  :

$$\tilde{\varphi}_\rho(z) = \frac{1}{\rho^{2n}} \int_{\zeta \in T_{X,z}} \varphi(\text{exp}_z(\zeta)) \chi\left(\frac{|\zeta|^2}{\rho^2}\right) d\lambda(\zeta), \quad \rho > 0.$$

Here the cut-off function  $\chi$  satisfies that

$$\chi(t) > 0 \quad \text{for } t < 1, \quad \chi(t) = 0 \quad \text{for } t \geq 1, \quad \int_{v \in \mathbb{C}^n} \chi(|v|^2) d\lambda(v) = 1$$

and  $d\lambda$  denotes the Lebesgue measure on the hermitian space  $(T_{X,z}, \omega(z))$  resp. on  $\mathbb{C}^n$ . Set then  $\Phi(z, w) = \tilde{\varphi}_\rho(z)$  for  $w \in \mathbb{C}$ ,  $|w| = \rho$  with

$$\Phi(z, w) = \int_{\zeta \in T_{X,z}} \varphi(\exp_z(w\zeta)) \chi(|\zeta|^2) d\lambda(\zeta).$$

If we change the variable  $y = \exp_z(w\zeta)$ , we can see that  $w\zeta$  is a smooth function of  $y, z$  in a neighborhood of the diagonal in  $X \times X$ . Hence  $\Phi$  is smooth over  $X \times \{0 < |w| < \rho_0\}$  for some  $\rho_0 > 0$ . The family of  $\varphi_\rho$  are then defined by the Legendre transform

$$\varphi_\rho = \inf_{|w| < 1} \left( \tilde{\Phi}(x, \rho w) + \frac{\rho}{1 - |w|^2} - \rho \log |w| \right),$$

where  $\tilde{\Phi}(x, w) = \Phi(x, w) + |w|$ . In conclusion, the construction above is independent of  $\gamma$ . We refer to [Dem94] for more details.

We apply these considerations to the current

$$T := \sqrt{-1} \Theta_{h_L}(L)$$

and formulate the conclusion as follows.

**Lemma 1.18.** *For each  $\rho > 0$ , there exists a metric  $h_{L,\rho} = e^{-\varphi_{L,\rho}}$  on  $L$  such that we have :*

$$\sqrt{-1} \Theta_{h_{L,\rho}}(L) \geq -\delta_\rho \omega, \quad \sqrt{-1} \Theta_{h_{L,\rho}}(L) \geq \frac{1}{\alpha} \sqrt{-1} \Theta(E) - \delta_\rho \omega.$$

Moreover, for each  $\rho > 0$ , there exists an analytic set  $Y_\rho \subset X$  such that  $\varphi_{L,\rho}$  is smooth on  $X \setminus Y_\rho$ .

Indeed, this is a consequence of the regularization theorem quoted above, together with the hypotheses (1) and (2) of the main theorem.

### 1.4.2 Construction of a $\mathcal{C}^\infty$ extension and its properties

We consider here a covering of  $X$  by a finite number of polydisks  $U_\alpha \subset X$ . For those  $U_\alpha$  such that  $U_\alpha \cap Z \neq \emptyset$ , fix local coordinates  $z_\alpha = (z_\alpha^1, \dots, z_\alpha^n)$  such that  $Z \cap U_\alpha = (z_\alpha^1 = 0)$ . Let  $e_\alpha$  be a non vanishing holomorphic section of  $\wedge^n T_X^* \otimes L \otimes Z|_{U_\alpha}$ . On  $Z \cap U_\alpha$ , we can write the form  $u$  as  $u = f_\alpha \otimes e_\alpha|_{Z \cap U_\alpha}$ , where  $f_\alpha$  is a holomorphic function on  $Z \cap U_\alpha$ .

In what follows, we will define a smooth extension of our function  $u$  according to the type of hypotheses we assume in the main theorems on the singularities of the metric  $h_L$ . We will treat separately two cases.

• **Case (I).**

In this case, by hypothesis, we have  $u \in \mathcal{I}(h_L|_Z)$  which in local terms translates as

$$\int_{Z \cap U_\alpha} |f_\alpha \otimes e_\alpha|^2 e^{-\varphi_L} < \infty.$$

Since  $h_L$  has logarithmic poles, we have

$$\int_{Z \cap U_\alpha} |f_\alpha \otimes e_\alpha|^2 e^{-(1+\varepsilon_0)\varphi_L} < \infty;$$

here  $\varepsilon_0$  is a small, positive real number.

By the version of Ohsawa-Takegoshi theorem in [Dem00], for each index  $\alpha$  there exists a holomorphic section  $\tilde{f}_\alpha$  defined on  $U_\alpha$ , such that  $\tilde{f}_\alpha|_{Z \cap U_\alpha} = f_\alpha \otimes e_\alpha$ , and that

$$\int_{U_\alpha} \frac{|\tilde{f}_\alpha|^2 e^{-(1+\varepsilon_0)\varphi_L}}{|s|^2 \log^2 \frac{1}{|s|^2}} \leq C_0 \int_{Z \cap U_\alpha} |f_\alpha \otimes e_\alpha|^2 e^{-(1+\varepsilon_0)\varphi_L} \quad (1.1)$$

where  $C_0$  above is a numerical constant.

Let  $(\theta_\alpha)$  be a partition of unity corresponding to the covering  $(U_\alpha)$ ; we define the section  $U_\infty$  of  $K_X + Z + L$  as follows

$$U_\infty := \sum_\alpha \theta_\alpha \tilde{f}_\alpha$$

where  $\tilde{f}_\alpha$  are local holomorphic sections of the bundle above, which are globalized via the partition of unity. Of course,  $U_\infty$  is only a  $\mathcal{C}^\infty$  section of  $K_X + Z + L$ .

We will use  $U_\infty$  in order to construct the holomorphic extension  $U$  required by the main theorem, but prior to this, we analyze next the  $L^2$  properties of  $\bar{\partial}U_\infty$ . To this end, we remark that for any index  $\beta$  we have

$$\begin{aligned} \bar{\partial}U_\infty|_{U_\beta} &= \sum_\alpha \tilde{f}_\alpha \bar{\partial}\theta_\alpha|_{U_\beta} = \\ &= \sum_\alpha (\tilde{f}_\alpha - \tilde{f}_\beta) \bar{\partial}\theta_\alpha|_{U_\beta}; \end{aligned} \quad (1.2)$$

here we use the fact that  $\sum \bar{\partial}\theta_\alpha = 0$ .

Let us consider an index  $\gamma$  such that  $U_\gamma \cap U_\beta \neq \emptyset$ . Then there exists a holomorphic section  $g_{\gamma\beta}$  defined on  $U_\gamma \cap U_\beta$  such that

$$\tilde{f}_\gamma - \tilde{f}_\beta = sg_{\gamma\beta} \quad (1.3)$$

since  $\tilde{f}_\gamma$  and  $\tilde{f}_\beta$  are both local extensions of  $u$ . Moreover, the next relation holds

$$\int_{U_\gamma \cap U_\beta} \frac{|g_{\gamma\beta}|^2 e^{-(1+\varepsilon_0)\varphi_L}}{\log^2 \frac{1}{|s|^2}} < \infty$$

as a consequence of (1.1) and (1.3).

We denote by  $\varepsilon$  a positive number; during the proof of the main theorem, we will have to evaluate an expression of the following type

$$I_\varepsilon := \int_{\{|s|^2 < \varepsilon^2\}} \frac{|\bar{\partial}U_\infty|_\omega^2 e^{-\varphi_L}}{|s|^2} dV_\omega; \quad (1.4)$$

as  $\varepsilon \rightarrow 0$ . We remark that up to a constant which is independent of  $\varepsilon$ , the quantity  $I_\varepsilon$  is bounded from above by a finite sum of integrals of the following type

$$\int_{U_\alpha \cap U_\beta \cap \{|s|^2 < \varepsilon^2\}} |g_{\alpha\beta}|^2 e^{-\varphi_L}. \quad (1.5)$$

By Hölder inequality the quantity (1.4) is smaller than

$$\begin{aligned} & \left( \int_{U_\alpha \cap U_\beta} \frac{|g_{\alpha\beta}|^2 e^{-(1+\varepsilon_0)\varphi_L}}{\log^2 \frac{1}{|s|^2}} \right)^{\frac{1}{1+\varepsilon_0}} \left( \int_{U_\alpha \cap U_\beta \cap \{|s|^2 < \varepsilon^2\}} |g_{\alpha\beta}|^2 \log^{\frac{2}{\varepsilon_0}} \frac{1}{|s|^2} \right)^{\frac{\varepsilon_0}{1+\varepsilon_0}} \leq \\ & \leq C \left( \varepsilon^2 \log^{\frac{2}{\varepsilon_0}} \frac{1}{\varepsilon^2} \right)^{\frac{\varepsilon_0}{1+\varepsilon_0}}. \end{aligned}$$

Hence we have

$$I_\varepsilon \leq C \left( \varepsilon^2 \log^{\frac{2}{\varepsilon_0}} \frac{1}{\varepsilon^2} \right)^{\frac{\varepsilon_0}{1+\varepsilon_0}} \quad (1.6)$$

with the constant  $C$  depending on the special choice of the covering and  $\varepsilon_0$ .

Therefore under the hypothesis (I), we have the following statement :

**Lemma 1.19.** *There exists a  $C^\infty$  extension  $U_\infty$  of  $u$  such that*

$$\int_{\{|s|^2 < \varepsilon^2\}} \frac{|\bar{\partial}U_\infty|_\omega^2 e^{-\varphi_L}}{|s|^2} dV_\omega \leq C \left( \varepsilon^2 \log^{\frac{2}{\varepsilon_0}} \frac{1}{\varepsilon^2} \right)^{\frac{\varepsilon_0}{1+\varepsilon_0}}$$

with the constant  $C$  depending on the special choice of the covering and  $\varepsilon_0$ .

• **Case (II).**

For the second case, that is, the multiplier ideal sheaf associated with the metric  $h_L$  is trivial when restricted to  $Z$ , we first make the following observation (cf. [AS95], [PS00]) : Denote  $\Omega := \mathbb{D} \times \Delta \subset \mathbb{C}^n$  the domain equals to the product of the unit disk  $\mathbb{D}$  in  $\mathbb{C}$  with the unit disk  $\Delta$  in  $\mathbb{C}^{n-1}$ . We consider a function  $\varphi \in \text{Psh}(\Omega)$  in an open set containing the closure of  $\Omega$  such that  $\sup_\Omega \varphi = 0$ .

**Lemma 1.20.** *Assume that we have*

$$\int_\Delta e^{-\varphi(0,z)} d\lambda(z) < \infty. \quad (1.2.1)$$

Then for any  $\delta > 0$ , the following inequality holds :

$$\int_{\Omega'_\delta} e^{-\varphi(t,z)} \frac{d\lambda(t,z)}{\delta^2} \leq C_1 \int_\Delta e^{-\varphi(0,z)} d\lambda(z)$$

where we use the notation  $\Omega'_\delta := \{|t| < \delta\} \times \Delta$  intersected with the polydisk of radius  $\frac{1}{2}$  and  $C_1$  is a numerical constant.

**Proof.** For each  $\delta$  positive, we consider the function

$$(t, z) \rightarrow \varphi(\delta t, z);$$

it is psh, and its restriction to  $t = 0$  is independent of  $\delta$ . By the classical version of the Ohsawa-Takegoshi theorem (cf. [OT87]), we can extend the function identically equal to 1 on the domain  $\{t = 0\} \times \Delta$ ; We get a function  $f_\delta$ , holomorphic on  $\Omega$ , together with the estimate

$$\int_{\Omega} |f_\delta|^2 e^{-\varphi(\delta t, z)} d\lambda \leq C_0 \int_{\Delta} e^{-\varphi(0, z)} d\lambda(z). \quad (1.2.2)$$

By the above inequality, we infer that on the polydisk with radius  $\frac{1}{2}$ , say  $\Omega'$ , the  $C^1$  norm of  $f_\delta$  is uniformly bounded and it is independent of  $\delta$ . Indeed, by Cauchy inequality, for any  $w_0 \in \Omega'$ , we have

$$|df_\delta(w_0)| \leq 2 \sup |f_\delta| \leq 2 \sup \left( \int_{\Omega} |f_\delta|^2 e^{-\varphi(\delta t, z)} d\lambda \right)^{\frac{1}{2}} \leq 2 \left( C_0 \int_{\Delta} e^{-\varphi(0, z)} d\lambda(z) \right)^{\frac{1}{2}} := I_{\Delta}(\varphi).$$

Since  $f_\delta$  is equal to 1 on  $t = 0$ , by the mean value theorem, we infer that

$$|f_\delta(t, z) - 1| \leq I_{\Delta}(\varphi)|t|.$$

We get

$$|f_\delta(t, z) - 1| \leq 1/2 \quad (1.2.3)$$

provided that  $|t| \leq \frac{1}{2I_{\Delta}(\varphi)}$ . By the inequality (1.2.3), we obtain

$$|f_\delta(t, z)| \geq 1/2.$$

Combined this with (1.2.2), we have the definite bound

$$\int_{\{|t| \leq 1/2I_{\Delta}(\varphi)\} \times \Delta} e^{-\varphi(\delta t, z)} d\lambda \leq 4C_0 \int_{\Delta} e^{-\varphi(0, z)} d\lambda(z).$$

Finally, by the change of variables, first let  $v = \delta t$  then  $w = 2I_{\Delta}(\varphi)v$ , we get

$$\int_{\Omega'_\delta} e^{-\varphi(t, z)} \frac{d\lambda(t, z)}{\delta^2} \leq C_1 I_{\Delta}(\varphi)^2$$

with  $C_1$  a numerical constant. □

We will use this lemma to estimate (1.5) later. For the moment, let us first construct  $U_\infty$  following the method in [Dem00]. That is, construct a smooth section

$$U_\infty \in \mathcal{C}^\infty(X, \wedge^{n,0} T_X^* \otimes L) = \mathcal{C}^\infty(X, \wedge^{0,0} T_X^* \otimes \wedge^n T_X^* \otimes L)$$

such that  $U_\infty$  coincides with  $u$  on  $Z$  and  $\bar{\partial} U_\infty = 0$  at every point of  $Z$ . To this end, we consider the same covering of  $X$  established at the beginning of this section together with the holomorphic retractions  $\rho_\alpha : U_\alpha \rightarrow Z \cap U_\alpha$  of  $U_\alpha$  onto  $Z \cap U_\alpha$ . Hence  $\tilde{f}_\alpha = (\rho_\alpha^* f_\alpha) \otimes e_\alpha$  is an arbitrary  $\bar{\partial}$ -closed extension of  $u$  to  $U_\alpha$ . Let  $\theta_\alpha = \mathcal{D}(U_\alpha)$  be a partition of unity such that  $\sum \theta_\alpha = 1$  on the neighborhood of  $Z$ . Then

$$U_\infty = \sum \theta_\alpha \tilde{f}_\alpha$$

is the smooth extension which satisfies

$$\bar{\partial}U_\infty = \sum \bar{\partial}\theta_\alpha \wedge \tilde{f}_\alpha.$$

Especially, on  $Z$ , we have

$$\bar{\partial}U_\infty = \sum \bar{\partial}\theta_\alpha \wedge u = 0.$$

For the estimate of (1.5), the above lemma tells us that

$$\int_{U_\alpha \cap U_\beta \cap \{|s|^2 < \varepsilon^2\}} e^{-\varphi_L} d\lambda \leq \varepsilon^2 \int_{U_\alpha \cap U_\beta \cap Z} e^{-\varphi_L} d\lambda.$$

Therefore, we have

$$\begin{aligned} \int_{U_\alpha \cap U_\beta \cap \{|s|^2 < \varepsilon^2\}} |g_{\alpha\beta}|^2 e^{-\varphi_L} &\leq \sup_{U_\alpha \cap U_\beta} |g_{\alpha\beta}|^2 \int_{U_\alpha \cap U_\beta \cap \{|s|^2 < \varepsilon^2\}} e^{-\varphi_L} \leq \\ &\leq \varepsilon^2 \sup_{U_\alpha \cap U_\beta} |g'_{\alpha\beta}|^2 |e_{\alpha\beta}|^2 \int_{U_\alpha \cap U_\beta \cap Z} e^{-\varphi_L} d\lambda. \end{aligned}$$

Here  $g_{\alpha\beta} = g'_{\alpha\beta} \otimes e_{\alpha\beta}$  where  $g'_{\alpha\beta}$  is a holomorphic function and  $e_{\alpha\beta}$  a non vanishing holomorphic section of  $\wedge^n T_X^* \otimes L \otimes Z|_{U_\alpha \cap U_\beta}$ .

For further reference, we formulate next the result we have obtained in this paragraph.

**Lemma 1.21.** *We suppose that under suitable coordinates, the hypersurface  $Z$  can be denoted by  $Z = \{z = (z_1, \dots, z_n) \in X, z_1 = 0\}$ . Then we have*

$$\frac{1}{\varepsilon^2} \int_{\{|z_1|^2 < \varepsilon^2\} \times z'} e^{-\varphi_L(z_1, z')} d\lambda(z_1, z') \leq C_2 \int_Z e^{-\varphi_L} d\lambda_Z$$

provided that

$$\int_Z e^{-\varphi_L} d\lambda_Z < \infty,$$

here  $z' = (z_2, \dots, z_n)$  and  $C_2$  is a numerical constant. In this case, the  $C^\infty$  extension  $U_\infty$  of  $u$  satisfies the inequality

$$\int_{\{|s|^2 < \varepsilon^2\}} \frac{|\bar{\partial}U_\infty|_\omega^2 e^{-\varphi_L}}{|s|^2} dV_\omega \leq \varepsilon^2 C_X \int_Z e^{-\varphi_L} d\lambda_Z$$

where  $C_X$  depends on  $X$  but independent of  $\varepsilon$ .

In the next paragraph, we follow the usual approach for the Ohsawa-Takegoshi type theorems by recalling the twisted Bochner formula in some details.

### 1.4.3 Twisted a-priori inequality

Let  $X$  be a complex manifold, endowed with a hermitian metric  $\omega$ . If  $z^1, \dots, z^n$  are local coordinates on some open set  $U \subset X$ , then the metric  $\omega$  viewed as a (1,1)-form can be expressed as

$$\omega = \sqrt{-1} \sum_{j,k} \omega_{j,\bar{k}} dz^j \wedge d\bar{z}^{\bar{k}}.$$

We are particularly interested in the tensor bundles  $\Lambda^{p,q}T_X^* := \Lambda^p T_X^* \otimes \Lambda^q \overline{T}_X^*$ ; we can use  $\omega$  to endow each of them with a hermitian structure (see e.g. [Dem82b], [MK71]).

More generally, let  $E \rightarrow X$  be a holomorphic vector bundle of rank  $r$  and let  $h$  be a hermitian metric on  $E$ . Then we have a metric induced on the space of  $(p, q)$ -forms with values in  $E$  with measurable coefficients and let us set

$$\|u\|^2 := \int_X |u|^2 dV_\omega;$$

we can equally define in a similar fashion an inner product

$$\langle\langle u, v \rangle\rangle := \int_X \langle u, v \rangle dV_\omega.$$

We obtain in this way a Hilbert space usually denoted by  $L^2(X, \Lambda^{p,q}T_X^* \otimes E)$ ; the space of smooth  $(p, q)$  forms with values in  $E$  and compact support is a dense subspace of the above Hilbert space, and will be denoted by  $\mathcal{D}(X, \Lambda^{p,q}T_X^* \otimes E)$ .

The holomorphic structure of  $E$  allows us to extend the  $\bar{\partial}$  to the space of sections of  $E$ ; the reason is that the transition functions of  $E$  belong to the kernel of  $\bar{\partial}$ .

In a similar fashion, we can define the  $\partial$  of a section (or more generally, of a  $(p, q)$  form) of a vector bundle whose transition functions are *anti-holomorphic*. This observation can be used to define an operator acting on the space of the  $(p, q)$  forms with values in a *holomorphic* vector bundle which is the analog of the full exterior derivative  $d$ .

Indeed, given a holomorphic hermitian bundle  $(E, h)$ , we have a metric identification  $\rho : E \rightarrow \overline{E}^*$ , defined by  $\rho(v)(\bar{w}) := \langle v, w \rangle_h$  and we can use this identification in order to define the next differential operator :

$$\partial_h(u) := \rho^{-1} \partial (\rho(u)).$$

In local coordinates, the operator  $D'$  can be described as follows. If  $(e_\alpha)_{\alpha=1, \dots, r}$  is a holomorphic local frame of  $E|_\Omega$ , let us define

$$h_{\alpha\bar{\beta}} := \langle e_\alpha, e_\beta \rangle.$$

The section  $u$  can be written locally as  $u = \sum_\alpha u^\alpha e_\alpha$  and then

$$\rho(u) = \sum u^\alpha h_{\alpha\bar{\beta}} e^{*\bar{\beta}}$$

where the  $(e^{*\bar{\beta}})$  is the induced frame on  $\overline{E}^*$ . Next we compute

$$\partial \rho(u) = \sum (\partial u^\alpha h_{\alpha\bar{\beta}} + u^\alpha \partial h_{\alpha\bar{\beta}}) e^{*\bar{\beta}}$$

and finally we obtain

$$\begin{aligned} \partial_h(u) &= \sum h^{\bar{\beta}\gamma} (\partial u^\alpha h_{\alpha\bar{\beta}} + u^\alpha \partial h_{\alpha\bar{\beta}}) e_\gamma = \\ &= \sum (\partial u^\alpha + u^\alpha \partial h_{\alpha\bar{\beta}} h^{\bar{\beta}\gamma}) e_\gamma. \end{aligned}$$

Of course, this is nothing but the  $(1, 0)$  part of the Chern connection

$$D : \mathcal{C}^\infty(X, E) \rightarrow \mathcal{C}_1^\infty(X, E)$$

which can be extended as a linear differential operator of order 1 on the space of  $(p, q)$  forms with values in  $E$ . Maybe the most familiar characterization of it is as follows :



- • We have the decomposition  $D = \partial_h + \bar{\partial}$  in types  $(1, 0)$  and respectively  $(0, 1)$ ; the  $(0, 1)$  part of  $D$  is the extension of the usual  $\bar{\partial}$  operator;
- ••  $D$  is compatible with the metric  $h$ .

Unlike the operators  $\bar{\partial}$ ,  $\partial$  acting on  $(p, q)$  forms, the components of  $D$  do not commute in general. We measure the non commutativity defect by the *curvature* tensor, as follows.

We denote by  $[A, B] := AB - (-1)^{\deg A \deg B} BA$  the graded commutator bracket of operators  $A, B$ ; then it turns out that given  $(E, h)$  a hermitian holomorphic vector bundle the commutator

$$\sqrt{-1}\Theta_h(E) := [\partial_h, \bar{\partial}] \in \mathcal{C}^\infty(X, \Lambda^{1,1}T_X^* \otimes \text{Hom}(E, E))$$

is of order zero and it is called the curvature tensor of  $(E, h)$ .

We define the operator  $L_\omega : \mathcal{C}^\infty(X, \Lambda^{p,q}T_X^* \otimes E) \rightarrow \mathcal{C}^\infty(X, \Lambda^{p+1,q+1}T_X^* \otimes E)$  as follows

$$Lu := \omega \wedge u$$

and its adjoint is denoted by  $\Lambda_\omega$ . It is easy to verify that with respect to some local trivialization of  $E$ , the expression of the adjoint operator is

$$(\Lambda_\omega u)_{I\bar{J}}^\alpha = -\sqrt{-1} \sum_{i,j} \omega^{\bar{j}i} u_{i\bar{j}I\bar{J}}^\alpha$$

where  $|I| = p - 1, |J| = q - 1$  and the components  $(u_{K\bar{L}}^\alpha)$  of  $u$  are assumed to be skew-symmetric with respect to the ordered multi-index  $K, L$ .

Now the operators  $\partial_h$  and  $\bar{\partial}$  can be viewed as closed and densely defined operators on the Hilbert space  $L^2(X, \Lambda^{p,q}T_X^* \otimes E)$ . Their *formal* adjoints  $\partial_h^*$  and  $\bar{\partial}^*$  are densely defined and they admit extensions in the sense of distributions. Remark that in general they differ from the Hilbert space adjoints in the sense of von Neumann, as simple examples show. However, it is well known that if the Kähler metric  $\omega$  is geodesically complete, then the two operators coincide. We will use this fundamental fact in a moment, in the context of the pseudoconvex domains.

Next we recall the fundamental Bochner-Kodaira-Nakano identity ([Dem94]); the context in which we will use is the following.

We are given a Kähler weakly pseudoconvex manifold  $(X, \omega)$  and a holomorphic line bundle  $(L, h)$ . Then for any  $(n, 1)$ -form with compact support  $u \in \mathcal{D}(X, \Lambda^{n,1}T_X^* \otimes L)$  we have the next relation

$$\int_X |\bar{\partial}u|_h^2 dV_\omega + \int_X |\bar{\partial}^*u|_h^2 dV_\omega = \int_X |\partial_h^*u|_h^2 dV_\omega + \int_X \langle [\sqrt{-1}\Theta_h(L), \Lambda_\omega]u, u \rangle_h dV_\omega.$$

Since this relation holds true for any metric  $h$  on  $L$ , we analyze next how does it change when we consider a new metric  $h_1 := \eta h$ , where  $\eta > 0$  is a smooth strictly positive function on  $X$ . We have

$$\begin{aligned} \int_X |\bar{\partial}u|_h^2 \eta dV_\omega + \int_X |\bar{\partial}^*u|_h^2 \eta dV_\omega &= \int_X |\partial_\eta^*u|_h^2 \eta dV_\omega + \\ &+ \int_X \langle [\sqrt{-1}\Theta_h(L) - \sqrt{-1}\partial\bar{\partial}\log\eta, \Lambda_\omega]u, u \rangle_h \eta dV_\omega. \end{aligned}$$

We use the identity

$$\partial\bar{\partial}\log\eta = \frac{\partial\bar{\partial}\eta}{\eta} - \frac{\partial\eta \wedge \bar{\partial}\eta}{\eta^2}$$

and then the curvature term above becomes

$$\int_X \langle [\eta\sqrt{-1}\Theta_h(L) - \sqrt{-1}\partial\bar{\partial}\eta + \sqrt{-1}\frac{\partial\eta \wedge \bar{\partial}\eta}{\eta}, \Lambda_\omega]u, u \rangle_h dV_\omega.$$

The term  $\int_X |\partial_\eta^* u|_h^2 \eta dV_\omega$  is non-negative, therefore good enough for our future purposes; let us expand the term  $\int_X |\bar{\partial}^* u|_h^2 \eta dV_\omega$  (the reason is that later on we will choose  $\eta$  converging to some function with logarithmic poles and we have to control in a very precise manner what will happen with this term under the convergence procedure). For any  $u \in \text{Dom}(\bar{\partial}^*)$  and  $v$  with compact support we have

$$\begin{aligned} \int_X \langle \bar{\partial}^* u, v \rangle_h \eta dV_\omega &= \int_X \langle u, \bar{\partial}v \rangle_h \eta dV_\omega = \\ &= \int_X \langle u, \bar{\partial}(\eta v) \rangle_h dV_\omega - \int_X \langle u, \bar{\partial}\eta \wedge v \rangle_h dV_\omega = \\ &= \int_X \langle \bar{\partial}^* u, v \rangle_h \eta dV_\omega - \int_X \langle (\bar{\partial}\eta)^* u, v \rangle_h dV_\omega. \end{aligned}$$

In conclusion, the formal adjoint operator corresponding to the twisted metric  $h_1$  is equal to

$$\bar{\partial}^{*\eta} u = \bar{\partial}^* u - 1/\eta(\bar{\partial}\eta)^* u$$

and let us now compute the corresponding term in the Bochner formula

$$\begin{aligned} \int_X |\bar{\partial}^{*\eta} u|_h^2 \eta dV_\omega &= \int_X |\bar{\partial}^* u - 1/\eta(\bar{\partial}\eta)^* u|_h^2 \eta dV_\omega = \\ &= \int_X |\bar{\partial}^* u|_h^2 \eta dV_\omega + \int_X \frac{1}{\eta} |(\bar{\partial}\eta)^* u|_h^2 dV_\omega - 2\text{Re} \int_X \langle \bar{\partial}^* u, (\bar{\partial}\eta)^* u \rangle_h dV_\omega. \end{aligned}$$

We claim next that we have the following identity

$$|(\bar{\partial}\eta)^* u|_h^2 = \langle [\sqrt{-1}\partial\eta \wedge \bar{\partial}\eta, \Lambda_\omega]u, u \rangle_h \quad (1.7)$$

at each point of  $X$ . Indeed, let us check it by a computation in local coordinates. We take  $x \in X$  an arbitrary point and let  $(z^j)$  be local coordinates on some open set  $\Omega$  centered at  $x$ , which are geodesics for the Kähler metric  $\omega$  at this point. Locally we have

$$u = \sum_j u_{\bar{j}} dz \wedge dz^{\bar{j}} \otimes e_\Omega$$

where  $dz := dz^1 \wedge \dots \wedge dz^n$  and  $e_\Omega$  is a local holomorphic frame of  $L$ . Then by the definition of  $(\bar{\partial}\eta)^*$  we have

$$(\bar{\partial}\eta)^*(u) = (-1)^n \sum_j u_{\bar{j}} \frac{\partial\eta}{\partial z^j} dz \otimes e_\Omega$$

at  $x$ , and therefore we get

$$|(\bar{\partial}\eta)^* u|_h^2 = \sum_{j,k} u_{\bar{j}} \bar{u}_{\bar{k}} \frac{\partial\eta}{\partial z^j} \frac{\partial\eta}{\partial z^{\bar{k}}} \exp(-\varphi_L(x)). \quad (1.8)$$

On the other hand, the expression of the contraction operator  $\Lambda_\omega$  on  $u$  reads as

$$\Lambda_\omega(u) = \sqrt{-1} \sum_j (-1)^{n+j+1} u_{\bar{j}} dz^1 \wedge \widehat{dz^j} \wedge dz^n \otimes e_\Omega$$

and we have

$$\begin{aligned} \sqrt{-1} \partial \eta \wedge \bar{\partial} \eta \wedge \Lambda_\omega(u) &= \sum_{j,p,q} (-1)^{n+j} u_{\bar{j}} \frac{\partial \eta}{\partial z^p} \frac{\partial \eta}{\partial z^q} dz^p \wedge dz^q \wedge dz^1 \wedge \widehat{dz^j} \wedge dz^n \otimes e_\Omega = \\ &= \sum_{j,k} u_{\bar{j}} \frac{\partial \eta}{\partial z^j} \frac{\partial \eta}{\partial z^k} dz \wedge dz^{\bar{k}} \otimes e_\Omega. \end{aligned}$$

The relation (1.7) follows, as by the previous identity we have

$$\langle [\sqrt{-1} \partial \eta \wedge \bar{\partial} \eta, \Lambda_\omega] u, u \rangle_h = \sum_{j,k} u_{\bar{j}} \bar{u}_k \frac{\partial \eta}{\partial z^j} \frac{\partial \eta}{\partial z^k} \exp(-\varphi_L(x)). \quad (1.9)$$

Summing up what we have obtained so far, the twisted Bochner identity becomes

$$\begin{aligned} \int_X |\bar{\partial} u|_h^2 \eta dV_\omega + \int_X |\bar{\partial}^* u|_h^2 \eta dV_\omega &= \int_X |\partial_\eta^* u|_h^2 \eta dV_\omega + \\ &+ \int_X \langle [\eta \sqrt{-1} \Theta_h(L) - \sqrt{-1} \partial \bar{\partial} \eta, \Lambda_\omega] u, u \rangle_h dV_\omega + \\ &+ 2\text{Re} \int_X \langle \bar{\partial}^* u, (\bar{\partial} \eta)^* u \rangle_h dV_\omega. \end{aligned}$$

Let  $\lambda$  be another positive function on  $X$ ; by completing the square in the above identity, that is to say, when added two terms  $\int_X \lambda |\bar{\partial}^* u|_h^2 dV_\omega$  and  $\int_X \frac{1}{\lambda} |(\bar{\partial} \eta)^* u|^2 dV_\omega$  to the term  $2\text{Re} \int_X \langle \bar{\partial}^* u, (\bar{\partial} \eta)^* u \rangle_h dV_\omega$  gives the integral of a square, and neglecting the positive terms in the right hand side term above, we have proved the next inequality (see e.g. [Kim08], [Koz11], [MV07], [OT87]).

**Lemma 1.22.** *Let  $\eta, \lambda$  be smooth functions on  $X$ , and let  $u \in \mathcal{D}(X, \Lambda^{n,1} T_X^* \otimes L)$ . Then we have the next inequality*

$$\begin{aligned} \int_X |\bar{\partial} u|_h^2 \eta dV_\omega + \int_X (\eta + \lambda) |\bar{\partial}^* u|_h^2 dV_\omega &\geq \\ &\geq \int_X \langle [\eta \sqrt{-1} \Theta_h(L) - \sqrt{-1} \partial \bar{\partial} \eta - \sqrt{-1} \frac{\partial \eta \wedge \bar{\partial} \eta}{\lambda}, \Lambda_\omega] u, u \rangle_h dV_\omega. \end{aligned} \quad (1.10)$$

#### 1.4.4 Functional analysis

We consider the modified  $\bar{\partial}$  operators

$$Tu := \bar{\partial} \left( \sqrt{\eta + \lambda} u \right) \quad \text{and} \quad Su := \sqrt{\eta} (\bar{\partial} u)$$

acting on  $(p, q)$ -forms with values in a bundle  $F$ . They are densely defined, and we obviously have  $S \circ T = 0$ .

Then the previous lemma can be reformulated as follows : for any  $(n, 1)$ -form  $u$  with values in a line bundle  $(F, h)$  which belongs to the domains of  $S$  and  $T^*$  we have

$$\|T^*u\|^2 + \|Su\|^2 \geq \langle [\eta\sqrt{-1}\Theta_h(F) - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}\frac{\partial\eta \wedge \bar{\partial}\eta}{\lambda}, \Lambda_\omega]u, u \rangle. \quad (1.11)$$

We stress on the fact that this was only proved for compactly supported forms  $u$ ; in general, we use a standard density argument—remark that it is at this point we need the fact that  $X$  carries a complete Kähler metric (see [Dem82b]).

Now assume that for some specific choice of the functions  $\eta$  and  $\lambda$  above we have

$$\eta\sqrt{-1}\Theta_h(F) - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}\frac{\partial\eta \wedge \bar{\partial}\eta}{\lambda} \geq \sqrt{-1}\tau\partial\mu \wedge \bar{\partial}\mu - \delta\omega \quad (1.12)$$

on  $X$ ; here  $\tau$  is a positive function,  $\mu$  is an arbitrary one, and  $\delta \geq 0$  is a real number. Then the curvature term in the twisted Bochner inequality above verifies the next relation

$$\langle [\eta\sqrt{-1}\Theta_h(F) - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}\frac{\partial\eta \wedge \bar{\partial}\eta}{\lambda}, \Lambda_\omega]u, u \rangle \geq \|\sqrt{\tau}(\bar{\partial}\mu)^*u\|^2 - \delta\|u\|^2$$

and therefore the relation (1.11) becomes

$$\|T^*u\|^2 + \|Su\|^2 + \delta\|u\|^2 \geq \|\sqrt{\tau}(\bar{\partial}\mu)^*u\|^2. \quad (1.13)$$

As usual, we want to solve the equation  $Tv = g$  for some closed  $(n, 1)$ -form  $g$  with values in  $F := Z + L$ , such that  $Sg = 0$ . Given the lower bound (1.13) above (the corresponding operator is *never* positively definite, even if  $\delta = 0$ ), we cannot hope to solve the previous equation for all  $g$  in the kernel of  $\bar{\partial}$ , but nevertheless let us consider  $g = \bar{\partial}\tau \wedge g_0 + g_1$ , where  $g_0$  is an  $L^2$  form of  $(n, 0)$  type, and  $g_1$  is a  $(n, 1)$ -form.

Then if  $\delta > 0$ , we have

$$\begin{aligned} \frac{1}{2} \left| \int_X \langle g, u \rangle_h dV_\omega \right|^2 &\leq \left| \int_X \langle \bar{\partial}\tau \wedge g_0, u \rangle_h dV_\omega \right|^2 + \left| \int_X \langle g_1, u \rangle_h dV_\omega \right|^2 = \\ &= \left| \int_X \langle g_0, (\bar{\partial}\tau)^*u \rangle_h dV_\omega \right|^2 + \left| \int_X \langle g_1, u \rangle_h dV_\omega \right|^2 \leq \\ &\leq \|\sqrt{\tau}(\bar{\partial}\tau)^*u\|^2 \int_X 1/\tau |g_0|^2 dV_\omega + \delta\|u\|^2 \cdot \frac{1}{\delta} \|g_1\|^2 \leq \\ &\leq C(g, \tau, \delta) (\|T^*u\|^2 + \delta\|u\|^2); \end{aligned} \quad (1.14)$$

here we use the standard trick—see [Ber96], [Hör65]—in order to avoid the term  $\|Su\|^2$  in the above inequality, we decompose  $u$  into two parts :  $u_1$  and  $u_2$ , where  $u_1$  belongs to the kernel of  $\bar{\partial}$  while  $u_2$  is in the orthogonal complement space. So  $\|Su_1\|^2 = 0$ . The inner product  $\langle g, u \rangle_h$  in the above estimate is then in fact  $\langle g, u_1 \rangle$  since  $g$  is in the kernel of  $\bar{\partial}$ . We have also employed the notation

$$C(g, \tau, \delta) := \int_X 1/\tau |g_0|^2 e^{-\varphi_F} + \frac{1}{\delta} \int_X |g_1|_\omega^2 e^{-\varphi_F} dV_\omega.$$

In conclusion, under the precise circumstances described in this paragraph the map

$$(T^*u, \sqrt{\delta}u) \rightarrow \langle g, u \rangle$$

is continuous, hence we can solve the approximate  $\bar{\partial}$  equation

$$Tv + \sqrt{\delta}w = \bar{\partial}\tau \wedge g_0 + g_1$$

together with the estimate

$$\int_X |v|^2 e^{-\varphi_F} + \int_X |w|_\omega^2 e^{-\varphi_F} dV_\omega \leq C(g, \tau, \delta). \quad (1.15)$$

If  $\delta = 0$ , we have

$$\begin{aligned} \frac{1}{2} \left| \int_X \langle g, u \rangle_h dV_\omega \right|^2 &\leq \left| \int_X \langle \bar{\partial}\tau \wedge g_0, u \rangle_h dV_\omega \right|^2 + \left| \int_X \langle g_1, u \rangle_h dV_\omega \right|^2 = \\ &= \left| \int_X \langle g_0, (\bar{\partial}\tau)^* u \rangle_h dV_\omega \right|^2 + \left| \int_X \langle g_1, u \rangle_h dV_\omega \right|^2 \leq \\ &\leq \|\sqrt{\tau}(\bar{\partial}\tau)^* u\|^2 \int_X 1/\tau |g_0|^2 dV_\omega + \|u\|^2 \cdot \|g_1\|^2 \leq \\ &\leq C(g_0, \tau) \left( \|T^* u\|^2 + \frac{\|g_1\|^2}{C(g_0, \tau)} \|u\|^2 \right); \end{aligned} \quad (1.14')$$

Denote  $C(g_0, g_1, \tau) := \frac{\|g_1\|}{\sqrt{C(g_0, \tau)}}$ , we see that the map

$$(T^* u, C(g_0, g_1, \tau)u) \rightarrow \langle g, u \rangle$$

is continuous and therefore we can solve the approximate  $\bar{\partial}$  equation

$$Tv + C(g_0, g_1, \tau)w = \bar{\partial}\tau \wedge g_0 + g_1$$

together with the estimate

$$\int_X |v|^2 e^{-\varphi_F} + \int_X |w|_\omega^2 e^{-\varphi_F} dV_\omega \leq C(g_0, \tau),$$

where

$$C(g_0, \tau) = \int_X 1/\tau |g_0|^2 e^{-\varphi_F} dV_\omega.$$

We remark that all the computations/considerations above are done under the assumption that the metric  $\omega$  is complete.

### 1.4.5 End of the proof

Since we want the final estimates of the extension to depend on the restriction of  $U_\infty$  to  $Z$  only, it is natural to truncate the section  $U_\infty$  as follows : let  $\theta : \mathbb{R} \rightarrow [0, 1]$  be a smooth function with support in  $]-\infty, 1[$  such that  $\theta \equiv 1$  on the interval  $]-\infty, 1/2]$ , and  $\sup_{\mathbb{R}} |\theta'| \leq 4$ . Then we define

$$g_\varepsilon := \bar{\partial}(\theta(|s|^2/\varepsilon^2) U_\infty).$$

It is an  $(n, 1)$ -form with values in  $F := E + L$ , which is  $L^2$  with respect to the original metric  $h_L$  of the bundle  $L$ , twisted with the singular weight function  $\log |s|^2$  (we will verify this

assertion in a moment). Let  $\chi_0$  be any function on  $\mathbb{R}_-$  whose derivative is strictly positive in absolute value (we will make a specific choice later). We compute

$$\begin{aligned} g_\varepsilon &= \frac{1}{\varepsilon^2} \theta' \left( \frac{|s|^2}{\varepsilon^2} \right) \bar{\partial} |s|^2 \wedge U_\infty + \theta (|s|^2/\varepsilon^2) \bar{\partial} U_\infty = \\ &= \left( 1 + \frac{|s|^2}{\varepsilon^2} \right) \frac{\theta' \left( \frac{|s|^2}{\varepsilon^2} \right)}{\chi'_0(\log(\varepsilon^2 + |s|^2))} \bar{\partial} (\chi_0(\log(\varepsilon^2 + |s|^2))) \wedge U_\infty + \theta (|s|^2/\varepsilon^2) \bar{\partial} U_\infty := \\ &= \bar{\partial} \eta_\varepsilon \wedge g_{\varepsilon,0} + g_{\varepsilon,1}. \end{aligned}$$

The motivation for introducing the logarithmic term above is that in this way the function in front of the  $\bar{\partial}$  in the second line is bounded; besides, it is coherent with the "functional analysis" discussion above. In the last line of the above equality we have used the function

$$\eta_\varepsilon := -\chi_0(\log(\varepsilon^2 + |s|^2)) \quad (1.16)$$

and the  $(n,0)$ -form  $g_{\varepsilon,0}$  which is essentially  $U_\infty$  up to a bounded function.

According to the previous section 1.4.4 (see especially the relations (1.12), (1.13)), the lower bound we have to obtain for the curvature term in (1.12) must be something of the form  $\tau \sqrt{-1} \partial \bar{\partial} \eta_\varepsilon \wedge \bar{\partial} \eta_\varepsilon$ , so we are going to expand the term

$$\eta_\varepsilon (\sqrt{-1} \Theta_h(F) + \frac{\sqrt{-1}}{\pi} \partial \bar{\partial} \log |s|^2) - \sqrt{-1} \partial \bar{\partial} \eta_\varepsilon$$

and choose an appropriate function  $\lambda_\varepsilon$  in order to obtain the correct lower bound of the curvature.

To this end, we define the function

$$\sigma_\varepsilon := \log(\varepsilon^2 + |s|^2);$$

then a straightforward computation shows that

$$\sqrt{-1} \partial \bar{\partial} \sigma_\varepsilon \geq \sqrt{-1} \frac{\varepsilon^2}{|s|^2} \partial \sigma_\varepsilon \wedge \bar{\partial} \sigma_\varepsilon - \frac{\langle \sqrt{-1} \Theta(E) s, s \rangle}{\varepsilon^2 + |s|^2} \quad (1.17)$$

and we clearly have

$$- \partial \bar{\partial} \eta_\varepsilon = \chi'_0(\sigma_\varepsilon) \partial \bar{\partial} \sigma_\varepsilon + \chi''_0(\sigma_\varepsilon) \partial \sigma_\varepsilon \wedge \bar{\partial} \sigma_\varepsilon. \quad (1.18)$$

We want to use the relation (1.17) in order to get a lower bound for the hessian of  $-\eta_\varepsilon$ , so we need some definite positive bounds for the derivative of  $\chi_0$ ; we assume  $1 \leq \chi'_0 \leq 2$ . Then we have

$$- \sqrt{-1} \partial \bar{\partial} \eta_\varepsilon \geq \left( \frac{\varepsilon^2}{2|s|^2} + \frac{\chi''_0(\sigma_\varepsilon)}{\chi'_0(\sigma_\varepsilon)^2} \right) \sqrt{-1} \partial \eta_\varepsilon \wedge \bar{\partial} \eta_\varepsilon - \frac{\chi'_0(\sigma_\varepsilon)}{\varepsilon^2 + |s|^2} \langle \sqrt{-1} \Theta(E) s, s \rangle. \quad (1.19)$$

• **End of the proof for the case (I).**

We now end up with the first case, recall here the curvature properties of  $h_L$  in this case :

– •  $\sqrt{-1} \Theta_{h_{L,p}}(L) \geq 0;$

– •  $\sqrt{-1}\Theta_{h_{L,\rho}}(L) \geq \frac{1}{\alpha}\sqrt{-1}\Theta(E)$  for some  $\alpha \geq 1$ .

By the normalization condition of the section  $s$  we have  $\eta_\varepsilon \geq 2\alpha$ ; in particular we get  $\eta_\varepsilon \geq \frac{\alpha\chi'_0(\sigma_\varepsilon)|s|^2}{\varepsilon^2 + |s|^2}$  and therefore we obtain the inequality

$$\begin{aligned} \eta_\varepsilon(\sqrt{-1}\Theta_{h_\rho}(L) + \sqrt{-1}\Theta_h(E) + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|s|^2) &\geq \\ &\geq \frac{\chi'_0(\sigma_\varepsilon)}{\varepsilon^2 + |s|^2}\sqrt{-1}\Theta(E) \end{aligned} \quad (1.20)$$

(remark that we have used at this point both curvature assumptions above).

By the relation (1.18) combined with (1.20), it is clear that the ideal candidate for the function  $\lambda_\varepsilon$  would be

$$\lambda_\varepsilon := \frac{\chi'_0(\sigma_\varepsilon)^2}{\chi''_0(\sigma_\varepsilon)}$$

provided that the denominator is non-zero. A choice of function  $\chi_0$  which will be convenient for our purpose is as in [Dem00]

$$\chi_0(t) := t - \log(1 - t) \quad (1.21)$$

(we refer to the article [Man93] for a very complete treatment of other possible choices of  $\chi_0$ ).

By collecting all the relations above we get

$$\begin{aligned} \eta_\varepsilon(\sqrt{-1}\Theta_{h_\rho}(L) + \sqrt{-1}\Theta_h(E) + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|s|^2) - \\ - \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon - \frac{\sqrt{-1}}{\lambda_\varepsilon}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon \geq \frac{\varepsilon^2}{2|s|^2}\sqrt{-1}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon. \end{aligned} \quad (1.22)$$

The above inequality holds on  $X' := X \setminus (E(\sqrt{-1}\Theta_{h_L}(L)) \cup Z)$ , but it involves the metric  $\omega$  which is not complete on  $X'$ . We remark that the completeness of the metric is necessary in order to use the results obtained in the functional analysis paragraph 1.4.4, hence we will use next the following lemma (cf. [Dem82b]) :

**Lemma 1.23.** *Let  $(X, \omega)$  be a compact Kähler manifold, and let  $W \subset X$  be an analytic subset. Then  $X \setminus W$  carries a complete Kähler metric.*

Let  $\omega'$  be a complete metric on  $X'$ ; for each  $k \geq 1$ , we define the metric

$$\omega_k := \omega + \frac{1}{k}\omega'.$$

It is equally complete on  $X'$ , and we have  $\omega_k > \omega$ .

We see that all the conditions corresponding to  $\delta = 0$  in the "Functional analysis" subsection 1.4.4 are fulfilled and as a consequence, for each  $\varepsilon > 0$  we can solve the following approximate  $\bar{\partial}$  equation :

$$\bar{\partial}\left(\sqrt{\eta_\varepsilon + \lambda_\varepsilon v_{\varepsilon,k}}\right) + C(\varepsilon, k)w_{\varepsilon,k} = \bar{\partial}\eta_\varepsilon \wedge g_{\varepsilon,0} + g_{\varepsilon,1};$$

where

$$C(\varepsilon, k) = \frac{\int_X \theta\left(\frac{|s|^2}{\varepsilon^2}\right)^2 \frac{|\bar{\partial}U_\infty|^2 \omega_k e^{-\varphi_L - \varphi_E} dV_{\omega_k}}{|s|^2}}{\left(\int_X \theta'\left(\frac{|s|^2}{\varepsilon^2}\right)^2 \frac{|U_\infty|^2 e^{-\varphi_L - \varphi_E}}{|s|^2}\right)^{1/2}};$$

in addition, we have the estimates

$$\begin{aligned} \int_X \frac{|v_{\varepsilon,k}|^2}{|s|^2} e^{-\varphi_L - \varphi_E} + \int_X \frac{|w_{\varepsilon,k}|_{\omega_k}^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_{\omega_k} &\leq \\ &\leq C \int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_L - \varphi_E}. \end{aligned} \quad (1.23)$$

We intend to remove the dependence on  $k$  from the relation (1.23), by extracting a limit, as follows (the procedure is completely similar to the one employed in [Dem82b]) :

For each  $\varepsilon > 0$  fixed, the  $(n, 0)$ -forms  $(v_{\varepsilon,k})$  are defined on the manifold  $X'$  and bounded in  $L^2$  norms with respect to the metric  $\varphi_L + \log |s|^2$  by a constant which is independent of  $k$ . Therefore we can extract a weak limit, say

$$v_\varepsilon \in L^2(X')$$

from the sequence  $(v_{\varepsilon,k})$ .

The arguments we will use in order to derive a similar conclusion for the sequence of  $(n, 1)$ -forms  $(w_{\varepsilon,k})$  are slightly more technical. Indeed, we need the following lemma :

**Lemma 1.24.** *Let  $\omega_1 \leq \omega_2$  be two Kähler metrics and let  $\|\cdot\|_{1,2}$  be the corresponding norms. Then, if  $f$  is an  $(n, q)$ -form we have the pointwise inequality*

$$\|f\|_2^2 dV_{\omega_2} \leq \|f\|_1^2 dV_{\omega_1}.$$

Let  $k \geq l$  be two integers ; since  $\omega_l \geq \omega_k$ , we have

$$\begin{aligned} \int_X \frac{|w_{\varepsilon,k}|_{\omega_l}^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_{\omega_l} &\leq \int_X \frac{|w_{\varepsilon,k}|_{\omega_k}^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_{\omega_k} \leq \\ &\leq C \int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_L - \varphi_E}, \end{aligned} \quad (1.24)$$

as one can infer thanks to (1.23). Therefore, we can extract a weak limit  $w_\varepsilon$  of  $(w_{\varepsilon,k})$  ; for any compact  $K \subset X'$ , the metric  $\omega_l$  is comparable with the non-complete metric  $\omega$ , so in conclusion, we let  $k \rightarrow \infty$  in (1.23) to obtain the estimate

$$\begin{aligned} \int_X \frac{|v_\varepsilon|^2}{|s|^2} e^{-\varphi_L - \varphi_E} + \int_X \frac{|w_\varepsilon|_{\omega}^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_\omega &\leq \\ &\leq C_1 \int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_L - \varphi_E} \end{aligned} \quad (1.25)$$

together with the equation

$$\bar{\partial} \left( \sqrt{\eta_\varepsilon + \lambda_\varepsilon v_\varepsilon} \right) + C_\varepsilon w_\varepsilon = \bar{\partial} \left( \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_\infty \right) \quad (1.26)$$

in the sense of distribution on  $X'$  if we denote

$$C_\varepsilon := \lim_{k \rightarrow \infty} C(\varepsilon, k) = \frac{\int_X \theta \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|\bar{\partial} U_\infty|_{\omega}^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_\omega}{\left( \int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_L - \varphi_E} \right)^{\frac{1}{2}}}.$$



We analyze next the limit of  $C_\varepsilon$  as  $\varepsilon \rightarrow 0$ . Let us first seek for an upper bound for the denominator

$$\int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_L - \varphi_E}. \quad (1.27)$$

We have to use here the hypothesis that the metric  $\varphi_L$  is not identically  $-\infty$  on  $Z$  and that its singularities are logarithmic : it implies that  $Z$  is not entirely contained in the singularities of  $\varphi_L$  and the limit of (1.27) as  $\varepsilon \rightarrow 0$  is equal to

$$\int_Z |u|^2 e^{-\varphi_L} \quad (1.28)$$

up to a universal constant. Indeed, after a change of variables we may assume that  $Z$  could be written as  $Z = \{z^1 = 0\}$ , then according to the main theorem in [PS00], the function

$$t \rightarrow \int_X \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_L(t, z')} dV$$

is continuous with  $z' = (z^2, \dots, z^n)$ .

As for the numerator, the relation (1.6) of the paragraph 1.4.2 shows that we have

$$\int_X \theta \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|\bar{\partial} U_\infty|_\omega^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_\omega \leq C \varepsilon^{\frac{\varepsilon_0}{1+\varepsilon_0}} \log^{\frac{2}{1+\varepsilon_0}} \frac{1}{\varepsilon^2}. \quad (1.29)$$

Combining (1.28) and (1.29) we obtain that  $C_\varepsilon \rightarrow 0$  when  $\varepsilon \rightarrow 0$  and

$$\begin{aligned} \int_X \frac{|v_\varepsilon|^2}{|s|^2} e^{-\varphi_L - \varphi_E} + \int_X \frac{|w_\varepsilon|_\omega^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_\omega &\leq \\ &\leq C_1 \int_Z |u|^2 e^{-\varphi_L} \end{aligned} \quad (1.30)$$

with  $C_1$  a numerical constant.

The following lemma (cf. [Dem82b] and the references therein) shows that the equation (1.26) is verified on the manifold  $X$  in the sense of distributions.

**Lemma 1.25.** *Let  $\Omega$  be an open subset of  $\mathbb{C}^n$  and let  $Y$  be a complex analytic subset of  $\Omega$ . Assume that  $v$  is a  $(p, q-1)$ -form with  $L^2_{loc}$  coefficients and  $w$  a  $(p, q)$ -form with  $L^1_{loc}$  coefficients such that  $\bar{\partial}v = w$  on  $\Omega \setminus Y$  (in the sense of distributions). Then  $\bar{\partial}v = w$  on  $\Omega$ .*

Besides the fact that  $v_\varepsilon$  and  $w_\varepsilon$  are in  $L^2$ , we do not have further informations about their regularity. Indeed, from the estimate (1.30) one would like to infer that  $v_\varepsilon$  is equal to zero when restricted to  $Z$ , but in the  $L^2$  setting, this makes no sense. To tackle this difficulty, we do as follows : the equality (1.26) shows that we have

$$\bar{\partial}w_\varepsilon = 0;$$

so locally on each coordinate polydisk  $\Omega \subset X$  we can solve the equation

$$\bar{\partial}f_{\Omega, \varepsilon} = w_\varepsilon$$

such that

$$\int_{\Omega} \frac{|f_{\Omega, \varepsilon}|^2}{|s|^2} e^{-\varphi_L - \varphi_E} d\lambda \leq C \int_X \frac{|w_{\varepsilon}|_{\omega}^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_{\omega} \quad (1.31)$$

by Hörmander  $L^2$  estimates (cf. [Dem82b], [Hör65]). Now consider a finite covering of  $X$  with polydisks  $\Omega_j$ , and we denote by  $f_{j, \varepsilon}$  the solutions of the corresponding equation (1.31). Combined with (1.26), we have

$$\bar{\partial} \left( \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_{\infty} - \sqrt{\eta_{\varepsilon} + \lambda_{\varepsilon}} v_{\varepsilon} - C_{\varepsilon} f_{j, \varepsilon} \right) = 0.$$

In other words, the local section

$$U_{j, \varepsilon} := \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_{\infty} - \sqrt{\eta_{\varepsilon} + \lambda_{\varepsilon}} v_{\varepsilon} - C_{\varepsilon} f_{j, \varepsilon}$$

of the bundle  $K_X + Z + L$  defined on  $\Omega_j$  is holomorphic. In particular, this implies that the function

$$\sqrt{\eta_{\varepsilon} + \lambda_{\varepsilon}} v_{\varepsilon} + C_{\varepsilon} f_{j, \varepsilon}$$

is in  $C^{\infty}(\Omega_j)$ , and the estimates (1.30) and (1.31) show that this function is equal to zero when restricted to  $Z \cap \Omega_j$ . Hence we have

$$U_{j, \varepsilon}|_{Z \cap \Omega_j} = u$$

and we will show next that :

- As  $\varepsilon \rightarrow 0$ , the local sections piece together, and give an extension of  $u$ .
- The extension obtained as described in the previous bullet verifies the estimates required by the main theorem.

The verification is quite easy : in the expression of  $U_{j, \varepsilon}$ , the only "local" component is the term

$$C_{\varepsilon} f_{j, \varepsilon};$$

by the estimate (1.31), there exists  $f_j \in L^2(\Omega_j, e^{-\varphi_L - \log|s|^2})$  such that  $f_{j, \varepsilon} \rightarrow f_j$  weakly. But then we have

$$C_{\varepsilon} f_{j, \varepsilon} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ , this implies that we will have

$$U_j|_{\Omega_j \cap \Omega_l} = U_l|_{\Omega_j \cap \Omega_l}$$

for any pair of indexes  $(j, l)$ , where  $U_j$  is a limit of  $U_{j, \varepsilon}$ . We denote by  $U$  the resulting section.

In order to evaluate the  $L^2$  norm of  $U$ , remark that we have

$$\eta_{\varepsilon} + \lambda_{\varepsilon} \leq 5 \log(|s|^2 + \varepsilon^2)^2$$

for  $\varepsilon \ll 1$ ; therefore we obtain

$$\begin{aligned} \int_{\Omega_j} \frac{|U_{j, \varepsilon}|^2 e^{-\varphi_L - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2} &\leq C_1 \int_Z |u|^2 e^{-\varphi_L - \varphi_E} + \\ &+ \int_X \frac{|\theta(|s|^2/\varepsilon^2) U_{\infty}|^2 e^{-\varphi_L - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2} \end{aligned} \quad (1.32)$$

(cf. (1.31) combined with (1.30)).

The proof of Theorem 1.12 will be finished if we can show that the term

$$\int_X \frac{|\theta(|s|^2/\varepsilon^2)U_\infty|^2 e^{-\varphi_L - \varphi_E}}{(|s|^2 + \varepsilon^2)(\log(|s|^2 + \varepsilon^2))^2}$$

converges to zero. This is a consequence of the Hölder inequality, as follows. In the first place, given the expression of the section  $U_\infty$ , we see that it would be enough to show that we have

$$\int_{\Omega_\alpha} \frac{|\theta(|s|^2/\varepsilon^2)\tilde{f}_\alpha|^2 e^{-\varphi_L - \varphi_E}}{(|s|^2 + \varepsilon^2)(\log(|s|^2 + \varepsilon^2))^2} \rightarrow 0 \quad (1.33)$$

as  $\varepsilon \rightarrow 0$ . By Hölder inequality,

$$\begin{aligned} \int_{\Omega_\alpha} \frac{|\theta(|s|^2/\varepsilon^2)\tilde{f}_\alpha|^2 e^{-\varphi_L - \varphi_E}}{(|s|^2 + \varepsilon^2)(\log(|s|^2 + \varepsilon^2))^2} &\leq \left( C \int_{\Omega_\alpha} \frac{|\tilde{f}_\alpha|^2 e^{-(1+\varepsilon_0)\varphi_L}}{|s|^2 (\log(|s|^2))^2} \right)^{\frac{1}{1+\varepsilon_0}} \\ &\quad \left( \int_{\Omega_\alpha \cap \{|s|^2 < \varepsilon^2\}} \frac{d\lambda}{(|s|^2 + \varepsilon^2)(\log(|s|^2 + \varepsilon^2))^2} \right)^{\frac{\varepsilon_0}{1+\varepsilon_0}}. \end{aligned} \quad (1.34)$$

As a consequence, we get the inequality

$$\int_X \frac{|\theta(|s|^2/\varepsilon^2)U_\infty|^2 e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E}}{(|s|^2 + \varepsilon^2)(\log(|s|^2 + \varepsilon^2))^2} \leq \frac{C}{\log^{\frac{2\varepsilon_0}{1+\varepsilon_0}}(\frac{1}{\varepsilon^2})}.$$

Now let  $\varepsilon \rightarrow 0$  in relation (1.32) in order to obtain precisely the estimate claimed in the main statement. Therefore, we are done with the first case.

• **End of the proof for the case (II).**

For the second case, we use now the regularized metric  $\varphi_{L,\rho}$  of  $L$  obtained in the subsection 1.4.1; recall also its curvature properties

- •  $\sqrt{-1}\Theta_{h_{L,\rho}}(L) \geq -\delta_\rho\omega$ ;
- •  $\sqrt{-1}\Theta_{h_{L,\rho}}(L) \geq \frac{1}{\alpha}\sqrt{-1}\Theta(E) - \delta_\rho\omega$  for some  $\alpha \geq 1$ .

Compared with the case (I), the inequality we get now is

$$\begin{aligned} \eta_\varepsilon(\sqrt{-1}\Theta_{h_\rho}(L) + \sqrt{-1}\Theta_h(E) + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|s|^2) &\geq \\ &\geq \frac{\chi'_0(\sigma_\varepsilon)}{\varepsilon^2 + |s|^2}\sqrt{-1}\Theta(E) - \delta_\rho\eta_\varepsilon\omega \end{aligned} \quad (1.2.20)$$

(both curvature assumptions above are also used here).

We choose  $\lambda_\varepsilon$  and  $\chi_0$  as in the first case, i.e.

$$\lambda_\varepsilon := \frac{\chi'_0(\sigma_\varepsilon)^2}{\chi''_0(\sigma_\varepsilon)}$$

and

$$\chi_0(t) := t - \log(1 - t). \quad (1.2.21)$$

By collecting all the relations above we get

$$\begin{aligned} & \eta_\varepsilon(\sqrt{-1}\Theta_{h_\rho}(L) + \sqrt{-1}\Theta_h(E) + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|s|^2) - \\ & - \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon - \frac{\sqrt{-1}}{\lambda_\varepsilon}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon \geq \frac{\varepsilon^2}{2|s|^2}\sqrt{-1}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon - \delta_\rho\eta_\varepsilon\omega. \end{aligned} \quad (1.2.22)$$

The above inequality holds on  $X_\rho := X \setminus (E_\rho(\sqrt{-1}\Theta_{h_L}(L)) \cup Z)$ , and it involves the metric  $\omega$  which is not complete on  $X_\rho$ . Treated as in the first case, let  $\omega_\rho$  be a complete metric on  $X_\rho$ ; for each  $k \geq 1$ , we define the metric

$$\omega_k := \omega + \frac{1}{k}\omega_\rho. \quad (1.2.23)$$

It is equally complete on  $X_\rho$ , and satisfies  $\omega_k > \omega$ . Therefore the inequality (1.2.22) is still valid if we use  $\omega_k$  instead of  $\omega$ , i.e. we have

$$\begin{aligned} & \eta_\varepsilon(\sqrt{-1}\Theta_{h_\rho}(L) + \sqrt{-1}\Theta_h(E) + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|s|^2) - \\ & - \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon - \frac{\sqrt{-1}}{\lambda_\varepsilon}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon \geq \frac{\varepsilon^2}{2|s|^2}\sqrt{-1}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon - \delta_\rho\eta_\varepsilon\omega_k \end{aligned} \quad (1.2.24)$$

for each  $k \geq 1$ .

For each  $\varepsilon > 0$ , we can choose  $\rho_\varepsilon > 0$  such that

$$\delta_{\rho_\varepsilon} \log^2 \frac{1}{\varepsilon^2} \leq \varepsilon$$

(see the inequality (1.2.4) in the paragraph 1.4.2) and then the inequality (1.2.24) implies

$$\begin{aligned} & \eta_\varepsilon(\sqrt{-1}\Theta_{h_{\rho_\varepsilon}}(L) + \sqrt{-1}\Theta_h(E) + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\log|s|^2) - \\ & - \sqrt{-1}\partial\bar{\partial}\eta_\varepsilon - \frac{\sqrt{-1}}{\lambda_\varepsilon}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon \geq \frac{\varepsilon^2}{2|s|^2}\sqrt{-1}\partial\eta_\varepsilon \wedge \bar{\partial}\eta_\varepsilon - \varepsilon\omega_k, \end{aligned} \quad (1.2.25)$$

given the expression of  $\eta_\varepsilon$ .

This time, all the conditions correspond to  $\delta > 0$  in the "Functional analysis" paragraph 1.4.4 are fulfilled and as a consequence, for each  $\varepsilon > 0$  we can solve the following approximate  $\bar{\partial}$  equation

$$\bar{\partial}\left(\sqrt{\eta_\varepsilon + \lambda_\varepsilon v_{\varepsilon,k}}\right) + \varepsilon^{\frac{1}{2}}w_{\varepsilon,k} = \bar{\partial}\eta_\varepsilon \wedge g_{\varepsilon,0} + g_{\varepsilon,1};$$

in addition, we have the estimates

$$\begin{aligned} & \int_X \frac{|v_{\varepsilon,k}|^2}{|s|^2} e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E} + \int_X \frac{|w_{\varepsilon,k}|_{\omega_k}^2}{|s|^2} e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E} dV_{\omega_k} \leq \\ & \leq C \int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E} + \\ & + \varepsilon^{-1} \int_X \theta \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|\bar{\partial}U_\infty|_{\omega_k}^2}{|s|^2} e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E} dV_{\omega_k}. \end{aligned} \quad (1.2.26)$$

We analyze next the limit as  $\varepsilon \rightarrow 0$  of the right hand side terms of the inequality (1.2.26). For the first integral, by the properties of the regularization family  $(\varphi_{L,\rho})$ , we obtain the relation

$$\int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_{L,\rho\varepsilon} - \varphi_E} \leq \int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_L - \varphi_E}. \quad (1.2.27)$$

The right hand side term of (1.2.27) can be written as

$$\int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{\varepsilon^2}{|s|^2} \frac{|U_\infty|^2}{\varepsilon^2} e^{-\varphi_L - \varphi_E} \leq 2 \int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|U_\infty|^2}{\varepsilon^2} e^{-\varphi_L - \varphi_E}$$

since the support of  $\theta'$  is contained in  $\{\frac{\varepsilon^2}{2} < |s|^2 < \varepsilon^2\}$ . By the lemma in the second subsection of section 1.4.2, the right hand side of (1.2.27) converges to

$$\sup_Z |u|^2 C_X \int_Z e^{-\varphi_L} d\lambda_Z \quad (1.2.28)$$

with  $C_X$  a constant which depends on  $X$  but not on  $\varepsilon$ .

As for the remaining term, observe that we have

$$\int_X \theta \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|\bar{\partial}U_\infty|_{\omega_k}^2}{|s|^2} e^{-\varphi_{L,\rho\varepsilon} - \varphi_E} dV_{\omega_k} \leq \int_X \theta \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|\bar{\partial}U_\infty|_{\omega}^2}{|s|^2} e^{-\varphi_{L,\rho\varepsilon} - \varphi_E} dV_{\omega}$$

since  $\omega_k > \omega$ . Then the relation (1.2.4) of the paragraph 1.4.2 shows that

$$\int_X \theta \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \frac{|\bar{\partial}U_\infty|_{\omega}^2}{|s|^2} e^{-\varphi_L - \varphi_E} dV_{\omega} \leq C_X \varepsilon^2 \int_Z e^{-\varphi_L} d\lambda_Z \quad (1.2.29)$$

holds on  $X$ .

Combining the inequalities (1.2.26), (1.2.27) (1.2.29) together with (1.2.28), we obtain

$$\begin{aligned} & \int_X \frac{|v_{\varepsilon,k}|^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} + \int_X \frac{|w_{\varepsilon,k}|_{\omega_k}^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} dV_{\omega_k} \leq \\ & \leq \sup |u|^2 C_X \int_Z e^{-\varphi_L} d\lambda_Z + C_{X,\varepsilon} \int_Z e^{-\varphi_L} d\lambda_Z := C'_{X,\varepsilon,\varphi_L} \end{aligned} \quad (1.2.30)$$

where  $C_{X,\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

As in the first case, we intend to remove the dependence on  $k$  from the relation (1.2.30), by doing exactly in the same way as in the first case. For the coherence of the text, we repeat them in the following :

For each  $\varepsilon > 0$  fixed, the  $(n,0)$ -forms  $(v_{\varepsilon,k})$  are defined on the manifold  $X \setminus (Z \cup Y_\varepsilon)$  and bounded in  $L^2$  norm with respect to the metric  $\varphi_{L,\rho\varepsilon} + \log |s|^2$  by a constant which is independent of  $k$ . Therefore we can extract a weak limit say

$$v_\varepsilon \in L^2(X \setminus (Z \cup Y_\varepsilon))$$

from the sequence  $(v_{\varepsilon,k})$ .

As for the sequence  $(\overline{w_{\varepsilon,k}})$ , let  $k \geq l$  be two integers; since  $\omega_l \geq \omega_k$ , we have

$$\int_X \frac{|w_{\varepsilon,k}|_{\omega_l}^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} dV_{\omega_l} \leq \int_X \frac{|w_{\varepsilon,k}|_{\omega_k}^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} dV_{\omega_k} \leq C'_{X,\varepsilon,\varphi_L} \quad (1.2.31)$$

as one can infer thanks to (1.2.30). Therefore, we can extract a weak limit  $w_\varepsilon$  of  $(w_{\varepsilon,k})$ ; for any compact  $K \subset X \setminus (Z \cup Y_\varepsilon)$ , the metric  $\omega_l$  is comparable with the non-complete metric  $\omega$ , so in conclusion, we let  $k \rightarrow \infty$  in (1.2.30) and obtain the estimate

$$\begin{aligned} \int_X \frac{|v_\varepsilon|^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} + \int_X \frac{|w_\varepsilon|_\omega^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} dV_\omega &\leq \\ &\leq \sup_Z |u|^2 C_X \int_Z e^{-\varphi_L} d\lambda_Z + C_{X,\varepsilon} \int_Z e^{-\varphi_L} d\lambda_Z \end{aligned} \quad (1.2.32)$$

together with the equation

$$\bar{\partial} \left( \sqrt{\eta_\varepsilon + \lambda_\varepsilon v_\varepsilon} \right) + \varepsilon^{\frac{1}{2}} w_\varepsilon = \bar{\partial} \left( \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_\infty \right) \quad (1.2.33)$$

in the sense of distributions on  $X \setminus (Z \cup Y_\varepsilon)$ . Always like in the first case, the equation (1.2.33) is verified on the whole manifold  $X$  in the sense of distributions. Now we do the same process for the regularization of  $v_\varepsilon$  and  $w_\varepsilon$  as in the first case : the equation (1.2.33) shows that we have

$$\bar{\partial} w_\varepsilon = 0;$$

so locally on each coordinate polydisk  $\Omega \subset X$  we can solve the equation

$$\bar{\partial} f_{\Omega,\varepsilon} = w_\varepsilon$$

such that

$$\int_\Omega \frac{|f_{\Omega,\varepsilon}|^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} d\lambda \leq C \int_X \frac{|w_\varepsilon|_\omega^2}{|s|^2} e^{-\varphi_E - \varphi_{L,\rho\varepsilon}} dV_\omega \quad (1.2.34)$$

holds by Hörmander  $L^2$  estimates (cf. [Dem82b], [Hör65]). We consider a finite covering of  $X$  with polydisks  $\Omega_j$ , and denote by  $f_{j,\varepsilon}$  the solutions of the corresponding equation (1.2.34). On each  $\Omega_j$ , we have

$$\bar{\partial} \left( \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_\infty - \sqrt{\eta_\varepsilon + \lambda_\varepsilon v_\varepsilon} - \varepsilon^{\frac{1}{2}} f_{j,\varepsilon} \right) = 0.$$

In other words, the local section

$$U_{j,\varepsilon} := \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_\infty - \sqrt{\eta_\varepsilon + \lambda_\varepsilon v_\varepsilon} - \varepsilon^{\frac{1}{2}} f_{j,\varepsilon}$$

of the bundle  $K_X + Z + L$  defined on  $\Omega_j$  is holomorphic. In particular, this implies that the function

$$\sqrt{\eta_\varepsilon + \lambda_\varepsilon v_\varepsilon} + \varepsilon^{\frac{1}{2}} f_{j,\varepsilon}$$

is in  $C^\infty(\Omega_j)$ , and the estimates (1.2.32) and (1.2.34) show that this function is equal to zero when restricted to  $Z \cap \Omega_j$ . Hence we have

$$U_{j,\varepsilon}|_{Z \cap \Omega_j} = u.$$

The end part is also like in the first case, that is, we will show in the next :

- As  $\varepsilon \rightarrow 0$ , the local sections piece together, and give an extension of  $u$ .
- The extension obtained as described in the previous bullet verifies the estimates required by the main theorem.

The verification follows the same routine as in the first case : the only "local" component in the expression of  $U_{j,\varepsilon}$  is the term

$$\varepsilon^{\frac{1}{2}} f_{j,\varepsilon};$$

by the estimate (1.2.34), there exists  $f_j \in L^2(\Omega_j, e^{-\varphi_L - \log|s|^2})$  such that  $f_{j,\varepsilon} \rightarrow f_j$  weakly. Then we have

$$\varepsilon^{\frac{1}{2}} f_{j,\varepsilon} \rightarrow 0$$

as  $\varepsilon \rightarrow 0$  and

$$U_j|_{\Omega_j \cap \Omega_l} = U_l|_{\Omega_j \cap \Omega_l}$$

for any pair of indexes  $(j, l)$ , where  $U_j$  is a limit of  $U_{j,\varepsilon}$ . We denote by  $U$  the resulting section.

In order to evaluate the  $L^2$  norm of  $U$ , remember that

$$\eta_\varepsilon + \lambda_\varepsilon \leq 5 \log(|s|^2 + \varepsilon^2)^2$$

for  $\varepsilon \ll 1$ ; we obtain this time

$$\begin{aligned} \int_{\Omega_j} \frac{|U_{j,\varepsilon}|^2 e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2} &\leq \sup_Z |u|^2 C_X \int_Z e^{-\varphi_L} d\lambda_Z + \\ &+ C_{X,\varepsilon} \int_Z e^{-\varphi_L} d\lambda_Z + \int_X \frac{|\theta(|s|^2/\varepsilon^2) U_\infty|^2 e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2} \end{aligned} \quad (1.2.35)$$

where  $C_{X,\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$  (cf. (1.2.34) combined with (1.2.30)).

Again, the proof will be finished if we can show that the term

$$\int_X \frac{|\theta(|s|^2/\varepsilon^2) U_\infty|^2 e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2}$$

converges to zero. As in the first case, given the expression of the section  $U_\infty$ , it would be enough if we show that

$$\int_{\Omega_\alpha} \frac{|\theta(|s|^2/\varepsilon^2) \tilde{f}_\alpha|^2 e^{-\varphi_L - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2} \rightarrow 0 \quad (1.2.36)$$

as  $\varepsilon \rightarrow 0$ . By Cauchy-Schwarz inequality, we have

$$\begin{aligned} \int_{\Omega_\alpha} \frac{|\theta(|s|^2/\varepsilon^2) \tilde{f}_\alpha|^2 e^{-\varphi_L - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2} &\leq \left( C \int_{\Omega_\alpha \cap \{|s|^2 < \varepsilon^2\}} \frac{|\tilde{f}_\alpha|^2 e^{-\varphi_L}}{\varepsilon^2} \right)^{\frac{1}{2}} \\ &\left( \int_{\Omega_\alpha \cap \{|s|^2 < \varepsilon^2\}} \frac{d\lambda}{(\log(|s|^2 + \varepsilon^2))^2} \right)^{\frac{1}{2}}. \end{aligned} \quad (1.2.37)$$

Apply again the lemma in subsection 1.4.2, we get

$$\int_X \frac{|\theta(|s|^2/\varepsilon^2) U_\infty|^2 e^{-\varphi_{L,\rho_\varepsilon} - \varphi_E}}{(|s|^2 + \varepsilon^2) (\log(|s|^2 + \varepsilon^2))^2} \leq \frac{\varepsilon^2 C_X}{\log(\frac{1}{\varepsilon^2})}.$$

Finally, let  $\varepsilon \rightarrow 0$  in relation (1.2.35), we obtain precisely the estimate claimed in the main statement 1.13. Therefore, we are done.  $\square$

## 1.5 Comments about the general case

The extension result 1.12 is of course expected to hold without the hypothesis (I). So far, we are unable to obtain this result in full generality; however, the same proof as above gives an extension theorem which we describe next :

• We recall first the notion of upper regularized multiplier ideal sheaf  $\mathcal{I}_+(\varphi)$ , it defines like follows :

$$\mathcal{I}_+(\varphi) = \lim_{\varepsilon \rightarrow 0} \mathcal{I}((1 + \varepsilon)\varphi_L).$$

Suppose that our section  $u$  satisfies

$$u \in \mathcal{I}_+(\varphi_L|_Z),$$

that is,  $u \in \mathcal{I}((1 + \varepsilon_0)\varphi_L|_Z)$  for some  $\varepsilon_0$ . Conjecturally (cf. [DP03]), we have  $\mathcal{I}(\varphi_L) = \mathcal{I}_+(\varphi_L)$ , but so far this equality is only established up to the dimension 2 (and it seems to be a difficult problem);

• The following assumption was formulated in [Pal08] :

**Hypothesis 1.26.** *Let  $x_0 \in Z$  be an arbitrary point, and let  $z = (z_1, \dots, z_n)$  be a coordinate system of  $X$  defined on an open set  $U$ , such that  $Z \cap U = \{z_1 = 0\}$ . Let  $f \in \mathcal{I}(\varphi_L|_Z)_{x_0}$  be an element of the multiplier ideal sheaf associated to  $\varphi_L|_Z$  defined on  $Z \cap U$ , and let  $\tilde{f}$  be an extension of  $f$  to  $U$ . Then the function*

$$t \rightarrow \int_{U'} |\tilde{f}(t, z')|^2 e^{-\varphi_L(t, z')} d\lambda(z')$$

is continuous at  $t = 0$ , where  $z' := (z^2, \dots, z^n)$ .

We remark that by applying the methods developed in the paper to solve Theorem 1.12 (resp. Theorem 1.13), both hypotheses are needed because we cannot allow the parameters  $\rho$  (appearing in the regularization process) and  $\varepsilon$  (whose effect is to concentrate the mass along  $Z$  in the estimate (1.32) (and (1.2.35) respectively) to vary independently. We describe next the problems we encountered while trying to bypass this difficulty.

By using the regularization in [Dem94] with just the application of the convolution kernel (i.e. without the Legendre transform), we obtain a family of non-singular metrics  $h_{L, \rho}$  on  $L$  with the following properties

- (1)  $\sqrt{-1}\Theta_{h_{L, \rho}}(L) \geq -\mu_\rho\omega$ ;
- (2)  $\sqrt{-1}\Theta_{h_{L, \rho}}(L) \geq \frac{1}{\alpha}\sqrt{-1}\Theta(E) - \mu_\rho\omega$  for some  $\alpha \geq 1$ ,  $\alpha \in \mathcal{C}^\infty(X)$ .

In the inequality above,  $(\mu_\rho)_{\rho > 0}$  is a family of positive functions on  $X$ , such that :

- there exists a constant  $C > 0$  such that we have  $\mu_\rho(x) \leq C$  for any  $x \in X$  and  $\rho > 0$ .
- we have  $\lim_{\rho \rightarrow 0} \mu_\rho(x) = \nu(\sqrt{-1}\Theta_{h_L}(L), x)$  for each  $x \in X$ .

The discussion in the functional analysis paragraph still applies, but in the actual circumstances we only obtain

$$\bar{\partial} \left( \sqrt{\eta_\varepsilon + \lambda_\varepsilon v_{\varepsilon, \rho}} \right) + \sqrt{\mu_\rho \eta_\varepsilon} w'_{\varepsilon, \rho} + \bar{\partial}^* (\sqrt{\eta_\varepsilon} w''_{\varepsilon, \rho}) = \bar{\partial} \left( \theta \left( \frac{|s|^2}{\varepsilon^2} \right) U_\infty \right) \quad (1.35)$$

together with estimates for  $v_{\varepsilon, \rho}$ ,  $w'_{\varepsilon, \rho}$  and  $w''_{\varepsilon, \rho}$  similar to (1.2.26). The adjoint operator  $\bar{\partial}^*$  is with respect to the metric  $\varphi_{L, \rho} + \log |s|^2$  on  $X \setminus Z$ .



Since we want to use the fact that the metric  $h_{L,\rho}$  is non-singular, we have to take first  $\varepsilon \rightarrow 0$ , and then  $\rho \rightarrow 0$  in the relation above. The trouble comes from the presence of the factor  $\bar{\partial}^*(\sqrt{\eta_\varepsilon} w'_{\varepsilon,\rho})$ : in order to remove it, one can project on the kernel of the  $\bar{\partial}$  (similar ideas were used in [Dem82b]). Indeed, the arguments invoked in [Dem82b] would give the desired result, provided that one can obtain estimates for the  $L^2$  norm of

$$\sqrt{\mu_\rho \eta_\varepsilon} w'_{\varepsilon,\rho}$$

and not just for the  $L^2$  norm of  $w'_{\varepsilon,\rho}$  as it is the case here. The puzzling fact is that if one tries to obtain these estimates by incorporating the factor

$$\log \left( \log \frac{1}{\varepsilon^2 + |s|^2} \right)$$

in the expression of the metric of  $L$ , the curvature hypotheses needed are basically unchanged (which is very good!), but then the right hand side term of (1.32) (and (1.2.35) respectively) becomes

$$\int_X \theta' \left( \frac{|s|^2}{\varepsilon^2} \right)^2 \log \frac{1}{\varepsilon^2 + |s|^2} \frac{|U_\infty|^2}{|s|^2} e^{-\varphi_{L,\rho} - \varphi_E}$$

which is unbounded as  $\varepsilon \rightarrow 0$ . In conclusion, we are one estimate far from the general case.

## 1.6 Two open problems

We formulate in this section two open problems. The first problem is concerned with the second hypothesis in the main theorem, that is, if  $\mathcal{I}(\varphi_{L|Z}) = \mathcal{O}_Z$ . We hope that Hypothesis 1.26 to be verified. To this end, we propose the following problem.

**Conjecture 1.27.** *Denote  $\Omega := \mathbb{D} \times \Delta \subset \mathbb{C}^n$  the domain equals to the product of the unit disk  $\mathbb{D}$  in  $\mathbb{C}$  with the unit disk  $\Delta$  in  $\mathbb{C}^{n-1}$ , we consider a function  $\varphi \in Psh(\Omega)$  defined on an open set containing the closure of  $\Omega$ , such that  $\sup_\Omega \varphi = 0$ . Assume that we have*

$$\int_\Delta e^{-\varphi(0,z)} d\lambda(z) < \infty.$$

Then the function

$$t \rightarrow \int_\Delta e^{-\varphi_L(t,z)} d\lambda(z)$$

is continuous at  $t = 0$ , and for any  $0 < \delta \ll 1$ , the following inequality holds :

$$\int_{\Omega'_\delta} e^{-\varphi(t,z)} \frac{d\lambda(t,z)}{\delta^2} \leq C_0 \int_\Delta e^{-\varphi(0,z)} d\lambda(z)$$

where we use the notation  $\Omega'_\delta := (|t| < \delta) \times \Delta$  intersected with the polydisk of radius  $\frac{1}{2}$ .

Remark that if this is true, estimate (1.2.4) would be like

$$I_\varepsilon \leq C\varepsilon^2$$

with  $C$  a numerical constant. As a consequence, the right hand side of (1.2.27) goes to

$$\int_Z |u|^2 e^{-\varphi_L}$$

when  $\varepsilon \rightarrow 0$ . In conclusion, we can prove exactly the same estimate as in Theorem 1.12 for Theorem 1.13.

The next problem was formulated by Y.-T. Siu in his seminal article [Siu02].

**Conjecture 1.28.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a Kähler family over the unit disk  $\mathbb{D}$ . Then any section of  $mK_{\mathcal{X}_0}$  extends to  $\mathcal{X}$ .*

It is rather safe to predict that the Ohsawa-Takegoshi theorem in the Kähler setting will be necessary for the solution of this problem : this is a very strong motivation for our research in this chapter. Given the proof of the conjecture above in the case  $X$  projective, the hypothesis

$$u \in H^0(Z, (K_Z + L|_Z) \otimes \mathcal{I}_+(\varphi_L|_Z))$$

is not as restrictive as it looks : the construction of the metric for the bundle  $L$  in [Ohs03] shows that the initial section is in fact *bounded* with respect to this metric.

## 1.7 Appendix

### 1.7.1 Remark 1.14

As we mentioned in Remark 1.14, the holomorphic line bundle  $(E, h_E)$  in the main theorem can be replaced by a hermitian holomorphic vector bundle of rank  $r$  ( $r \geq 1$ ). The curvature assumptions and the conclusions are modified relatively (see the settings in the Ohsawa-Takegoshi-Manivel's theorem in Section 1.1.1). Apart from the first part of the proof, that is, the approximation process, the demonstration of the main theorem is almost unchanged. We discuss the modifications in the first part.

In the first place, notice that

$$\sqrt{-1}\Theta_{h_L}(L) + r \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} \log |s|_{h_E}^2$$

can be written as

$$\beta + \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}(\phi + r \log |s|_{h_E}^2)$$

where  $\beta$  is a smooth closed  $(1, 1)$ -form and  $\phi$  an almost psh function on  $X$ . If we denote

$$\psi = \phi + r \log |s|^2$$

and

$$T = \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\psi.$$

We see that  $\psi$  is almost psh. The curvature assumptions can be interpreted

- (1)  $T \geq -\beta$ ;
- (2)  $T \geq \frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)s, s\}}{|s|^2} - \beta$ .

What obstructs us to apply the regularization theorem in section 1.4.1 is that in the second curvature assumption, it's not clear whether the current

$$\frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)s, s\}}{|s|^2}$$

is continuous near  $Z$ . We then have to do our computation first on the blow-up manifold  $\widehat{X}$ . Denote the holomorphic map

$$\sigma : \widehat{X} \rightarrow X$$

and

$$H = \sigma^{-1}(Z)$$

the exceptional divisor.  $\widehat{X}$  has an induced metric

$$\widehat{\omega} = \sigma^* \omega + \sqrt{-1} \Theta(\mathcal{O}(-H)).$$

Denote  $U_j$  the relative compact balls on  $X$  s.t.  $\cup_j (U_j) \supset X$  and  $\widehat{U}_j = \sigma^{-1}(U_j)$ . After changing the coordinate charts on  $\widehat{X}$ , we obtain that for such  $U_j$ s which intersect with  $Z$ ,  $(\widehat{U}_j^i)_{1 \leq i \leq r}$  form a covering of  $\widehat{U}_j$  on  $\widehat{X}$ . Moreover,  $H$  can be characterized by

$$H \cap \widehat{U}_j^i = \{p_i = 0, (p_1, \dots, p_i, \dots, p_n) \in \widehat{X}\}.$$

Suppose  $(e_1, \dots, e_r)$  be a  $\mathcal{C}^\infty$  orthonormal frame of  $E$  over the coordinate patch  $\widehat{U}_j^i \subset \widehat{X}$  with complex coordinates  $(p_1, \dots, p_n)$ . On  $\widehat{U}_j^i$ , the curvature current  $\sqrt{-1} \Theta(E)$  can be written as

$$\sqrt{-1} \Theta(E) = \sqrt{-1} \sum_{1 \leq j, k \leq n, 1 \leq \lambda, \mu \leq r} c_{j,k,\lambda,\mu} dp_j \wedge d\bar{p}_k \otimes e_\lambda^* \otimes e_\mu, \quad c_{j,k,\lambda,\mu} \in \mathbb{C},$$

while  $s$  can be written as

$$s = \sum_{1 \leq \lambda \leq r} \sigma_\lambda \otimes e_\lambda$$

where  $\sigma_\lambda$  are holomorphic functions on  $\widehat{U}_j^i$ .

Now since  $s$  vanishes on  $H$ , every  $\sigma_\mu$  can be written like  $\sigma_\mu = p_i \cdot f_\mu$ , where  $a_\lambda \in \mathbb{N}$  and  $f_\mu$  are holomorphic functions which satisfy that  $\sum_{1 \leq \mu \leq r} |f_\mu|^2 \neq 0$ . Thus on  $\widehat{U}_j^i$ , we have

$$\begin{aligned} \{\sqrt{-1} \Theta(E)s, s\} &= \left\{ \sqrt{-1} \sum_{1 \leq j, k \leq n; 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} dp_j \wedge d\bar{p}_k \otimes e_\lambda^* \otimes e_\mu \cdot \left( \sum_{1 \leq \lambda \leq r} \sigma_\lambda \otimes e_\lambda \right), \sum_{1 \leq \eta \leq r} \sigma_\eta \otimes e_\eta \right\} \\ &= \left\{ \sqrt{-1} \sum_{1 \leq j, k \leq n; 1 \leq \lambda, \mu \leq r} c_{jk\lambda\mu} \sigma_\lambda dp_j \wedge d\bar{p}_k \otimes e_\mu, \sum_{1 \leq \eta \leq r} \sigma_\eta \otimes e_\eta \right\} \\ &= \sqrt{-1} \sum_{1 \leq j, k \leq n; 1 \leq \lambda, \mu, \eta \leq r} c_{jk\lambda\mu} \sigma_\lambda \bar{\sigma}_\eta dp_j \wedge d\bar{p}_k \langle e_\mu, e_\eta \rangle. \end{aligned}$$

On the other hand, we have

$$|s|^2 = \sum_{1 \leq \mu, \eta \leq r} \sigma_\mu \bar{\sigma}_\eta \langle e_\mu, e_\eta \rangle.$$

Hence the pull back of

$$\frac{1}{\alpha} \frac{\{\sqrt{-1} \Theta(E)s, s\}}{|s|^2}$$

on  $\widehat{X}$  can be written like

$$\begin{aligned}
 \sigma^* \frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)s, s\}}{|s|^2} &= \frac{\sqrt{-1}}{\alpha} \frac{\sum_{1 \leq j, k \leq n; 1 \leq \lambda, \mu, \eta \leq r} c_{jk\lambda\mu} \sigma_\lambda \bar{\sigma}_\eta dp_j \wedge d\bar{p}_k \langle e_\mu, e_\eta \rangle}{\sum_{1 \leq \mu, \eta \leq r} \sigma_\mu \bar{\sigma}_\eta \langle e_\mu, e_\eta \rangle} \\
 &= \frac{\sqrt{-1}}{\alpha} \frac{\sum_{1 \leq j, k \leq n; 1 \leq \lambda, \mu, \eta \leq r} c_{jk\lambda\mu} p_i \cdot f_\lambda \bar{p}_i \cdot \overline{f_\eta} \langle e_\mu, e_\eta \rangle dp_j \wedge d\bar{p}_k}{\sum_{1 \leq \mu, \eta \leq r} p_i \cdot f_\mu \bar{p}_i \cdot \overline{f_\eta} \langle e_\mu, e_\eta \rangle} \\
 &= \frac{\sqrt{-1}}{\alpha} \frac{\sum_{1 \leq j, k \leq n; 1 \leq \lambda, \mu, \eta \leq r} c_{jk\lambda\mu} |p_i|^2 \cdot f_\lambda \bar{f}_\eta \langle e_\mu, e_\eta \rangle dp_j \wedge d\bar{p}_k}{\sum_{1 \leq \mu, \eta \leq r} |p_i|^2 \cdot f_\mu \bar{f}_\eta \langle e_\mu, e_\eta \rangle}.
 \end{aligned}$$

Since  $\sum_{1 \leq \mu \leq r} |f_\mu|^2 \neq 0$ , we can see that the last equality is a  $\mathcal{C}^\infty$ -form. The approximation theorem can be applied to  $T \circ \sigma$  and therefore we obtain that

$$T_\rho \circ \sigma \geq \sigma^* \frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)s, s\}}{|s|^2} - \delta_\rho \widehat{\omega} - \beta \circ \sigma.$$

We now want to go down to the original manifold  $X$ , recall that we have

$$\widehat{\omega} = \sigma^* \omega + \sqrt{-1}\Theta(\mathcal{O}(-H)),$$

and  $\mathcal{O}(-H)|_H$  is isomorphic to  $\mathcal{O}_{\mathbb{P}(NZ)}(1)$ . Then Lelong-Poincaré equation tells us that

$$\widehat{\omega} + [H] = \sigma^* \omega + \sqrt{-1}\partial\bar{\partial} \log |p_i|^2$$

with the relation  $\log |p_i|^2 \sigma^{-1} = \log |s|^2$ . Hence when we go back to  $X$ , we obtain

$$\sigma_*(\widehat{\omega}) = \omega + \sqrt{-1}\partial\bar{\partial}\xi$$

where  $\xi$  has the same singularities as  $\log |s|^2$  and  $\xi - \log |s|^2 \in \mathcal{C}^\infty$ .

In conclusion, when we go down to the original manifold  $X$ , the next inequality holds :

$$T_\delta \geq \frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)s, s\}}{|s|^2} - \delta_\rho \omega - \delta_\rho \partial\bar{\partial}\xi - \beta.$$

Remind here that we can choose  $\delta_\rho$  such that

$$\delta_\rho \omega + \delta_\rho \sqrt{-1}\partial\bar{\partial}\xi \geq 0. \tag{1.36}$$

We proved actually

$$\sqrt{-1}\Theta_{h_\delta}(L) + r \frac{\sqrt{-1}}{\pi} \partial\bar{\partial} |s|_{h_E}^2 \geq \frac{1}{\alpha} \frac{\{\sqrt{-1}\Theta(E)s, s\}}{|s|^2} - (\delta_\rho \omega + \delta_\rho \sqrt{-1}\partial\bar{\partial}\xi).$$

This is the regularization we want.

### 1.7.2 Evaluation of (1.5).

In the evaluation of (1.5), we deduce that

$$\int_{U_\alpha \cap U_\beta \cap \{|s|^2 < \varepsilon^2\}} |g_{\alpha\beta}|^2 \log^{\frac{2}{\varepsilon_0}} \frac{1}{|s|^2} \leq \varepsilon \log^{\frac{2}{\varepsilon_0}} \frac{1}{\varepsilon^2}.$$

We now explain this in detail. First of all, notice that we can write the domain

$$\{|s|^2 < \varepsilon^2\} = \bigcup_{k=1}^{+\infty} \left\{ \frac{\varepsilon^2}{2^k} \leq |s|^2 < \frac{\varepsilon^2}{2^{k-1}} \right\}.$$

So we have

$$\begin{aligned} & \int_{U_\alpha \cap U_\beta \cap \{|s|^2 < \varepsilon^2\}} |g_{\alpha\beta}|^2 \log^{\frac{2}{\varepsilon_0}} \frac{1}{|s|^2} \\ &= \sum_{k=1}^{+\infty} \int_{U_\alpha \cap U_\beta \cap \{\frac{\varepsilon^2}{2^k} \leq |s|^2 < \frac{\varepsilon^2}{2^{k-1}}\}} |g_{\alpha\beta}|^2 \log^{\frac{2}{\varepsilon_0}} \frac{1}{|s|^2} \\ &\leq C \sum_{k=1}^{+\infty} \int_{\{\frac{\varepsilon^2}{2^k} \leq |s|^2 < \frac{\varepsilon^2}{2^{k-1}}\}} \log^{\frac{2}{\varepsilon_0}} \frac{1}{|s|^2} \\ &\leq C \sum_{k=1}^{+\infty} \frac{\varepsilon}{2^k} \log^{\frac{2}{\varepsilon_0}} \frac{2^k}{\varepsilon^2}. \end{aligned}$$

Denote

$$a_k = \frac{\varepsilon^2}{2^k} \log^{\frac{2}{\varepsilon_0}} \frac{2^k}{\varepsilon^2} = \frac{\varepsilon}{2^k} (k \log 2 - 2 \log \varepsilon)^{\frac{2}{\varepsilon_0}},$$

we have

$$\frac{a_{k+1}}{a_k} = \frac{((k+1) \log 2 - 2 \log \varepsilon)^{\frac{2}{\varepsilon_0}}}{2(k \log 2 - 2 \log \varepsilon)^{\frac{2}{\varepsilon_0}}} = \frac{1}{2} \left( 1 + \frac{\log 2}{k \log 2 - 2 \log \varepsilon} \right)^{\frac{2}{\varepsilon_0}}.$$

Since  $\forall k$ ,

$$k \log 2 - \log \varepsilon \geq -2 \log \varepsilon,$$

then

$$\left( 1 + \frac{\log 2}{k \log 2 - 2 \log \varepsilon} \right)^{\frac{2}{\varepsilon_0}} \geq \left( 1 + \frac{\log 2}{-2 \log \varepsilon} \right)^{\frac{2}{\varepsilon_0}}.$$

If

$$\log \varepsilon < -\frac{\log 2}{2(2^{\frac{\varepsilon_0}{2}} - 1)},$$

we get

$$1 + \frac{\log 2}{-2 \log \varepsilon} < 2^{\frac{\varepsilon_0}{2}},$$

and hence

$$\frac{a_{k+1}}{a_k} < 1.$$

Denote  $S_a = \sum_{k=1}^{+\infty} a_k$ , we have

$$\begin{aligned}
 S_a &\leq \frac{\frac{\varepsilon^2}{2}(\log 2 - 2 \log \varepsilon)^{\frac{2}{\varepsilon_0}}}{1 - \frac{1}{2} \left(1 + \frac{\log 2}{-2 \log \varepsilon}\right)^{\frac{2}{\varepsilon_0}}} \\
 &= \frac{\varepsilon^2}{2} \left(\log \frac{1}{\varepsilon^2}\right)^{\frac{2}{\varepsilon_0}} \frac{e^{\frac{2}{\varepsilon_0} \log\left(1 + \frac{\log 2}{-2 \log \varepsilon}\right)}}{1 - \frac{1}{2} e^{\frac{2}{\varepsilon_0} \log\left(1 + \frac{\log 2}{-2 \log \varepsilon}\right)}} \\
 &\sim \frac{\varepsilon^2}{2} \left(\log \frac{1}{\varepsilon^2}\right)^{\frac{2}{\varepsilon_0}}
 \end{aligned}$$

and this ends the proof.

## Chapitre 2

# Invariance of plurigenera

As we have seen in the introduction of Chapter 1, the Ohsawa-Takegoshi theorem has many applications in complex analysis and geometry. In many respects, the most spectacular consequence of this result is the solution obtained by Y.-T. Siu in 1998 of the long-standing problem of the invariance of plurigenera. Here we will obtain a generalization of Siu's result in the context of closed positive currents. Our main theorem states the sub-extension of closed positive currents whose cohomology class is the adjoint of the Chern class of a  $\mathbb{Q}$ -line bundle having certain positivity properties.

### 2.1 Introduction

We recall first that the  $m$ -th plurigenus of a compact complex manifold  $X$  with canonical bundle  $K_X$  is the number

$$h^0(X, mK_X).$$

If

$$\mathcal{X} \rightarrow \mathbb{D}$$

is a projective family over the unit disk  $\mathbb{D}$ , it was expected that the function

$$t \rightarrow h^0(\mathcal{X}_t, mK_{\mathcal{X}_t})$$

is constant on  $\mathbb{D}$ . The first result in this direction was established by Iitaka [Iit70] :

**Theorem 2.1.** *The plurigenera are invariant under a deformation of surfaces.*

In his proof, Iitaka used many facts and results from the Enriques-Kodaira classification of surfaces, so his arguments seem very difficult to generalize in higher dimensions.

In higher dimensions, it was conjectured that the next statement is true :

**Conjecture 2.2.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{B}$  be a holomorphic proper map over a connected base  $\mathbb{B}$  such that for every  $t \in \mathbb{B}$ , the fiber  $\mathcal{X}_t$  is Kähler, then  $h^0(\mathcal{X}_t, mK_{\mathcal{X}_t})$  is a constant.*

We remark that a counterexample of Nakamura [Nak72] showed that the plurigenera might have jumps during a deformation of a manifold of dimension  $\geq 3$ . On the other hand, the fibers of the deformation constructed in his counterexample are not Kähler. So the hypothesis on  $\pi$  in the conjecture is the minimum requirement. The first breakthrough is obtained by Levine [Lev83] who showed that for every positive integer  $m$ , every element of  $H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$  can be extended to the fiber of  $\mathcal{X}$  over the double point of  $\mathbb{B}$  at  $t = 0$  :

**Theorem 2.3.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{B}$  be a deformation of complex manifolds and  $0$  a point in  $\mathbb{B}$ . Suppose that the fiber  $\mathcal{X}_0 = \pi^{-1}(0)$  is in the Fujiki class  $\mathcal{C}$ . Let  $m$  be a positive integer and suppose that the divisor of some non-zero section in  $H^0(\mathcal{X}_0, \mathcal{O}(mK_{\mathcal{X}_0}))$  is smooth. Then there is a Zariski open neighborhood  $U$  of  $0$  in  $\mathbb{B}$  such that the  $m$ -genus  $P_m(\pi^{-1}(t))$  is a constant for all  $t$  in  $U$ .*

Recall here a complex manifold is called in the Fujiki class  $\mathcal{C}$  if it is bimeromorphic to a compact Kähler manifold. Such manifolds preserve certain properties of compact Kähler manifolds, like, for example, the Hodge decomposition theorem as well as the  $E_1$  degeneration of the Hodge-Frölicher spectral sequence. Levine's proof is to calculate explicitly the obstruction of extending a section defined on the central fiber  $\mathcal{X}_0$  to a neighborhood fiber. With the hypotheses in the theorem and the degeneration of the Frölicher spectral sequence of a finite covering of  $\mathcal{X}$ , he showed that this obstruction in fact vanishes.

Levine's theorem illustrates that the question of the invariance of plurigenera can be translated to a problem of extending sections. In fact, by the theorems of direct image of H. Grauert, the function

$$\mathbb{B} \ni t \rightarrow P_m(\pi^{-1}(t))$$

is upper semi-continuous, in particular, for  $t$  in a neighborhood of  $0$ , we have

$$P_m(\mathcal{X}_t) \leq P_m(\mathcal{X}_0).$$

Therefore, if every section  $s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$  can be extended to a neighborhood fiber  $\mathcal{X}_t$  of  $\mathcal{X}_0$ , we will have that

$$P_m(\mathcal{X}_t) = P_m(\mathcal{X}_0)$$

in the neighborhood of  $0$ .

In the context of a holomorphic family of complex projective algebraic manifolds, the first relevant result was given by Siu [Siu98] under the assumption that the fibers are of general type :

**Theorem 2.4.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a projective family of compact manifolds parametrized by the open unit 1-disk  $\mathbb{D}$ . Assume that the family  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  is of general type. Then for every positive integer  $m$  the plurigenus  $h^0(\mathcal{X}_t, mK_{\mathcal{X}_t})$  is independent of  $t \in \mathbb{D}$ , where  $\mathcal{X}_t = \pi^{-1}(t)$  and  $K_{\mathcal{X}_t}$  is the canonical line bundle of  $\mathcal{X}_t$ .*

Two ingredients are used in the proof : the global generation of multiplier ideal sheaves (by the method of Skoda [Sko72b]) and an extension theorem of Ohsawa-Takegoshi type. [OT87]. Several generalizations of Theorem 2.4 were made later by Kawamata [Kaw99], Nakayama [Nak98] and Tsuji [Tsu02], where the first two authors generalized the transcendently formulated methods in [Siu98] into a completely algebraic setting while the last author proved the theorem for a projective family whose fibers are not necessarily of general type. Several years later, Siu proved a generalized case of his previous result, this new theorem implies the invariance of semipositively twisted plurigenera (cf. [Siu02]) :

**Theorem 2.5.** ([Siu02]) *Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a projective family and  $(L, h)$  be a pseudo-effective line bundle on  $\mathcal{X}$  with  $h_{\mathcal{X}_0}$  well defined and  $\mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0}) = \mathcal{O}_{\mathcal{X}_0}$ . Let  $m \geq 1$  be an integer. Then every section  $s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0} + L)$  satisfies  $\|s\|_{\infty, h} < +\infty$  admits an extension  $\tilde{s} \in H^0(\mathcal{X}, mK_{\mathcal{X}} + L)$ .*

The proof of Theorem 2.5 has been considerably simplified by Păun, the result in [Pău07] is as follows :



**Theorem 2.6.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a family of projective manifolds and  $(L, h_L)$  a pseudo-effective line bundle on  $\mathcal{X}$  with  $h_{L|_{\mathcal{X}_0}}$  well defined. Let  $m \geq 1$  be an integer. Then every section  $s \in H^0((\mathcal{X}_0, mK_{\mathcal{X}_0} + L) \otimes \mathcal{I}(\mathcal{X}_0, h_{L|_{\mathcal{X}_0}}))$  extends to a section  $\tilde{s} \in H^0(\mathcal{X}, mK_{\mathcal{X}} + L)$ .*

In addition to the techniques used by Siu, Păun introduced an "One-Tower" method in his proof. This method is turned out to be very flexible because it adapts to many situations. Actually, we will use it in the proof of the main theorem we established in this chapter. It is therefore worthwhile to draw an outline of its general principle :

• **The "One-Tower" method :**

Let  $\mathcal{X}$  be a family of complex manifolds and  $(A, h_A)$  a hermitian line bundle with strictly positive curvature (in the case that  $\mathcal{X}$  is a projective family,  $(A, h_A)$  corresponds to an ample line bundle). Let  $\mathcal{X}_0$  be the central fiber. Given a section  $s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$  we would like to extend. In order to run the inductive process we need an Ohsawa-Takegoshi type's result (corresponding to the case  $m = 1$ ), like follows :

**Theorem 2.7.** ([Siu02]) *Let  $L$  be a line bundle on  $\mathcal{X}$  equipped with a metric  $h$  whose curvature form  $\sqrt{-1}\Theta_h(L)$  is positive. Then every section  $\sigma \in H^0(\mathcal{X}_0, K_{\mathcal{X}_0} + L)$  satisfying*

$$\int_{\mathcal{X}_0} \|\sigma\|_h^2 < +\infty$$

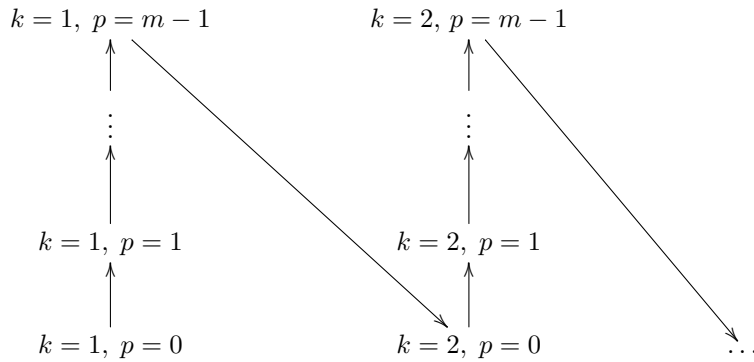
extends to the whole  $\mathcal{X}$ .

Replace  $A$  by a sufficiently big multiply of itself, we impose two conditions as follows :

- (A<sub>1</sub>) Every section of  $mK_{\mathcal{X}_0} + A_{\mathcal{X}_0}$  extends to  $\mathcal{X}$ ;
- (A<sub>2</sub>) For every  $p = 0, \dots, m - 1$ , the bundle  $pK_{\mathcal{X}} + A$  is generated by a family of sections  $(s_j^{(p)})_{j=1, \dots, N_p}$  in a neighborhood of  $\mathcal{X}_0$ .

Since there are just a finite number of conditions imposed, such a bundle  $A$  that fulfills the conditions (A<sub>1</sub>) and (A<sub>2</sub>) can always be found. We are now ready to start out the inductive process, it consists of two steps :

- Extensions of sections  $s^k \otimes s_j^{(p)}$  step by step : by Theorem 2.7 together with (A<sub>1</sub>) and (A<sub>2</sub>), we produce an extension  $s_j^{(km+p)}$  of  $s^k \otimes s_j^{(p)}$ . we proceed it by induction and use every time the family of extensions  $(s_j^{(km+p-1)})_{j=1, \dots, N_{p-1}}$  constructed in the precedent induction step to concentrate the singularities along the zeros of  $s$ . For  $p = m - 1$ , we use the family  $(s_j^{(k-1)m+m-1})_{j=1, \dots, N_{m-1}}$  instead. Starting from  $k = 1$  and  $p = 0$ , the picture of this inductive process presents in the following manner :



- Construction of a metric on  $K_{\mathcal{X}}$  by considering the limit of  $m$ -th root  $\sum |s_j^{(km)}|^{\frac{2}{k}}$  : the family of sections  $(s_j^{(km)})_{j=1, \dots, N_0}$  constructed in the first step induces then a metric on the fiber  $(km)K_{\mathcal{X}} + A$ , its  $k$ -th root is therefore a metric on the  $\mathbb{Q}$ -fiber  $mK_{\mathcal{X}} + \frac{1}{k}A$ . Let then  $k$  go to infinity, we hope to produce a metric on  $(m-1)K_{\mathcal{X}}$  with respect to which the norm square  $|s|^2$  is integrable. The proof ends by applying for the last time Theorem 2.7 in writing  $mK_{\mathcal{X}} = K_{\mathcal{X}} + (m-1)K_{\mathcal{X}}$ .

To understand more clearly why this method simplifies the proof in [Siu02] (and how the name "One-Tower" comes into the picture), we review briefly the techniques of Siu : to extend the section  $s^k \otimes s_A$  where  $s$  is the section we want to extend and  $s_A$  a section of a sufficiently ample bundle  $A$ , the author proceeds first the contractions of this section with local sections of  $-K_{\mathcal{X}}$  till we obtain local sections of  $K_{\mathcal{X}_0} + A$  : indeed, if  $\alpha$  is a local section of  $-K_{\mathcal{X}}$ ,  $\alpha \otimes s^k \otimes s_A$  is a one of  $(km-1)K_{\mathcal{X}_0} + A$ . Apply then the Ohsawa-Takegoshi theorem to extend these local sections of  $K_{\mathcal{X}_0} + A$  to  $K_{\mathcal{X}} + A$ . With these new extensions, we can climb up step by step until arrive to  $(mk)K_{\mathcal{X}} + A$ , hence obtain an extension of  $s^k \otimes s_A$ . To conclude by graphs, the method of Siu can be seen like follows :

$$\begin{array}{ccc}
 kmK_{\mathcal{X}_0} + A & \longrightarrow & (km)K_{\mathcal{X}} + A \\
 \downarrow & & \uparrow \\
 (km-1)K_{\mathcal{X}_0} + A & \dashrightarrow & (km-1)K_{\mathcal{X}} + A \\
 \downarrow & & \uparrow \\
 \vdots & & \vdots \\
 \downarrow & & \uparrow \\
 2K_{\mathcal{X}_0} + A & \dashrightarrow & 2K_{\mathcal{X}} + A \\
 \downarrow & & \uparrow \\
 K_{\mathcal{X}_0} + A & \dashrightarrow & K_{\mathcal{X}} + A
 \end{array}$$

As we mentioned a little bit earlier, the "One-Tower" method allows us to obtain numbers of versions of the invariance of plurigenera. We quote quite a few of them ; the first result is due to B. Claudon [Cla07] :

**Theorem 2.8.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a projective family and  $(L, h_L)$  a pseudo-effective line bundle on  $\mathcal{X}$  with  $h_{\mathcal{X}_0}$  well defined. Suppose that  $\mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0}) = \mathcal{O}_{\mathcal{X}_0}$ . Then for every integer  $m \geq 1$ , the restriction map*

$$H^0(\mathcal{X}, m(K_{\mathcal{X}} + L)) \rightarrow H^0(\mathcal{X}_0, m(K_{\mathcal{X}_0} + L))$$

*is surjective.*

We remark that although Theorem 2.8 looks very similar to Theorem 2.6, it is not a consequence of the latter : indeed, Theorem 2.6 allows to extend sections in  $\mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0}^{\otimes m})$ . In general, we have

$$\mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0}^{\otimes m}) \subset \mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0})^{\otimes m} (= \mathcal{O}_{\mathcal{X}_0})$$

with strict conclusion. Theorem 2.8 states that under the assumption  $\mathcal{I}(\mathcal{X}_0, h_{\mathcal{X}_0}) = \mathcal{O}_{\mathcal{X}_0}$ , we can extend all sections of  $m(K_{\mathcal{X}} + L)$ .

Combined Theorem 2.6 with Theorem 2.8, J-P. Demailly obtained the following result [Dem06] :

**Theorem 2.9.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a projective family and  $(L_j, h_j)$  ( $0 \leq j \leq m-1$ ) are pseudo-effective line bundles on  $\mathcal{X}$ . Suppose that*

- (i) *For all  $0 \leq j \leq m-1$ , the restriction of  $h_j$  to  $\mathcal{X}_0$  is well defined.*
- (ii) *For all  $1 \leq j \leq m-1$ , the multiplier ideal sheaf  $\mathcal{I}(h_j|_{\mathcal{X}_0})$  is trivial.*

*Then, every section*

$$s \in H^0 \left( \mathcal{X}_0, (mK_{\mathcal{X}_0} + \sum_{j=0}^{m-1} L_j) \otimes \mathcal{I}(h_0|_{\mathcal{X}_0}) \right)$$

*admits an extension to  $\mathcal{X}$ .*

Inspired by an algebro-geometric set-up, in [BP10], the authors generalized this kind of extension theorems in the context of twisted pluricanonical forms and metrics of adjoint  $\mathbb{R}$ -bundles. We quote this result to end the introduction :

Given an ideal  $\mathcal{I} \subset \mathcal{O}_X$  and a positive integer  $k$ . Define  $\overline{\mathcal{I}}^{(k)}$  the integral closure of the  $k$ -th power of  $\mathcal{I}$ : if  $x \in X$  and  $(f_1, \dots, f_r)$  are generators of  $\mathcal{I}$  at  $(X, x)$ ,  $\overline{\mathcal{I}}^{(k)}$  can be described locally at  $x$  as

$$\overline{\mathcal{I}}_x^{(k)} := \{g \in \mathcal{O}_{X,x}; |g|^{\frac{2}{k}} \leq \sum_{\alpha} |f_{\alpha}|^2\}.$$

Their result is the following :

**Theorem 2.10.** *Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a projective family over the unit disk and let  $(L, h)$  be a hermitian line bundle with the properties*

- (i) *The curvature current of  $(L, h_L)$  is positive, i.e.  $\sqrt{-1}\Theta_{h_L}(L) \geq 0$ ;*
- (ii) *The restriction of the metric  $h_L$  to the central fiber is well defined, i.e.  $h_L|_{\mathcal{X}_0} \neq -\infty$ .*

*Then there exists a universal constant  $C_0 > 0$  such that for any positive integer  $p \geq q$  and for any section*

$$u \in H^0 \left( \mathcal{X}_0, (pK_{\mathcal{X}_0} + qL) \otimes \overline{\mathcal{I}}(h_L|_{\mathcal{X}_0})^{(q)} \right),$$

*there exists a section*

$$U \in H^0(\mathcal{X}, pK_{\mathcal{X}} + qL)$$

*such that :*

- (i) *Over the central fiber we have  $U|_{\mathcal{X}_0} = u \otimes d\pi^{\otimes p}$ ;*
- (ii) *The next  $L^{\frac{2}{p}}$  integrability condition holds :*

$$\int_{\mathcal{X}} |U|^{\frac{2}{p}} e^{-\frac{q}{p}\varphi_L} \leq C_0 \int_{\mathcal{X}_0} |u|^{\frac{2}{p}} e^{-\frac{q}{p}\varphi_L}.$$

We see that if  $p = q = 1$ , this is precisely the Ohsawa-Takegoshi extension theorem. For  $p \geq 2$ , the qualitative part of the theorem is a generalized invariance of plurigenera.

## 2.2 Preliminary

In the current chapter and the next one, we will use intensively the notion of closed positive currents. Their cohomology classes may be transcendental, i.e. they are not necessarily the curvature form of some metric on a line bundle. We start by recalling a few positivity notions in this context, they are established by J-P. Demailly in [Dem92] :

### 2.2.1 Positive cones

If  $X$  is a compact Kähler manifold, recall that an integral class in  $H^2(X, \mathbb{Z})$  is the *first Chern class of a holomorphic (algebraic) line bundle* if and only if it lies in the *Neron-Severi* group

$$\mathrm{NS}(X) = \ker(H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathcal{O}_X));$$

or equivalently, if and only if it is of  $(1, 1)$  type with respect to the Hodge composition. The real vector space  $\mathrm{NS}_{\mathbb{R}}(X) = \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$  can be seen as a subspace of  $H^{1,1}(X, \mathbb{R})$  of real  $(1, 1)$  cohomology classes. Remark that a  $(1, 1)$  class is not necessarily integral or rational. Nevertheless, the positivity concepts for line bundles (see Subsection 1.2.2) have natural generalizations to these  $(1, 1)$  classes defined on compact Kähler manifolds. We recall next their definitions (cf. [Dem09]) :

**Definition 2.11.** *Let  $(X, \omega)$  be a compact Kähler manifold.*

- (i) *The Kähler cone is the set  $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$  of cohomology classes  $\{\omega\}$  of Kähler forms. This is an open convex cone.*
- (ii) *The closure  $\overline{\mathcal{K}}$  of the Kähler cone consists of classes  $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$  such that for every  $\varepsilon > 0$  the sum  $\{\alpha + \varepsilon\omega\}$  is Kähler, or equivalently, for every  $\varepsilon > 0$ , there exists a smooth function  $\varphi_\varepsilon$  on  $X$  such that  $\alpha + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon \geq -\varepsilon\omega$ . We say that  $\overline{\mathcal{K}}$  is the cone of nef  $(1, 1)$ -classes.*
- (iii) *The pseudo-effective cone is the set  $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$  of cohomology classes  $\{T\}$  of closed positive currents of type  $(1, 1)$ . This is a closed convex cone.*
- (iv) *The interior  $\mathcal{E}^0$  of  $\mathcal{E}$  consists of classes which still contain a closed positive current after one subtracts  $\varepsilon\{\omega\}$  for  $\varepsilon > 0$  small, in other words, they are classes of closed  $(1, 1)$ -currents  $T$  such that  $T \geq \varepsilon\omega$ . Such a current will be called a Kähler current, and we say that  $\{T\} \in H^{1,1}(X, \mathbb{R})$  is a big  $(1, 1)$ -class.*

By the above definition, we have  $\overline{\mathcal{K}} \subset \mathcal{E}$  and they are in general different. For example, a Kähler class  $\{\alpha\}$  satisfies  $\int_Y \alpha^p > 0$  for every  $p$ -dimensional analytic set  $Y$ . On the other hand, if  $X$  is the surface obtained by blowing-up  $\mathbb{P}^2$  in one point, then the exceptional divisor  $E \simeq \mathbb{P}^1$  possesses a cohomology class  $\{\alpha\}$  such that  $\int_E \alpha = E^2 = -1$ . Therefore  $\{\alpha\} \notin \overline{\mathcal{K}}$  but  $\{\alpha\} = \{(E)\} \in \mathcal{E}$ .

If  $X$  is projective, all Chern classes  $c_1(L)$  of line bundles lie in  $\mathrm{NS}(X)$ , and all classes of real divisors  $D = \sum c_j D_j$ ,  $c_j \in \mathbb{R}$ , lie in  $\mathrm{NS}_{\mathbb{R}}(X)$ . We refer to classes of  $H^{1,1}(X, \mathbb{R})$  contained in  $\mathrm{NS}_{\mathbb{R}}(X)$  as *algebraic classes*, otherwise, *transcendental classes* for those who are not contained in  $\mathrm{NS}_{\mathbb{R}}(X)$ . The above definitions have natural algebraic interpretations. Indeed, denote

$$\mathcal{K}_{\mathrm{NS}} = \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X), \quad \mathcal{E}_{\mathrm{NS}} = \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X),$$

we have the following result ([Dem09]) :

**Theorem 2.12.** *Let  $X$  be a projective manifold. Then*

- (i)  $\mathcal{K}_{\text{NS}}$  is equal to the open cone  $\text{Amp}(X)$  generated by classes of ample (or very ample) divisors  $A$ .
- (ii) The interior  $\mathcal{E}_{\text{NS}}^0$  is the cone  $\text{Big}(X)$  generated by classes of big divisors, namely, divisors  $D$  such that  $h^0(X, \mathcal{O}(kD)) \geq ck^{\dim X}$  for  $k$  large.
- (iii)  $\mathcal{E}_{\text{NS}}$  is the closure  $\overline{\text{Eff}(X)}$  of the cone generated by classes of effective divisors, i.e.  $D = \sum c_j D_j$ ,  $c_j \in \mathbb{R}_+$ .
- (iv) The closed cone  $\overline{\mathcal{K}}_{\text{NS}}$  consists of the closure  $\overline{\text{Nef}(X)}$  of the cone generated by nef divisors  $D$  (or nef line bundles  $L$ ), namely effective integral divisors  $D$  such that  $D \cdot C \geq 0$  for every curve  $C$ , also equal to  $\text{Amp}(X)$ .

## 2.3 Special settings and the main theorem

Our main theorem of this chapter is to establish an extension result for the closed positive currents in the class  $c_1(K_{\mathcal{X}} + L)|_{\mathcal{X}_0}$ . This theorem can be seen as a natural generalization of the invariance of plurigenera result in [Siu02].

We present in the following the context in which we are going to work, it is inspired by the usual hypotheses in algebraic geometry.

Let  $\pi : \mathcal{X} \rightarrow \mathbb{D}$  be a smooth projective family over the unit disk  $\mathbb{D}$  and  $L$  be a  $\mathbb{Q}$ -line bundle over  $\mathcal{X}$ . We will consider the following set-up :

- (i) We assume that there exists an integer  $m_0 \in \mathbb{Z}_+$  such that  $m_0 L$  is a line bundle, and such that we have the next decomposition

$$m_0 L = L_1 + L_2 + \dots + L_{m_0-1},$$

here  $L_j$  are line bundles. For each  $j$ , we assume that the bundle  $L_j$  is endowed with a (possibly singular) metric  $h_j$  such that the curvature current

$$\sqrt{-1}\Theta_{h_j}(L_j) \geq 0$$

is positive. We equally assume that the restriction  $h_j|_{\mathcal{X}_0}$  is well defined, in the sense that the local weights  $e^{-\varphi_{L_j}}$  of  $h_j$  are not identically equal to  $-\infty$  when restricted to  $\mathcal{X}_0$ .

- (ii) Let  $T \in c_1(K_{\mathcal{X}} + L)$  be a closed positive current such that  $T|_{\mathcal{X}_0}$  is well defined.
- (iii) There exists a  $\varepsilon > 0$  such that for any  $\delta$  positive and any  $j = 1, \dots, m_0 - 1$ , we have

$$(\mathcal{T}_j) \quad \mathcal{I}((1 - \delta)\varphi_{L_j} + \varepsilon\varphi_{T|_{\mathcal{X}_0}}) = \mathcal{O}_{\mathcal{X}_0}.$$

It is worth mentioning that a similar setting is also appeared in [BP08], where the authors proved an extension theorem in order to show the Kawamata subadjunction theorem in an analytic way. To compare the two settings, we cite here their frameworks :

- We consider a projective, non-singular manifold  $X$ , and let  $\Delta_0 \rightarrow X$  be a  $\mathbb{Q}$ -line bundle, such that for any integer  $m$  large enough and divisible, we have

$$m\Delta_0 = L_1 + \dots + L_{m-1}$$

where for each  $1 \leq j \leq m-1$ ,  $L_j$  is a line bundle endowed with a metric  $h_j$  whose curvature current is assumed to be positive.

- $\Delta_1 \rightarrow X$  is a  $\mathbb{Q}$ -line bundle, whose first Chern class contains a closed positive current  $\Xi$ . We introduce the notation

$$\Delta := \Delta_0 + \Delta_1$$

- $S \subset X$  is a non-singular hypersurface such that  $h_j|_S$  and  $\Xi|_S$  are well defined, in the sense that the local potentials of  $h_j$  are not identically equal to  $-\infty$  when restricted to  $S$ . We assume the existence of a closed positive current

$$T \in c_1(K_X + S + \Delta)$$

such that :

- $(\mathcal{R})$  The restriction of  $T$  to  $S$  is well defined ;
- $(\mathcal{T})$  There is some positive number  $\varepsilon$  such that for any  $\delta > 0$  and any  $j = 1, \dots, m-1$  we have

$$(\mathcal{T}_j) \quad \mathcal{I}((1-\delta)\varphi_{L_j} + \varepsilon\varphi_{T|_S}) = \mathcal{O}_S.$$

- There exists a metric  $h_{K_S+\Delta}$  on the  $\mathbb{Q}$ -line bundle  $K_S + \Delta$  such that the corresponding curvature current is positive and for any positive integer  $m$  we have

$$\mathcal{I}(m\varphi_{K_S+\Delta}) \subset \mathcal{I}(m\varphi_{\Xi|_S}).$$

In a more geometric context, their setting is equivalent to the following :

- (1)  $\Delta_0$  is an effective klt  $\mathbb{Q}$ -divisor ;
- (2)  $\Delta_1, \Delta_2$  are effective, respectively ample  $\mathbb{Q}$ -bundles ;
- (3) The support of  $S + \Delta_0 + \Delta_1$  has normal crossing and  $S$  is not contained in the support of  $\Delta_0 + \Delta_1$  ;
- (4) Let  $\Delta := \Delta_0 + \Delta_1 + \Delta_2$  ; We assume that there exists a closed positive current

$$T \in c_1(K_X + S + \Delta)$$

such that the restriction of  $T$  to any intersection of components of the support of the divisor  $S + \Delta_0 + \Delta_1$  is well defined.

Observe that the difference between our context and that in [BP08] is that the  $\mathbb{Q}$ -line bundle  $\Delta$  in their setting consists of two parts :  $\Delta_0$  and  $\Delta_1$ , where  $\Delta_0$  has similar properties as the  $\mathbb{Q}$ -line bundle  $L$  in our setting while  $\Delta_1$  contains an ample  $\mathbb{Q}$ -line bundle (respectively the  $\Delta_2$  in the condition (2)). By contrast, there is no ample assumption in our setting, therefore our result is more general.

The hypothesis  $(\mathcal{T}_j)$  in both settings implies that the multiplier ideal sheaf of the metric  $(1-\delta)\varphi_j$  is automatically trivial for any  $\delta > 0$ . This assumption intuitively means that the singularities of  $\varphi_j$  and  $\varphi_T$  are transversal.

Within this framework, our main theorem states as follows : denote for  $0 < \eta \ll 1$ ,  $\pi_\eta : \mathcal{X}_\eta \rightarrow \mathbb{D}_\eta$  the map defined by restricting  $\pi$  to the disk  $\mathbb{D}_\eta$  of radius  $\eta$ . The main extension theorem we are going to prove is the following :

**Theorem 2.13.** *We consider a non-singular, projective family  $\mathcal{X} \rightarrow \mathbb{D}$  over the unit disk  $\mathbb{D}$ , together with a  $\mathbb{Q}$ -line bundle  $L \rightarrow \mathcal{X}$  with the properties (i)-(iii) above. Let  $\Theta_0$  be a closed positive current on  $\mathcal{X}_0$  such that*

$$\Theta_0 \in c_1(K_{\mathcal{X}} + L)|_{\mathcal{X}_0},$$

the first Chern class of the adjoint bundle  $K_{\mathcal{X}} + L$  restricted to  $\mathcal{X}_0$ . Suppose that there exists a  $\delta_0 > 0$  such that

$$\int_{\mathcal{X}_0} e^{(1+\delta_0)(\varphi_{\Theta_0} - \varphi_L)} dV_{\omega_0} < \infty.$$

Then there exists a closed positive current  $\Theta$  on  $\mathcal{X}$  in the cohomology class of the  $\mathbb{Q}$ -bundle  $K_{\mathcal{X}} + L$  and a constant  $C > 0$  such that

$$\int_{\mathcal{X}_\eta} e^{\varphi_\Theta - \varphi_L} dV_{\omega_\eta} \leq C \int_{\mathcal{X}_0} e^{\varphi_{\Theta_0} - \varphi_L} dV_{\omega_0}.$$

In the statement above, we denote by  $\varphi_\Theta$  a well-chosen local potential of the current  $\Theta$  and  $dV_{\omega_0}, dV_{\omega_\eta}$  are Lebesgue volume forms on  $\mathcal{X}_0$  and  $\mathcal{X}_\eta$  respectively. So our result shows that there exists a closed positive current in the first Chern class of  $K_{\mathcal{X}} + L$ , whose restriction to the central fiber is well defined, and less singular than the given current  $\Theta_0$ . We observe that even if one ignores the quantitative part (i.e. the inequality), the theorem above implies that the class  $K_{\mathcal{X}} + L$  is pseudo-effective, provided that its restriction to the central fiber has this property. We will offer a few comments concerning the main techniques needed for the proof of this result in a moment; we mention next that we expect the following transcendental version of Theorem 2.13 to be true. Actually this problem was the main motivation for the questions we explore in this chapter.

**Conjecture 2.14.** *Let  $\alpha \geq 0$  be a semi-positive  $(1,1)$ -form on Kähler family  $\mathcal{X}$ , and let  $\Theta_0 \in \{K_{\mathcal{X}} + \alpha\}|_{\mathcal{X}_0}$  be a closed positive current. Then there exists a current  $\Theta \in \{K_{\mathcal{X}} + \alpha\}$  such that :*

- (1) *The current  $\Theta|_{\mathcal{X}_0}$  is well-defined, and it is less singular than  $\Theta_0$ .*
- (2) *There exists a numerical constant  $C > 0$  such that the following estimate holds true*

$$\int_{\mathcal{X}} e^{\varphi_\Theta - \varphi_\alpha} dV_\omega \leq C \int_{\mathcal{X}_0} e^{\varphi_{\Theta_0} - \varphi_\alpha} dV_{\omega_0}.$$

Indeed, it would be very nice to establish such a result; besides its intrinsic interest, as an application one would obtain the Kähler case of the ‘‘invariance of plurigenera’’, conjectured by Y.-T. Siu in [Siu02]. If  $\mathcal{X} \rightarrow \mathbb{D}$  is a Kähler family over the unit disk  $\mathbb{D}$ , then given a section  $u$  of the bundle  $m_0 K_{\mathcal{X}}|_{\mathcal{X}_0}$ , the conjecture above implies the existence of a metric  $h$  on the bundle  $K_{\mathcal{X}}$ , with positive curvature, and such that the section  $u$  is bounded with respect to  $h^{m_0}$ . Then a result slightly more general than the one we obtain in [Yi11] would show that the section  $u$  extends to the whole family.

## 2.4 Proof of the main theorem

Our arguments for Theorem 2.13 split into two parts; we first show that the next qualitative extension theorem holds.

**Theorem 2.15.** *Let  $\mathcal{X} \rightarrow \mathbb{D}$  be a smooth projective family,  $L$  and  $T$  be respectively a pseudo-effective  $\mathbb{Q}$ -line bundle and a closed positive current with the properties (i)-(iii) mentioned above. Then every section  $\sigma$  of the bundle  $m_0(K_{\mathcal{X}} + L)|_{\mathcal{X}_0}$  extends to  $\mathcal{X}_\eta$ .*

Theorem 2.15 is yet another application of the ‘‘invariance of plurigenera techniques’’ invented by Y.-T. Siu in [Siu98]. We start by showing that there exists an ample line bundle  $A$  on  $\mathcal{X}$

such that the section  $\sigma^k \otimes s_A$  (defined only on the central fiber of  $\pi$ ) extends to  $\mathcal{X}$  for each integer  $k \geq 0$ ; here we denote by  $s_A$  some non-zero section of the bundle  $A$ . Similar ideas are used in [BP08b], [BP10],[BP10b], [Cla07], [Tsu07] and [Tsu11]. Next, our contribution here is to show that one can perform the extensions in an effective manner, in the following sense. Given a bound for the  $L^{2/m_0}$ -norm of the section  $\sigma$  to be extended, we can extend the sections  $\sigma^k \otimes s_A$  with an uniform control of their  $L^{2/km_0}$  norms.

Once this is achieved, the proof of our main result goes as follows. By a result of J.-P. Demailly in [Dem93], the current  $\Theta_0$  can be approximated by the current associated to a certain space of sections of the bundle

$$k(K_{\mathcal{X}} + L)|_{\mathcal{X}_0} + A,$$

where  $A$  is any positive enough, fixed line bundle defined over  $\mathcal{X}$ . Each of the approximating section is extended via the effective version of Theorem 2.13 above (with  $m_0 := k$ ), and the current  $\Theta$  is finally obtained by considering the curvature current associated to the metric given by the  $-$ appropriate root of the  $-$ extensions.

We mention here that similar results are obtained in [BP08b], [BP10b], where the authors are using the convexity properties of the Bergman kernels; here we avoid this technique by using a more direct argument. Also, Theorem 2.13 can be seen as a metric version of the extension results due to Ein-Popa [EP12] and Hacon- McKernan [HM06].

### 2.4.1 Main tools

The first main tool we will use is an extension theorem of Ohsawa-Takegoshi type. The version which is most adapted for our needs was established by Y.-T. Siu (cf. [Siu02]).

**Theorem 2.16.** *Let  $\mathcal{X} \rightarrow \mathbb{D}$  be a smooth projective family and  $L \rightarrow \mathcal{X}$  be a line bundle endowed with a possibly singular metric  $h$  with semi-positive curvature current. Then there exists a (universal) constant  $C_0$  such that for every section  $\sigma \in H^0(\mathcal{X}_0, K_{\mathcal{X}_0} + L)$  satisfying :*

$$\int_{\mathcal{X}_0} \|\sigma\|_h^2 < +\infty,$$

there exists  $\tilde{\sigma} \in H^0(\mathcal{X}, K_{\mathcal{X}} + L)$  with  $\tilde{\sigma}|_{\mathcal{X}_0} = \sigma \wedge dt$  and moreover :

$$\int_{\mathcal{X}} \|\tilde{\sigma}\|_h^2 \leq C_0 \int_{\mathcal{X}_0} \|\sigma\|_h^2.$$

We intend to apply the previous result in order to extend the section  $\sigma$  in Theorem 2.15, hence we write

$$m_0(K_{\mathcal{X}} + L) = K_{\mathcal{X}} + (m_0 - 1)(K_{\mathcal{X}} + L) + L.$$

Therefore we have to produce a metric on the bundle  $(m_0 - 1)(K_{\mathcal{X}} + L) + L$ , such that :

- it has semi-positive curvature;
- the section  $\sigma$  belongs to the multiplier ideal sheaf defined by its restriction to  $\mathcal{X}_0$  (the said restriction is therefore assumed to be well defined).

The algorithm used to produce this metric was invented by Y.-T. Siu; actually, we remark that except for the work by M. Levine in [Lev83], all the extension results concerning the twisted pluricanonical forms are using this type of arguments—cf. proof of the claim  $\mathcal{P}_{k,p}$  below.

Another main tool we will use is the regularization theorem for closed positive currents (cf. Theorem 1.17).

We end this section by the following remark.



**Remark 2.17.** *The hypotheses (i) – (iii) at the beginning of the paper are motivated by algebraic geometry. Let  $D = \sum_{j=1}^N d^j Y_j$  be an effective divisor with  $d^j \in \mathbb{Q}$  such that for every  $j$ ,  $0 < d^j < 1$ . If we denote  $m_0 = \text{lcm}(d^j)$  and assume that the hypersurfaces  $(Y_j)$  have simple normal crossings, then we have*

$$m_0 D = L_1 + \cdots + L_{m_0-1},$$

as observed by L. Ein and M. Popa in [EP12], with  $L_j = \sum_{i \in I_j} Y_i$ . In this context, the hypothesis  $(\mathcal{T}_j)$  in (iii) means that the algebraic singularities of  $T|_{\mathcal{X}_0}$  are transversal to the restrictions on  $(Y_i|_{\mathcal{X}_0})$ .

### 2.4.2 Proof of Theorem 2.15.

We will extend the section  $\sigma$  by constructing a metric on the bundle  $m_0(K_{\mathcal{X}} + L)$  with the properties specified in the bullets of the previous section. This is done by the usual procedure (cf. for example, [Pău07], [Cla07]), i.e. by showing first that such a metric can be constructed if one introduces an auxiliary positive enough bundle  $A$  : for each  $k \geq 1$ , we obtain our metric on the bundle

$$km_0(K_{\mathcal{X}} + L) + A,$$

where  $A$  does not depend on  $k$ . The metric we seek is obtained by removing the bundle  $A$  via a limiting procedure ; it is for this reason that we have to perform the intermediate steps in an effective manner.

To start with, we fix a notation : for each integer  $p \in [0, m_0 - 1]$ , we denote  $L^{(p)}$  by

$$L^{(p)} := pK_{\mathcal{X}} + L_1 + \cdots + L_p.$$

Let  $A$  be an ample line bundle over  $\mathcal{X}$  with the following properties :

- (1) For each  $0 \leq p \leq m_0 - 1$ ,  $L^{(p)} + A$  is generated by its global sections  $(s_j^{(p)})_{j=1, \dots, N_p}$ .
- (2) Every section of  $m_0(K_{\mathcal{X}} + L) + A$  extends to  $\mathcal{X}$ .

We formulate next an inductive statement :

$\mathcal{P}_{k,p}$  : For  $k \in \mathbb{N}$  and  $0 \leq p \leq m_0 - 1$ , the section

$$\sigma^k \otimes s_j^{(p)} \in H^0(\mathcal{X}_0, L^{(p)} + km_0(K_{\mathcal{X}} + L) + A)$$

admits an extension to  $\mathcal{X}$ , for every  $j = 1, \dots, N_p$ .

We remark that the inductive procedure depends on the integers  $k$  and  $p$ . The statement above will be obtained by applying several times Theorem 2.16. Once the statement  $\mathcal{P}_{k,p}$  is shown to be true, we prove that we can choose a special extension of

$$\sigma^k \otimes s_j^{(0)}$$

whose  $L^{\frac{2}{km_0}}$  is bounded by a quantity which is independent of  $k$ . This will enable us to construct a metric on  $km_0(K_{\mathcal{X}_\eta} + L) + A$  whose singularities over the central fiber are given by the zeros of the section  $\sigma$ . The next step is to extract its  $k$ th root and pass to the limit to produce a metric  $h_\infty$  on  $m_0(K_{\mathcal{X}_\eta} + L)$ . A last application of Ohsawa-Takegoshi theorem for

$h_\infty$  twisted with  $h_{L_j}$  enables us to obtain the extension of  $\sigma$ .

We proceed now to the proof of the fact that  $\mathcal{P}_{k,p}$  is verified for all  $k, p$ ; this will be done by iterating the following steps.

- (I) The section  $\sigma \otimes s_j^{(0)} \in H^0(\mathcal{X}_0, m_0(K_{\mathcal{X}} + L) + A)$  admits an extension  $U_j^{(m_0)}$  for each  $j = 1, \dots, N_0$  by the choice of the bundle  $A$ .
- (II) We use the set of sections  $(U_j^{(m_0)})$  in order to construct a metric

$$\varphi^{(m_0)} = \log \sum |U_j^{(m_0)}|^2$$

on the bundle  $m_0(K_{\mathcal{X}} + L) + A$ .

- (III) We consider next the section

$$\sigma \otimes s_j^{(1)} \in H^0(\mathcal{X}_0, L^{(1)} + m_0(K_{\mathcal{X}} + L) + A|_{\mathcal{X}_0}).$$

Remark that the bundle

$$\begin{aligned} &L^{(1)} + m_0(K_{\mathcal{X}} + L) + A \\ &= K_{\mathcal{X}} + L_1 + m_0(K_{\mathcal{X}} + L) + A \end{aligned}$$

can be written as  $K_{\mathcal{X}} + F$  with

$$F = L_1 + m_0(K_{\mathcal{X}} + L) + A.$$

Let  $0 < \delta \ll \varepsilon \ll 1$  be real numbers. We endow the bundle  $F$  with the metric given by

$$\varphi_{\delta, \varepsilon}^{(m_0+1)} := (1 - \delta)\varphi_{L_1} + \delta\tilde{\varphi}_{L_1} + (1 - \varepsilon)\varphi^{(m_0)} + \varepsilon(\varphi_A + m_0\varphi_T) \quad (2.1)$$

where the metric  $\tilde{\varphi}_{L_1}$  is smooth (no curvature requirements) and  $\varphi_{L_1}$  is the singular metric with semi-positive curvature whose existence is given by Hypothesis (i). We have to introduce the parameters  $\delta$  and  $\varepsilon$  in the expression of the metric  $\varphi_{\delta, \varepsilon}^{(m_0+1)}$  for the following reasons. In the first place, the section  $\sigma$  is not assumed to be  $L^2$  with respect to  $h_{L_1}$ . Indeed, even if  $\mathcal{I}((1 - \delta)\varphi_{L_1}|_{\mathcal{X}_0}) = \mathcal{O}_{\mathcal{X}_0}$ , it may very well happen that  $\mathcal{I}(\varphi_{L_1}) \not\subseteq \mathcal{O}_{\mathcal{X}}$ ; as a consequence, we have to use  $\delta$  in order to attenuate the singularities of  $\varphi_{L_1}$ . But then a negative part appears in the curvature from the factor  $\delta\tilde{\varphi}_{L_1}$ , so we have to introduce  $\varepsilon$  to compensate the negativity. By adjusting the parameters, we see that  $\varphi_{\delta, \varepsilon}^{(m_0+1)}$  has semi-positive curvature.

On the central fiber, the  $L^2$  norm of the section  $\sigma \otimes s_j^{(1)}$  with respect to the metric constructed in (2.1) reads as

$$\begin{aligned}
 & \int_{\mathcal{X}_0} \left| \sigma \otimes s_j^{(1)} \right|^2 e^{-\varphi_{\delta, \varepsilon}^{(m_0+1)}} dV \\
 &= \int_{\mathcal{X}_0} \left| \sigma \otimes s_j^{(1)} \right|^2 e^{-(1-\delta)\varphi_{L_1} - \delta\tilde{\varphi}_{L_1} - (1-\varepsilon)\varphi^{(m_0)} - \varepsilon(\varphi_A + m_0\varphi_T)} dV \\
 &= \int_{\mathcal{X}_0} \frac{\left| \sigma \otimes s_j^{(1)} \right|^2}{\sum \left| U_q^{(m_0)} \right|^{2(1-\varepsilon)}} e^{-(1-\delta)\varphi_{L_1} - \delta\tilde{\varphi}_{L_1} - \varepsilon\varphi_A - \varepsilon m_0\varphi_T} dV \\
 &= \int_{\mathcal{X}_0} \frac{\left| \sigma \otimes s_j^{(1)} \right|^2}{\sum_{q=1}^{N_0} \left| \sigma \otimes s_q^{(0)} \right|^{2(1-\varepsilon)}} e^{-(1-\delta)\varphi_{L_1} - \delta\tilde{\varphi}_{L_1} - \varepsilon\varphi_A - \varepsilon m_0\varphi_T} dV \\
 &= \int_{\mathcal{X}_0} \frac{|\sigma|^{2\varepsilon} \left| s_j^{(1)} \right|^2}{\sum \left| s_q^{(0)} \right|^{2(1-\varepsilon)}} < +\infty,
 \end{aligned}$$

by transversality hypothesis ( $\mathcal{T}_1$ ) together with the choice of the bundle  $A$ .

(IV) Applying the extension theorem 2.16 we obtain the section  $U_j^{(m_0+1)}$  on  $\mathcal{X}$ , extending  $\sigma \otimes s_j^{(1)}$ , together with the estimate

$$\int_{\mathcal{X}} \left| U_j^{(m_0+1)} \right|^2 e^{-\varphi_{\delta, \varepsilon}^{(m_0+1)}} dV < +\infty.$$

The statement  $\mathcal{P}_{k,p}$  is proved by repeating several times the algorithms (I)-(IV). Indeed, assume that  $\mathcal{P}_{k,p}$  has been checked for some pair of  $(k, p)$  and consider the set of global sections

$$U_j^{(km_0+p)} \in H^0(\mathcal{X}, L^{(p)} + km_0(K_{\mathcal{X}} + L) + A)$$

which extend  $\sigma^k \otimes s_j^{(p)}$ . They induce a metric on the above bundle, which we denote by  $\varphi^{(km_0+p)}$ .

- If  $p < m_0 - 1$ , then we define the family of sections

$$\sigma^k \otimes s_j^{(p+1)} \in H^0(\mathcal{X}_0, L^{(p+1)} + km_0(K_{\mathcal{X}} + L) + A|_{\mathcal{X}_0})$$

on  $\mathcal{X}_0$ . Like in step (III), we have that

$$L^{(p+1)} = K_{\mathcal{X}} + L_{p+1} + L^{(p)}.$$

To apply Theorem 2.16, we have to exhibit a metric on the bundle

$$F := L_{p+1} + L^{(p)} + km_0(K_{\mathcal{X}} + L) + A.$$

To this end, define

$$\varphi_{\delta, \varepsilon}^{(km_0+p+1)} := (1-\delta)\varphi_{L_{p+1}} + \delta\tilde{\varphi}_{L_{p+1}} + (1-\varepsilon)\varphi^{(km_0+p)} + \varepsilon(\tilde{\varphi}_{L^{(p)}} + m_0k\varphi_T + \varphi_A) \quad (2.2)$$

and see that the curvature requirement is satisfied once we choose the parameters  $\varepsilon, \delta$  satisfying the following conditions :

- (A)  $m_0k\varepsilon \ll 1$ .

(B) Absorb the negativity in the smooth curvature terms in (2.2), thus we are forced to have  $\delta \ll \varepsilon$ .

We may clearly find such  $\delta$  and  $\varepsilon$  with the properties that fit the requirement of (A) and (B). Therefore the curvature requirement is satisfied. For the  $L^2$  requirement, check that on the central fiber, we have the following estimate :

$$\begin{aligned}
 & \int_{\mathcal{X}_0} \left| \sigma^k \otimes s_j^{(p+1)} \right|^2 e^{-\varphi_{\delta, \varepsilon}^{(km_0+p+1)}} dV \\
 &= \int_{\mathcal{X}_0} \left| \sigma^k \otimes s_j^{(p+1)} \right|^2 e^{-(1-\delta)\varphi_{L_{p+1}} - \delta\tilde{\varphi}_{L_{p+1}} - (1-\varepsilon)\varphi^{(km_0+p)} - \varepsilon(\tilde{\varphi}_{L^{(p)}} + \varphi_A + km_0\varphi_T)} dV \\
 &= \int_{\mathcal{X}_0} \frac{\left| \sigma^k \otimes s_j^{(p+1)} \right|^2}{\sum \left| U_q^{(km_0+p)} \right|^{2(1-\varepsilon)}} e^{-(1-\delta)\varphi_{L_{p+1}} - \delta\tilde{\varphi}_{L_{p+1}} - \varepsilon\tilde{\varphi}_{L^{(p)}} - \varepsilon\varphi_A - \varepsilon km_0\varphi_T} dV \\
 &= \int_{\mathcal{X}_0} \frac{\left| \sigma^k \otimes s_j^{(p+1)} \right|^2}{\sum_{q=1}^{N_p} \left| \sigma^k \otimes s_q^{(p)} \right|^{2(1-\varepsilon)}} e^{-(1-\delta)\varphi_{L_{p+1}} - \delta\tilde{\varphi}_{L_{p+1}} - \varepsilon\tilde{\varphi}_{L^{(p)}} - \varepsilon\varphi_A - \varepsilon km_0\varphi_T} dV \\
 &< +\infty,
 \end{aligned}$$

thanks again to the transversality hypothesis ( $\mathcal{T}_{p+1}$ ) together with the fact that the sections  $(s_q^{(p)})_q$  do not have common zeroes.

In conclusion, we get an extension  $U_j^{(km_0+p+1)}$  of  $\sigma^k \otimes s_j^{(p+1)}$  on  $\mathcal{X}$  with the estimate

$$\int_{\mathcal{X}} \left| U_j^{(km_0+p+1)} \right|^2 e^{-\varphi_{\varepsilon, \delta}^{(km_0+p+1)}} dV < +\infty.$$

- If  $p = m_0 - 1$ , we consider the family of sections

$$\sigma^k \otimes s_j^{(0)} \in H^0(\mathcal{X}_0, (k+1)m_0(K_{\mathcal{X}} + L) + A|_{\mathcal{X}_0})$$

where  $(k+1)m_0(K_{\mathcal{X}} + L) + A$  can be written like

$$\begin{aligned}
 & (k+1)m_0(K_{\mathcal{X}} + L) + A \\
 &= m_0(K_{\mathcal{X}} + L) + km_0(K_{\mathcal{X}} + L) + A \\
 &= K_{\mathcal{X}} + (m_0 - 1)K_{\mathcal{X}} + m_0L + km_0(K_{\mathcal{X}} + L) + A \\
 &= K_{\mathcal{X}} + (m_0 - 1)K_{\mathcal{X}} + L_1 + \dots + L_{m_0-1} + km_0(K_{\mathcal{X}} + L) + A \\
 &= K_{\mathcal{X}} + L^{(m_0-1)} + km_0(K_{\mathcal{X}} + L) + A.
 \end{aligned}$$

Consider the metric  $e^{-\varphi^{(km_0+m_0-1)}}$  on the line bundle

$$F := L^{(m_0-1)} + km_0(K_{\mathcal{X}} + L) + A$$

defined by  $U_j^{(km_0+m_0-1)}$ . The curvature requirement is satisfied by the inductive hypothesis. It remains to check the  $L^2$  integral on the central fiber. We have,

$$\begin{aligned}
 & \int_{\mathcal{X}_0} \left| \sigma^{k+1} \otimes s_j^{(0)} \right|^2 e^{-\varphi^{(km_0+m_0-1)}} dV \\
 &= \int_{\mathcal{X}_0} \frac{\left| \sigma^{k+1} \otimes s_j^{(0)} \right|^2}{\sum_q \left| \sigma^k \otimes s_q^{(m_0-1)} \right|^2} dV \\
 &\leq C' \int_{\mathcal{X}_0} \frac{\left| \sigma \otimes s_j^{(0)} \right|^2}{\sum_q \left| s_q^{(m_0-1)} \right|^2} dV \\
 &< +\infty.
 \end{aligned}$$

Therefore, by Theorem 2.16, we get an extension  $U_j^{m_0(k+1)}$  of  $\sigma^{k+1} \otimes s_j^{(0)}$  on  $\mathcal{X}$  with the estimate

$$\int_{\mathcal{X}} \left| U_j^{(k+1)m_0} \right|^2 e^{-\varphi^{(km_0+m_0-1)}} dV < +\infty.$$

The inductive statement is therefore proved and this ends the first part of the proof of Theorem 2.15. In conclusion, we have a sequence of metrics  $e^{-\varphi^{(km_0)}}$  with semi-positive curvature on the bundle

$$km_0(K_{\mathcal{X}} + L) + A$$

whose singularities when restricted on the central fiber are equivalent to the zero set of the section  $\sigma$ . We would like next to consider the metric  $\lim_{k \rightarrow \infty} \frac{1}{k} \varphi^{(km_0)}$ , but a-priori this is not possible since we do not have an upper bound for the weights  $\varphi^{(km_0)}$ . Indeed, the inductive procedure requires that the parameter  $\varepsilon$  used at the step “ $k$ ” to be of order  $\frac{c}{k}$  with  $c \ll 1$  fixed. Hence, because of this reason we cannot conclude that the family of functions

$$\frac{1}{k} \varphi^{(km_0)}$$

is uniformly bounded from above, even if the estimate we obtain thanks to Theorem 2.16 is far more precise than the one used above. Indeed, one can see that we cannot use Hölder inequality as in [Siu98]. In order to proceed, we have to find some special extension for which the  $L^{\frac{2}{km_0}}$  norm is uniformly bounded.

To this end, we first remark that the set

$$\mathcal{E}_{k,j} = \{U \in H^0(\mathcal{X}_\eta, km_0(K_{\mathcal{X}_\eta} + L) + A), U|_{\mathcal{X}_0} = \sigma^k \otimes s_j^{(0)}\}$$

is non-empty : this is precisely what  $\mathcal{P}_{k,0}$  states.

Moreover, we show next the following statement holds true : *there exists a positive number  $\eta$  such that for any element  $U$  of the set  $\mathcal{E}_{k,j}$  we have :*

$$\|U\|_L^{\frac{2}{km_0}} := \int_{\mathcal{X}_\eta} |U|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0} \varphi_A} < +\infty. \quad (2.3)$$

Indeed, by the definition of  $\varphi_L$ ,

$$\varphi_L := \frac{1}{m_0} \sum_{j=1}^{m_0-1} \varphi_{L_j}.$$

Hypothesis  $(\mathcal{T}_j)$  as well as Hölder inequality yield

$$\left( \int_{\mathcal{X}_0} e^{-\varphi_L} \right)^{m_0-1} \leq \prod_{j=1}^{m_0-1} \int_{\mathcal{X}_0} e^{-\frac{m_0-1}{m_0} \varphi_{L_j}} < \infty,$$

with  $\delta = \frac{1}{m_0}$  in Hypothesis  $(\mathcal{T}_j)$ .

In other words, the multiplier ideal sheaf associated to the metric  $e^{-\varphi_L}|_{\mathcal{X}_0}$  is equal to the structural sheaf. As a consequence of the local version of Ohsawa-Takegoshi theorem, there exists a small family  $\mathcal{X}_\eta \subset \mathcal{X}$  such that

$$e^{-\varphi_L} \in L^1(\mathcal{X}_\eta)$$

meaning that the multiplier ideal sheaf of the metric  $e^{-\varphi_L}$  is trivial when restricted to  $\mathcal{X}_\eta$ .

By Hölder inequality, we obtain

$$\begin{aligned} & \int_{\mathcal{X}_\eta} |U|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0} \varphi_A} \\ & \leq C \left( \int_{\mathcal{X}_\eta} |U|^2 e^{-\varphi_L} \right)^{\frac{1}{km_0}} \left( \int_{\mathcal{X}_\eta} e^{-\varphi_L} \right)^{\frac{km_0-1}{km_0}} \\ & < +\infty \end{aligned}$$

thus the inequality (2.3) is proved.

In conclusion, the set  $(\mathcal{E}_{k,j}, \|\cdot\|_L^{\frac{2}{km_0}})$  becomes a semi-normed space. The semi-norm defined in (2.3) is a positive function defined on  $(\mathcal{E}_{k,j}, \|\cdot\|_L^{\frac{2}{km_0}})$ , and we denote by  $x_{\min} \geq 0$  the infimum of the values that this function assume. By definition, there exists a sequence

$$\{U_n^j\} \in (\mathcal{E}_{k,j}, \|\cdot\|_L^{\frac{2}{km_0}})$$

which verifies the following relation

$$\lim_{n \rightarrow \infty} \|U_n^j\|_L^{\frac{2}{km_0}} = x_{\min}.$$

By the standard properties of the holomorphic functions, we see that we can extract a limit  $U_{\min}^j$  from the sequence  $(U_n^j)_n$ ; clearly we have  $U_{\min}^j \in \mathcal{E}_{k,j}$ . Fatou's lemma shows that

$$\int_{\mathcal{X}_\eta} |U_{\min}^j|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0} \varphi_A} dV \leq \lim_{n \rightarrow \infty} \int_{\mathcal{X}_\eta} |U_n^j|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0} \varphi_A} dV = x_{\min}.$$

That is to say,  $U_{\min}^j$  is the section in  $\mathcal{E}_{j,k}$  which minimize the above seminorm.

We claim next that there exists a constant  $C > 0$  such that for every  $k$  and  $j$ , the minimizing section  $U_{\min}^j$  satisfies the uniform estimate

$$\int_{\mathcal{X}_\eta} |U_{\min}^j|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0}\varphi_A} dV \leq C$$

with  $C$  a universal constant which does not depend on  $k$ .

In order to prove the claim above—which despite of its simplicity, is one of the important steps in the proof—we consider the bundle

$$km_0(K_{\mathcal{X}_\eta} + L) + A = K_{\mathcal{X}_\eta} + (km_0 - 1)(K_{\mathcal{X}_\eta} + L) + L + A.$$

It is the adjoint bundle of  $(km_0 - 1)(K_{\mathcal{X}_\eta} + L) + L + A$ , and we may endow the latter with the metric induced by  $U_{\min}^j$  raised to the power  $\frac{km_0-1}{km_0}$  and twisted with the metric of  $L + \frac{A}{km_0}$ . For every  $k$ , the metric obtained in this way has semi-positive curvature and can be restricted to the central fiber, as it is the case for the metric of  $L$ , and the section  $U_{\min}^j$  is not identically zero on  $\mathcal{X}_0$ .

The section  $\sigma^k \otimes s_j^{(0)} \in H^0(\mathcal{X}_0, km_0(K_{\mathcal{X}_\eta} + L) + A)$  is square integrable with respect to the metric we have just defined. Indeed, we have

$$\begin{aligned} & \int_{\mathcal{X}_0} \frac{|\sigma^k \otimes s_j^{(0)}|^2}{|\sigma^k \otimes s_j^{(0)}|^{\frac{2(km_0-1)}{km_0}}} e^{-\varphi_L - \frac{1}{km_0}\varphi_A} \\ &= \int_{\mathcal{X}_0} |\sigma^k \otimes s_j^{(0)}|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0}\varphi_A} \\ &\leq C \int_{\mathcal{X}_0} |\sigma|^{\frac{2}{m_0}} e^{-\varphi_L} \\ &\leq C \left( \int_{\mathcal{X}_0} |\sigma|^2 e^{-\varphi_L} \right)^{\frac{1}{m_0}} \left( \int_{\mathcal{X}_0} e^{-\varphi_L} \right)^{\frac{m_0-1}{m_0}} \\ &\leq C_2 \end{aligned}$$

where  $C_2$  does not depend on  $k$ . By the standard Ohsawa-Takegoshi theorem, there exist extensions

$$U_1^j \in H^0(\mathcal{X}_\eta, km_0(K_{\mathcal{X}_\eta} + L) + A)$$

such that

$$\int_{\mathcal{X}_\eta} \frac{|U_1^j|^2}{|U_{\min}^j|^{\frac{2(km_0-1)}{km_0}}} e^{-\varphi_L - \frac{1}{km_0}\varphi_A} \leq C_0 \int_{\mathcal{X}_0} |\sigma^k \otimes s_j^{(0)}|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0}\varphi_A}.$$

Then we necessarily have

$$\int_{\mathcal{X}_\eta} |U_{\min}^j|^{\frac{2}{km_0}} e^{-\varphi_L - \frac{1}{km_0}\varphi_A} \leq \int_{\mathcal{X}_\eta} \frac{|U_1^j|^2}{|U_{\min}^j|^{\frac{2(km_0-1)}{km_0}}} e^{-\varphi_L - \frac{1}{km_0}\varphi_A} \leq C_0 C_2.$$

Indeed, if the previous inequality fails to hold, then we argue as follows :

$$\begin{aligned}
 & \int_{\mathcal{X}_\eta} |U_1^j|_{k m_0}^{\frac{2}{k m_0}} e^{-\varphi_L - \frac{1}{k m_0} \varphi_A} \\
 & \leq \int_{\mathcal{X}_\eta} \frac{|U_1^j|_{k m_0}^{\frac{2}{k m_0}} e^{-\frac{\varphi_L}{k m_0} - \frac{\varphi_A}{(k m_0)^2}}}{|U_{\min}^j|_{k m_0}^{\frac{2(k m_0 - 1)}{(k m_0)^2}}} |U_{\min}^j|_{k m_0}^{\frac{2(k m_0 - 1)}{k m_0}} e^{-\frac{k m_0 - 1}{k m_0} \varphi_L - \frac{k m_0 - 1}{(k m_0)^2} \varphi_A} \\
 & \leq \left( \int_{\mathcal{X}_\eta} \frac{|U_1^j|_{k m_0}^2 e^{-\varphi_L - \frac{\varphi_A}{k m_0}}}{|U_{\min}^j|_{k m_0}^{\frac{2(k m_0 - 1)}{k m_0}}} \right)^{\frac{1}{k m_0}} \left( \int_{\mathcal{X}_\eta} |U_{\min}^j|_{k m_0}^{\frac{2}{k m_0}} e^{-\varphi_L - \frac{1}{k m_0} \varphi_A} \right)^{\frac{k m_0 - 1}{k m_0}} \\
 & < \int_{\mathcal{X}_\eta} |U_{\min}^j|_{k m_0}^{\frac{2}{k m_0}} e^{-\varphi_L - \frac{1}{k m_0} \varphi_A}
 \end{aligned}$$

which contradicts to the fact that  $U_{\min}^j \in \mathcal{E}_{k,j}$  has the minimal seminorm. Hence, the claim is proved. We will use next this property of  $U_{\min}^j$  to finish the proof of Theorem 2.15.

To this end, consider first the weight function

$$f_k = \log \sum_j \left( |U_{\min}^j|^2 \right).$$

Shrink  $\mathbb{D}$  if necessary, the above discussions and the concavity of the logarithm function imply that

$$\frac{1}{k} \int_{\mathcal{X}_\eta} f_k dV \leq C_3$$

where  $C_3$  is a constant independent of  $k$ . Moreover, the Hessian of the function  $f_k$  satisfies the inequality

$$\sqrt{-1} \Theta(m_0(K_{\mathcal{X}_\eta} + L)) + \frac{\sqrt{-1}}{k} \partial \bar{\partial} f_k + \frac{\sqrt{-1}}{k} \Theta(A) \geq 0. \quad (2.4)$$

Thus, there exist uniform local upper bounds for  $\frac{1}{k} f_k$  on each relatively compact subset of  $\mathcal{X}_\eta$ . We can consider therefore

$$f_\infty = \overline{\lim}_{k \rightarrow \infty} \text{Reg} \frac{1}{k} f_k$$

the upper semi-continuous envelope of the family  $\left( \frac{1}{k} f_k \right)_{k \geq 1}$ . It is a quasi-psh function on  $\mathcal{X}_\eta$ . On the central fiber  $\mathcal{X}_0$ , we have

$$\frac{1}{k} f_k|_{\mathcal{X}_0} \geq \log(|\sigma|^2) + \frac{1}{k} \log \left( \sum_{j=1}^{N_0} |s_j^{(0)}|^2 \right).$$

Therefore on the central fiber  $\mathcal{X}_0$ , we have the pointwise estimate

$$|\sigma|^2 e^{-f_\infty} \leq 1. \quad (2.5)$$

Hence the metric  $h_\infty$  has a semi-positive curvature when we take the limit in (2.4) and the norm of  $\sigma$  is bounded with respect to this metric. To conclude the proof, consider the metric  $h_\infty^{\frac{m_0-1}{m_0}} \otimes h_{L_1}^{\frac{1}{m_0}} \otimes \cdots \otimes h_{L_{m_0-1}}^{\frac{1}{m_0}}$  on the line bundle  $(m_0 - 1)(K_{\mathcal{X}} + L) + \frac{1}{m_0}(L_1 + \cdots + L_{m_0-1})$ .



It defines a metric with semi-positive curvature. It remains only to check the  $L^2$  requirement on the central fiber. The quantity we have to evaluate reads as

$$\int_{\mathcal{X}_0} |\sigma|^2 e^{-\frac{m_0-1}{m_0} f_\infty} e^{-\frac{1}{m_0}(\varphi_{L_1} + \dots + \varphi_{L_{m_0-1}})}$$

and we first observe that by (2.5) we have

$$\begin{aligned} & \int_{\mathcal{X}_0} |\sigma|^2 e^{-\frac{m_0-1}{m_0} f_\infty} e^{-\frac{1}{m_0}(\varphi_{L_1} + \dots + \varphi_{L_{m_0-1}})} \\ & \leq \int_{\mathcal{X}_0} e^{-\frac{1}{m_0}(\varphi_{L_1} + \dots + \varphi_{L_{m_0-1}})}. \end{aligned}$$

We recall that we have already proved that the integral on the right hand side satisfies

$$\int_{\mathcal{X}_0} e^{-\frac{1}{m_0}(\varphi_{L_1} + \dots + \varphi_{L_{m_0-1}})} < +\infty.$$

In conclusion, the integral

$$\int_{\mathcal{X}_0} |\sigma|^2 e^{-\frac{m_0-1}{m_0} f_\infty} e^{-\frac{1}{m_0}(\varphi_{L_1} + \dots + \varphi_{L_{m_0-1}})}$$

is convergent. The Ohsawa-Takegoshi Theorem allows us to extend the section  $\sigma$ ; this ends the proof of Theorem 2.15.  $\square$

### 2.4.3 Proof of Theorem 2.13.

To end the proof of Theorem 2.13, we need first the following result which states that a closed positive  $(1, 1)$ -current can be approximated by a sequence of divisors (cf. [Dem93]) :

**Theorem 2.18.** *Let  $A \rightarrow \mathcal{X}$  be an ample line bundle, then there is a sequence of sections  $u_k \in H^0(\mathcal{X}, k(K_{\mathcal{X}} + L) + l_k A)$  with  $l_k > 0$  and  $\lim_{k \rightarrow +\infty} \frac{l_k}{k} = 0$ , such that the divisors  $D_k = \frac{1}{k} \text{div}(u_k)$  satisfy*

$$\Theta_0 = \lim_{k \rightarrow +\infty} D_k$$

in the weak topology.

The construction of  $u_k$  is like follows : let  $(\sigma_j^{(k)})_{j=1, \dots, N_k}$  be an orthonormal basis of the space

$$H^0(\mathcal{X}_0, k(K_{\mathcal{X}} + L)|_{\mathcal{X}_0} + A)$$

endowed with the metric whose local weights are given by the functions  $k\varphi_{\Theta_0} + \varphi_A$  and an arbitrary smooth volume form on  $\mathcal{X}_0$ . Select sections

$$(g_1, \dots, g_{N_k}) \in H^0(\mathcal{X}, m_0 A)$$

with  $m_0$  so large such that  $m_0 A$  is very ample and set

$$u_{mk} = \left(\sigma_1^{(k)}\right)^m g_1 + \dots + \left(\sigma_{N_k}^{(k)}\right)^m g_{N_k} \in H^0(\mathcal{X}_0, mk(K_{\mathcal{X}} + L)|_{\mathcal{X}_0} + (m + m_0)A).$$

The following lemma implies Theorem 2.18 for some subsequence

$$D_k = D_{m(k), k}.$$

**Lemma 2.19.** *Let  $f_j \in H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  be nonzero holomorphic functions and  $G \subset H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  be a finite dimensional subspace whose elements generate all 1-jets at any point of  $\mathcal{X}$ . Moreover, set  $v = \sup \log |f_j|$  and*

$$h_m = f_1^m g_1 + \cdots + f_N^m g_N, \quad g_j \in G \setminus \{0\}.$$

*Then for all  $(g_1, \dots, g_N)$  in  $(G \setminus \{0\})^N$  except a set of measure 0, the sequence  $\frac{1}{m} \log |h_m|$  converges to  $v$  in  $L^1_{\text{loc}}(\mathcal{X})$ .*

We show next that

$$\int_{\mathcal{X}_0} |u_{mk}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} \leq C. \quad (2.6)$$

This will be a consequence of the hypothesis

$$\int_{\mathcal{X}_0} e^{(1+\delta_0)(\varphi_{\Theta_0} - \varphi_L)} dV_{\omega_0} < +\infty.$$

In the first place, we can assume that for every  $j = 1, \dots, N_k$ ,

$$|g_j|^2 \leq C_0,$$

since every  $(g_j)$  is a linear combination with coefficients less or equal than 1 in absolute value of a finite set of sections. Then we have the immediate inequality

$$|u_{mk}|^2 \leq C \cdot N_k \sum_j |\sigma_j^{(k)}|^{2m} \leq C \cdot N_k^2 \left( \sum_j |\sigma_j^{(k)}|^2 \right)^m,$$

here

$$N_k = \dim H^0(\mathcal{X}_0, k(K_X + L) + A) \leq C \cdot k^n$$

for some  $C > 0$  independent of  $k$ . This implies that

$$\begin{aligned} & \int_{\mathcal{X}_0} |u_{km}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} \leq \\ & \leq C (N_k)^{\frac{2}{mk}} \int_{\mathcal{X}_0} \left( \sum_j |\sigma_j^{(k)}|^2 \right)^{\frac{1}{k}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} \\ & \leq C \int_{\mathcal{X}_0} \left( \sum_j |\sigma_j^{(k)}|^2 \right)^{\frac{1}{k}} e^{-\varphi_{\Theta_0} - \frac{1}{k} \varphi_A} \cdot e^{\varphi_{\Theta_0} - \varphi_L - \frac{m_0}{mk} \varphi_A} \\ & \leq C \left( \int_{\mathcal{X}_0} \sum_j |\sigma_j^{(k)}|^2 e^{-k\varphi_{\Theta_0} - \varphi_A} \right)^{\frac{1}{k}} \cdot \left( \int_{\mathcal{X}_0} e^{\frac{k}{k-1}(\varphi_{\Theta_0} - \varphi_L) - \frac{m_0}{m(k-1)} \varphi_A} dV_{\omega_0} \right)^{\frac{k-1}{k}}. \end{aligned}$$

In the above calculation we used the fact that  $k^{\frac{2n}{mk}} \leq C$ . Now the first term of the product in the last line is bounded by  $(N_k)^{\frac{1}{k}} \leq C$  and the second one is bounded by hypothesis. In conclusion, (2.6) holds.

We use the result obtained in the proceeding section in order to extend  $u_{mk}$ ; we obtain a section

$$U_{mk} \in H^0(\mathcal{X}_\eta, mk(K_X + L) + (m + m_0)A)$$

together with the estimate

$$\int_{\mathcal{X}_\eta} |U_{mk}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} \leq C \int_{\mathcal{X}_0} |u_{mk}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} \leq C_1 \quad (2.7)$$

with  $C_1$  independent of  $m$  and  $k$ . Therefore we can apply the Lebesgue's dominated convergence theorem to (2.7) and get that

$$\begin{aligned} & \lim_{k \rightarrow +\infty, m(k) \rightarrow +\infty} \int_{\mathcal{X}_\eta} |U_{mk}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} = \int_{\mathcal{X}_\eta} \lim_{k \rightarrow +\infty, m(k) \rightarrow +\infty} \left( |U_{mk}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} \right) \\ & \leq C \lim_{k \rightarrow +\infty, m(k) \rightarrow +\infty} \int_{\mathcal{X}_0} |u_{mk}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} = \int_{\mathcal{X}_0} \lim_{k \rightarrow +\infty, m(k) \rightarrow +\infty} \left( |u_{mk}|^{\frac{2}{mk}} e^{-\varphi_L - \frac{m+m_0}{mk} \varphi_A} \right). \end{aligned}$$

On the other hand, Theorem 2.18 implies the existence of

$$\lim_{k \rightarrow +\infty, m(k) \rightarrow +\infty} \frac{1}{mk} \operatorname{div}(U_{mk}) := \Theta.$$

So if we denote  $\varphi_\Theta$  by its local potential, the above calculation shows that

$$\int_{\mathcal{X}_\eta} e^{\varphi_\Theta - \varphi_L} dV_{\omega_\eta} \leq C \int_{\mathcal{X}_0} e^{\varphi_{\Theta_0} - \varphi_L} dV_{\omega_0}.$$

□

## 2.5 Comments

At the end of Section 2.3, we mentioned that as an application of Conjecture 2.14, we can deduce the invariance of plurigenera in the Kähler case (see Conjecture 1.28). Indeed, given a section  $u$  of the bundle  $m_0 K_{\mathcal{X}|\mathcal{X}_0}$ , the conjecture implies that

$$\varphi_{\Theta|\mathcal{X}_0} \geq \log |u|^{\frac{2}{m_0}} + C_0.$$

In other words, we have

$$\int_{\mathcal{X}_0} |u|^2 e^{-m_0 \varphi_{\Theta|\mathcal{X}_0}} \leq C.$$

Then the Ohsawa-Takegoshi theorem in Kähler case gives the extension.



Deuxième partie

**Generalized Kähler-Einstein  
metrics.**



# Chapitre 3

## A Bando-Mabuchi uniqueness theorem

In this chapter we establish a uniqueness result of Bando-Mabuchi type, this concerns generalized Monge-Ampère equation. The main steps of our proof are similar to Berndtson's approach for a similar (but somehow less general) problem. However, there are serious technical difficulties to overcome.

### 3.1 Introduction

#### 3.1.1 Kähler-Einstein equation and twisted Kähler-Einstein equation

Given a smooth complex manifold  $X$  and a Kähler metric  $\omega$  on  $X$ , the *Ricci curvature* of  $\omega$  is given by

$$\text{Ricci}(\omega) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\det\omega).$$

Of course, by this we do not mean that  $\text{Ricci}(\omega)$  is an exact  $(1, 1)$ -form : locally, we have

$$\omega = \sum \sqrt{-1} \omega_{j\bar{k}} dz^j dz^{\bar{k}}$$

and the potential of the closed form  $\text{Ricci}(\omega)$  is

$$\text{Ricci}(\omega) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\det(\omega_{j\bar{k}})).$$

Recall that a Chern curvature  $\omega_{L,h}$  of a holomorphic line bundle  $L$  on  $X$  associated to a hermitian metric  $h$  is given by a real closed differentiable  $(1, 1)$ -form

$$\omega_{L,h} = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log h.$$

The De Rham cohomology class  $[\omega_{L,h}] \in H^2(X, \mathbb{R})$  does not depend on the choice of the metric  $h$  and equals to  $c_1(L) \in H^2(X, \mathbb{Z})$ , the first Chern class of  $L$ . Since a metric  $g$  on  $X$  is a hermitian metric on the tangent bundle  $T_X$ , we see that the De Rham cohomology  $[\text{Ricci}(\omega)]$  is equal to  $c_1(X)$ , the first Chern class of the anti-canonical bundle  $-K_X$ . One of

the important problems in Kähler geometry is the following :

**Question.** Let  $X$  be a smooth Kähler manifold, does there exist a metric  $g$  on  $X$  such that for  $\lambda \in \mathbb{R}$ ,

$$\text{Ricci}(g) = \lambda\omega_g, \tag{3.1}$$

where  $\omega_g$  is the Kähler form associated to  $g$ ?

Equation (3.1) is the *Kähler-Einstein equation*, solutions of this equation are called *Kähler-Einstein metrics*. The Kähler form associated to the metric  $g$  is of course positive, therefore the existence of a Kähler-Einstein metric is possible only when  $c_1(X)$  is strictly positive, strictly negative or zero. After a change of scale, we can suppose that  $\lambda \in \{-1, 0, 1\}$ . We remark that the existence of a Kähler-Einstein metric imposes considerable restrictions on  $c_1(X)$  : for example, if a metric  $\omega$  verifies (3.1) with  $\lambda = -1$ , this implies that  $K_X$  is ample. In this context, a fundamental result was established by Aubin and Yau [Aub76], [Aub78], [Yau77], [Yau78] :

**Theorem 3.1.** (i) *Suppose that  $c_1(X) = 0$ . Then for any Kähler class  $\{\omega_0\}$ , there exists a Kähler metric  $\omega \in \{\omega_0\}$  such that  $\text{Ricci}(\omega) = 0$ .*

(ii) *If  $c_1(X) < 0$ , then there exists a unique Kähler metric  $\omega \in -c_1(X)$  such that*

$$\text{Ricci}(\omega) = -\omega.$$

The manifolds with  $c_1(X) < 0$  are of general type, in this case, the above theorem shows the existence and uniqueness of a Kähler-Einstein metric in  $c_1(K_X)$ . For the manifolds with  $c_1(X) = 0$ , Kähler-Einstein equation is just  $\text{Ricci}(\omega) = 0$ . Theorem 3.1 shows that every positive cohomology class of type  $(1, 1)$  has a representative whose Ricci curvature is equal to zero. This is a special case of the Calabi's conjecture [Cal54], [Cal57]. Among the manifolds with  $c_1(X) = 0$ , there exists a special kind named after Calabi and Yau. A *Calabi-Yau manifold* is a compact Kähler manifold which satisfies that the canonical bundle  $K_X$  is trivial and that  $H^i(X, \mathcal{O}_X) = 0$  for all  $0 < i < \dim X$ . The second property distinguishes the Calabi-Yau manifolds from the abelian manifolds.

At last, in the case that  $c_1(X) > 0$ , i.e.  $X$  is a Fano manifold, things are more complicated. It was known for a long time that there exists obstructions to the existence of the Kähler-Einstein metrics in this case. We know some examples of Fano manifolds who do not admit Kähler-Einstein metrics, but we don't have at our disposal an algebro-geometric criterion for the existence of this type of metrics. These obstructions were formulated, first conjectured by Yau and later extended by Tian and Donaldson [Tia00], [Don04], in terms of an algebro-geometric notion of *stability*, in the sense of geometry invariant theory. As shown by Tian [Tia00], there is a stronger analytic notion of stability which is equivalent to the existence of Kähler-Einstein metrics in the first Chern class, namely the properness of *Mabuchi's K-energy functional*  $\mathcal{K}$ . This functional  $\mathcal{K}$  is defined on the subspace of all Kähler metrics in the first Chern class and its critical points are precisely the Kähler-Einstein metrics.

On the other hand, the uniqueness of Kähler-Einstein metrics on Fano manifolds was established by Bando and Mabuchi in 1987 [BM87] :

**Theorem 3.2.** *Let  $\omega_0, \omega_1 \in c_1(X)$  be two metrics such that*

$$\text{Ricci}(\omega_j) = \omega_j, \quad \forall j = 0, 1.$$



Then there exists an automorphism  $\varphi : X \rightarrow X$  such that

$$\varphi^* \omega_1 = \omega_0.$$

In the context of metrics with conic singularities along a divisor  $D$  such that  $(X, D)$  is klt, the generalization of the Kähler-Einstein condition reads as

$$\text{Ricci}(\omega) = \omega + [D] \tag{3.2}$$

where  $\omega \in -c_1(K_X + D)$ .

Even more generally, we have the concept of *twisted Kähler-Einstein equation*. If  $\theta$  is a given closed  $(1, 1)$ -form on  $X$ , the twisted Kähler-Einstein equation for a Kähler metric  $\omega$  is defined by

$$\text{Ricci}(\omega) - \theta = \omega$$

where the left hand side is called the *twisted Ricci curvature* of  $\omega$ .

### 3.1.2 Homogenous Monge-Ampère equation

In this subsection, let us consider an  $n$ -dimensional projective manifold  $X$  with seminegative canonical bundle and a domain  $\Omega$  in the complex plane. Suppose we have a curve of metrics  $t \rightarrow \phi_t$  on  $-K_X$ , the local representatives  $\phi_t$  are bounded (see the definition in subsection 3.2.2) and satisfies  $\sqrt{-1}\partial\bar{\partial}_{t,X}\phi \geq 0$  with  $t$  varies in  $\Omega$ . Then the complex Monge-Ampère operator is well defined. We say that  $\phi$  solves the homogenous Monge-Ampère equation if

$$(\sqrt{-1}\partial\bar{\partial}\phi)^{n+1} = 0. \tag{3.3}$$

If  $\phi_t$  is moreover smooth with nonnegative curvature on  $X$  for  $t$  fixed on  $\partial\Omega$ , by adapting some of the methods from Yau's proof of the Calabi's conjecture, Chen [Che00] showed that, for any given  $\phi_0$  defined on the boundary of  $\Omega$ , we can find a solution of (3.3) with  $\phi_0$  as boundary values. The solution he found equals to the supremum of all subsolutions, that is, all metrics with semipositive curvature that are dominated by  $\phi_0$  on the boundary. Therefore it is in general not smooth, but Chen showed that it may have the property that all the mixed complex derivatives  $\sqrt{-1}\partial\bar{\partial}_{t,X}\phi$  are bounded on  $X \times \Omega$ . If we assume further that  $\phi$  is smooth and  $\sqrt{-1}\partial\bar{\partial}_X\phi > 0$  on  $X$  for any  $t$  fixed, calculation yields that

$$(\sqrt{-1}\partial\bar{\partial}\phi)^{n+1} = nc(\phi)(\sqrt{-1}\partial\bar{\partial}\phi)^n \wedge \sqrt{-1}dt \wedge d\bar{t}$$

with

$$c(\phi) = \frac{\partial^2\phi}{\partial t\partial\bar{t}} - \left| \frac{\partial\phi}{\partial t} \right|_{\sqrt{-1}\partial\bar{\partial}_X\phi}^2$$

where the norm is with respect to the Kähler metric  $\sqrt{-1}\partial\bar{\partial}_X\phi$ . Then the equation (3.3) is equivalent to  $c(\phi) = 0$ .

### 3.1.3 Brunn-Minkowski type inequality

Further observations by Semmes [Sem92] and Donaldson [Don99] pointed out that  $c(\phi) = 0$  is the equation for a geodesic in the space of Kähler potentials. Then apart from the smoothness and strict positivity being kept along all the curves, Chen's theorem implies that any two points in the space of Kähler potentials can be jointed by a geodesic. As we will

see in the next section, a geodesic between two Kähler metrics can be obtained as the limit superior of subgeodesics. We mention here an interesting property of these objects and their interpretation in convex geometry (cf. [Ber 09]).

**Theorem 3.3.** *Assume that  $-K_X \geq 0$  and let  $\phi_t$  be a curve of metrics on  $-K_X$  such that*

$$\sqrt{-1}\partial\bar{\partial}_{t,X}\phi \geq 0$$

*in the sense of currents. Then*

$$\mathcal{F}(t) = -\log \int_X e^{-\phi_t}$$

*is subharmonic in  $\Omega$ . In particular, if  $\phi_t$  does not depend on the imaginary part of  $t$ ,  $\mathcal{F}$  is convex.*

The volume form  $e^{-\phi_t}$  in the integral can be written under the local coordinates  $z^j$  by

$$c_n e^{-\phi_t^j} dz^j \wedge d\bar{z}^j,$$

where  $c_n = \sqrt{-1}^{n^2}$  is a unimodular constant to make the form positive,  $\phi_t^j$  is the local representative of  $\phi$  on a neighborhood  $U_j$ . It is independent of the choice of local coordinates.

**Remark 3.4.** *Theorem 3.3 can be seen as the complex counterpart of Prékopa's theorem [Pré73], which states that*

$$f(t) = -\log \int_{\mathbb{R}^n} e^{-\phi_t}$$

is convex if  $\phi$  is a convex function on  $\mathbb{R}^{n+1}$ . Convexity in Prékopa's theorem corresponds to the positivity, or at least semipositivity of the curvature  $\phi_t$  on  $-K_X$  in the complex case. Prékopa's theorem can be seen, in its turn, as the functional version of Brunn-Minkowski theorem. If we suppose  $K$  is a compact convex set in  $\mathbb{R}^{n+1}$  and  $\phi = 0$  in  $K$  and  $+\infty$  elsewhere, the Brunn-Minkowski inequality states that if  $K$  and  $L$  are compact convex sets with nonempty interiors in  $\mathbb{R}^n$  and  $0 < \lambda < 1$ , then

$$(1 - \lambda)\text{Vol}(K)^{1/n} + \lambda\text{Vol}(L)^{1/n} \leq \text{Vol}(((1 - \lambda)K + \lambda L)^{1/n})$$

with the equality holds if  $K$  and  $L$  are equal up to translation and dilatation ([Gar02]). The above inequality shows that the logarithm of the volumes of  $n$ -dimensional slices,  $K_t$  of convex sets are concave :

$$|K_{(t+s)/2}|^2 \leq |K_t||K_s| \tag{3.4}$$

with the equality holds if all the slices  $K_t$  and  $K_s$  satisfy

$$K_t = K_s + (t - s)\mathbf{v}$$

with  $\mathbf{v}$  some vector in  $\mathbb{R}^n$  ([Ber11]). This last equality can be translated, in some extent, into the fact that one slice can move to another via the flow of a constant vector field.

From this point of view, Berndtsson proved, first in [Ber09b], a similar conclusion in the complex setting under the assumption that the curvature is smooth and strictly plurisubharmonic on  $X$  for  $t$  fixed, then in [Ber11] under a less regular situation, like follows :

**Theorem 3.5.** *Assume that  $H^{0,1}(X) = 0$  and that the curve of metrics  $\phi_t$  is independent of the imaginary part of  $t$ . Assume moreover that the metrics  $\phi_t$  are uniformly bounded in the sense that for some smooth metric  $\psi$  on  $-K_X$ ,*

$$|\phi_t - \psi| \leq C.$$

*Then, if the function  $\mathcal{F}$  in Theorem 3.3 is affine in a neighborhood of 0 in  $\Omega$ , there is a (possibly time independent) holomorphic vector field  $V$  on  $X$  with flow  $F_t$  such that*

$$F_t^*(\partial\bar{\partial}\phi_t) = \partial\bar{\partial}\phi_0.$$

### 3.1.4 Bando-Mabuchi uniqueness theorem

We assume next that  $\phi_t$  is a geodesic connecting two solutions of the equation

$$\text{Ricci}(\omega) = \omega.$$

Then one would like to use a statement like Theorem 3.5 in order to show the existence of a biholomorphism  $F$  such that

$$F^*(\omega_{\phi_1}) = \omega_{\phi_0}.$$

We give here a brief introduction of Berndtsson's method (cf. [Ber11]). Consider a compact Kähler  $n$ -dimensional manifold  $(X, \omega)$  normalized by  $\int_X \omega^n = 1$ . The original *Aubin-Mabuchi energy functional* for smooth  $\omega$ -psh function  $\phi$  is defined by

$$\mathcal{E} := \frac{1}{n+1} \sum_{j=0}^n \int_X \phi (\omega + \sqrt{-1}\partial\bar{\partial}\phi)^j \wedge \omega^{n-j}$$

[Aub 84], [Mab 86]. Using the integration by parts, we see that the Gâteaux derivative of  $\mathcal{E}$  at  $\phi$  is given by

$$\mathcal{E}'(\psi) \cdot (\phi - \psi) = \int_X (\phi - \psi) \cdot (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^n$$

where the restriction  $t \rightarrow \mathcal{E}(t\phi + (1-t)\psi)$  to line segments is a polynomial map of degree  $n+1$ . Further computation shows that

$$\mathcal{E}''(\psi) \cdot (\phi - \psi, \phi - \psi) = -n \int_X \partial(\phi - \psi) \wedge \bar{\partial}(\phi - \psi) \wedge (\omega + \sqrt{-1}\partial\bar{\partial}\psi)^{n-1}.$$

They imply in particular that  $\mathcal{E}$  is non-decreasing on smooth  $\omega$ -psh functions and  $\mathcal{E}$  is concave. This functional can be extended by monotonicity to arbitrary  $\omega$ -psh functions by setting

$$\mathcal{E}(\phi) := \inf\{\mathcal{E}(\psi) \mid \psi \text{ smooth } \omega\text{-psh}, \psi \geq \phi\} \in [-\infty, +\infty[.$$

Now for  $\phi_0, \phi_1$  two metrics on a line bundle  $L$  over  $X$ , we consider their relative energy

$$\mathcal{E}(\phi_0, \phi_1),$$

it is well defined if  $\phi_j$  are bounded with the property that  $\sqrt{-1}\partial\bar{\partial}\phi_j \geq 0$ . By integration by parts as above, we see that if  $\phi_t$  is smooth in  $t$  for  $t$  in  $\Omega \in \mathbb{R}$  (i.e.,  $\phi_t$  depends only on the real part of  $t$ ),

$$\frac{\partial}{\partial t} \mathcal{E}(\phi_t, \phi_1) = \int_X \dot{\phi}_t (\sqrt{-1}\partial\bar{\partial}\phi_t)^n / \text{Vol}(L)$$

and

$$\sqrt{-1}\partial\bar{\partial}_t\mathcal{E}(\phi_t, \phi_1) = p_*((\sqrt{-1}\partial\bar{\partial}_{X,t}\phi)^{n+1})/\text{Vol}(L),$$

where  $p$  is the projection map from  $X \times \Omega$  to  $\Omega$  and  $\text{Vol}(L)$  is the normalizing factor

$$\text{Vol}(L) = \int_X (\sqrt{-1}\partial\bar{\partial}_X\phi)^n,$$

chosen so that the derivative of  $\mathcal{E}$  becomes 1 if  $\phi_t = \phi + t$ . For the general family who is only bounded, these formulas hold in the sense of distributions. From the above calculations we see that if  $\phi$  solves the homogenous Monge-Ampère equation, that is, according to the subsection 3.1.2,

$$(\sqrt{-1}\partial\bar{\partial}_{X,t}\phi)^{n+1} = 0 \tag{3.5}$$

(even if the said geodesics are not smooth, the fact that they are bounded and verify the equation (3.5) in the sense of Bedford-Taylor [BT76] is enough),  $\mathcal{E}(\phi_t, \phi_1)$  is harmonic in  $t$ . In other words, this function is linear along geodesics.

Now define

$$\mathcal{G}(t) = \mathcal{F}(t) - \mathcal{E}(\phi_t, \psi)$$

where  $\mathcal{F}(t) = -\log \int_X e^{-\phi_t}$  and  $\psi$  is arbitrary. Recall that we say a metric  $\omega_\psi = \sqrt{-1}\partial\bar{\partial}\psi$  solves the Kähler-Einstein equation if

$$\text{Ricci}(\omega_\psi) = \omega_\psi$$

where  $\psi$  is a metric on  $-K_X$ , or, in terms of volume forms in the sense of Bedford and Taylor [BT76], [BT82], if for some positive  $a > 0$ ,

$$e^{-\psi} = a(\sqrt{-1}\partial\bar{\partial}\psi)^n$$

where  $e^{-\psi}$  is a volume form. Then  $\phi_0$  solves the Kähler-Einstein equation if and only if  $\mathcal{G}'(0) = 0$  for any smooth curve  $\phi_t$ . If  $\phi_0$  and  $\phi_1$  are two Kähler-Einstein metrics we can connect them, according to Chen's theorem, by a geodesic  $\phi_t$ . Since  $\phi_t$  depends only on the real part of  $t$ ,  $\mathcal{G}$  is convex. Now by assumption both end points are Kähler-Einstein metrics, so 0 and 1 are stationary points for  $\mathcal{G}$ . We stress on the fact that this holds under very weak regularity assumptions, in the case that geodesic is either smooth or bounded and uniformly Lipschitz in  $t$ . At the same time, by definition, the function  $\mathcal{F}$  is convex, hence it has one sided derivatives at the endpoints. It's easy to see that the sided derivatives are equal to

$$\int \dot{\phi}_t e^{-\phi} / \int e^{-\phi}$$

where  $\dot{\phi}_t$  now stands for the one sided derivatives. We proved that the function  $\mathcal{E}(\phi_t, \psi)$  is linear along geodesics, so its distributional derivative

$$\int_X \dot{\phi}_t (\sqrt{-1}\partial\bar{\partial}\phi_t)^n / \text{Vol}(L)$$

is constant. By simple convergence theorems for the Monge-Ampère operator, the derivative is equal to its values at the endpoints. It induces that both end points are critical points for  $\mathcal{G}$ . Finally, the convexity implies that  $\mathcal{G}$  is constant. Hence we obtain that  $\mathcal{F}$  is linear. Theorem 3.5 says that  $\partial\bar{\partial}\phi_t$  are related via a holomorphic family of automorphisms, in particular,  $\partial\bar{\partial}\phi_0$  and  $\partial\bar{\partial}\phi_1$  correspond to each other via an automorphism homotopic to the identity.

These are the main steps in the proof of Theorem 3.2.

From the above discussions, we see that Theorem 3.5 is a crucial step in the method of Berndtsson to illustrate the Bando-Mabuchi Theorem. Its proof relies on 3 main steps : Regularization of the singular metrics, solving the  $\partial\bar{\partial}$ -equations (see the definition of the operator  $\partial^{\phi_t}$  in the proof of our main theorem) and the application of a curvature formula. It is hence independent of Chen's theorem and avoids showing that the geodesic joining two Kähler-Einstein metrics should be of  $\mathcal{C}^{1,1}$ .

A further question arises from Theorem 3.5 is about the minimal assumption on the behavior of the metrics  $\phi_t$ , instead being bounded. An alike answer for this question would be that  $e^{-\phi_t}$  being integrable but it is unclear for the moment whether Theorem 3.5 still hold under this assumption. An intermediate way is to consider  $\phi_t$  as a sum of certain  $\chi_t$  and  $\psi$ , where  $\chi_t$  is bounded and  $\psi$  is such that  $e^{-\psi}$  is integrable, so that the singularities don't change with  $t$ . A consequent result of Theorem 3.5 is given in [Ber11] under this assumption together with some positivity assumptions :

**Theorem 3.6.** *Assume that  $-(K_X + S) \geq 0$  and that  $H^{0,1}(X) = 0$ . Let  $\tau_t = \phi_t + \psi$  be a curve of metrics on  $-K_X$  where*

- (i)  $\phi_t$  are metrics on  $-(K_X + S)$  with  $\sqrt{-1}\partial\bar{\partial}\phi_t \geq 0$ ,
- (ii)  $\psi$  is a metric on  $S$  with  $\sqrt{-1}\partial\bar{\partial}\psi = [\Delta]$ , where  $\Delta$  is a klt divisor with simple normal crossings.

Assume that

$$\mathcal{F}(t) = -\log \int_X e^{-\tau_t}$$

is affine. Then there is a holomorphic vector field  $V$  on  $X$  with flow  $F_t$  such that

$$F_t^*(\partial\bar{\partial}\tau_t) = \partial\bar{\partial}\tau_0.$$

The decomposition of  $\tau_t$  as  $\tau_t = \phi_t + \psi$  is very well adapted to the twisted Kähler-Einstein equation

$$\text{Ricci}(\omega) = \omega + \Theta$$

with  $\Theta$  a fixed positive  $(1, 1)$ -current. In the same time, the linearity of  $\mathcal{F}$  is related to the twisted Kähler-Einstein equation. As a consequence of Theorem 3.5, in [Ber11], the author showed the followings result :

**Theorem 3.7.** *Assume that  $-K_X$  is semipositive and that  $H^{0,1}(X) = 0$ . Assume that  $\sqrt{-1}\partial\bar{\partial}\psi = \Theta$ , where  $e^{-\psi}$  is integrable. Let  $\phi_0$  and  $\phi_1$  be two bounded solutions of equation*

$$\text{Ricci}(\omega) = \omega + \Theta$$

with  $\sqrt{-1}\partial\bar{\partial}\phi_j \geq 0$ . Then there is a holomorphic automorphism,  $F$ , of  $X$ , homotopic to the identity, such that

$$F^*(\partial\bar{\partial}\phi_1) = \partial\bar{\partial}\phi_0$$

and

$$F^*(\Theta) = \Theta.$$

We remark that the automorphism  $F$  is in fact identity if we assume  $\Theta$  to be strictly positive. This can be concluded from the proof of Theorem 3.6, where both  $\phi_t$  and  $\phi_t + \psi$  are geodesics and hence forces  $\phi_t$  to be linear in  $t$  if  $\sqrt{-1}\partial\bar{\partial}\psi > 0$ . We remark as well that in Theorem 3.7, one has to assume that  $-K_X \geq 0$  and  $-(K_X + S) \geq 0$  in order to conclude.

## 3.2 Preliminaries

### 3.2.1 Notations

Like mentioned in the preliminaries in the first chapter, given a line bundle  $L$  over a complex manifold  $X$ , we measure the size of a section  $s$  on  $L$  by introducing the notion of metric

$$\|s\|^2 = |s_j|^2 e^{-\phi_j}$$

with  $\phi_j$  the local weight of the metric  $e^{-\phi}$  on the local coordinate patch  $U_j$ . It is globally defined. The metric  $e^{-\phi}$  induces an  $L^2$ -norm on the adjoint bundle  $K_X + L$ . Indeed, a section  $u$  on  $K_X + L$  can be written locally as

$$u = dz \otimes s$$

where  $dz = dz_1 \wedge \cdots \wedge dz_n$  with  $(z_1, \dots, z_n)$  local coordinates and  $s$  a section on  $L$ . Define a volume form

$$|u|^2 e^{-\phi} := c_n \|s\|^2 dz \wedge d\bar{z}$$

with  $c_n$  a unimodular constant to make the volume form positive. The  $L^2$ -norm of  $u$  is therefore

$$\|u\|^2 = \int_X |u|^2 e^{-\phi}.$$

This norm depends only on the weight  $\phi$ .

We specialize our discussion to the case  $L = -K_X$ , the anti-canonical bundle. Then  $K_X + L$  is trivial and hence it has a section  $u$  which equals to 1 identically. By definition, the norm of 1 is

$$\|1\|^2 = \int_X |1|^2 e^{-\phi} = \int_X e^{-\phi}.$$

In other words, the volume form  $e^{-\phi}$  can be interpreted locally by

$$e^{-\phi} = c_n e^{-\phi_j} dz_j \wedge d\bar{z}_j$$

with  $e^{-\phi_j} = |(dz_j)^{-1}|_\phi^2$  the local representative of the metric. It is concordant with the notations in the introduction.

### 3.2.2 Bounded geodesics

The following definitions are global versions of standard notions in the pluripotential theory. Following Bedford and Taylor [BT76], if  $\phi$  is a psh *bounded* function defined on an open subset  $\Omega \subset \mathbb{C}^n$ , then the quantity

$$(\sqrt{-1} \partial \bar{\partial} \phi)^n$$

is a positive measure with locally finite mass on  $\Omega$ . Let  $L$  be a line bundle over a complex manifold  $X$  and  $t \rightarrow \phi_t$  a curve of metrics on  $L$  with  $t \in \Omega \subset \mathbb{C}$ . Although  $t$  is complex, we consider mainly those curves who depend only on the real part of  $t$ . We say that  $\phi_t$  is a *subgeodesic* if it is upper semicontinuous and satisfies that  $\sqrt{-1} \partial \bar{\partial}_{t,X} \phi_t \geq 0$ , that is, if the local representatives are plurisubharmonic with respect to  $t$  and  $X$  jointly. The curve  $\phi_t$  is *bounded* if

$$|\phi_t - \psi| \leq C$$

holds for some constant  $C$  and any fixed smooth metric  $\psi$  on  $L$ . For this  $\phi_t$ , the complex Monge-Ampère operator is well defined. Finally, we say that  $\phi_t$  is a (generalized) *geodesic* if

$$(\sqrt{-1}\partial\bar{\partial}_{t,X}\phi_t)^{n+1} = 0.$$

The existence of a generalized geodesic which is bounded in the sense of Theorem 3.5 is in fact almost trivial. To see this, let  $\phi_0$  and  $\phi_1$  be two bounded metrics on  $L$  over  $X$  satisfying  $\sqrt{-1}\partial\bar{\partial}\phi_{0,1} \geq 0$ , define  $\phi_t$  by

$$\phi_t = \sup\{\psi_t\} \tag{3.6}$$

where the supremum is taken over all psh  $\psi_t$  with

$$\lim_{t \rightarrow 0,1} \psi_t \leq \phi_{0,1}.$$

We say that  $\phi_t$  defined in this way is a bounded geodesic. To show this, we first construct a barrier like follows :

$$\chi_t = \max(\phi_0 - A\text{Ret}, \phi_1 + A(\text{Ret} - 1)).$$

Then  $\chi_t$  is plurisubharmonic and satisfies that

$$\lim_{t \rightarrow 0,1} \chi_t = \phi_{0,1}$$

once  $A$  is sufficiently large. Take those  $\psi$  in (3.6) which are larger than  $\chi$ , the supremum satisfies the same property as  $\chi$ . By definition, the onesided derivatives of such  $\psi$  are larger than  $-A$  at 0 and smaller than  $A$  at 1. Since  $\psi$  is moreover independent of the imaginary part of  $t$ , it is convex in  $t$  and hence the derivatives with respect to  $t$  increase, therefore must lie between  $-A$  and  $A$ . Hence  $\phi_t$  satisfies that

$$\phi_0 - A\text{Ret} \leq \phi_t \leq \phi_0 + A\text{Ret}$$

at 0 and a similar estimate at 1. This shows that

$$\lim_{t \rightarrow 0,1} \phi_t = \phi_{0,1}$$

uniformly on  $X$ . Moreover, the upper semicontinuous regularization  $\phi_t^*$  of  $\phi_t$  must satisfy the same estimate. Since  $\phi_t^*$  is plurisubharmonic it belongs to the class of competitors for  $\phi_t$  and must coincide with  $\phi_t$ , so  $\phi_t$  is plurisubharmonic. Finally  $\phi_t$  solves the homogenous Monge-Ampère equation follows from the fact that, according to [Che00], it is maximal with given boundary values.

**Remark 3.8.** *The above proof shows that the geodesic joining two bounded metrics is uniformly Lipschitz in  $t$ .*

### 3.3 Main theorem

Our main theorem can be seen as a generalization of Theorem 3.7. We first fix some notations. Let  $X$  be an  $n$ -dimensional projective manifold; we denote by  $c_1(X)$  its first Chern class, i.e. the Chern class corresponding to the anti-canonical bundle  $-K_X$ . Let  $\{\alpha\}$  be a  $(1, 1)$  class on  $X$ ; here we denote by  $\alpha$  a non-singular representative. We assume that the following requirements are satisfied.

- (i) The class  $c_1(X) - \{\alpha\}$  is Kähler, i.e. it contains a Kähler metric which we denote by  $\omega$ ;

- (ii) The class  $\{\alpha\}$  is pseudo-effective; let  $\Theta \in \{\alpha\}$  be a closed positive current;
- (iii) If we denote  $\text{Ricci}(\omega)$  the Ricci curvature associated to the metric  $\omega$ , then we have

$$\text{Ricci}(\omega) = \omega + \Theta - \sqrt{-1}\partial\bar{\partial}f_\Theta$$

for some function  $f_\Theta$  which is unique up to an additive constant. We assume that there exists a real number  $\varepsilon_0 > 0$  such that

$$\int_X e^{-(1+\varepsilon_0)\frac{n}{2}f_\Theta} dV < \infty.$$

In other words, the singularities of the current  $\Theta$  are not too wide.

Now we consider the equation

$$(\star) \quad (\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = e^{-\varphi - f_\Theta} \omega^n$$

where the solution  $\varphi$  is assumed to be bounded, and such that

$$\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi$$

is a Kähler current (i.e. it is greater than a Kähler metric).

We remark that if  $\Theta$  is non-singular, then the equation  $(\star)$  is equivalent with the following identity

$$\text{Ricci}(\omega_\varphi) = \omega_\varphi + \Theta.$$

In this chapter we show that the following generalization of the result in [Ber11] holds.

**Theorem 3.9.** *Let  $\varphi_0$  and  $\varphi_1$  be two bounded solutions of the equation  $(\star)$ . Then there exists a biholomorphism  $F : X \rightarrow X$  such that*

$$F^*(\omega_{\varphi_1}) = \omega_{\varphi_0} \quad \text{and} \quad F^*(\Theta) = \Theta.$$

If we assume that the anti-canonical class has further positivity properties, for example,  $-K_X \geq 0$ , then one proves in [Ber11] that the proceeding result holds under the assumption  $e^{-f_\Theta} \in L^1_{\text{loc}}(X)$ . However, we remark that in order to have this result, we have to assume two strong positivity properties :  $c_1(X) - \{\alpha\}$  is a Kähler class, and  $c_1(X)$  is a semi-positive class.

The arguments we will invoke in order to prove this result generalize the approach by Berndtsson in [Ber11]. It is based on the convexity properties of Bergman kernel, which we recall next.

Let  $L \rightarrow X$  be a line bundle, which is endowed with a family of non-singular metrics  $h_t = e^{-\varphi_t}$ ,  $t \in \mathbb{D}$ , the unit disk. We assume that the function

$$\varphi(t, x) = \varphi_t(x)$$

is non-singular and psh. We can see  $L$  as a bundle over the family  $X \times \mathbb{D}$ ; the bundle structure is independent of  $t \in \mathbb{D}$ , but the hermitian structure varies with respect to  $t$ . Let

$$E := p_*(K_{X \times \mathbb{D}/\mathbb{D}} + L)$$

be the direct image of the relative adjoint bundle of  $L$ , and let  $(u_t)_{t \in \mathbb{D}}$  be a holomorphic section of  $E$ . Of course, in general  $(u_t)$  does not correspond to a holomorphic  $n$ -form on  $X \times \mathbb{D}$  : we can only represent  $(u_t)_{t \in \mathbb{D}}$  by an  $n$ -form  $U$  on  $X \times \mathbb{D}$  such that  $\bar{\partial}U$  is equal to some multiple of  $dt$ , and such that  $U|_{X \times \{t\}} = u_t$ . In order to evaluate the curvature  $\Theta_E$  of  $E$  in the direction of the section  $(u_t)$ , we introduce a family  $(v_t)$  of  $(n-1)$ -forms with values in  $L$  such that



- \*  $\bar{\partial}v_t \wedge \omega = 0$ ,
- \*  $\partial^{\varphi_t}v_t = P_{\perp}(\dot{\varphi}_t u_t)$

where  $P$  is the projection on the orthogonal complement of the holomorphic  $n$ -forms on each fiber, then we have the following formula (cf. [Ber11]) :

$$\langle \Theta_E u_t, u_t \rangle = p_*(c_n \sqrt{-1} \partial \bar{\partial} \varphi_t \wedge \hat{u} \wedge \bar{\hat{u}} e^{-\varphi_t}) + \int_X \|\bar{\partial}v_t\|^2 e^{-\varphi_t} \sqrt{-1} dt \wedge d\bar{t}$$

where we denote by  $\hat{u} = (u_t - dt \wedge v_t)_t$ , another representation of our initial section  $(u_t)$ .

We assume now that for some geometric reasons as evoked in subsection 3.1.4, the curvature of  $E$  vanishes. As a consequence,  $v_t$  will be holomorphic, and the first term on the right hand side of the formula above will be zero. But then we can define a vector field by the formula

$$-v_t = V_t \lrcorner u_t;$$

it will be holomorphic outside  $u_t = 0$ . Actually, under the condition  $\Theta_E = 0$  one can see that  $V_t$  is the complex gradient of  $\sqrt{-1} \bar{\partial} \varphi_t$ ; finally, a simple computation shows that the flow associated to  $V_t$ ,  $F$ , is the holomorphic map we seek.

While trying to implement this scheme in our setting, we have to face two important difficulties. In our case, the bundle  $L$  will be just  $-K_X$ , endowed with the following family of metrics. We connect  $\varphi_0$  and  $\varphi_1$  with a geodesic, (this is known to exist), and this will give us a metric  $\varphi_t$  whose curvature

$$\omega_t = \omega + \sqrt{-1} \partial \bar{\partial}_X \varphi_t$$

belongs to the class  $c_1(X) - \{\alpha\}$ . Then the current

$$\omega_t + \Theta$$

with  $\Theta \in \{\alpha\}$  closed and positive, belongs to  $c_1(X)$  for each  $t$ . Our main concerns are :

- \*  $\Theta$  is not necessarily smooth ;
- \* the current  $\omega_t = \omega + \sqrt{-1} \partial \bar{\partial}_X \varphi_t$  is not necessarily Kähler, it is only known to be positive.

The plan of the next section will be as follows : we approximate  $\omega_t$  and  $\Theta$  with non-singular objects (subsection 3.4.1), and then we construct the analogue  $v_t'$  of the  $(n-1)$ -form  $v_t$  mentioned above. Thanks to the fact that  $\varphi_0$  and  $\varphi_1$  are solutions of  $(\star)$ , we show that the curvature of  $E$ , endowed with the regularized metric, converges to zero. An uniformity argument (which is quite subtle, given that the Lelong set of  $\Theta$  could be quite complicated) shows that  $v_t'$  converges to a holomorphic  $(n-1)$ -form. Then the vector field will be defined as above.

## 3.4 Proof of the main theorem

### 3.4.1 Regularization process

Fix a non-singular real  $(1,1)$ -form  $\omega$  in the Kähler class  $c_1(X) - \{\alpha\}$ , a quasi-psh function  $\varphi$  is said to be  $\omega$ -psh if  $\omega + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0$ . We denote  $\text{PSH}(X, \omega) \cap L^\infty$  the convex space of all  $\omega$ -psh functions which are bounded on  $X$ . Let  $\varphi_0$  and  $\varphi_1 \in \text{PSH}(X, \omega) \cap L^\infty$ , we recall that the path of semi-positive forms

$$\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t \quad \text{with } t \in [0, 1]$$

is a (generalized) geodesic in  $\text{PSH}(X, \omega)$ . The following theorem implies the existence of a continuous geodesic which connects  $\omega_0$  and  $\omega_1$  (cf. [BD08]) :

**Theorem 3.10.** *Assume that the semi-positive closed  $(1, 1)$ -forms  $\omega_0$  and  $\omega_1$  belong to the same Kähler class  $c_1(X) - \{\alpha\}$  and have bounded coefficients. Then there exists a geodesic  $\omega_t$  connecting  $\omega_0$  and  $\omega_1$  with the properties that it is continuous on  $[0, 1] \times X$  and that there is a constant  $C$  such that  $\omega_t \leq C\omega$ , i.e.  $\omega_t$  has uniformly bounded coefficients.*

By subsection 3.2.2, the geodesic can be constructed to be Lipschitz in  $t$ .

Our proof begins with the regularization for  $\varphi_t$ , the ampleness of  $c_1(X) - \{\alpha\}$  allows us to write  $\varphi_t$  as a uniform limit of a decreasing sequence of non-singular functions  $\varphi'_t$  with the property that

$$\omega'_t := \omega + \sqrt{-1}\partial\bar{\partial}\varphi'_t \geq 0.$$

In order to regularize the current

$$\Theta = \alpha + \sqrt{-1}\partial\bar{\partial}\theta,$$

we use the following result (cf. [Dem94]) :

**Theorem 3.11.** *Let  $T$  be a closed almost positive  $(1, 1)$ -current and let  $\alpha$  be a non-singular real  $(1, 1)$ -form in the same  $\partial\bar{\partial}$ -cohomology class as  $\Theta$ , i.e.  $\Theta = \alpha + \sqrt{-1}\partial\bar{\partial}\theta$  where  $\theta$  is an almost psh function. Let  $\gamma$  be a continuous real  $(1, 1)$ -form such that  $\Theta \geq \gamma$ . Suppose that  $\mathcal{O}_{T_X}(1)$  is equipped with a non-singular hermitian metric such that the Chern curvature form satisfies*

$$c(\mathcal{O}_{T_X}(1)) + \pi_X^*u \geq 0$$

with  $\pi_X : P(T^*X) \rightarrow X$  and with some nonnegative non-singular  $(1, 1)$ -form  $u$  on  $X$ . Fix a hermitian metric  $\omega^0$  on  $X$ . Then for every  $c > 0$ , there is a sequence of closed almost positive  $(1, 1)$ -current  $\Theta_{c,k} = \alpha + \sqrt{-1}\partial\bar{\partial}\theta_{c,k}$  such that  $\theta_{c,k}$  is non-singular on  $X \setminus E_c(\Theta)$  and decreases to  $\theta$  as  $k$  tends to  $+\infty$  (in particular, the current  $\Theta_{c,k}$  is non-singular on  $X \setminus E_c(\Theta)$  and converges weakly to  $\Theta$  on  $X$ ) and such that

- (1)  $\Theta_{c,k} \geq \gamma - \min\{\lambda_k, c\}u - \varepsilon_k\omega^0$  where :
- (2)  $\lambda_k(x)$  is a decreasing sequence of continuous functions on  $X$  such that  $\lim_{k \rightarrow +\infty} \lambda_k(x) = \nu(\Theta, X)$  at every point.
- (3)  $\varepsilon_k$  is positive decreasing and  $\lim_{k \rightarrow +\infty} \varepsilon_k = 0$ .
- (4)  $\nu(\Theta_{c,k}, x) = (\nu(\Theta, x) - c)_+$  at every point  $x \in X$ .

Recall that

$$E_c(\Theta) = \{x \in X; \nu(\Theta, x) \geq c\}$$

is the  $c$ -upperlevel set of Lelong numbers of  $\Theta$ . A well-known theorem of Siu [Siu74] asserts that  $E_c(\Theta)$  is an analytic subset of  $X$  for any  $c > 0$ .

Choose  $u = A\omega^0$  where  $A$  is a positive constant large enough and choose  $c = \frac{1}{\nu}$  with  $\nu \in \mathbb{N}$  a decreasing sequence, we get a sequence of functions  $\theta^\nu$  such that they are non-singular on  $X \setminus E_{\frac{1}{\nu}}(\Theta)$  and

$$\Theta^\nu := \alpha + \sqrt{-1}\partial\bar{\partial}\theta^\nu \geq -A \min\{\lambda_\nu, \frac{1}{\nu}\}\omega^0 - \varepsilon_\nu\omega^0$$

holds on  $X$  for every  $\nu$ , where  $\lambda_\nu(x)$  is a decreasing sequence of continuous functions on  $X$  such that  $\lim_{\nu \rightarrow +\infty} \lambda_\nu(x) = \nu(\Theta, x)$  at every point and  $\varepsilon_\nu$  are positive decreasing such that  $\lim_{\nu \rightarrow +\infty} \varepsilon_\nu = 0$ .

During the proof we need to use the "metric" form "e<sup>-</sup>" concerning the currents  $\omega_t$  and  $\Theta$ . So we revise next the notations. Denote  $\phi_t = \phi^0 + \varphi_t$  with  $\sqrt{-1}\partial\bar{\partial}\phi^0 = \omega$ . So  $\omega_t$  can be written as

$$\omega_t = \sqrt{-1}\partial\bar{\partial}\phi_t.$$

Similarly, denote  $\psi = \psi^0 + \theta$  with  $\sqrt{-1}\partial\bar{\partial}\psi^0 = \alpha$ . Then  $\Theta$  can be written as

$$\Theta = \sqrt{-1}\partial\bar{\partial}\psi.$$

Hence if we denote  $\tau_t = \phi_t + \psi$  and respectively  $\tau_t^\nu = \phi_t^\nu + \psi^\nu$ , the above discussions imply that the sequence of currents

$$\omega_t^\nu + \Theta^\nu = \sqrt{-1}\partial\bar{\partial}\tau_t^\nu$$

decreases to

$$\omega_t + \Theta = \sqrt{-1}\partial\bar{\partial}\tau_t$$

as  $\nu$  tends to  $+\infty$  and is non-singular on  $X \setminus E_{\frac{1}{\nu}}(\Theta)$ . We evaluate next the lower bounds of  $\omega_t^\nu + \Theta^\nu$ . To this end, denote first a neighborhood set of  $E_{\frac{1}{\nu}}(\Theta)$  by

$$U_{\delta_\nu} = \{x \in X : e^{\psi^\nu} < \delta_\nu\}$$

(the choice of  $\delta_\nu$  to be precise in a moment). Remark that according to Theorem 3.10,  $\omega_t$  has uniformly bounded coefficients, since  $\omega_t^\nu$  converge uniformly to  $\omega_t$ , we have actually  $e^{\tau_t^\nu} < C\delta_\nu$  for  $x \in U_{\delta_\nu}$  with  $C > 0$  a numerical constant. Then by Theorem 3.11, we have

$$\sqrt{-1}\partial\bar{\partial}\tau_t^\nu \geq - \left( A \min\{\lambda_\nu, \frac{1}{\nu}\} + \varepsilon_\nu \right) \omega^0 \quad (3.7)$$

on  $X \setminus U_{\delta_\nu}$ . As  $\nu \rightarrow +\infty$ , the quantity  $- \left( A \min\{\lambda_\nu, \frac{1}{\nu}\} + \varepsilon_\nu \right) \omega^0$  tends to zero; Globally on  $X$ , we have

$$\sqrt{-1}\partial\bar{\partial}\tau_t^\nu \geq - \left( 2A \frac{1}{\nu} \right) \omega^0 \geq -3A\omega^0. \quad (3.8)$$

In the following proof we need to solve some  $\partial^{\tau_t^\nu}$ -equations, to insure that the solutions of such equations we will deal with satisfy certain  $L^2$ -properties, we need to modify further our approximations in order that the new "metrics" are non-singular on  $X$ . To this end, define

$$\tilde{\tau}_t^\nu = \log \left( e^{\tau_t^\nu} + \frac{1}{\nu} \right).$$

Since  $e^{\tau_t^\nu} + \frac{1}{\nu} > 0$ , functions  $\tilde{\tau}_t^\nu$  constructed in this way are therefore non-singular on  $X$ . We

analyze next their Hessian properties.

$$\begin{aligned}
 & \sqrt{-1} \partial \bar{\partial} \log \left( e^{\tau_t^\nu} + \frac{1}{\nu} \right) \\
 &= \sqrt{-1} \partial \frac{e^{\tau_t^\nu} \bar{\partial} \tau_t^\nu}{e^{\tau_t^\nu} + \frac{1}{\nu}} \\
 &= \sqrt{-1} \frac{(e^{\tau_t^\nu} \partial \bar{\partial} \tau_t^\nu + e^{\tau_t^\nu} \partial \tau_t^\nu \wedge \bar{\partial} \tau_t^\nu) (e^{\tau_t^\nu} + \frac{1}{\nu}) - e^{2\tau_t^\nu} \partial \tau_t^\nu \wedge \bar{\partial} \tau_t^\nu}{(e^{\tau_t^\nu} + \frac{1}{\nu})^2} \\
 &= \sqrt{-1} \frac{(e^{\tau_t^\nu} + \frac{1}{\nu}) e^{\tau_t^\nu} \partial \bar{\partial} \tau_t^\nu + \frac{1}{\nu} e^{\tau_t^\nu} \partial \tau_t^\nu \wedge \bar{\partial} \tau_t^\nu}{(e^{\tau_t^\nu} + \frac{1}{\nu})^2} \\
 &= \sqrt{-1} \frac{e^{\tau_t^\nu}}{e^{\tau_t^\nu} + \frac{1}{\nu}} \cdot \partial \bar{\partial} \tau_t^\nu + \sqrt{-1} \frac{\frac{1}{\nu} e^{\tau_t^\nu}}{(e^{\tau_t^\nu} + \frac{1}{\nu})^2} \cdot \partial \tau_t^\nu \wedge \bar{\partial} \tau_t^\nu.
 \end{aligned}$$

We remark that Hessians of these new functions  $\tilde{\tau}_t^\nu$  have similar properties as  $\tau_t^\nu$ , that is,

$$\sqrt{-1} \partial \bar{\partial} \tilde{\tau}_t^\nu$$

would be uniformly bounded from below on  $X$  and almost positive (i.e. the lower bounds tend to zero as  $\nu$  tends to infinity) on  $X \setminus U_{\delta_\nu}$ . Indeed, this will be true if we choose properly the value of  $\delta_\nu$ . Observe first that we have

$$0 < \frac{e^{\tau_t^\nu}}{e^{\tau_t^\nu} + \frac{1}{\nu}} \leq 1,$$

so the first term in the last equality satisfies our requirements. We now analyze the second term. Notice that we have

$$\sqrt{-1} \partial \tau_t^\nu \wedge \bar{\partial} \tau_t^\nu \geq 0$$

and

$$\frac{\frac{1}{\nu} e^{\tau_t^\nu}}{(e^{\tau_t^\nu} + \frac{1}{\nu})^2} \geq 0 \tag{3.9}$$

holds on  $X$ . So it is sufficient to check that the function in (3.9) goes to zero on  $X \setminus U_{\delta_\nu}$ . Indeed, for  $x \in X \setminus U_{\delta_\nu}$ , we have

$$\frac{\frac{1}{\nu} e^{\tau_t^\nu}}{(e^{\tau_t^\nu} + \frac{1}{\nu})^2} \leq \frac{1}{\nu} \frac{e^{\tau_t^\nu}}{e^{2\tau_t^\nu}} = \frac{1}{\nu} \frac{1}{e^{\tau_t^\nu}} \leq \frac{C}{\nu \delta_\nu}.$$

Hence if we choose for example  $\delta_\nu = \frac{1}{\sqrt{\nu}}$ ,

$$\frac{C}{\nu \delta_\nu} = \frac{C}{\sqrt{\nu}}$$

tends to zero as  $\nu$  tends to infinity. Therefore the second term fulfills also our requirements. We can conclude from (3.7), (3.8) and the above discussions that

$$\sqrt{-1} \partial \bar{\partial} \tilde{\tau}_t^\nu \geq -A_1 \omega^0 \tag{3.10}$$

on  $X$  with  $A_1 > 0$  a numerical constant as well as

$$\sqrt{-1} \partial \bar{\partial} \tilde{\tau}_t^\nu \geq -A_{\delta_\nu, \nu} \omega^0 \tag{3.11}$$

on  $X \setminus U_{\delta_\nu}$  with the property that  $A_{\delta_\nu, \nu}$  tends to zero as  $\nu \rightarrow +\infty$ . For the sake of simplifying the notations, we denote from now on  $\tilde{\tau}_t^\nu$  still by  $\tau_t^\nu$  with the understanding that it is now a sequence of non-singular potentials on  $X$  satisfying (3.10) and (3.11) on the relevant sets.

### 3.4.2 Constructions of $v_t^\nu$ .

As explained in the introduction, finding solutions of the  $\partial^{\tau_t^\nu}$ -equations

$$\partial^{\tau_t^\nu} v_t^\nu = P_{\perp}^{\nu,t}(\dot{\tau}_t^\nu u_t) \quad (3.12)$$

is important in the search of holomorphic vector fields. In this expression,  $u_t$  is the section of  $E$  described in the introduction, the operator  $\partial^{\tau_t^\nu}$  is defined by

$$\partial^{\tau_t^\nu} := e^{\tau_t^\nu} \partial e^{-\tau_t^\nu} = \partial - \partial \tau_t^\nu \wedge$$

and  $P_{\perp}^{\nu,t}$  is the projection on the orthogonal complement of the holomorphic  $n$ -forms on each fiber, and

$$\dot{\tau}_t^\nu = \frac{\partial \tau_t^\nu}{\partial t}.$$

We will explain why equation (3.12) can always be solved for  $\tau_t^\nu$  fixed in a moment. In order to analyze the uniformity properties of these solutions, the following lemma will be important (it is a slight variation of Lemma 6.2 in [Ber11]).

**Lemma 3.12.** *Let  $L$  be a holomorphic line bundle over  $X$  with a metric  $\xi$  satisfying  $\sqrt{-1}\partial\bar{\partial}\xi \geq -c_0\omega$  with  $c_0 > 0$  a numerical constant. Let  $\xi_0$  be a smooth metric on  $L$  with  $\xi \leq \xi_0$  and assume*

$$I := \int_X e^{\xi_0 - \xi} < \infty.$$

*Then there is a constant  $B$ , only depending on  $I$  and  $\xi_0$  such that if  $f$  is a  $\bar{\partial}$ -exact  $L$  valued  $(n, 1)$ -form with*

$$\int_X |f|^2 e^{-\xi} \leq 1, \quad (3.13)$$

*there is a solution  $u$  to  $\bar{\partial}u = f$  with*

$$\int_X |u|^2 e^{-\xi} \leq B.$$

**Proof.** The assumptions imply that

$$\|f\|_{\xi_0}^2 = \int_X |f|^2 e^{-\xi_0} \leq 1.$$

Since  $\bar{\partial}$  has closed range for  $L^2$ -norms defined by smooth metrics, we can solve  $\bar{\partial}u = f$  with the estimate

$$\int_X |u|^2 e^{-\xi_0} \leq C_1$$

for some constant  $C_1$  depending only on  $X$  and  $\xi_0$ . This can be shown by contradiction. Indeed, if this is not true, we have a non-zero sequence of  $\bar{\partial}$ -exact  $L$  valued  $(n, 1)$ -forms  $f_n$  with

$$\|f_n\|_{\xi_0}^2 \leq 1,$$

and a sequence of  $L$  valued  $(n, 0)$ -forms which solve the  $\bar{\partial}$ -equation

$$\bar{\partial}u_n = f_n$$

with

$$\|u_n\|_{\xi_0}^2 = n.$$

Since  $f_n$  are  $\bar{\partial}$ -exact and non-zero,  $u_n$  are in the space which is orthogonal to the space of  $\bar{\partial}$ -exact forms. Since the space of  $\bar{\partial}$ -exact forms is closed, its orthogonal space is also closed. Let us now consider the  $(n, 0)$ -forms

$$\frac{u_n}{\|u_n\|_{\xi_0}},$$

they lie in the space which is orthogonal to  $\bar{\partial}$ -exact forms, so the limit

$$\lim_{n \rightarrow +\infty} \frac{u_n}{\|u_n\|_{\xi_0}} \quad (3.14)$$

must stay in the same space by its closeness. On the other hand, we have

$$\frac{f_n}{\|u_n\|_{\xi_0}} = \bar{\partial} \frac{u_n}{\|u_n\|_{\xi_0}}$$

with the left hand side goes to zero when  $n \rightarrow +\infty$ . In other words, the term in (3.14) is  $\bar{\partial}$ -exact. That is to say, it lies in the intersection of the two spaces and hence must be equal to zero. But clearly the norm of  $\frac{u_n}{\|u_n\|_{\xi_0}}$  is equal to 1 for every  $n$ , hence its limit cannot be zero and this leads to the contradiction.

Return to the proof of the lemma, in subsection 3.4.1 we proved that

$$\sqrt{-1} \partial \bar{\partial} \tau_t^\nu \geq -A_1 \omega^0$$

on  $X$ . Choose a collection of coordinate balls  $B_j$  such that  $B_j/2$  cover  $X$  and that on the local coordinate sets  $z = (z_1, \dots, z_n) \in B_j$ , the curvature form of  $\tau_t^\nu$  satisfies

$$\sqrt{-1} \partial \bar{\partial} (\tau_t^\nu + A_1 |z|^2) \geq 0.$$

By Hörmander's  $L^2$ -estimate (cf. [Hör73]), on each  $B_j$  we can solve  $\bar{\partial} u_j = f$  with the estimate

$$\int_{B_j} |u_j|^2 e^{-(\tau_t^\nu + A_1 |z|^2)} \leq C_2 \int_{B_j} |f|^2 e^{-(\tau_t^\nu + A_1 |z|^2)} \leq C_2 \quad (3.15)$$

where  $C_2$  depends only on the size of the balls. Here we used again the assumption (3.13) on  $B_j$  together with the fact that  $|z|^2 \leq \text{Vol}(B_j)^2$ . Rewrite inequality (3.15), we have

$$\int_{B_j} |u_j|^2 e^{-\tau_t^\nu} \cdot e^{-A_1 |z|^2} \leq C_2,$$

so on each  $B_j$ , we have

$$\int_{B_j} |u_j|^2 e^{-\tau_t^\nu} \leq C_2 e^{A_1 \text{Vol}(B_j)^2} := C'_2$$

with  $C'_2$  depends only on the size of the balls. Denote

$$h_j := u - u_j,$$

then it is holomorphic on  $B_j$  and we have for  $\xi_0$  some arbitrary non-singular metric on  $L$ ,

$$\int_{B_j} |h_j|^2 e^{-\xi_0} \leq C_3.$$

By Cauchy's estimates, this induces in particular that

$$\sup_{B_j/2} |h_j|^2 e^{-\xi_0} \leq C_4.$$

Therefore we get

$$\int_{B_j/2} |h_j|^2 e^{-\tau_t^\nu} \leq \sup_{B_j/2} |h_j|^2 e^{-\xi_0} \cdot I \leq C_4 I$$

and finally

$$\int_{B_j/2} |u|^2 e^{-\tau_t^\nu} \leq \int_{B_j/2} |h_j|^2 e^{-\tau_t^\nu} + \int_{B_j/2} |u_j|^2 e^{-\tau_t^\nu} \leq C_5 I, \quad (3.16)$$

where  $C_4$  and  $C_5$  depend only on the size of balls. Since  $B_j/2$  cover  $X$ , Summing the inequality (3.16) up, we then get uniform estimates for solutions of  $\bar{\partial}$ -equations, independent of  $\nu$  and  $t$ .  $\square$

Since the  $\bar{\partial}$ -operator has closed range, it follows that for every  $\nu$  and  $t$ , the adjoint operator  $\bar{\partial}_{\tau_t^\nu}^*$  also has closed range. Since  $P_\perp^{\nu,t}(\dot{\tau}_t^\nu u)$  is orthogonal to holomorphic forms, we can solve the equation

$$\bar{\partial}_{\tau_t^\nu}^* \alpha_t^\nu = P_\perp^{\nu,t}(\dot{\tau}_t^\nu u)$$

with the property that  $\bar{\partial} \alpha_t^\nu = 0$ . Write  $\alpha_t^\nu = v_t^\nu \wedge \omega^1$  where  $\omega^1$  is the fixed Kähler form on  $X$ , then modulo a sign we have

$$\partial^{\tau_t^\nu} v_t^\nu = \bar{\partial}_{\tau_t^\nu}^* \alpha_t^\nu.$$

In other words,  $v_t^\nu$  solve the equation (3.12). The fact that the  $L^2$ -norm of  $v_t^\nu$  is uniformly bounded can be concluded from the above lemma together with some functional analysis, as we will show in the next. Indeed, given an  $(n, 1)$ -form  $\alpha_t^\nu$  such that  $\|\alpha\|_{\tau_t^\nu} < +\infty$ , by the above lemma there exists an  $(n, 0)$ -form  $f_t^\nu$  which solves the  $\bar{\partial}$ -equation

$$\bar{\partial} f_t^\nu = \alpha_t^\nu$$

together with the estimate

$$\|f_t^\nu\|_{\tau_t^\nu} \leq C \|\alpha_t^\nu\|_{\tau_t^\nu}. \quad (3.17)$$

with  $C$  a constant independent of  $t$  and  $\nu$ . Denote

$$\langle \langle \alpha_t^\nu, \alpha_t^\nu \rangle \rangle_{\tau_t^\nu} = \int_X |\alpha_t^\nu|^2 e^{-\tau_t^\nu},$$

then

$$\langle \langle \alpha_t^\nu, \alpha_t^\nu \rangle \rangle_{\tau_t^\nu} = \langle \langle \bar{\partial}_{\tau_t^\nu}^* \alpha_t^\nu, f_t^\nu \rangle \rangle_{\tau_t^\nu} \leq C \|\alpha_t^\nu\|_{\tau_t^\nu} \|\bar{\partial}_{\tau_t^\nu}^* \alpha\|_{\tau_t^\nu}.$$

Set  $\alpha_t^\nu = v_t^\nu \wedge \omega^1$  with  $\omega^1$  a fixed Kähler form on  $X$ , we obtain that

$$\|v_t^\nu\|_{\tau_t^\nu} \leq C \|\bar{\partial}_{\tau_t^\nu}^* \alpha_t^\nu\|_{\tau_t^\nu} = C \|P_\perp^{\nu,t}(\dot{\tau}_t^\nu u_t)\|_{\tau_t^\nu} \leq C \|\dot{\tau}_t^\nu u_t\|_{\tau_t^\nu}$$

with the same constant  $C$  as in (3.17). By definition,

$$\dot{\tau}_t^\nu = \frac{\partial}{\partial t}(\phi_t^\nu + \psi^\nu) = \dot{\phi}_t^\nu$$

Since  $\psi^\nu$  is independent of  $t$ . By construction,  $\phi_t^\nu$  is uniformly bounded with respect to  $\nu$  and Lipschitz in  $t$ , so  $\dot{\phi}_t^\nu$  is uniformly bounded with respect to both  $\nu$  and  $t$  and  $|\tau_t^\nu - \tau_0^\nu| \leq C$ . Hence

$$\|\dot{\tau}_t^\nu u_t\|_{\tau_t^\nu}$$

is uniformly bounded and this implies that the  $L^2$ -norm of  $v_t^\nu$  is uniformly bounded.

### 3.4.3 Holomorphic vector fields.

We show here that the limit of  $v_t^\nu$  to be holomorphic, as presented in the introduction, this is an important step towards the proof of Theorem 3.9 to construct the holomorphic vector field. So we are going to show that the sequence of  $L^2$ -norms of  $\bar{\partial}v_t^\nu$  tends to zero on  $X$ . To this end, notice that for every  $\nu$ , we can establish a formula concerning  $v_t^\nu$  thanks to the following theorem :

**Theorem 3.13.** *Let  $\Theta_E$  be a curvature form on  $E$  and let  $u_t$  be a holomorphic section of  $E$ . For each  $t$  in  $\mathbb{D}$  let  $v_t$  solve*

$$\partial^{\tau_t} v_t = P_\perp(\dot{\tau}_t u_t)$$

and be such that  $\bar{\partial}_X v_t \wedge \omega = 0$ . Put

$$\hat{u} = u_t - dt \wedge v_t.$$

Then

$$\langle \Theta_E u_t, u_t \rangle_t = p_*(c_n \sqrt{-1} \partial \bar{\partial} \tau \wedge \hat{u} \wedge \bar{u} e^{-\tau}) + \int_X \|\bar{\partial} v_t\|^2 e^{-\tau_t} \sqrt{-1} dt \wedge d\bar{t}.$$

□

Indeed, define

$$\mathcal{F}(t) := -\log \int_X e^{-\tau_t} d\lambda,$$

and respectively

$$\mathcal{F}_\nu(t) := -\log \int_X e^{-\tau_t^\nu} d\lambda,$$

apply Theorem 3.13 to  $u_t = u$ , we obtain the following formula

$$\|u\|_{t,\nu}^2 \sqrt{-1} \partial \bar{\partial} \mathcal{F}_\nu = \langle \Theta_E^\nu u, u \rangle_t = p_* \left( c_n \sqrt{-1} \partial \bar{\partial} \tau_t^\nu \wedge \hat{u} \wedge \bar{u} e^{-\tau_t^\nu} \right) + \int_X \|\bar{\partial} v_t^\nu\|^2 e^{-\tau_t^\nu} \sqrt{-1} dt \wedge d\bar{t}. \quad (3.18)$$

This formula gives a relationship between the  $L^2$ -norm of  $\bar{\partial}v_t^\nu$  and  $\sqrt{-1} \partial \bar{\partial} \mathcal{F}_\nu(t)$ . If  $\mathcal{F}(t)$  is affine along the geodesic, then we have

$$\sqrt{-1} \partial \bar{\partial} \mathcal{F}(t) = 0.$$

As a consequence, we infer that

$$\sqrt{-1} \partial \bar{\partial} \mathcal{F}_\nu(t)$$

tends to zero weakly on  $\mathbb{D}$ . It is the starting point to deduce that  $\bar{\partial}v_t^\nu$  goes to zero. Before going further, we first show that  $\mathcal{F}(t)$  is affine provided that  $\varphi_0$  and  $\varphi_1$  are two solutions of equation  $(\star)$ .

If  $\varphi_0$  and  $\varphi_1$  are two such local potentials of the currents  $\omega_0$  and  $\omega_1$  respectively, we connect them by a continuous geodesic

$$\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t = \sqrt{-1} \partial \bar{\partial} \phi_t.$$

Since  $\omega_t$  is semi-positive, the relative energy

$$\mathcal{E}(\phi_0, \phi_1)$$



is well defined. If  $\phi_t$  is non-singular in  $t$ , further calculations give that

$$\frac{\partial}{\partial t} \mathcal{E}(\phi_t, \phi_1) = \int_X \dot{\phi}_t (\sqrt{-1} \partial \bar{\partial} \phi_t)^n / \|T\|$$

and

$$\sqrt{-1} \partial \bar{\partial}_t \mathcal{E}(\phi_t, \phi_1) = \frac{p_*((\sqrt{-1} \partial \bar{\partial}_{X,t} \phi)^{n+1})}{\|T\|}.$$

In the above expression,  $p$  is the projection map from  $X \times \mathbb{D}$  to  $\mathbb{D}$  and  $\|T\|$  is the normalizing factor

$$\|T\| = \int_X (\sqrt{-1} \partial \bar{\partial}_X \phi)^n,$$

chosen so that the derivative of  $\mathcal{E}$  becomes 1 if  $\phi_t = \phi + t$ . So in particular, if  $\phi$  solves the homogenous Monge-Ampère equation

$$(\sqrt{-1} \partial \bar{\partial}_{X,t} \phi)^{n+1} = 0,$$

the function  $\mathcal{E}(\phi_t, \phi_1)$  is harmonic in  $t$ . Hence this function is linear along geodesics.

For the family  $\phi_t$  which is only bounded, the above calculations are done by defining for every  $t$ ,

$$\mathcal{E}(\phi_t, \phi_1) = \inf\{\mathcal{E}(\tilde{\phi}_t, \phi_1) \mid \tilde{\phi}_t \text{ are non-singular, } \tilde{\phi}_t \geq \phi_t\}.$$

Define

$$\mathcal{G}_\psi(\phi_t) = -\log \int_X e^{-(\phi_t + \psi)} d\lambda - \mathcal{E}(\phi_t, \chi) = \mathcal{F}(t) - \mathcal{E}(\phi_t, \chi)$$

with  $\chi$  arbitrary and  $\psi$  fixed. Then  $\phi_0$  and  $\phi_1$  are critical points of  $\mathcal{G}$ . Indeed, the fact that  $\phi_t$  depends only on the real part of  $t$  implies that  $\mathcal{G}_\psi(\phi_t)$  is convex. Now by the result in [Ber09b],  $\mathcal{F}$  is convex, and since  $\phi$  is convex in  $t$ , we see that the onesided derivatives of  $\mathcal{F}(t)$  at the endpoints equal

$$\int \dot{\phi}_t e^{-\tau} d\lambda / \int e^{-\tau} d\lambda,$$

with  $\dot{\phi}_t$  now stands for the onesided derivatives. On the other hand, the function  $\mathcal{E}(\phi_t, \chi)$  is linear, so its distributional derivative

$$\int_X \dot{\phi}_t (\sqrt{-1} \partial \bar{\partial} \phi_t)^n / \|T\|$$

is constant, and hence is equal to its values at the end points by applying the simple convergence theorems for the Monge-Ampère operator. Hence both endpoints are critical points and therefore the convexity of  $\mathcal{G}_\psi$  implies that it is constant. Finally, since  $\mathcal{E}$  is affine along the geodesic, it follows that

$$t \rightarrow \mathcal{F}(t)$$

is also affine along the geodesic.

We now show that  $\bar{\partial}_X v_t'$  goes to zero weakly over  $X \times K$  for any compact  $K \subset \mathbb{D}$ . Firstly, we know that for every  $t$ ,  $\tau_t'$  decrease to  $\tau_t$  and hence

$$\sqrt{-1} \partial \bar{\partial} \mathcal{F}_t'$$

tends weakly to

$$\sqrt{-1}\partial\bar{\partial}\mathcal{F} = 0$$

on  $\mathbb{D}$ . By curvature formula (3.18), we see that the right hand side

$$p_*(c_n\sqrt{-1}\partial\bar{\partial}\tau_t^\nu \wedge \widehat{u} \wedge \bar{\widehat{u}}e^{-\tau_t^\nu}) + \int_X \|\bar{\partial}v_t^\nu\|^2 e^{-\tau_t^\nu} \sqrt{-1}dt \wedge d\bar{t} \quad (3.19)$$

tends to zero weakly as well. We aim to show that the second quantity in (3.19),

$$\int_X \|\bar{\partial}v_t^\nu\|^2 e^{-\tau_t^\nu} \sqrt{-1}dt \wedge d\bar{t} \quad (3.20)$$

converges to zero. To this end, we examine first the quantity

$$p_*(c_n\sqrt{-1}\partial\bar{\partial}\tau_t^\nu \wedge \widehat{u} \wedge \bar{\widehat{u}}e^{-\tau_t^\nu}).$$

As shown in subsection 3.4.1, it is uniformly bounded from below on  $X$  by

$$-A_1\|\widehat{u}\|^2.$$

This implies in particular that the second quantity (3.20) is at least uniformly bounded from above. More explicit estimates give that for any  $t$ , the first quantity is bounded from below by

$$-A_{\delta_\nu,\nu}\|\widehat{u}\|^2 - A_1 \int_{U_{\delta_\nu}} |v_t^\nu|^2 e^{-\tau_t^\nu} \quad (3.21)$$

with  $A_{\delta_\nu,\nu}$  tends to zero as  $\nu \rightarrow +\infty$ . So we need to evaluate the quantity

$$\int_{U_{\delta_\nu}} |v_t^\nu|^2 e^{-\tau_t^\nu} \quad (3.22)$$

when  $\nu \rightarrow +\infty$ . We remark that the difference from the case analyzed in [Ber11] is that here the domain of integration  $U_{\delta_\nu}$  changes with  $\nu$ . Nevertheless, as we have discussed in subsection 3.4.2, we have

$$\int_X |v_t^\nu|^2 e^{-\tau_t^\nu} \leq D \quad (3.23)$$

with  $D$  a constant independent of  $t$  and  $\nu$ ; we have just proved that the quantity

$$\int_X |\bar{\partial}v_t^\nu|^2 e^{-\tau_t^\nu}$$

is uniformly bounded from above, together with (3.23) implies that

$$\int_X |v_t^\nu|^{\frac{2n}{n-2}} dV_\omega \leq E$$

holds on  $X$  for some uniform constant (at this point, we are using the classical Sobolev type estimates, see for example, [Joy00]). Recall that

$$U_{\delta_\nu} = \{x \in X : e^{\tau_t^\nu} < \frac{C}{\sqrt{\nu}}\}.$$

By Hölder inequality, we have

$$\begin{aligned} \int_{U_{\delta_\nu}} |v_t^\nu|^2 e^{-\tau_t^\nu} &\leq \nu \int_{U_{\delta_\nu}} |v_t^\nu|^2 dV_\omega = \nu \int_X 1_{U_{\delta_\nu}} |v_t^\nu|^2 dV_\omega \\ &\leq \nu \left( \int_X |v_t^\nu|^{\frac{2n}{n-2}} dV_\omega \right)^{\frac{n-2}{n}} \text{Vol}(U_{\delta_\nu})^{\frac{2}{n}} \leq \nu \cdot F \cdot \text{Vol}(U_{\delta_\nu})^{\frac{2}{n}}. \end{aligned} \quad (3.24)$$

In order to evaluate the volume of  $U_{\delta_\nu}$ , we quote the following result :

**Proposition 3.14.** ([DK01]) Let  $X$  be a complex manifold and  $\varphi$  a psh function of  $x$ . Let  $K \subset X$  be a compact set,  $U \Subset X$  a relatively compact neighborhood of  $K$ , and let  $\mu_U$  be the Riemannian measure on  $U$  associated with some choice of hermitian metrics  $\omega$  on  $X$ . Then for any  $c \geq 0$ , the volume of sublevel sets  $\{\varphi < \log r\}$  satisfies that

$$r^{-2c} \mu_U(\{\varphi < \log r\}) \leq \int_U e^{-2c\varphi} dV_\omega.$$

According to the above proposition, the volume of  $U_{\delta_\nu}$  satisfies the following inequality : For  $\beta \geq 0$ , we have

$$\text{Vol}(U_{\delta_\nu})^{\frac{2}{n}} \leq \left( \frac{C}{\sqrt{\nu}} \right)^{2\frac{\beta}{n}} \left( \int_X e^{-2\beta\tau_t^\nu} dV_\omega \right)^{\frac{1}{n}} \leq \left( \frac{C}{\sqrt{\nu}} \right)^{2\frac{\beta}{n}} \left( \int_X e^{-2\beta\tau_t} dV_\omega \right)^{\frac{1}{n}}.$$

We take  $\beta = \frac{n}{2}(1 + \varepsilon_0)$ , by hypothesis,

$$\int_X e^{-(1+\varepsilon_0)\frac{n}{2}\tau} < +\infty$$

and by the proceeding inequalities,

$$\int_{U_{\delta_\nu}} |v_t^\nu|^2 e^{-\tau_t^\nu} \leq C\nu \frac{1}{\nu^{(1+\varepsilon_0)}} \rightarrow 0$$

as  $\nu$  tends to infinity and hence by (3.24) the quantity (3.22) tends to zero. As a consequence, (3.21) tends to zero. We have proved actually that the first quantity in (3.19) is positive when  $\nu$  tends to infinity. By the fact that the second quantity in (3.19) is positive and  $\sqrt{-1}\partial\bar{\partial}\mathcal{F}_\nu$  tends to zero, we conclude that both quantities in (3.19) tend to zero. In particular, the  $L^2$ -norm of  $\bar{\partial}_X v_t^\nu$  goes to zero weakly. Since the  $L^2$ -norm of  $v_t^\nu$  is uniformly bounded we can select a subsequence of  $v_t^\nu$  that converges weakly to a form  $v_t$  in  $L^2$ . Therefore we get

$$\bar{\partial}_X v_t = 0$$

and

$$\partial_X^{\tau_t} v_t = P_\perp^t(\dot{\tau}_t u)$$

in the weak sense that

$$\int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge v_t \wedge \overline{\partial W} e^{-\tau_t} = \int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge P_\perp^t(\dot{\tau}_t u) \wedge \overline{W} e^{-\tau_t}$$

for any compact supported non-singular form  $W$  of appropriate degree. Indeed, by integration by parts, for every  $\nu$ , we have

$$\int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge v_t^\nu \wedge \overline{\partial W} e^{-\tau_t^\nu} = \int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge \partial^{\tau_t^\nu}(v_t^\nu) \wedge \overline{W} e^{-\tau_t^\nu} = \int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge P_\perp^{t,\nu}(\dot{\tau}_t^\nu u) \wedge \overline{W} e^{-\tau_t^\nu}.$$

The left hand side integral in the above equalities converges to

$$\int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge v_t \wedge \overline{\partial W} e^{-\tau_t}.$$

We now deal with the third integral, since  $\tau_t^\nu$  is uniformly Lipschitz in  $t$ ,  $\dot{\tau}_t^\nu$  is uniformly bounded; since moreover  $\tau_t^\nu$  is convex with respect to  $t$ ,  $\dot{\tau}_t^\nu$  increase. The following lemma implies that  $\dot{\tau}_t^\nu$  converge to  $\dot{\tau}_t$ .

**Lemma 3.15.** *Let  $f_\nu$  be a sequence of non-singular convex functions on an interval in  $\mathbb{R}$  that converge uniformly to the convex function  $f$ . Let  $a$  be a point in the interval such that  $f'(a)$  exists. Then  $f'_\nu(a)$  converge to  $f'(a)$ . Since a convex function is differentiable almost everywhere it follows that  $f'_\nu$  converges to  $f'$  almost everywhere, with dominated convergence on any compact subinterval.*

By the above lemma,  $\dot{\tau}_t^\nu u$  converge to  $\dot{\tau}_t u$ . Since the sequence  $\dot{\tau}_t^\nu u$  are increasing, for any  $t$ , we have

$$|P_\perp^{t,\nu}(\dot{\tau}_t^\nu u)| \leq |\dot{\tau}_t^\nu u| \leq |\dot{\tau}_t u|.$$

That is to say,  $P_\perp^{t,\nu}(\dot{\tau}_t^\nu u)$  is uniformly bounded. By Banach-Alaoglu-Bourbaki theorem, there exists a subsequence of  $P_\perp^{t,\nu}(\dot{\tau}_t^\nu u)$  which converges weakly, say, to  $P_\perp^t(\dot{\tau}_t u)$ . In other words, there exists a subsequence of  $P_\perp^{t,\nu}(\dot{\tau}_t^\nu u)$  such that

$$\int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge P_\perp^{t,\nu}(\dot{\tau}_t^\nu u) \wedge \overline{W} e^{-\tau_t^\nu} \rightarrow \int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge P_\perp^t(\dot{\tau}_t u) \wedge \overline{W} e^{-\tau_t}.$$

Hence

$$\int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge v_t \wedge \overline{\partial W} e^{-\tau_t} = \int_{X \times \mathbb{D}} dt \wedge d\bar{t} \wedge P_\perp^t(\dot{\tau}_t u) \wedge \overline{W} e^{-\tau_t}.$$

Now we define a holomorphic vector field  $V_t$  by

$$v_t = -V_t \lrcorner u,$$

we see that the proof ends if there are no nontrivial holomorphic vector fields on  $X$ . Then  $v$  must be zero, so  $\dot{\tau}_t$  is holomorphic, hence constant. Therefore

$$\partial \bar{\partial} \dot{\tau}_t = 0$$

so  $\partial \bar{\partial} \dot{\tau}_t$  does not depend on  $t$ . In the general case, we show that  $v_t$  is holomorphic in  $t$ . Since we don't know any regularity of  $v_t$  except that it belongs to  $L^2$ , we need to formulate holomorphicity weakly. That is, we are going to prove

$$\int_{X \times \mathbb{D}} dt \wedge v_t \wedge \partial_t \bar{\gamma} e^{-\tau_t} = 0$$

for any non-singular  $\tau_t'$  on  $-K_X$  and  $\gamma$  an  $(n, 1)$  form on  $X \times \mathbb{D}$  which does not contain  $dt$ . For this, we follow the same steps as shown in section 4 of [Ber11].

Denote

$$a_\nu := \int_{X \times \mathbb{D}'} \sqrt{-1} \partial \bar{\partial} \tau_t^\nu \wedge \hat{u} \wedge \bar{u} e^{-\tau_t^\nu},$$

we have shown that it goes to zero if  $\mathbb{D}'$  is a relatively compact subdomain in  $\mathbb{D}$ . Shrinking  $\mathbb{D}$  slightly we assume that this actually holds with  $\mathbb{D}' = \mathbb{D}$ . Choose  $W$  to contain no differential  $dt$ , so it is an  $(n, 0)$ -form on  $X$  with coefficients depending on  $t$ , then we have

$$\int_{X \times \mathbb{D}} \sqrt{-1} \partial \bar{\partial} \tau_t^\nu \wedge \hat{u} \wedge \bar{W} e^{-\tau_t^\nu} = \int_{X \times \mathbb{D}} \sqrt{-1} \partial \bar{\partial}_t \tau_t^\nu \wedge \hat{u} \wedge \bar{W} e^{-\tau_t^\nu}.$$

Since

$$\sqrt{-1} \partial \bar{\partial}_t \tau_t^\nu \geq 0,$$

by Cauchy-Schwarz inequality, we have

$$\left| \int_{X \times \mathbb{D}} \sqrt{-1} \partial \bar{\partial}_t \tau_t^\nu \wedge \hat{u} \wedge \bar{W} e^{-\tau_t^\nu} \right|^2 \leq a_\nu \int_{X \times \mathbb{D}} \sqrt{-1} \partial \bar{\partial}_t \tau_t^\nu \wedge W \wedge \bar{W} e^{-\tau_t^\nu}. \quad (3.25)$$

Assume now  $W$  has compact support. We apply the following one variable Hörmander's theorem to the right hand side integral in (3.25) to get that it is dominated by

$$\int_{X \times \mathbb{D}} |\partial_t^\nu W|^2 e^{-\tau_t^\nu}. \quad (3.26)$$

**Proposition 3.16.** *(One variable Hörmander's theorem). Let  $\Omega$  be a domain in  $\mathbb{C}$  and let  $\phi \in C^2(\Omega)$ ,  $\alpha \in C_c^2(\Omega)$ . Then*

$$\int \Delta |\alpha|^2 e^{-\phi} + \int \left| \frac{\partial}{\partial \bar{z}} \alpha \right|^2 e^{-\phi} = \int |\bar{\partial}_\phi^* \alpha|^2 e^{-\phi}.$$

□

Assume moreover that  $W$  is Lipschitz with respect to  $t$  as a map from  $\mathbb{D}$  into  $L^2(X)$ . Then (3.26) is uniformly bounded by Lemma 3.15. Hence the left hand side integral in (3.25) goes to zero, in particular, the part which contains  $dt \wedge d\bar{t}$

$$\int_{X \times \mathbb{D}} \sqrt{-1} dt \wedge d\bar{t} \wedge \left( \partial \bar{\partial}_t \tau_t^\nu - \partial_X \left( \frac{\partial \tau_t^\nu}{\partial t} \right) (V_t) \right) \bar{W} e^{-\tau_t^\nu}$$

goes to zero, where  $V_t$  is a vector field defined by

$$-v_t = V_t \lrcorner u.$$

Denote

$$\mu_t^\nu = \partial \bar{\partial}_t \tau_t^\nu - \partial_X \left( \frac{\partial \tau_t^\nu}{\partial t} \right) (V_t).$$

The next lemma shows that

$$\int_{X \times \mathbb{D}} \sqrt{-1} dt \wedge d\bar{t} \wedge \mu_t^\nu \wedge P_\perp \bar{W} e^{-\tau_t^\nu}$$

also goes to zero.

**Lemma 3.17.** *Let  $\alpha_t$  be forms on  $X$  with coefficients depending on  $t$  on  $\mathbb{D}$ . Assume that  $\alpha_t$  is Lipschitz with respect to  $t$  as a map from  $\mathbb{D}$  to  $L^2(X)$ . Let  $P^t$  be the orthogonal projection on  $\bar{\partial}$ -closed forms with respect to the metric  $\tau_t$  and the fixed Kähler metric  $\omega$ . Then  $P^t(\alpha_t)$  is also Lipschitz, with a Lipschitz constant depending only on that of  $\alpha$  and the Lipschitz constant of  $\tau_t$  with respect to  $t$ .*

In other words, this lemma gives that the integral

$$\int_{X \times \mathbb{D}} \sqrt{-1} dt \wedge d\bar{t} \wedge P_{\perp}^t(\mu_t^{\nu}) \wedge \overline{W} e^{-\tau_t^{\nu}}$$

goes to zero. Recall that

$$\partial^{\tau_t^{\nu}} v_t^{\nu} = \dot{\tau}_t^{\nu} \wedge u + h_t^{\nu}$$

where each  $h_t$  is holomorphic on  $X$  for each  $t$  fixed. Since  $\tau_t^{\nu}$  are non-singular,  $v_t^{\nu}$  depend smoothly on  $t$ . Differentiating with respect to  $\bar{t}$  yields

$$\partial^{\tau_t^{\nu}} \frac{\partial v_t^{\nu}}{\partial \bar{t}} = \left[ \frac{\partial^2 \tau_t^{\nu}}{\partial t \partial \bar{t}} - \partial_X \left( \frac{\partial \tau_t^{\nu}}{\partial \bar{t}} \right) (V_t) \right] \wedge u + \frac{\partial h_t^{\nu}}{\partial \bar{t}}.$$

We obtain therefore

$$P_{\perp}^t(\mu_t^{\nu}) = \partial^{\tau_t^{\nu}} \left( \frac{\partial v_t^{\nu}}{\partial \bar{t}} \right)$$

since the left hand side is automatically orthogonal to holomorphic forms. Hence

$$\int_{X \times \mathbb{D}} \sqrt{-1} dt \wedge d\bar{t} \wedge \frac{\partial v_t^{\nu}}{\partial \bar{t}} \wedge \overline{\partial_X W} e^{-\tau_t^{\nu}} = \int_{X \times \mathbb{D}} \sqrt{-1} dt \wedge d\bar{t} \wedge P_{\perp}^t(\mu_t^{\nu}) \wedge \overline{W} e^{-\tau_t^{\nu}}$$

vanishes as  $\nu$  tends to infinity.

Now let  $\gamma$  be a form of bidegree  $(n, 1)$  on  $X \times \mathbb{D}$  that does not contain any differential  $dt$ . Assume that it is Lipschitz with respect to  $t$  and decompose it into two parts,  $\overline{\partial}_X W$ , which is  $\overline{\partial}_X$ -exact and the other part orthogonal to  $\overline{\partial}_X$ -exact forms. We make such orthogonal decomposition for each  $t$  separately, by Lemma 3.17 each term in the decomposition is still Lipschitz in  $t$  and uniformly in  $\nu$ . Since  $v_t^{\nu} \wedge \omega^0$  is  $\overline{\partial}_X$ -closed, it is also the case for  $\partial v_t^{\nu} / \partial \bar{t}$ . So  $\partial v_t^{\nu} / \partial \bar{t}$  is  $\overline{\partial}$ -exact, this is due to Nadel's vanishing theorem [Nad90]. Indeed, since

$$L := -K_X = c_1(X) - \{\alpha\} + \{\alpha\}$$

and  $c_1(X) - \{\alpha\}$  is a Kähler class, we can find a Kähler form  $T^0$  in  $c_1(X) - \{\alpha\}$  and a closed positive current  $T^1$  such that

$$T^0 + T^1 \geq \varepsilon_1 \omega^0$$

for some  $\varepsilon_1 > 0$ . The fact that  $e^{-(1+\varepsilon_0)\psi} \in L_{\text{loc}}^1(X)$  implies the multiplier ideal sheaf associated to the local potential of  $T^0 + T^1$  is trivial. Therefore, by Nadel's vanishing theorem,

$$H^1(X) = H^1(X, K_X + L) = 0.$$

In the sequel, we obtain

$$\int_{X \times \mathbb{D}} \sqrt{-1} dt \wedge d\bar{t} \wedge \frac{\partial v_t^{\nu}}{\partial \bar{t}} \wedge \overline{\gamma} e^{-\tau_t^{\nu}} = \int_{X \times \mathbb{D}} \sqrt{-1} dt \wedge d\bar{t} \wedge \frac{\partial v_t^{\nu}}{\partial \bar{t}} \wedge \overline{\partial_X W} e^{-\tau_t^{\nu}}.$$

Hence

$$\int_{X \times \mathbb{D}} dt \wedge v_t \wedge \overline{\partial_t^{\tau_t^{\nu}} \gamma} e^{-\tau_t^{\nu}}$$

goes to zero. Again by Lemma 3.15 we can pass to the limit and get

$$\int_{X \times \mathbb{D}} dt \wedge v_t \wedge \overline{\partial_t^{\tau_t} \gamma} e^{-\tau_t} = 0 \tag{3.27}$$

under the assumption that  $\gamma$  is of compact support and Lipschitz in  $t$ . We almost show that

$$\bar{\partial}_t v_t = 0$$

except that  $\tau_t$  is not smooth. By replacing  $\gamma$  by  $e^{\tau_t - \tau'_t} \gamma$ , where  $\tau'_t$  is another potential on  $L$ . If (3.27) holds for some  $\tau_t$ , Lipschitz in  $t$ , it holds for any such potential. So we can replace  $\tau_t$  in (3.27) by some other smooth potential. It shows that  $v_t$  is holomorphic in  $t$  and therefore  $v_t$  is holomorphic in  $X \times \mathbb{D}$ .

Finally, since by definition

$$-v_t = V_t \lrcorner u,$$

$V_t$  is holomorphic on  $X \times \mathbb{D}$ . Further calculation yields

$$\bar{\partial} \dot{\tau}_t \wedge u = \partial \bar{\partial} \tau_t \wedge v_t = -\partial \bar{\partial} \tau_t \wedge (V_t \lrcorner u) = (V_t \lrcorner \partial \bar{\partial} \tau_t) \wedge u.$$

Since  $u$  never vanishes, it implies that

$$V_t \lrcorner \sqrt{-1} \partial \bar{\partial} \tau_t = \sqrt{-1} \partial \bar{\partial} \dot{\tau}_t.$$

The Lie derivative of  $\partial \bar{\partial} \tau_t$  along  $V_t$  is

$$L_{V_t} \partial \bar{\partial} \tau_t = \partial V_t \lrcorner \partial \bar{\partial} \tau_t = \partial \bar{\partial} \dot{\tau}_t = \frac{\partial}{\partial t} \partial \bar{\partial} \tau_t. \quad (3.28)$$

Define a holomorphic vector field  $\mathcal{V}$  on  $X \times \omega$  by

$$\mathcal{V} := V_t - \frac{\partial}{\partial t}.$$

Let  $\eta$  be the form  $\partial \bar{\partial}_X \tau_t$  on  $\mathcal{X}$ . The formula (3.28) says that

$$L_{\mathcal{V}} \eta = 0$$

on  $X$ . It follows that  $\eta$  is invariant under the flow of  $\mathcal{V}$  so  $\partial \bar{\partial} \tau_t$  moves by the flow of a holomorphic family of automorphisms of  $X$ . In other words, we have shown that

$$F^*(\omega_{\varphi_1} + \Theta) = \omega_{\varphi_0} + \Theta.$$

The equality

$$\text{Ricci}(\omega_{\varphi_1}) = \omega_{\varphi_1} + \Theta$$

implies that

$$F^*(\text{Ricci}(\omega_{\varphi_1})) = F^*(\omega_{\varphi_1} + \Theta) = \omega_{\varphi_0} + \Theta = \text{Ricci}(\omega_{\varphi_0}). \quad (3.29)$$

Since  $F^*(\text{Ricci}(\omega_{\varphi_1})) = \text{Ricci}(F^*(\omega_{\varphi_1}))$ , by (3.29), we have

$$\text{Ricci}(F^*(\omega_{\varphi_1})) = \text{Ricci}(\omega_{\varphi_0}). \quad (3.30)$$

That is to say, there exists a bounded psh function  $\zeta$  such that

$$F^*(\omega_{\varphi_1}) = \omega_{\varphi_0} + \sqrt{-1} \partial \bar{\partial} \zeta. \quad (3.31)$$

(3.30) and (3.31) then induce that

$$(\omega_{\varphi_0} + \sqrt{-1} \partial \bar{\partial} \zeta)^n = \omega_{\varphi_0}^n$$

and hence

$$(\omega + \sqrt{-1} \partial \bar{\partial}(\varphi_0 + \zeta))^n = (\omega + \sqrt{-1} \partial \bar{\partial} \varphi_0)^n. \quad (3.32)$$

The following theorem implies that  $\zeta = 0$  (cf. [EGZ09]).

**Theorem 3.18.** ([EGZ09]) Let  $X$  be a compact Kähler manifold,  $\phi$  a smooth closed form of bidegree  $(1, 1)$  which is non-negative and big, i.e. the smooth measure  $\phi^n$  is not identical to zero. Let  $\omega$  be a semi positive  $(1, 1)$ -form such that  $\int_X \omega^n > 0$  and  $0 \leq f \in L^p(X, \omega^n)$ ,  $p > 1$ , a density such that  $\int_X f \phi^n = \int_X \phi^n$ . Then there is a unique bounded function  $\varphi$  on  $X$  such that  $\phi + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0$  and

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = f \phi^n \quad \text{with} \quad \sup_X \varphi = -1.$$

Therefore (3.31) yields

$$F^*(\text{Ricci}(\omega_{\varphi_1})) = \text{Ricci}(\omega_{\varphi_0})$$

and hence

$$F^*(\Theta) = \Theta.$$

□

### 3.5 Comments.

The curvature formula in Theorem 3.13 is based on a special choice of the auxiliary  $(n-1, 0)$ -form  $v_t$  as the solution of the equation

$$\partial^{\tau_t} v_t = \pi_{\perp}(\dot{\tau}_t v_t).$$

A more general formula was given in [Ber11]. More precisely, let  $p : \mathcal{X} \rightarrow Y$  be a smooth proper holomorphic fibration of complex manifolds of relative dimension  $n$ , and  $\mathcal{L} \rightarrow \mathcal{X}$  be a holomorphic line bundle equipped with a smooth metric of semipositive curvature. The direct image sheaf of the relative canonical bundle twisted with  $\mathcal{L}$

$$p_*(\mathcal{L} + K_{\mathcal{X}/Y}),$$

is then associated to a vector bundle,  $E$ , over  $Y$  with fibers (we identify  $\mathcal{X}_y = X_y$ )

$$E_y = H^0(X_y, K_{X_y} + L|_{X_y}).$$

Let  $\omega$  be some Kähler form on  $\mathcal{X}$  and take the base to be  $\mathbb{D}$ , the unit disk in  $\mathbb{C}$ . Choose a representative  $\mathbf{u}$  of a section  $u_t$  of the bundle  $E$ , since  $\bar{\partial} \mathbf{u} = 0$  on fibers,

$$\bar{\partial} \mathbf{u} = d\bar{t} \wedge \nu + dt \wedge \eta$$

where  $\nu$  is of bidegree  $(n, 0)$  and  $\eta$  is of  $(n-1, 1)$ . The forms  $\nu$  and  $\eta$  are not uniquely determined but their restrictions to fibers are, since

$$dt \wedge d\bar{t} \wedge \bar{\partial} \nu = dt \wedge \bar{\partial}^2 \mathbf{u} = 0,$$

$\nu$  is holomorphic on fibers. Then

$$D'' u = \nu d\bar{t}.$$

In particular,  $u$  is a holomorphic section of  $E$  if and only if  $\bar{\partial} \mathbf{u} \wedge dt = 0$ , i.e. if and only if

$$\bar{\partial} \mathbf{u} = dt \wedge \eta$$



for some  $\eta$ . The norm of the section  $u_t$  can be expressed by

$$\|u_t\|_t^2 = p_*(c_n \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\phi})$$

and Laplacian of  $\|u_t\|_t^2$  can be computed by  $\sqrt{-1} \partial \bar{\partial} p_*(c_n \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\phi})$ ,

$$\sqrt{-1} \partial \bar{\partial} p_*(c_n \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\phi}) = -p_*(c_n \sqrt{-1} \partial \bar{\partial} \phi \wedge \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\phi}) + c_n \int_{X_t} \eta \wedge \bar{\eta} e^{-\phi} dV_t + \|\mu\|_t^2 dV_t$$

where  $\partial \phi \mathbf{u} = e^\phi \partial e^\phi \mathbf{u} = dt \wedge \mu$ , if  $u_t$  is a holomorphic section of  $E$ . Combine this formula with the standard one (always under the assumption that  $u_t$  is holomorphic)

$$\Delta \|u_t\|_t^2 = -\langle \Theta_E u_t, u_t \rangle + \|D' u_t\|_t^2$$

where  $(D' u)_t = P(\mu) dt$ , here  $P(\mu)$  is the orthogonal projection of the  $(n, 0)$ -form  $\mu$  on the space of holomorphic  $(n, 0)$ -forms. We get the curvature of  $E$

$$\begin{aligned} \langle \Theta_E u_t, u_t \rangle_t &= \|D' u_t\|_t^2 - \|\mu\|_t^2 + p_*(c_n \sqrt{-1} \partial \bar{\partial} \phi \wedge \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\phi}) / dV_t - c_n \int_{X_t} \eta \wedge \bar{\eta} e^{-\phi} \\ &= -\|P_\perp \mu\|_t^2 + p_*(c_n \sqrt{-1} \partial \bar{\partial} \phi \wedge \mathbf{u} \wedge \bar{\mathbf{u}} e^{-\phi}) / dV_t - c_n \int_{X_t} \eta \wedge \bar{\eta} e^{-\phi} \end{aligned} \quad (3.33)$$

with  $P_\perp$  the orthogonal projection on the orthogonal complement of holomorphic forms. Every term in the right hand side depends on the choice of the representative whereas the left hand side does not. A special choice of  $\mathbf{u}$  leads  $\mu$  to be holomorphic on  $X_t$  and  $\eta$  be primitive on it. (i.e.,  $\eta \wedge \omega = 0$  on  $X_t$  with  $\omega$  the Kähler form on  $X$ .) This is the curvature formula in Theorem 3.13.

Another suitable choice of  $\omega$  and  $\mathbf{u}$  induce (3.33) to another curvature formula, as follows (cf. [Ber11]) :

**Theorem 3.19.** *Assume  $\phi$  is a metric on  $\mathcal{L}$  with  $\sqrt{-1} \partial \bar{\partial} \phi$  strictly positive on fibers. The curvature of the  $L^2$ -metric on  $E$  is then given at  $t$  by*

$$\langle \Theta_E u, u \rangle = \int_{X_t} c(\phi) |u|^2 e^{-\phi} + \langle (\square' + 1)^{-1} \eta, \eta \rangle.$$

with  $\eta = k_t^\phi \cdot u$ ,  $k_t^\phi$  is a representative of the Kodaira-Spencer class  $\mathcal{K}_t$  and  $\square' = \nabla'(\nabla')^* + (\nabla')^* \nabla'$  is the Laplacian on  $\mathcal{L}|_{X_t}$ -valued forms on  $X_t$  defined by the  $(1, 0)$ -part of the Chern connection on  $\mathcal{L}|_{X_t}$ .

Before going further, let us recall first the definition of the Kodaira-Spencer class  $\mathcal{K}_t$  (cf. [Ber11]). Let  $V$  be a smooth vector field of type  $(1, 0)$  on  $X$  which maps to the field  $\partial/\partial t$  on  $\mathbb{D}$ , the unit disk in  $\mathbb{C}$  under the derivative of the map  $p$  from  $X$  to  $\mathbb{D}$ . Then  $\bar{\partial} V$  is a  $(0, 1)$ -form on  $X$  with values in the bundle of tangent vectors tangential to the fiber. It's restriction to a fiber  $X_t(i_t^*(\bar{\partial} V))$ ,  $k_t$  is a  $\bar{\partial}$ -closed form with values in  $T^{1,0}$  of the fiber. The cohomology class it defines is the Kodaira-Spencer class  $\mathcal{K}_t$  at  $t$ . If  $u_t$  is an element in  $E_t$ , the Kodaira-Spencer  $\mathcal{K}_t$  class operate on  $u_t$  is defined as the cohomology class of  $k_t \cdot u_t$ , where the vectorvalued  $(0, 1)$  form  $k_t$  acts on  $u_t$  by contraction :  $k_t$  can be decomposed locally by

$$k_t = w \otimes v,$$

where  $w$  is scalar valued and  $v$  is a vector field, then

$$k_t \cdot u_t = w \wedge \delta_\nu u_t.$$

This is then a cohomology class in  $H^{n-1,1}(X_t, \mathcal{L}|_{X_t})$ . When  $n = 1$  and  $\mathcal{L}$  is trivial,  $k_t$  is a  $(0, 1)$ -form with values in  $-K_{X_t}$ , and if  $u$  is a holomorphic 1-form on  $X_t$ , then  $k_t \cdot u$  is just the product defining a scalar  $(0, 1)$ -form.

The curvature formula in Theorem 3.19 was obtained by the choice of the form  $\tilde{v}_t$  as

$$\tilde{v}_t = V_t \lrcorner u,$$

where  $V_t$  is the complex gradient of  $\dot{\phi}_t$  defined by

$$V_t \lrcorner \partial \bar{\partial}_X \phi_t = \bar{\partial} \dot{\phi}_t.$$

The definition of  $\tilde{v}_t$  and  $v_t$  coincide in the case that the curvature vanishes. Moreover, from Theorem 3.19, we see that  $\Theta_E u = 0$  for some  $u$  in  $E_t$ , then  $c(\phi) = 0$  and  $k_t^\phi \cdot u = 0$ . The second condition implies that  $k_t^\phi = 0$ , this is because that  $k_t^\phi$  is a fixed smooth form, but not a cohomology class, so if it vanishes outside the zero divisor of  $u$ , it vanishes identically. Hence the Kodaira-Spencer class vanishes, even if  $\mathcal{L}$  is not necessarily trivial. It also follows that  $\Theta_E u = 0$  for all  $u$  in  $E_t$ .

In our more general setting, the definition of  $\tilde{v}_t$  makes no sense since we have no metric at hand on  $X$  to define a complex gradient. Nevertheless, as mentioned in [Ber11], the method used in the proof of the main theorem can be adapted to define a 'complex gradient' in a nonsingular situation. The basic principal is the following proposition.

**Proposition 3.20.** *Let  $L$  be a holomorphic line bundle over the compact Kähler manifold  $X$ , and let  $\phi$  be a smooth metric on  $L$ , not necessarily with positive curvature. Assume  $V$  is a holomorphic vector field on  $X$  such that*

$$V \lrcorner \partial \bar{\partial} \phi = 0.$$

Then  $V = 0$  provided that

$$H^{0,1}(X, K_X + L) = 0$$

and

$$H^0(X, K_X + L) \neq 0.$$

It means that the holomorphic solutions of

$$V \lrcorner \partial \bar{\partial} \phi = \bar{\partial} \xi$$

are unique, if they exist.

Finally, we predict that our method might be useful to conclude the uniqueness of solutions of the equations appeared in the continuity method, proposed by Yau to prove the Calabi's conjecture. We present in the following this method.

The goal is to prove that a particular nonlinear equation

$$(\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^{-\varphi+f} \omega^n \tag{3.34}$$

has a solution  $\varphi$ . The function  $f$  satisfies that

$$\text{Ricci}_\omega = \omega + \sqrt{-1} \partial \bar{\partial} f. \tag{3.35}$$

The continuity method is to consider a 1-parameter family of equations depending continuously on  $t \in [0, 1]$ ,

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{-t\varphi_t + f}\omega^n. \quad (3.36)$$

Denote

$$\omega_{\varphi_t} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t,$$

rewrite equation (3.36), we have

$$\begin{aligned} \text{Ricci}_{\omega_{\varphi_t}} &= \sqrt{-1}\partial\bar{\partial}t\varphi_t - \sqrt{-1}\partial\bar{\partial}f + \text{Ricci}_{\omega} \\ &= \omega + \sqrt{-1}\partial\bar{\partial}t\varphi_t \\ &= t\omega_{\varphi_t} + (1-t)\omega. \end{aligned}$$

When  $t = 0$ , the problem has been solved by Yau; and  $t = 1$  the equation is the one must be solved.

Now for  $(t_k) \in I$ , denote

$$\omega_{\varphi_{t_k}} = \omega + \sqrt{-1}\partial\bar{\partial}\varphi_{t_k}.$$

The sequence  $(\omega_{\varphi_{t_k}})$  may have several weak limits. One problem we hope to analyze in a near future is to show the uniqueness of the weak limit. To this end, the methods used and generalized in this chapter should be useful.



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## Théorèmes d'extension et métriques de Kähler-Einstein généralisées

Cette thèse comporte deux parties :

- Dans la première partie, nous traitons d'abord une version kählérienne du célèbre théorème d'extension d'Ohsawa-Takegoshi, puis, un problème de prolongement des courants positifs fermés. Notre motivation provient de la conjecture de Siu sur l'invariance des plurigenres dans le cas d'une famille kählérienne. En effet, dans la preuve du célèbre théorème d'invariance des plurigenres de Siu, le théorème d'extension d'Ohsawa-Takegoshi joue un rôle important. Il est donc naturel de penser que la preuve de la conjecture fera également intervenir un théorème d'extension de type Ohsawa-Takegoshi dans le cas kählérien. Suite aux difficultés techniques qui proviennent de la régularisation des fonctions quasi-psh sur les variétés kählériennes compactes, nous obtenons seulement deux cas particuliers du résultat espéré. Pour ce qui est du prolongement des courants positifs fermés, notre résultat est un cas particulier de la conjecture qui prédit que tout courant positif fermé défini sur le fibré central d'une classe de cohomologie kählérienne tordue par la classe de Chern du fibré canonique admet un prolongement.

- Dans la deuxième partie, nous nous intéressons à l'unicité des solutions des équations de type Monge-Ampère généralisées. Il s'agit d'une généralisation d'un théorème de Bando-Mabuchi concernant les métriques de Kähler-Einstein sur les variétés de Fano. Nous suivons la méthode introduite par Berndtsson et généralisons son résultat en travaillant avec un courant positif fermé à la place d'une paire klt dans son contexte. Les propriétés de convexité des métriques de Bergman jouent un rôle important dans cette partie.

**Mots clés :** variétés kählériennes, théorème d'extension d'Ohsawa-Takegoshi, l'invariance des plurigenres, courants positifs fermés, équation de Monge-Ampère, théorème de Bando-Mabuchi, variétés de Fano, noyau de Bergman

## Extension theorems and generalized Kähler-Einstein metrics

This thesis consists in two parts :

-In the first part, we first deal with a Kähler version of the famous Ohsawa-Takegoshi extension theorem ; then, a problem of extending the closed positive currents. Our motivation comes from the Siu's conjecture on the invariance of plurigenera over a Kähler family. Indeed, in the proof of his famous theorem, the Ohsawa-Takegoshi theorem plays an important role. It is, therefore, natural to think that the proof for the conjecture involves an extension theorem of Ohsawa-Takegoshi type in the Kähler case. Because of the technical difficulties coming from the regularization process of quasi-psh functions over the compact Kähler manifolds, we only obtain two special cases of the hoped result. As for the extension of closed positive currents, our result is a special case of the conjecture which predicts that every closed positive current defined over the central fiber in a Kähler cohomology class twisted by the first Chern class of the canonical bundle admits an extension.

-In the second part, we are interested in the uniqueness of the solutions of the equations of generalized Monge-Ampère type, a generalized Bando-Mabuchi theorem concerning the Kähler-Einstein metrics over Fano manifolds. We follow the method introduced by Berndtsson and generalize his result by working with a closed positive current in place of a klt pair in his context. The properties of the convexity of the Bergman metrics play an important role in this part.

**Key words :** Kähler manifolds, Ohsawa-Takegoshi extension theorem, invariance of plurigenera, closed positive currents, Monge-Ampère equation, Bando-Mabuchi theorem, Fano manifolds, Bergman kernel