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Rim Touibi

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Sur le comportement qualitatif des solutions de certaines équations aux dérivées partielles stochastiques de type parabolique

THÈSE

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par

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*Je dédie cette thèse
à ma maman.*

On the qualitative behavior of
solutions to certain stochastic
partial differential equations of
parabolic type

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Chapter 1

Introduction et résumé des résultats

Cette thèse est une contribution à l'étude du comportement des solutions d'une classe d'équations aux dérivées partielles stochastiques (EDPS) semilinéaires de type parabolique. La partie stochastique de ces équations tient compte de facteurs exogènes, qui ne sont généralement pas complètement connus et donc aléatoires, et qui affectent le comportement du système décrit par les équations déterministes correspondantes. La théorie des EDPS est un domaine de recherche très actif depuis la fin des années 1960 et connaît actuellement un développement important, faisant intervenir des concepts et des résultats aussi bien de la théorie des probabilités que de l'analyse proprement dite. Cette théorie est particulièrement intéressante en raison de son caractère interdisciplinaire et de la richesse des applications potentielles actuelles.

De nombreuses applications allant du monde de l'ingénierie et de la finance aux sciences naturelles nécessitent une modélisation mathématique en termes d'EDPS. En particulier, de nombreux travaux ont été consacrés à l'analyse du comportement des solutions d'EDPS de type parabolique qui apparaissent spécifiquement dans la propagation des impulsions nerveuses ([76], [74]) et dans la dynamique des populations pour la description de certains phénomènes de migration ([14], [8]) ou pour la génétique des populations. En physique, la littérature est très importante et couvre une très large classe d'équations aux dérivées partielles (EDP) (équation de la chaleur, des ondes, Navier-Stokes,...) qui modélisent l'évolution spatio-temporelle du système étudié et contiennent une partie stochastique pour tenir compte de l'aléa, c'est-à-dire des perturbations non prévisibles agissant sur ce système physique. Citons dans ce contexte les travaux sur l'équation de sine-Gordon ([74], [75]) ou l'équation de Fitzhugh-Nagumo [9]. En finance et contrairement aux domaines précédents, les EDPS ne modélisent en général pas une évolution spatio-temporelle, mais une évolution par rapport au temps et par rapport à d'autres paramètres financiers ([52]). Ces dernières années, on s'est intéressé de plus en plus à l'étude des EDPS dirigées par

un bruit gaussien ayant la structure de covariance du mouvement brownien fractionnaire (mbf). Cet intérêt vient du grand nombre d'applications du mbf dans les sciences. Pour ne citer que quelques exemples de l'application de bruits fractionnaires, nous mentionnons [38] pour la biophysique et [18] pour la physique.

Le mbf de paramètre de Hurst $H \in (0, 1)$, introduit par A. N. Kolmogorov en 1940 et développé par B. B. Mandelbrot et J. W. Van Ness en 1968 [51], est un processus gaussien centré et continu $(B_t^H)_{t \geq 0}$ dont la covariance est donnée par

$$\mathbb{E}(B_t^H B_s^H) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

Il est caractérisé par la propriété d'auto-similarité

$$\forall a > 0, (a^{-H} B_{at}^H)_{t \geq 0} \stackrel{\mathcal{L}}{=} (B_t^H)_{t \geq 0},$$

la stationnarité de ses accroissements

$$\forall h > 0, (B_{t+h}^H - B_h^H)_{t \geq 0} \stackrel{\mathcal{L}}{=} (B_t^H)_{t \geq 0}$$

et, dans le cas $H > 1/2$, la propriété de dépendance à longue portée. Le mbf est l'une des généralisations les plus naturelles du mouvement brownien. En effet, pour le cas $H = \frac{1}{2}$, le mBf n'est autre qu'un mouvement brownien classique (mb). Un aspect important de l'étude du mbf réside dans le domaine de l'analyse stochastique. Puisque ce processus n'est ni une semi-martingale ni un processus de Markov (si $H \neq \frac{1}{2}$), le calcul stochastique de Itô ne s'applique pas. Pour les valeurs du paramètre de Hurst H supérieur à $1/2$, les intégrales de Young et des techniques du calcul fractionnaire ont été appliquées ([80], [81]). Bien que la plupart des travaux ait été concentrée principalement sur des questions d'existence globale, d'unicité et d'explosion de la solution en temps fini (citons par exemple [65], [60], [50], [4], [5], [20], [21], [22]), des recherches ont également été consacrées à l'analyse d'équations impliquant un mélange de bruit brownien et de bruit fractionnaire, dans le domaine des équations différentielles stochastiques (EDS) ordinaires et partielles (EDPS). Un premier pas dans cette direction a été fait dans les articles de J. Guerra et D. Nualart [30] et par Y. Mishura, K. Ralchenko et G. Shevchenko ([55], [56], [57]). La motivation pour un tel type de perturbation stochastique mixte provient notamment des mathématiques financières. En fait, afin de modéliser le hasard sur les marchés de la finance, il est utile de distinguer les sources principales de ces aléas. La première source est la bourse elle-même avec ses milliers d'agents. Le bruit provenant de cette source peut être modélisé par un mb. La deuxième source est la situation financière et économique. Le bruit aléatoire provenant de cette source a généralement une propriété de dépendance à longue portée et peut être modélisée par un mbf de paramètre de Hurst $H > 1/2$.

Notons bien que dans la plus grande partie de la littérature, l'étude des EDPS a été faite

soit sur un domaine D borné ([60], [65], [50]), soit sur l'espace entier [5].

Le but de ce projet de thèse est, dans un premier temps, de contribuer davantage à l'analyse de certaines questions liées à l'existence et l'unicité de solutions variationnelles globales et locales pour des EDPS avec des conditions aux bords de type Neumann, définies sur une classe de domaines non bornés de l'espace euclidien, et avec une perturbation fractionnaire plus riche. Il s'agit d'une extension des résultats de D. Nualart et P-A. Vuillermot [60]. Puis, nous nous intéresserons au comportement en temps long de la solution d'une classe plus particulière d'EDPS paraboliques avec des conditions aux bords de type Dirichlet, perturbées par un mélange de mb et mbf et dirigées par des opérateurs différentiels contenant les générateurs infinitésimaux d'une classe assez large de processus markoviens. Nous avons traité des équations définies sur un domaine borné, mais certaines méthodes développées dans cette partie s'appliquent également à des équations définies sur l'espace entier. Une partie des résultats de cette seconde partie prolongent ceux obtenus dans [20], [21], [22], [4] et [48], d'autres résultats sont démontrés par de nouvelles méthodes.

On se propose dans l'organisation de ce manuscrit de reprendre, dans un premier temps, l'article élaboré en collaboration avec M. Dozzi et P-A. Vuillermot [23], et dans la seconde partie le travail avec M. Dozzi et J.A. López-Mimbela sous une forme plus détaillée et en version anglaise. Dans la suite, nous donnons un résumé de ces deux travaux:

Dans le deuxième chapitre, nous prouvons de nouveaux résultats concernant l'existence et l'unicité de solutions variationnelles globales et locales pour une classe d'EDPS paraboliques semi-linéaires non autonomes, données formellement par

$$\begin{cases} du(x, t) = (\operatorname{div}(k(x, t)\nabla u(x, t)) + g(u(x, t)))dt + h(u(x, t))W^H(x, dt), \\ (x, t) \in D \times (0, T], \\ u(x, 0) = \varphi(x), \quad x \in D, \\ \frac{\partial u(x, t)}{\partial n(k)} = 0, \quad (x, t) \in \partial D \times (0, T], \end{cases} \quad (1.0.1)$$

où $D \subset \mathbb{R}^d$ est un ouvert non borné satisfaisant à certaines conditions géométriques, la dernière relation représente la dérivée conormale de u relativement au champ matriciel k et W^H est un processus de Wiener fractionnaire de dimension infinie, caractérisé par une suite de paramètres de Hurst $H = (H_i)_{i \in \mathbb{N}^+} \subset (\frac{1}{2}, 1)$. Ces paramètres sont soumis à d'autres contraintes intimement liées à la nature de la non-linéarité h du terme stochastique, et au choix des espaces fonctionnels dans lesquels le problème à résoudre est bien posé. Signalons que le choix de ce type de bruit nécessite une extension de quelques aspects de la théorie développée dans [58], plus précisément des outils techniques concernant les intégrales de Stieltjes généralisées (sous-section 2.1). Sous des conditions de Lipschitz sur g et h et sous des conditions de Hölder sur h' , nous montrons l'existence d'une solution variationnelle de (1.0.1) sur les intervalles $(0, T]$ pour tout $T > 0$, ainsi que l'unicité dans le cas où h est affine. Notre méthode de preuve repose essentiellement sur des arguments d'injection compacte qui imposent des conditions de nature géométrique sur la forme de

D : nous supposons que D est à volume fini et décroît rapidement et régulièrement en dehors des boules centrées à l'origine et de rayon r , quand r tend vers l'infini. Dans la sous-section 2.4 nous illustrons notre résultat à l'aide de quelques exemples de domaines D non bornés. Finalement nous supposons que la fonction g vérifie une condition de Lipschitz locale seulement et, par conséquent, une explosion en temps fini de la solution ne peut pas être exclue. Dans ce contexte, nous introduisons la notion de solution variationnelle locale de (1.0.1), et nous démontrons son existence et son unicité jusqu'à un éventuel temps d'explosion dans le cas où h est affine.

Le troisième chapitre est consacré à l'étude du comportement en temps long de la solution du problème de Dirichlet parabolique

$$\begin{cases} du(x, t) = \left(\frac{1}{2}k^2(t)Lu(x, t) + g(u(x, t))\right)dt + u(x, t)dN_t, \\ x \in D, t > 0, \\ u(x, 0) = \varphi(x) \geq 0, x \in D, \\ u(x, t) = 0, x \in \partial D, t \geq 0, \end{cases} \quad (1.0.2)$$

où $D \subset \mathbb{R}^d$ est un domaine borné, g satisfait à une condition de Lipschitz locale et L fait partie d'une classe d'opérateurs représentant les générateurs infinitésimaux associés à des processus de diffusion et à des processus de Lévy non nécessairement symétriques. Ces processus sont précisés dans la section 3.2. Le coefficient k^2 de L permet de contrôler l'influence de l'évolution temporelle de l'opérateur sur le comportement en temps long de la solution. La perturbation est donnée par un mélange

$$N_t := \int_0^t a(s)dW(s) + b^H(s)dB^H(s)$$

où W est un mb et B^H est un mbf avec paramètre de Hurst $H > 1/2$. Nous nous intéressons à l'étude d'une éventuelle explosion de la solution de (1.0.2) en temps fini et aux conditions suffisantes qui permettent de les éviter et de disposer donc d'une solution globale. Plus précisément, nous donnons des expressions explicites pour des bornes supérieures et inférieures du temps d'explosion τ de (1.0.2) si $\tau < \infty$. Comme ces bornes s'appliquent au niveau des trajectoires individuelles de la solution, elles sont souvent difficiles à évaluer en présence d'une partie stochastique. Pour cette raison nous donnons des estimations de la loi de τ et de la probabilité que $\tau < \infty$. Pour obtenir des énoncés aussi explicites que possible sur les explosions de la solution, nous nous sommes restreints dans ce chapitre au cas où la partie stochastique dépend linéairement de la solution. Ceci et le fait de considérer des perturbations qui agissent uniquement dans le temps permettent de transformer la forme faible de (1.0.2) en la forme faible d'une équation aux dérivées partielles (EDP) aléatoire où W et B^H n'interviennent plus comme intégrateurs dans une intégrale stochastique, mais font partie des coefficients de l'équation, ce qui facilite la description du

comportement en temps long de la solution. Cette EDP aléatoire est donnée formellement par:

$$\begin{cases} \frac{\partial v}{\partial t}(x, t) = \frac{1}{2}k^2(t)Lv(x, t) - \frac{1}{2}a^2(t)v(x, t) + \exp(-N_t)g(\exp(N_t)v(x, t)), \\ \quad x \in D, t > 0, \\ v(x, 0) = \varphi(x), x \in D, \\ v(x, t) = 0, x \in \partial D, t \geq 0. \end{cases} \quad (1.0.3)$$

Le lien entre (1.0.2) et (1.0.3) se fait à l'aide d'une formule d'Itô qui est applicable à la fois au mb et mbf et s'exprime par l'égalité

$$v(x, t) = \exp(-N_t)u(x, t).$$

Elle peut s'interpréter comme un cas particulier de la transformation connue sous le nom de Doss-Sussman (pour le cas $D = \mathbb{R}^d$ voir S. Jing [37]). Si une explosion en temps fini a lieu, elle se produit au même temps t et au même point x pour la solution faible de (1.0.2) que pour la solution faible de (1.0.3). Certains résultats de ce chapitre sur l'absence resp. la présence d'une explosion en temps fini sont démontrés avec la forme faible de (1.0.3), d'autres avec la forme évolutive de (1.0.3). Il est donc important d'établir l'équivalence des solutions faibles et des solutions évolutives de (1.0.3), ce qui est fait en sous-section 3.2 à l'aide des équations de Kolmogorov. Nous établissons également un résultat d'existence et d'unicité de solution évolutive de (1.0.3).

Les résultats obtenus à l'aide de la solution faible sont valables pour des équations définies sur un domaine borné de \mathbb{R}^d . En fait, les fonctions propres de l'opérateur L jouent un rôle important dans les preuves et on a besoin d'une fonction propre positive. C'est le cas pour la première fonction propre des opérateurs L étudiés en section 2, définis sur un domaine borné $D \subset \mathbb{R}^d$. Si D est non borné ou si l'équation est définie sur l'espace \mathbb{R}^d entier, de telles fonctions propres ne sont pas nécessairement disponibles. Dans la section 3.4, nous étudions le comportement de la solution avec la forme faible de (1.0.3). Afin d'obtenir une borne supérieure pour le temps d'explosion (s'il est fini), nous considérons dans la sous-section 3.4.1 une sous-solution de (1.0.3) qui est la solution d'une équation différentielle dont le temps d'explosion τ^* est déterminé explicitement en termes d'une fonctionnelle exponentielle de N . A l'aide d'un résultat de N.T. Dung [25] nous estimons dans la sous-section 3.4.2 la loi de cette borne supérieure. Une attention particulière est donnée au cas $H \in (3/4, 1)$ dans la sous-section 3.4.3. Il est bien connu que, dans ce cas, N est équivalent au mb [16]. Cela permet d'appliquer un résultat sur la loi des fonctionnelles exponentielles du mouvement brownien du à D. Dufresne [24] et M. Yor [79] afin d'obtenir une borne inférieure de la probabilité d'explosion en temps fini de (1.0.3).

Les résultats de la sous-section 3.3.5 sont démontrés à l'aide de la forme évolutive de (1.0.3). Les méthodes utilisées dans cette sous-section sont en fait applicables aux équations

tions définies sur des domaines bornés ainsi qu'aux équations définies sur \mathbb{R}^d . Nous nous limitons aux domaines bornés et donnons une condition suffisante pour la solution de (1.0.3) d'exploser en temps fini (sous-section 5.1) et une borne inférieure du temps d'explosion (sous-section 5.2). Cette borne inférieure permet d'obtenir une condition suffisante pour l'existence d'une solution globale.

Notons que la question de l'explosion en temps fini des EDP déterministes, c'est-à-dire sans partie stochastique ($a = b^H = 0$ dans (1.0.2)) a été étudiée de façon très approfondie pour une grande variété d'EDP et systèmes d'EDP. Les premiers résultats ont été obtenus pour $L = \Delta$, le Laplacien sur $D \subseteq \mathbb{R}^d$ par H. Fujita ([28], [29]). Il a étudié l'équation

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + u(x, t)^{1+\beta}, & t > 0, x \in D \\ u(x, 0) = \varphi(x) \geq 0. \end{cases} \quad (1.0.4)$$

Si D est un domaine borné, et $\phi_0 > 0$ est la fonction propre associée à la première valeur propre $\lambda_0 > 0$ de Δ sur D , H. Fujita a montré en 1968 [29] que la condition $\int_D \varphi(x)\phi_0(x)dx > \lambda_0^{1/\beta}$ est suffisante pour l'explosion en temps fini de la solution, condition qu'on peut facilement récupérer comme cas particulier de la borne supérieure donnée dans la sous-section 3.4.1. Si on remplace, dans l'équation de H. Fujita (1.0.4), Δ par $\Delta - V$, où V est une constante positive, une condition suffisante pour l'existence d'une solution globale est donnée par

$$\Lambda\beta \int_0^\infty \exp(-Vr) \|S(r)\varphi\|_\infty^\beta dr < 1,$$

où $S(r)$ est le semigroupe associé au Laplacien sur D et Λ une constante qui est liée à une condition de croissance de la non-linéarité g . Nous récupérerons cette condition suffisante en adaptant l'énoncé du corollaire 3.5.1 de la sous-section 3.5.2. Notons qu'on arrive toujours à satisfaire cette condition en choisissant V assez grand.

Evaluation des résultats et perspectives:

À notre connaissance, le problème parabolique de type Neumann (1.0.1) n'a pas été traité dans la littérature jusqu'à présent sur des domaines non bornés et pour un mélange infini de mbf de paramètres de Hurst $H_i > 1/2$, $i \in \mathbb{N}$. Notre travail a permis de généraliser à une classe de domaines non bornés un résultat d'existence de solutions variationnelles de l'EDPS étudiée dans [60]. Cette étude est basée sur un résultat d'injection compacte qui a mené à imposer certaines conditions sur la forme géométrique de la classe des domaines. De même, le problème de l'unicité de la solution variationnelle au cas où h n'est pas affine reste ouvert. Sous une condition de Lipschitz locale sur g dans (1.0.1) nous montrons, pour h affine, l'existence et l'unicité de la solution variationnelle locale de (1.0.1).

Les bornes inférieure τ_* (3.5.4) et supérieure τ^* (3.4.2) du temps d'explosion τ de la so-

lution de (1.0.2) et (1.0.3) permettent de connaître l'influence de chacun des paramètres k , a , b^H et φ de ces équations sur τ si $\tau < +\infty$. Notons que l'opérateur L n'agit sur τ que par sa première valeur propre λ_0 et sa première fonction propre ϕ_0 , supposées positives. L'évaluation quantitative de ces formules n'est cependant facile que pour les EDP déterministes, càd pour $a = b^H = 0$ (voir notre commentaire ci-dessus sur l'équation de Fujita). En présence d'une partie stochastique cette évaluation est beaucoup plus difficile, d'une part parce que les valeurs de λ_0 , de ϕ_0 et la forme explicite du semigroupe associé à L ne sont connues que pour certains domaines $D \subset \mathbb{R}^d$ particuliers ([78], [34]), d'autre part, parce que les formules pour τ_* et τ^* s'évaluent trajectoire par trajectoire. Pour ces raisons des estimations de λ_0 , de ϕ_0 et de la fonction de Green, ainsi que de la loi de τ et de la probabilité que $\tau < +\infty$ sont très utiles. Notons que les formules pour τ_* et τ^* sont données en termes de fonctionnelles exponentielles de N et que leur loi n'est connue que pour $N = W$, le mb (sous-section 4.3 du chapitre 3). Pour notre mélange de mb et mbf, nous avons obtenu une estimation de la loi de τ^* (sous-section 4.2 du chapitre 3).

Un constat similaire est valable pour les résultats obtenus dans la dernière section du chapitre 3, à savoir la condition suffisante pour l'explosion en temps fini de la solution de (1.0.2) et (1.0.3) (sous-section 5.1) et la condition suffisante pour l'existence et l'unicité d'une solution globale (sous-section 3.5.2), déduite de la formule pour τ_* .

Concernant les perspectives apparaissant à l'issue de ce travail, nous pouvons envisager les points suivants:

- Généraliser le résultat de l'existence d'une solution variationnelle au problème de type Neumann, étudié dans le deuxième chapitre, à une classe plus large de fonctions tests spatio-temporelles en vue d'établir le lien avec la solution évolutive du problème.
- Étendre l'étude effectuée au deuxième chapitre à des EDPS avec un bruit multiplicatif dérivé d'un mélange de processus de Wiener et de processus de Wiener fractionnaire.
- Remplacer dans le chapitre 3 les conditions de Dirichlet par des conditions de Neumann pour comprendre l'influence des conditions aux bords sur le comportement en temps long de la solution.
- Apprendre davantage en ce qui concerne la loi des fonctionnelles exponentielles qui apparaissent dans les formules de τ_* et τ^* pour le mbf et pour un mélange de mb et de mbf.
- Étudier la possibilité d'une explosion pour les solutions d'autres types d'EDPS, par exemple des EDPS hyperboliques, l'équation de Burgers, l'équation de Navier-Stokes...

Chapter 2

Global and local variational solutions to a class of fractional SPDE's on unbounded domains

2.1 Introduction

Many applications ranging from the worlds of engineering and finance to the natural sciences call for a mathematical modelling in terms of stochastic differential equations. In particular, there have been many works devoted to the analysis of the ultimate behavior of solutions to stochastic partial differential equations of parabolic type which specifically occur in population dynamics, population genetics, nerve pulse propagation and related topics, to name only a few (see, e.g., [77] for a brief account of some of those works and the references therein). Moreover, there have also been several more recent articles dealing with the analysis of solutions to various types of semilinear parabolic stochastic partial differential equations driven either by a Brownian noise, or by a fractional noise with Hurst parameter $H \in (\frac{1}{2}, 1)$ (see, e.g., [4], [5], [20]-[22], and the plethora of references therein, particularly [49]). While these works have been primarily centered around questions of global existence, uniqueness and blowup in finite time, there have also been investigations essentially motivated by issues in financial mathematics devoted to the analysis of problems that involve a mixture of a Brownian noise with a fractional noise, within the realm of both ordinary and partial stochastic differential equations (see, e.g., [30], [55]-[57] and the references therein).

It is our purpose here to contribute further to the analysis of some of the above questions by proving the existence, and in some cases the uniqueness, of global and local variational solutions to Neumann initial-boundary value problems associated with a class of non-autonomous stochastic parabolic partial differential equations defined on certain regions

of Euclidean space. In what follows we assume that all the functional spaces are real, use the standard notations for the usual spaces of Lebesgue integrable functions and their norms, and begin by defining the Wiener process that will generate the fractional noise we need.

Let $D \in \mathbb{R}^d$ be an unbounded open domain where $d \in \mathbb{N}^+$, and let us consider a linear, self-adjoint, positive, non-degenerate trace-class operator C in $L^2(D)$ whose eigenfunctions and eigenvalues we write $(e_i)_{i \in \mathbb{N}^+}$ and $(\lambda_i)_{i \in \mathbb{N}^+}$, respectively. Let $H = (H_i)_{i \in \mathbb{N}^+}$ be a sequence of Hurst parameters satisfying $H_i \in (\frac{1}{2}, 1)$ for every $i \in \mathbb{N}^+$, and for $t \in \mathbb{R}_0^+$ let $(B_i^{H_i}(t))_{t \in \mathbb{R}^+}$ be a sequence of one-dimensional, independent fractional Brownian motions defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and starting at the origin. Assuming that $(e_i)_{i \in \mathbb{N}^+}$ constitutes an orthonormal basis of $L^2(D)$, we then define the $L^2(D)$ -valued fractional Wiener process $W^H(., t)$ by

$$W^H(., t) := \sum_{i=1}^{+\infty} \sqrt{\lambda_i} e_i(.) B_i^{H_i}(t). \quad (2.1.1)$$

This series converges strongly in $L^2(D)$ \mathbb{P} -*a.s.* by virtue of the basic properties of $B_i^{H_i}(t)$'s, the boundedness of the sequence H and the fact that C is trace-class. From this we conclude that (2.1.1) defines a centered Gaussian process whose covariance is entirely determined by C , that is,

$$\begin{aligned} & \mathbb{E}((W^H(., s), v)_2 (W^H(., t), \hat{v})_2) \\ &= \frac{1}{2} \sum_{i=1}^{+\infty} (s^{2H_i} + t^{2H_i} - |s - t|^{2H_i}) (v, C e_i)_2 (\hat{v}, e_i)_2 \end{aligned}$$

for all $s, t \in \mathbb{R}_0^+$ and all $v, \hat{v} \in L^2(D)$, where $(., .)_2$ denotes the standard inner product in this space and \mathbb{E} the expectation functional on $(\Omega, \mathcal{F}, \mathbb{P})$ (see, e.g., [58] for a discussion of various basic properties of fractional Brownian motion).

For $T \in \mathbb{R}_0^+$ arbitrary, we then consider the class of stochastic initial-boundary value problems formally given by

$$\left\{ \begin{array}{l} du(x, t) = (\operatorname{div}(k(x, t) \nabla_x u(x, t)) + g(u(x, t))) dt + h(u(x, t)) W^H(x, dt), \\ \quad (x, t) \in D \times (0, T], \\ u(x, 0) = \varphi(x), \quad x \in D, \\ \frac{\partial u(x, t)}{\partial n(k)} = 0, \quad (x, t) \in \partial D \times (0, T] \end{array} \right. \quad (2.1.2)$$

where $\partial D = \bar{D} \setminus D$ stands for the boundary of D and where the last relation stands for the conormal derivative of u relative to the matrix-valued vector field k . Moreover, g and

h are real-valued while φ is a random initial condition. Regarding these functions we will need the following hypotheses:

- (K) The function $k : D \times (0, T] \mapsto \mathbb{R}^{d^2}$ is Lebesgue-measurable and we have $k_{i,j}(\cdot) = k_{j,i}(\cdot)$ for every $i, j \in \{1, \dots, d\}$. Moreover, there exist constants $\underline{k}, \bar{k} \in \mathbb{R}^+$ such that the inequalities

$$\underline{k}|y|^2 \leq (k(x, t)y, y) \leq \bar{k}|y|^2 \quad (2.1.3)$$

hold uniformly in $(x, t) \in D \times (0, T]$ for all $y \in \mathbb{R}^d$ where (\cdot, \cdot) and $|\cdot|$ denote the standard Euclidean inner product and the related norm in \mathbb{R}^d , respectively.

- (L) The function $h, g : \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz continuous. Moreover, the derivative $h' : \mathbb{R} \mapsto \mathbb{R}$ of h exists and is Hölder continuous with exponent $\gamma \in (0, 1]$ and bounded. In addition, we also impose that

$$\underline{H} \in \left(\frac{1}{\gamma + 1}, 1 \right) \quad (2.1.4)$$

where $\underline{H} := \inf_{i \in \mathbb{N}^+} H_i$.

- (I) the initial condition φ is an $L^2(D)$ -valued random variable.

Finally, whereas the properties of the operator C are sufficient to define W^H , they are not quite strong enough to allow us to prove the result we are looking for. Recalling that C is necessarily an integral transform whose generating kernel we denote by κ , we still impose the following spectral condition:

- (C) We have

$$x \mapsto \int_D dy |\kappa(x, y)|^2 \in L^\infty(D) \quad (2.1.5)$$

and

$$\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \|e_i\|_\infty < +\infty. \quad (2.1.6)$$

As to the consistency of this hypothesis, we simply remark that we have indeed $e_i \in L^\infty(D)$ as a consequence of (2.1.5), which follows easily from the eigenvalue equation $Ce_i = \lambda_i e_i$, Schwarz inequality and the fact that $\|e_i\|_2 = 1$.

We will also need some functional spaces in order to define the notion of variational solutions we are interested in. To this end we fix once and for all an $\alpha \in (1 - \underline{H}, \frac{\gamma}{\gamma+1})$ where \underline{H} satisfies (2.1.4), and introduce the Banach space $\mathcal{B}^{\alpha, 2}([0, T]; L^2(D))$ consisting of all measurable mappings $u : [0, T] \mapsto L^2(D)$ endowed with the norm

$$\|u\|_{\alpha, 2, T}^2 := \left(\sup_{t \in [0, T]} \|u\|_2 \right)^2 + \int_0^T dt \left(\int_0^t d\tau \frac{\|u(\cdot, t) - u(\cdot, \tau)\|_2}{(t - \tau)^{\alpha+1}} \right)^2 < +\infty. \quad (2.1.7)$$

For the first notion of variational solutions to (2.1.2) the admissible test functions are supposed to be independent of time.

Definition 2.1.1. *We say the $H^1(D)$ -valued random field $(u_{I,\varphi}(\cdot, t))_{t \in [0, T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a global variational solution of type I to the Neumann problem (2.1.2) if the following two conditions hold:*

(1) $u_{I,\varphi} \in L^2(0, T; H^1(D)) \cap \mathcal{B}^{\alpha, 2}([0, T]; L^2(D))$ \mathbb{P} -a.s. which means that

$$\int_0^T dt \|u_{I,\varphi}(\cdot, t)\|_{1,2}^2 = \int_0^T dt (\|u_{I,\varphi}(\cdot, t)\|_2^2 + \|\nabla u_{I,\varphi}(\cdot, t)\|_2^2) < +\infty \quad (2.1.8)$$

and

$$\|u_{I,\varphi}\|_{\alpha, 2, T} < +\infty$$

hold \mathbb{P} -a.s.

(2) *The integral relation*

$$\begin{aligned} (v, u_{I,\varphi}(\cdot, t))_2 &= (v, \varphi)_2 - \sum_{i,j=1}^d \int_0^t d\tau (v_{x_i}, k_{i,j}(\cdot, \tau) u_{I,\varphi, x_j}(\cdot, \tau))_2 \\ &\quad + \int_0^t d\tau \int_D dx v(x) g(u_{I,\varphi}(x, \tau)) \\ &\quad + \int_0^t \int_D dx v(x) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau) \end{aligned} \quad (2.1.9)$$

holds \mathbb{P} -a.s. for every $v \in H^1(D)$ and every $t \in [0, T]$, where we have defined the stochastic integral as

$$\int_0^t \int_D dx v(x) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau) := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (v, h(u_{I,\varphi}(\cdot, \tau)) e_i)_2 B_i^{H_i}(d\tau). \quad (2.1.10)$$

Furthermore, let $H^1(D \times (0, T))$ be the isotropic Sobolev space consisting of all real-valued functions $v \in L^2(D \times (0, T))$ that possess distributional derivatives $v_{x_i}, v_\tau \in L^2(D \times (0, T))$, endowed with the inner product

$$\begin{aligned} (v_1, v_2)_{1,2,T} &:= \int_{D \times (0, T)} dx d\tau v_1(x, \tau) v_2(x, \tau) \\ &\quad + \sum_{i=1}^d \int_{D \times (0, T)} dx d\tau v_{1, x_i}(x, \tau) v_{2, x_i}(x, \tau) \\ &\quad + \int_{D \times (0, T)} dx d\tau v_{1, \tau}(x, \tau) v_{2, \tau}(x, \tau) \end{aligned} \quad (2.1.11)$$

and the corresponding norm

$$\|v\|_{1,2,T} = (v, v)_{1,2,T}^{\frac{1}{2}}. \quad (2.1.12)$$

We note that for any function $v \in H^1(D \times (0, T))$ which does not depend on time we have $v \in H^1(D)$, the usual Sobolev space on D whose norm we denote by $\|\cdot\|_{1,2}$. Moreover let $H^1((0, T))$ be the Sobolev space of functions defined on the time interval $(0, T)$. From these definitions it follows immediately that if $v \in H^1(D)$ and $\eta \in H^1((0, T))$ then $v \otimes \eta \in H^1(D \times (0, T))$. This allows us to consider the vector space of all finite linear combinations of such tensor products as an inner product space with respect to (2.1.11), and we write $\mathcal{H}(D \times (0, T))$ for its completion in $H^1(D \times (0, T))$ with respect to (2.1.12). About this Hilbert space we shall prove later on the existence of the trace embedding

$$\mathcal{H}(D \times (0, T)) \rightarrow L^2(D \times \{ \tau = t \}) \quad (2.1.13)$$

valid for each $t \in [0, T]$ and every $d \in \mathbb{N}^+$, which is important to formulate the following notion of variational solution:

Definition 2.1.2. *We say the $H^1(D)$ -valued random field $(u_{II,\varphi}(\cdot, t))_{t \in [0, T]}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is a global variational solution of type II to the Neumann problem (2.1.2) if the first condition of Definition 2.1.1 holds, and if the integral relation*

$$\begin{aligned} (v(\cdot, t), u_{II,\varphi}(\cdot, t))_2 &= (v(\cdot, 0), \varphi)_2 + \int_0^t d\tau (v_\tau(\cdot, \tau), u_{II,\varphi}(\cdot, \tau))_2 \\ &\quad - \sum_{i,j=1}^d \int_0^t d\tau (v_{x_i}(\cdot, \tau), k_{i,j}(\cdot, \tau) u_{II,\varphi, x_j}(\cdot, \tau))_2 \\ &\quad + \int_0^t d\tau (v(\cdot, \tau), g(u_{II,\varphi}(\cdot, \tau)))_2 \\ &\quad + \int_0^t \int_D dx v(x, \tau) h(u_{II,\varphi}(x, \tau)) W^H(x, d\tau) \end{aligned} \quad (2.1.14)$$

holds \mathbb{P} -a.s. for every $v \in \mathcal{H}(D \times (0, T))$ and every $t \in [0, T]$, where $x \mapsto v(x, t) \in L^2(D)$ stands for the trace of v in the sense of (2.1.13) and where the stochastic integral is defined as

$$\int_0^t \int_D dx v(x, \tau) h(u_{II,\varphi}(x, \tau)) W^H(x, d\tau) := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (v(\cdot, \tau), h(u_{II,\varphi}(\cdot, \tau)) e_i)_2 B_i^{H_i}(d\tau). \quad (2.1.15)$$

In order to make sense out of each term in (2.1.14) and prove the existence of such a solution we will still need the following geometric hypothesis on the domain D :

- (D) There exists a sequence $(D_N)_{N \in \mathbb{N}^+}$ of open sets such that for every N we have $D_N \subset D_{N+1} \subset D$ and the compactness of the embedding

$$H^1(D_N) \hookrightarrow L^2(D_N). \quad (2.1.16)$$

Moreover, let $\mathcal{B} \subset H^1(D)$ be any bounded set. Then for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}^+$

such that

$$\int_{D \setminus D_{N_\epsilon}} dx |u(x)|^2 < \epsilon \tag{2.1.17}$$

for every $u \in \mathcal{B}$.

Remark. It is essential to observe here that any domain satisfying (D) is necessarily of finite Lebesgue measure as a consequence of the theory of tessellations developed in Chapter 6 of [1], and more importantly that (D) implies the compactness of the embedding $H^1(D) \hookrightarrow L^2(D)$ as it amounts to a geometric condition implying that D becomes rapidly narrow at infinity, as the two examples discussed at the end of this section will show. It is also easily verified that each term in (2.1.14) is well defined and finite \mathbb{P} -a.s. as a consequence of all of the above hypotheses. In particular, we may conclude that (2.1.15) is an infinite sum of one-dimensional, pathwise, generalized Stieltjes integrals which defines a real-valued random variable as a consequence of Hypothesis (C) and of the fact that h is Lipschitz continuous. We shall dwell a bit more on this and on related properties of the stochastic integral in subsection 2.2.1. We note that Problem (2.1.2) was thoroughly analyzed in [60] in case D is a bounded domain satisfying the cone condition and with a single Hurst parameter in (2.1.1).

We organize this chapter in the following way: In Section 2.2 we state our main result concerning the existence of a global variational solution along with its uniqueness when h is an affine function. In subsection 2.2.1 we prove some crucial estimates for the stochastic integral by means of a necessary modification of the theory set forth in [58] due to the fact that we are dealing here with an infinite sequence of Hurst parameters satisfying (2.1.4). The method rests in an essential way on a particular case of an inequality proved by Garsia, Rodemich and Rumsey in [31], and on Minkowski's integral inequality (see, e.g., Appendix A in [71]). In subsection 2.2.2 we give a sketch of the proof for the existence of a variational solution of type I. The proof follows closely the first part in [60], the crucial ingredient being the compactness of the embedding $H^1(D) \hookrightarrow L^2(D)$. In fact, the compactness of this embedding remains true for unbounded domains which satisfy condition (D). In subsection 2.2.3 we extend the class of test functions to functions depending on time, first to finite linear combinations of tensor products $v \otimes \eta$ of functions $v \in H^1(D)$ and $\eta \in H^1((0, T))$, and then to all test functions in $\mathcal{H}(D \times (0, T))$ by a suitable density argument and by invoking the trace embedding of the form (2.1.13) which we prove by elementary means. In subsection 2.2.4 we illustrate our results with two examples of unbounded domains which satisfy condition (D). In Section 2.3, we consider functions g which satisfy a local Lipschitz condition. Since the blowup in finite time of the solution to (2.1.2) can not be excluded in this case, we define the notion of a local variational solution to (2.1.2) and show, for h affine, its existence and uniqueness up to an eventual blowup time by applying a truncation method. The proof contains an explicit construction of the

blowup time.

2.2 Existence and uniqueness of global variational solutions

The main result of this section is gathered in the next theorem.

Theorem 2.2.1. *Assume that the hypotheses (K), (L), (I), (C) and (D) hold. Then*

- (A) *The Neumann problem (2.1.2) possesses a variational solution $u_{I,\varphi}$ of type I. Moreover, if h is an affine function, $u_{I,\varphi}$ is the only variational solution of type I to (2.1.2).*
- (B) *Any variational solution of type I of (2.1.2) is a variational solution of type II to (2.1.2).*

In what follows, we write c for all the irrelevant constants that occur in the various estimates unless we specify these constants otherwise.

2.2.1 Some remarks on generalized Stieltjes integrals

The following considerations constitute a necessary modification of the theory developed in [58], needed to take into account the fact that we are dealing here with an infinite sequence of Hurst parameters. For every $i \in \mathbb{N}^+$, let us introduce the random variable

$$\Lambda_\alpha^{H_i} = \frac{\sin(\pi\alpha)}{\pi} \sup_{0 \leq t < t^* \leq T} \left| \frac{B^{H_i}(t) - B^{H_i}(t^*)}{(t^* - t)^{1-\alpha}} + (1 - \alpha) \int_t^{t^*} d\tau \frac{B^{H_i}(t) - B^{H_i}(\tau)}{(\tau - t)^{2-\alpha}} \right| \quad (2.2.1)$$

where α is the fixed parameter chosen in Section 2.1, for which we have the inequality

$$\Lambda_\alpha^{H_i} \leq \sup_{0 \leq t < t^* \leq T} \left(\frac{|B_i^H(t^*) - B_i^H(t)|}{(t^* - t)^{1-\alpha}} + (1 - \alpha) \int_t^{t^*} d\tau \frac{|B_i^H(\tau) - B_i^H(t)|}{(\tau - t)^{2-\alpha}} \right). \quad (2.2.2)$$

For reasons that will appear soon, we first need to get appropriate estimates for the moments of (2.2.1) that are uniform in $i \in \mathbb{N}^+$, and, to achieve this, suitable upper bounds for the increments of $B^{H_i}(t)$.

Proposition 2.2.1. *Let $(B^{H_i})_{i \in \mathbb{N}^+}$ be the one-dimensional, independent fractional Brownian motions introduced in Section 2.1, where the sequence $(H_i)_{i \in \mathbb{N}^+}$ satisfies (2.1.4). Then we have*

$$\sup_{i \in \mathbb{N}^+} \mathbb{E} |\Lambda_\alpha^{H_i}|^p < +\infty \quad (2.2.3)$$

for every $p \in [1, +\infty)$.

The proof of this proposition rests on the following particular case of the Garsia-Rodemich-Rumsey inequality, obtained by rescaling the basic estimate stated in Lemma 1.1 of [31] to establish its validity on $[0, T]$ rather than just on $[0, 1]$, and by applying the particular choice of the functions involved made at the very beginning of [35]:

Lemma 2.2.1. *Let $f : [0, T] \mapsto \mathbb{R}$ be continuous, let $q \in [1, +\infty)$ and $\beta \in (\frac{1}{q}, +\infty)$. If the integral between parentheses in (2.2.4) is finite, then the inequality*

$$|f(t^*) - f(t)| \leq c_{\beta,q,T} \left(\int_0^T \int_0^T d\sigma d\tau \frac{|f(\sigma) - f(\tau)|^q}{|\sigma - \tau|^{\beta q + 1}} \right)^{\frac{1}{q}} |t^* - t|^{\beta - \frac{1}{q}} \quad (2.2.4)$$

holds for all $t, t^* \in [0, T]$, where

$$c_{\beta,q,T} = c_T \left(\beta + \frac{1}{q} \right) \left(\beta - \frac{1}{q} \right)^{-1} \quad (2.2.5)$$

and where $c_T > 0$ depends only on T .

The upper bounds for the increments of B^{H_i} we need turn out to be provided by (2.2.4) for particular values of the parameters. In all that follows we write ϵ for a fixed auxiliary quantity which we can eventually express in terms of the constants α, γ and \underline{H} :

Lemma 2.2.2. *Let us fix $\epsilon \in (0, \frac{1}{\gamma+1})$ where $\gamma \in (0, 1]$ is the constant of Hypothesis (L). Then for every $i \in \mathbb{N}^+$ there exists a positive, \mathbb{P} -a.s. finite random variable $\Theta_{i,\epsilon,T}$ such that the inequality*

$$|B^{H_i}(t^*) - B^{H_i}(t)| \leq c_{\gamma,\epsilon,T} \Theta_{i,\epsilon,T} |t^* - t|^{\underline{H} - \epsilon} \quad (2.2.6)$$

holds \mathbb{P} -a.s. for all $t, t^* \in [0, T]$, where $c_{\gamma,\epsilon,T} > 0$ depends only on γ, ϵ and T . Moreover we have

$$\sup_{i \in \mathbb{N}^+} \mathbb{E} |\Theta_{i,\epsilon,T}|^p < +\infty \quad (2.2.7)$$

for every $p \in [1, +\infty)$.

Proof. Let us first define the random variable

$$\tilde{\Theta}_{i,\epsilon,T} := \int_0^T \int_0^T d\sigma d\tau \frac{|B_i^H(\sigma) - B_i^H(\tau)|^{\frac{2}{\epsilon}}}{|\sigma - \tau|^{\frac{2H_i}{\epsilon}}}$$

and prove that

$$\sup_{i \in \mathbb{N}^+} \mathbb{E} |\tilde{\Theta}_{i,\epsilon,T}|^p < +\infty \quad (2.2.8)$$

for every $p \in [1, +\infty)$. For any $r \in [\frac{2}{\epsilon}, +\infty)$ we have $p := \frac{r\epsilon}{2} \in [1, +\infty)$ so that on the one

hand we obtain

$$\begin{aligned} \mathbb{E}|\tilde{\Theta}_{i,\epsilon,T}|^p &= \mathbb{E} \left(\int_0^T \int_0^T d\sigma d\tau \frac{|B_i^H(\sigma) - B_i^H(\tau)|^{\frac{2}{\epsilon}}}{|\sigma - \tau|^{\frac{2H_i}{\epsilon}}} \right)^{\frac{r\epsilon}{2}} \\ &\leq \left(\int_0^T \int_0^T d\sigma d\tau \left(\mathbb{E} \frac{|B_i^H(\sigma) - B_i^H(\tau)|^r}{|\sigma - \tau|^{rH_i}} \right)^{\frac{2}{r\epsilon}} \right)^{\frac{r\epsilon}{2}} \end{aligned} \quad (2.2.9)$$

from Minkowski's integral inequality. On the other hand, from the basic properties of the one-dimensional fractional Brownian motion we have

$$\mathbb{E}|B_i^H(\sigma) - B_i^H(\tau)|^r \leq c_r |\sigma - \tau|^{rH_i}$$

for some constant c_r depending only on r , so that the substitution of this expression into (2.2.9) annihilates the dependence in H_i , thus leading to

$$\mathbb{E}|\tilde{\Theta}_{i,\epsilon,T}|^p \leq c_r T^{2p} < +\infty$$

uniformly in i , which is (2.2.8). Let us now choose $q = \frac{2}{\epsilon}$ and $\beta_i = H_i - \frac{\epsilon}{2}$ for every $i \in \mathbb{N}^+$ in Lemma 2.2.1. Then clearly $q \in [1, +\infty)$ and $\beta_i \in (\frac{\epsilon}{2}, +\infty)$ by virtue of (2.1.4), and since (2.2.8) implies in particular that $\tilde{\Theta}_{i,\epsilon,T} < +\infty$ \mathbb{P} -a.s. we may thus apply (2.2.4) to obtain

$$\begin{aligned} &|B_i^H(t^*) - B_i^H(t)| \\ &\leq c_{\beta_i,q,T} \left(\int_0^T \int_0^T d\sigma d\tau \frac{|B_i^H(\sigma) - B_i^H(\tau)|^{\frac{2}{\epsilon}}}{|\sigma - \tau|^{\frac{2H_i}{\epsilon}}} \right)^{\frac{\epsilon}{2}} |t^* - t|^{H_i - \epsilon} \\ &\leq \tilde{c}_{\beta_i,q,T} \left(\int_0^T \int_0^T d\sigma d\tau \frac{|B_i^H(\sigma) - B_i^H(\tau)|^{\frac{2}{\epsilon}}}{|\sigma - \tau|^{\frac{2H_i}{\epsilon}}} \right)^{\frac{\epsilon}{2}} |t^* - t|^{\underline{H} - \epsilon} \\ &= \tilde{c}_{\beta_i,q,T} \tilde{\Theta}_{i,\epsilon,T}^{\frac{\epsilon}{2}} |t^* - t|^{\underline{H} - \epsilon} \end{aligned}$$

\mathbb{P} -a.s. for all $t, t^* \in [0, T]$, where $\tilde{c}_{\beta_i,q,T}$ differs from $c_{\beta_i,q,T}$ by a trivial factor depending only on T and \underline{H} . In order to get (2.2.6) it is thus sufficient to take

$$\Theta_{i,\epsilon,T} := \tilde{\Theta}_{i,\epsilon,T}^{\frac{\epsilon}{2}} \quad (2.2.10)$$

and prove that

$$\sup_{i \in \mathbb{N}^+} \tilde{c}_{\beta_i,q,T} < +\infty \quad (2.2.11)$$

along with (2.2.7). Ignoring the trivial dependence in T and \underline{H} in $\tilde{c}_{\beta_i,q,T}$ we first infer from (2.2.5) that the simple estimate

$$\left(\beta_i + \frac{\epsilon}{2} \right) \left(\beta_i - \frac{\epsilon}{2} \right)^{-1} = H_i (H_i - \epsilon)^{-1} \leq \left(\frac{1}{\gamma + 1} - \epsilon \right)^{-1} < +\infty$$

holds uniformly in i thanks to (2.1.4) and our choice of ϵ , which gives (2.2.11). Finally,

let us partition the probability space as

$$\Omega = \left\{ \omega \in \Omega : \tilde{\Theta}_{i,\epsilon,T}(\omega) \leq 1 \right\} \cup \left\{ \omega \in \Omega : \tilde{\Theta}_{i,\epsilon,T}(\omega) > 1 \right\}$$

and split the expectation functional accordingly. From (2.2.10) it is then plain that

$$\sup_{i \in \mathbb{N}^+} \mathbb{E} |\Theta_{i,\epsilon,T}|^p = \sup_{i \in \mathbb{N}^+} \mathbb{E} |\tilde{\Theta}_{i,\epsilon,T}|^{\frac{p\epsilon}{2}} \leq 1 + \sup_{i \in \mathbb{N}^+} \mathbb{E} |\tilde{\Theta}_{i,\epsilon,T}|^p < +\infty$$

according to (2.2.8) since $\frac{\epsilon}{2} \leq 1$, which is the desired result. \square

The proof of (2.2.3) then follows from (2.2.7) and from the substitution of (2.2.6) into (2.2.2) provided we impose a further restriction on the parameter ϵ to make the singularities integrable:

Proof of Proposition 2.2.1. We first notice that

$$\underline{H} - 1 + \alpha < \frac{1}{\gamma + 1}$$

as a consequence of the conditions we imposed on these parameters in Section 2.1. Then, fixing $\epsilon \in (0, \underline{H} - 1 + \alpha)$ we may substitute (2.2.6) into (2.2.2) to obtain

$$\Lambda_\alpha^{H_i} \leq c_{\gamma,\epsilon,T} T^{\underline{H}-\epsilon-1+\alpha} \frac{\underline{H}-\epsilon+\alpha}{\underline{H}-\epsilon-1+\alpha} \Theta_{i,\epsilon,T} < +\infty$$

\mathbb{P} -a.s. after an explicit integration, where the prefactor is uniform in $i \in \mathbb{N}^+$. Therefore, (2.2.3) indeed follows from (2.2.7). \square

Remark. To follow up on our remark preceding the statement of Lemma 2.2.2, we see a posteriori that we could have chosen for instance $\epsilon = \frac{1}{2}(\underline{H} - 1 + \alpha)$ throughout.

We are now ready for the following:

Lemma 2.2.3. *Let us consider the stochastic integral as defined in (2.1.15). Then, there exists a \mathbb{P} -a.s. finite, positive random variable r_α^H such that the estimate*

$$\begin{aligned} & \left| \int_t^{t^*} \int_D dx v(x, \tau) h(u_{II,\varphi}(x, \tau)) W^H(x, d\tau) \right| \\ & \leq r_\alpha^H (1 + \|u_{II,\varphi}\|_{\alpha,2,T}) \|v\|_{\alpha,2,T} |t^* - t|^{\frac{1}{2}} \end{aligned} \tag{2.2.12}$$

holds \mathbb{P} -a.s. for every $v \in \mathcal{B}^{\alpha,2}([0, T]; L^2(D))$ and all $t, t^* \in [0, T]$.

Proof. We begin by estimating each integral on the right-hand side of (2.1.15). For every $i \in \mathbb{N}^+$ let us set

$$f_i(\tau) := (v(\cdot, \tau), h(u_{II,\varphi}(\cdot, \tau) e_i))_2.$$

Then, from the basic estimate (4.11) in Proposition 4.1 of [58] regarding generalized Stieltjes integrals we infer that the inequality

$$\left| \int_{t^*}^t f_i(\tau) B^{H_i}(d\tau) \right| \leq \Lambda_\alpha^{H_i} \int_{t^*}^t d\tau \left(\frac{|f_i(\tau)|}{(\tau - t^*)^\alpha} + \alpha \int_{t^*}^\tau d\rho \frac{|f_i(\tau) - f_i(\rho)|}{(\tau - \rho)^{\alpha+1}} \right) \quad (2.2.13)$$

holds \mathbb{P} -a.s., where $\Lambda_\alpha^{H_i}$ is given by (2.2.1). Furthermore, from Schwarz inequality and the basic hypotheses of Section 2.1 we have

$$\begin{aligned} |f_i(\tau)| &\leq c \|e_i\|_\infty (1 + \|u_{II,\varphi}(\cdot, \tau)\|_2) \|v(\cdot, \tau)\|_2 \\ &\leq c \|e_i\|_\infty (1 + \|u_{II,\varphi}\|_{\alpha,2,T}) \|v\|_{\alpha,2,T} \end{aligned} \quad (2.2.14)$$

where the second inequality follows from (2.1.7), and similarly

$$\begin{aligned} |f_i(\tau) - f_i(\rho)| &\leq c \|e_i\|_\infty \|u_{II,\varphi}(\cdot, \tau) - u_{II,\varphi}(\cdot, \rho)\|_2 \|v\|_{\alpha,2,T} \\ &\quad + c \|e_i\|_\infty (1 + \|u_{II,\varphi}\|_{\alpha,2,T}) \|v(\cdot, \tau) - v(\cdot, \rho)\|_2 \end{aligned} \quad (2.2.15)$$

for all $\rho, \tau \in [0, T]$, for some constant $c > 0$. Consequently, on the one hand the substitution of (2.2.14) into the first integral on the right-hand side of (2.2.13) gives

$$\int_{t^*}^t d\tau \frac{|f_i(\tau)|}{(\tau - t^*)^\alpha} \leq c \|e_i\|_\infty (1 + \|u_{II,\varphi}\|_{\alpha,2,T}) \|v\|_{\alpha,2,T} |t^* - t|^{1-\alpha} \quad (2.2.16)$$

by direct integration. On the other hand, the substitution of (2.2.15) into the second integral on the right-hand side of (2.2.13), Schwarz inequality relative to the measure $d\tau$ and (2.1.7) lead to

$$\int_{t^*}^t d\tau \int_{t^*}^\tau d\rho \frac{|f_i(\tau) - f_i(\rho)|}{(\tau - \rho)^{\alpha+1}} \leq c \|e_i\|_\infty (1 + \|u_{II,\varphi}\|_{\alpha,2,T}) \|v\|_{\alpha,2,T} |t^* - t|^{\frac{1}{2}} \quad (2.2.17)$$

Therefore, with (2.2.16) and (2.2.17) into (2.2.13) we obtain

$$\left| \int_{t^*}^t f_i(\tau) B^{H_i}(d\tau) \right| \leq c \|e_i\|_\infty \Lambda_\alpha^{H_i} (1 + \|u_{II,\varphi}\|_{\alpha,2,T}) \|v\|_{\alpha,2,T} |t^* - t|^{\frac{1}{2}} \quad (2.2.18)$$

for every $i \in \mathbb{N}^+$ since $1 - \alpha > \frac{1}{2}$ according to our original choice of this parameter. In order to prove (2.2.12) it is therefore sufficient to show that

$$\sum_{i=1}^{+\infty} \sqrt{\lambda_i} \|e_i\|_\infty \Lambda_\alpha^{H_i} < +\infty \quad (2.2.19)$$

\mathbb{P} -a.s., for then

$$\begin{aligned} & \left| \int_{t^*}^t \int_D dx v(x, \tau) h(u_{II, \varphi}(\cdot, \tau)) W^H(x, d\tau) \right| \\ & \leq \left(c \sum_{i=1}^{+\infty} \sqrt{\lambda_i} \|e_i\|_{\infty} \Lambda_{\alpha}^{H_i} \right) (1 + \|u_{II, \varphi}\|_{\alpha, 2, T}) \|v\|_{\alpha, 2, T} |t^* - t|^{\frac{1}{2}} \end{aligned}$$

according to (2.1.15) and (2.2.18), with the obvious choice for r_{α}^H . But (2.2.19) follows from Relation (2.2.3) of Proposition 2.2.1 with $p = 1$ since spectral condition (2.1.6) holds. \square

Remark. By replacing the test function $v \in \mathcal{B}^{\alpha, 2}([0, T]; L^2(D))$ in Lemma 2.2.3 by a time independent test function $v \in H^1(D)$, we get, by a similar proof as above, the following estimate for the stochastic integral in the variational solution of type I (see (2.1.10)):

$$\begin{aligned} & \left| \int_t^{t^*} \int_D dx v(x) h(u_{I, \varphi}(x, \tau)) W^H(x, d\tau) \right| \\ & \leq r_{\alpha}^H (1 + \|u_{I, \varphi}\|_{\alpha, 2, T}) \|v\|_2 |t^* - t|^{\frac{1}{2}} \end{aligned} \tag{2.2.20}$$

\mathbb{P} -a.s. for every $v \in H^1(D)$ and all $t, t^* \in [0, T]$.

2.2.2 Sketch of the proof of the existence and uniqueness of $u_{I, \varphi}$ (Theorem 2.2.1 (A))

The proof of statement (A) in Theorem 2.2.1 is a straightforward adaptation of the arguments in the first part of [60]. In particular, the compactness of the embedding $H^1(D) \hookrightarrow L^2(D)$, the crucial tool in [60], still holds for unbounded domains D that satisfy condition (D) (see Chapter 6 of [1]). For this reason we mention here just the main steps of the proof.

Let $(w_n)_{n \in \mathbb{N}^+}$ be an orthonormal basis of $L^2(D)$ such that $(c_n w_n)_{n \in \mathbb{N}^+}$ be an orthonormal basis of $H^1(D)$ for some suitably chosen coefficients c_n . The existence of such a basis follows from an extension of the construction of orthonormal bases of $L^2(D)$ and $H^1(D)$ investigated by V. Mikhaïlov in [54], Ch. IV, p.172 - 178. For each $N \in \mathbb{N}^+$, let V_N be the finite-dimensional subspace of $L^2(D)$ generated by w_1, \dots, w_N , and let φ_N the orthogonal projection of the initial condition φ onto V_N . Let $\mathcal{B}^{\alpha, \infty}([0, T]; V_N)$ be the Banach space of all Lebesgue-measurable mappings $u : [0, T] \mapsto V_N$ endowed with the norm $\|\cdot\|_{\alpha, \beta}$ defined by

$$\|u\|_{\alpha, \beta} = \sup_{t \in [0, T]} \exp[-\beta t] \left(\|u(\cdot, t)\|_2 + \int_0^t d\tau \frac{\|u(\cdot, t) - u(\cdot, \tau)\|_2}{(t - \tau)^{\alpha+1}} \right)$$

for every $\beta \in \mathbb{R}^+$.

The idea of the proof is to construct a variational solution $u_{I, \varphi}$ of (2.1.2) by *Faedo-Galerkin*

approximations in the spaces V_N for all $N \in \mathbb{N}^+$. This is possible due to the following result which is an immediate consequence of the definition of the variational solution of type I: Let $u_{I,\varphi} \in L^2(0, T; H^1(D)) \cap \mathcal{B}^{\alpha,2}([0, T]; L^2(D))$ and assume that hypotheses (C), (L), (I), (K) and (D) hold. Then $u_{I,\varphi}$ is a variational solution of type I to (2.1.2) if, and only if, the relation

$$\begin{aligned} (w_n, u_{I,\varphi})_2 &= (w_n, \varphi)_2 - \sum_{i,j=1}^d \int_0^t d\tau (w_{n,x_i}, k_{i,j}(\cdot, \tau) u_{I,\varphi,x_j}(\cdot, \tau))_2 \\ &\quad + \int_0^t d\tau (w_n, g(u_{I,\varphi}(\cdot, \tau)))_2 \\ &\quad + \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (w_n, h(u_{I,\varphi}(\cdot, \tau)) e_i)_2 B_i^{H_i}(d\tau) \end{aligned} \quad (2.2.21)$$

holds for every $n \in \mathbb{N}^+$ and every $t \in [0, T]$.

The *Faedo-Galerkin approximating scheme* associated with (2.1.2) reads as follows:

$$\begin{aligned} (w_n, u_N)_2 &= (w_n, \varphi)_2 - \sum_{i,j=1}^d \int_0^t d\tau (w_{n,x_i}, k_{i,j}(\cdot, \tau) u_{N,x_j}(\cdot, \tau))_2 \\ &\quad + \int_0^t d\tau (w_n, g(u_N(\cdot, \tau)))_2 \\ &\quad + \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t (w_n, h(u_N(\cdot, \tau)) e_i)_2 B_i^{H_i}(d\tau) \end{aligned} \quad (2.2.22)$$

\mathbb{P} -a.s. for every $n \in \{1, \dots, N\}$ and every $t \in [0, T]$.

The existence of a variational solution of type I of (2.1.2) rests upon the existence and uniqueness of a solution $u_N \in \mathcal{B}^{\alpha,\infty}([0, T]; V_N)$ for all $N \in \mathbb{N}^+$ to (2.2.22) and on the convergence of the sequence $(u_N(\cdot, t))_{t \in [0, T]}_{N \in \mathbb{N}^+}$. We define mappings Φ_N by

$$\Phi_N(u)(\cdot, t) := \sum_{n=1}^N (w_n, \Phi_N(u)(\cdot, t))_2 w_n$$

for $u \in \mathcal{B}^{\alpha,\infty}([0, T]; V_N)$ where $(w_n, \Phi_N(u)(\cdot, t))_2$ is defined to be the term on the right side of (2.2.22). To prove the existence and uniqueness of u_N , we construct a fixed point, again denoted by u_N , of Φ_N , for every $N \in \mathbb{N}^+$, in the closed ball B_N defined by

$$B_N := \{u \in \mathcal{B}^{\alpha,\infty}([0, T]; V_N) : \|u\|_{\alpha, \beta_0} \leq 2(1 + N\|\varphi_N\|_2)\}, \quad (2.2.23)$$

where $\beta_0 \in [1, +\infty)$ is a sufficiently large random variable. An estimation of $\|\Phi_N(u)\|_{\alpha, \beta_0}$ leads to the upper bound

$$\|\Phi_N(u)\|_{\alpha, \beta_0} \leq N\|\varphi_N\|_2 + \frac{1}{2}(1 + \|u\|_{\alpha, \beta_0}).$$

Therefore $\Phi_N(B_N) \subset B_N$ \mathbb{P} -a.s., i.e. B_N is invariant under Φ_N . Moreover, for $u, u^* \in B_N$ one shows

$$\begin{aligned} & \|\Phi_N(u) - \Phi_N(u^*)\|_{\alpha, \beta} \\ & \leq c(1 + r_\alpha^H)^{\gamma+1} \exp[\beta_0 \gamma T] \beta^{2\alpha-1} (1 + N\|\Phi_N\|_2)^\gamma \|u - u^*\|_{\alpha, \beta}. \end{aligned}$$

Consequently there exists a random variable $\beta_1 \in [1, +\infty)$, sufficiently large and depending on N , such that

$$\|\Phi_N(u) - \Phi_N(u^*)\|_{\alpha, \beta_1} \leq \frac{1}{2} \|u - u^*\|_{\alpha, \beta_1} \quad (2.2.24)$$

holds \mathbb{P} -a.s. since $\alpha < \frac{1}{2}$. We then define a sequence $(u_{m,N})_{m \in \mathbb{N}^+}$ by $u_{m,N} := \Phi_N(u_{m-1,N})$ for every $m \in \mathbb{N}^+$ with $u_{0,N}$ fixed in B_N , and show that $(u_{m,N})_{m \in \mathbb{N}^+}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_{\alpha, \beta_1}$. We denote by u_N its limit and prove that u_N is the only fixed point of Φ_N in $\mathcal{B}^{\alpha, \infty}([0, T]; V_N)$, using the above invariance property and the fact that B_N is closed. We argue exactly as in the last part of the proof of Proposition 2 in [60].

Next we turn to the convergence of the Faedo-Galerkin approximation

$$((u_N(\cdot, t))_{t \in [0, T]})_{N \in \mathbb{N}^+}.$$

The convergence relies upon a priori estimate for u_N that is uniform in N . We prove that the sequence $(u_N)_{N \in \mathbb{N}^+}$ is bounded in $L^2(0, T; H^1(D)) \cap \mathcal{B}^{\alpha, 2}([0, T]; L^2(D))$, i.e.

$$\sup_{N \in \mathbb{N}^+} \|u_N\|_{1, \alpha, 2} < +\infty \quad (2.2.25)$$

\mathbb{P} -a.s., where

$$\|u_N\|_{1, \alpha, 2}^2 := \int_0^T dt \|u_N(\cdot, t)\|_{1, 2}^2 + \|u_N\|_{\alpha, 2}^2. \quad (2.2.26)$$

We remark that (2.2.25) is equivalent to

$$\sup_{N \in \mathbb{N}^+} \left(\sup_{t \in [0, T]} \|u_N(\cdot, t)\|_2^2 + \int_0^T dt \|\nabla u_N(\cdot, t)\|_2^2 \right) < +\infty \quad (2.2.27)$$

and

$$\sup_{N \in \mathbb{N}^+} \int_0^T dt \left(\int_0^t d\tau \frac{\|u_N(\cdot, t) - u_N(\cdot, \tau)\|_2}{(t - \tau)^{\alpha+1}} \right)^2 < +\infty \quad (2.2.28)$$

\mathbb{P} -a.s. Exactly the same proof as for proposition 3 in [60] may be carried out since all the estimates in Lemma 6-12 in [60] are still true in our case.

Consequently there exists $u_{I, \varphi}$ such that $u_N \rightharpoonup u_{I, \varphi}$ weakly in $L^2(0, T; H^1(D))$ as $N \rightarrow +\infty$ \mathbb{P} -a.s., by passing to a suitable subsequence if necessary. One shows that

$$u_{I, \varphi} \in L^2(0, T; H^1(D)) \cap \mathcal{B}^{\alpha, 2}([0, T], L^2(D)).$$

It is therefore natural to think that the random field $u_{I,\varphi}$ provides a solution to (2.1.2). This is indeed the case by exactly the same proof as for Proposition 4 in [60]. An essential ingredient for the estimates is the compactness of the embedding

$$L^2(0, T; V_N) \cap \mathcal{B}^{\alpha,2}([0, T]; V_N) \hookrightarrow L^2(0, T; L^2(D)).$$

In order to prove that $u_{I,\varphi}$ satisfies condition (2) of Definition 2.2.1, we need to show that (2.2.21) holds. This is done by showing the weak convergence in $L^2(0, T; \mathbb{R})$ of each term in (2.2.22) considered as a function of t to the corresponding term in (2.2.21) when $N \rightarrow +\infty$.

Finally, it turns out that there is no variational solution of type I to (2.1.2) different from $u_{I,\varphi}$ when h is affine. In fact, let u_I and u_I^* be variational solutions of type I to (2.1.2) and define $Z_I = u_I - u_I^*$. The idea is to show that there exists a random variable $r \in \mathbb{R}^+$ such that the estimate

$$\|Z_I(\cdot, t)\|_2^2 \leq r \int_0^t d\tau \frac{\sup_{\sigma \in [0, \tau]} \|Z_I(\cdot, \sigma)\|_2^2}{\tau^\alpha}$$

holds \mathbb{P} -a.s. for every $t \in [0, T]$, and hence that $Z_I = 0$ \mathbb{P} -a.s. by Gronwall's inequality since $\tau \mapsto \tau^{-\alpha} \sup_{\sigma \in [0, \tau]} \|Z_I(\cdot, \sigma)\|_2^2$ is integrable on $(0, T)$. It should be pointed out here that the above inequality requires very fine estimates of the form

$$\|h(u(\cdot, t)) - h(u^*(\cdot, t)) - h(u(\cdot, \tau)) + h(u^*(\cdot, \tau))\|_2$$

where $u, u^* \in \mathcal{B}^{\alpha,2}([0, T]; L^2(D))$, which are not available unless h is an affine function or u, u^* take values in a finite-dimensional space.

We refer the reader to [60] for more details and for the complete proofs of the above results that are still true in our case for domains D that satisfy condition (D).

2.2.3 Proof of existence and uniqueness of $u_{II,\varphi}$ (Theorem 2.2.1 (B))

We show that $(u_{I,\varphi}(\cdot, t))_{t \in [0, T]}$ is itself of type II. We obtain the statement (B) by proving continuity property of the stochastic integral with respect to $\mathcal{B}^{\alpha,2}([0, T]; L^2(D))$ and combining this with a suitable approximation of the test functions that appear in Definition 2.1.2. We begin with the following intermediary result:

Lemma 2.2.4. *Assume that Hypotheses (K), (L), (I), (C) and (D) hold and let $u_{I,\varphi}$ be the random field of Theorem 2.2.1 (A). Then for all finite linear combinations \hat{v} of functions*

of the form $v \otimes \eta$ where $v \in H^1(D)$ and $\eta \in H^1((0, T))$ we have

$$\begin{aligned}
 (\hat{v}(\cdot, t), u_{I,\varphi}(\cdot, t))_2 &= (\hat{v}(\cdot, 0), \varphi)_2 \\
 &\quad - \sum_{i,j=1}^d \int_0^t d\tau (\hat{v}_{x_i}(\cdot, \tau), k_{i,j}(\cdot, \tau) u_{I,\varphi, x_j}(\cdot, \tau))_2 \\
 &\quad + \int_0^t d\tau (\hat{v}, g(u_{I,\varphi}(\cdot, \tau)))_2 \\
 &\quad + \int_0^t \int_D dx \hat{v}(x, \tau) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau)
 \end{aligned} \tag{2.2.29}$$

P – a.s. for every $t \in [0, T]$.

Proof. By linearity it is sufficient to prove that (2.2.29) holds for $\hat{v} = v \otimes \eta$.

The easiest way out is to start with the first integral on the right-hand side. Using (2.1.9) we obtain

$$\begin{aligned}
 &\int_0^t d\tau (\hat{v}_\tau(\cdot, \tau), u_{I,\varphi}(\cdot, \tau))_2 \\
 &= \int_0^t d\tau \eta'(\tau) (v, u_{I,\varphi}(\cdot, \tau))_2 \\
 &= (\hat{v}(\cdot, t), \varphi)_2 - (\hat{v}(\cdot, 0), \varphi)_2 \\
 &\quad - \sum_{i,j=1}^d \int_0^t d\tau \eta'(\tau) \int_0^\tau d\sigma (v_{x_i}, k_{i,j}(\cdot, \sigma) u_{I,\varphi, x_j}(\cdot, \sigma))_2 \\
 &\quad + \int_0^t d\tau \eta'(\tau) \int_0^\tau d\sigma (v, g(u_{I,\varphi}(\cdot, \sigma)))_2 \\
 &\quad + \int_0^t d\tau \eta'(\tau) \int_0^\tau \int_D dx v(x) h(u_{I,\varphi}(x, \sigma)) W^H(x, d\sigma).
 \end{aligned} \tag{2.2.30}$$

We then integrate by parts the last three terms with respect to τ and reintroduce \hat{v} whenever possible to obtain respectively

$$\begin{aligned}
 &\sum_{i,j=1}^d \int_0^t d\tau \eta'(\tau) \int_0^\tau d\sigma (v_{x_i}, k_{i,j}(\cdot, \sigma) u_{I,\varphi, x_j}(\cdot, \sigma))_2 \\
 &= \eta(t) \sum_{i,j=1}^d \int_0^t d\tau d\sigma (v_{x_i}, k_{i,j}(\cdot, \tau) u_{I,\varphi, x_j}(\cdot, \tau))_2 \\
 &\quad - \sum_{i,j=1}^d \int_0^t d\tau d\sigma (\hat{v}_{x_i}, k_{i,j}(\cdot, \tau) u_{I,\varphi, x_j}(\cdot, \tau))_2
 \end{aligned} \tag{2.2.31}$$

and

$$\begin{aligned}
 &\int_0^t d\tau \eta'(\tau) \int_0^\tau d\sigma (v, g(u_{I,\varphi}(\cdot, \sigma)))_2 \\
 &= \eta(t) \int_0^t d\tau (v, g(u_{I,\varphi}(\cdot, \tau)))_2 - \int_0^t d\tau (\hat{v}(\cdot, \tau), g(u_{I,\varphi}(\cdot, \tau)))_2
 \end{aligned} \tag{2.2.32}$$

for the deterministic integrals, while we get

$$\begin{aligned}
& \int_0^t d\tau \eta'(\tau) \int_0^\tau \int_D dx v(x) h(u_{I,\varphi}(x, \sigma)) W^H(x, d\sigma) \\
&= \eta(t) \int_0^t \int_D dx v(x) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau) \\
&\quad - \int_0^t \int_D dx \hat{v}(x, \tau) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau)
\end{aligned} \tag{2.2.33}$$

for the stochastic integral. The substitution of (2.2.31)-(2.2.33) into (2.2.30) then leads to the desired result, after having lumped together the three terms containing the factor $\eta(t)$ and used there (2.1.9) once again. \square

We infer from the preceding considerations that for any $v \in \mathcal{H}(D \times (0, T))$ there exist functions \hat{v}_n satisfying (2.2.29) for every $n \in \mathbb{N}^+$ such that

$$\|v - \hat{v}_n\|_{1,2,T} \rightarrow 0 \tag{2.2.34}$$

as $n \rightarrow +\infty$ where $\|\cdot\|_{1,2,T}$ is given by (2.1.12). The first consequence of this is that we can already approximate the three deterministic integrals in (2.1.14), deferring to separate propositions the analysis of the remaining terms:

Proposition 2.2.2. *Let $v \in \mathcal{H}(D \times (0, T))$ and let $(\hat{v}_n)_{n \in \mathbb{N}^+}$ be as in (2.2.34).*

Then we have

$$\lim_{n \rightarrow +\infty} \int_0^t d\tau (\hat{v}_{n,\tau}(\cdot, \tau), u_{I,\varphi}(\cdot, \tau))_2 = \int_0^t d\tau (v_\tau(\cdot, \tau), u_{I,\varphi}(\cdot, \tau))_2, \tag{2.2.35}$$

$$\begin{aligned}
& \lim_{n \rightarrow +\infty} \sum_{i,j=1}^d \int_0^t d\tau (\hat{v}_{n,x_i}(\cdot, \tau), k_{i,j}(\cdot, \tau) u_{I,\varphi,x_j}(\cdot, \tau))_2 \\
&= \sum_{i,j=1}^d \int_0^t d\tau (v_{x_i}(\cdot, \tau), k_{i,j}(\cdot, \tau) u_{I,\varphi,x_j}(\cdot, \tau))_2
\end{aligned} \tag{2.2.36}$$

and

$$\lim_{n \rightarrow +\infty} \int_0^t d\tau (\hat{v}_n(\cdot, \tau), g(u_{I,\varphi}(\cdot, \tau)))_2 = \int_0^t d\tau (v(\cdot, \tau), g(u_{I,\varphi}(\cdot, \tau)))_2 \tag{2.2.37}$$

\mathbb{P} -a.s. for every $t \in [0, T]$.

Proof. Regarding (2.2.35) we have

$$\begin{aligned}
& \int_0^t d\tau |(v_\tau(\cdot, \tau) - \hat{v}_{n,\tau}(\cdot, \tau), u_{I,\varphi}(\cdot, \tau))_2| \\
&\leq \|u_{I,\varphi}\|_{L^2(0,T;H^1(D))} \left(\int_{D \times (0,T)} dx d\tau |v_\tau(x, \tau) - \hat{v}_{n,\tau}(x, \tau)|^2 \right)^{\frac{1}{2}} \\
&\leq \|u_{I,\varphi}\|_{L^2(0,T;H^1(D))} \|v - \hat{v}_n\|_{1,2,T} \rightarrow 0
\end{aligned}$$

as $n \rightarrow +\infty$ \mathbb{P} -a.s. by using successively Schwarz inequalities in $L^2(D)$ and on the time interval $(0, T)$ along with (2.1.8) and (2.2.34). The proof of (2.2.36) follows from similar arguments. Thus, noting that the matrix elements $k_{i,j}$ are uniformly bounded for all $i, j \in \{1, 2, \dots, d\}$ as a consequence of (2.1.3), we eventually get

$$\begin{aligned} & \sum_{i,j=1}^d \int_0^t d\tau |(v_{n,x_i}(\cdot, \tau) - \hat{v}_{n,x_i}(\cdot, \tau), k_{i,j}(\cdot, \tau)u_{I,\varphi,x_j}(\cdot, \tau))_2| \\ & \leq c \|u_{I,\varphi}\|_{L^2(0,T;H^1(D))} \sum_{i,j=1}^d \left(\int_{D \times (0,T)} dx d\tau |v_{x_j}(x, \tau) - \hat{v}_{n,x_j}(x, \tau)|^2 \right)^{\frac{1}{2}} \\ & \leq c \|u_{I,\varphi}\|_{L^2(0,T;H^1(D))} \|v - \hat{v}_n\|_{1,2,T} \rightarrow 0 \end{aligned}$$

as $n \rightarrow +\infty$ \mathbb{P} -a.s., where $c > 0$ is a constant depending only on \underline{k} , \bar{k} and d . In a similar way for (2.2.37) we have

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_0^t d\tau |(v(\cdot, \tau) - \hat{v}_n(\cdot, \tau), g(u_{I,\varphi}(\cdot, \tau)))_2| \\ & \leq (c_1 + c_2 \|u_{I,\varphi}\|_{L^2(0,T;H^1(D))}) \lim_{n \rightarrow +\infty} \|v - \hat{v}_n\|_{1,2,T} = 0 \end{aligned}$$

\mathbb{P} -a.s. with $c_{1,2} > 0$, which follows from the fact that g is Lipschitz continuous and $|D| < +\infty$ where $|D|$ stands for the Lebesgue measure of D . □

In the sequel we will also need approximation properties for the remaining terms in (2.1.14) that are similar to those of the above proposition. However, the proof of those properties and eventually of the main theorem will require two more preparatory results. The first one is:

Lemma 2.2.5. *Let $\mathcal{H}(D \times (0, T))$ be the Hilbert space defined in Section 2.1. Then there exists the continuous trace embedding*

$$\mathcal{H}(D \times (0, T)) \rightarrow L^2(D \times \{\tau = t\}) \tag{2.2.38}$$

valid for each $t \in [0, T]$ and every $d \in \mathbb{N}^+$. Moreover, there also exists the continuous embedding

$$\mathcal{H}(D \times (0, T)) \rightarrow \mathcal{B}^{\alpha,2}([0, T]; L^2(D)) \tag{2.2.39}$$

where $\mathcal{B}^{\alpha,2}([0, T]; L^2(D))$ is endowed with norm (2.1.7).

Proof. It is sufficient to prove the result for $H^1(D \times (0, T))$. Let $v \in H^1(D \times (0, T))$ and let us write momentarily $\|\cdot\|_{1,2,H^1((0,T))}$ for the norm in $H^1((0, T))$. From (2.1.12) and Fubini's theorem we infer that $x \mapsto v(x, t) \in L^2(D)$ for almost every t , and more importantly that $t \mapsto v(x, t) \in H^1((0, T))$ for almost every $x \in D$. Therefore, writing $\mathcal{C}([0, T])$ for the space

of all continuous functions defined on $[0, T]$ endowed with the uniform norm $\|\cdot\|_{\infty, T}$ we have $t \mapsto v(x, t) \in \mathcal{C}([0, T])$ by virtue of the embedding

$$H^1((0, T)) \hookrightarrow \mathcal{C}([0, T]).$$

Consequently we obtain

$$|v(x, t)| \leq \|v(x, \cdot)\|_{\infty, T} \leq c \|v(x, \cdot)\|_{1,2, H^1((0, T))}$$

for each $t \in [0, T]$ and almost every $x \in D$ for some $c > 0$, and thereby the desired result

$$\int_D dx |v(x, t)|^2 \leq c^2 \int_D dx \|v(x, \cdot)\|_{1,2, H^1((0, T))}^2 \leq c^2 \|v\|_{1,2, T}^2$$

according to (2.1.12). As for the proof of the second embedding, we note that for every $v \in H^1(D \times (0, T))$ and $t \geq \tau$ we have

$$\begin{aligned} & \|v(\cdot, t) - v(\cdot, \tau)\|_2 \\ & \leq (t - \tau)^{\frac{1}{2}} \left(\int_{D \times (0, T)} dx d\sigma |v_\sigma(x, \sigma)|^2 \right)^{\frac{1}{2}} \leq (t - \tau)^{\frac{1}{2}} \|v\|_{1,2, T} \end{aligned}$$

by (2.1.12), and therefore

$$\begin{aligned} \|v\|_{\alpha, 2, T}^2 &= \left(\sup_{t \in [0, T]} \|v\|_2 \right)^2 + \int_0^T dt \left(\int_0^t d\tau \frac{\|v(\cdot, t) - v(\cdot, \tau)\|_2}{(t - \tau)^{\alpha+1}} \right)^2 \\ &\leq c \|v\|_{1,2, T}^2 + \int_0^T dt \left(\int_0^t d\tau (t - \tau)^{-\alpha - \frac{1}{2}} \right)^2 \|v\|_{1,2, T}^2 \leq c \|v\|_{1,2, T}^2 \end{aligned}$$

by virtue of (2.1.12) and the fact that $\alpha < \frac{1}{2}$, changing the value of c whenever necessary. \square

The expected approximation property of the stochastic integral is the following:

Proposition 2.2.3. *Let $v \in \mathcal{H}(D \times (0, T))$ and $(\hat{v}_n)_{n \in \mathbb{N}^+}$ be as in Proposition 2.2.2. Then we have*

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \int_0^t d\tau \int_D dx \hat{v}_n(x, \tau) h(u_{I, \varphi}(x, \tau)) W^H(x, d\tau) \\ &= \int_0^t d\tau \int_D dx v(x, \tau) h(u_{I, \varphi}(x, \tau)) W^H(x, d\tau) \end{aligned}$$

\mathbb{P} -a.s. for every $t \in [0, T]$.

Proof. From Relation (2.2.12) and embedding (2.2.39) we have

$$\begin{aligned} & \left| \int_0^t d\tau \int_D dx (v(x, \tau) - \hat{v}_n(x, \tau)) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau) \right| \\ & \leq cr_\alpha^H (1 + \|u_{I,\varphi}\|_{\alpha,2,T}^2) \|v - \hat{v}_n\|_{1,2,T} \end{aligned}$$

\mathbb{P} -a.s. for some $c > 0$, hence the desired result from (2.2.34). \square

Proof of Theorem 2.2.1 (B). Let $v \in \mathcal{H}(D \times (0, T))$ and $(\hat{v}_n)_{n \in \mathbb{N}^+}$ be as in (2.2.34). According to Lemma 2.2.4 we have

$$\begin{aligned} (\hat{v}_n(\cdot, t), u_{I,\varphi}(\cdot, t))_2 &= (\hat{v}_n(\cdot, 0), \varphi)_2 + \int_0^t d\tau (\hat{v}_{n,\tau}(\cdot, \tau), u_{I,\varphi}(\cdot, \tau))_2 \\ &\quad - \sum_{i,j=1}^d \int_0^t d\tau (\hat{v}_{n,x_i}(\cdot, \tau), k_{i,j}(\cdot, \tau) u_{I,\varphi,x_j}(\cdot, \tau))_2 \\ &\quad + \int_0^t d\tau (\hat{v}_n(\cdot, \tau), g(u_{I,\varphi}(\cdot, \tau)))_2 \\ &\quad + \int_0^t \int_D dx \hat{v}_n(x, \tau) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau) \end{aligned} \tag{2.2.40}$$

holds \mathbb{P} -a.s. for every $t \in [0, T]$ and every n , and we already know that each integral on the right-hand side of this expression converges to the desired integral in (2.1.14) according to Propositions 2.2.2 and 2.2.3. Let us now investigate the term on the left-hand side. From Schwarz inequality in $L^2(D)$ and (2.1.7) we have

$$\begin{aligned} & | (v(\cdot, t) - \hat{v}_n(\cdot, t), u_{I,\varphi}(\cdot, t))_2 | \\ & \leq \|u_{I,\varphi}\|_{\alpha,2,T}^2 \sup_{t \in [0,T]} \|v(\cdot, t) - \hat{v}_n(\cdot, t)\|_2 \\ & \leq \|u_{I,\varphi}\|_{\alpha,2,T}^2 \|v - \hat{v}_n\|_{\alpha,2,T}^2 \leq c \|u_{I,\varphi}\|_{\alpha,2,T}^2 \|v - \hat{v}_n\|_{1,2,T}^2 \rightarrow 0 \end{aligned}$$

\mathbb{P} -a.s. as $n \rightarrow +\infty$ for some $c > 0$ by virtue of (2.2.34) and (2.2.39). Therefore we get

$$\lim_{n \rightarrow +\infty} (\hat{v}_n(\cdot, t), u_{I,\varphi}(\cdot, t))_2 = (v(\cdot, t), u_{I,\varphi}(\cdot, t))_2$$

\mathbb{P} -a.s. uniformly in $t \in [0, T]$, where $v(\cdot, t)$ stands for the trace of v in the sense of (2.2.38).

Passing to the limit in (2.2.40) we then obtain for the remaining term

$$\begin{aligned} \lim_{n \rightarrow +\infty} (\hat{v}_n(\cdot, 0), \varphi)_2 &= (v(\cdot, 0), \varphi)_2 + \int_0^t d\tau (v_\tau(\cdot, \tau), u_{I,\varphi}(\cdot, \tau))_2 \\ &\quad - \sum_{i,j=1}^d \int_0^t d\tau (v_{x_i}(\cdot, \tau), k_{i,j}(\cdot, \tau) u_{I,\varphi,x_j}(\cdot, \tau))_2 \\ &\quad + \int_0^t d\tau (v(\cdot, \tau), g(u_{I,\varphi}(\cdot, \tau)))_2 \\ &\quad + \int_0^t \int_D dx v(x, \tau) h(u_{I,\varphi}(x, \tau)) W^H(x, d\tau) \end{aligned} \tag{2.2.41}$$

\mathbb{P} -a.s. for every $t \in [0, T]$, hence

$$\lim_{n \rightarrow +\infty} (\hat{v}_n(\cdot, 0), \varphi)_2 = (v(\cdot, 0), u_{I,\varphi}(\cdot, 0))_2$$

by choosing $t = 0$. But from (2.1.9) at $t = 0$ we have $(v, u_{I,\varphi}(\cdot, 0) - \varphi)_2 = 0$ for every $v \in H^1(D)$ and thereby for every smooth and compactly supported $v \in \mathcal{C}_c^1(D)$, the latter space being dense in $L^2(D)$. Therefore $u_{I,\varphi}(\cdot, 0) - \varphi$ is orthogonal to $L^2(D)$, hence $u_{I,\varphi}(\cdot, 0) = \varphi$ so that

$$\lim_{n \rightarrow +\infty} (\hat{v}_n(\cdot, 0), \varphi)_2 = (v(\cdot, 0), \varphi)_2.$$

The substitution of the preceding expression into (2.2.41) then proves (2.1.14) for every $v \in \mathcal{H}(D \times (0, T))$.

Finally, the proof that $u_{II,\varphi}$ is the unique variational solution to (2.1.2) satisfying (2.1.9) when h is an affine function is identical to that carried out in [60]. In fact, let $\tilde{u}_{II,\varphi}$ be another variational solution of type II to (2.1.2) satisfying (2.1.14).

Since $H^1(D) \subset \mathcal{H}(D \times (0, T))$, it is clear that $\tilde{u}_{II,\varphi}$ is also a variational solution of type I. Then, from the uniqueness statement in part (A) of Theorem 2.2.1, we have $\tilde{u}_{II,\varphi} = u_{I,\varphi}$ \mathbb{P} -a.s. as a solution to (2.1.9), hence also as a solution to (2.1.14). \square

Remarks. (1) If D is a bounded domain that satisfies the cone condition, the natural space of test functions for variational solutions of type II is $H^1(D \times (0, T))$, as was shown in [60] by first using (2.1.9) to prove (2.1.14) for polynomial test functions, then for smooth test functions by invoking the \mathcal{C}^1 -version of Weierstrass' approximation theorem, and eventually for functions in $H^1(D \times (0, T))$ by a density argument. It is not possible to argue in that way when D is unbounded and merely satisfies Hypothesis (D). Thus, it is still true that the space $\mathcal{C}^1(\bar{D} \times [0, T])$ of all uniformly continuous and uniformly bounded functions on $D \times (0, T)$ is dense in $H^1(D \times (0, T))$ if we assume in addition that D and therefore $D \times (0, T)$ satisfy the so-called segment condition (see, e.g. [1]). However, we can no longer approximate functions in $\mathcal{C}^1(\bar{D} \times [0, T])$ by polynomials in a suitable topology to make such approximations worthwhile in the context of this chapter (an approximation of the functions in $\mathcal{C}^1(\bar{D} \times [0, T])$ by polynomials is still possible if one endows $\mathcal{C}^1(D \times (0, T))$ with the structure of a Fréchet space, see, e.g., section 3 of Chapter 3 in [43], but this did not turn out to be strong enough in our case). We therefore bypassed the difficulty by constructing the space $\mathcal{H}(D \times (0, T))$ in section 2.1, but the determination of the relative size of this space within $H^1(D \times (0, T))$ remains an open problem at the present time.

(2) To the best of our knowledge, the question of uniqueness of the variational solution in the case of a general nonlinearity h in the stochastic term remains open.

2.2.4 Some examples

In this subsection, we consider classes of domains D that satisfy condition (D) and enter therefore into the framework of our existence and uniqueness result.

Proposition 2.2.4. *Let us consider the two-dimensional domain*

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, 0 < x_2 < b(x_1)\} \tag{2.2.42}$$

where the positive boundary curve $b \in C^1(\mathbb{R}_0^+)$ is decreasing, has a bounded derivative and satisfies

$$\lim_{x \rightarrow \infty} \frac{b(x + \epsilon)}{b(x)} = 0 \tag{2.2.43}$$

for every $\epsilon > 0$. Then the conditions of Hypothesis (D) hold.

Proof. For every $N \in \mathbb{N}^+$ let us define the bounded open set

$$D_N = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < N\} \cap D.$$

Then we have $D_N \subset D_{N+1} \subset D$ and D_N satisfies the cone condition, so that compact embedding (2.1.16) holds by the Rellich-Kondrachov theorem. Furthermore, the properties of the flow associated with D in the sense of Example 6.51 and Theorem 6.52 in [1] by means of the function b satisfying (2.2.43) imply that for every $u \in H^1(D)$, the inequality

$$\int_{D \setminus D_N} dx |u(x)|^2 \leq c_N \|u\|_{1,2}^2$$

is valid with $\lim_{N \rightarrow +\infty} c_N = 0$. Consequently, for each $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}^+$ such that (2.1.17) holds for each u belonging to a bounded set of $H^1(D)$. \square

We conclude that for a domain of the form (2.2.42) the embedding $H^1(D) \hookrightarrow L^2(D)$ is compact, so that our main theorem holds true in this case. By [3] the condition

$$\int_N^\infty b(x) dx = o(b(N)) \text{ as } N \rightarrow \infty \tag{2.2.44}$$

is equivalent to (2.2.43). Generally speaking, it is the finiteness of the Lebesgue measure of D , along with the rapid decrease of the measure of the part of D situated outside the disk of radius R centered at the origin as $R \rightarrow +\infty$, which makes all this possible.

Typical examples of boundary functions satisfying the above conditions include $b(x) = \exp(-x^{1+\delta})$ where $\delta > 0$, whereas the mere exponential $b(x) = \exp(-x)$ does not belong to that class of curves. In fact, since b is decreasing it follows easily from (2.2.43) that $\lim_{x \rightarrow +\infty} \exp(x)b(x) = 0$. We can slightly slow down the decay of b by choosing $b(x) = x^a \exp(-x^{1+\delta})$, $a > 0$ which is decreasing for x large enough.

We conjecture that such domains D can be extended to star-shaped domains by considering $D' = \{(x, y) \in \mathbb{R}^2, |y| < b(|x|)\}$ and by adding a congruent copy D'' of D' in the vertical direction, that is by considering the domain $D' \cup D''$, where $D'' = \{(x, y) \in \mathbb{R}^2, |x| < b^{-1}(|y|)\}$ and this time it is $y \rightarrow x = b(y)$ that satisfies the (equivalent) conditions (2.2.43) and (2.2.44). Let us notice that in each branch of $D' \cup D''$ the function b or b^{-1} is again the length of the slices $S_N = \{z \in D' \cup D'', |z| = N\}$ for N large enough. Further copies of D' can be added to the domain in various directions without losing the compactness of the embedding.

Our second example refers to the horn-shaped region discussed in [2], for which our main theorem is also valid.

Proposition 2.2.5. *Let us consider the three-dimensional domain*

$$D = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0, 0 < x_1^2 + x_2^2 < b^2(x_3)\}$$

where the boundary curve b satisfies the same hypotheses as in Proposition 2.2.4. Then the conditions of Hypothesis (D) hold.

Proof. The argument is similar to that given in the proof of Proposition 2.2.4 if we define

$$D_N = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3 < N\} \cap D$$

for every $N \in \mathbb{N}^+$. Thus the compactness of embedding (2.1.16) along with (2.1.17) hold. \square

Remark. An extension to domains $D \subset \mathbb{R}^3$ of the technique applied above to domains $D \subset \mathbb{R}^2$ is carried out more easily in cylindrical polar coordinates (r, θ, x) , $r, x > 0$, $\theta \in [0, 2\pi)$. The norm is, as in \mathbb{R}^2 , the maximum norm $|z| \equiv |(r, \theta, x)| = \max\{r, \theta, x\}$. The horn-shaped domain in Proposition 2.2.5 is then given by $D = \{(r, \theta, x), x > 0, r < b(x)\}$. The equivalence of conditions (2.2.43) and (2.2.44) has, as far as we know, not been established for domains $D \subset \mathbb{R}^3$. Since, in \mathbb{R}^3 , the surface measure A_N of the slices $S_N = \{z = (r, \theta, x) \in D, |z| = N\}$ is given, for sufficiently large x , by $A_N = \pi b(N)^2$, we can apply theorem 6.45 in [1] (see also theorem 4 in [2]) to see that condition (2.2.44) on b is necessary for the compactness of the embedding. The compactness is a consequence of theorem 6.52 in [1] (see also theorem 5 in [2]) where, in a more general context, a sufficient condition for the compactness of the embedding is given in terms of a flow. As a matter of fact, the explicit form of the flow given in [2] for horn-shaped domains shows that the embedding is compact for functions b that satisfy condition (2.2.44), in particular for the examples of b given above. For the details on the flow we refer to [2].

2.3 Local variational solutions

In this section we suppose that h is affine. We extend the existence and uniqueness result in Theorem 2.2.1 to the case where only a local Lipschitz condition is required to hold for the nonlinearity g in (2.1.2). Since the blowup of a solution in finite time can not be excluded in this case, we start with the definition of a local variational solution of (2.1.2) up to a blowup time.

We recall that hypothesis (L) is the classical Lipschitz condition for functions $g : \mathbb{R} \rightarrow \mathbb{R}$ and implies the Lipschitz condition on $L^2(D)$ that is used in the proof of Theorem 2.2.1. It may be formulated in the context of this proof as follows: There exists a constant $C > 0$ such that, for all $t \in [0, T]$ and all $u(\cdot, t), v(\cdot, t) \in L^2(D)$

$$\|g(u(\cdot, t)) - g(v(\cdot, t))\|_2 \leq C \|u(\cdot, t) - v(\cdot, t)\|_2. \tag{2.3.1}$$

For the proof of the existence of a local variational solution of (2.1.2) we need the following local Lipschitz condition in $L^2(D)$.

(L_m) For every $m \in \mathbb{N}^+$, there exists a constant $C_m > 0$ such that

$$\|g(u) - g(v)\|_2 \leq C_m \|u - v\|_2 \tag{2.3.2}$$

for all $u, v \in L^2(D)$ with $\|u\|_2 \leq m$ and $\|v\|_2 \leq m$.

Since we are interested in $L^2(D)$ -valued solutions $u(\cdot, t)$ of (2.1.2), we define the blowup time as the first time where $u(\cdot, t)$ is not in $L^2(D)$ anymore. Another possible definition of the blowup time would be the first time where $u(\cdot, t)$ is not in $L^\infty(D)$ anymore (by supposing that the initial condition $\varphi \in L^\infty(D)$). Notice that the two definitions are not equivalent since $u(\cdot, t) \in L^2(D)$ does not imply $u(\cdot, t) \in L^\infty(D)$, while the converse is true for domains $D \subset \mathbb{R}^d$ with finite Lebesgue measure.

Definition 2.3.1. A random time $\tau : \Omega \rightarrow \mathbb{R}^+$ is called blow up time of $u \in L^2(D)$ if

$$\limsup_{t \nearrow \tau} \sup_{s \in (0, t)} \|u(\cdot, s)\|_2 = +\infty \quad \mathbb{P} - a.s. \text{ on } \{\omega \in \Omega, \tau(\omega) < +\infty\}.$$

Remark. We set $\tau(\omega) = +\infty$ if the solution of (2.1.2) does not blow up, i.e. if a global solution exists.

We give only the definition of a local variational solution of type II of (2.1.2), the definition of the local variational solution of type I being an easy modification of it.

Definition 2.3.2. Let τ a strictly positive random time. An $L^2(D)$ -valued random field

$(u_{II,\varphi}(\cdot, t))_{t \in [0, T \wedge \tau]}$ is a local variational solution of type II on $[0, T]$ to the Neumann problem (2.1.2) if τ is a blowup time for $u_{II,\varphi}$ and

$$(1) \quad u_{II,\varphi} \in L^2(0, t; H^1(D)) \cap \mathcal{B}^{\alpha,2}([0, t]; L^2(D)) \quad \mathbb{P}\text{-a.s. for every } t \in [0, T \wedge \tau),$$

(2) The integral relation (2.1.14) holds \mathbb{P} -a.s. for every $v \in \mathcal{H}(D \times (0, T))$ and every $t \in [0, T \wedge \tau)$.

Notice that a global variational solution of (2.1.2) on $[0, T]$ in the sense of Definitions 2.1.1 and 2.1.2 is a local variational solution with $\tau > T$ \mathbb{P} -a.s.

Theorem 2.3.1. *Assume that conditions (C), (L_m) , (I), (K) and (D) are satisfied and that h is an affine function. Then, for all $T > 0$, (2.1.2) possesses a unique local variational solution $u_{I,\varphi}$ of type I on $[0, T]$, and $u_{I,\varphi}$ is also the unique local variational solution of type II of (2.1.2) on $[0, T]$.*

Proof. We apply a truncation technique which allows one to reduce the proof to a sequence of Neuman problems of the type of (2.1.2) to which Theorem 2.2.1 may be applied. The proof makes use of mollifiers η_m , ($m \in \mathbb{N}$) (see for instance [17]) defined as \mathbb{C}^∞ -functions $\eta_m : [0, +\infty) \rightarrow [0, 1]$ as follows:

$$\eta_m(s) = \begin{cases} 1, & \text{for } 0 \leq s \leq m \\ 0 & \text{for } s > 2m. \end{cases} \quad (2.3.3)$$

Instead of the equation (2.1.2), we consider the system of truncated Neumann problems given by

$$\begin{cases} du(x, t) = (\operatorname{div}(k(x, t)\nabla u(x, t)) + g_m(u(x, t)))dt + h(u(x, t))W^H(x, dt), \\ \quad (x, t) \in D \times (0, T), \\ u(x, 0) = \varphi(x), \forall x \in D, \\ \frac{\partial u(x, t)}{\partial n(k)} = 0, (x, t) \in \partial D \times (0, T], \end{cases} \quad (2.3.4)$$

where $g_m(u) = g(J_m u)$ with $J_m u = \eta_m(\|u\|_2)u$.

Let us show now that g_m verifies the global Lipschitz condition in $L^2(D)$.

Let $u(\cdot, t), v(\cdot, t) \in L^2(D)$ for all $t \in [0, T]$, and assume without loss of generality that

$\|u(\cdot, t)\|_2 > \|v(\cdot, t)\|_2$. Then, by using condition (L_m) , we have

$$\begin{aligned} & \|g_m(u(\cdot, t)) - g_m(v(\cdot, t))\|_2^2 \\ &= \|g(J_m u(\cdot, t)) - g(J_m v(\cdot, t))\|_2 \\ &= \|g(\eta_m(\|u(\cdot, t)\|_2)u(\cdot, t)) - g(\eta_m(\|v(\cdot, t)\|_2)v(\cdot, t))\|_2 \\ &\leq K_m \|\eta_m(\|u(\cdot, t)\|_2)u(\cdot, t) - \eta_m(\|v(\cdot, t)\|_2)v(\cdot, t)\|_2 \\ &= K_m \|\eta_m(\|u(\cdot, t)\|_2)(u - v)(\cdot, t) + v(\cdot, t)(\eta_m(\|u(\cdot, t)\|_2) - \eta_m(\|v(\cdot, t)\|_2))\|_2 \\ &\leq K_m (\|u(\cdot, t) - v(\cdot, t)\|_2 + \|v(\cdot, t)\|_2 |\eta_m(\|u(\cdot, t)\|_2) - \eta_m(\|v(\cdot, t)\|_2)|). \end{aligned}$$

We discuss two cases according to the value of $\|v(\cdot, t)\|_2$:

- If $\|v(\cdot, t)\|_2 > 2m$, we have $\|u(\cdot, t)\|_2 > 2m$ and therefore

$$\eta_m(\|v(\cdot, t)\|_2) = \eta_m(\|u(\cdot, t)\|_2) = 0.$$

Hence, we get

$$\|g_m(u(\cdot, t)) - g_m(v(\cdot, t))\|_2 \leq K_m \|u(\cdot, t) - v(\cdot, t)\|_2.$$

- If $\|v(\cdot, t)\|_2 \leq 2m$, we have

$$\|g_m(u(\cdot, t)) - g_m(v(\cdot, t))\|_2 \leq K_m (\|u(\cdot, t) - v(\cdot, t)\|_2 + 2m |\eta_m(\|u(\cdot, t)\|_2) - \eta_m(\|v(\cdot, t)\|_2)|).$$

Furthermore, for $s < r$, by the mean-value theorem, there exists $\rho \in (s, r)$ such that

$$|\eta_m(r) - \eta_m(s)| \leq |\eta'_m(\rho)| |r - s| \leq \max_{m \leq \rho \leq 2m} |\eta'_m(\rho)| |r - s|,$$

since the derivative $\eta'_m(\rho) = 0$ for $\rho < m$ or $\rho > 2m$. So, by putting this inequality in the above one, we get

$$\begin{aligned} \|g_m(u(\cdot, t)) - g_m(v(\cdot, t))\|_2 &\leq K_m (\|u(\cdot, t) - v(\cdot, t)\|_2 + 2m \gamma_m (\|u(\cdot, t)\|_2 - \|v(\cdot, t)\|_2)) \\ &\leq (K_m + 2m \gamma_m) \|u(\cdot, t) - v(\cdot, t)\|_2 \end{aligned}$$

with $\gamma_m := \max_{m \leq \rho \leq 2m} |\eta'_m(\rho)|$.

Therefore g_m satisfies the global Lipschitz condition (2.3.1). Hence, by Theorem 2.2.1, the system (2.3.4) has a variational solution of type I and II $u_{I,\varphi}^m(\cdot, t)$ respectively $u_{II,\varphi}^m(\cdot, t)$ on $[0, T]$ such that $u_{I,\varphi}^m(\cdot, t) = u_{II,\varphi}^m(\cdot, t)$ for all $t \in [0, T]$.

Let the random time τ_m be given by

$$\tau_m = \inf \left\{ t > 0, \sup_{s \in (0, t)} \|u_{I,\varphi}^m(\cdot, s)\|_2 > m \right\} \wedge T.$$

Then, for any m and for any $t < \tau_m$, $u_{I,\varphi}^m(\cdot, t)$ is the unique variational solution of type I

and of type II of the Neumann problem (2.1.2). In fact, for $m < m'$ the solutions $u_{I,\varphi}^m(\cdot, t)$ and $u_{I,\varphi}^{m'}(\cdot, t)$ coincide for $t < \tau_m$ by the unicity statement in Theorem 2.2.1.

Let $\tau_\infty = \lim_{m \rightarrow \infty} \tau_m$. This limit exists since τ_m is increasing. For $t < \tau_\infty$, we have $t < \tau_m$ for some $m > 0$, and $u_{I,\varphi}(\cdot, t) := u_{I,\varphi}^m(\cdot, t) = u_{II,\varphi}^m(\cdot, t)$ is the unique variational solution. For $\tau_\infty < T$,

$$\lim_{t \rightarrow \tau_\infty} \sup_{s \in (0,t)} \|u_{I,\varphi}(\cdot, s)\|_2 = \infty,$$

and hence $u_{I,\varphi}(\cdot, t)$ is the unique local variational solution of type I and II on $[0, T]$.

□

Chapter 3

On the large time behavior of solutions to semilinear SPDE's driven by a mixture of Brownian and fractional Brownian motions

3.1 Introduction

We investigate the blowup behavior of solutions to stochastic parabolic differential equations of the form

$$\left\{ \begin{array}{l} du(x, t) = \left(\frac{1}{2} k^2(t) Lu(x, t) + g(u(x, t)) \right) dt + u(x, t) dN_t, \\ x \in D, t > 0, \\ u(x, 0) = \varphi(x) \geq 0, x \in D, \\ u(x, t) = 0, x \in \partial D, t \geq 0, \end{array} \right. \quad (3.1.1)$$

where $D \subset \mathbb{R}^d$ is a bounded connected domain, L is the infinitesimal generator of a strongly continuous semigroup to be precised below, g is a locally Lipschitz function and N modelizes the randomness in the time evolution of the system.

The blowup behavior of (3.1.1) has been studied for $L = \Delta$, the Laplace operator on D , in [22], [4] for N being the Brownian motion and in [21] for N being the fractional Brownian motion with Hurst parameter $H > 1/2$. In this paper we extend this study to a larger class of differential operators, namely to (not necessarily symmetric) Lévy and diffusion processes, and to the presence of both, possibly time-changed, Brownian and fractional Brownian motion which, due to different analytic and probabilistic properties, modelize

different aspects of the random evolution in time of (3.1.1). More precisely we consider

$$N_t = \int_0^t a(s)dB_s + \int_0^t b^H(s)dB_s^H,$$

where B and B^H are Brownian and fractional Brownian motions, and a is continuous and b^H is Hölder continuous of order $\alpha > 1 - H$. The second integral can then be understood as a fractional integral, a fact which will be essential when applying the Itô formula. Both, B and B^H are supposed to be defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), P)$ and adapted to the filtration $(\mathcal{F}_t, t \geq 0)$. Such models have recently been studied under the name of 'mixed models' in the context of stochastic differential equations [55], see also [57] for the mild solution of an equation of the type of (3.1.1).

Our aim is to understand the influence of the stochastic part and of the differential operator in (3.1.1) (and of the parameters which characterize them) on the blowup behavior of the solution. In order to include inhomogeneous differential operators, we consider, as in [4], a function k of time in front of L and study its influence on the blowup behavior of u .

We will use the following notion of blowup time.

Definition 3.1.1. *A stopping time $\tau : \Omega \rightarrow (0, \infty)$ with respect to the filtration $(\mathcal{F}_t, t \geq 0)$ is a blowup time of a solution u of (3.1.1) if*

$$\limsup_{t \nearrow \tau, x \in D} |u(x, t)| = +\infty \quad P\text{-a.s.}$$

By this definition $\tau < +\infty$ P-a.s., but we will write $\tau = +\infty$ if $\tau > t$ for all $t > 0$.

In Sections 3.4 and 3.5 we give

- explicit expressions for upper and lower bounds of τ if $\tau < \infty$,
- sufficient conditions for $\tau < \infty$ and for $\tau = +\infty$, i.e. for the existence of a global solution,
- estimates for the probability $P(\tau < \infty)$ and for the tail probabilities of an upper bound of τ .

To carry out this study, (3.1.1) is transformed into a random partial differential equation (RPDE) where B and B^H do not enter as stochastic integrators but make part of the coefficients of a partial differential equation which is understood trajectorywise. Denoting the solution of the RPDE by v , the transformation is given by

$$v(x, t) = \exp(-N_t)u(x, t).$$

We show that v satisfies the partial differential equation formally given by

$$\begin{cases} \frac{\partial v}{\partial t}(x, t) = \frac{1}{2}k^2(t)Lv(x, t) - \frac{1}{2}a^2(t)v(x, t) + \exp(-N_t)g(\exp(N_t)v(x, t)), \\ \quad x \in D, t > 0 \\ v(x, 0) = \varphi(x), x \in D, \\ v(x, t) = 0, t \geq 0, x \in \partial D. \end{cases} \quad (3.1.2)$$

Evidently v blows up at the same time τ and at the same place $x \in D$ as u . In fact, such transformations are known for more general stochastic partial differential equations (SPDE) than (3.1.1), including equations where the stochastic term does not depend linearly on u , see [47]. But these transformations are not as simple as the exponential transformation above, and explicit expressions for upper and lower bounds for τ seem to be difficult to obtain. Another reason for having chosen the relatively simple form of (3.1.1) and (3.1.2) is that we consider the blowup trajectorywise which is a relatively strong notion compared, e.g., to blowup of the moments of the solution (see, e.g. [17]). Lower bounds for τ and sufficient conditions for the existence of a global solution are obtained with the mild form of (3.1.2). Upper bounds for τ are derived from the weak form of (3.1.2), and the existence of a positive eigenvalue and an eigenfunction with constant sign of the adjoint of L , denoted by L^* , are crucial ingredients for the proofs. The equivalence of both notions of solutions is therefore essential in this context.

The paper is organized as follows. In the next section we discuss shortly some classes of differential operators L on a bounded domain D whose adjoint admits an eigenfunction with constant sign and enter therefore into the framework for upper bounds for the blowup time. In Section 3.3 we show the transformation of the weak form of (3.1.1) into the weak form of (3.1.2) and the equivalence of weak and mild solutions of (3.1.2). In Section 3.4 the blowup behavior is studied with the weak form of (3.1.2). In order to get an upper bound for the blowup time, we consider in subsection 3.4.1 a subsolution of (3.1.2) which is the solution of a differential equation whose blowup time is determined explicitly in terms of an exponential functional of N . By means of a result by N.T. Dung [25], we estimate in subsection 3.4.2 the tail probabilities of this upper bound. Special attention is given in subsection 3.4.3 to the case $H \in (\frac{3}{4}, 1)$. It is well known that, in this case, N is equivalent to Brownian motion (see [16]). This allows to apply a result by Dufresne and Yor [79] on the law of exponential functionals of Brownian motion to get a lower bound for the probability of blowup in finite time of (3.1.2). In Section 3.5 we consider the mild form of (3.1.2) and methods which are in fact applicable to equations on bounded domains as well as to equations on \mathbb{R}^d . We restrict ourselves to bounded domains D and give a sufficient condition for blowup in finite time (subsection 3.5.1) and a lower bound for the blowup time (subsection 3.5.2). This lower bound is applied to get a sufficient condition

for the existence of a global solution.

3.2 Differential operators on a bounded domain with an eigenfunction of constant sign

In this section we show that rather general classes of infinitesimal generators L^D on bounded domains $D \subset \mathbb{R}^d$ admit an eigenfunction of constant sign on D , a property that will be essential for deriving an upper bound of the blowup time in Section 3.4. The fundamental result in this context, known as Theorem of Jentsch (see Theorem V.6.6 in [67]) will be applied to the semigroup $((P_t^D)^*, t \geq 0)$ of the Markov process associated to the adjoint $(L^D)^*$ of L^D in $L^2(D)$.

We start with two semigroups $(P_t^D, t \geq 0)$ and $((P_t^D)^*, t \geq 0)$ of bounded linear operators on $L^2(D)$ such that

$$\int_D f(x)P_t^D g(x)dx = \int_D g(x)(P_t^D)^* f(x)dx,$$

for all $f, g \in L^2(D)$. We assume that $(P_t^D, t \geq 0)$ and $((P_t^D)^*, t \geq 0)$ are strongly continuous (i.e. $\lim_{t \searrow 0} P_t^D f = \lim_{t \searrow 0} (P_t^D)^* f = f$ for all $f \in L^2(D)$), and that there exists a family of continuous functions $(p^D(t, \cdot, \cdot), t > 0)$ on $D \times D$ such that, for any $(t, x) \in (0, \infty) \times D$ we have

$$P_t^D f(x) = \int_D p^D(t, x, y)f(y)dy \quad \text{and} \quad (P_t^D)^* f(x) = \int_D p^D(t, y, x)f(y)dy. \quad (3.2.1)$$

The above mentioned theorem of Jentsch then allows to state that if, for each $t > 0$, $p^D(t, \cdot, \cdot)$ is bounded and strictly positive, the top eigenvalue $e^{-\lambda_0 t}$ of both P_t^D and $(P_t^D)^*$ (where $\lambda_0 > 0$) is associated to a $D - a.e.$ strictly positive eigenfunction ψ_0 of P_t^D with $\int_D \psi_0(x)dx = 1$ and to a $D - a.e.$ strictly positive eigenfunction ϕ_0 of $(P_t^D)^*$ with $\int_D \phi_0(x)dx = 1$, i.e. $D - a.e.$

$$\begin{aligned} P_t^D \psi_0(x) &= \int_D p^D(t, x, y)\psi_0(y)dy = e^{-\lambda_0 t} \psi_0(x), \\ (P_t^D)^* \phi_0(x) &= \int_D p^D(t, y, x)\phi_0(y)dy = e^{-\lambda_0 t} \phi_0(x). \end{aligned}$$

It can easily be shown (see proposition 2.3 in [41]) that ψ_0 and ϕ_0 are in fact strictly positive and continuous on D , and the equalities above are therefore valid for all $x \in D$.

We will now state classes of infinitesimal generators to which the above considerations can

be applied. Notice that

$$\begin{aligned} L^D \psi_0(x) &= \lim_{t \rightarrow 0} \frac{1}{t} (P_t^D \psi_0(x) - \psi_0(x)) = -\lambda_0, \\ (L^D)^* \phi_0(x) &= \lim_{t \rightarrow 0} \frac{1}{t} ((P^D)_t^* \phi_0(x) - \phi_0(x)) = -\lambda_0. \end{aligned}$$

λ_0 is therefore eigenvalue of L^D and of $(L^D)^*$, with associated eigenfunctions ψ_0 and ϕ_0 , i.e.

$$(L^D + \lambda_0)\psi_0 = ((L^D)^* + \lambda_0)\phi_0 = 0.$$

3.2.1 Lévy processes

Let us recall that a Lévy process $(X_t, t \geq 0)$ on \mathbb{R}^n is defined as a stochastically continuous process with independent and stationary increments which starts at $x = 0$ at time $t = 0$. Its semigroup $(P_t, t \geq 0)$ is then given by

$$P_t f(x) = \int p(t, x, y) f(y) dy = \int p(t, y - x) f(y) dy = \int p(t, y') f(x + y') dy' = \mathbb{E} f(x + X_t).$$

$(P_t, t \geq 0)$ is a strongly continuous semigroup (see Theorem 31.5 in [66]). Its infinitesimal generator L is given by

$$\begin{aligned} Lf(x) &= \frac{1}{2} \sum_{j,k=1}^d a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} f(x) + \sum_{j=1}^d b_j \frac{\partial}{\partial x_j} f(x) \\ &\quad + \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \sum_{j=1}^d y_j \frac{\partial}{\partial x_j} f(x) 1_{\{|y| \leq 1\}}(y) \right) \nu(dy), \end{aligned}$$

where $A = (a_{jk}, j, k = 1, \dots, d)$ is a symmetric nonnegative definite matrix, $b = (b_1, \dots, b_d) \in \mathbb{R}^d$, ν is a Lévy measure (i.e. ν is defined on $\mathbb{R}^d \div \{0\}$ and satisfies $\int (1 \wedge |y|^2) \nu(dy) < \infty$) and $f \in \mathcal{C}_0^2(\mathbb{R}^d)$, the space of twice continuously differentiable functions from \mathbb{R}^d in \mathbb{R} satisfying $\lim_{|x| \rightarrow \infty} f(x) = 0$.

The dual process $(X_t^*, t \geq 0)$ is given by $X_t^* = -X_t$. In fact (see, e.g. [10], chapter 2.1)

$$\begin{aligned} \int f(x) P_t g(x) dx &= \int \int f(x) p(t, y) g(x+y) dy dx = \int \int f(x' - y) p(t, y) g(x') dy dx' \\ &= \int g(x') \int p(t, -y) f(x' + y) dy dx' = \int g(x') P_t^* f(x') dx'. \end{aligned}$$

Its infinitesimal generator is given by

$$L^*f(x) = \frac{1}{2} \sum_{j,k=1}^d a_{jk} \frac{\partial^2}{\partial x_j \partial x_k} f(x) - \sum_{j=1}^d b_j \frac{\partial}{\partial x_j} f(x) + \int_{\mathbb{R}^d} \left(f(x+y) - f(x) - \sum_{j=1}^d y_j \frac{\partial}{\partial x_j} f(x) 1_{\{|y| \leq 1\}}(y) \right) \nu(-dy).$$

Therefore the dual of X is again a Lévy process with characteristics $(A, -b, \bar{\nu})$, where $\bar{\nu}(B) = \nu(-B)$ for any $B \in \mathcal{B}(\mathbb{R}^d)$. In particular, the Lévy process is symmetric (i.e. its law is equal to the law of its dual) if $b = 0$ and $\bar{\nu} = \nu$.

Let us prove that ν in L is replaced by $\bar{\nu}$ in L^* . Denote by \tilde{L} the jump part of the operator L , and by \tilde{L}^* the dual of \tilde{L} . We have

$$\begin{aligned} \langle \tilde{L}f, g \rangle &= \int \tilde{L}f(x)g(x)dx = \int dx \int \nu(dy) f(x+y)g(x) - \int dx f(x)g(x) \int \nu(dy) \\ &\quad - \int dx \int \nu(dy) \sum_{j=1}^d y_j \frac{\partial}{\partial x_j} f(x) 1_{\{|y| \leq 1\}}(y) g(x) = I_1 - I_2 - I_3 \end{aligned}$$

By the change of variables $x' = x + y$, $y' = -y$, we get

$$\begin{aligned} I_1 &= \int dx \int \nu(dy) f(x+y)g(x) = \int dx' \int \bar{\nu}(dy') f(x')g(x'+y') \\ I_2 &= \int dx f(x)g(x) \int \bar{\nu}(dy') \\ I_3 &= - \int dx \int \bar{\nu}(dy') \sum_{j=1}^d (-y'_j) \frac{\partial}{\partial x_j} g(x) 1_{\{|y'| \leq 1\}}(y') f(x). \end{aligned}$$

Therefore

$$\begin{aligned} \int \tilde{L}f(x)g(x)dx &= \int dx f(x) \int \left[g(x+y) - g(x) - \sum_{j=1}^d y_j \frac{\partial}{\partial x_j} g(x) 1_{\{|y| \leq 1\}}(y) \right] \bar{\nu}(dy) \\ &= \int dx f(x) L^*g(x) \end{aligned}$$

where

$$\tilde{L}^*g(x) = \int \left[g(x+y) - g(x) - \sum_{j=1}^d y_j \frac{\partial}{\partial x_j} g(x) 1_{\{|y| \leq 1\}}(y) \right] \bar{\nu}(dy).$$

Particular cases of L are the Laplacian Δ and its fractional powers $(-\Delta)^{\alpha/2}$ with $0 < \alpha < 2$. The associated processes are Brownian motion and stable processes. The

Lévy measure of a stable process is given by

$$v(B) = \int_S \lambda(\xi) \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}}, \quad B \in \mathcal{B}(\mathbb{R}^d),$$

where λ is a finite measure on the unit sphere S of \mathbb{R}^d . Let us notice that fractional powers of the Laplacian can be defined for any $\alpha > 0$ (see [62], Chapter 2.6), but we restrict ourselves to $\alpha \in (0, 2]$ in order to have stable processes associated to them (Brownian motion if $\alpha = 2$).

We cite now some results on Lévy processes which are killed when exiting the domain D and to which the Jentsch theorem can be applied to state that the smallest eigenvalue λ_0 is associated to a strictly positive eigenfunction.

T. Grzywny [33] considers symmetric Lévy processes and defines the process killed on exiting from D at time $\tau_D = \inf\{t > 0, X_t \notin D\}$ by the transition density given by

$$p^D(t, x, y) = p(t, x, y) - r_D(t, x, y),$$

where

$$r_D(t, x, y) = \mathbb{E}^x[p(t - \tau_D, X_{\tau_D}, y); t > \tau_D]$$

and shows that $p^D(t, \cdot, \cdot)$ is strictly positive and bounded on $D \times D$. P. P. Kim and R. Song [42], and X. Chen and J. Wang [15] generalize these results to rather general classes of non symmetric Lévy processes and to certain not necessarily bounded and connected domains D in such a way that the eigenfunction of the first eigenvalue is still strictly positive. These studies are usually carried out in the context of the intrinsic ultracontractivity property.

Other ways to define the fractional Laplacian on a bounded domain D have been investigated, see e.g. [11], [69]. Contrary to the authors cited above, these authors first kill the Brownian motion when exiting D and then raise its infinitesimal generator to a fractional power. More precisely, they define, for $x \in D$,

$$(-\Delta|_D)^s f(x) = \sum_{j=1}^{\infty} \lambda_j^s f_j \phi_j(x), \quad s \in (0, 1),$$

where the sum is understood in $L^2(D)$, λ_j ($j \in \mathbb{N}$), are the eigenvalues of the Dirichlet Laplacian on D , ϕ_j are the associated normalized eigenfunctions ($\phi(x) = 0$ for $x \in \partial D$, $\|\phi\|_{L^1(D)} = 1$) and $f_j = \int_D f(x) \phi_j(x) dx$. The infinitesimal generator defined in this way does not coincide with the infinitesimal generator associated to P_t^D . The eigenvalues of $(-\Delta|_D)^s$ are of course equal to the s -th power of the eigenvalues of the Dirichlet Laplacian on D , written down in increasing order by taking into account their multiplicity, and the eigenfunctions are the same as those for the Dirichlet Laplacian on D . In particular, the

first eigenfunction is strictly positive on D . Notice that this operator is symmetric in $L^2(D)$.

3.2.2 Diffusion processes

Diffusion processes on bounded domains $D \subset \mathbb{R}^d$ are usually given by linear partial differential operators of the form

$$Lf(x) = \sum_{j,k=1}^d a_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k} f(x) + \sum_{j=1}^d b_j(x) \frac{\partial}{\partial x_j} f(x) - c(x)f(x),$$

where $a_{j,k}$, b_j , $j, k = 1, \dots, d$ are bounded smooth functions on D and c is bounded and continuous. We assume that the matrix $(a_{j,k}, j, k = 1, \dots, d)$ is symmetric and uniformly elliptic, that is, there exists a positive number ς such that $\sum_{j,k=1}^d a_{jk}(x) \xi_j \xi_k \geq \varsigma \|\xi\|^2$ for all $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. We adopt a slightly different, but equivalent form for L which has the advantage that the adjoint operator L^* has a simpler form. We write

$$\bar{L}f(x) = \sum_{j,k=1}^d \frac{\partial}{\partial x_j} (a'_{jk}(x) \frac{\partial}{\partial x_k} f(x)) + \sum_{j=1}^d b'_j(x) \frac{\partial}{\partial x_j} f(x) - c(x)f(x).$$

The adjoint operator \bar{L}^* is then given, for $f \in \mathcal{C}_c^2(D)$, by

$$\begin{aligned} \bar{L}^*f(x) &= \sum_{j,k=1}^d \frac{\partial}{\partial x_j} (a'_{jk}(x) \frac{\partial}{\partial x_k} f(x)) - \sum_{j=1}^d b'_j(x) \frac{\partial}{\partial x_j} f(x) \\ &\quad - (c(x) + \sum_{j=1}^d \frac{\partial}{\partial x_j} b'_j(x))f(x). \end{aligned}$$

In particular \bar{L} is selfadjoint if $b' = 0$. If

$$c(x) \geq 0 \quad \text{and} \quad c(x) + \sum_{j=1}^d \frac{\partial}{\partial x_j} b'_j(x) \geq 0 \quad \text{for all } x \in D,$$

we can conclude again that the first eigenvalue of \bar{L} and $(\bar{L}^*$ has positive eigenfunctions on D . For the (strict) positivity of the transition density functions $p^D(t, x, y)$, $(x, y) \in D \times D$, we refer to chapter 2, theorem 11 in [26]. The same conclusion can be obtained by applying the (strong) maximum principle for uniformly elliptic operators, see e.g. [27], chapter 34 or [64], chapter 3.5.4.

A particular class of infinitesimal generators is given for $b' = 0$ and c a nonnegative function

in the Kato class of potentials K_d , i.e. c satisfies

$$\limsup_{r \searrow 0} \int_x \int_{B(x,r)} \frac{|c(y)|}{|x-y|^d} dy = 0,$$

where $B(x, r)$ is the ball centered at x and with radius r (see [6]). We get

$$\bar{L} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left(a'_{ij}(x) \frac{\partial}{\partial x_j} \right) - c(x),$$

and \bar{L} is known under the name of Schrödinger operator.

3.3 The weak solution of the associated random partial differential equation, equivalence with the mild solution

3.3.1 Weak solutions

In this subsection we transform the weak form of (3.1.1) into the weak form of (3.1.2) by means of an Itô-type formula due to Y. Mishura [54]. The blowup behavior of the solution of (3.1.2) is in fact easier to derive, and, by the linearity in u of the stochastic integral, the transformation is given by $v(x, t) = \exp(-N_t)u(x, t)$ ($x \in D, t \geq 0$). If blowup takes place in finite time, it occurs of course at the same time and at the same place for the solutions of both equations. At the moment of writing down this paper there are, to our best knowledge, no existence and uniqueness results for the weak solution of (3.1.1). For the existence and the uniqueness of a mild solution of (3.1.1) driven by a non autonomous, but symmetric differential operator of second order we refer to [57]. We prefer therefore to discuss the existence and uniqueness of the solution in the context of (3.1.2), and to obtain a weak solution of (3.1.1) by means of the transformation mentioned above.

We denote by $dom(L^*)$ the functions in the domain of the adjoint of L which vanish on D^c . It is a subspace of $L^2(D)$. We write $\langle \cdot, \cdot \rangle_D$ for the scalar product in $L^2(D)$.

Definition 3.3.1. *An $(\mathcal{F}_t, t \geq 0)$ -adapted random field $v = (v(x, t), t \in [0, T], x \in D)$ is a weak solution of (3.1.2) if, for all $t \in [0, T]$ and all $f \in dom(L^*)$,*

$$\begin{aligned} & \langle v(\cdot, t), f \rangle_D \\ = & \langle \varphi, f \rangle_D + \int_0^t \left(\frac{1}{2} k^2(s) \langle v(\cdot, s), L^* f \rangle_D - \frac{1}{2} a^2(s) \langle v(\cdot, s), f \rangle_D \right) ds \\ & + \int_0^t \exp(-N_s) \langle g(\exp(N_s)v(\cdot, s)), f \rangle_D ds. \end{aligned} \tag{3.3.1}$$

The definition is for fixed $\omega \in \Omega$. Since g is supposed to be locally Lipschitz, a blowup in finite time of v may occur, and this blowup time depends in general on ω . A slightly different way to define the weak solution v is therefore to define it as an $(\mathcal{F}_t, t \geq 0)$ -adapted random field v which satisfies (3.3.1) P -a.s. for all $t \in (0, \tau)$, where τ is the random blowup time of v . If ω is such that $v(\omega, \cdot, \cdot)$ does not blowup in finite time, we set $\tau(\omega) = \infty$.

We proceed now to the relation between the weak solutions of (3.1.1) and of (3.1.2).

Proposition 3.3.1. *Let v be a weak solution of (3.1.2) up to a random time τ P -a.s.. Then $u(x, t) = \exp(N_t)v(x, t)$ is a weak solution of (3.1.1) up to τ P -a.s., that is,*

$$\begin{aligned} \langle u(\cdot, t), f \rangle_D &= \langle \varphi, f \rangle_D + \int_0^t \left(\frac{1}{2} k^2(s) \langle u(\cdot, s), L^* f \rangle_D + \langle g(u(\cdot, s), f) \rangle_D \right) ds \\ &\quad + \int_0^t \langle u(\cdot, s), f \rangle_D dN_s \end{aligned} \quad (3.3.2)$$

P -a.s for all $t \in (0, \tau)$.

Remark. The stochastic integral in (3.3.2) is well defined on $[0, T \wedge \tau)$ for any $T > 0$ by the assumptions made so far. In fact, for $t < T \wedge \tau(\omega)$ the Itô integral

$$\begin{aligned} \int_0^t \langle u(\cdot, s), f \rangle_D a(s) dB_s &= \int_0^t \langle v(\cdot, s), f \rangle_D \exp(N_s) a(s) dB_s \\ &= \int_0^t \int_D v(x, s) f(x) dx \exp(N_s) a(s) dB_s \end{aligned}$$

is well defined since

$$\int_0^t \left(\int_D v(x, s) f(x) dx \right)^2 \exp(2N_s) a^2(s) ds < \infty \quad P - a.s.$$

Notice that the definition of the weak solution of (3.1.2) implies that $v(\cdot, s) \in L^2(D)$ for all $s \in [0, t]$ and $\langle v(\cdot, s), f \rangle_D$ is absolutely continuous in s .

The integral

$$\int_0^t \langle u(\cdot, s), f \rangle_D b^H(s) dB_s^H = \int_0^t \langle v(\cdot, s), f \rangle_D \exp(N_s) b^H(s) dB_s^H \quad (3.3.3)$$

is well defined trajectorywise as a fractional integral for $t \in (0, T \wedge \tau(\omega))$.

In fact, $N_t = \int_0^t (a(s) dB_s + b^H(s) dB_s^H)$ is P -a.s. Hölder continuous of order $1/2 - \epsilon$ for all $\epsilon > 0$ by the theorem of Kolmogorov and Proposition 4.1 in [58]. Therefore the integrand on the right side of (3.3.3) is Hölder continuous of order $> 1 - H$ and since $H > 1/2$ the integral is well defined as a generalized Riemann-Stieltjes integral [80].

Proof. Let $T > 0$. It suffices to prove the assertion for $t \in (0, T \wedge \tau)$. The proof consists of

an application of the Itô fomula in [54], page 184. Let

$$\begin{aligned} Y_t^1 &= \int_0^t a(s)dB_s, \quad Y_t^2 = \int_0^t b^H(s)dB_s^H, \\ Y_t^3 = \langle v(\cdot, t), f \rangle_D &= \langle \varphi, f \rangle_D + \int_0^t h(s)ds, \quad \text{where} \\ h(s) &= \frac{1}{2}k^2(s) \langle v(\cdot, s), L^* f \rangle_D \\ &\quad - \frac{1}{2}a^2(s) \langle v(\cdot, s), f \rangle_D + \exp(-N_s) \langle g(\exp(N_s)v(\cdot, s)), f \rangle_D. \end{aligned}$$

and $F(y, y_2, y_3) = \exp(y_1 + y_2)y_3$. Then

$$F(Y_t^1, Y_t^2, Y_t^3) = \exp(N_t) \langle v(\cdot, t), f \rangle_D = \langle u(\cdot, t), f \rangle_D = \int_D u(x, s)f(x)dx.$$

The above mentioned Itô formula then reads

$$\begin{aligned} F(Y_t^1, Y_t^2, Y_t^3) &= F(Y_0^1, Y_0^2, Y_0^3) + \sum_{i=1}^3 \int_0^t \frac{\partial F}{\partial y_i}(Y_s^1, Y_s^2, Y_s^3) dY_s^i \\ &\quad + \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial y_1^2}(Y_s^1, Y_s^2, Y_s^3) d\langle Y_s^1 \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \langle u(\cdot, t), f \rangle_D &= \langle \varphi, f \rangle_D + \int_0^t \exp(N_s) \langle v(\cdot, s), f \rangle_D (a(s)dB_s + b^H(s)dB_s^H) \\ &\quad + \int_0^t \exp(N_s) \left[\frac{1}{2}k^2(s) \langle v(\cdot, s), L^* f \rangle_D - \frac{1}{2}a^2(s) \langle v(\cdot, s), f \rangle_D \right. \\ &\quad \left. + \exp(-N_s) \langle g(\exp(N_s)v(\cdot, s)), f \rangle_D \right] ds \\ &\quad + \frac{1}{2} \int_0^t \exp(N_s) \langle v(\cdot, s), f \rangle_D a^2(s) ds \\ &= \langle \varphi, f \rangle_D + \int_0^t \langle u(\cdot, s), f \rangle_D dN_s + \int_0^t \left[\frac{1}{2}k^2(s) \langle u(\cdot, s), L^* f \rangle_D \right. \\ &\quad \left. + \int_0^t \langle g(u(\cdot, s)), f \rangle_D \right] ds. \end{aligned}$$

Therefore u is a weak solution of (3.1.1). □

3.3.2 Mild solutions

The definition of the mild solution is more delicate since the operator $k^2(t)L$ is not autonomous. Nethertheless the dependence on t is simple enough to allow for a two-parameter evolution semigroup of contractions which is based on the semigroup associated to L on D . We follow the procedure developed for symmetric Lévy processes in [48]. As in

Section 3.2 we assume that the semigroups $(P_t^D, t \geq 0)$ and $((P^D)_t^*, t \geq 0)$ associated to L and L^* on D are strongly continuous, and that there exists a family of continuous functions $(p^D(t, \cdot, \cdot), t > 0)$ on $D \times D$ such that, for any $(t, x) \in (0, \infty) \times D$ (3.2.1) holds. Let $K(t, s) = \frac{1}{2} \int_s^t k^2(u) du$ and set $p^D(s, x; t, y) = p^D(K(t, s), x, y)$, $x, y \in D \times D$, $0 \leq s < t$. For $f \in L^2(D)$ let

$$\begin{aligned} U^D(t, s)f(x) &= \int_D p^D(s, x; t, y)f(y)dy = P_{K(t,s)}^D f(x), \\ (U^D)^*(t, s)f(x) &= \int_D p^D(s, y; t, x)f(y)dy = (P^D)_{K(t,s)}^* f(x). \end{aligned}$$

Since $(P_t^D, t \geq 0)$ and $((P^D)_t^*, t \geq 0)$ are strongly continuous semigroups of contractions on $L^2(D)$, $(U^D(t, s), 0 \leq s < t)$ and $((U^D)^*(t, s), 0 \leq s < t)$ are evolution families of contractions on $L^2(D)$. Notice that the semigroup property now reads for $0 \leq r < s < t$

$$\begin{aligned} U^D(t, s)U^D(s, r)f(x) &= P_{K(t,s)}^D P_{K(s,r)}^D f(x) \\ &= \int_D p^D(K(t, s), x, y) \int_D p^D(K(s, r), y, z)f(z)dzdy \\ &= \int_D p^D(K(t, s) + K(s, r), x, z)f(z)dz \\ &= \int_D p^D(K(t, r), x, z)f(z)dz = U^D(t, r)f(x). \end{aligned}$$

Suppose now that, for any y fixed, $p^D(u, x, y)$ satisfies the Kolmogorov backward equation, i.e. $\frac{\partial}{\partial u} p^D(u, x, y) = Lp^D(u, x, y)$. Then $(s, x) \rightarrow p^D(s, x; t, y)$ satisfies, for (t, y) fixed, the following Kolmogorov equation with the operator $\frac{1}{2}k^2(s)L$:

$$\begin{aligned} -\frac{\partial}{\partial s} p^D(s, x; t, y) &= -\frac{\partial}{\partial s} p^D(K(t, s), x, y) = -\frac{\partial}{\partial u} p^D(u, x, y) \Big|_{u=K(t,s)} \frac{\partial}{\partial s} K(t, s) \\ &= \frac{1}{2}k^2(s)Lp^D(K(t, s), x, y) = \frac{1}{2}k^2(s)Lp^D(s, x; t, y). \end{aligned} \tag{3.3.4}$$

Similarly, suppose that, for any x fixed, $p^D(u, x, y)$ satisfies the Kolmogorov forward equation, i.e. $\frac{\partial}{\partial u} p^D(u, x, y) = L^*p^D(u, x, y)$. Then $(t, y) \rightarrow p^D(s, x; t, y)$ satisfies, for (s, x) fixed, the following Kolmogorov equation with the operator $\frac{1}{2}k^2(t)L^*$:

$$\begin{aligned} \frac{\partial}{\partial t} p^D(s, x; t, y) &= \frac{\partial}{\partial t} p^D(K(t, s), x, y) = \frac{\partial}{\partial u} p^D(u, x, y) \Big|_{u=K(t,s)} \frac{\partial}{\partial t} K(t, s) \\ &= \frac{1}{2}k^2(t)L^*p^D(K(t, s), x, y) = \frac{1}{2}k^2(t)L^*p^D(s, x; t, y). \end{aligned} \tag{3.3.5}$$

Definition 3.3.2. An $(\mathcal{F}_t, t \geq 0)$ -adapted random field $v = (v(x, t), t \geq 0, x \in D)$ is a

mild solution of (3.1.2) on $[0, T]$ if, for all $t \in [0, T]$,

$$\begin{aligned} v(x, t) &= U^D(t, 0)\varphi(x) - \frac{1}{2} \int_0^t a^2(s)U^D(t, s)v(x, s)ds \\ &\quad + \int_0^t \exp(-N_s)U^D(t, s)(g(\exp N_s)v(x, s)))ds. \end{aligned}$$

The mild form of (3.1.2) may be written as follows (see also [4], Lemma 4.1).

Proposition 3.3.2. *The mild form of (3.1.2) can be written as*

$$v(x, t) = \exp(-A(t))U^D(t, 0)\varphi(x) + \int_0^t ds \exp(-N_s - A(t, s))U^D(t, s)g(\exp(N_s)v(\cdot, s))(x), \quad (3.3.6)$$

where $A(t, s) = \frac{1}{2} \int_s^t a^2(r)dr$ and $A(t) = A(t, 0)$.

This is a slightly different, but equivalent form of the mild solution of (3.1.2), obtained by considering the infinitesimal generator $L_t = \frac{1}{2}k^2(t)L - \frac{1}{2}a^2(t)$.

In particular $v(x, t) \geq \exp(-A(t))U^D(t, 0)\varphi(x) \geq 0$ since g is supposed to be nonnegative.

Proof. Let $w(x, t) = \exp(A(t))v(x, t)$. For $f \in L(D)$, we get from the definition of the mild solution

$$\begin{aligned} \frac{d}{dt} \langle w(\cdot, t), f \rangle_D &= \frac{1}{2}a^2(t) \exp(A(t)) \langle v(\cdot, t), f \rangle_D + \exp(A(t)) \frac{d}{dt} \langle v(\cdot, t), f \rangle_D \\ &= \frac{1}{2}a^2(t) \exp(A(t)) \langle v(\cdot, t), f \rangle_D \\ &\quad + \exp(A(t)) \left(\frac{1}{2}k^2(t) \langle v(\cdot, t), L^* f \rangle_D - \frac{1}{2}a^2(t) \langle v(\cdot, t), f \rangle_D \right) \\ &\quad + \exp(A(t)) \exp(-N_t) \langle g(\exp(N_t)v(\cdot, t)), f \rangle_D \\ &= \frac{1}{2} \exp(A(t))k^2(t) \langle v(\cdot, t), L^* f \rangle_D + \exp(A(t) - N_t) \langle g(\exp(N_t)v(\cdot, t)), f \rangle_D \\ &= \frac{1}{2}k^2(t) \langle w(\cdot, t), L^* f \rangle_D + \exp(A(t) - N_t) \langle g(\exp(N_t - A(t))w(\cdot, t)), f \rangle_D \end{aligned}$$

with boundary conditions $w(x, 0) = \varphi(x)$ for $x \in D$ and $w(x, t) = 0$ for $x \in \partial D$. Therefore w is a weak solution of the RPDE formally given by

$$\frac{d}{dt} w(x, t) = \frac{1}{2}k^2(t)Lw(x, t) + \exp(A(t) - N_t)g(\exp(N_t - A(t))w(x, t)).$$

By the definition of the mild solution

$$w(x, t) = U^D(t, 0)\varphi(x) + \int_0^t \exp(A(s) - N_s)U^D(t, s)g(\exp(N_s - A(s))w(\cdot, s))(x)ds.$$

Consequently,

$$\begin{aligned}
 v(x, t) &= \exp(-A(t))w(x, t) \\
 &= \exp(-A(t)) \left(U^D(t, 0)\varphi(x) + \int_0^t \exp(A(s) - N_s)U^D(t, s)g(\exp(N_s)v(\cdot, s))(x)ds \right) \\
 &= \exp(-A(t))U^D(t, 0)\varphi(x) + \int_0^t ds \exp(-A(t, s) - N_s)U^D(t, s)g(\exp(N_s)v(\cdot, s))(x).
 \end{aligned}$$

□

3.3.3 Equivalence of weak and mild solutions

Next we show the equivalence of weak and mild solutions of (3.1.2) on $[0, T]$.

Theorem 3.3.1. *Consider the RPDE formally given by (3.1.2). Suppose that*

$$\int_0^T (a^2(t) \| v(x, t) \|_D + \exp(-N_t) \| g(\exp(N_t)v(x, t)) \|_D) dt < \infty.$$

Suppose also that $\mathcal{C}_0^2(D)$, the class of twice continuously differentiable functions which vanish outside of D , is contained in $\text{dom}(L^)$.*

- a) Suppose that $(p^D)^*$ satisfies the Kolmogorov backward equation. Then any weak solution of (3.1.2) on $[0, T]$ is a mild solution of (3.1.2) on $[0, T]$.
- b) Suppose that $(p^D)^*$ satisfies the Kolmogorov forward equation. Then any mild solution of (3.1.2) on $[0, T]$ is a weak solution of (3.1.2) on $[0, T]$.

Remark. The operators of Section 3.2 admit $\mathcal{C}_0^2(D)$ as $\text{dom}(L^*)$. Therefore $\text{dom}(L^*) \subset L^2(D)$ is dense in $L^2(D)$ (with respect to $\langle \cdot, \cdot \rangle_D$, see [2], Corollary 2.30).

Proof. a) Assume that v is a weak solution of (3.1.2). Let $h \in C^1(\mathbb{R})$, $f \in \text{dom}(L^*)$, and let $G(x, t) = -\frac{1}{2}a^2(t)v(x, t) + \exp(-N_t)g(\exp(N_t)v(x, t))$. The integration by parts formula is applicable since $h \in C^1(\mathbb{R})$ (see [63] Proposition 9.16) and yields

$$\begin{aligned}
 \langle v(\cdot, t), h(t)f(\cdot) \rangle_D &= \langle v(\cdot, 0), h(0)f(\cdot) \rangle_D + \int_0^t \langle v(\cdot, s), h'(s)f(\cdot) \rangle_D ds \\
 &\quad + \int_0^t \langle v(\cdot, s), \frac{1}{2}h(s)k^2(s)L^*f(\cdot) \rangle_D ds + \int_0^t \langle G(\cdot, s), h(s)f(\cdot) \rangle_D ds.
 \end{aligned} \tag{3.3.7}$$

In order to proceed to the mild solution, the class of test functions of the type $h(s)f(x)$ is not sufficient. We have in fact to apply the Kolmogorov backward equation to $(U^D)^*$ and need therefore test functions ψ of the following form

$$\psi(x, s) := (U^D)^*(t, s)f(x) = \begin{cases} \langle p^D(s, \cdot; t, x), f(\cdot) \rangle_D & \text{if } s < t, \\ f(x) & \text{if } s = t. \end{cases}$$

This can be achieved by a density argument as in ([63], page 152) which allows for test functions in the class $\mathcal{C}^1([0, T], \text{Dom}(L^*))$ in (3.3.7). Alternatively one may approximate functions in $\mathcal{C}^1([0, T], \text{Dom}(L^*))$ by polynomials in (s, x) which satisfy again (3.3.7) as test functions (see [60]). Replacing in (3.3.7) $h(s)f(x)$ by $\psi(s, x)$, we get, for any $t \in [0, T]$ fixed,

$$\begin{aligned} \langle v(\cdot, t), \psi(\cdot, t) \rangle_D &= \langle v(\cdot, 0), \psi(\cdot, 0) \rangle_D + \int_0^t \left\langle v(\cdot, s), \frac{d}{ds} \psi(\cdot, s) + \frac{1}{2} k^2(s) L^* \psi(\cdot, s) \right\rangle_D ds \\ &\quad + \int_0^t \langle G(\cdot, s), \psi(\cdot, s) \rangle_D ds. \end{aligned}$$

Evaluating successively the three terms above we get

$$\begin{aligned} \langle v(\cdot, 0), \psi(\cdot, 0) \rangle_D &= \int_D v(x, 0) \int_D p^D(0, y; t, x) f(y) dy dx \\ &= \int_D f(y) \int_D p^D(0, y; t, x) v(x, 0) dx dy \\ &= \langle U^D(t, 0) v(\cdot, 0), f(\cdot) \rangle_D. \end{aligned}$$

Since $(p^D)^*$ is supposed to satisfy the Kolmogorov backward equation, $(s, x) \rightarrow (p^D)^*(s, x; t, y)$ satisfies (3.3.4) and we get

$$\begin{aligned} -\frac{d}{ds} \psi(\cdot, s) &= -\frac{\partial}{\partial s} \langle (p^D)^*(s, x; t, \cdot), f(\cdot) \rangle_D \\ &= \frac{1}{2} k^2(s) L^* \langle (p^D)^*(s, x; t, \cdot), f(\cdot) \rangle_D = \frac{1}{2} k^2(s) L^* \psi(\cdot, s). \end{aligned}$$

Moreover

$$\begin{aligned} \langle G(\cdot, s), \psi(\cdot, s) \rangle_D &= \int_D G(x, s) \int_D p^D(s, y; t, x) f(y) dy dx \\ &= \int_D f(y) \int_D p^D(s, y; t, x) G(x, s) dx dy \\ &= \langle U^D(t, s) G(\cdot, s), f(\cdot) \rangle_D. \end{aligned}$$

Therefore

$$\langle v(\cdot, t), f(\cdot) \rangle_D = \langle U^D(t, 0) v(\cdot, 0), f(\cdot) \rangle_D + \int_0^t \langle U^D(t, s) G(\cdot, s), f(\cdot) \rangle_D ds.$$

Since, by assumption, $\text{dom}(L^*)$ is dense in $L^2(D)$ v is a mild solution of (3.1.2) on $[0, T]$.

b) To prove the converse let v be a mild solution of (3.1.2) on $[0, T]$, and write

$$\begin{aligned}
 & \int_0^t \langle v(\cdot, s), \frac{1}{2}k^2(s)L^*f(\cdot) \rangle_D ds \\
 &= \int_0^t \langle U^D(s, 0)v(\cdot, 0), \frac{1}{2}k^2(s)L^*f(\cdot) \rangle_D ds \\
 &+ \int_0^t \left\langle \int_0^s \chi_{[0,s]}(r)U^D(s, r)G(\cdot, r) dr, \frac{1}{2}k^2(s)L^*f(\cdot) \right\rangle_D ds \quad (3.3.8) \\
 &= \int_0^t \langle v(\cdot, 0), (U^D)^*(s, 0)\frac{1}{2}k^2(s)L^*f(\cdot) \rangle_D ds \\
 &+ \int_0^t \int_r^t \left\langle U^D(s, r)G(\cdot, r), \frac{1}{2}k^2(s)L^*f(\cdot) \right\rangle_D ds dr.
 \end{aligned}$$

Since $(p^D)^*$ is supposed to satisfy the Kolmogorov forward equation, $(t, y) \rightarrow (p^D)^*(s, x; t, y)$ satisfies (3.3.5), and we get for the first integral on the right side of (3.3.8)

$$\begin{aligned}
 (U^D)^*(s, 0)\left(\frac{1}{2}k^2(s)L^*f\right)(x) &= \int_D (p^D)^*(0, x; s, y)\frac{1}{2}k^2(s)L^*f(y)dy \\
 &= \int_D \left(\frac{1}{2}k^2(s)L\right)(p^D)^*(0, x; s, y)f(y)dy \\
 &= \int_D \frac{\partial}{\partial s}(p^D)^*(0, x; s, y)f(y)dy,
 \end{aligned}$$

and therefore

$$\begin{aligned}
 \int_0^t \langle v(\cdot, 0), (U^D)^*(s, 0)\left(\frac{1}{2}k^2(s)L^*f\right) \rangle_D ds &= \int_0^t \langle v(\cdot, 0), \int_D \frac{\partial}{\partial s}(p^D)^*(0, \cdot; s, y)f(y)dy \rangle_D ds \\
 &= \langle v(\cdot, 0), \int_D (p^D)^*(0, \cdot; t, y)f(y)dy - f(\cdot) \rangle_D \\
 &= \langle v(\cdot, 0), (U^D)^*(t, 0)f(\cdot) \rangle_D - \langle v(\cdot, 0), f(\cdot) \rangle_D
 \end{aligned}$$

In the same way we get for the second integral on the right side of (3.3.8)

$$\begin{aligned}
 \left\langle U^D(s, r)G(\cdot, r), \frac{1}{2}k^2(s)L^*f(\cdot) \right\rangle_D &= \left\langle G(\cdot, r), (U^D)^*(s, r)\left(\frac{1}{2}k^2(s)L^*f\right)(\cdot) \right\rangle_D \\
 &= \left\langle G(\cdot, r), \int_D \frac{\partial}{\partial s}(p^D)^*(r, \cdot; s, y)f(y)dy \right\rangle_D
 \end{aligned}$$

and therefore

$$\begin{aligned}
 \int_r^t \left\langle U^D(s, r)G(\cdot, r), \frac{1}{2}k^2(s)L^*f(\cdot) \right\rangle_D ds &= \int_r^t \left\langle G(\cdot, r), \int_D \frac{\partial}{\partial s}(p^D)^*(r, \cdot; s, y)f(y)dy \right\rangle_D ds \\
 &= \left\langle G(\cdot, r), \int_D (p^D)^*(r, \cdot; t, y)f(y)dy - f(\cdot) \right\rangle_D \\
 &= \langle G(\cdot, r), (U^D)^*(t, r)f(\cdot) - f(\cdot) \rangle_D \\
 &= \langle U^D(t, r)G(\cdot, r), f(\cdot) \rangle_D - \langle G(\cdot, r), f(\cdot) \rangle_D.
 \end{aligned}$$

We have obtained

$$\begin{aligned}
 & \int_0^t \langle v(\cdot, s), \frac{1}{2}k^2(s)L^*f(\cdot) \rangle_D ds \\
 &= \langle U^D(t, 0)v(\cdot, 0) + \int_0^t U^D(t, r)G(\cdot, r)dr, f(\cdot) \rangle_D - \langle v(\cdot, 0), f(\cdot) \rangle_D \\
 & - \int_0^t \langle G(\cdot, r), f(\cdot) \rangle_D dr \\
 &= \langle v(\cdot, t), f(\cdot) \rangle_D - \langle v(\cdot, 0), f(\cdot) \rangle_D - \int_0^t \langle G(\cdot, r), f(\cdot) \rangle_D dr
 \end{aligned}$$

since v is a mild solution on $[0, T]$. It follows that v is a weak solution on $[0, T]$. □

Concerning the existence and the uniqueness of the weak and the mild solutions of (3.1.1) and (3.1.2) we restrict ourselves to cite existing results, because we think that these questions are better treated in the context of more general equations than (3.1.1) and (3.1.2). For the existence and uniqueness of the (local) mild solution of (3.1.2) for general autonomous equations (that is, for time independent differential operators and the associated one-parameter semigroup), we refer to Chapter 6 of [62]. For the existence and the uniqueness of the (local) mild solution of equations with two-parameter semigroups we refer to [48], where equations with respect to Brownian motion have been treated. See also Proposition 3.3.3 below. This result and Theorem 3.3.1 imply the (local) existence and uniqueness of the weak solution of (3.1.2), and Proposition 3.3.1 shows the existence of a weak solution of (3.1.1). The question of uniqueness of the weak solution of (3.1.1) is more delicate, since it requires a more precise description of the function space where it lives. We refer to [60] for an existence and uniqueness theorem of the variational solution of (3.1.1) in the case $a = 0$. For the existence and the uniqueness of the mild solution of (3.1.1) with the same differential operator as in [60] we refer to [57]. At the moment of writing this paper, there is, at our best knowledge, no uniqueness theorem for the weak solution of (3.1.1).

As mentioned above, [48] treats the existence and uniqueness of mild solutions for equations with respect to Brownian motion. We give therefore a proof which is adapted to the context of (3.3.6), even if it is similar to the proof in [48]. We assume that $\varphi \in L^\infty(D)$, where $L^\infty(D)$ is the space of real-valued essentially bounded functions defined on D , and that g satisfies a local Lipschitz condition. For $t > 0$, let

$$E_t := \{v : [0, t] \rightarrow L^\infty(D), |||v||| < \infty\}, \text{ where } |||v||| := \sup_{0 \leq s \leq t} \|v(\cdot, s)\|_\infty.$$

Then $(E_t, |||\cdot|||)$ is a Banach space. In the proof below we use the following closed subsets of E_t

$$C_{R,t} := \{v \in E_t : v \geq 0, |||v||| \leq R\}, \text{ where } R > 0.$$

The spaces $C_{R,t}$ are complete metric spaces.

Proposition 3.3.3. *Let $\varphi \in L^\infty(D)$, $\varphi \geq 0$. Then (3.3.6) has a unique non-negative local mild solution, i.e. there exists $t > 0$ such that (3.3.6) has a mild solution in $L^\infty([0, t] \times D)$.*

Proof. Let $v \in E_t$, where t will be specified later, and let $\Psi(v)$ be given by the right side of (3.3.6), i.e.

$$\begin{aligned} \Psi(v)(x, t) &:= \exp(-A(t))U^D(t, 0)\varphi(x) \\ &+ \int_0^t ds \exp(-N_s - A(t, s))U^D(t, s)g(\exp(N_s)v(\cdot, s))(x). \end{aligned}$$

Since $v \in E_t$ and since N is continuous, there exists $R > 0$ such that

$$||| \exp(N_\cdot)v ||| = \sup_{0 \leq r \leq t} \{ \exp(N_r) \| v(\cdot, r) \|_\infty \} < R.$$

Therefore, denoting by K_R the constant of the local Lipschitz property of g , we get

$$\begin{aligned} ||| \Psi(v) ||| &\leq \| \varphi \|_\infty + \sup_{0 \leq s \leq t} \left\| \int_0^s dr \exp(-N_r - A(s, r))U^D(s, r)g(\exp(N_r)v(\cdot, r)) \right\|_\infty \\ &\leq \| \varphi \|_\infty + \sup_{0 \leq s \leq t} \int_0^s dr \exp(-N_r - A(s, r)) \| U^D(s, r)g(\exp(N_r)v(\cdot, r)) \|_\infty \\ &\leq \| \varphi \|_\infty + \sup_{0 \leq s \leq t} \int_0^s dr \exp(-N_r) \| g(\exp(N_r)v(\cdot, r)) \|_\infty \\ &\leq \| \varphi \|_\infty + K_R \int_0^t dr \exp(-N_r) \| g(0) + \exp(N_r) | v(\cdot, r) | \|_\infty \\ &\leq \| \varphi \|_\infty + K_R \left(g(0) \int_0^t dr \exp(-N_r) + t \sup_{0 \leq r \leq t} \| v(\cdot, r) \|_\infty \right) \\ &= \| \varphi \|_\infty + K_R \left(g(0) \int_0^t dr \exp(-N_r) + t ||| v ||| \right) < +\infty. \end{aligned}$$

Therefore $\Psi(v) \in E_t$. By a similar calculation one shows that Ψ is a contraction in $C_{R,t}$ for suitably chosen R, t . In fact, for $v_1, v_2 \in E_t$ and for $R > 0$ such that $||| \exp(N_\cdot)v_i ||| = \sup_{0 \leq r \leq t} \{ \exp(N_r) \| v_i(\cdot, r) \|_\infty \} < R$ for $i = 1, 2$, we have

$$\begin{aligned} &||| \Psi(v_1) - \Psi(v_2) ||| \\ &= \sup_{0 \leq s \leq t} \left\| \int_0^s dr \exp(-N_r - A(s, r))U^D(s, r) (g(\exp(N_r)v_1(\cdot, r)) - g(\exp(N_r)v_2(\cdot, r))) \right\|_\infty \\ &\leq \sup_{0 \leq s \leq t} \int_0^s dr \exp(-N_r) \| g(\exp(N_r)v_1(\cdot, r)) - g(\exp(N_r)v_2(\cdot, r)) \|_\infty \\ &\leq K_R \int_0^t dr \| v_1(\cdot, r) - v_2(\cdot, r) \|_\infty \\ &\leq K_R t \sup_{0 \leq r \leq t} \| v_1(\cdot, r) - v_2(\cdot, r) \|_\infty = K_R t ||| v_1 - v_2 |||. \end{aligned}$$

For any $v_1, v_2 \in E_t$ it is therefore possible to find R large enough and then t small enough, such that $K_R t \leq \frac{1}{2}$. Let now $N^* := \sup_{0 \leq r \leq t} \exp(N_r)$, and choose $\|v_i\| \leq R/N^*$, $i = 1, 2$. Then $\|\exp(N)v_i\| \leq N^* \|v_i\| \leq R$. This means that Ψ is a contraction on $C_{R/N^*, t}$ for t small enough. By the Banach fixed-point theorem Ψ has exactly one fixed point which is the local mild solution of (3.3.6). □

3.4 An upper bound for the blowup time and probability estimates by means of the weak solution

By Proposition 3.3.1, the weak solutions of (3.1.1) and (3.1.2) have the same blowup time which we denote by τ . In this section we derive and evaluate an upper bound for τ on $\{\omega \in \Omega, \tau(\omega) < \infty\}$ by means of the blowup time τ^* of a subsolution of (3.1.2). This subsolution is obtained by transforming (3.1.2) into a differential inequality in time and by solving the associated differential equality explicitly. The explicit expression for τ^* is given in (3.4.2), and an upper bound for the probability that τ^* occurs before a fixed time T is given in (3.4.7) for sufficiently small initial conditions φ . For this result we restrict ourselves to the case where k, a and b^H are constants. In subsection 3.4.3 we suppose $H > 3/4$ which implies that $M := B + cB^H$ ($c \in \mathbb{R}$) is equivalent to a Brownian motion. This allows for a lower bound of the probability that $\tau < \infty$. The lower bound is shown under the hypothesis (3.4.9) which controls the decrease of an exponential functional of $\tilde{A}(t) := \int_0^t a^2(s) ds = 2A(t)$ by an exponential functional of $K(t) := \frac{1}{2} \int_0^t k^2(s) ds$. The standing hypotheses for this section are

$$(H_1) \quad g(z) \geq Cz^{1+\beta} \tag{3.4.1}$$

for all $z > 0$, where $C > 0, \beta > 0$ are given constants.

$$(H_2) \quad L^* \text{ is supposed to have an eigenfunction } \phi_0 > 0, \text{ associated to an eigenvalue } \lambda_0 > 0 \text{ and normalized by } \int_D \phi_0(x) dx = 1.$$

We refer to Section 3.2 for classes of operators which satisfy (H_2) . In particular, (H_2) implies

$$\langle v(\cdot, t), L^* \phi_0 \rangle_D = -\lambda_0 \langle v(\cdot, t), \phi_0 \rangle_D$$

for any $v(\cdot, t) \in L^2(D)$.

3.4.1 An upper bound for the blowup time

Theorem 3.4.1. *Let*

$$\tau^* = \inf\{t > 0 : \int_0^t dr \exp[-\beta(\lambda_0 K(r) + A(r)) + \beta N_r] \geq \frac{1}{C\beta} \langle \varphi, \phi_0 \rangle^{-\beta}\}. \quad (3.4.2)$$

On $\{\tau^* < \infty\}$ the solution $v(\cdot, t)$ of (3.1.2) and the solution $u(\cdot, t) = \exp(N_t)v(\cdot, t)$ of (3.1.1) blow up in finite time, and the blowup time is upper bounded by τ^* $P - a.s.$

Proof. By hypothesis $(H)_1$ and Jensen's inequality, we get

$$\int_D dx \exp(-N_r) g(\exp(N_r)v(x, r)) \phi_0(x) \geq C \int_D dx \exp(\beta N_r) v^{1+\beta}(x, r) \phi_0(x),$$

and

$$\int_D dx v^{1+\beta}(x, r) \phi_0(x) \geq \left(\int_D dx v(x, r) \phi_0(x) \right)^{1+\beta} = \langle v(\cdot, r), \phi_0 \rangle_D^{1+\beta}.$$

Since v is the weak solution of (3.1.2).

$$\begin{aligned} \langle v(\cdot, t), \phi_0 \rangle_D &= \langle \varphi, \phi_0 \rangle_D + \frac{1}{2} \int_0^t dr (k^2(r) \langle v(\cdot, r), L^* \phi_0 \rangle_D - a^2(r) \langle v(\cdot, r), \phi_0 \rangle_D) \\ &\quad + \int_0^t dr \exp(-N_r) \langle g(\exp(N_r)v(\cdot, r)), \phi_0 \rangle_D \\ &\geq \langle \varphi, \phi_0 \rangle_D - \frac{1}{2} \int_0^t dr (\lambda_0 k^2(r) + a^2(r)) \langle v(\cdot, r), \phi_0 \rangle_D \\ &\quad + C \int_0^t dr \exp(\beta N_r) \langle v(\cdot, r), \phi_0 \rangle_D^{1+\beta}. \end{aligned}$$

Applying this inequality to $\langle v(\cdot, t + \varepsilon), \phi_0 \rangle_D - \langle v(\cdot, t), \phi_0 \rangle_D$ ($\varepsilon > 0$), we get

$$\begin{aligned} \frac{1}{\varepsilon} (\langle v(\cdot, t + \varepsilon), \phi_0 \rangle_D - \langle v(\cdot, t), \phi_0 \rangle_D) &\geq -\frac{1}{2\varepsilon} \int_t^{t+\varepsilon} dr (\lambda_0 k^2(r) + a^2(r)) \langle v(\cdot, r), \phi_0 \rangle_D \\ &\quad + \frac{C}{\varepsilon} \int_t^{t+\varepsilon} dr \exp(\beta N_r) \langle v(\cdot, r), \phi_0 \rangle_D^{1+\beta}. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we get the differential inequality

$$\frac{d}{dt} \langle v(\cdot, t), \phi_0 \rangle_D \geq -\frac{1}{2} (\lambda_0 k^2(t) + a^2(t)) \langle v(\cdot, t), \phi_0 \rangle_D + C \exp(\beta N_t) \langle v(\cdot, t), \phi_0 \rangle_D^{1+\beta}. \quad (3.4.3)$$

Let us consider the corresponding differential equality

$$\frac{d}{dt} I(t) = -\frac{1}{2} (\lambda_0 k^2(t) + a^2(t)) I(t) + C \exp(\beta N_t) I(t)^{1+\beta}.$$

Its solution, denoted again by $I(t)$, is therefore a subsolution of (3.4.3),

i.e. $\langle v(\cdot, t), \phi_0 \rangle_D \geq I(t)$. $I(t)$ can be calculated explicitly (see [4] for a proof) :

$$I(t) = \exp(-(\lambda_0 K(t) + A(t))) \left(\langle \varphi, \phi_0 \rangle_D^{-\beta} - \beta C \int_0^t dr \exp[-\beta(\lambda_0 K(r) + A(r)) + \beta N_r] \right)^{-1/\beta}$$

for all $t \in [0, \tau^*)$, where τ^* is given by (3.4.2). Therefore τ^* is an upper bound for the blowup time of $\langle v(\cdot, t), \phi_0 \rangle_D$, and the function

$$t \mapsto \|v(\cdot, t)\|_\infty = \exp(-N_t) \|u(\cdot, t)\|_\infty$$

can not stay finite on $[0, \tau^*]$ if $\tau^* < \infty$. Therefore u and v blow up before τ^* if $\tau^* < \infty$. \square

Remarks.

1. τ^* depends on L only by the positive eigenvalue λ_0 and the associated eigenfunction ϕ_0 . τ^* is a decreasing function of φ , ϕ_0 and C , and it is an increasing function of functions A , K and λ_0 . Therefore small functions φ , ϕ_0 and a small constant C , as well as huge functions K and A may postpone or prevent the blowup of I .
2. The behavior as $t \rightarrow \infty$ of the exponential functional in (3.4.2) can be estimated, for K and A fixed, by the laws of iterated logarithm for B and B^H . In particular, if $a = 0$, b^H constant and $k(t) > \sqrt{Ht}^{\frac{H-1}{2}}$, K dominates B^H at the limit as $t \rightarrow \infty$, and this may postpone or prevent the blowup of I . If a , b^H and k are constants, the exponent in (3.4.2) reads

$$-\beta(\lambda_0 k^2 t + a^2 t)/2 + \beta N_t$$

and, by the law of the iterated logarithm again, the behavior of the exponent at the limit as $t \rightarrow \infty$ is determined by $(\lambda_0 k^2 + a^2)t$.

3.4.2 A tail probability estimate for the upper bound of the blowup time

The preceding subsection shows that the presence of a stochastic part in (3.1.2) renders difficult the trajectorywise evaluation of the blowup behavior of the solution. Therefore estimates of the probability of blowup may be useful. In this subsection we apply a tail probability estimate for exponential functionals due to N.T. Dung [25] to estimate the probability that τ^* occurs before a fixed time. We assume here that k , a and b^H are constants.

Let us recall the (Volterra) kernel representation of fractional Brownian given by

$$B_t^H = \int_0^t K^H(t, s) dW_s,$$

where the kernel K^H is given for $H > 1/2$ by

$$K^H(t, s) = \begin{cases} C_H s^{1/2-H} \int_s^t (\sigma - s)^{H-3/2} \sigma^{H-1/2} d\sigma & \text{if } t > s, \\ 0 & \text{if } t \leq s, \end{cases}$$

where

$$C_H = \sqrt{\frac{H(2H-1)}{\mathcal{B}(2-2H, H-\frac{1}{2})}}, \quad \text{and } \mathcal{B}(a, b) = \int_0^1 (1-x)^{a-1} x^{b-1} dx, \quad a, b > 0$$

is the *beta* function. Let $X_t = \alpha_1 t + \alpha_2 B_t + \alpha_3 B_t^H$, where $\alpha_1 < 0, \alpha_2$ and α_3 are constants. The estimates for the tail probabilities of the exponential functional $\int_0^T e^{X_t} dt$ in [25] are given in terms of the Malliavin derivative of X_t , usually denoted by $D_r X_t, r > 0$. In particular, $D_r B_t^H = K^H(t, r)$ for all $r < t$ and $D_r B_t^H = 0$ for $r \geq t$.

We recall briefly the notions we need from this calculus and refer for more details to [59]. For $h \in L^2[0, T]$ let $B^H(h) = \int_0^T h(t) dB^H(t)$. Let \mathcal{S} be the set of random variables $F = f(B^H(h_1), B^H(h_2), \dots, B^H(h_n))$, where $n \in \mathbb{N}, f \in \mathcal{C}_b^\infty(\mathbb{R}^n), h_1, h_2, \dots, h_n \in L^2[0, T]$. Let $\mathbb{D}^{1,2}$ denote the closure of \mathcal{S} in $L^2(\Omega, \mathcal{F}, P)$ with respect to the norm

$$\| F \|_{1,2}^2 := \mathbb{E} | F |^2 + \mathbb{E} \int_0^T | D_u F |^2 du.$$

Theorem 3.4.2. ([25]) *Let $(X_t)_{t \in [0, T]}$ be a continuous stochastic process in $\mathbb{D}^{1,2}$. Assume that one of the following two conditions holds for some $M > 0$:*

$$\sup_{t \in [0, T]} \int_0^T | D_r X_t |^2 dr \leq M^2 \quad P - a.s., \quad (3.4.4)$$

$$\int_0^T E \left[\sup_{t \in [0, T]} | D_r X_t |^2 | \mathcal{F}_r \right] dr \leq M^2 \quad P - a.s.$$

Then

$$P \left(\int_0^T e^{X_t} dt \geq x \right) \leq 2 \exp \left(- \frac{(\ln x - \ln \mu)^2}{2M^2} \right) \quad (3.4.5)$$

for all $x > \mu$, where $\mu = \int_0^T \mathbb{E} [e^{X_t}] dt$.

Lemma 3.4.1. *Let $(X_t)_{t \in [0, T]}$ be given by $X_t = \alpha_1 t + \alpha_2 B_t + \alpha_3 B_t^H$. Then $X_t \in \mathbb{D}^{1,2}$ for all $t \in [0, T]$ and satisfies (3.4.4) with*

$$M^2 = \alpha_2^2 T + \alpha_3^2 T^{2H} + 2\alpha_2 \alpha_3 C_H (H - \frac{1}{2})^{-1} \mathcal{B}(\frac{3}{2} - H, H + \frac{1}{2}) T^{H+\frac{1}{2}}. \quad (3.4.6)$$

Proof. It is sufficient to verify that $(X_t)_{t \in [0, T]}$ is in $\mathbb{D}^{1,2}$, and (3.4.4) holds. For $r \leq t$, we

have $D_r X_t = \alpha_2 + \alpha_3 K^H(t, r)$. Furthermore

$$\begin{aligned} \sup_{t \in [0, T]} \int_0^T |D_r X_t|^2 dr &= \sup_{t \in [0, T]} \int_0^T |\alpha_2 + \alpha_3 K^H(t, r)|^2 dr \\ &= \sup_{t \in [0, T]} \int_0^T (\alpha_2^2 + 2\alpha_2\alpha_3 K^H(t, r) + \alpha_3^2 K^H(t, r)^2) dr, \end{aligned}$$

where

$$\sup_{t \in [0, T]} \int_0^T (K^H(t, r))^2 dr = \sup_{t \in [0, T]} \int_0^t (K^H(t, r))^2 dr = \sup_{t \in [0, T]} \mathbb{E}|B_t^H|^2 = T^{2H},$$

and, for $r \leq t$,

$$\begin{aligned} \int_0^t K^H(t, r) dr &= C_H \int_0^t r^{\frac{1}{2}-H} \int_r^t (\sigma - r)^{H-\frac{3}{2}} \sigma^{H-\frac{1}{2}} d\sigma dr \\ &\leq C_H \int_0^t r^{\frac{1}{2}-H} \int_r^t (\sigma - r)^{H-\frac{3}{2}} d\sigma dr t^{H-\frac{1}{2}} \\ &= C_H (H - \frac{1}{2})^{-1} \int_0^t r^{\frac{1}{2}-H} (t - r)^{H-\frac{1}{2}} dr t^{H-\frac{1}{2}} \\ &= C_H (H - \frac{1}{2})^{-1} \mathcal{B}(\frac{3}{2} - H, \frac{1}{2} + H) t^{H+\frac{1}{2}}. \end{aligned}$$

Therefore

$$\sup_{t \in [0, T]} \int_0^T K^H(t, r) dr \leq C_H (H - \frac{1}{2})^{-1} T^{H+\frac{1}{2}} \mathcal{B}(\frac{3}{2} - H, \frac{1}{2} + H).$$

We get

$$\sup_{t \in [0, T]} \int_0^T |D_r X_t|^2 dr \leq \alpha_2^2 T + \alpha_3^2 T^{2H} + 2\alpha_2\alpha_3 C_H (H - \frac{1}{2})^{-1} \mathcal{B}(\frac{3}{2} - H, H + \frac{1}{2}) T^{H+\frac{1}{2}}.$$

□

Let

$$\xi := \frac{1}{C\beta} \langle \varphi, \phi_0 \rangle^{-\beta}.$$

Then

$$P \{ \tau_* \leq T \} = P \left\{ \int_0^T dr \exp(-\beta r(\lambda_0 k^2 + a^2)/2 + \beta(aB_t + b^H B_t^H)) \geq \xi \right\}.$$

By applying (3.4.5) with

$$\alpha_1 = -\beta(\lambda_0 k^2 + a^2)/2, \quad \alpha_2 = \beta a, \quad \alpha_3 = \beta b^H$$

we get

Proposition 3.4.1. *Let*

$$\mu := \int_0^T \mathbb{E} [e^{X_t}] dt = \int_0^T \exp(-\beta t(\lambda_0 k^2 + a^2)/2) \mathbb{E} [\exp(\beta(aB_t + b^H B_t^H))] dt.$$

If $\xi > \mu$, then

$$\mathbb{P} \{ \tau^* \leq T \} \leq 2 \exp \left(-\frac{(\ln \xi - \ln \mu)^2}{2M^2} \right). \quad (3.4.7)$$

where

$$M^2 = a^2 \beta^2 T + (b^H)^2 \beta^2 T^{2H} + 2ab^H \beta^2 C_H \mathcal{B} \left(\frac{3}{2} - H, H + \frac{1}{2} \right) T^{H+\frac{1}{2}}. \quad (3.4.8)$$

Remark. If B and B^H are supposed to be independent

$$\mu = \int_0^T \exp(-\beta t(\lambda_0 k^2 + a^2)/2) \exp((a\beta)^2 t/2 + (b^H \beta)^2 t^{2H}/2) dt.$$

Again, the estimate in (3.4.7) depends on L only via the positive eigenvalue λ_0 and the associated eigenfunction ϕ_0 . The estimate is applicable if the initial condition φ is sufficiently small.

3.4.3 A lower bound for the probability of finite time blowup in the case $H > 3/4$

In this subsection we assume that

- i) $H \in (3/4, 1)$,
- ii) $b^H(r) = ca(r)$, for all $r \in \mathbb{R}_+$, where c is a constant.

Then, we have

$$N_t = \int_0^t a(r) dM_r, \quad \text{where } M_r := B_r + cB_r^H.$$

Since $H > 3/4$, the mixed model M is equivalent to a Brownian motion \tilde{W} (see [16]), and therefore N_t is equivalent to $\tilde{N}_t := \int_0^t a(r) d\tilde{W}_r$. Notice that equivalence here means equality, for all $T > 0$, of the laws of the processes on $(\mathcal{C}[0, T], \mathcal{B})$, the space of continuous functions defined on $[0, T]$ with the σ -algebra generated by the cylinder sets. Furthermore, $(\tilde{N}_t)_{t \geq 0}$ is a continuous martingale and therefore a time-changed Brownian motion: $\tilde{N}_t = \tilde{B}_{\tilde{A}(t)}$, where $(\tilde{B}_t)_{t \geq 0}$ is a Brownian motion and $\tilde{A}(t) = \int_0^t a^2(r) dr$. The proof of next result follows closely [4].

Theorem 3.4.3. *We assume that k and a are positive continuous functions on \mathbb{R}_+ and that there exist constants $\eta \in]0, +\infty]$ and $c_1 > 0$ such that*

$$\frac{1}{a^2(t)} \exp(-\beta \lambda_0 K(t)) \geq c_1 \exp(-\beta \frac{\tilde{A}(t)}{\eta}), \quad t \in \mathbb{R}_+. \quad (3.4.9)$$

Then

$$P(\tau < +\infty) \geq \mathbb{P}(\tilde{Z}_{\tilde{\mu}} \leq \theta) = \int_0^\theta y^{\tilde{\mu}-1} e^{-y} dy \quad (3.4.10)$$

where τ is the blowup time of (3.1.2) and (3.1.1), $\tilde{Z}_{\tilde{\mu}}$ is a gamma distributed random variable with parameter $\tilde{\mu} := \frac{2}{\beta}(\frac{1}{\eta} + \frac{1}{2})$, $\theta := \frac{2c_1}{\beta^2 \xi}$ and $\xi := \frac{1}{C\beta} \langle \varphi, \phi_0 \rangle^{-\beta}$.

Proof. We need to know explicitly the law of the functional

$$\int_0^\infty dr \exp[-\beta(\lambda_0 K(r) + A(r)) + \beta \tilde{N}_r]. \quad (3.4.11)$$

The hypothesis (3.4.9) is chosen in such a way that this law is the gamma distribution with parameter $\tilde{\mu}$. In fact

$$\begin{aligned} P(\tau^* = +\infty) &= P\left(\int_0^t dr \exp[-\beta(\lambda_0 K(r) + A(r)) + \beta \tilde{N}_r] < \xi, \text{ for all } t > 0\right) \\ &= P\left(\int_0^\infty dr \exp[-\beta(\lambda_0 K(r) + A(r)) + \beta \tilde{N}_r] \leq \xi\right), \end{aligned}$$

where $\xi := \frac{1}{C\beta} \langle \varphi, \phi_0 \rangle^{-\beta}$.

By the change of variable $q = \tilde{A}(r) \Leftrightarrow dq = a^2(r)dr$, we get

$$\begin{aligned} P(\tau^* = +\infty) &= P\left(\int_0^\infty dr \exp[-\beta(\lambda_0 K(r) + \frac{1}{2} \tilde{A}(r)) + \beta \tilde{B}_{\tilde{A}(r)}] \leq \xi\right) \\ &= P\left(\int_0^\infty \frac{dq}{a^2(\tilde{A}^{-1}(q))} \exp[-\beta(\lambda_0 K(\tilde{A}^{-1}(q)) + \frac{1}{2} \tilde{A}(\tilde{A}^{-1}(q))) + \beta \tilde{B}_{\tilde{A}(\tilde{A}^{-1}(q))}] \leq \xi\right) \\ &= P\left(\int_0^\infty \frac{dq}{a^2(\tilde{A}^{-1}(q))} \exp[-\beta(\lambda_0 K(\tilde{A}^{-1}(q)) + \frac{1}{2} q) + \beta \tilde{B}_q] \leq \xi\right). \end{aligned}$$

Applying (3.4.9) to $\tilde{A}^{-1}(q)$, we obtain

$$\frac{1}{a^2(\tilde{A}^{-1}(q))} \exp[-\beta(\lambda_0 K(\tilde{A}^{-1}(q)))] \geq c_1 \exp(-\frac{\beta}{\eta} q), \quad q \in \mathbb{R}_+.$$

Therefore, we get

$$\begin{aligned} P(\tau^* = +\infty) &\leq P\left(c_1 \int_0^\infty dq \exp[-\beta q(\frac{1}{\eta} + \frac{1}{2}) + \beta \tilde{B}_q] \leq \xi\right) \\ &= P\left(\int_0^\infty dq \exp[\beta(\tilde{B}_q - \mu q)] \leq \frac{\xi}{c_1}\right) \end{aligned}$$

with $\mu := \frac{1}{\eta} + \frac{1}{2}$.

By a second change of variable $q = \frac{4s}{\beta^2} \Leftrightarrow dq = \frac{4}{\beta^2} ds$, we get

$$\begin{aligned} P(\tau^* = +\infty) &\leq P\left(\int_0^\infty ds \frac{4}{\beta^2} \exp[\beta(\tilde{B}_{\frac{4s}{\beta^2}} - \mu \frac{4s}{\beta^2})] \leq \frac{\xi}{c_1}\right) \\ &= P\left(\int_0^\infty ds \exp[2(\tilde{B}_s - \tilde{\mu}s)] \leq \frac{\beta^2 \xi}{4c_1}\right) \end{aligned}$$

with $\tilde{\mu} := \mu \frac{2}{\beta}$, where the scaling property of Brownian motion has been applied, i.e

$$\tilde{B}_{\frac{4s}{\beta^2}} \stackrel{\mathcal{L}}{=} \frac{2}{\beta} \tilde{B}_s.$$

By corollary 1.2, page 95 in [79],

$$\int_0^\infty ds e^{2(\tilde{B}_s - \tilde{\mu}s)} \stackrel{\mathcal{L}}{=} \frac{1}{2\tilde{Z}_{\tilde{\mu}}}.$$

Therefore

$$\begin{aligned} P(\tau^* = +\infty) &\leq P\left(\frac{1}{2\tilde{Z}_{\tilde{\mu}}} \leq \frac{\beta^2 \xi}{4c_1}\right) = P(\tilde{Z}_{\tilde{\mu}} \geq \frac{2c_1}{\beta^2 \xi}) \\ &= 1 - P(\tilde{Z}_{\tilde{\mu}} \leq \theta) = 1 - \frac{1}{\Gamma(\tilde{\mu})} \int_0^\theta y^{\tilde{\mu}-1} e^{-y} dy, \end{aligned}$$

where $\theta := \frac{2c_1}{\beta^2 \xi}$. Then, we get

$$P(\tau^* < +\infty) = 1 - P(\tau^* = +\infty) \geq \frac{1}{\Gamma(\tilde{\mu})} \int_0^\theta y^{\tilde{\mu}-1} e^{-y} dy.$$

Since $\tau \leq \tau^*$ where τ is the blowup time of u in (3.1.1),

$$P(\tau < +\infty) \geq P(\tau^* < +\infty)$$

and the probability for u to blow up in finite time is lower bounded by $\frac{1}{\Gamma(\tilde{\mu})} \int_0^\theta y^{\tilde{\mu}-1} e^{-y} dy$. \square

3.5 Methods based on the mild form of the equation

In this section we consider the mild form of (3.1.2) obtained in Proposition 3.3.2 of subsection 3.3.2 (see (3.3.6)). Recall the twoparameter semigroup

$$(\exp(-A(t, r))U^D(t, r))_{0 \leq r < t}$$

and the associated infinitesimal generator $L_t = \frac{1}{2}k^2(t)L - \frac{1}{2}a^2(t)$.

3.5.1 A sufficient condition for blowup in finite time

The aim of this subsection is to derive a sufficient condition for the blowup in finite time of the solution of (3.3.6). It stems from [46], where the blowup behavior of (deterministic) non-linear heat equations are studied with non-linearities of the form $h(t)g(v(x, t))$, $t > 0$, $x \in D$, where $h \in \mathcal{C}[0, \infty)$ and g is supposed to be locally Lipschitz. The authors' sufficient condition for finite time blowup of the solution v is the impossibility to control

$$t \mapsto \int_0^t h(s) ds \quad \text{by} \quad t \mapsto \int_{\|\exp(-A(t))U^D(t,0)\varphi\|_\infty}^\infty \frac{dz}{g(z)}.$$

In this subsection we assume again that (H_1) holds. This allows to lower bound the nonlinear term in (3.1.2) as follows

$$\exp(-N_t)g(\exp(N_t)v(x, t)) \geq C \exp(\beta N_t)v(x, t)^{1+\beta}.$$

Theorem 3.5.1. *Suppose that (H_1) holds, and suppose that there exists $t > 0$ with*

$$\int_{\|\exp(-A(t))U^D(t,0)\varphi\|_\infty}^\infty \frac{dz}{z^{1+\beta}} < C \int_0^t \exp(\beta N_s) ds. \quad (3.5.1)$$

Then the solutions $v(\cdot, t)$ of (3.1.2) and $u(\cdot, t) = \exp(N_t)v(\cdot, t)$ of (3.1.1) blow up in finite time.

Remarks: a) An interesting feature of the inequality (3.5.1) is that the influences of the differential operator and of the stochastic term on the blowup behavior are separated and appear on different sides of (3.5.1). (3.5.1) is understood trajectorywise. As in subsection 3.4.1 we see that, in the presence of a stochastic term, the trajectorywise evaluation of the blowup behavior of the solution is difficult. It would therefore be of interest to know the law of the right side of the inequality.

The left side shows that the function a (but not b^H) has an explicit influence on the blowup behavior of the solution. In fact, huge values of a keep the lower bound of the integration interval low and the value of the integral high. The influence of φ is opposite: large values are in favour of finite time blowup. This confirms the statement after Theorem 3.4.2 that high values of a (small values of φ) may increase the blowup time τ^* or may even prevent the blowup in finite time. Notice also that high values of C are in favour of finite time blowup.

b) (3.5.1) is of course simple to evaluate in the absence of a stochastic part, i.e. when $N = 0$. If we assume that (H_2) holds and we choose $\varphi = \psi_0$ (the positive eigenfunction of L), then (3.5.1) reads as follows

$$\frac{1}{\beta} \exp[-\beta(\lambda_0 K(t) + A(t))] \|\psi_0\|_\infty^{-\beta} < Ct.$$

Proof. Suppose that $(v(x, t), x \in D, t \in [0, T])$ is a global solution of (3.1.2), and let $0 < t < t' \leq T$. Then, by the semigroup property of $(U^D(t, r))_{0 \leq r < t \leq T}$ (see subsection 3.3.2)

$$\begin{aligned}
 & \exp(-A(t', t))U^D(t', t)v(x, t) \\
 = & \exp(-A(t', t))U^D(t', t) \exp(-A(t))U^D(t, 0)\varphi(x) \\
 & + \exp(-A(t', t))U^D(t', t) \int_0^t \exp(-N_r) \exp(-A(t, r))U^D(t, r)g(\exp(N_r)v(\cdot, r))(x)dr \\
 = & \exp(-A(t'))U^D(t', 0)\varphi(x) \\
 & + \int_0^t \exp(-N_r) \exp(-A(t', r))U^D(t', r)g(\exp(N_r)v(\cdot, r))(x)dr \\
 \geq & \exp(-A(t'))U^D(t', 0)\varphi(\cdot)(x) \\
 & + C \int_0^t \exp(\beta N_r) \exp(-A(t', r))U^D(t', r)v(\cdot, r)^{1+\beta}(x)dr
 \end{aligned}$$

By Jensen's inequality

$$\begin{aligned}
 U^D(t', r)v(\cdot, r)^{1+\beta}(x) &= \int_D p^D(r, x; t', y)v(y, r)^{1+\beta}dy \\
 &\geq \left(\int_D p^D(r, x; t', y)v(y, r)dy \right)^{1+\beta} = (U^D(t', r)v(\cdot, r)(x))^{1+\beta}
 \end{aligned}$$

Therefore

$$\begin{aligned}
 \exp(-A(t', t))U^D(t', t)v(\cdot, t)(x) &\geq \exp(-A(t'))U^D(t', 0)\varphi(x) \\
 &+ C \int_0^t \exp(\beta N_r) (\exp(-A(t', r))U^D(t', r)v(\cdot, r)(x))^{1+\beta} dr
 \end{aligned} \tag{3.5.2}$$

Let $\psi(t)$ be the right side of the inequality above, i.e.

$$\begin{aligned}
 \psi(t) &: = \exp(-A(t'))U^D(t', 0)\varphi(x) \\
 &+ C \int_0^t \exp(\beta N_r) (\exp(-A(t', r))U^D(t', r)v(\cdot, r)(x))^{1+\beta} dr
 \end{aligned}$$

Since $z \mapsto z^{1+\beta}$ is increasing, (3.5.2) implies

$$\begin{aligned}
 \psi'(t) &= C \exp(\beta N_t) (\exp(-A(t', t))U^D(t', t)v(\cdot, t)(x))^{1+\beta} \\
 &\geq C \exp(\beta N_t)(\psi(t))^{1+\beta}
 \end{aligned}$$

Let now $\Psi(t) := \int_t^\infty dz/z^{1+\beta}, t > 0$. Then

$$\frac{d}{dt}\Psi(\psi(t)) = -\frac{\psi'(t)}{(\psi(t))^{1+\beta}} \leq -C \exp(\beta N_t)$$

Therefore

$$\begin{aligned} C \int_0^{t'} \exp(\beta N_s) ds &\leq \Psi(\psi(0)) - \Psi(\psi(t')) \\ &= \int_{\psi(0)}^{\psi(t')} dz/z^{1+\beta} < \int_{\exp(-A(t'))U^D(t',0)\varphi(\cdot)(x)}^{\infty} dz/z^{1+\beta} \end{aligned}$$

for all $x \in D$ and all $t' > 0$. Therefore

$$C \int_0^{t'} \exp(\beta N_s) ds \leq \int_{\|\exp(-A(t'))U^D(t',0)\varphi\|_\infty}^{\infty} dz/z^{1+\beta}$$

for all $t' > 0$. This contradicts (3.5.1). \square

3.5.2 A lower bound for the blowup time and a sufficient condition for the existence of a global solution

A rather widely applicable method to provide a lower bound for the blowup time and a sufficient condition for the existence of a global solution of (3.1.2) and (3.1.1) can be found in [22] and [4]. The condition is in terms the semigroup $(\exp(-A(t,r))U^D(t,r))_{0 \leq r < t}$ appearing in (3.3.6) and requires a hypothesis on the locally Lipschitz function $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which differs from the hypothesis made so far. It reads

(H₁)' $g(0) = 0$, $z \rightarrow g(z)/z$ is increasing, and

$$g(z) \leq \Lambda z^{1+\beta}, \quad z > 0, \quad \text{for some constants } \Lambda, \beta > 0. \quad (3.5.3)$$

Since the proofs follow closely those in [4] (where $b^H = 0$), we only state the results here.

Theorem 3.5.2. *Let*

$$J(t) = \left(1 - \Lambda \beta \int_0^t \exp(\beta(N_r - A(r))) \|U^D(r,0)\varphi\|_\infty^\beta dr \right)^{-1/\beta}$$

for all $0 \leq t < \tau_*$, where

$$\tau_* = \inf\left\{ t > 0, \int_0^t \exp(\beta(N_r - A(r))) \|U^D(r,0)\varphi\|_\infty^\beta dr \geq \frac{1}{\Lambda\beta} \right\}. \quad (3.5.4)$$

Then the solutions v of (3.1.2) and $u(\cdot, t) = \exp(N_t)v(\cdot, t)$ of (3.1.1) satisfy, for $x \in D$, $0 \leq t < \tau_*$, P -a.s.

$$0 \leq u(x, t) \leq J(t) \exp(N_t - A(t))U^D(t,0)\varphi(x),$$

$$0 \leq v(x, t) \leq J(t) \exp(-A(t))U^D(t,0)\varphi(x). \quad (3.5.5)$$

Remark. More precisely, the proof of this theorem shows that the mild solution of (3.1.2) satisfies (3.5.5). By the results of Section 3.3 v is also the weak solution of (3.1.2), and the weak solution $u(\cdot, t) = \exp(N_t)v(\cdot, t)$ of (3.1.1) satisfies (3.5.5).

Corollary 3.5.1. *Assume that*

$$\Lambda\beta \int_0^\infty \exp(\beta(N_r - A(r))) \|U^D(r, 0)\varphi\|_\infty^\beta dr < 1.$$

Then the solutions u and v of (3.1.1) and (3.1.2) satisfy (3.5.5) P - a.s. for all t .

Remark. Suppose that p^D is symmetric. For the special case $\varphi = p\psi_0$, $p > 0$, the integrals appearing in (3.4.2) and (3.5.4) are the same exponential functionals of N . In fact,

$$U^D(r, 0)\psi_0 = \exp(-\lambda_0 K(r))\psi_0,$$

and τ_* becomes

$$\tau_* = \inf\{t > 0, \int_0^t \exp(\beta(N_r - \lambda_0 K(r) - A(r)))dr \geq \frac{p^{-\beta}}{\Lambda\beta} \|\psi_0\|_\infty^{-\beta}\}, \quad (3.5.6)$$

whereas

$$\tau^* = \inf\{t > 0 : \int_0^t \exp(\beta(N_r - \lambda_0 K(r) - A(r)))dr \geq \frac{p^{-\beta}}{C\beta} \|\psi_0\|^{-2\beta}\}. \quad (3.5.7)$$

In fact $\tau_* \leq \tau^*$ if $C \leq \Lambda$, since

$$\|\psi_0\|^2 = \int_D \psi_0^2(x)dx \leq \|\psi_0\|_\infty \int_D \psi_0(x)dx = \|\psi_0\|_\infty.$$

In order to apply both bounds simultaneously, we have to suppose that both, (3.4.1) and (3.5.3), are satisfied, i.e. $Cz^{1+\beta} \leq g(z) \leq \Lambda z^{1+\beta}$, $z > 0$.

It is therefore of interest to know the law of the integral appearing in (3.5.6) and (3.5.7). This seems possible only for $b^H = 0$, since, to our best knowledge, the law of exponential functionals of fractional Brownian motion is still unknown. For the moment it seems that only estimates of the type of those in subsection 3.4.2 are available. See also subsection 3.4.3 for $H > 3/4$.

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Abstract

This thesis is concerned with stochastic partial differential equations of parabolic type. In the first part we prove new results regarding the existence and the uniqueness of global and local variational solutions to a Neumann initial-boundary value problem for a class of non-autonomous stochastic parabolic partial differential equations. The equations we consider are defined on unbounded open domains in Euclidean space satisfying certain geometric conditions, and are driven by a multiplicative noise derived from an infinite-dimensional fractional Wiener process characterized by a sequence of Hurst parameters $H = (H_i)_{i \in \mathbb{N}^+} \subset (\frac{1}{2}, 1)$. These parameters are in fact subject to further constraints that are intimately tied up with the nature of the nonlinearity in the stochastic term of the equations, and with the choice of the functional spaces in which the problem at hand is well-posed. Our method of proof rests on compactness arguments in an essential way.

The second part is devoted to the study of the blowup behavior of solutions to semilinear stochastic partial differential equations with Dirichlet boundary conditions driven by a class of differential operators including (not necessarily symmetric) Lévy processes and diffusion processes, and perturbed by a mixture of Brownian and fractional Brownian motions. Our aim is to understand the influence of the stochastic part and that of the differential operator on the blowup behavior of the solutions. In particular we derive explicit expressions for an upper and a lower bound of the blowup time of the solution and provide a sufficient condition for the existence of global positive solutions. Furthermore, we give estimates of the probability of finite time blowup and for the tail probabilities of an upper bound for the blowup time of the solutions.

Keywords:

Stochastic partial differential equations, variational solutions, unbounded domains, mixture of Brownian and fractional Brownian motions, blowup time of semilinear equations, Lévy processes, diffusion processes, weak and mild solutions.

