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Noncubic Dirac Operators for finite-dimensional modules

Spyridon Afentoulidis Almpanis

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Noncubic Dirac Operators for finite-dimensional modules

THÈSE

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pour l'obtention du

Doctorat de l'Université de Lorraine

(mention **Mathématiques**)

par

Spyridon Afentoulidis-Almpanis

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Mis en page avec la classe thesul.

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SUMMARY

Introduction

1	Dolbeault and Lie algebra cohomologies	1
2	Dirac cohomology	4
3	Slebarski's Dirac operators	7
4	Introduction to the results of the thesis	10
4.1	Algebraic Dirac operators	10
4.2	Geometric Dirac operators	16
4.3	Dirac Indices	18

Introduction en français

1	Cohomologies de Dolbeault et des algèbres de Lie	21
2	Cohomologie de Dirac	24
3	Opérateurs de Dirac de Slebarski	27
4	Introduction aux résultats de la thèse	30
4.1	Opérateurs de Dirac algébriques	30
4.2	Opérateurs de Dirac géométriques	37
4.3	Indices de Dirac	39

Chapter 1

Preliminaries

1.1	Lie groups and Lie algebras	41
1.1.1	Lie groups	41
1.1.2	Lie group morphisms	42

1.1.3	Lie algebras	43
1.1.4	Lie algebra morphisms	45
1.1.5	The Lie algebra of a Lie group	46
1.1.6	Exponential map	47
1.1.7	Killing form	50
1.1.8	Universal enveloping algebra	51
1.2	Complex semisimple Lie algebras and root decomposition	52
1.2.1	Classical Lie algebras	56
1.2.2	Weyl group	60
1.2.3	Dynkin diagrams and Cartan matrices	62
1.2.4	Reductive Lie algebras	65
1.3	Real semisimple Lie algebras	66
1.4	Representations of complex Lie algebras	69
1.4.1	Basic definitions	69
1.4.2	Representations of reductive Lie algebras	70
1.4.3	Casimir element	75
1.4.4	Weyl dimension formula and Kostant multiplicity formula	76
1.5	Representations of Lie groups	79
1.6	(\mathfrak{g}, K) -modules	81
1.7	Harish-Chandra isomorphism and infinitesimal characters	83

Chapter 2

Clifford Algebras and Spin Modules

2.1	Clifford Algebras	87
2.2	Spaces of spinors	89
2.2.1	Hermitian form on S	91
2.2.2	Spin module of $\mathfrak{so}(V)$	92
2.3	Spin Groups	93
2.4	Clifford algebra and spinors for Lie algebras	96
2.5	Weights of spin modules	99

Chapter 3

Kostant's cubic Dirac Operator

3.1	Definition and basic properties	103
3.2	Cubic Dirac operators for finite dimensional modules	105

3.3	Cubic Dirac operators for (\mathfrak{g}, K) -modules	111
-----	--	-----

Chapter 4

Noncubic Dirac Operators

4.1	Definition and basic properties of noncubic Dirac operators	117
4.2	The square of \widehat{D}	118
4.3	Some examples	120
4.4	Inclusion of the kernels	124
4.5	Weyl's inequalities	128
4.6	Kernel of noncubic Dirac operators	129
4.6.1	A necessary condition	129
4.6.2	A sufficient condition	134
4.6.3	Isotypic components	139
4.7	Explicit description of the set $A(\lambda)$	141
4.8	t -cubic Dirac operators	144

Chapter 5

Kernel of noncubic Dirac Operators for the Classical Lie Algebras

5.1	Preliminaries	149
5.2	The Lie Algebra $A_n = \mathfrak{sl}(n + 1, \mathbb{C})$	151
5.3	The Lie Algebra $B_n = \mathfrak{so}(2n + 1, \mathbb{C})$	153
5.4	The Lie Algebra $C_n = \mathfrak{sp}(n, \mathbb{C})$	156
5.5	The Lie Algebra $D_n = \mathfrak{so}(2n, \mathbb{C})$	159

Chapter 6

Kernel of noncubic Dirac Operators for the Exceptional Lie Algebras

6.1	Preliminaries	163
6.2	The Lie Algebra E_6	165
6.3	The Lie Algebra E_7	166
6.4	The Lie Algebra E_8	167
6.5	The Lie Algebra F_4	168
6.6	The Lie Algebra G_2	170

Chapter 7

Geometric Dirac Operators

Summary

7.1	Homogeneous spaces and homogeneous vector bundles	171
7.2	Peter-Weyl Theorem for homogeneous vector bundles	173
7.3	Twisted spin bundles and geometric cubic Dirac operators	174
7.4	Geometric noncubic Dirac operators	182
7.5	Geometric t -cubic Dirac operators	185
7.6	Dirac indices	187
7.6.1	Algebraic Dirac index	187
7.6.2	Geometric Dirac index	189

Bibliography	193
---------------------	------------

Index	201
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INTRODUCTION

At the end of the XIXth century the Norwegian mathematician Sophus Lie introduced a continuous analog to Galois groups. While a Galois group encodes the solvability in radicals of an algebraic equation as well as the behaviour of the solutions under permutations, a Lie group carries information about the solvability of a differential equation and the behaviour of the solutions, if any, under the action of a continuous family of symmetries. In most cases, by the superposition principle, the solutions form a topological vector space. The invariance of the solutions under the symmetries encoded by the Lie group is then expressed in terms of some action by linear transformations on the vector space of solutions. Representation theory connects the structure of a Lie group, or the symmetries of a set, with linear actions on vector spaces. Important issues in representation theory are the classification, the unitarity and the explicit realization of representations of a given Lie group. In this regard, invariant differential operators and cohomology spaces, in particular Dirac operators, will provide a great deal of information.

1 Dolbeault and Lie algebra cohomologies

Let G_0 be a connected noncompact semisimple real linear Lie group with Lie algebra \mathfrak{g}_0 and complexified Lie algebra \mathfrak{g} and K_0 be a maximal compact subgroup of G_0 with complexified Lie algebra \mathfrak{k} associated with a Cartan involution θ of G_0 . Let G be a connected semisimple complexification of G_0 with Lie algebra \mathfrak{g} and G_c a compact real form of G . Let H_0 be a closed connected subgroup of G_c with Lie algebra \mathfrak{h}_0 and complexified Lie algebra \mathfrak{h} . We assume that the Killing form of \mathfrak{g} remains nondegenerate when restricted to \mathfrak{h} , i.e \mathfrak{h} is quadratic in \mathfrak{g} . Then the homogeneous space G_c/H_0 is reductive, in the sense that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp \quad \text{with} \quad [\mathfrak{h}, \mathfrak{h}^\perp] \subseteq \mathfrak{h}^\perp. \quad (1)$$

The representation theory of G_c is well understood; in particular every irreducible unitary representation of G_c is finite-dimensional, and every finite-dimensional representation of G_c is equivalent to a unitary representation. The celebrated highest weight Theorem 1.5.17 classifies irreducible finite-dimensional representations as follows:

- (i) a finite-dimensional irreducible representation of G_c has a unique analytically integral dominant highest weight,
- (ii) two finite-dimensional irreducible representations of G_c with the same highest weight are isomorphic.

Moreover, if V_λ denotes the finite-dimensional irreducible representation of G_c with analytically integral dominant highest weight λ and V_λ^* stands for its dual representation, Peter-Weyl Theorem 7.2.2 decomposes the biregular representation of G_c into the orthogonal direct sum:

$$L^2(G_c) \simeq \bigoplus_{\lambda} V_\lambda \otimes V_\lambda^*. \quad (2)$$

A similar classification follows for the finite-dimensional irreducible representations of complex semisimple Lie algebras in Theorem 1.4.22, i.e. finite-dimensional irreducible representations of \mathfrak{g} are in bijection with integral dominant highest weights.

It turns out that the highest weight Theorem also includes an existence statement. Namely, for each dominant analytically integral element λ , there exists a finite-dimensional irreducible representation with highest weight λ . To construct such a representation, consider a complex flag variety G/S , where S is a θ -stable parabolic subgroup of G with Lie algebra \mathfrak{s} . Write $S = LU$ (resp. $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$) for the Levi decomposition of S (resp. \mathfrak{s}) and $\bar{\mathfrak{u}}$ for the opposite of \mathfrak{u} . Then G_c acts transitively on the complex flag variety G/S and one has a diffeomorphism of complex manifolds :

$$G/S \simeq G_c/G_c \cap S.$$

It turns out that the connected subgroup $L_c := G_c \cap S$ is a compact real form of L , it coincides with the centralizer of a torus in G_c and has maximal rank in G_c . Following [Zie00], denote by $\Omega^p(G_c/L_c)$ the space of smooth sections of the bundle $\bigwedge^p(T^{(0,1)}G_c/L_c)^*$ over the compact complex manifold G_c/L_c , i.e. the space of forms of type $(0, p)$ equipped with the G_c -equivariant operator:

$$\bar{\partial}_{G_c/L_c} : \Omega^p(G_c/L_c) \rightarrow \Omega^{p+1}(G_c/L_c)$$

defined by

$$\begin{aligned} \bar{\partial}_{G_c/L_c} \omega(X_0, \dots, X_p) &= \sum_{k=0}^p (-1)^k X_k [\omega(X_0, \dots, \widehat{X}_k, \dots, X_p)] \\ &+ \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_p) \end{aligned}$$

where the hat indicates that the corresponding term is omitted. Let $\mathcal{E} \rightarrow G_c/L_c$ be a finite-dimensional holomorphic vector bundle over G_c/L_c induced by a finite-dimensional representation E of L_c . The G_c -equivariant operator $\bar{\partial} := 1_{\mathcal{E}} \otimes \bar{\partial}_{G_c/L_c}$ defines a complex

$$\begin{aligned} \bar{\partial} : \Omega^p(G_c/L_c, E) &\rightarrow \Omega^{p+1}(G_c/L_c, E) \\ \bar{\partial}^2 &= 0. \end{aligned}$$

on the spaces

$$\begin{aligned}\Omega^p(G_c/L_c, E) &\simeq \{C^\infty(G_c) \otimes \wedge^p \mathfrak{u}^* \otimes E\}^{L_c} \\ &= \{f : G_c \xrightarrow{C^\infty} \wedge^p \mathfrak{u}^* \otimes E \mid f(gl) = l^{-1}f(g), \forall g \in G_c, \forall l \in L_c\}\end{aligned}$$

of smooth \mathcal{E} -valued forms of type $(0, p)$, $p \geq 0$. The quotient spaces

$$H^p(G_c/L_c, E) = \ker(\bar{\partial})/\text{Im}(\bar{\partial})$$

are finite-dimensional G_c -modules, the G_c -action being by left translations on $C^\infty(G_c)$, known as the Dolbeault cohomology spaces.

Dolbeault cohomology is closely related to Lie algebra cohomology. More precisely, let V be a finite-dimensional G_c -module. Denote by

$$\mathcal{C}^p(\mathfrak{u}, V) := \text{Hom}_{\mathbb{C}}(\wedge^p \mathfrak{u}, V)$$

the space of p -linear alternating maps from \mathfrak{u} to V (with $\mathcal{C}^0(\mathfrak{u}, V) := V$). For $T \in \mathcal{C}^p(\mathfrak{u}, V)$, define $\delta T \in \mathcal{C}^{p+1}(\mathfrak{u}, V)$ to be

$$\begin{aligned}(\delta T)(\xi_0, \dots, \xi_p) &:= \sum_{i=0}^p (-1)^k \xi_i [T(\xi_0, \dots, \widehat{\xi}_i, \dots, \xi_p)] \\ &+ \sum_{i < j} (-1)^{i+j} T([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi}_i, \dots, \widehat{\xi}_j, \dots, \xi_p).\end{aligned}$$

The complex $(\mathcal{C}(\mathfrak{u}, V), \delta)$ computes the p -th Lie algebra cohomology space of \mathfrak{u} with coefficients in V :

$$\mathcal{H}^p(\mathfrak{u}, V) = \ker(\delta)/\text{Im}(\delta).$$

Note that L_c normalizes \mathfrak{u} and so acts on $\wedge^p \mathfrak{u}^*$ by the adjoint map. Furthermore, L_c acts on V and thus \mathfrak{l} acts on $\mathcal{C}^p(\mathfrak{u}, V)$. This action commutes with δ , so the spaces $\mathcal{H}^p(\mathfrak{u}, V)$ are (finite-dimensional) L_c -modules. Moreover, one has the isomorphism:

$$\begin{aligned}\Omega^p(G_c/L_c, E) &\simeq \{C^\infty(G_c) \otimes \wedge^p \mathfrak{u}^* \otimes E\}^{L_c} \\ &\simeq \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{V_\lambda^* \otimes \wedge^p \mathfrak{u}^* \otimes E\}^{L_c} \text{ by (2)} \\ &\simeq \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\text{Hom}_{\mathbb{C}}(\wedge^p \mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c}.\end{aligned}$$

Now, the commutativity of the following diagram

$$\begin{array}{ccc}\Omega^p(G_c/L_c, E) & \xrightarrow{\quad \bar{\partial} \quad} & \Omega^{p+1}(G_c/L_c, E) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\text{Hom}_{\mathbb{C}}(\wedge^p \mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c} & \xrightarrow{\quad 1 \otimes \delta \otimes 1 \quad} & \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\text{Hom}_{\mathbb{C}}(\wedge^{p+1} \mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c}\end{array}$$

implies that

$$H^p(G_c/L_c, E) \simeq \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\mathcal{H}^p(\mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c}. \quad (3)$$

The Lie algebra cohomology was computed by Kostant in [Kos61]. More precisely, let $W_{\mathfrak{g}}$ be the Weyl group for \mathfrak{g} , $W_{\mathfrak{l}} \subset W_{\mathfrak{g}}$ the Weyl group for \mathfrak{l} , and ρ half the sum of the positive roots in a positive system $\Delta^+ \subset \Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ defined by \mathfrak{u} . If $\Delta_{\mathfrak{l}}^+ \subseteq \Delta^+$ is the positive system for $\Delta_{\mathfrak{l}} := \Delta(\mathfrak{l}, \mathfrak{t})$ and

$$W^1 := \{w \in W_{\mathfrak{g}} \mid \Delta_{\mathfrak{l}}^+ \subseteq w\Delta^+\},$$

then one has the following multiplicity free decomposition into L_c -modules:

$$\mathcal{H}^p(\mathfrak{u}, V_\lambda^*) \simeq \bigoplus_{\substack{w \in W^1 \\ \ell(w)=p}} F_{-ww_0(\lambda+\rho)-\rho} \quad (4)$$

where $\ell(w)$ stands for the length of w , w_0 is the longest element of $W_{\mathfrak{g}}$ and $F_{-ww_0(\lambda+\rho)-\rho}$ the finite-dimensional irreducible representation of L_c with highest weight $-ww_0(\lambda+\rho)-\rho$. From (3), we deduce that

$$H^p(G_c/L_c, E) \simeq \bigoplus_{\lambda \in \widehat{G}_c} \bigoplus_{\substack{w \in W^1 \\ \ell(w)=\#\{\Delta_{\mathfrak{u}}\}-p}} V_\lambda \otimes \text{Hom}_{L_c}(F_{w(\lambda+\rho)-\rho_{\mathfrak{l}}+\rho_{\mathfrak{u}}}, E) \quad (5)$$

where $\Delta_{\mathfrak{u}} := \Delta^+ \setminus \Delta_{\mathfrak{l}}^+$. In particular, in the case where $E = E_\mu$ is irreducible with highest weight μ , one recovers Borel-Weil-Bott Theorem of [Bot57]:

$$H^{\#\{\Delta_{\mathfrak{u}}\}-\ell(w)}(G_c/L_c, E_\mu) \simeq \begin{cases} V_{w^{-1}(\mu+\rho_{\mathfrak{l}}-\rho_{\mathfrak{u}})-\rho} & \text{if } w^{-1}(\mu+\rho_{\mathfrak{l}}-\rho_{\mathfrak{u}})-\rho \text{ is } \Delta^+ \text{-dom. for some } w \in W^1 \\ \{0\} & \text{if no such } w \text{ exists.} \end{cases}$$

2 Dirac cohomology

Dolbeault cohomology and Lie algebra cohomology are related to cubic Dirac operators as follows. The subalgebras \mathfrak{u} and $\bar{\mathfrak{u}}$ are maximally dual isotropic subspaces, and $\mathfrak{q} := \mathfrak{u} \oplus \bar{\mathfrak{u}}$ is a \mathfrak{l} -invariant subspace of \mathfrak{g} . Moreover, \mathfrak{g} can be decomposed into

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q} \quad \text{and} \quad \text{Ad}(L_c)(\mathfrak{q}) \subset \mathfrak{q}, \quad (6)$$

in analogy to (1). Let S be the space of spinors for the Clifford algebra $\mathbf{C}(\mathfrak{q})$ of \mathfrak{q} , equipped with the Clifford multiplication γ . Then \mathfrak{l} embeds in $\mathbf{C}(\mathfrak{q})$ so that S becomes a module for \mathfrak{l} . Let \mathfrak{c} be the degree 3 element in $\mathbf{C}(\mathfrak{q})$ corresponding, via the Chevalley isomorphism, to the alternating trilinear form $\omega : (X, Y, Z) \mapsto B(X, [Y, Z])$ on \mathfrak{q} , where B is the Killing form of \mathfrak{g} , given by

$$(\omega : (X, Y, Z) \mapsto B(X, [Y, Z])) \mapsto \mathfrak{c}. \quad (7)$$

In [Kos99], Kostant associates with each \mathfrak{g} -module (π, V) , an operator $D_{\mathfrak{g}, \mathfrak{l}}(V)$

$$D_{\mathfrak{g}, \mathfrak{l}}(V) : V \otimes S \rightarrow V \otimes S \quad (8)$$

defined by

$$D_{\mathfrak{g},\mathfrak{l}}(V) = \sum_j \pi(X_j) \otimes \gamma(X_j) - 1 \otimes \gamma(c)$$

where $\{X_j\}$ is an orthonormal basis of \mathfrak{q} . Note that, according to Lemma 3.1.4, $D_{\mathfrak{g},\mathfrak{l}}(V)$ does not depend on the choice of the basis $\{X_j\}$ while, when the pair $(\mathfrak{g}, \mathfrak{l})$ is symmetric, the cubic term c vanishes. Moreover, \mathfrak{l} acts on $V \otimes S$, the action being given by the restriction of π to \mathfrak{l} and the spin action on S , and the operator $D_{\mathfrak{g},\mathfrak{l}}(V)$ is \mathfrak{l} -equivariant with respect to this action. In [Kos99], Kostant computes the square of $D_{\mathfrak{g},\mathfrak{l}}(V)$, showing that $(D_{\mathfrak{g},\mathfrak{l}}(V))^2$ differs from the Casimir by a scalar:

$$(D_{\mathfrak{g},\mathfrak{l}}(V))^2 = \pi(\Omega_{\mathfrak{g}}) \otimes 1 - (\pi \otimes \gamma)(\Omega_{\mathfrak{l}})_{\Delta} + (\|\rho\|^2 - \|\rho_{\mathfrak{l}}\|^2)1 \otimes 1 \quad (9)$$

where $\Omega_{\mathfrak{g}}$ (respectively $\Omega_{\mathfrak{l}}$) denotes the Casimir element in the enveloping algebra $U(\mathfrak{g})$ (respectively $U(\mathfrak{l})$) of \mathfrak{g} (respectively \mathfrak{l}), $\rho_{\mathfrak{l}}$ is half the sum of the positive roots in a positive system $\Delta_{\mathfrak{l}}^+ \subset \Delta^+$ for roots in \mathfrak{l} , $\|\cdot\|$ is the norm induced by the Killing form of \mathfrak{g} , and $(\cdot)_{\Delta}$ is the diagonal embedding

$$\begin{aligned} (\cdot)_{\Delta} : \mathfrak{l} &\longrightarrow U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q}) \\ X &\mapsto X \otimes 1 + 1 \otimes \varphi(\text{ad}X) \end{aligned}$$

of \mathfrak{l} into $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$, given by the embedding $\mathfrak{l} \subset \mathfrak{g}$ in $U(\mathfrak{g})$ and the embedding (2.11) of \mathfrak{l} in $\mathbf{C}(\mathfrak{q})$. $D_{\mathfrak{g},\mathfrak{l}}(V)$ is known as Kostant's cubic Dirac operator, generalizing Parthasarathy Dirac operators and his formula for the square, appearing in [Par72]. Cubic Dirac operator can be defined similarly in the more general setting of the pairs $(\mathfrak{g}, \mathfrak{h})$, including the unequal rank case. For more details, one can see, for example, [HKP09], [Kos03], [MP11], [MZ03] and [MZ14b]. In the late 1990's, in a series of talks at MIT [Vog97], Vogan defined the Dirac cohomology of a (\mathfrak{g}, K_0) -module X to be the \tilde{K}_0 -module

$$H_D^{\mathfrak{g},\mathfrak{k}}(X) = \ker(D_{\mathfrak{g},\mathfrak{k}}(X)) / \text{Im}(D_{\mathfrak{g},\mathfrak{k}}(X)) \cap \ker(D_{\mathfrak{g},\mathfrak{k}}(X)), \quad (10)$$

where \tilde{K}_0 denotes the spin double cover of K_0 . Assuming that X is irreducible and has an infinitesimal character, he conjectured that if $H_D^{\mathfrak{g},\mathfrak{k}}(X)$ contains a \tilde{K}_0 -module with highest weight β then X has infinitesimal character $\beta + \rho_{\mathfrak{k}}$. In other words, the infinitesimal character of a module can be recovered from its Dirac cohomology. Vogan's conjecture was first proved by Huang and Pandžić for equal rank pairs $(\mathfrak{g}, \mathfrak{k})$ in [HP02] and by Kostant for cubic Dirac operators for equal rank pairs $(\mathfrak{g}, \mathfrak{h})$ in [Kos03]. It was extended to modules with generalized infinitesimal character by Pandžić and Somberg in [PS16]. Dirac cohomology has been computed for various families of modules, including highest weight modules in [HX12], $A_{\mathfrak{q}}(\lambda)$ modules in [HKP09], generalized Enright-Varadarajan modules in [MP11], unipotent representations in [BP15] and [BP11], Jacquet modules in [DH11] or (\mathfrak{g}, K_0) -modules with generalized infinitesimal characters in [PS16].

In the case where X is a unitary (\mathfrak{g}, K_0) -module (respectively finite-dimensional), $D_{\mathfrak{g},\mathfrak{k}}(X)$ is a skew-adjoint (respectively self-adjoint) operator and Dirac cohomology reduces to the kernel

$$\ker(D_{\mathfrak{g},\mathfrak{k}}(X)) = \ker(D_{\mathfrak{g},\mathfrak{k}}(X)^2).$$

For more details, see [HP06, page 63]. Moreover, for an equal rank pair $(\mathfrak{g}, \mathfrak{k})$, if X is a unitary (\mathfrak{g}, K_0) -module with infinitesimal character Λ and β is a \widetilde{K}_0 -type occurring in $X \otimes S$, then the Parthasarathy-Dirac inequality

$$\|\beta + \rho_{\mathfrak{k}}\| \geq \|\Lambda\|$$

of [Par80a] must be satisfied, providing a test for the unitarizability of (\mathfrak{g}, K_0) -modules.

Kostant proved in [Kos99], that in the case of an equal rank pair $(\mathfrak{g}, \mathfrak{h})$ and for $V = V_{\lambda}$ a finite-dimensional irreducible \mathfrak{g} -module with highest weight λ , $H_D^{\mathfrak{g}, \mathfrak{h}}(V_{\lambda})$ admits the following multiplicity free decomposition

$$H_D^{\mathfrak{g}, \mathfrak{h}}(V_{\lambda}) = \ker(D_{\mathfrak{g}, \mathfrak{h}}(V_{\lambda})) = \sum_{w \in W^1} F_{w(\lambda + \rho) - \rho_{\mathfrak{h}}} \quad (11)$$

where $F_{w(\lambda + \rho) - \rho_{\mathfrak{h}}}$ is the finite-dimensional irreducible \mathfrak{h} -representation of highest weight $w(\lambda + \rho) - \rho_{\mathfrak{h}}$ and W^1 is defined in an analogous way as in the case of \mathfrak{l} . In fact, a similar formula was proved in the unequal rank situation, in [HKP09] for the pair $(\mathfrak{g}, \mathfrak{k})$ and in [MZ14b] for pairs $(\mathfrak{g}, \mathfrak{h})$. In particular, in the unequal rank case, the decomposition of $H_D^{\mathfrak{g}, \mathfrak{k}}(V_{\lambda})$ (respectively $H_D^{\mathfrak{g}, \mathfrak{h}}(V_{\lambda})$) into irreducibles is no longer multiplicity free.

Now, a connection between Dolbeault and Dirac cohomologies can be deduced from (5) and (11) when $G_c/H_0 = G_c/L_c$. First, note that the action of \mathfrak{l} on the first factor of the u -cohomology is given by the adjoint action so that the action of \mathfrak{l} on the u -cohomology differs from the action of \mathfrak{l} on $V_{\lambda} \otimes S$ only by a twist with the one dimensional \mathfrak{l} -module \mathbb{C}_{ρ_u} (for more details, see [Kos61]). Then, one has:

$$H(G_c/L_c, E \otimes \mathbb{C}_{\rho_u}) \simeq \bigoplus_{\lambda \in \widehat{G}_c} V_{\lambda} \otimes \text{Hom}_{L_c}(H_D^{\mathfrak{g}, \mathfrak{l}}(V_{\lambda}), E) = \bigoplus_{\lambda \in \widehat{G}_c} V_{\lambda} \otimes \text{Hom}_{L_c}(\ker(D_{\mathfrak{g}, \mathfrak{l}}(V_{\lambda})), E)$$

so that the Borel-Weil-Bott theorem may be viewed as a particular case of the Dirac cohomology.

Certain features of the Dirac cohomology were studied by Mehdi and Parthasarathy in [MP08], [MP10] and [MP11]. Moreover, it was proved by Mehdi and Pandžić in [MP13] that Dirac cohomology does not commute with translation functors. To circumvent this problem, Mehdi, Pandžić and Vogan introduced the Dirac index $I_D(X)$ of a (not necessarily finite-dimensional) (\mathfrak{g}, K_0) -module X in [MPV17a]. More precisely, suppose G_0 and K_0 have the same (complex) rank, so that the spin module S for the spin double cover \widetilde{K}_0 (or equivalently for \mathfrak{k}) splits into the direct sum of the half-spin modules S^+ and S^- . Then, the Dirac cohomology $H_D^{\mathfrak{g}, \mathfrak{k}}(X)$ of X decomposes accordingly into $H_D^{\mathfrak{g}, \mathfrak{k}}(X)^+$ and $H_D^{\mathfrak{g}, \mathfrak{k}}(X)^-$:

$$H_D^{\mathfrak{g}, \mathfrak{k}}(X) = H_D^{\mathfrak{g}, \mathfrak{k}}(X)^+ \oplus H_D^{\mathfrak{g}, \mathfrak{k}}(X)^-.$$

If X is admissible with infinitesimal character, the Dirac index of X is then the virtual \widetilde{K}_0 -module:

$$I_D(X) = H_D^{\mathfrak{g}, \mathfrak{k}}(X)^+ - H_D^{\mathfrak{g}, \mathfrak{k}}(X)^-.$$

We mention that this definition was extended by Pandžić and Somberg in [PS16] to finite length modules not necessarily with infinitesimal character using higher Dirac cohomology. Mehdi, Pandžić and Vogan proved in [MPV17a, Section 3] that

$$I_D(X) = X \otimes S^+ - X \otimes S^- \quad (12)$$

and that, as a functor from finite length (\mathfrak{g}, K_0) -modules to virtual \widetilde{K}_0 -modules, I_D is additive with respect to short exact sequences and behaves well with respect to translation functors:

$$I_D(X \otimes F) = I_D(X) \otimes F \text{ for any finite-dimensional } (\mathfrak{g}, K_0)\text{-module } F.$$

They also prove, in [MPV17a, Section 5], that the map

$$\lambda \mapsto \dim(I_D(X_\lambda))$$

is polynomial on the Cartan subalgebra whenever $\{X_\lambda\}$ is a coherent family of (\mathfrak{g}, K_0) -modules. Recently, Mehdi, Pandžić, Vogan and Zierau in [MPVZ20] related in a precise way this Dirac index polynomial with nilpotent orbits and associated cycles of Harish-Chandra modules. An analogous definition for the Dirac index for pairs $(\mathfrak{g}, \mathfrak{h})$ will be given in Section 7.6.

3 Slebarski's Dirac operators

It appears that Kostant's cubic Dirac operator (8) is actually an algebraic version of a twisted geometric Dirac operator introduced by Slebarski in a series of papers, [Sle87a], [Sle87b] and [Sle88], from the mid 1980's. More precisely, the homogeneous space G_c/H_0 is (naturally) reductive and the tangent space $T_{eH_0}(G_c/H_0)$ identifies with \mathfrak{h}_0^\perp . Denote by $\Gamma(T(G_c/H_0))$ the space of sections of the tangent bundle $T(G_c/H_0)$ of G_c/H_0 . Recall that a connection ∇ on the tangent bundle $T(G_c/H_0)$ of G_c/H_0 is a differential linear operator

$$\nabla : \Gamma(T(G_c/H_0)) \rightarrow \Gamma(T^*(G_c/H_0) \otimes T(G_c/H_0))$$

satisfying Leibniz's rule. Identifying vector fields on G_c/H_0 with elements in \mathfrak{h}_0^\perp , ∇ is H_0 -invariant if

$$\nabla_{\text{Ad}(h)(X)}(\text{Ad}(h)(Y)) = \text{Ad}(h)(\nabla_X(Y)), \forall h \in H_0, \forall X, Y \in \mathfrak{h}_0^\perp.$$

Slebarski introduced in [Sle87a, Section 3] the following one-parameter family $\{\nabla^t\}_{t \in [0, 1/2]}$ of connections:

$$\begin{aligned} \nabla^0 &= \text{canonical connection} \\ \nabla_X^t(Y) &= \nabla_X^0(Y) + t[X, Y]_{\mathfrak{h}_0^\perp} \text{ for all } (X, Y) \in \mathfrak{h}_0^\perp \times \mathfrak{h}_0^\perp \end{aligned} \quad (13)$$

where $[\cdot, \cdot]_{\mathfrak{h}_0^\perp}$ is the projection onto \mathfrak{h}_0^\perp of the Lie bracket on \mathfrak{g}_0 . Note that when G_c/H_0 is a symmetric space, the bracket $[\cdot, \cdot]_{\mathfrak{h}_0^\perp}$ is identically zero and the family $\{\nabla^t\}_{t \in [0, 1/2]}$ reduces to the torsion-free canonical connection ∇^0 . However, if the bracket $[\cdot, \cdot]_{\mathfrak{h}_0^\perp}$ is not identically zero then ∇^t is torsion-free if, and only if, $t = 1/2$, where $\nabla^{1/2}$ is the Levi-Civita connection. Actually, according to [Agr03, Lemma 2.1], the torsion T^t of ∇^t can be expressed in terms of the cubic term \mathfrak{c} :

$$T^t : (X, Y, Z) \in \mathfrak{h}_0^\perp \times \mathfrak{h}_0^\perp \times \mathfrak{h}_0^\perp \mapsto 2(2t - 1)B([X, Y]_{\mathfrak{h}_0^\perp}, Z). \quad (14)$$

In other words, the cubic term \mathfrak{c} coincides, up to a constant, with the torsion of the canonical connection ∇^0 on G_c/H_0 .

Next, in [Sle87a], Slebarski considers a family $\{D_{G_c/H_0}^t\}_{t \in [0,1/2]}$ of twisted Dirac operators defined by the family $\{\nabla^t\}$. Dirac operators on Riemannian manifolds are discussed extensively by Friedrich in [Fri00]. Namely, suppose that W is a finite-dimensional \mathfrak{h} -module such that the tensor product $S \otimes W$ lifts to a representation of H_0 ; observe that H_0 need not be simply connected. Then, one gets a homogeneous vector bundle $S \otimes W \rightarrow G_c/H_0$ over G_c/H_0 . The space of smooth sections of this bundle can be identified with the space of $(S \otimes W)$ -valued smooth functions on G_c :

$$C^\infty(G_c/H_0, S \otimes W) := \{\varphi : G_c \rightarrow S \otimes W \mid \varphi(gh) = h^{-1} \cdot (\varphi(g)) \ \forall (g, h) \in G_c \times H_0\}$$

equipped with the G_c -action by left translations. It is convenient to identify $C^\infty(G_c/H_0, S \otimes W)$ with the space of H_0 -invariants:

$$\{C^\infty(G_c) \otimes S \otimes W\}^{H_0}$$

where G_c acts by left translations on $C^\infty(G_c)$, while H_0 acts by right translations on $C^\infty(G_c)$ and the lifted action on $W \otimes S$. If $f \in C^\infty(G_c)$ and $Y \in \mathfrak{h}_0^\perp$, $R(Y)f$ will denote the differentiation

$$(R(Y)f)(g) := \frac{d}{ds} f(g \exp(sY))|_{s=0} \quad \forall g \in G_c.$$

If $\{Y_j\}$ is an orthonormal basis of \mathfrak{h}_0^\perp , Slebarski defines in [Sle87a, Section 1] the twisted Dirac operator associated with ∇^t to be the operator

$$\mathcal{D}_{G_c/H_0}^t(W) : \{C^\infty(G_c) \otimes S \otimes W\}^{H_0} \rightarrow \{C^\infty(G_c) \otimes S \otimes W\}^{H_0} \quad (15)$$

given by

$$\mathcal{D}_{G_c/H_0}^t(W) := \sum_j R(Y_j) \otimes \gamma(X_j) \otimes 1 - 3t(1 \otimes \gamma(c) \otimes 1)$$

where γ stands for the Clifford multiplication and c is the degree 3 element defined by (7). The operators $\mathcal{D}_{G_c/H_0}^t(W)$ belong to the algebra $\{U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \text{End}(S \otimes W)\}^{H_0}$ of G_c -invariant differential operators acting on smooth sections of $S \otimes W \rightarrow G_c/H_0$. This algebra need not be commutative; Deitmar, in [Dei90], gives a commutativity criterion in some cases. Note that the constant in front of the cubic term comes from our convention for the Clifford algebra (see Remark 2.1.2). As in the case of the algebraic cubic Dirac operator, one can check that $\mathcal{D}_{G_c/H_0}^t(W)$ does not depend on the choice of the basis $\{Y_j\}$ of \mathfrak{h}_0^\perp . Slebarski also computes the square of the operators $\mathcal{D}_{G_c/H_0}^t(W)$ in [Sle87a, Section 1]. He then observes in [Sle87a, Theorem 1] that among the family $\{\mathcal{D}_{G_c/H_0}^t(W)\}$, only the square of $\mathcal{D}_{G_c/H_0}^{1/3}(W)$ differs from the Casimir in $U(\mathfrak{g})$ by a constant. A careful computation and discussion is provided by Agricola in the more general setting of naturally reductive homogeneous spaces [Agr03, Theorem 3.2]. She also deduces useful information about certain equations of string theory in [Agr03, Section 4]. We will simply write $\mathcal{D}_{G_c/H_0}(W)$, or just \mathcal{D} when there is no confusion, for $\mathcal{D}_{G_c/H_0}^{1/3}(W)$ and $\widehat{\mathcal{D}}_{G_c/H_0}(W)$, or just $\widehat{\mathcal{D}}$ for $\mathcal{D}_{G_c/H_0}^0(W)$. While $\widehat{\mathcal{D}}_{G_c/H_0}(W)$ is the twisted Dirac operator associated with the canonical connection ∇^0 on G_c/H_0 , the operator $\mathcal{D}_{G_c/H_0}^{1/3}(W)$ may be viewed as the geometric version of Kostant's cubic Dirac operator. A precise relationship between algebraic and geometric Dirac operators through the kernels can be stated following [MZ14a].

Namely, as above, suppose V is a finite-dimensional representation for G_c and W is a finite-dimensional representation of \mathfrak{h} such that $S \otimes W$ lifts to a H_0 -representation. Then, by [MZ14a, Proposition 1.8], one has the following isomorphism of vector spaces:

$$\mathrm{Hom}_{G_c}(V, \ker(\mathcal{D}_{G_c/H_0}(W))) \simeq \mathrm{Hom}_{\widetilde{H}_0}(W, \ker(D_{\mathfrak{g},\mathfrak{h}}(V)))$$

This reciprocity law is actually valid in a more general setting; when V is a smooth admissible representation of G_0 . In Proposition 7.4.2, we see that noncubic Dirac operators satisfy analogous reciprocity for homogeneous reductive spaces G_c/H_0 :

$$\mathrm{Hom}_{G_c}(V, \ker(\mathcal{D}_{G_c/H_0}^t(W))) \simeq \mathrm{Hom}_{\widetilde{H}_0}(W, \ker(D_{\mathfrak{g},\mathfrak{h}}^t(V))) \quad \forall t \in [0, 1/2]. \quad (16)$$

We continue with the case where \mathfrak{h} is of maximal rank in \mathfrak{g} . The kernel and the index of the geometric cubic Dirac operator were also computed by Slebarski in [Sle87a, page 299]. Namely, suppose that $W = W_\mu$ is a finite-dimensional irreducible \mathfrak{h} -representation of highest weight μ such that $S \otimes W_\mu$ lifts to a H_0 -representation. Then, according to [Sle87a, Theorem 1], one has

$$\ker(\mathcal{D}_{G_c/H_0}(W_\mu)) = \begin{cases} V_{w(\mu+\rho_{\mathfrak{h}})-\rho} & \text{if } w(\mu+\rho_{\mathfrak{h}})-\rho \text{ is } \Delta^+\text{-dominant for some } w \in W_{\mathfrak{g}} \\ \{0\} & \text{if such a } w \text{ does not exist.} \end{cases} \quad (17)$$

For the index, we follow the description by Landweber in [Lan00]. Consider, as above, the two half-spin representations S^+ and S^- so that

$$D_{G_c/H_0}(W)^\pm : L^2(G_c/H_0, S^+ \otimes W) \rightarrow L^2(G_c/H_0, S^- \otimes W)$$

is an elliptic G_c -invariant differential operator. Here $L^2(G_c/H_0, S \otimes W)$ denotes the space of L^2 -sections of the bundle $S \otimes W \rightarrow G_c/H_0$. Using [Bot57, Theorem I], Landweber shows in [Lan00, Theorem 2] that the G_c -equivariant index of $\mathcal{D}_{G_c/H_0}(W)$ is the finite virtual G_c -module

$$\mathrm{Index}_{G_c}(\mathcal{D}_{G_c/H_0}(W)) = L^2(G_c/H_0, S^+ \otimes W) - L^2(G_c/H_0, S^- \otimes W), \quad (18)$$

while [Lan00, Theorem 3] says that

$$\mathrm{Index}_{G_c}(\mathcal{D}_{G_c/H_0}(W_\mu)) = \begin{cases} (-1)^{\ell(w)} F_{w(\mu+\rho_{\mathfrak{h}})-\rho} & \text{if } w(\mu+\rho_{\mathfrak{h}})-\rho \text{ is } \Delta^+\text{-dom. for some } w \in W_{\mathfrak{g}} \\ \{0\} & \text{if such a } w \text{ does not exist.} \end{cases}$$

In the case where G_c/H_0 is replaced by G_0/K_0 (equal rank), Parthasarathy in [Par72] and Atiyah-Schmid in [AS77] proved that the L^2 -kernel of the twisted geometric Dirac operator is a discrete series representation of G_0 and any discrete series representation occurs as the L^2 -kernel of some twisted geometric Dirac operator. In this setting the Dirac index is studied in [AS77]. We mention that in the late 1900's, Deitmar used Selberg trace formula to compute the Dirac index on arithmetic manifolds $\Gamma \backslash G_0/K_0$ where $\Gamma \subset G_0$ is a torsion-free discrete subgroup [Dei97]. Note that, more recently, Mehdi and Zierau proved in [MZ03], [MZ06] and [MZ14a] that the kernel of the twisted cubic Dirac operator on reductive homogeneous spaces contains principal series representations. Principal series are also involved in the study by Clerc and Ørsted of the conformal covariance of powers of the Dirac operator [CØ20]. On the other hand,

the kernel of the symplectic Dirac operator introduced by Kostant in [Kos74] has been described for the Segal-Shale-Weil representation by De Bie, Somberg and Souček in [DSS14]. We also mention that representation theoretic aspects of the spectrum of the Laplacian, somehow the square of the Dirac operator, has been extensively studied in the last decades. For example, resonances of the Laplacian on certain Riemannian symmetric spaces are described in terms of Langlands parameters and wave front sets of representations by Hilgert, Pasquale and Przebinda in [HPP17a] and [HPP17b]. More recently, Mehdi and Olbrich describe G_0 -representations involved in the spectrum of certain locally symmetric spaces $\Gamma \backslash G_0/H_0$ [MO20].

4 Introduction to the results of the thesis

4.1 Algebraic Dirac operators

Following (8) and (15), attached with a \mathfrak{g} -module (π, V) and the connections $\{\nabla^t\}$ is the following family of algebraic Dirac operators $\{D_{\mathfrak{g},\mathfrak{h}}^t(V)\}_{t \in [0,1/2]}$:

$$\begin{aligned} D_{\mathfrak{g},\mathfrak{h}}^t(V) &: V \otimes S \rightarrow V \otimes S \\ D_{\mathfrak{g},\mathfrak{h}}^t(V) &:= \sum_j \pi(Y_j) \otimes \gamma(Y_j) - 3t(1 \otimes \gamma(c)) \end{aligned}$$

We will write simply $D_{\mathfrak{g},\mathfrak{h}}(V)$ for $D_{\mathfrak{g},\mathfrak{h}}^{1/3}(V)$ and $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ for $D_{\mathfrak{g},\mathfrak{h}}^0(V)$. We call $D_{\mathfrak{g},\mathfrak{h}}(V)$ and $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ cubic and noncubic Dirac operators, respectively. For $t \in (0, \frac{1}{3}]$, we call $D_{\mathfrak{g},\mathfrak{h}}^t(V)$ t -cubic Dirac operator. If there is no ambiguity, we will write D , \widehat{D} and D^t instead of $D_{\mathfrak{g},\mathfrak{h}}(V)$, $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ and $D_{\mathfrak{g},\mathfrak{h}}^t(V)$, respectively. As we mentioned before, both $D_{\mathfrak{g},\mathfrak{h}}(V)$ and $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ coincide when G_c/H_0 is a symmetric space, and, in particular, $\ker(D_{\mathfrak{g},\mathfrak{h}}(V)) = \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$. We are interested in the kernel of noncubic Dirac operators in the case when G_c/H_0 is not symmetric and V is a finite-dimensional irreducible highest weight representation.

Let us consider some examples where F is the standard representation of \mathfrak{g} , $\mathfrak{h} = \mathfrak{l}$ and $\mathfrak{h}^\perp = \mathfrak{q} = \mathfrak{u} + \bar{\mathfrak{u}}$. Take \mathfrak{g} to be the 21-dimensional simple complex Lie algebra $\mathfrak{sp}(3, \mathbb{C})$. Fix, as in Section 4.3, a (3-dimensional) Cartan subalgebra \mathfrak{t} in \mathfrak{g} and let

$$\Delta^+ = \{\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_3, 2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3\}$$

be a positive system for $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$. Choose \mathfrak{s} to be the parabolic subalgebra $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$ with

$$\begin{aligned} \mathfrak{l} &= \mathfrak{t} + \mathfrak{g}_{\varepsilon_1 + \varepsilon_2} + \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} + \mathfrak{g}_{2\varepsilon_1} + \mathfrak{g}_{2\varepsilon_2} + \mathfrak{g}_{-\varepsilon_1 - \varepsilon_2} + \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} + \mathfrak{g}_{-2\varepsilon_1} + \mathfrak{g}_{-2\varepsilon_2} \\ \mathfrak{u} &= \mathfrak{g}_{\varepsilon_1 + \varepsilon_3} + \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} + \mathfrak{g}_{\varepsilon_2 + \varepsilon_3} + \mathfrak{g}_{\varepsilon_2 - \varepsilon_3} + \mathfrak{g}_{2\varepsilon_3}. \end{aligned}$$

Then $V = V_{\varepsilon_1}$ is the irreducible highest weight representation with highest weight ε_1 . Pick a highest weight vector v in V_{ε_1} so that $v \otimes 1$ is a highest vector in $V_{\varepsilon_1} \otimes S$ with highest weight $\varepsilon_1 + \rho_{\mathfrak{q}}$, where $\rho_{\mathfrak{q}}$ denotes half the sum of the positive \mathfrak{t} -roots in $\mathfrak{q} = \mathfrak{u} + \bar{\mathfrak{u}}$. The cubic term c in (7) acts as:

$$\begin{aligned} \gamma(c) &= \frac{\sqrt{2}}{4} \gamma(e_{\varepsilon_1 + \varepsilon_3}) \gamma(e_{-\varepsilon_1 + \varepsilon_3}) \gamma(e_{-2\varepsilon_3}) + \frac{\sqrt{2}}{4} \gamma(e_{\varepsilon_2 + \varepsilon_3}) \gamma(e_{-\varepsilon_2 + \varepsilon_3}) \gamma(e_{-2\varepsilon_3}) \\ &\quad - \frac{\sqrt{2}}{4} \gamma(e_{-\varepsilon_1 - \varepsilon_3}) \gamma(e_{\varepsilon_1 - \varepsilon_3}) \gamma(e_{2\varepsilon_3}) - \frac{\sqrt{2}}{4} \gamma(e_{-\varepsilon_2 - \varepsilon_3}) \gamma(e_{\varepsilon_2 - \varepsilon_3}) \gamma(e_{2\varepsilon_3}). \end{aligned}$$

and one checks that $1 \otimes \gamma(c)$ annihilates $v \otimes 1$. Changing the positive system Δ^+ to $(\Delta^+)'$ in such a way that $\Delta_{\mathfrak{h}}^+ \subset (\Delta^+)'$ would lead to another vector $v' \otimes 1$ with v' being a highest weight vector with respect to $(\Delta^+)'$. The collection of these vectors for all positive systems containing $\Delta_{\mathfrak{h}}^+$ generates the kernel of the cubic Dirac operator $D_{\mathfrak{g},\mathfrak{t}}(V)$ by Kostant's formula (11). We deduce that the kernel of the noncubic Dirac operator contains the kernel of the cubic Dirac operator. Moreover, one can check that the vector $v \otimes e_{-\varepsilon_2-\varepsilon_3} \in V \otimes S$ constructed in Example 4.3.3 is annihilated by $\widehat{D}_{\mathfrak{g},\mathfrak{t}}(V)$ but not by $D_{\mathfrak{g},\mathfrak{t}}(V)$. In other words, for this example we have:

$$\ker(D_{\mathfrak{g},\mathfrak{t}}(V)) \subset \ker(\widehat{D}_{\mathfrak{g},\mathfrak{t}}(V)) \text{ but } \ker(D_{\mathfrak{g},\mathfrak{t}}(V)) \neq \ker(\widehat{D}_{\mathfrak{g},\mathfrak{t}}(V)).$$

Suppose now that \mathfrak{g} is the 8-dimensional complex simple Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. Fix, as in Example 4.3.2, a 2-dimensional Cartan subalgebra \mathfrak{t} and choose \mathfrak{s} to be the Borel subalgebra $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$ with

$$\begin{aligned} \mathfrak{l} &= \mathfrak{t} \\ \mathfrak{u} &= \mathfrak{g}_{\varepsilon_1-\varepsilon_2} + \mathfrak{g}_{\varepsilon_1-\varepsilon_2} + \mathfrak{g}_{\varepsilon_2-\varepsilon_3}. \end{aligned}$$

Let $V = V_{\varepsilon_1}$ be the irreducible finite-dimensional representation with highest weight ε_1 . The cubic term acts as:

$$\gamma(c) = \frac{\sqrt{6}}{6} \gamma(e_{\varepsilon_1-\varepsilon_3}) \gamma(e_{-\varepsilon_1+\varepsilon_2}) \gamma(e_{-\varepsilon_2+\varepsilon_3}) - \frac{\sqrt{6}}{6} \gamma(e_{-\varepsilon_1+\varepsilon_3}) \gamma(e_{\varepsilon_1-\varepsilon_2}) \gamma(e_{\varepsilon_2-\varepsilon_3}).$$

and annihilates the highest weight vector $v \otimes 1$. For this example, we have

$$\ker(D_{\mathfrak{g},\mathfrak{t}}(V)) = \ker(\widehat{D}_{\mathfrak{g},\mathfrak{t}}(V)).$$

The above examples are actually special cases of the following theorem, connecting kernels of cubic and noncubic Dirac operators.

Theorem 4.4.1. *As above, suppose H_0 is a maximal rank connected real closed subgroup of $G_{\mathbb{C}}$. Then, for every irreducible finite-dimensional representation V of \mathfrak{g} , one has*

$$\ker(D_{\mathfrak{g},\mathfrak{h}}(V)) \subseteq \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)).$$

The converse is not true. For example, for a compact connected semisimple real Lie group G' , consider the symmetric space $G' \times G' / \Delta(G' \times G')$, where $\Delta(G' \times G')$ is the diagonal in $G' \times G'$. Then both the cubic and the noncubic Dirac operators coincide while the ranks of $G' \times G'$ and $\Delta(G' \times G')$ differ. Moreover, Example 4.4.4 implies that one cannot omit the equal rank condition. Indeed, take \mathfrak{g} to be the 8-dimensional simple complex Lie algebra $\mathfrak{sl}(3, \mathbb{C})$. If

$$\mathfrak{t} = \mathbb{C}H_{\varepsilon_1-\varepsilon_2} + \mathbb{C}H_{\varepsilon_2-\varepsilon_3},$$

where

$$H_{\varepsilon_1-\varepsilon_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H_{\varepsilon_2-\varepsilon_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

is the (2-dimensional) Cartan subalgebra in \mathfrak{g} , fix

$$\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$$

to be the positive system of $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$, choose

$$\mathfrak{h} = \mathbb{C}(5H_{\varepsilon_1 - \varepsilon_2} + 4H_{\varepsilon_2 - \varepsilon_3})$$

and

$$\mathfrak{q} = \mathbb{C}(H_{\varepsilon_1 - \varepsilon_2} - 2H_{\varepsilon_2 - \varepsilon_3}) + \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} + \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} + \mathfrak{g}_{\varepsilon_2 - \varepsilon_3} + \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} + \mathfrak{g}_{-\varepsilon_1 + \varepsilon_3} + \mathfrak{g}_{-\varepsilon_2 + \varepsilon_3}.$$

If V is the finite-dimensional irreducible representation of highest weight $\frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3$, one can check that the noncubic Dirac operator attached with the pair $(\mathfrak{g}, \mathfrak{h})$ does not annihilate the highest weight vector $v \otimes 1$. In other words, for this example, we have:

$$\ker(D_{\mathfrak{g}, \mathfrak{h}}(V)) \not\subseteq \ker(\widehat{D}_{\mathfrak{g}, \mathfrak{h}}(V)).$$

Motivated by Theorem 4.4.1, we restrict ourselves to the case where the subalgebra \mathfrak{l} is a Cartan subalgebra of \mathfrak{g} , i.e. \mathfrak{s} is a Borel subalgebra. The main result in Section 4.6 is the following theorem.

Theorem 4.6.13. *Let \mathfrak{g} be a complex semisimple Lie algebra, $\mathfrak{h} := \mathfrak{t}$ a Cartan subalgebra of \mathfrak{g} , Δ the corresponding root system and Δ^+ a positive system of Δ . For a Δ^+ -dominant algebraically integral element $\lambda \in \mathfrak{t}^*$, let V be the corresponding finite-dimensional irreducible representation of highest weight λ and $A(\lambda)$ be the subset of \mathfrak{t}^* defined in (4.11). The kernel of the noncubic Dirac operator $\widehat{D}_{\mathfrak{g}, \mathfrak{h}}(V)$ acting on $V \otimes S$ is*

$$\ker(\widehat{D}_{\mathfrak{g}, \mathfrak{h}}(V)) = \bigoplus_{\mu \in A(\lambda)} (V \otimes S)^{[\mu]}. \quad (19)$$

Here $(V \otimes S)^{[\mu]}$ stands for the direct sum

$$(V \otimes S)^{[\mu]} := \bigoplus_{w \in W/W_\mu} (V \otimes S)_{w\mu},$$

with $(V \otimes S)_{w\mu}$ being the $w\mu$ -isotypic component of $V \otimes S$ and W_μ the stabilizer of μ in W . The set $A(\lambda)$ depends on λ and is studied in Section 4.7. Crucial ingredients for the proof of Theorem 4.6.13 are Weyl's inequalities discussed in Section 4.5. More precisely, if A and B are self-adjoint operators acting on a finite-dimensional vector space, Weyl's inequalities (4.3) relate the eigenvalues of A and B with the eigenvalues of the operator sum $A + B$.

Note that, in the above context, i.e. for \mathfrak{h} being a Cartan subalgebra, an isotypic component $(V \otimes S)_\mu$, $\mu \in \mathfrak{t}^*$, either is contained in $\ker(\widehat{D}_{\mathfrak{g}, \mathfrak{h}}(V))$ or has trivial intersection with $\ker(\widehat{D}_{\mathfrak{g}, \mathfrak{h}}(V))$. This is also true for cubic Dirac operators $D_{\mathfrak{g}, \mathfrak{h}}(V)$ even if \mathfrak{h} is not a Cartan subalgebra of \mathfrak{g} . Nevertheless, this is not the case for $\widehat{D}_{\mathfrak{g}, \mathfrak{h}}(V)$ when \mathfrak{h} is not a Cartan subalgebra \mathfrak{l} of \mathfrak{g} . In Example 4.6.18, we consider $\mathfrak{g} := \mathfrak{sl}(4, \mathbb{C})$, with

$$\mathfrak{h} = \mathfrak{t} + \mathfrak{g}_{\varepsilon_3 - \varepsilon_4} + \mathfrak{g}_{-\varepsilon_3 + \varepsilon_4}$$

and $V := \mathbb{C}^4$ to be the standard representation of \mathfrak{g} , i.e. the finite-dimensional irreducible representation of \mathfrak{g} with highest weight $\lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4$. Let

$$\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V) : V \otimes S \rightarrow V \otimes S$$

be the corresponding noncubic Dirac operator and consider the vectors

$$\begin{aligned} x_1 &:= v_3 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4} \\ x_2 &:= v_4 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3}. \end{aligned}$$

of $V \otimes S$, where v_3 and v_4 are the third and fourth, respectively, standard basis vectors of \mathbb{C}^4 . Then

$$\begin{aligned} \widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)x_1 &= v_2 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \neq 0 \\ \widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)x_2 &= -v_2 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \neq 0 \end{aligned}$$

and

$$\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)(x_1 + x_2) = 0.$$

The elements x_1 , x_2 and $x_1 + x_2$ belong to the same isotypic component of $V \otimes S$, the element $x_1 + x_2$ belonging to $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$, while x_1 and x_2 do not. In other words, only a piece of this isotypic component belongs to $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$.

Finally, for the t -cubic Dirac operators $D_{\mathfrak{g},\mathfrak{h}}^t(V)$, $t \in (0, \frac{1}{3}]$, we prove the following theorem.

Theorem 4.8.2. *Let \mathfrak{g} be a complex semisimple Lie algebra and \mathfrak{h} a Lie subalgebra of \mathfrak{g} such that \mathfrak{h} satisfies condition (2.10) and $\text{rank}_{\mathbb{C}}(\mathfrak{g}) = \text{rank}_{\mathbb{C}}(\mathfrak{h})$. Let \mathfrak{t} be a common Cartan subalgebra of \mathfrak{g} and \mathfrak{h} , and V an irreducible finite-dimensional \mathfrak{g} -representation of highest weight $\lambda \in \mathfrak{t}^*$. For $t \in (0, \frac{1}{3}]$, consider the t -cubic Dirac operator*

$$D_{\mathfrak{g},\mathfrak{h}}^t(V) : V \otimes S \rightarrow V \otimes S$$

of Definition 4.8.1 and the cubic Dirac operator

$$D_{\mathfrak{g},\mathfrak{h}}(V) := D_{\mathfrak{g},\mathfrak{h}}^{1/3}(V) : V \otimes S \rightarrow V \otimes S$$

of Definition 3.1.1. Then, for every $t \in (0, \frac{1}{3}]$,

$$\ker(D_{\mathfrak{g},\mathfrak{h}}^t(V)) = \ker(D_{\mathfrak{g},\mathfrak{h}}(V)).$$

In particular, according to Theorem 3.2.4,

$$\ker(D_{\mathfrak{g},\mathfrak{h}}^t(V)) = \bigoplus_{w \in W^1} F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}},$$

where F_{μ} is the finite-dimensional irreducible \mathfrak{h} -representation of highest weight $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$ and

$$W^1 := \{w \in W \mid \Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+\}.$$

In other words, despite the fact that the square of the operators $D_{\mathfrak{g},\mathfrak{h}}^t(V)$, $t \in (0, \frac{1}{3})$, do not have the "nice" form of $D_{\mathfrak{g},\mathfrak{h}}(V)^2$ described in Proposition 3.2.6, the statement of Theorem 3.2.4 formulated for $D_{\mathfrak{g},\mathfrak{h}}^t(V)$, $t \in (0, 1/3)$ is still true and provides us with a complete description of the decomposition of their kernels in terms of irreducible representations.

In Chapters 5 and 6, we apply our results for noncubic Dirac operators in the case where $\mathfrak{h} \subseteq \mathfrak{g}$ is a Cartan subalgebra \mathfrak{t} of \mathfrak{g} and for V the standard representation of \mathfrak{g} .

Let us illustrate the situation when \mathfrak{g} is a classical Lie algebra with an example. Suppose that $\mathfrak{g} := \mathfrak{sl}(4, \mathbb{C})$ and $\mathfrak{h} := \mathfrak{t}$ is a Cartan subalgebra of \mathfrak{g} . The 4-dimensional standard representation (π, V) of \mathfrak{g} is of highest weight $\lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4$. Actually, it can be realized as $V := \mathbb{C}^4$ equipped with the matrix multiplication action of $\mathfrak{sl}(4, \mathbb{C})$ on V . The weights of V are

$$\begin{aligned}\mu_1 &:= \lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_2 &:= \lambda - \varepsilon_1 + \varepsilon_2 = -\frac{1}{4}\varepsilon_1 + \frac{3}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_3 &:= \lambda - \varepsilon_1 + \varepsilon_3 = -\frac{1}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 + \frac{3}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_4 &:= \lambda - \varepsilon_1 + \varepsilon_4 = -\frac{1}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 + \frac{3}{4}\varepsilon_4\end{aligned}$$

and the corresponding weight vectors v_i , $1 \leq i \leq 4$, are the standard basis vectors of \mathbb{C}^4 . For every $\alpha \in \Delta$, choose e_α in \mathfrak{g} such that $B(e_\alpha, e_{-\alpha}) = 1$. Then, the noncubic Dirac operator $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ is given by

$$\begin{aligned}\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V) &= \sqrt{2}\pi(e_{\varepsilon_1-\varepsilon_2}) \otimes \gamma(e_{-\varepsilon_1+\varepsilon_2}) + \sqrt{2}\pi(e_{\varepsilon_1-\varepsilon_3}) \otimes \gamma(e_{-\varepsilon_1+\varepsilon_3}) + \sqrt{2}\pi(e_{\varepsilon_1-\varepsilon_4}) \otimes \gamma(e_{-\varepsilon_1+\varepsilon_4}) \\ &+ \sqrt{2}\pi(e_{\varepsilon_2-\varepsilon_3}) \otimes \gamma(e_{-\varepsilon_2+\varepsilon_3}) + \sqrt{2}\pi(e_{\varepsilon_2-\varepsilon_4}) \otimes \gamma(e_{-\varepsilon_2+\varepsilon_4}) + \sqrt{2}\pi(e_{\varepsilon_3-\varepsilon_4}) \otimes \gamma(e_{-\varepsilon_3+\varepsilon_4}) \\ &+ \sqrt{2}\pi(e_{-\varepsilon_1+\varepsilon_2}) \otimes \gamma(e_{\varepsilon_1-\varepsilon_2}) + \sqrt{2}\pi(e_{-\varepsilon_1+\varepsilon_3}) \otimes \gamma(e_{\varepsilon_1-\varepsilon_3}) + \sqrt{2}\pi(e_{-\varepsilon_1+\varepsilon_4}) \otimes \gamma(e_{\varepsilon_1-\varepsilon_4}) \\ &+ \sqrt{2}\pi(e_{-\varepsilon_2+\varepsilon_3}) \otimes \gamma(e_{\varepsilon_2-\varepsilon_3}) + \sqrt{2}\pi(e_{-\varepsilon_2+\varepsilon_4}) \otimes \gamma(e_{\varepsilon_2-\varepsilon_4}) + \sqrt{2}\pi(e_{-\varepsilon_3+\varepsilon_4}) \otimes \gamma(e_{\varepsilon_3-\varepsilon_4}).\end{aligned}$$

According to Lemma 4.6.15, the kernel of $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ is monomial, i.e. is generated by vectors $v_i \otimes u$ where v_i is as above and $u = e_{-j_1} \wedge \dots \wedge e_{-j_k}$ is a monomial of S . Let $v_i \otimes u$ be such a vector in $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$. Then, according to Proposition 4.6.17, $v_i \otimes u$ belongs to

$$\bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha})).$$

If $i = 1$, i.e. is the weight vector $v_1 = (1, 0, 0, 0)$ of $V = \mathbb{C}^4$, then v_1 is annihilated by all $\pi(e_\beta)$, $\beta \in \Delta$, except for $\beta = -\varepsilon_1 + \varepsilon_2$, $\beta = -\varepsilon_1 + \varepsilon_3$ and $\beta = -\varepsilon_1 + \varepsilon_4$. Therefore, $v_1 \otimes u$ belongs to $\bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha}))$ if and only if $\gamma(e_{-\beta})$ annihilates u for these values of β . In other words, $e_{-\varepsilon_1+\varepsilon_2}$, $e_{-\varepsilon_1+\varepsilon_3}$ and $e_{-\varepsilon_1+\varepsilon_4}$ should not appear in u . Hence, one deduces that the only monomial vectors of the form $v_1 \otimes u$ belonging to $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$ are

$$\begin{aligned}&v_1 \otimes 1, v_1 \otimes e_{-\varepsilon_2+\varepsilon_3}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_3+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4}, \\ &v_1 \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4}.\end{aligned}$$

In the same way, one finds the other vectors $v_i \otimes u$ of $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$ for $i = 2, 3, 4$. In particular, $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$ is 32-dimensional while, using Theorem 3.2.4, one deduces that the dimension of $\ker(D_{\mathfrak{g},\mathfrak{h}}(V))$ is equal to $|W| = 4! = 24$.

The next table computes the size of $\ker(D_{\mathfrak{g},\mathfrak{h}}(V))$ and $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$ for classical Lie algebras in the case where \mathfrak{h} is a Cartan subalgebra and V is the standard representation.

\mathfrak{g}	$\dim \ker(D_{\mathfrak{g},\mathfrak{h}}(V)) = W $	$\dim \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$
$\mathfrak{sl}(n+1, \mathbb{C})$	$(n+1)!$	$(n+1) \times 2^{\frac{n(n-1)}{2}}$
$\mathfrak{so}(2n+1, \mathbb{C})$	$2^n n!$	$2n \times 2^{(n-1)^2}$
$\mathfrak{sp}(n, \mathbb{C})$	$2^n n!$	$2n \times 2^{(n-1)^2}$
$\mathfrak{so}(2n, \mathbb{C})$	$2^{n-1} n!$	$2n \times 2^{(n-1)(n-2)}$

Table 5.1: Dimensions of the kernels of cubic and noncubic Dirac operators

For the exceptional Lie algebras, \mathfrak{h} a Cartan subalgebra \mathfrak{t} of \mathfrak{g} and V the standard representation of \mathfrak{g} of highest weight λ , in order to describe $\ker(\widehat{D})$, we determine the set $A(\lambda)$ of Theorem 4.6.13. $A(\lambda)$ is defined to be the set of equivalence classes

$$A(\lambda) = \widetilde{A}(\lambda) / \sim,$$

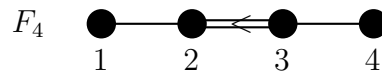
where

$$\widetilde{A}(\lambda) := \{\lambda + \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+\}$$

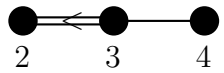
and \sim is the equivalence relation identifying the elements μ and $w\mu$ of $\widetilde{A}(\lambda)$ for every element w of the stabilizer W_λ of λ in W . In Proposition 4.7.1, we give the following characterization for $A(\lambda)$:

$$A(\lambda) = \{\lambda + \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq \Delta_1^+ \text{ and } \rho_1 - \sum_{\alpha \in A} \alpha \text{ is } \Delta_1^+ \text{-dominant}\},$$

with $\Delta_1^+ := (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+$ and ρ_1 being half the sum of all elements in Δ_1^+ . We prove that $\Delta_1 := \Delta_1^+ \cup (-\Delta_1^+)$ is a root system, the root system obtained by the Dynkin diagram of Δ if we delete the vertex corresponding to the first simple root α_1 of Δ . For instance, if \mathfrak{g} is the exceptional Lie algebra F_4 , its Dynkin diagram is



and deleting the edge corresponding to α_1 , gives the Dynkin diagram of Δ_1 :



Weight	Vectors
$\frac{5}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	1
$\frac{5}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_3}$ $e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{3}{2}\varepsilon_4$	$e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2}$ $e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3}$ $e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2}\wedge e_{-\varepsilon_3}$ $e_{-\varepsilon_3}\wedge e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_4}$ $e_{-\varepsilon_2}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_4}$ $e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2-\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_3}\wedge e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}$
$\frac{1}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2}\wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2}\wedge e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_3}\wedge e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_3}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2}\wedge e_{-\varepsilon_3}\wedge e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_4}$ $e_{-\varepsilon_3}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_3}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2}\wedge e_{-\varepsilon_3}\wedge e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_3+\varepsilon_4}$ $e_{-\varepsilon_4}\wedge e_{-\varepsilon_2+\varepsilon_3}\wedge e_{-\varepsilon_2+\varepsilon_4}\wedge e_{-\varepsilon_3+\varepsilon_4}\wedge e_{-\varepsilon_3-\varepsilon_4}$

Table 6.1: Δ_1^+ -dominant weights of S_1 for $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$ and $\mathfrak{h} = \mathfrak{t}$

This is exactly the Dynkin diagram of the classical Lie algebra B_3 . Then, we construct a new space of spinors S_1 for the Clifford algebra of the direct sum of all rootspaces of Δ_1 . In order to describe $A(\lambda)$, it suffices to find all Δ_1^+ -dominant weights of S_1 . Table 6.1 gives these weights for the space of spinors S_1 of Δ_1^+ for the above example.

4.2 Geometric Dirac operators

In Chapter 7, motivated by the connection between the algebraic cubic Dirac operator $D_{\mathfrak{g}, \mathfrak{h}}$ and the geometric cubic Dirac operator \mathcal{D}_{G_c/H_0} established in [MZ14a], following [Lan00], we use

Peter-Weyl Theorem to define an isomorphism:

$$\Phi : [L^2(G_c) \otimes E]^H \xrightarrow{\sim} \widehat{\bigoplus_{\lambda \in \widehat{G}_c} V_\lambda} \otimes [V_\lambda^* \otimes E]^{H_0}$$

which generalizes this connection in the case of geometric noncubic and t -cubic Dirac operators.

Following [Lan00, Theorem 4] and [Sle87a], we obtain the following description of the kernel of the noncubic Dirac operator in the case where L_c is a maximal torus T of G_c .

Theorem 7.4.4. *Let G_c be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , T a maximal torus of G_c such that its complexified Lie algebra \mathfrak{t} is θ -stable. Fix E to be the 1-dimensional representation of \mathfrak{t} determined by a Δ^+ -dominant element μ of \mathfrak{t}^* with respect to a fixed positive root system Δ^+ of $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ and such that $S \otimes E$ lifts to a representation of T . For every element λ of the set Λ of all Δ^+ -dominant and analytically integral elements of \mathfrak{t}^* , set*

$$K_\lambda := \left\{ \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+ \text{ and } \rho - \sum_{\alpha \in A} \alpha \text{ is } \Delta^+ \text{-dominant} \right\}$$

and

$$\mathcal{A}_\mu := \{(\lambda, \kappa) \in \Lambda \times K_\lambda \mid \lambda + \kappa = \mu\}.$$

Then, the kernel of the noncubic Dirac operator

$$\widehat{\mathcal{D}}_{G_c/T}(E) : [L^2(G_c) \otimes (S \otimes E)]^T \rightarrow [L^2(G_c) \otimes (S \otimes E)]^T$$

is

$$\ker \widehat{\mathcal{D}}_{G_c/T}(E) \simeq \dim(V \otimes S)_\mu \sum_{(\lambda, \kappa) \in \mathcal{A}_\mu} V_{\mu - \kappa}$$

where $V_{\mu - \kappa}$ is the finite-dimensional irreducible G_c -representation of highest weight $\mu - \kappa$.

Moreover, the following theorem generalizes [Lan00, Theorem 4] in the unequal rank case.

Theorem 7.3.10. *Let G_c be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H_0 be a closed subgroup of G_c such that its complexified Lie algebra \mathfrak{h} satisfies condition (2.10). Moreover, suppose that $\mathfrak{t}_\mathfrak{h}$ is a Cartan subalgebra of \mathfrak{h} and $\mathfrak{t}_\mathfrak{q} \subseteq \mathfrak{q}$ is such that $\mathfrak{t} := \mathfrak{t}_\mathfrak{h} + \mathfrak{t}_\mathfrak{q}$ is a Cartan subalgebra of \mathfrak{g} . Let $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ be the corresponding root system of \mathfrak{g} , fix Δ^+ to be a positive system for Δ satisfying the $\mathfrak{t}_\mathfrak{h}$ -property of Definition 3.2.3, and $\Delta_\mathfrak{h}^+$ to be the corresponding positive system for $\Delta_\mathfrak{h}$. Fix E to be an irreducible finite-dimensional \mathfrak{h} -representation of highest weight $\mu \in \mathfrak{t}_\mathfrak{h}^*$ with respect to $\Delta_\mathfrak{h}^+$ and such that $S \otimes E$ lifts to a representation of H_0 . Set Λ to be the set of all analytically integral and Δ^+ -dominant elements of \mathfrak{t}^* and, for every $\lambda \in \Lambda$, set*

$$W_\lambda^1 := \{w \in W \mid w(\lambda + \rho)|_{\mathfrak{t}_\mathfrak{q}} = 0 \text{ and } w(\lambda + \rho)|_{\mathfrak{t}_\mathfrak{h}} \text{ is } \Delta_\mathfrak{h}^+ \text{-dominant}\}.$$

If there are $\lambda \in \Lambda$ and $w \in W_\lambda^1$ such that $w^{-1}(\mu + \rho_\mathfrak{h}) - \rho$ is analytically integral and Δ^+ -dominant, the kernel of the geometric cubic Dirac operator

$$\mathcal{D}_{G_c/H_0}(E) : [L^2(G_c) \otimes (S \otimes E)]^{H_0} \rightarrow [L^2(G_c) \otimes (S \otimes E)]^{H_0}$$

is equivalent with 2^d copies of the G_c -representation $V_{w^{-1}(\mu+\rho_{\mathfrak{h}})-\rho}$ of highest weight $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$ where $d := \lceil \frac{\dim \mathfrak{t}_{\mathfrak{h}}}{2} \rceil$. Otherwise, $\ker \mathcal{D}_{G_c/H_0}(E) = 0$.

Finally, by analogy to Theorem 4.8.2, we describe $\ker(\mathcal{D}_{G_c/H_0}^t(E))$ for $t \in (0, \frac{1}{3}]$, showing that it coincides with $\ker(\mathcal{D}_{G_c/H_0}(E))$.

Theorem 7.5.2. *Let G_c be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H_0 be a closed subgroup of G_c such that its complexified Lie algebra \mathfrak{h} satisfies condition (2.10). Moreover, suppose that there is a common Cartan subalgebra \mathfrak{t} of \mathfrak{g} and \mathfrak{h} , and E is an irreducible finite-dimensional \mathfrak{h} -representation of highest weight $\mu \in \mathfrak{t}^*$ with respect to a fixed positive root system $\Delta_{\mathfrak{h}}^+$ such that $S \otimes E$ lifts to a representation of H_0 . Then, for $t \in (0, \frac{1}{3}]$, the kernel $\ker(\mathcal{D}_{G_c/H_0}^t(E))$ of the geometric t -cubic Dirac operator*

$$\mathcal{D}_{G_c/H_0}^t(E) : [L^2(G_c) \otimes (S \otimes E)]^{H_0} \rightarrow [L^2(G_c) \otimes (S \otimes E)]^{H_0}$$

coincides with the kernel $\ker(\mathcal{D}_{G_c/H_0}(E))$ of the geometric cubic Dirac operator $\mathcal{D}_{G_c/H_0}(E)$. In particular, if there is $w \in W$ such that $w(\mu + \rho_{\mathfrak{h}}) - \rho$ is analytically integral and dominant with respect to a positive system $\Delta^+ \supseteq \Delta_{\mathfrak{h}}^+$ of Δ , $\ker(\mathcal{D}_{G_c/H_0}^t(E))$ is equivalent with the finite-dimensional irreducible G_c -representation $V_{w(\mu+\rho_{\mathfrak{h}})-\rho}$ of highest weight $w(\mu + \rho_{\mathfrak{h}}) - \rho$. Otherwise, $\ker(\mathcal{D}_{G_c/H_0}(E))$ is trivial.

4.3 Dirac Indices

In Section 7.6, in the case of equal rank \mathfrak{g} and \mathfrak{h} , we consider the \mathbb{Z}_2 -grading $S = S^+ \otimes S^-$ (see Remark 2.2.3) and the operators

$$\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)^{\pm} : V \otimes S^{\pm} \rightarrow V \otimes S^{\mp}.$$

Then, the (algebraic) noncubic Dirac index of a finite-length \mathfrak{g} -module V is the virtual \mathfrak{h} -module

$$I_{\widehat{D}}(V) = \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)^+) - \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)^-).$$

Following [MPV17a], we prove the following lemma.

Lemma 7.6.6. *If V is finite-dimensional, then*

$$I_{\widehat{D}}(V) = V \otimes S^+ - V \otimes S^-.$$

Furthermore, for a finite-dimensional \mathfrak{h} -representation E such that $S \otimes E$ lifts to an H_0 -representation, we consider the operators

$$\widehat{\mathcal{D}}_{G_c/H_0}(E)^{\pm} : [L^2(G_c) \otimes (S^{\pm} \otimes E)]^{H_0} \rightarrow [L^2(G_c) \otimes (S^{\mp} \otimes E)]^{H_0}$$

of (7.21) and we define the (geometric) noncubic Dirac index of E to be the virtual G_c -module

$$\text{Index}_{G_c} \widehat{\mathcal{D}}_{G_c/H_0}(E) := \ker(\widehat{\mathcal{D}}_{G_c/H_0}(E)^+) - \ker(\widehat{\mathcal{D}}_{G_c/H_0}(E)^-).$$

Then, following [Lan00], we calculate the noncubic Dirac index of E in the following theorem.

Theorem 7.6.10. *Let G_c be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H_0 be a closed subgroup of G_c with complexified Lie algebra \mathfrak{h} such that \mathfrak{h} satisfies condition (2.10). Suppose that there is a common Cartan subalgebra \mathfrak{t} of \mathfrak{g} and \mathfrak{h} and fix E to be an irreducible finite-dimensional \mathfrak{h} -representation of highest weight $\mu \in \mathfrak{t}^*$ with respect to a fixed positive root system $\Delta_{\mathfrak{h}}^+$ and such that $S \otimes E$ lifts to a representation of H_0 . Extend $\Delta_{\mathfrak{h}}^+$ to a positive system Δ^+ of Δ , set Λ to be the set of all analytically integral and Δ^+ -dominant elements of \mathfrak{t}^* , and*

$$W^1 := \{w \in W \mid \Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+\}.$$

If there is $w \in W^1$ such that $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$ is analytically integral and Δ^+ -dominant, then

$$\text{Index}_G \widehat{\mathcal{D}}_{G_c/H_0}(E) = (-1)^{\ell(w)} V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho},$$

where $V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho}$ is the irreducible finite-dimensional G_c -representation of highest weight $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$. Otherwise, $\text{Index}_G \widehat{\mathcal{D}}_{G_c/H_0}(E) = 0$. In particular,

$$\text{Index}_G \widehat{\mathcal{D}}_{G_c/H_0}(E) = \text{Index}_G \mathcal{D}_{G_c/H_0}(E).$$

INTRODUCTION EN FRANÇAIS

À la fin du XIXe siècle, le mathématicien norvégien Sophus Lie introduisit un analogue continu aux groupes galoisiens. Alors qu'un groupe de Galois encode la solvabilité par radicaux d'une équation algébrique ainsi que le comportement des solutions par permutations, un groupe de Lie porte informé sur la solvabilité d'une équation différentielle et le comportement des solutions sous l'action d'une famille continue de symétries. Dans la plupart des cas, par le principe de superposition, les solutions forment un espace vectoriel topologique. L'invariance des solutions sous les symétries encodées par le groupe de Lie peut se traduire en termes d'une action par transformations linéaires sur l'espace vectoriel des solutions. La théorie des représentations relie la structure d'un groupe de Lie, ou les symétries d'un ensemble, avec des actions linéaires sur des espaces vectoriels. Deux problèmes importants de la théorie des représentations sont l'unitarité et la réalisation explicite des représentations d'un groupe de Lie donné. A cet égard, les opérateurs différentiels invariants et les espaces de cohomologie, en particulier les opérateurs de Dirac, fournissent des informations cruciales.

1 Cohomologies de Dolbeault et des algèbres de Lie

Soit G_0 un groupe de Lie connexe semi-simple non compact linéaire réel d'algèbre de Lie complexifiée \mathfrak{g} et K_0 un sous-groupe compact maximal de G_0 d'algèbre de Lie complexifiée \mathfrak{k} associé à une involution de Cartan θ de G_0 . Soit G une complexification connexe semi-simple de G_0 d'algèbre de Lie \mathfrak{g} et G_c une forme réelle compacte de G . Soit H_0 un sous-groupe fermé connexe de G_c d'algèbre de Lie complexifiée \mathfrak{h} . Nous supposons que la forme de Killing de \mathfrak{g} reste non-dégénérée lorsqu'elle est restreinte à \mathfrak{h} , c'est-à-dire que \mathfrak{h} est quadratique dans \mathfrak{g} . L'espace homogène G/H_0 est alors réductif, en ce sens que:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp \quad \text{où} \quad [\mathfrak{h}, \mathfrak{h}^\perp] \subseteq \mathfrak{h}^\perp. \quad (20)$$

La théorie des représentations de G_c est bien comprise; en particulier chaque représentation unitaire irréductible de G_c est de dimension finie, et toute représentation de dimension finie de G_c est équivalente à une représentation unitaire. Le célèbre théorème de plus haut poids 1.5.17 classifie les représentations irréductibles de dimension finie comme suit:

- (i) une représentation irréductible de dimension finie de G_c a un plus haut poids unique qui est dominant et analytiquement intégral,
- (ii) deux représentations irréductibles de dimension finie de G_c ayant le même plus haut poids sont isomorphes.

De plus, si V_λ désigne la représentation irréductible de dimension finie de G_c de plus haut poids λ (relativement à un système positif fixé) et V_λ^* sa représentation duale, le Théorème de Peter-Weyl 7.2.2 décompose la représentation bi-régulière de G_c en somme directe orthogonale:

$$L^2(G_c) \simeq \bigoplus_{\mu} V_\mu \otimes V_\mu^*. \quad (21)$$

Il en découle une classification similaire découle pour les représentations irréductibles de dimension finie des algèbres de Lie complexes semi-simples dans le théorème 1.4.22: les représentations irréductibles de dimension finie de \mathfrak{g} sont en bijection avec des plus haut poids dominants intégraux.

Il s'avère que le théorème de plus haut poids porte également sur l'existence de représentation de plus haut poids. A savoir, pour chaque élément dominant analytiquement intégral λ , il existe une représentation irréductible de dimension finie de plus haut poids λ . Une construction explicite d'une telle représentation se fait comme suit. Considérons une variété de drapeaux complexe G/S , où S est un sous-groupe parabolique θ -stable de G d'algèbre de Lie \mathfrak{s} . Écrivons $S = LU$ (respectivement $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$) pour la décomposition de Levi de S (respectivement \mathfrak{s}) et \bar{u} pour l'opposé de u . Alors G_c agit transitivement sur G/S et l'on obtient un difféomorphisme de variétés complexes:

$$G/S \simeq G_c/G_c \cap S.$$

Il s'avère que le sous-groupe connexe $L_c := G_c \cap S$ est une forme réelle compacte de L , L_c coïncide avec le centralisateur d'un tore dans G_c et est de rang maximal dans G_c . Selon [Zie00], notons $\Omega^p(G_c/L_c)$ l'espace des sections lisses du fibré vectoriel $\bigwedge^p(T^{(0,1)}G_c/L_c)^*$ au-dessus de la variété complexe compacte G_c/L_c , muni de l'opérateur G_c -equivariant:

$$\bar{\partial}_{G_c/L_c} : \Omega^p(G_c/L_c) \rightarrow \Omega^{p+1}(G_c/L_c)$$

défini par

$$\begin{aligned} \bar{\partial}_{G_c/L_c} \omega(X_0, \dots, X_p) &= \sum_{k=0}^p (-1)^k X_k [\omega(X_0, \dots, \hat{X}_k, \dots, X_p)] \\ &+ \sum_{i < j} (-1)^{i+j} \omega([X_i, X_j], X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p) \end{aligned}$$

où le chapeau indique que le terme correspondant est omis. Soit $\mathcal{E} \rightarrow G_c/L_c$ un fibré vectoriel holomorphe de dimension finie sur G_c/L_c induit par une représentation de dimension finie E de L_c . L'opérateur G_c -equivariant $\bar{\partial} := 1_{\mathcal{E}} \otimes \bar{\partial}_{G_c/L_c}$ définit un complexe

$$\begin{aligned} \bar{\partial} : \Omega^p(G_c/L_c, E) &\rightarrow \Omega^{p+1}(G_c/L_c, E) \\ \bar{\partial}^2 &= 0. \end{aligned}$$

sur les espaces

$$\begin{aligned}\Omega^p(G_c/L_c, E) &\simeq \{C^\infty(G_c) \otimes \wedge^p \mathfrak{u}^* \otimes E\}^{L_c} \\ &= \{f : G_c \xrightarrow{C^\infty} \wedge^p \mathfrak{u}^* \otimes E \mid f(gl) = l^{-1}f(g), \forall g \in G_c, \forall l \in L_c\}\end{aligned}$$

des formes lisses de type $(0, p)$, $p \geq 0$, à valeurs dans \mathcal{E} . Les espaces quotients

$$H^p(G_c/L_c, E) = \ker(\bar{\partial})/\text{Im}(\bar{\partial}),$$

munis d'une G_c -action par translations à gauche sur $C^\infty(G_c)$, sont des G_c -modules de dimension finie, appelés espaces de cohomologie de Dolbeault.

La cohomologie de Dolbeault est étroitement liée à la cohomologie des algèbre de Lie. Plus précisément, soit V un G_c -module de dimension finie. Notons

$$\mathcal{C}^p(\mathfrak{u}, V) := \text{Hom}_{\mathbb{C}}(\wedge^p \mathfrak{u}, V)$$

l'espace de p -formes linéaires alternées à valeurs dans V (avec $\mathcal{C}^0(\mathfrak{u}, V) := V$). Pour $T \in \mathcal{C}^p(\mathfrak{u}, V)$, définissons $\delta T \in \mathcal{C}^{p+1}(\mathfrak{u}, V)$ par:

$$\begin{aligned}(\delta T)(\xi_0, \dots, \xi_p) &:= \sum_{i=0}^p (-1)^k \xi_i [T(\xi_0, \dots, \widehat{\xi}_i, \dots, \xi_p)] \\ &+ \sum_{i < j} (-1)^{i+j} T([\xi_i, \xi_j], \xi_0, \dots, \widehat{\xi}_i, \dots, \widehat{\xi}_j, \dots, \xi_p).\end{aligned}$$

Le complexe $(\mathcal{C}(\mathfrak{u}, V), \delta)$ calcule le p -th espace de \mathfrak{u} -cohomologie à coefficients dans V :

$$\mathcal{H}^p(\mathfrak{u}, V) = \ker(\delta)/\text{Im}(\delta).$$

Observons que L_c normalise \mathfrak{u} et agit ainsi sur $\wedge^p \mathfrak{u}^*$ par l'application adjointe. De plus, L_c agit sur V et donc l agit sur $\mathcal{C}^p(\mathfrak{u}, V)$. Cette action commute avec δ , donc les espaces $\mathcal{H}^p(\mathfrak{u}, V)$ sont des L_c -modules (de dimension finie). De plus, on a l'isomorphisme de G_c -modules:

$$\begin{aligned}\Omega^p(G_c/L_c, E) &\simeq \{C^\infty(G_c) \otimes \wedge^p \mathfrak{u}^* \otimes E\}^{L_c} \\ &\simeq \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{V_\lambda^* \otimes \wedge^p \mathfrak{u}^* \otimes E\}^{L_c} \text{ by (21)} \\ &\simeq \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\text{Hom}_{\mathbb{C}}(\wedge^p \mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c}.\end{aligned}$$

A présent, la commutativité du diagramme suivant

$$\begin{array}{ccc}\Omega^p(G_c/L_c, E) & \xrightarrow{\quad \bar{\partial} \quad} & \Omega^{p+1}(G_c/L_c, E) \\ \downarrow \wr & & \downarrow \wr \\ \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\text{Hom}_{\mathbb{C}}(\wedge^p \mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c} & \xrightarrow{\quad 1 \otimes \delta \otimes 1 \quad} & \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\text{Hom}_{\mathbb{C}}(\wedge^{p+1} \mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c}\end{array}$$

implique que

$$H^p(G_c/L_c, E) \simeq \bigoplus_{\lambda \in \widehat{G}_c} V_\lambda \otimes \{\mathcal{H}^p(\mathfrak{u}, V_\lambda^*) \otimes E\}^{L_c}. \quad (22)$$

La \mathfrak{u} -cohomologie a été calculée par Kostant dans [Kos61]. Plus précisément, soient $W_{\mathfrak{g}}$ le groupe de Weyl pour \mathfrak{g} , $W_{\mathfrak{l}} \subset W_{\mathfrak{g}}$ le groupe de Weyl pour \mathfrak{l} , et ρ la demi-somme des racines positives pour un système positif $\Delta^+ \subset \Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ défini par \mathfrak{u} . Si, de plus, $\Delta_{\mathfrak{l}}^+ \subseteq \Delta^+$ est le système positif pour $\Delta_{\mathfrak{l}} := \Delta(\mathfrak{l}, \mathfrak{t})$ et

$$W^1 := \{w \in W_{\mathfrak{g}} \mid \Delta_{\mathfrak{l}}^+ \subseteq w\Delta^+\},$$

alors on a la décomposition sans multiplicité suivante de L_c -modules:

$$\mathcal{H}^p(\mathfrak{u}, V_\lambda^*) \simeq \bigoplus_{\substack{w \in W^1 \\ \ell(w)=p}} F_{-ww_0(\lambda+\rho)-\rho} \quad (23)$$

où $\ell(w)$ est la longueur de w , w_0 est l'élément de longueur maximale de $W_{\mathfrak{g}}$ et $F_{w(\lambda+\rho)-\rho}$ la représentation irréductible de dimension finie de plus haut poids $-ww_0(\lambda + \rho) - \rho$ de L_c . De (22), nous déduisons que

$$H^p(G_c/L_c, E) \simeq \bigoplus_{\lambda \in \widehat{G}_c} \bigoplus_{\substack{w \in W^1 \\ \ell(w)=\#\{\Delta_{\mathfrak{u}}\}-p}} V_\lambda \otimes \text{Hom}_{L_c}(F_{w(\lambda+\rho)-\rho_{\mathfrak{l}}+\rho_{\mathfrak{u}}}, E) \quad (24)$$

où $\Delta_{\mathfrak{u}} := \Delta^+ \setminus \Delta_{\mathfrak{l}}^+$. En particulier, dans le cas où μ est un élément analytiquement intégral de l'espace dual de la sous-algèbre de Cartan de \mathfrak{l} et $E = E_\mu$ est la représentation irréductible de dimension finie de L_c de plus haut poids μ , on retrouve le Théorème de Borel-Weil-Bott [Bot57]:

$$H^{\#\{\Delta_{\mathfrak{u}}\}-\ell(w)}(G_c/L_c, E_\mu) \simeq \begin{cases} V_{w^{-1}(\mu+\rho_{\mathfrak{l}}-\rho_{\mathfrak{u}})-\rho} & \text{si } w^{-1}(\mu+\rho_{\mathfrak{l}}-\rho_{\mathfrak{u}})-\rho \text{ est } \Delta^+\text{-dom. pour un } w \in W^1 \\ \{0\} & \text{si un tel } w \text{ n'existe pas.} \end{cases}$$

2 Cohomologie de Dirac

La cohomologie de Dolbeault et la \mathfrak{u} -cohomologie sont liées aux opérateurs de Dirac cubiques comme suit. Les sous-algèbres \mathfrak{u} et $\bar{\mathfrak{u}}$ sont des sous-espaces isotropes duaux maximaux, et $\mathfrak{q} := \mathfrak{u} \oplus \bar{\mathfrak{u}}$ est un sous-espace L_c -invariant de \mathfrak{g} . De plus, \mathfrak{g} peut être décomposée comme

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{q} \quad \text{and} \quad \text{Ad}(L_c)(\mathfrak{q}) \subset \mathfrak{q}, \quad (25)$$

par analogie avec (20). Soit S l'espace des spineurs de l'algèbre de Clifford $\mathbf{C}(\mathfrak{q})$ de \mathfrak{q} , munie de la multiplication de Clifford γ . \mathfrak{l} se plonge dans $\mathbf{C}(\mathfrak{q})$ de sorte que S devienne un module pour \mathfrak{l} . Soit \mathfrak{c} l'élément de degré 3 dans $\mathbf{C}(\mathfrak{q})$ correspondant, via l'isomorphisme de Chevalley, à la forme trilinéaire alternée $\omega : (X, Y, Z) \mapsto B(X, [Y, Z])$ de \mathfrak{q} , où B est la forme Killing de \mathfrak{g} :

$$(\omega : (X, Y, Z) \mapsto B(X, [Y, Z])) \mapsto \mathfrak{c}. \quad (26)$$

Dans [Kos99], Kostant associe à chaque \mathfrak{g} -module (π, V) , un opérateur $D_{\mathfrak{g},\mathfrak{l}}(V)$

$$D_{\mathfrak{g},\mathfrak{l}}(V) : V \otimes S \rightarrow V \otimes S \quad (27)$$

défini par

$$D_{\mathfrak{g},\mathfrak{l}}(V) = \sum_j \pi(X_j) \otimes \gamma(X_j) - 1 \otimes \gamma(c)$$

où $\{X_j\}$ est une base orthonormée de \mathfrak{q} . Notons que, d'après le lemme 3.1.4, $D_{\mathfrak{g},\mathfrak{l}}(V)$ ne dépend pas du choix de la base $\{X_j\}$ tandis que, quand la paire $(\mathfrak{g}, \mathfrak{l})$ est symétrique, le terme cubique c s'annule. De plus, \mathfrak{l} agit sur $V \otimes S$, l'action étant donnée par la restriction de π à \mathfrak{l} et l'action spinorielle sur S , et l'opérateur $D_{\mathfrak{g},\mathfrak{l}}(V)$ est \mathfrak{l} -équivariant par rapport à cette action. Dans [Kos99], Kostant calcule le carré de $D_{\mathfrak{g},\mathfrak{l}}(V)$, montrant que $(D_{\mathfrak{g},\mathfrak{l}}(V))^2$ diffère du Casimir par un scalaire:

$$(D_{\mathfrak{g},\mathfrak{l}}(V))^2 = \pi(\Omega_{\mathfrak{g}}) \otimes 1 - (\pi \otimes \gamma)(\Omega_{\mathfrak{l}})_{\Delta} + (\|\rho\|^2 - \|\rho_{\mathfrak{l}}\|^2)1 \otimes 1, \quad (28)$$

où $\Omega_{\mathfrak{g}}$ (respectivement $\Omega_{\mathfrak{l}}$) désigne l'élément de Casimir dans l'algèbre enveloppante $U(\mathfrak{g})$ (respectivement $U(\mathfrak{l})$) de \mathfrak{g} (respectivement \mathfrak{l}), $\rho_{\mathfrak{l}}$ est la demi-somme des racines positives dans un système positif $\Delta_{\mathfrak{l}}^+ \subset \Delta^+$ pour les racines dans \mathfrak{l} , $\|\cdot\|$ est la norme induite par la forme de Killing de \mathfrak{g} , et $(\cdot)_{\Delta}$ est l'inclusion diagonale

$$\begin{aligned} (\cdot)_{\Delta} : \mathfrak{l} &\longrightarrow U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q}) \\ X &\mapsto X \otimes 1 + 1 \otimes \varphi(\text{ad}X) \end{aligned}$$

de \mathfrak{l} dans $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$, donnée par le plongement de $\mathfrak{l} \subset \mathfrak{g}$ dans $U(\mathfrak{g})$ et le plongement (2.11) de \mathfrak{l} dans $\mathbf{C}(\mathfrak{q})$. $D_{\mathfrak{g},\mathfrak{l}}(V)$ est connu comme l'opérateur de Dirac cubique de Kostant, généralisant les opérateurs de Dirac de Parthasarathy et sa formule pour le carré, apparaissant dans [Par72]. L'opérateur de Dirac cubique peut être défini de la même manière dans le cadre plus général des paires $(\mathfrak{g}, \mathfrak{h})$, y compris pour le cas de rangs inégaux. Pour plus de détails, on peut consulter, par exemple, [HKP09], [Kos03], [MP11], [MZ03] et [MZ14b]. À la fin des années 1990, dans une série d'exposés au MIT [Vog97], Vogan a défini la cohomologie de Dirac d'un (\mathfrak{g}, K_0) -module X comme étant le \tilde{K}_0 -module

$$H_D^{\mathfrak{g},\mathfrak{k}}(X) = \ker(D_{\mathfrak{g},\mathfrak{k}}(X)) / \text{Im}(D_{\mathfrak{g},\mathfrak{k}}(X)) \cap \ker(D_{\mathfrak{g},\mathfrak{k}}(X)), \quad (29)$$

où \tilde{K}_0 est le revêtement double spinoriel de K_0 . En supposant que X est irréductible et possède un caractère infinitésimal, il a conjecturé que si $H_D^{\mathfrak{g},\mathfrak{k}}(X)$ contient un \tilde{K}_0 -module de plus haut poids β alors X possède un caractère infinitésimal $\beta + \rho_{\mathfrak{k}}$. En d'autres termes, le caractère infinitésimal d'un module peut être obtenu à partir de sa cohomologie de Dirac. La conjecture de Vogan a été prouvée pour la première fois par Huang et Pandžić dans [HP02] pour des paires de rangs égaux $(\mathfrak{g}, \mathfrak{k})$ et par Kostant pour les opérateurs de Dirac cubiques pour des paires de rangs égaux $(\mathfrak{g}, \mathfrak{h})$ in [Kos03]. Il a été étendu aux modules avec un caractère infinitésimal généralisé par Pandžić et Somberg dans [PS16]. La cohomologie de Dirac a été calculée pour diverses familles de modules, y compris les modules de plus haut poids dans [HX12], les modules $A_{\mathfrak{q}}(\lambda)$ dans [HKP09], les modules d'Enright-Varadarajan généralisés dans [MP11], des

représentations unipotentes dans [BP15] et [BP11], les modules de Jacquet dans [DH11] ou (\mathfrak{g}, K_0) -modules avec un caractère infinitésimal généralisé dans [PS16].

Dans le cas où X est un (\mathfrak{g}, K_0) -module unitaire (respectivement de dimension finie), $D_{\mathfrak{g},\mathfrak{k}}(X)$ est un opérateur anti-auto-adjoint (respectivement auto-adjoint) et la cohomologie de Dirac se réduit au noyau [HP06, page 63]:

$$\ker(D_{\mathfrak{g},\mathfrak{k}}(X)) = \ker(D_{\mathfrak{g},\mathfrak{k}}(X)^2).$$

De plus, pour une paire $(\mathfrak{g}, \mathfrak{k})$ de rangs égaux, si X est un (\mathfrak{g}, K_0) -module unitaire avec un caractère infinitésimal Λ et β est un \widetilde{K}_0 -type apparaissant dans $X \otimes S$, alors l'inégalité de Parthasarathy-Dirac [Par80a]

$$\|\beta + \rho_{\mathfrak{k}}\| \geq \|\Lambda\|$$

doit être satisfaite, fournissant aussi un test d'unitarisabilité pour les (\mathfrak{g}, K_0) -modules.

Kostant a prouvé dans [Kos99], que dans le cas d'une paire $(\mathfrak{g}, \mathfrak{h})$ de rangs égaux et pour $V = V_{\lambda}$ un \mathfrak{g} -module irréductible de dimension finie et de plus haut poids λ , $H_D^{\mathfrak{g},\mathfrak{h}}(V_{\lambda})$ admet la décomposition sans multiplicité suivante:

$$H_D^{\mathfrak{g},\mathfrak{h}}(V_{\lambda}) = \ker(D_{\mathfrak{g},\mathfrak{h}}(V_{\lambda})) = \sum_{w \in W^1} F_{w(\lambda+\rho) - \rho_{\mathfrak{h}}} \quad (30)$$

où $F_{w(\lambda+\rho) - \rho_{\mathfrak{h}}}$ est la \mathfrak{h} -représentation irréductible de dimension finie de plus haut poids $w(\lambda + \rho) - \rho_{\mathfrak{h}}$ et W^1 est défini de manière analogue comme dans le cas de \mathfrak{l} . En fait, une formule similaire a été prouvée dans la situation de rangs inégaux, dans [HKP09] pour la paire $(\mathfrak{g}, \mathfrak{k})$ et dans [MZ14b] pour des paires $(\mathfrak{g}, \mathfrak{h})$. En particulier, dans le cas de rangs inégaux, la décomposition de $H_D^{\mathfrak{g},\mathfrak{k}}(V_{\lambda})$ (respectivement $H_D^{\mathfrak{g},\mathfrak{h}}(V_{\lambda})$) n'est plus sans multiplicité.

Une relation entre les cohomologies de Dolbeault et de Dirac peut être déduite de (24) et (30) quand $G_c/H_0 = G_c/L_c$. Tout d'abord, notons que l'action de \mathfrak{l} sur le premier facteur de la \mathfrak{u} -cohomologie est donnée par l'action adjointe de sorte que l'action de \mathfrak{l} sur la \mathfrak{u} -cohomologie diffère de l'action de \mathfrak{l} sur $V_{\lambda} \otimes S$ uniquement par une torsion avec le \mathfrak{l} -module $\mathbb{C}_{\rho_{\mathfrak{u}}}$ de dimension 1 (pour plus de détails, voir [Kos61]). Alors, on a:

$$H(G_c/L_c, E \otimes \mathbb{C}_{\rho_{\mathfrak{u}}}) \simeq \bigoplus_{\lambda \in \widehat{G}_c} V_{\lambda} \otimes \text{Hom}_{L_c}(H_D^{\mathfrak{g},\mathfrak{l}}(V_{\lambda}), E) = \bigoplus_{\lambda \in \widehat{G}_c} V_{\lambda} \otimes \text{Hom}_{L_c}(\ker(D_{\mathfrak{g},\mathfrak{l}}(V_{\lambda})), E)$$

de sorte que le théorème de Borel-Weil-Bott peut être considéré comme un cas particulier de la cohomologie de Dirac.

Certaines propriétés de la cohomologie de Dirac ont été étudiées par Mehdi et Parthasarathy dans [MP08], [MP10] et [MP11]. De plus, il a été prouvé par Mehdi et Pandžić dans [MP13] que la cohomologie de Dirac ne commute pas avec les foncteurs de translation. Pour contourner ce problème, Mehdi, Pandžić et Vogan ont introduit l'indice de Dirac $I_D(X)$ d'un (\mathfrak{g}, K_0) -module X (pas nécessairement de dimension finie) dans [MPV17a]. Plus précisément, supposons que G_0 et \widetilde{K}_0 ont le même rang (complexe), de sorte que le module spinoriel S pour le revêtement double \widetilde{K}_0 (ou de manière équivalente pour \mathfrak{k}) est la somme directe de S^+ et S^- . Alors, la cohomologie de Dirac $H_D^{\mathfrak{g},\mathfrak{k}}(X)$ de X se décompose en $H_D^{\mathfrak{g},\mathfrak{k}}(X)^+$ et $H_D^{\mathfrak{g},\mathfrak{k}}(X)^-$:

$$H_D^{\mathfrak{g},\mathfrak{k}}(X) = H_D^{\mathfrak{g},\mathfrak{k}}(X)^+ \oplus H_D^{\mathfrak{g},\mathfrak{k}}(X)^-.$$

Si X est admissible avec un caractère infinitésimal, alors l'indice de Dirac de X est le \widetilde{K}_0 -module virtuel :

$$I_D(X) = H_D^{\mathfrak{g}, \mathfrak{k}}(X)^+ - H_D^{\mathfrak{g}, \mathfrak{k}}(X)^-.$$

On note que cette définition a été prolongée par Pandžić et Somberg dans [PS16] à des modules de longueur finie pas nécessairement avec un caractère infinitésimal en utilisant une extension de la cohomologie de Dirac. Mehdi, Pandžić et Vogan ont prouvé dans [MPV17a, Section 3] que

$$I_D(X) = X \otimes S^+ - X \otimes S^- \quad (31)$$

et qu'en tant que foncteur des (\mathfrak{g}, K_0) -modules de longueur finie vers les \widetilde{K}_0 -modules virtuels, I_D est additif par rapport aux suites exactes courtes et se comporte bien par rapport aux foncteurs de translation:

$$I_D(X \otimes F) = I_D(X) \otimes F \text{ pour tout } (\mathfrak{g}, K_0)\text{-module de dimension finie } F.$$

Ils ont aussi démontré, dans [MPV17a, Section 5], que l'application

$$\lambda \mapsto \dim(I_D(X_\lambda))$$

est polynômiale sur la sous-algèbre de Cartan lorsque que $\{X_\lambda\}$ est une famille cohérente de (\mathfrak{g}, K_0) -modules. Récemment, Mehdi, Pandžić, Vogan et Zierau dans [MPVZ20] ont lié de façon précise ce polynôme de l'indice de Dirac à des orbites nilpotentes et cycles associés de modules de Harish-Chandra. Une définition analogue pour l'indice de Dirac pour les paires $(\mathfrak{g}, \mathfrak{h})$ sera donnée dans la Section 7.6.

3 Opérateurs de Dirac de Slebarski

Il apparaît que l'opérateur de Dirac cubique de Kostant (27) soit en fait une version algébrique d'un opérateur de Dirac géométrique tordu introduit par Slebarski dans une série d'articles, [Sle87a], [Sle87b] et [Sle88], au milieu des années 80. Plus précisément, l'espace homogène G_c/H_0 est (naturellement) réductif et l'espace tangent $T_{eH_0}(G_c/H_0)$ s'identifie à \mathfrak{h}_0^\perp . Notons $\Gamma(T(G_c/H_0))$ l'espace des sections du fibré tangent $T(G_c/H_0)$ de G_c/H_0 . Rappelons qu'une connexion ∇ sur le fibré tangent $T(G_c/H_0)$ de G_c/H_0 est un opérateur différentiel linéaire

$$\nabla : \Gamma(T(G_c/H_0)) \rightarrow \Gamma(T^*(G_c/H_0) \otimes T(G_c/H_0))$$

satisfaisant la règle de Leibniz. Identifiant les champs vectoriels sur G_c/H_0 avec des éléments dans \mathfrak{h}_0^\perp , ∇ est H_0 -invariant si

$$\nabla_{\text{Ad}(h)(X)}(\text{Ad}(h)(Y)) = \text{Ad}(h)(\nabla_X(Y)), \forall h \in H_0, \forall X, Y \in \mathfrak{h}_0^\perp.$$

Slebarski introduit dans [Sle87a, Section 3] la famille $\{\nabla^t\}_{t \in [0, 1/2]}$ à un paramètre suivante de connexions:

$$\begin{aligned} \nabla^0 &= \text{connexion canonique} \\ \nabla_X^t(Y) &= \nabla_X^0(Y) + t[X, Y]_{\mathfrak{h}_0^\perp} \text{ pour tous } (X, Y) \in \mathfrak{h}_0^\perp \times \mathfrak{h}_0^\perp \end{aligned} \quad (32)$$

où $[\cdot, \cdot]_{\mathfrak{h}_0^\perp}$ est la projection dans \mathfrak{h}_0^\perp du crochet de Lie de \mathfrak{g}_0 . Notons que lorsque G_c/H_0 est un espace symétrique, le crochet $[\cdot, \cdot]_{\mathfrak{h}_0^\perp}$ est nul et la famille $\{\nabla^t\}_{t \in [0, 1/2]}$ se réduit à la connexion canonique sans torsion ∇^0 . Cependant, si le crochet $[\cdot, \cdot]_{\mathfrak{h}_0^\perp}$ n'est pas identiquement nul alors ∇^t est sans torsion si et seulement si $t = 1/2$. Dans ce cas, $\nabla^{1/2}$ est la connexion de Levi-Civita. En fait, selon [Agr03, Lemme 2.1], la torsion T^t de ∇^t peut être exprimée en fonction du terme cubique c :

$$T^t : (X, Y, Z) \in \mathfrak{h}_0^\perp \times \mathfrak{h}_0^\perp \times \mathfrak{h}_0^\perp \mapsto 2(2t - 1)B([X, Y]_{\mathfrak{h}_0^\perp}, Z). \quad (33)$$

En d'autres termes, le terme cubique c coïncide, à un coefficient près, avec la torsion de la connexion canonique ∇^0 sur G_c/H_0 . Les opérateurs de Dirac sur des variétés riemanniennes sont discutés par Friedrich dans [Fri00].

Ensuite, dans [Sle87a], Slebarski considère une famille $\{D_{G_c/H_0}^t\}_{t \in [0, 1/2]}$ d'opérateurs de Dirac tordus définis par la famille $\{\nabla^t\}$. A savoir, supposons que W soit un \mathfrak{h} -module de dimension finie tel que le produit tensoriel $S \otimes W$ s'étende à une représentation de H_0 ; on note que H_0 n'est pas nécessairement simplement connexe. On obtient un fibré vectoriel homogène $S \otimes W \rightarrow G_c/H_0$ au-dessus de G_c/H_0 . L'espace des sections lisses de ce fibré peut être identifié avec l'espace des fonctions lisses à valeurs dans $S \otimes W$ de G_c :

$$C^\infty(G_c/H_0, S \otimes W) := \{\varphi : G_c \xrightarrow{C^\infty} S \otimes W \mid \varphi(gh) = h^{-1} \cdot (\varphi(g)) \forall (g, h) \in G_c \times H_0\}$$

muni de l'action G_c par translations à gauche. il est pratique d'identifier $C^\infty(G_c/H_0, S \otimes W)$ avec sous-espace des vecteurs H_0 -invariants:

$$\{C^\infty(G_c) \otimes S \otimes W\}^{H_0}$$

où G_c agit par translations à gauche sur $C^\infty(G_c)$, tandis que H_0 agit par translations à droite sur $C^\infty(G_c)$ et le relèvement de l'action sur $W \otimes S$. Si $f \in C^\infty(G_c)$ et $Y \in \mathfrak{h}_0^\perp$, $R(Y)f$ dénotera le différenciation

$$(R(Y)f)(g) := \frac{d}{ds} f(g \exp(sY))|_{s=0}, \quad \forall g \in G_c.$$

Si $\{Y_j\}$ est une base orthonormée de \mathfrak{h}_0^\perp , Slebarski définit dans [Sle87a, Section 1] l'opérateur de Dirac tordu associé à ∇^t comme l'opérateur

$$\mathcal{D}_{G_c/H_0}^t(W) : \{C^\infty(G_c) \otimes S \otimes W\}^{H_0} \rightarrow \{C^\infty(G_c) \otimes S \otimes W\}^{H_0} \quad (34)$$

donné par

$$\mathcal{D}_{G_c/H_0}^t(W) := \sum_j R(Y_j) \otimes \gamma(X_j) \otimes 1 - 3t(1 \otimes \gamma(c) \otimes 1)$$

où γ représente la multiplication de Clifford et c est l'élément de degré 3 défini par (26). Les opérateurs $\mathcal{D}_{G_c/H_0}^t(W)$ appartiennent à l'algèbre $\{U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} \text{End}(S \otimes W)\}^{H_0}$ des opérateurs différentiels G_c -invariants agissant sur des sections lisses de $S \otimes W \rightarrow G_c/H_0$. Cette algèbre n'est pas nécessairement commutative; Deitmar, dans [Dei90], donne un critère de commutativité dans certains cas. Notons que la constante devant le terme cubique provient de notre convention pour l'algèbre de Clifford (voir Remarque 2.1.2). Comme dans le cas de l'opérateur de Dirac cubique algébrique, on peut vérifier que $\mathcal{D}_{G_c/H_0}^t(W)$ ne dépend pas du choix de la

base $\{Y_j\}$ de \mathfrak{h}_0^\perp . Slebarski calcule également le carré des opérateurs $\mathcal{D}_{G_c/H_0}^t(W)$ dans [Sle87a, Section 1]. Il observe ensuite dans [Sle87a, Théorème 1] que parmi la famille $\{\mathcal{D}_{G_c/H_0}^t(W)\}$, seul le carré de $\mathcal{D}_{G_c/H_0}^{1/3}(W)$ diffère du Casimir de $U(\mathfrak{g})$ par une constante. Un calcul explicite et une discussion précise sont fournis par Agricola dans le cadre plus général des espaces homogènes naturellement réductifs [Agr03, Théorème 3.2]. Elle en déduit également des informations intéressantes sur certaines équations de la théorie des cordes dans [Agr03, Section 4]. Nous écrirons simplement $\mathcal{D}_{G_c/H_0}(W)$, ou simplement \mathcal{D} quand il n'y a pas de confusion, pour $\mathcal{D}_{G_c/H_0}^{1/3}(W)$ et $\widehat{\mathcal{D}}_{G_c/H_0}(W)$, ou simplement $\widehat{\mathcal{D}}$ pour $\mathcal{D}_{G_c/H_0}^0(W)$. Alors que $\widehat{\mathcal{D}}_{G_c/H_0}(W)$ est l'opérateur de Dirac tordu associé à la connexion canonique ∇^0 sur G_c/H_0 , l'opérateur $\mathcal{D}_{G_c/H_0}^{1/3}(W)$ peut être considéré comme la version géométrique de l'opérateur de Dirac cubique de Kostant. Une relation précise entre les opérateurs de Dirac algébriques et géométriques via les noyaux est proposée dans [MZ14a]. Comme précédemment, supposons que V soit une représentation de dimension finie de G_c et W est une représentation de dimension finie de \mathfrak{h} telle que $S \otimes W$ se relève en une représentation de H_0 . Alors, selon [MZ14a, Proposition 1.8], on a l'isomorphisme suivant des espaces vectoriels:

$$\mathrm{Hom}_{G_c}(V, \ker(\mathcal{D}_{G_c/H_0}(W))) \simeq \mathrm{Hom}_{\widetilde{H}_0}(W, \ker(D_{\mathfrak{g},\mathfrak{h}}(V))).$$

Cette loi de réciprocité est en fait valable dans un cadre plus général; quand V est une représentation lisse et admissible de G_0 . Dans la Proposition 7.4.2, on voit que les opérateurs de Dirac non-cubiques satisfont une réciprocité analogue pour des espaces réductifs homogènes G_c/H_0 :

$$\mathrm{Hom}_{G_c}(V, \ker(\mathcal{D}_{G_c/H_0}^t(W))) \simeq \mathrm{Hom}_{\widetilde{H}_0}(W, \ker(D_{\mathfrak{g},\mathfrak{h}}^t(V))) \quad \forall t \in [0, 1/2]. \quad (35)$$

Nous continuons avec le cas où \mathfrak{h} est de rang maximal dans \mathfrak{g} . Le noyau et l'indice de l'opérateur de Dirac cubique géométrique ont également été calculés par Slebarski dans [Sle87a, page 299]. À savoir, supposons que $W = W_\mu$ soit une \mathfrak{h} -représentation irréductible de dimension finie et de plus haut poids μ tel que $S \otimes W_\mu$ se relève en une représentation de H_0 . Alors, d'après [Sle87a, Theorem 1], on a

$$\ker(\mathcal{D}_{G_c/H_0}(W_\mu)) = \begin{cases} V_{w(\mu+\rho_{\mathfrak{h}})-\rho} & \text{si } w(\mu+\rho_{\mathfrak{h}})-\rho_{\mathfrak{g}} \text{ est } \Delta^+ \text{-dominant pour un } w \in W_{\mathfrak{g}} \\ \{0\} & \text{si un tel } w \text{ n'existe pas.} \end{cases} \quad (36)$$

Pour l'indice, nous suivons la description de Landweber dans [Lan00]. Considérons, comme ci-dessus, les deux représentations S^+ et S^- de sorte que

$$D_{G_c/H_0}(W)^\pm : L^2(G_c/H_0, S^+ \otimes W) \rightarrow L^2(G_c/H_0, S^- \otimes W)$$

soit un opérateur différentiel elliptique G_c -invariant. Ici $L^2(G_c/H_0, S \otimes W)$ désigne l'espace des sections L^2 du fibré $S \otimes W \rightarrow G_c/H_0$. En utilisant [Bot57, Theorem I], Landweber montre dans [Lan00, Theorem 2] que l'indice G_c -equivariant de $\mathcal{D}_{G_c/H_0}(W)$ est le G_c -module virtuel de dimension finie

$$\mathrm{Index}_{G_c}(\mathcal{D}_{G_c/H_0}(W)) = L^2(G_c/H_0, W \otimes S^+) - L^2(G_c/H_0, W \otimes S^-) \quad (37)$$

puis [Lan00, Theorem 3]:

$$\text{Index}_{G_c}(\mathcal{D}_{G_c/H_0}(W_\mu)) = \begin{cases} (-1)^{\ell(w)} F_{w(\mu+\rho_{\mathfrak{h}})-\rho} & \text{si } w(\mu + \rho_{\mathfrak{h}}) - \rho \text{ est } \Delta^+ \text{-dom. pour un } w \in W_{\mathfrak{g}} \\ \{0\} & \text{si un tel } w \text{ n'existe pas.} \end{cases}$$

Dans le cas où G_c/H_0 est remplacé par G_0/K_0 (rangs égaux), Parthasarathy dans [Par72] et Atiyah-Schmid dans [AS77] ont prouvé que le noyau L^2 de l'opérateur de Dirac géométrique tordu est une représentation de la série discrète de G_0 et toute représentation de la série discrète se produit comme le noyau L^2 d'un opérateur de Dirac géométrique tordu sur G_0/K_0 . Dans ce cadre, l'indice de Dirac est étudié dans [AS77]. On mentionne qu'à la fin des années 1990, Deitmar a utilisé la formule des traces de Selberg pour calculer l'indice de Dirac sur des variétés arithmétiques $\Gamma \backslash G_0/K_0$ où $\Gamma \subset G_0$ est un sous-groupe discret sans torsion [Dei97]. Nous mentionnons que, plus récemment, Mehdi et Zierau ont prouvé dans [MZ03], [MZ06] et [MZ14a] que le noyau de l'opérateur de Dirac cubique pour des espaces homogènes réductifs contient des représentations de la série principale. Les séries principales sont également impliquées dans l'étude par Clerc et Ørsted de la covariance conforme des puissances d'opérateurs de Dirac [CØ20]. Notons également que le noyau de l'opérateur de Dirac symplectique introduit par Kostant [Kos74] est décrit, dans le cas de la représentation de Segal-Shale-Weil, par De Brie, Somberg et Souček [DSS14]. Nous mentionnons également que des propriétés, liées à la théorie des représentations, du spectre du Laplacien, en quelque sorte le carré de l'opérateur de Dirac, ont été beaucoup étudiées ces dernières décennies. Par exemple, les résonances du Laplacien sur certains espaces symétriques riemanniens sont décrites en termes des paramètres de Langlands et de fronts d'ondes de représentations par Hilgert, Pasquale et Przebinda dans [HPP17a] et [HPP17b]. Plus récemment, Mehdi et Olbrich étudient des représentations de G_0 qui apparaissent dans le spectre d'espaces localement symétriques $\Gamma \backslash G_0/H_0$ [MO20].

4 Introduction aux résultats de la thèse

4.1 Opérateurs de Dirac algébriques

D'après (27) et (34), à un \mathfrak{g} -module (π, V) et aux connexions $\{\nabla^t\}$, est associée la famille suivante d'opérateurs de Dirac algébriques $\{D_{\mathfrak{g},\mathfrak{h}}(V)^t\}_{t \in [0,1/2]}$:

$$\begin{aligned} D_{\mathfrak{g},\mathfrak{h}}^t(V) &: V \otimes S \rightarrow V \otimes S \\ D_{\mathfrak{g},\mathfrak{h}}^t(V) &:= \sum_j \pi(Y_j) \otimes \gamma(Y_j) - 3t(1 \otimes \gamma(c)). \end{aligned}$$

Nous écrirons simplement $D_{\mathfrak{g},\mathfrak{h}}(V)$ pour $D_{\mathfrak{g},\mathfrak{h}}^{1/3}(V)$ and $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ pour $D_{\mathfrak{g},\mathfrak{h}}^0(V)$. De plus, s'il n'y a pas d'ambiguïté, nous écrirons D et \widehat{D} au lieu de $D_{\mathfrak{g},\mathfrak{h}}(V)$ et $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$, respectivement. Comme nous l'avons déjà mentionné, $D_{\mathfrak{g},\mathfrak{h}}(V)$ et $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ coïncident lorsque G_c/H_0 est un espace symétrique, et, en particulier, $\ker(D_{\mathfrak{g},\mathfrak{h}}(V)) = \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$. Nous nous intéressons au noyau des opérateurs de Dirac non-cubiques dans le cas où G_c/H_0 n'est pas symétrique et $V = V_\lambda$ est une représentation irréductible de dimension finie et de plus haut poids λ .

Considérons quelques exemples, où V est la représentation standard de \mathfrak{g} , $\mathfrak{h} = \mathfrak{l}$ et $\mathfrak{h}^\perp = \mathfrak{q} = \mathfrak{u} + \bar{\mathfrak{u}}$. Soit \mathfrak{g} l'algèbre de Lie complexe simple $\mathfrak{sp}(3, \mathbb{C})$ de dimension 21. Choisissons comme dans 4.3, une sous-algèbre de Cartan (de dimension 3) \mathfrak{t} dans \mathfrak{g} et

$$\Delta^+ = \{\varepsilon_1 \pm \varepsilon_2, \varepsilon_1 \pm \varepsilon_3, \varepsilon_2 \pm \varepsilon_3, 2\varepsilon_1, 2\varepsilon_2, 2\varepsilon_3\}$$

un système positif pour $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$. Choisissons \mathfrak{s} une sous-algèbre parabolique $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$ telle que

$$\begin{aligned} \mathfrak{l} &= \mathfrak{t} + \mathfrak{g}_{\varepsilon_1 + \varepsilon_2} + \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} + \mathfrak{g}_{2\varepsilon_1} + \mathfrak{g}_{2\varepsilon_2} + \mathfrak{g}_{-\varepsilon_1 - \varepsilon_2} + \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} + \mathfrak{g}_{-2\varepsilon_1} + \mathfrak{g}_{-2\varepsilon_2} \\ \mathfrak{u} &= \mathfrak{g}_{\varepsilon_1 + \varepsilon_3} + \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} + \mathfrak{g}_{\varepsilon_2 + \varepsilon_3} + \mathfrak{g}_{\varepsilon_2 - \varepsilon_3} + \mathfrak{g}_{2\varepsilon_3}. \end{aligned}$$

Soit $V = V_{\varepsilon_1}$ la représentation irréductible de dimension finie de plus haut poids ε_1 . Choisissons un vecteur v dans V_{ε_1} de sorte que $v \otimes 1$ soit un vecteur de plus haut poids de $V_{\varepsilon_1} \otimes S$, le poids étant $\varepsilon_1 + \rho_{\mathfrak{q}}$, où $\rho_{\mathfrak{q}}$ désigne la demi-somme des \mathfrak{t} -racines positives dans $\mathfrak{q} = \mathfrak{u} + \bar{\mathfrak{u}}$. Le terme cubique c dans (26) agit par:

$$\begin{aligned} \gamma(c) &= \frac{\sqrt{2}}{4} \gamma(e_{\varepsilon_1 + \varepsilon_3}) \gamma(e_{-\varepsilon_1 + \varepsilon_3}) \gamma(e_{-2\varepsilon_3}) + \frac{\sqrt{2}}{4} \gamma(e_{\varepsilon_2 + \varepsilon_3}) \gamma(e_{-\varepsilon_2 + \varepsilon_3}) \gamma(e_{-2\varepsilon_3}) \\ &\quad - \frac{\sqrt{2}}{4} \gamma(e_{-\varepsilon_1 - \varepsilon_3}) \gamma(e_{\varepsilon_1 - \varepsilon_3}) \gamma(e_{2\varepsilon_3}) - \frac{\sqrt{2}}{4} \gamma(e_{-\varepsilon_2 - \varepsilon_3}) \gamma(e_{\varepsilon_2 - \varepsilon_3}) \gamma(e_{2\varepsilon_3}). \end{aligned}$$

On vérifie que $1 \otimes \gamma(c)$ agit trivialement sur $v \otimes 1$. Changer le système positif Δ^+ en $(\Delta^+)'$ de telle sorte que $\Delta_{\mathfrak{h}}^+ \subset (\Delta^+)'$ conduirait à un autre vecteur $v' \otimes 1$ avec v' un vecteur de plus haut poids par rapport au système positif $(\Delta^+)'$. La collection de ces vecteurs pour tous les systèmes positifs contenant $\Delta_{\mathfrak{h}}^+$ engendre en tant que \mathfrak{l} -module le noyau de l'opérateur de Dirac cubique $D_{\mathfrak{g}, \mathfrak{t}}(V)$ par la formule de Kostant (30). On en déduit que le noyau de l'opérateur de Dirac non-cubique contient le noyau de l'opérateur de Dirac cubique. De plus, on peut vérifier que le vecteur $v \otimes e_{-\varepsilon_2 - \varepsilon_3} \in F \otimes S$ construit dans l'exemple 4.3.3 est annulé par $\widehat{D}_{\mathfrak{g}, \mathfrak{t}}(V)$. En d'autres termes, pour cet exemple, nous avons:

$$\ker(D_{\mathfrak{g}, \mathfrak{t}}(V)) \subset \ker(\widehat{D}_{\mathfrak{g}, \mathfrak{t}}(V)) \text{ et } \ker(D_{\mathfrak{g}, \mathfrak{t}}(V)) \neq \ker(\widehat{D}_{\mathfrak{g}, \mathfrak{t}}(V)).$$

Supposons maintenant que \mathfrak{g} soit l'algèbre de Lie complexe simple $\mathfrak{sl}(3, \mathbb{C})$ de dimension 8. Fixons comme dans l'exemple 4.3.2, une sous-algèbre de Cartan (de dimension 2) \mathfrak{t} et choisissons \mathfrak{s} comme la sous-algèbre de Borel $\mathfrak{s} = \mathfrak{l} + \mathfrak{u}$ telle que

$$\begin{aligned} \mathfrak{l} &= \mathfrak{t} \\ \mathfrak{u} &= \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} + \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} + \mathfrak{g}_{\varepsilon_2 - \varepsilon_3}. \end{aligned}$$

Soit $V = V_{\varepsilon_1}$ la représentation irréductible de dimension finie de plus haut poids ε_1 . Le terme cubique agit par:

$$\gamma(c) = \frac{\sqrt{6}}{6} \gamma(e_{\varepsilon_1 - \varepsilon_3}) \gamma(e_{-\varepsilon_1 + \varepsilon_2}) \gamma(e_{-\varepsilon_2 + \varepsilon_3}) - \frac{\sqrt{6}}{6} \gamma(e_{-\varepsilon_1 + \varepsilon_3}) \gamma(e_{\varepsilon_1 - \varepsilon_2}) \gamma(e_{\varepsilon_2 - \varepsilon_3}).$$

et annule le vecteur de plus haut poids $v \otimes 1$. Pour cet exemple, nous avons

$$\ker(D_{\mathfrak{g},\mathfrak{l}}(V)) = \ker(\widehat{D}_{\mathfrak{g},\mathfrak{l}}(V)).$$

Les exemples ci-dessus sont, en fait, des cas particuliers du résultat suivant, reliant les noyaux d'opérateurs de Dirac cubiques et non-cubiques.

Theorem 4.4.1. *Comme précédemment, supposons que H_0 est un sous-groupe fermé connexe réel de rang maximal de G_c . Alors, pour toute représentation irréductible de dimension finie V de \mathfrak{g} , on a*

$$\ker(D_{\mathfrak{g},\mathfrak{h}}(V)) \subseteq \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)).$$

La réciproque n'est pas vraie. Par exemple, pour un groupe de Lie connexe compact réel semi-simple G' , considérons l'espace symétrique $G' \times G' / \Delta(G' \times G')$, où $\Delta(G' \times G')$ est la diagonale en $G' \times G'$. Alors les opérateurs de Dirac cubique et non-cubique coïncident alors que les rangs de $G' \times G'$ et $\Delta(G' \times G')$ sont différents. De plus, l'exemple 4.4.4 implique que l'on ne peut pas omettre la condition de rangs égaux. En effet, considérons \mathfrak{g} comme l'algèbre de Lie complexe simple $\mathfrak{sl}(3, \mathbb{C})$ de dimension 8. Si

$$\mathfrak{t} = \mathbb{C}H_{\varepsilon_1 - \varepsilon_2} + \mathbb{C}H_{\varepsilon_2 - \varepsilon_3},$$

où

$$H_{\varepsilon_1 - \varepsilon_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } H_{\varepsilon_2 - \varepsilon_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

est la sous-algèbre de Cartan (de dimension 2) dans \mathfrak{g} , soit

$$\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$$

le système positif pour $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$, choisissons

$$\mathfrak{h} = \mathbb{C}(5H_{\varepsilon_1 - \varepsilon_2} + 4H_{\varepsilon_2 - \varepsilon_3})$$

et

$$\mathfrak{q} = \mathbb{C}(H_{\varepsilon_1 - \varepsilon_2} - 2H_{\varepsilon_2 - \varepsilon_3}) + \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} + \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} + \mathfrak{g}_{\varepsilon_2 - \varepsilon_3} + \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} + \mathfrak{g}_{-\varepsilon_1 + \varepsilon_3} + \mathfrak{g}_{-\varepsilon_2 + \varepsilon_3}.$$

Si V est la représentation irréductible de dimension finie de plus haut poids $\frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3$, on peut vérifier que l'opérateur de Dirac non-cubique associé à la paire $(\mathfrak{g}, \mathfrak{h})$ n'annule pas le vecteur de plus haut poids $v \otimes 1$. En d'autres termes, pour cet exemple, nous avons:

$$\ker(D_{\mathfrak{g},\mathfrak{h}}(V)) \not\subseteq \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)).$$

Motivé par le Théorème 4.4.1, on se restreint au cas où la sous-algèbre \mathfrak{l} est une sous-algèbre de Cartan de \mathfrak{g} , c'est-à-dire \mathfrak{s} est une sous-algèbre de Borel. Le résultat principal de la Section 4.6 est le théorème suivant.

Theorem 4.6.13. Soient \mathfrak{g} une algèbre de Lie complexe semi-simple, $\mathfrak{h} := \mathfrak{t}$ une sous-algèbre de Cartan de \mathfrak{g} , Δ le système radiciel associé et Δ^+ un système positif pour Δ . Pour un élément intégral Δ^+ -dominant $\lambda \in \mathfrak{t}^*$, soient V la représentation irréductible de dimension finie de plus haut poids λ et $A(\lambda)$ le sous-ensemble de \mathfrak{t}^* défini dans (4.11). Alors, le noyau de l'opérateur de Dirac non-cubique $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ agissant sur $V \otimes S$ est

$$\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)) = \bigoplus_{\mu \in A(\lambda)} (V \otimes S)^{[\mu]}. \quad (38)$$

Ici $(V \otimes S)^{[\mu]}$ est la somme directe

$$(V \otimes S)^{[\mu]} := \bigoplus_{w \in W/W_\mu} (V \otimes S)_{w\mu},$$

avec $(V \otimes S)_{w\mu}$ étant la $w\mu$ -composante isotypique de $V \otimes S$ et W_μ le stabilisateur de μ dans W . L'ensemble $A(\lambda)$ dépend de λ et est étudié dans la Section 4.7. Des ingrédients cruciaux pour la preuve du Théorème 4.6.13 sont les inégalités de Weyl discutées dans la Section 4.5. Plus précisément, si A et B sont des opérateurs auto-adjoints agissant sur un espace vectoriel de dimension finie, les inégalités de Weyl (4.3) relient les valeurs propres de A et B aux valeurs propres de l'opérateur $A + B$.

Notons que, dans le contexte ci-dessus, i.e. pour \mathfrak{h} étant une sous-algèbre de Cartan, une composante isotypique $(V \otimes S)_\mu$, $\mu \in \mathfrak{t}^*$, est soit contenue dans $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$ soit a une intersection triviale avec $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$. Ceci est également vrai pour les opérateurs de Dirac cubiques $D_{\mathfrak{g},\mathfrak{h}}(V)$ même si \mathfrak{h} n'est pas une sous-algèbre de Cartan de \mathfrak{g} . Néanmoins, ce n'est plus le cas pour $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ quand \mathfrak{h} n'est pas une sous-algèbre de Cartan \mathfrak{t} de \mathfrak{g} . Dans l'exemple 4.6.18, nous considérons $\mathfrak{g} := \mathfrak{sl}(4, \mathbb{C})$ et choisissons \mathfrak{h} comme étant $\mathfrak{h} = \mathfrak{t} + \mathfrak{g}_{\varepsilon_3 - \varepsilon_4} + \mathfrak{g}_{-\varepsilon_3 + \varepsilon_4}$ et $V := \mathbb{C}^4$ la représentation standard de \mathfrak{g} , c'est-à-dire la représentation irréductible de dimension finie de \mathfrak{g} de plus haut poids $\lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4$. Soit

$$\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V) : V \otimes S \rightarrow V \otimes S$$

l'opérateur de Dirac non-cubique correspondant et considérons les vecteurs

$$\begin{aligned} x_1 &:= v_3 \otimes e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_1 + \varepsilon_4} \wedge e_{-\varepsilon_2 + \varepsilon_4} \\ x_2 &:= v_4 \otimes e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_1 + \varepsilon_4} \wedge e_{-\varepsilon_2 + \varepsilon_3} \end{aligned}$$

de $V \otimes S$, où v_3 et v_4 sont le troisième et quatrième, respectivement, vecteurs de la base canonique de \mathbb{C}^4 . Alors

$$\begin{aligned} \widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)x_1 &= v_2 \otimes e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_1 + \varepsilon_4} \wedge e_{-\varepsilon_2 + \varepsilon_3} \wedge e_{-\varepsilon_2 + \varepsilon_4} \neq 0 \\ \widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)x_2 &= -v_2 \otimes e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_1 + \varepsilon_4} \wedge e_{-\varepsilon_2 + \varepsilon_3} \wedge e_{-\varepsilon_2 + \varepsilon_4} \neq 0 \end{aligned}$$

mais

$$\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)(x_1 + x_2) = 0.$$

Les éléments x_1 , x_2 et $x_1 + x_2$ appartiennent à la même composante isotypique de $V \otimes S$, l'élément $x_1 + x_2$ appartenant à $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$, contrairement à x_1 et x_2 . En d'autres termes, seule une partie de cette composante isotypique appartient à $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$.

Enfin, pour les opérateurs de Dirac t -cubiques $D_{\mathfrak{g},\mathfrak{h}}^t(V)$, $t \in (0, \frac{1}{3}]$, nous démontrons le théorème suivant.

Theorem 4.8.2. *Soient \mathfrak{g} une algèbre de Lie complexe semi-simple et $\mathfrak{h} := \mathfrak{t}$ une sous-algèbre de \mathfrak{g} telle qu'elle satisfasse la condition (2.10). Supposons que \mathfrak{t} soit une sous-algèbre de Cartan commune à \mathfrak{g} et \mathfrak{h} et soit V une \mathfrak{g} -représentation irréductible de dimension finie de plus haut poids $\lambda \in \mathfrak{t}^*$. Pour $t \in (0, \frac{1}{3}]$, considérons l'opérateur de Dirac t -cubique*

$$D_{\mathfrak{g},\mathfrak{h}}^t(V) : V \otimes S \rightarrow V \otimes S$$

de la Définition 4.8.1 et l'opérateur de Dirac cubique

$$D_{\mathfrak{g},\mathfrak{h}}(V) := D_{\mathfrak{g},\mathfrak{h}}^{1/3}(V) : V \otimes S \rightarrow V \otimes S$$

de la Définition 3.1.1. Alors, pour tout $t \in (0, \frac{1}{3}]$,

$$\ker(D_{\mathfrak{g},\mathfrak{h}}^t(V)) = \ker(D_{\mathfrak{g},\mathfrak{h}}(V)).$$

En particulier, selon le Théorème 3.2.4,

$$\ker(D_{\mathfrak{g},\mathfrak{h}}^t(V)) = \bigoplus_{w \in W^1} F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}},$$

où F_{μ} est la \mathfrak{h} -représentation irréductible de plus haut poids $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$ et

$$W^1 := \{w \in W \mid \Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+\}.$$

En d'autres termes, bien que le carré des opérateurs $D_{\mathfrak{g},\mathfrak{h}}^t(V)$, $t \in (0, \frac{1}{3})$, n'ait pas la "bonne" forme de $D_{\mathfrak{g},\mathfrak{h}}(V)^2$ décrit dans la Proposition 3.2.6, un analogue du Théorème 3.2.4 fournit une description complète de leurs noyaux en termes de représentations irréductibles.

Dans les Chapitres 5 et 6, nous appliquons nos résultats pour les opérateurs de Dirac non-cubiques dans le cas où $\mathfrak{h} \subseteq \mathfrak{g}$ est une sous-algèbre de Cartan \mathfrak{t} d'une algèbre de Lie complexe simple \mathfrak{g} et V la représentation standard de \mathfrak{g} .

Illustrons la situation où \mathfrak{g} est une algèbre de Lie classique avec un exemple. Supposons que $\mathfrak{g} := \mathfrak{sl}(4, \mathbb{C})$ et $\mathfrak{h} := \mathfrak{t}$ est une sous-algèbre de Cartan \mathfrak{t} de \mathfrak{g} . La \mathfrak{g} -représentation standard (π, V) de dimension 4 est de plus haut poids $\lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4$. En fait, elle peut être réalisée comme $V := \mathbb{C}^4$ équipé de l'action par multiplication matricielle de $\mathfrak{sl}(4, \mathbb{C})$ sur V . Les poids de V sont

$$\begin{aligned} \mu_1 &:= \lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_2 &:= \lambda - \varepsilon_1 + \varepsilon_2 = -\frac{1}{4}\varepsilon_1 + \frac{3}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_3 &:= \lambda - \varepsilon_1 + \varepsilon_3 = -\frac{1}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 + \frac{3}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_4 &:= \lambda - \varepsilon_1 + \varepsilon_4 = -\frac{1}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 + \frac{3}{4}\varepsilon_4 \end{aligned}$$

et les vecteurs de poids correspondants v_i , $1 \leq i \leq 4$, sont les vecteurs de la base canonique de \mathbb{C}^4 . Pour chaque $\alpha \in \Delta$, choisissons $e_\alpha \in \mathfrak{g}$ tel que $B(e_\alpha, e_{-\alpha}) = 1$. L'opérateur de Dirac non-cubique $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ est donné par

$$\begin{aligned} \widehat{D}_{\mathfrak{g},\mathfrak{h}}(V) = & \sqrt{2}\pi(e_{\varepsilon_1-\varepsilon_2}) \otimes \gamma(e_{-\varepsilon_1+\varepsilon_2}) + \sqrt{2}\pi(e_{\varepsilon_1-\varepsilon_3}) \otimes \gamma(e_{-\varepsilon_1+\varepsilon_3}) + \sqrt{2}\pi(e_{\varepsilon_1-\varepsilon_4}) \otimes \gamma(e_{-\varepsilon_1+\varepsilon_4}) \\ & + \sqrt{2}\pi(e_{\varepsilon_2-\varepsilon_3}) \otimes \gamma(e_{-\varepsilon_2+\varepsilon_3}) + \sqrt{2}\pi(e_{\varepsilon_2-\varepsilon_4}) \otimes \gamma(e_{-\varepsilon_2+\varepsilon_4}) + \sqrt{2}\pi(e_{\varepsilon_3-\varepsilon_4}) \otimes \gamma(e_{-\varepsilon_3+\varepsilon_4}) \\ & + \sqrt{2}\pi(e_{-\varepsilon_1+\varepsilon_2}) \otimes \gamma(e_{\varepsilon_1-\varepsilon_2}) + \sqrt{2}\pi(e_{-\varepsilon_1+\varepsilon_3}) \otimes \gamma(e_{\varepsilon_1-\varepsilon_3}) + \sqrt{2}\pi(e_{-\varepsilon_1+\varepsilon_4}) \otimes \gamma(e_{\varepsilon_1-\varepsilon_4}) \\ & + \sqrt{2}\pi(e_{-\varepsilon_2+\varepsilon_3}) \otimes \gamma(e_{\varepsilon_2-\varepsilon_3}) + \sqrt{2}\pi(e_{-\varepsilon_2+\varepsilon_4}) \otimes \gamma(e_{\varepsilon_2-\varepsilon_4}) + \sqrt{2}\pi(e_{-\varepsilon_3+\varepsilon_4}) \otimes \gamma(e_{\varepsilon_3-\varepsilon_4}). \end{aligned}$$

D'après le Lemme 4.6.15, le noyau de $\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)$ est monômial, c'est-à-dire qu'il est engendré par les vecteurs $v_i \otimes u$ où v_i est comme ci-dessus et $u = e_{-j_1} \wedge \dots \wedge e_{-j_k}$ est un monôme de S . Soit $v_i \otimes u$ un tel vecteur dans $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$. Alors, selon la Proposition 4.6.17, $v_i \otimes u$ appartient à

$$\bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha})).$$

Si $i = 1$, i.e. v_i est le vecteur $v_1 = (1, 0, 0, 0)$ de $V = \mathbb{C}^4$ de poids λ , alors v_1 est annulé par tout $\pi(e_\beta)$, $\beta \in \Delta$, sauf pour $\beta = -\varepsilon_1 + \varepsilon_2$, $\beta = -\varepsilon_1 + \varepsilon_3$ et $\beta = -\varepsilon_1 + \varepsilon_4$. Par conséquent, $v_1 \otimes u$ appartient à $\bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha}))$ si et seulement si $\gamma(e_{-\beta})$ annule u pour ces valeurs de β .

En d'autres termes, $e_{-\varepsilon_1+\varepsilon_2}$, $e_{-\varepsilon_1+\varepsilon_3}$ et $e_{-\varepsilon_1+\varepsilon_4}$ ne devraient pas apparaître dans u . On en déduit que les seuls vecteurs monomiaux de la forme $v_1 \otimes u$ appartenant à $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$ sont:

$$\begin{aligned} & v_1 \otimes 1, v_1 \otimes e_{-\varepsilon_2+\varepsilon_3}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_3+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4}, \\ & v_1 \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4}, v_1 \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4}. \end{aligned}$$

De la même manière, on trouve les autres vecteurs $v_i \otimes u$ de $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$ pour $i = 2, 3, 4$. En particulier, $\ker(\widehat{D})$ est de dimension 32 tandis que, d'après le Théorème 3.2.4, la dimension de $\ker(D)$ est égale à $|W| = 4! = 24$.

La table ci-dessous compare les tailles des noyaux cubique et non-cubiques pour toute algèbre de Lie classique dans le cas où \mathfrak{h} est une sous-algèbre de Cartan de \mathfrak{g} et V est la représentation standard.

\mathfrak{g}	$\dim \ker(D_{\mathfrak{g},\mathfrak{h}}(V)) = W $	$\dim \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$
$\mathfrak{sl}(n+1, \mathbb{C})$	$(n+1)!$	$(n+1) \times 2^{\frac{n(n-1)}{2}}$
$\mathfrak{so}(2n+1, \mathbb{C})$	$2^n n!$	$2n \times 2^{(n-1)^2}$
$\mathfrak{sp}(n, \mathbb{C})$	$2^n n!$	$2n \times 2^{(n-1)^2}$
$\mathfrak{so}(2n, \mathbb{C})$	$2^{n-1} n!$	$2n \times 2^{(n-1)(n-2)}$

Table 5.1: Dimensions des noyaux d'opérateurs de Dirac cubiques et non-cubiques

Pour les algèbres de Lie exceptionnelles, \mathfrak{h} étant toujours une sous-algèbre de Cartan \mathfrak{t} de \mathfrak{g} et V la représentation standard de \mathfrak{g} de plus haut poids λ , pour décrire $\ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V))$, nous déterminons l'ensemble $A(\lambda)$ du Théorème 4.6.13. $A(\lambda)$ est défini comme l'ensemble des classes d'équivalence

$$A(\lambda) = \widetilde{A}(\lambda) / \sim,$$

où

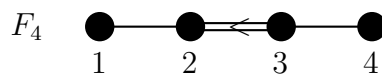
$$\widetilde{A}(\lambda) := \left\{ \lambda + \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+ \right\}$$

et \sim est la relation d'équivalence identifiant les éléments μ et $w\mu$ de $\widetilde{A}(\lambda)$ pour chaque élément w du stabilisateur W_λ de λ dans W . Dans la Proposition 4.7.1, nous donnons la caractérisation suivante pour $A(\lambda)$:

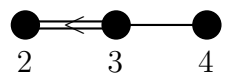
$$A(\lambda) = \left\{ \lambda + \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq \Delta_1^+ \text{ et } \rho_1 - \sum_{\alpha \in A} \alpha \text{ est } \Delta_1^+ \text{-dominant} \right\},$$

avec $\Delta_1^+ := (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+$ et ρ_1 étant la demi-somme de toutes les racines positives de Δ_1^+ . On montre que $\Delta_1 := \Delta_1^+ \cup (-\Delta_1^+)$ est un système radiciel, celui-ci obtenu par le diagramme de Dynkin de Δ si l'on supprime le sommet correspondant à la première racine simple α_1 de Δ .

Par exemple, si \mathfrak{g} est l'algèbre de Lie exceptionnelle F_4 , son diagramme de Dynkin est



et effacer le sommet correspondant à α_1 donne le diagramme de Dynkin de Δ_1 :



Celui-ci est exactement le diagramme de Dynkin de l'algèbre de Lie classique B_3 . Ensuite, nous construisons un nouvel espace de spineurs S_1 pour l'algèbre de Clifford de la somme directe de tous les espaces radiciels de Δ_1 . Pour décrire $A(\lambda)$, il suffit alors de trouver tous les poids Δ_1^+ -dominants de S_1 . Le Tableau 6.1 donne ces poids pour l'espace des spineurs S_1 de Δ_1^+ pour l'exemple ci-dessus.

Poids	Vecteurs
$\frac{5}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	1
$\frac{5}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_3}$ $e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{3}{2}\varepsilon_4$	$e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2}$ $e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3}$ $e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_4}$ $e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}$
$\frac{1}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$

 Table 6.1: Poids Δ_1^+ -dominants de S_1 pour $\mathfrak{g} = \mathfrak{so}(7, \mathbb{C})$ et $\mathfrak{h} = \mathfrak{t}$

4.2 Opérateurs de Dirac géométriques

Dans le Chapitre 7, motivés par la lien entre l'opérateur de Dirac cubique algébrique $D_{\mathfrak{g}, \mathfrak{h}}(V)$ et l'opérateur de Dirac cubique géométrique \mathcal{D}_{G_c/H_0} établie dans [MZ14a], en suivant [Lan00], nous utilisons le Théorème de Peter-Weyl pour définir un isomorphisme:

$$\Phi : [L^2(G_c) \otimes E]^H \xrightarrow{\sim} \widehat{\bigoplus_{\lambda \in \widehat{G}_c} V_\lambda} \otimes [V_\lambda^* \otimes E]^H$$

qui généralise cette relation dans le cas des opérateurs de Dirac non-cubiques et t -cubiques géométriques.

En suivant [Lan00, Theorem 4] et [Sle87a], on obtient la description suivante du noyau de l'opérateur de Dirac non-cubique dans le cas où L_c est un tore maximal T de G_c .

Theorem 7.4.4. *Soient G_c un groupe de Lie connexe compact semi-simple d'algèbre de Lie complexifiée \mathfrak{g} , T un tore maximal dans G_c tel que son algèbre de Lie complexifiée \mathfrak{t} soit θ -stable. Fixons E la \mathfrak{t} -représentation de dimension 1 déterminée par un élément Δ^+ -dominant μ de \mathfrak{t}^* par rapport à un système positif Δ^+ de $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ et telle que $S \otimes E$ se relève en une représentation de T . Pour tout élément λ de l'ensemble des éléments Δ^+ -dominants analytiquement intégraux de \mathfrak{t}^* , posons*

$$K_\lambda := \left\{ \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+ \text{ et } \rho - \sum_{\alpha \in A} \alpha \text{ est } \Delta^+\text{-dominant} \right\}$$

et

$$\mathcal{A}_\mu := \{(\lambda, \kappa) \in \Lambda \times K_\lambda \mid \lambda + \kappa = \mu\}.$$

Alors, le noyau de l'opérateur de Dirac non-cubique

$$\widehat{\mathcal{D}}_{G_c/T}(E) : [L^2(G) \otimes (S \otimes E)]^T \rightarrow [L^2(G) \otimes (S \otimes E)]^T$$

est

$$\ker \widehat{\mathcal{D}}_{G_c/T}(E) \simeq \dim(V \otimes S)_\mu \sum_{(\lambda, \kappa) \in \mathcal{A}_\mu} V_{\mu-k},$$

où $V_{\mu-k}$ est la G_c -représentation irréductible de dimension finie de plus haut poids $\mu - k$.

Le résultat suivant généralise [Lan00, Theorem 4] dans le cas de rangs inégaux.

Theorem 7.3.10. *Soient G_c un groupe de Lie connexe compact semi-simple d'algèbre de Lie complexifiée \mathfrak{g} , H_0 un sous-groupe fermé de G_c tel que son algèbre de Lie complexifiée \mathfrak{h} satisfasse la condition (2.10). En plus, supposons que $\mathfrak{t}_\mathfrak{h}$ est une sous-algèbre de Cartan de \mathfrak{h} et $\mathfrak{t}_\mathfrak{q} \subseteq \mathfrak{q}$ est un sous-espace de \mathfrak{q} tel que $\mathfrak{t} := \mathfrak{t}_\mathfrak{h} + \mathfrak{t}_\mathfrak{q}$ soit une sous-algèbre de Cartan de \mathfrak{g} . Soient $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ le système radiciel correspondant, Δ^+ un système positif pour Δ satisfaisant la $\mathfrak{t}_\mathfrak{h}$ -propriété de la Définition 3.2.3, et $\Delta_\mathfrak{h}^+$ le système positif correspondant pour $\Delta_\mathfrak{h} := \Delta(\mathfrak{h}, \mathfrak{t}_\mathfrak{h})$. Fixons E la \mathfrak{h} -représentation irréductible de dimension finie de plus haut poids $\mu \in \mathfrak{t}_\mathfrak{h}^*$ par rapport à $\Delta_\mathfrak{h}^+$ et telle que $S \otimes E$ se relève en une représentation de H_0 . Soit Λ l'ensemble de tous les éléments Δ^+ -dominants analytiquement intégraux de \mathfrak{t}^* . Pour tout $\lambda \in \Lambda$, posons*

$$W_\lambda^1 := \{w \in W \mid w(\lambda + \rho)|_{\mathfrak{t}_\mathfrak{q}} = 0 \text{ et } w(\lambda + \rho)|_{\mathfrak{t}_\mathfrak{h}} \text{ est } \Delta_\mathfrak{h}^+\text{-dominant}\}.$$

S'il existe $\lambda \in \Lambda$ et $w \in W_\lambda^1$ tels que $w^{-1}(\mu + \rho_\mathfrak{h}) - \rho$ soit Δ^+ -dominant et analytiquement intégral, le noyau de l'opérateur de Dirac cubique

$$\mathcal{D}_{G_c/H_0}(E) : [L^2(G_c) \otimes (S \otimes E)]^{H_0} \rightarrow [L^2(G_c) \otimes (S \otimes E)]^{H_0}$$

est équivalent à 2^d copies de la G_c -représentation $V_{w^{-1}(\mu + \rho_\mathfrak{h}) - \rho}$ de plus haut poids $w^{-1}(\mu + \rho_\mathfrak{h}) - \rho$ où $d := \lfloor \frac{\dim \mathfrak{t}_\mathfrak{q}}{2} \rfloor$. Sinon, $\ker \mathcal{D}_{G_c/H_0}(E) = 0$.

Enfin, par analogie avec le Théorème 4.8.2, nous décrivons $\ker(\mathcal{D}_{G_c/H_0}^t(E))$ pour $t \in (0, \frac{1}{3}]$, montrant qu'il coïncide avec $\ker(\mathcal{D}_{G_c/H_0}(E))$.

Theorem 7.5.2. *Soient G_c un groupe de Lie compact connexe semi-simple d'algèbre de Lie complexifiée \mathfrak{g} , H_0 un sous-groupe de Lie fermé de G_c tel que son algèbre de Lie complexifiée \mathfrak{h} satisfasse la condition (2.10). En plus, supposons qu'il existe une sous-algèbre de Cartan commune \mathfrak{t} de \mathfrak{g} et \mathfrak{h} . Soit E une \mathfrak{h} -représentation irréductible de dimension finie de plus haut poids $\mu \in \mathfrak{t}^*$ par rapport à un système positif fixé $\Delta_{\mathfrak{h}}^+$ telle que $S \otimes E$ se relève en une représentation de H_0 . Alors, pour $t \in (0, \frac{1}{3}]$, le noyau $\ker(\mathcal{D}^t)$ de l'opérateur de Dirac t -cubique géométrique*

$$\mathcal{D}_{G_c/H_0}^t(E) : [L^2(G_c) \otimes (S \otimes E)]^{H_0} \rightarrow [L^2(G_c) \otimes (S \otimes E)]^{H_0}$$

coïncide avec le noyau $\ker(\mathcal{D}_{G_c/H_0}(E))$ de l'opérateur de Dirac. En particulier, s'il existe $w \in W$ tel que $w(\mu + \rho_{\mathfrak{h}}) - \rho$ soit dominant par rapport à un système positif $\Delta^+ \supseteq \Delta_{\mathfrak{h}}^+$ de Δ , $\ker(\mathcal{D}_{G_c/H_0}^t(E))$ est équivalent à la G_c -représentation irréductible de dimension finie $V_{w(\mu+\rho_{\mathfrak{h}})-\rho}$ de plus haut poids $w(\mu + \rho_{\mathfrak{h}}) - \rho$. Sinon, $\ker(\mathcal{D}_{G_c/H_0}(E))$ est trivial.

4.3 Indices de Dirac

Dans la section 7.6, dans le cas où \mathfrak{g} et \mathfrak{h} ont le même rang complexe, nous considérons la \mathbb{Z}_2 -gradation $S = S^+ \oplus S^-$ et les opérateurs de Dirac :

$$\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)^{\pm} : V \otimes S^{\pm} \rightarrow V \otimes S^{\mp}.$$

Alors, l'indice de Dirac (algébrique) non-cubique d'un \mathfrak{g} -module de longueur finie est le \mathfrak{h} -module virtuel

$$I_{\widehat{D}}(V) = \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)^+) - \ker(\widehat{D}_{\mathfrak{g},\mathfrak{h}}(V)^-).$$

Suivant [MPV17a], nous démontrons le lemme suivant.

Lemma 7.6.6. *Si V est un \mathfrak{g} -module de dimension finie, alors*

$$I_{\widehat{D}}(V) = V \otimes S^+ - V \otimes S^-.$$

De plus, pour une \mathfrak{h} -représentation de dimension finie E telle que $S \otimes E$ se relève en une représentation de H_0 , nous considérons les opérateurs

$$\widehat{D}_{G_c/H_0}(E)^{\pm} : [L^2(G_c) \otimes (S^{\pm} \otimes E)]^{H_0} \rightarrow [L^2(G_c) \otimes (S^{\mp} \otimes E)]^{H_0}$$

de (7.21) et nous définissons l'indice de Dirac (géométrique) non-cubique de E comme étant le G_c -module virtuel

$$\text{Index}_{G_c} \widehat{D}_{G_c/H_0}(E) := \ker(\widehat{D}_{G_c/H_0}(E)^+) - \ker(\widehat{D}_{G_c/H_0}(E)^-).$$

Ensuite, suivant [Lan00], nous calculons l'indice de Dirac non-cubique de E .

Theorem 7.6.10. Soient G_c un groupe de Lie compact connexe semi-simple d'algèbre de Lie complexifiée \mathfrak{g} , H_0 un sous-groupe fermé de G_c d'algèbre de Lie complexifiée \mathfrak{h} telle que la condition (2.10) soit satisfaite. Supposons qu'il existe une sous-algèbre de Cartan commune \mathfrak{t} de \mathfrak{g} et \mathfrak{h} . Fixons une \mathfrak{h} -représentation irréductible de dimension finie E de plus haut poids $\mu \in \mathfrak{t}^*$ par rapport à un système positif $\Delta_{\mathfrak{h}}^+$ et telle que $S \otimes E$ se relève en une \mathfrak{h} -représentation de H_0 . Étendons $\Delta_{\mathfrak{h}}^+$ à un système positif Δ^+ de Δ et soient Λ l'ensemble de tous les éléments Δ^+ -dominants et analytiquement intégraux, et

$$W^1 := \{w \in W \mid \Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+\}.$$

S'il existe $w \in W^1$ tel que $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$ soit Δ^+ -dominant et analytiquement intégral, alors

$$\text{Index}_G \widehat{\mathcal{D}}_{G_c/H_0}(E) = (-1)^{\ell(w)} V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho},$$

où $V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho}$ est la G_c -représentation irréductible de dimension finie de plus haut poids $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$. Sinon, $\text{Index}_G \widehat{\mathcal{D}}_{G_c/H_0}(E) = 0$. En particulier,

$$\text{Index}_G \widehat{\mathcal{D}}_{G_c/H_0}(E) = \text{Index}_G \mathcal{D}_{G_c/H_0}(E).$$

CHAPTER 1

PRELIMINARIES

In this chapter we have collected basic notions of Lie algebras, Lie groups and their representations we use in this thesis. More precisely, Section 1.1 is devoted to the structure of Lie algebras and Lie groups. Sections 1.2-1.5 contain elementary facts from the representation theory of complex and real semisimple Lie algebras and Lie groups. In Section 1.6 we discuss (\mathfrak{g}, K) -modules while Section 1.7 describes Harish-Chandra morphism. The main references for this chapter are Paulin's electronic notes [Pau] and Knapp's book [Kna96].

1.1 Lie groups and Lie algebras

1.1.1 Lie groups

Definition 1.1.1 (Lie group). *A real (respectively complex) Lie group G is a real (respectively complex) smooth manifold equipped with a group structure such that group multiplication and inversion are smooth (respectively holomorphic) maps.*

Example 1.1.2 (Abelian Lie groups). The groups $(\mathbb{R}^n, +)$, (\mathbb{R}^*, \cdot) , $(\mathbb{C}^n, +)$, (\mathbb{C}^*, \cdot) and (S_1, \cdot) , where $S_1 := \{z \in \mathbb{C} \mid \|z\| = 1\}$, all are abelian Lie groups, i.e. Lie groups with abelian group structure. Moreover, S_1 is a compact Lie group, i.e. it is compact as a topological manifold.

Example 1.1.3 (General Linear Lie Group). Let V be a real or complex n -dimensional vector space. The group $GL(V)$ of all linear automorphisms of V is a Lie group called the general linear group of V . In the case where $V = \mathbb{K}^n$, with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we write $GL(n, \mathbb{K})$ instead of $GL(\mathbb{K}^n)$. By fixing a basis for V one can identify $GL(V)$ with the group of n -by- n invertible matrices with entries in \mathbb{K} . This group is an open subset of the topological vector space $M_n(\mathbb{K})$ of all n -by- n matrices with entries in \mathbb{K} . Note that $M_n(\mathbb{K})$ has a canonical manifold structure induced by its vector space structure. Since $GL(n, \mathbb{K})$ is an open subset of a manifold, it is a manifold as well. On the other hand, matrix multiplication and inversion involve only rational functions with nowhere zero denominators and thus they are smooth maps.

Example 1.1.4 (Product Lie Group). The product $G_1 \times G_2$ of two Lie groups G_1, G_2 (both real or complex) endowed with the product group structure and the product manifold structure is a Lie group called the product Lie group of G_1 and G_2 .

Definition 1.1.5 (Lie subgroup). Let G be a Lie group. A subset H of G is a Lie subgroup of G if it is a subgroup and an immersed submanifold of G .

Theorem 1.1.6 (See [Pau, Theorem 1.43]). If G is a real Lie group, a subgroup H of G is an embedded Lie subgroup if and only if it is closed in G . If G is a complex Lie group, a Lie subgroup H of G is an embedded Lie subgroup if and only if H is closed.

Proposition 1.1.7 (See [Pau, Proposition 1.3]). If G is a Lie group, the identity component G_0 , i.e. the connected component containing the identity element of G , is a Lie subgroup of G .

Theorem 1.1.6 provides us with many examples of matrix Lie groups. All groups of the following list are Lie groups, called classical Lie groups. They are Lie subgroups of $GL(N, \mathbb{K})$ for suitable N and $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .

$SL(n, \mathbb{K}) = \{x \in GL(n, \mathbb{K}) \mid \det(x) = 1\}$ $O(n) = \{x \in GL(n, \mathbb{R}) \mid x \cdot x^t = I_n\}$ $O(n, \mathbb{C}) = \{x \in GL(n, \mathbb{C}) \mid x \cdot x^t = I_n\}$ $SO(n) = O(n) \cap SL(n, \mathbb{R})$ $SO(n, \mathbb{C}) = O(n, \mathbb{C}) \cap SL(n, \mathbb{C})$ $U(n) = \{x \in GL(n, \mathbb{C}) \mid x \cdot x^* = I_n\}$ $SU(n) = U(n) \cap SL(n, \mathbb{C})$ $Sp(n, \mathbb{K}) = \{x \in GL(n, \mathbb{K}) \mid x^t J_n x = J_n\}$

Table 1.1: Classical Lie groups

Here I_n stands for the identity n -by- n matrix, J_n is the matrix $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ while \cdot^t (respectively \cdot^*) indicates the transpose (respectively conjugate-transpose) of the indicated matrix.

1.1.2 Lie group morphisms

Definition 1.1.8 (Lie group morphism). Let G_1 and G_2 be real (respectively complex) Lie groups. We call Lie group morphism a smooth (respectively holomorphic) group morphism $\Phi : G_1 \rightarrow G_2$. A Lie group morphism Φ is said to be a Lie group isomorphism if Φ is bijective and its inverse is a Lie group morphism. We will say that G_1 and G_2 are isomorphic if there is a Lie group isomorphism $\Phi : G_1 \rightarrow G_2$.

Example 1.1.9 (Adjoint map). Let G be a Lie group. For g in G , define a map

$$i_g : \begin{cases} G \rightarrow G \\ x \mapsto gxg^{-1}. \end{cases}$$

The map i_g is a Lie group automorphism of G . Set $\text{Ad}(g)$ to be the tangent map $T_e i_g : \mathfrak{g} \rightarrow \mathfrak{g}$, where $\mathfrak{g} := T_e G$ and e is the neutral element of G . The map Ad defined by

$$\text{Ad} : \begin{cases} G \rightarrow GL(\mathfrak{g}) \\ g \mapsto \text{Ad}(g) \end{cases}$$

is a Lie group morphism called the adjoint map of G .

1.1.3 Lie algebras

Definition 1.1.10 (Lie algebra). *A real (respectively complex) Lie algebra \mathfrak{g} is a finite dimensional vector space over \mathbb{R} (respectively \mathbb{C}) equipped with a Lie bracket, i.e. a bilinear skew-symmetric map*

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

satisfying the Jacobi identity:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \quad (1.1)$$

for every X, Y and Z in \mathfrak{g} . The Lie algebra \mathfrak{g} is said to be abelian if $[X, Y] = 0$ for every X and Y in \mathfrak{g} . We will say that a vector subspace \mathfrak{h} of \mathfrak{g} is a Lie subalgebra of \mathfrak{g} if $[X, Y] \in \mathfrak{h}$ for every X and Y in \mathfrak{h} .

Example 1.1.11 (Lie algebra of an associative algebra). Let A be an associative algebra over \mathbb{K} , where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , equipped with the bracket

$$[a, b] = ab - ba$$

for a and b in A . One can check that the conditions for A to be a Lie algebra are satisfied. Therefore, any associative algebra over \mathbb{K} admits a canonical structure of Lie algebra.

Example 1.1.12 (Classical Lie algebras). Let V be a vector space and $\text{End}(V)$ be the associative endomorphism algebra of V . Then $\text{End}(V)$ equipped with the bracket of the previous example is a Lie algebra called the general Lie algebra $\mathfrak{gl}(V)$ of V . In the case where $V = \mathbb{K}^n$, with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , we set $\mathfrak{gl}(n, \mathbb{K}) := \mathfrak{gl}(\mathbb{K}^n)$. Choosing the standard basis of \mathbb{K}^n , one can identify $\mathfrak{gl}(n, \mathbb{K})$ with $M_n(\mathbb{K})$, the space of n -by- n matrices with coefficients in \mathbb{K} .

The vector spaces of the following table all are Lie algebras, as Lie subalgebras of $\mathfrak{gl}(N, \mathbb{K})$ for suitable $N \in \mathbb{N}$, called classical Lie algebras.

$\mathfrak{sl}(n, \mathbb{K}) = \{X \in \mathfrak{gl}(n, \mathbb{K}) \mid \text{tr}(X) = 0\}$ $\mathfrak{so}(n, \mathbb{K}) = \{X \in \mathfrak{gl}(n, \mathbb{K}) \mid X + X^t = 0\}$ $\mathfrak{sp}(n, \mathbb{K}) = \{X \in \mathfrak{gl}(2n, \mathbb{K}) \mid X^t J_n + J_n X = 0\}$ $\mathfrak{u}(n) = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X^* + X = 0\}$ $\mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n, \mathbb{C})$

Table 1.2: Classical Lie algebras

The Lie algebras $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$ are called general linear Lie algebra, special linear Lie algebra, orthogonal Lie algebra and symplectic Lie algebra respectively.

Example 1.1.13 (Exceptional Lie algebras). There are 5 nonisomorphic complex Lie algebras, denoted E_6 , E_7 , E_8 , F_4 and G_2 , called exceptional Lie algebras. They are of dimension 78, 133, 248, 52 and 14, respectively. For instance, the Lie algebra G_2 is the complexification of the Lie algebra of all derivations of the octanion algebra. For an explicit realization of each exceptional Lie algebra, see [FH91, Chapter 21]. Later, in Subsection 1.2.3, we will consider the parametrization of exceptional Lie algebras by their Dynkin diagram and their role in the classification of complex semisimple Lie algebras (see Subsection 1.2.3).

Example 1.1.14 (Direct sum of Lie algebras). Let \mathfrak{g}_1 and \mathfrak{g}_2 be Lie algebras (both real or complex). We define the Lie algebra direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of \mathfrak{g}_1 and \mathfrak{g}_2 as the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ equipped with the Lie bracket

$$[X_1 + X_2, Y_1 + Y_2] = [X_1, Y_1] + [X_2, Y_2], \quad X_1, Y_1 \in \mathfrak{g}_1, X_2, Y_2 \in \mathfrak{g}_2.$$

Example 1.1.15 (Complexification of Lie algebra). Let \mathfrak{g} be a real Lie algebra. We define the complexification $\mathfrak{g}^{\mathbb{C}}$ of \mathfrak{g} as the complex Lie algebra being the complexified vector space $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ equipped with the Lie bracket

$$[X_1 \otimes c_1, X_2 \otimes c_2] := [X_1, X_2] \otimes c_1 c_2, \quad X_1, X_2 \in \mathfrak{g}, c_1, c_2 \in \mathbb{C}.$$

Recall that the complex scalars act on $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ by multiplication on the second factor. On the other hand, if \mathfrak{g} is a complex Lie algebra, we define $\mathfrak{g}^{\mathbb{R}}$ to be the real Lie algebra obtained by restricting the scalar multiplication of \mathfrak{g} on real scalars.

Definition 1.1.16 (Real form of complex Lie algebra). *Let \mathfrak{g} be a complex Lie algebra. A real subalgebra \mathfrak{h} of $\mathfrak{g}^{\mathbb{R}}$ is a real form of \mathfrak{g} if $\mathfrak{h}^{\mathbb{C}} = \mathfrak{g}$.*

Example 1.1.17 (Lie algebra of a manifold). Let M be a differential manifold and $\Gamma(TM)$ be the vector space of smooth sections of the tangent bundle TM of M . Considering the elements of $\Gamma(TM)$ as derivations acting on the space $\mathcal{C}^{\infty}(M)$ of complex smooth functions defined on M , $\Gamma(TM)$ has a canonical structure of associative algebra given by composition of derivations. In view of Example 1.1.11, $\Gamma(TM)$ equipped with the bracket

$$[X, Y] := X \circ Y - Y \circ X, \quad X, Y \in \Gamma(TM),$$

is a Lie algebra.

Definition 1.1.18 (Adjoint map of \mathfrak{g}). *Let \mathfrak{g} be a Lie algebra. The map*

$$\text{ad} : \begin{cases} \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g}) \\ X \mapsto \text{ad}X := [X, \cdot] \end{cases}$$

is called the adjoint map of \mathfrak{g} .

1.1.4 Lie algebra morphisms

Definition 1.1.19 (Lie algebra morphism). *Let \mathfrak{g}_1 and \mathfrak{g}_2 be Lie algebras. We call Lie algebra morphism a linear map $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ preserving the Lie algebra structures, i.e. satisfying*

$$[\phi(X), \phi(Y)] = \phi([X, Y])$$

for every X and Y in \mathfrak{g}_1 . If ϕ is a monomorphism (epimorphism, endomorphism etc) of the corresponding vector spaces, we say that ϕ is a monomorphism (epimorphism, endomorphism etc respectively) of Lie algebras. We say that two Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 are isomorphic if there is an isomorphism of Lie algebras $\phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$.

Example 1.1.20. Let \mathfrak{g} be the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of Table 1.2. Then, as vector space, \mathfrak{g} is generated by the matrices

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$[H, X] = 2X, \quad [H, Y] = -2Y \quad \text{and} \quad [X, Y] = H.$$

Moreover, consider the vector space \mathfrak{g}' , generated by the differential operators

$$D_1 := x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}, \quad D_2 := x \frac{\partial}{\partial y} \quad \text{and} \quad D_3 := y \frac{\partial}{\partial x}$$

defined on $C^\infty(\mathbb{R}^2)$. Equip \mathfrak{g}' with the Lie bracket inherited by the associative algebra structure of the vector space of differential operators defined on $C^\infty(\mathbb{R}^2)$ (see Example 1.1.11). Then,

$$\begin{aligned} [D_1, D_2] &= D_1 \circ D_2 - D_2 \circ D_1 = 2D_2 \\ [D_1, D_3] &= D_1 \circ D_3 - D_3 \circ D_1 = -2D_3 \\ [D_2, D_3] &= D_2 \circ D_3 - D_3 \circ D_2 = D_1. \end{aligned}$$

and one can see that the linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{g}'$ mapping H to D_1 , X to D_2 and Y to D_3 is an isomorphism of vector spaces. Furthermore, ϕ respects the Lie algebra structures of \mathfrak{g} and \mathfrak{g}' . Indeed,

$$\begin{aligned} \phi([H, X]) &= 2\phi(X) = 2D_2 \\ &= [D_1, D_2] = [\phi(H), \phi(X)], \\ \phi([H, Y]) &= -2\phi(Y) = -2D_3 \\ &= [D_1, D_3] = [\phi(H), \phi(Y)], \\ \phi([X, Y]) &= \phi(H) = D_1 \\ &= [D_2, D_3] = [\phi(X), \phi(Y)]. \end{aligned}$$

In other words, ϕ is an isomorphism of Lie algebras.

1.1.5 The Lie algebra of a Lie group

There is a relation between Lie groups and Lie algebras. Namely, to each Lie group we can associate a unique Lie algebra. More precisely, let G be a Lie group, \mathfrak{g} be the tangent space $T_e G$ of G at the neutral element $e \in G$, and consider its adjoint map $\text{Ad} : G \rightarrow GL(\mathfrak{g})$ defined in Example 1.1.9. Define ad to be the map

$$\text{ad} : \begin{cases} \mathfrak{g} \rightarrow \text{End } \mathfrak{g} \\ X \in \mathfrak{g} \mapsto \text{ad}X := T_e \text{Ad}X \end{cases} \quad (1.2)$$

and set $[X, Y] := \text{ad}(X)(Y)$. Then \mathfrak{g} equipped with the bracket $[\cdot, \cdot]$ turns out to be a Lie algebra (for more details, see [Pau, Proposition 1.6]).

Definition 1.1.21 (Lie algebra of a Lie group). *The Lie algebra $\mathfrak{g} := T_e G$ is called the Lie algebra of G , also denoted $\text{Lie}(G)$.*

Theorem 1.1.22 (See [Pau, Corollary 1.54 and Corollary 1.55]). *Every Lie algebra is the Lie algebra of a Lie group. This Lie group is not unique. Nevertheless, up to isomorphism, it is unique if the group is connected and simply connected.*

Example 1.1.23 (Lie algebras of abelian Lie groups). The Lie algebras of the abelian Lie groups $(\mathbb{R}^n, +)$, (\mathbb{R}^*, \cdot) , $(\mathbb{C}^n, +)$ and (\mathbb{C}^*, \cdot) , (S_1, \cdot) of Example 1.1.2 are \mathbb{R}^n , \mathbb{R} , \mathbb{C}^n , \mathbb{C} and $i\mathbb{R}$ respectively, all equipped with the trivial Lie bracket. In fact, the Lie algebra of any abelian Lie group is abelian.

Example 1.1.24 (Lie algebras of Classical Lie groups). The following list gives the Lie algebras for each classical Lie group of Table 1.1.

G	\mathfrak{g}
$GL(n, \mathbb{K})$	$\mathfrak{gl}(n, \mathbb{K})$
$SL(n, \mathbb{K})$	$\mathfrak{sl}(n, \mathbb{K})$
$O(n)$	$\mathfrak{so}(n, \mathbb{R})$
$O(n, \mathbb{C})$	$\mathfrak{so}(n, \mathbb{C})$
$SO(n)$	$\mathfrak{so}(n, \mathbb{R})$
$SO(n, \mathbb{C})$	$\mathfrak{so}(n, \mathbb{C})$
$U(n)$	$\mathfrak{u}(n)$
$SU(n)$	$\mathfrak{su}(n)$
$Sp(n, \mathbb{K})$	$\mathfrak{sp}(n, \mathbb{K})$

Table 1.3: Lie algebras of Classical Lie groups

For a Lie group G , we define the left and right translation L and R respectively to be the maps

$$L : \begin{cases} G \rightarrow G \\ x \mapsto (L_x : g \mapsto xg) \end{cases}$$

and

$$R : \begin{cases} G \rightarrow G \\ x \mapsto (R_x : g \mapsto gx^{-1}). \end{cases}$$

Consider the tangent bundle TG of G and let ${}^G\Gamma(TG)$ be the space of left invariant sections of $\Gamma(TG)$, i.e. the space of sections $\bar{X} \in \Gamma(TG)$ such that $L_g^*(\bar{X}) = \bar{X}$ for every g in G , equipped with the Lie algebra structure of Example 1.1.17. Recall that $L_g^*(\bar{X})$ stands for the pullback section, i.e. the section of TG whose value at $x \in G$ coincides with the value \bar{X}_{gx} of \bar{X} at gx .

Proposition 1.1.25 (See [Pau, Proposition 1.6]). *The Lie algebra ${}^G\Gamma(TG)$ is isomorphic to \mathfrak{g} , the isomorphism being that a left-invariant section \bar{X} of ${}^G\Gamma(TG)$ corresponds to $\bar{X}_e \in \mathfrak{g}$, i.e. \bar{X} evaluated at the neutral element e of G .*

Proposition 1.1.26 (See [Pau, Theorem 1.41 and Corollary 1.42]). *Let G be a Lie group and H be a Lie subgroup of G . If \mathfrak{g} and \mathfrak{h} are the Lie algebras of G and H respectively, \mathfrak{h} is a Lie subalgebra of \mathfrak{g} . On the other hand, if \mathfrak{h}' is a subalgebra of \mathfrak{g} , there is a unique connected subgroup H' of G such that \mathfrak{h}' is the Lie algebra of H' .*

1.1.6 Exponential map

There is a map connecting a Lie group G and its Lie algebra \mathfrak{g} . Let X be an element of \mathfrak{g} and \bar{X} be the corresponding left-invariant section of ${}^G\Gamma(TG)$. Suppose that ϕ^X is the flow of \bar{X} , i.e. the map sending $g \in G$ to $\phi_{(\cdot)}^X(g)$, with $\phi_{(\cdot)}^X(g)$ being the unique solution of the differential equation

$$\begin{cases} \frac{d}{dt}\phi_t^X(g) = \bar{X}(\phi_t^X(g)) \\ \phi_0^X(g) = g. \end{cases}$$

Proposition 1.1.27 (See [Pau, Proposition 1.37]). *Let G be a real (complex respectively) Lie group and \bar{X} be a left-invariant vector field of ${}^G\Gamma(TG)$. The vector field \bar{X} is of class C^∞ (respectively holomorphic) and it is complete, i.e. for every $g \in G$, the map*

$$t \mapsto \phi_t^X(g) \in G$$

is well-defined on \mathbb{R} .

Definition 1.1.28 (Exponential map). *We call exponential map of a Lie group G the map*

$$\exp_G : \begin{cases} \mathfrak{g} \rightarrow G \\ X \mapsto \phi_1^X(e), \end{cases}$$

where e stands for the neutral element of G .

When there is no ambiguity, we will simply write \exp instead of \exp_G .

Proposition 1.1.29. *The exponential map of G is a smooth map.*

Proof. Consider the map

$$F : \begin{cases} G \times \mathfrak{g} \times \mathbb{R} \rightarrow G \times \mathfrak{g} \\ (g, X, t) \mapsto (\phi_t^X(g), X), \end{cases}$$

where $\phi^X : \mathbb{R} \times G \rightarrow G$ is the flow of the left-invariant vector field \bar{X} corresponding to X . Then

$$\begin{aligned} \frac{d}{dt}F(g, X, t) &= \left(\frac{d}{dt}\phi_t^X(g), 0 \right) \\ &= (\bar{X}(\phi_t^X(g)), 0). \end{aligned}$$

In other words, the map F is the flow of the smooth left-invariant vector field $(\bar{X}, 0)$ of $G \times \mathfrak{g}$ and thus F is smooth, as well. The exponential evaluated at $X \in \mathfrak{g}$ is the projection on the first factor of $F(e, X, 1)$ and the result follows. \square

In the following lemma, we summarize the basic properties of the exponential map.

Lemma 1.1.30. *Let G be a Lie group and $\exp : \mathfrak{g} \rightarrow G$ be its exponential map.*

- (i) *The tangent map of \exp at $0 \in \mathfrak{g}$ is $T_0 \exp = I_{\mathfrak{g}}$. In particular, \exp is a local diffeomorphism at 0 .*
- (ii) *If $[X, Y] = 0$, $X, Y \in \mathfrak{g}$, then $\exp(X + Y) = \exp(X)\exp(Y)$.*

Proof. The first assertion is a special case of [Pau, Proposition 1.38 and Proposition 1.39]. The assertion (ii) is a corollary of Campbell-Baker-Hausdorff Formula (see [Kna96, Theorem B22]). \square

Remark 1.1.31. Using (ii) of Lemma 1.1.30, one deduces that for $s, t \in \mathbb{R}$ and $X \in \mathfrak{g}$, $\exp((s + t)X) = \exp(sX)\exp(tX)$ and $\exp(-X) = \exp(X)^{-1}$.

Proposition 1.1.32 (See [Pau, Proposition 1.7]). *Let G_1 and G_2 be Lie groups with \mathfrak{g}_1 and \mathfrak{g}_2 being the corresponding Lie algebras. A map $\Phi : G_1 \rightarrow G_2$ is a Lie group morphism if and only if its tangent map $\phi := T_e \Phi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra morphism. Moreover, one has:*

$$\Phi \circ \exp_{G_1} = \exp_{G_2} \circ \phi.$$

In other words, the following diagram is:

$$\begin{array}{ccc} \mathfrak{g}_1 & \xrightarrow{\phi} & \mathfrak{g}_2 \\ \exp_{G_1} \downarrow & & \downarrow \exp_{G_2} \\ G_1 & \xrightarrow{\Phi} & G_2 \end{array}$$

If $\psi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a Lie algebra morphism and G_1 is connected and simply connected, there is a unique Lie group morphism $\Psi : G_1 \rightarrow G_2$ such that $T_e \Psi = \psi$.

Corollary 1.1.33. *Let G be a Lie group of Lie algebra \mathfrak{g} , Ad be the adjoint map of G defined in Example 1.1.9, and ad be the adjoint map of \mathfrak{g} defined in (1.2). Then, for every X in \mathfrak{g} , one has that:*

$$\text{Ad} \circ \exp(X) = \exp \circ \text{ad}(X).$$

Proof. In Proposition 1.1.32, set $\Phi = \text{Ad}$ and $\phi = \text{ad}$. □

Corollary 1.1.34. *Let G be a Lie group with Lie algebra \mathfrak{g} , H be a Lie subgroup of G with Lie algebra \mathfrak{h} and $i : H \hookrightarrow G$ be the inclusion of H in G . Then one has that:*

$$(\exp_G)|_{\mathfrak{h}} = \exp_H,$$

i.e. \exp_H is the restriction $(\exp_G)|_{\mathfrak{h}}$ of \exp_G to \mathfrak{h} .

Proof. The result is an application of Proposition 1.1.32 for $\Phi = i$. □

Example 1.1.35 (Exponential map of \mathbb{R}^* and \mathbb{C}^*). The exponential map of the Lie group (\mathbb{R}^*, \times) (respectively (\mathbb{C}^*, \times)) coincides with the standard exponential function. Indeed, if X is a vector of $T_1\mathbb{R}^* \cong \mathbb{R}$ (respectively $T_1\mathbb{C}^* \cong \mathbb{C}$), the left-invariant vector field \bar{X} corresponding to X is given by $\bar{X}(x) = xX$ for every $x \in \mathbb{R}^*$ (respectively $x \in \mathbb{C}^*$) while the flow of \bar{X} is the unique solution of the differential equation

$$\begin{cases} \frac{d}{dt}\phi_t^X(x) = \phi_t^X(x)X \\ \phi_0^X(x) = x. \end{cases}$$

One can check that $\phi_t^X(x) = xe^{tX}$ and thus $\exp(X) := \phi_1^X(1) = e^X$.

Example 1.1.36 (Exponential map of S_1). Let G be the Lie group of the unit circle S_1 . Then G is a Lie subgroup of \mathbb{C}^* and, according to Corollary 1.1.34, the exponential map of S_1 is given by the restriction of the exponential map of \mathbb{C}^* to S_1 . In other words, using the Example 1.1.35, one deduces that the exponential map of S_1 is given by the standard complex exponential map.

Example 1.1.37 (Exponential of matrix Lie groups). The exponential maps of the matrix Lie groups of Table 1.1 coincide with the standard matrix exponential map

$$e^X := \sum_{k \geq 0} \frac{X^k}{k!},$$

where $X \in M_n(\mathbb{K})$, i.e. X is a square matrix with entries in \mathbb{K} , with $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Indeed, let $G := GL(n, \mathbb{K})$ and X be an element of $\text{Lie}(G) = \mathfrak{gl}(n, \mathbb{K})$. The left-invariant vector field corresponding to X is given by $\bar{X}(x) = xX$, $x \in G$, and the flow of \bar{X} is the unique solution of

$$\begin{cases} \frac{d}{dt}\phi_t^X(x) = \phi_t^X(x)X \\ \phi_0^X(x) = x. \end{cases}$$

One can check that $\phi_t^X(x) = xe^{tX}$ and thus $\exp(X) := \phi_1^X(1) = e^X$. If H is any group of Table 1.1, according to Corollary 1.1.34, its exponential map is given again by the standard matrix exponential restricted to $\text{Lie}(H)$.

Example 1.1.38 (Exponential map of a compact Lie group). Let G be a compact Lie group. Then G admits a bi-invariant Riemannian metric (see [Pau, page 202]). The Riemannian exponential map defined by this metric coincides with the exponential map coming from the Lie group structure (see [Pri77, Theorem 4.3.3]). Since G is compact, using Hopf-Rinow theorem (see [Pau, Theorem 3.24]), one deduce that G is geodesically complete and so \exp is surjective. For a different proof of the above statement, without using Riemannian geometry, see [Kna96, Corollary 4.48].

1.1.7 Killing form

Definition 1.1.39. Let \mathfrak{g} be a Lie algebra. We define its Killing form $B_{\mathfrak{g}}$ (or just B when there is no confusion) to be the bilinear form of \mathfrak{g} given by

$$B(X, Y) := \text{tr}(\text{ad}X \circ \text{ad}Y), \quad X, Y \in \mathfrak{g}.$$

Lemma 1.1.40. The Killing form B of a Lie algebra \mathfrak{g} is symmetric and \mathfrak{g} -invariant, i.e. it satisfies the identity

$$B(\text{ad}Z(X), Y) + B(X, \text{ad}Z(Y)) = 0, \quad \forall X, Y, Z \in \mathfrak{g}.$$

Proof. The first statement follows from the properties of trace. For the second one, suppose X, Y and Z are elements of \mathfrak{g} . Then

$$\begin{aligned} B(\text{ad}Z(X), Y) &= B([Z, X], Y) \\ &= \text{tr}(\text{ad}[Z, X] \circ \text{ad}Y) \\ &= \text{tr}([\text{ad}Z, \text{ad}X] \circ \text{ad}Y) \\ &= \text{tr}((\text{ad}Z \circ \text{ad}X - \text{ad}X \circ \text{ad}Z) \circ \text{ad}Y) \\ &= \text{tr}(\text{ad}Z \circ \text{ad}X \circ \text{ad}Y) - \text{tr}(\text{ad}X \circ \text{ad}Z \circ \text{ad}Y) \\ &= \text{tr}(\text{ad}X \circ \text{ad}Y \circ \text{ad}Z) - \text{tr}(\text{ad}X \circ \text{ad}Z \circ \text{ad}Y) \\ &= \text{tr}(\text{ad}X \circ (\text{ad}Y \circ \text{ad}Z - \text{ad}Z \circ \text{ad}Y)) \\ &= \text{tr}(\text{ad}X \circ [\text{ad}Y, \text{ad}Z]) \\ &= -\text{tr}(\text{ad}X \circ \text{ad}[Z, Y]) \\ &= -B(X, [Z, Y]) \\ &= -B(X, \text{ad}Z(Y)), \end{aligned}$$

as we wanted. □

Lemma 1.1.41. The Killing form B of \mathfrak{g} is invariant by Lie algebra automorphisms, i.e. if $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra automorphism, then

$$B(\phi(X), \phi(Y)) = B(X, Y), \quad \forall X, Y \in \mathfrak{g}.$$

Proof. If $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra automorphism, then

$$\text{ad}(\phi(X)) = \phi \circ \text{ad}X \circ \phi^{-1}, \quad \forall X \in \mathfrak{g}.$$

Therefore, for every $X, Y \in \mathfrak{g}$,

$$\begin{aligned}
 B(\phi(X), \phi(Y)) &= \text{tr}(\text{ad}(\phi(X)) \circ \text{ad}(\phi(Y))) \\
 &= \text{tr}(\phi \circ \text{ad}(X) \circ \phi^{-1} \circ \phi \circ \text{ad}(Y) \circ \phi^{-1}) \\
 &= \text{tr}(\phi \circ \text{ad}(X) \circ \text{ad}(Y) \circ \phi^{-1}) \\
 &= \text{tr}(\text{ad}(X) \circ \text{ad}(Y)) \\
 &= B(X, Y)
 \end{aligned}$$

□

Example 1.1.42 (Killing forms of Classical Lie algebras). The following table gives the Killing form for each of the classical Lie algebras of Table 1.2.

\mathfrak{g}	Killing form
$\mathfrak{gl}(n, \mathbb{K})$	$2n\text{tr}(XY) - 2\text{tr}(X)\text{tr}(Y)$
$\mathfrak{sl}(n, \mathbb{K})$	$2n\text{tr}(XY)$
$\mathfrak{so}(2n + 1, \mathbb{K})$	$(2n - 1)\text{tr}(XY)$
$\mathfrak{sp}(n, \mathbb{K})$	$(2n + 2)\text{tr}(XY)$
$\mathfrak{so}(2n, \mathbb{K})$	$2(n - 1)\text{tr}(XY)$

Table 1.4: Killing forms of Classical Lie algebras

1.1.8 Universal enveloping algebra

From the definition of a Lie algebra it is clear that, in general, it is neither associative nor unital. Nevertheless, we can always find an associative unital algebra containing \mathfrak{g} in a canonical way. More precisely, let $T(\mathfrak{g})$ be the tensor algebra of the vector space \mathfrak{g} , i.e. the unital associative algebra

$$T(\mathfrak{g}) := \bigoplus_{n \geq 0} (\bigotimes^n \mathfrak{g}),$$

and set J to be the ideal

$$J := \left(\begin{array}{l} \text{two-sided ideal generated by} \\ X \otimes Y - Y \otimes X - [X, Y] \\ \text{with } X, Y \in \mathfrak{g} \subset T(\mathfrak{g}) \end{array} \right)$$

of $T(\mathfrak{g})$.

Definition 1.1.43 (Universal enveloping algebra of \mathfrak{g}). We define the universal enveloping algebra of \mathfrak{g} to be the unital associative (quotient) algebra

$$U(\mathfrak{g}) = T(\mathfrak{g})/J.$$

We have a canonical map ι from \mathfrak{g} to $U(\mathfrak{g})$ consisting of the embedding of \mathfrak{g} in $T(\mathfrak{g})$ following by projection to the quotient $U(\mathfrak{g})$. The enveloping algebra $U(\mathfrak{g})$ is harder to work with than other algebras like the symmetric algebra $S(\mathfrak{g})$ or the exterior algebra $\bigwedge \mathfrak{g}$ of \mathfrak{g} . The reason is that in contrast to these algebras, $U(\mathfrak{g})$ is defined using an ideal which is not generated by homogeneous elements. Nevertheless, we have the following theorem.

Theorem 1.1.44 (Poincaré-Birkhoff-Witt, see [Dix77, Theorem 2.1.11]). *Let $\{X_i\}_{1 \leq i \leq n}$ be a vector space basis of \mathfrak{g} . The monomials*

$$\iota(X_1)^{j_1} \dots \iota(X_n)^{j_n}, \tag{1.3}$$

with $j_k \geq 0$, $1 \leq k \leq n$, form a basis of $U(\mathfrak{g})$.

In particular, the map ι is an injection and \mathfrak{g} is canonically embedded in $U(\mathfrak{g})$. The universal enveloping algebra of \mathfrak{g} is characterized by the following universal property. For every unital associative algebra A and for every linear map $\varphi : \mathfrak{g} \rightarrow A$ satisfying:

$$\varphi(X)\varphi(Y) - \varphi(Y)\varphi(X) - \varphi([X, Y]) = 0, \quad \forall X, Y \in \mathfrak{g}, \tag{1.4}$$

there is a unique morphism of unital algebras $\tilde{\varphi} : U(\mathfrak{g}) \rightarrow A$ such that the diagram

$$\begin{array}{ccc} U(\mathfrak{g}) & \xrightarrow{\tilde{\varphi}} & A \\ & \swarrow \iota & \nearrow \phi \\ & \mathfrak{g} & \end{array}$$

commutes. From now on, since the map ι is injective, we will write

$$X_1^{j_1} \dots X_n^{j_n} \tag{1.5}$$

instead of the monomial (1.3).

Corollary 1.1.45. *Let \mathfrak{h} be a Lie subalgebra of \mathfrak{g} and $\{H_1, \dots, H_m\}$ be a vector space basis of \mathfrak{h} . The subalgebra of $U(\mathfrak{g})$ generated by 1 and the elements H_1, \dots, H_m is canonically isomorphic to the universal enveloping algebra $U(\mathfrak{h})$ of \mathfrak{h} .*

Proof. If $\rho : \mathfrak{h} \rightarrow U(\mathfrak{g})$ is the linear map sending the element H of \mathfrak{h} to the element H of $U(\mathfrak{h})$, ρ satisfies the condition (1.4) and so, by the universal property of $U(\mathfrak{h})$, it can be extended to a morphism of unital associative algebras $\tilde{\rho} : U(\mathfrak{h}) \rightarrow U(\mathfrak{g})$. The image of this map is certainly the subalgebra of $U(\mathfrak{g})$ generated by 1 and the elements H_1, \dots, H_m . On the other hand, according to Theorem 1.1.44, the monomials of the form (1.5) generated by H_1, \dots, H_m are linearly independent both in $U(\mathfrak{h})$ and $U(\mathfrak{g})$ and thus $\tilde{\rho}$ is a monomorphism onto the subalgebra of $U(\mathfrak{g})$ generated by 1 and the elements H_1, \dots, H_m . \square

1.2 Complex semisimple Lie algebras and root decomposition

This section focuses on a special case of complex Lie algebras, the semisimple ones. We start with some definitions.

Definition 1.2.1 (Ideal of a Lie algebra). *Let \mathfrak{g} be a real or complex Lie algebra. A subspace \mathfrak{h} of \mathfrak{g} is said to be an ideal of \mathfrak{g} if, for every X in \mathfrak{g} and every Y in \mathfrak{h} , $[X, Y]$ lies in \mathfrak{h} .*

In other words, \mathfrak{h} is an ideal of \mathfrak{g} if it is \mathfrak{g} -stable, i.e. it is invariant by the action of every operator $\text{ad}X$, $X \in \mathfrak{g}$.

Definition 1.2.2 (Simple Lie algebra). *A nonabelian Lie algebra is said to be simple if it has no nonzero proper ideal. Recall that \mathfrak{g} is abelian if its adjoint map ad is the zero map.*

Definition 1.2.3 (Semisimple Lie algebra). *A Lie algebra is said to be semisimple if it is the direct sum of simple Lie algebras.*

Lemma 1.2.4 (See [Kna96, Corollary 1.53]). *Let \mathfrak{g} be a complex Lie algebra and \mathfrak{g}_0 be a real form of \mathfrak{g} . Then \mathfrak{g} is semisimple if and only if \mathfrak{g}_0 is semisimple.*

Example 1.2.5. The complex Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of all 2-by-2 matrices is simple. Indeed, the matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a vector space basis of $\mathfrak{sl}(2, \mathbb{C})$. The Lie brackets of X, Y and H are

$$[H, X] = 2X, [H, Y] = -2Y \text{ and } [X, Y] = H.$$

Suppose that I is a nonzero ideal of $\mathfrak{sl}(2, \mathbb{C})$ and let $L = aH + bX + cY$ be a nonzero element of I . Then

$$\begin{aligned} [H, L] &= [H, aH + bX + cY] \\ &= a[H, H] + b[H, X] + c[H, Y] \\ &= 2bX - 2cY \end{aligned}$$

and

$$\begin{aligned} [H, [H, L]] &= [H, 2bX - 2cY] \\ &= 2b[H, X] - 2c[H, Y] \\ &= 4bX + 4cY \end{aligned}$$

belong to I and this is also the case for aH . Therefore H belongs to I or $a = 0$. If H belongs to I , then this is the case for $X = \frac{1}{2}[H, X]$ and $Y = -\frac{1}{2}[H, Y]$ and so I coincides with $\mathfrak{sl}(2, \mathbb{C})$. If $a = 0$, then applying X to L gives $[X, L] = -c[X, Y] = -cH$. Thus H belongs to I or $c = 0$. If H belongs to I we are done. Otherwise, $L = bX$ and X belongs to I . In this case, $[X, Y] = H$ belongs to I and we are done again. Therefore $\mathfrak{sl}(2, \mathbb{C})$ is simple.

Definition 1.2.6 (Semisimple Lie group). *A connected Lie group G is said to be semisimple (respectively simple) if its Lie algebra \mathfrak{g} is simple (respectively semisimple).*

Proposition 1.2.7 (Cartan's criteria for semisimplicity, see [Pau, Section 1.3.5]). *A complex Lie algebra \mathfrak{g} is semisimple if and only if one of the following equivalent conditions holds:*

1. \mathfrak{g} is the direct sum of simple Lie algebras,
2. Every abelian ideal of \mathfrak{g} is trivial,
3. The Killing form B of \mathfrak{g} is nondegenerate.

Definition 1.2.8 (Cartan subalgebra of \mathfrak{g}). A Lie subalgebra \mathfrak{t} of a complex semisimple Lie algebra \mathfrak{g} is said to be a Cartan subalgebra of \mathfrak{g} if \mathfrak{t} is a maximal abelian subalgebra of \mathfrak{g} such that, for every $H \in \mathfrak{t}$, the operator $\text{ad}(H)$ is diagonalizable. By abuse of terminology, we will say that a real Lie subalgebra \mathfrak{t}_0 of a real semisimple Lie algebra \mathfrak{g}_0 is a Cartan subalgebra of \mathfrak{g}_0 if the corresponding complexified Lie algebra \mathfrak{t} of \mathfrak{t}_0 is a Cartan subalgebra of the complexified Lie algebra \mathfrak{g} of \mathfrak{g}_0 .

Theorem 1.2.9 (See [Pau, Theorem 1.27]). Any complex semisimple Lie algebra \mathfrak{g} contains a Cartan subalgebra. All Cartan subalgebras of \mathfrak{g} are conjugate under automorphism of \mathfrak{g} .

Since all Cartan subalgebras of \mathfrak{g} are conjugate, they have the same vector space dimension.

Definition 1.2.10 (Rank of \mathfrak{g}). The dimension of the Cartan subalgebras of a complex semisimple Lie algebra \mathfrak{g} , denoted by $\text{rk}_{\mathbb{C}}(\mathfrak{g})$, is called the (complex) rank of \mathfrak{g} .

For the rest of this chapter, \mathfrak{g} will stand for a complex semisimple Lie algebra and \mathfrak{t} for a Cartan subalgebra of \mathfrak{g} .

Definition 1.2.11 (Root subspaces). For every $\alpha \in \mathfrak{t}^*$, set

$$\mathfrak{g}_{\alpha} := \{X \in \mathfrak{g} \mid \text{ad}(H)X = \alpha(H)X, \forall H \in \mathfrak{t}\}.$$

Let $\Delta(\mathfrak{g}, \mathfrak{t})$ (or just Δ if there is no confusion) be the set of all nonzero $\alpha \in \mathfrak{t}^*$ with $\mathfrak{g}_{\alpha} \neq \{0\}$. The elements of Δ are called the roots (or the \mathfrak{t} -roots) of \mathfrak{g} with respect to \mathfrak{t} , Δ is the root system of \mathfrak{g} while \mathfrak{g}_{α} , $\alpha \in \Delta$, are the root subspaces of \mathfrak{g} .

Since \mathfrak{t} is by definition abelian, the operators $\text{ad}H$, $H \in \mathfrak{t}$, commute and thus they are simultaneously diagonalizable.

Definition 1.2.12 (Root decomposition). The algebra \mathfrak{g} admits the following spectral decomposition, called the root decomposition of \mathfrak{g} , with respect to \mathfrak{t} :

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}.$$

Lemma 1.2.13. Let α and β be roots of Δ .

- (i) Δ spans \mathfrak{t}^* ,
- (ii) $-\alpha \in \Delta$,
- (iii) $B(\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}) \neq 0$ if and only if $\alpha + \beta = 0$,
- (iv) If $\alpha + \beta \neq 0$, then $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha + \beta \in \Delta$ and $\{0\}$ otherwise,

- (v) $\dim(\mathfrak{g}_\alpha) = 1$,
- (vi) $B(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$ for every α in Δ ,
- (vii) $\frac{1}{2}\alpha \notin \Delta$.

Proof. The Lemma is a combination of Proposition 2.17, Proposition 2.21 and Corollary 2.35 of [Kna96]. \square

From (vi) of the above Lemma and the fact that B is nondegenerate, one gets that the restriction of B to $\mathfrak{t} \times \mathfrak{t}$ defines a nondegenerate form. Therefore, there is a natural duality between \mathfrak{t} and \mathfrak{t}^* . Namely, for every $\lambda \in \mathfrak{t}^*$, we define $H_\lambda \in \mathfrak{t}$ to be the unique element of \mathfrak{t} such that $\lambda(H) = B(H_\lambda, H)$ for every $H \in \mathfrak{t}$. Using this pairing, we define a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{t}^* given by

$$\langle \lambda, \mu \rangle := B(H_\lambda, H_\mu)$$

for every $\lambda, \mu \in \mathfrak{h}^*$. Let

$$\mathfrak{t}_0 := \langle H_\alpha \in \mathfrak{t} \mid \alpha \in \Delta \rangle_{\mathbb{R}}, \quad (1.6)$$

i.e. the real vector space generated by all H_α with α being in Δ . Then \mathfrak{t}_0 is a real form of \mathfrak{t} , i.e. $\mathfrak{t}_0^{\mathbb{C}} := \mathfrak{t}_0 \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{t}$ as complex Lie algebras.

Lemma 1.2.14 (See [Kna96, Corollary 2.38]). *The subalgebra \mathfrak{t}_0 coincides with the real subspace of $\mathfrak{t}^{\mathbb{R}}$ characterized by the property that every root is real valued on it. In other words, for every root α and every $H \in \mathfrak{t}_0$, $\alpha(H)$ is real. Moreover, the bilinear forms $B_{\mathfrak{t}_0} := B|_{\mathfrak{t}_0 \times \mathfrak{t}_0}$ and $\langle \cdot, \cdot \rangle$ are inner products of \mathfrak{t}_0 and \mathfrak{t}_0^* respectively.*

Using the inner product $\langle \cdot, \cdot \rangle$ of \mathfrak{t}_0^* , one obtains again a natural pairing between \mathfrak{t}_0 and \mathfrak{t}_0^* . Moreover, let $\|\cdot\|$ be the norm of \mathfrak{t}_0^* determined by the inner product $\langle \cdot, \cdot \rangle$. Let l be the rank of \mathfrak{g} and $\{H_1, \dots, H_l\}$ be a basis of \mathfrak{t}_0^* . On \mathfrak{t}_0^* we define a lexicographic ordering in the following way.

Definition 1.2.15 (Lexicographic ordering). *We will say that $\mu \in \mathfrak{t}_0^*$ is greater than $\nu \in \mathfrak{t}_0^*$, and write $\mu > \nu$ or $\nu < \mu$, if there is k , $1 \leq k \leq l - 1$ such that $\mu(H_i) = \nu(H_i)$ for every i in $\{1, \dots, k\}$ and $\mu(H_{k+1}) > \nu(H_{k+1})$.*

With this ordering, Δ splits into Δ^+ , the set of positive roots or the positive system of Δ , i.e. the set of all roots α with $\alpha > 0$, and Δ^- , the set of negative roots, i.e. the set of all roots α with $\alpha < 0$. We note that, by (ii) of Lemma 1.2.13, $\Delta^+ = -\Delta^-$.

Definition 1.2.16 (positive system). *Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{t} be a Cartan subalgebra of \mathfrak{g} and Δ be the corresponding root system. On $\mathfrak{t}_0 := \{H_\alpha \in \mathfrak{t} \mid \alpha \in \Delta\}_{\mathbb{R}}$, fix a lexicographic ordering. The set of all positive roots Δ^+ of Δ with respect to this lexicographic ordering is called positive system for Δ , corresponding to this ordering.*

Definition 1.2.17 (Half sum of positive roots). *The element*

$$\rho := \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

is called the half sum of the positive roots of Δ .

Definition 1.2.18 (Simple system). A subset Π of Δ^+ is called simple system for Δ if every root of Δ can be written as an integer sum of elements of Π having its coefficients either all nonnegative or all nonpositive. The elements of Π are called simple roots.

Definition 1.2.19 (Fundamental weights). The element $\bar{\omega}_i$ of $\mathfrak{t}_0^* \subset \mathfrak{t}^*$ given by

$$2 \frac{\langle \bar{\omega}_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij}$$

is called the i -th fundamental weight of \mathfrak{g} .

1.2.1 Classical Lie algebras

There are four families of classical Lie algebras, namely A_n , B_n , C_n and D_n which all are simple.

$$\underline{A_n = \mathfrak{sl}(n+1, \mathbb{C}), n \geq 1}$$

Let \mathfrak{g} be the Lie algebra

$$\mathfrak{sl}(n+1, \mathbb{C}) := \{X \in \mathfrak{gl}(n+1, \mathbb{C}) \mid \text{tr}(X) = 0\}.$$

The Lie subalgebra \mathfrak{t} of \mathfrak{g} consisting of all diagonal traceless matrices is a Cartan subalgebra of \mathfrak{g} . Define ε_i to be the linear functional given by

$$\varepsilon_i \left(\begin{pmatrix} h_1 & & \\ & \ddots & \\ & & h_{n+1} \end{pmatrix} \right) = h_i.$$

Then, the root space of \mathfrak{g} is

$$\Delta := \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n+1\}.$$

Indeed, for i and j in $\{1, \dots, n+1\}$, define $E_{\varepsilon_i - \varepsilon_j} := E_{ij}$ to be the matrix having all its entries zero except for the (i, j) -entry which is 1. Then $\text{ad}(H)(E_{ij}) = (\varepsilon_i - \varepsilon_j)(H)E_{ij}$ for every H in \mathfrak{t} . The subset

$$\Delta^+ := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\} \subset \Delta$$

is a set of positive roots of Δ while

$$\rho = \frac{n}{2}\varepsilon_1 + \frac{n-2}{2}\varepsilon_2 + \dots - \frac{n}{2}\varepsilon_{n+1}$$

is the half sum of the positive roots. The simple system of Δ is

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_n - \varepsilon_{n+1}\}.$$

$\mathbf{B}_n = \mathfrak{so}(2n+1, \mathbb{C}), n \geq 2$

Let \mathfrak{g} be the Lie algebra

$$\mathfrak{so}(2n+1, \mathbb{C}) := \{X \in \mathfrak{gl}(2n+1, \mathbb{C}) \mid X + X^t = 0\},$$

i.e. the orthogonal Lie algebra of $(2n+1)$ -by- $(2n+1)$ matrices. Let \mathfrak{t} be the Cartan subalgebra of \mathfrak{g} generated by all the matrices H of the form

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix},$$

where $h_1, \dots, h_n \in \mathbb{C}$. For every i in $\{1, \dots, n\}$, let ε_i be the element of the dual space \mathfrak{t}^* given by $\varepsilon_i(H) = h_i$, where H is the above matrix. The root space of \mathfrak{g} is

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\pm\varepsilon_i \mid 1 \leq i \leq n\}.$$

Indeed, for $i < j$ and $\alpha = \pm\varepsilon_i \pm \varepsilon_j$, let E_α be the matrix being everywhere 0 except for the entries corresponding to the i^{th} and j^{th} pairs which are given by

$$E_\alpha = \begin{pmatrix} & i & & j \\ 0 & & X_\alpha & \\ -X_\alpha^t & & & 0 \end{pmatrix} \begin{matrix} i \\ \\ j \end{matrix}$$

with

$$X_{\varepsilon_i - \varepsilon_j} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad X_{\varepsilon_i + \varepsilon_j} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},$$

$$X_{-\varepsilon_i + \varepsilon_j} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad X_{-\varepsilon_i - \varepsilon_j} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}.$$

For $\alpha = \pm\varepsilon_i$, we define E_α to be everywhere 0 except for the entries corresponding to the i^{th} pair and the $(2n+1)^{\text{th}}$ entry which are given by

$$E_\alpha = \begin{pmatrix} & k^{\text{th}} \text{ pair} & & (2n+1)^{\text{th}} \text{ entry} \\ 0 & & X_\alpha & \\ -X_\alpha^t & & & 0 \end{pmatrix} \begin{matrix} k^{\text{th}} \text{ pair} \\ \\ (2n+1)^{\text{th}} \text{ entry} \end{matrix}$$

with

$$X_{\varepsilon_i} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad X_{-\varepsilon_i} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

One can verify that, for every $H \in \mathfrak{t}$ and for every $\alpha \in \Delta$, $\text{ad}(H)E_\alpha = \alpha(H)E_\alpha$. We choose

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_i \mid 1 \leq i \leq n\}$$

to be the set of positive roots of Δ . The half sum ρ of the positive roots is

$$\rho = (n - \frac{1}{2})\varepsilon_1 + (n - \frac{3}{2})\varepsilon_2 + \dots + \frac{1}{2}\varepsilon_n$$

while the simple system is

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}.$$

$\mathbf{C}_n = \mathfrak{sp}(n, \mathbb{C}), n \geq 3$

Let \mathfrak{g} be the Lie algebra

$$\mathfrak{sp}(n, \mathbb{C}) := \{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid X^t J + JX = 0\},$$

i.e. the symplectic Lie algebra of $2n$ -by- $2n$ matrices. Let \mathfrak{t} be the Lie subalgebra generated by all matrices H of the form

$$H = \begin{pmatrix} h_1 & & & & & \\ & \ddots & & & & \\ & & h_n & & & \\ & & & -h_1 & & \\ & & & & \ddots & \\ & & & & & -h_n \end{pmatrix},$$

with h_1, \dots, h_n in \mathbb{C} . It turns out \mathfrak{t} to be a Cartan subalgebra of \mathfrak{g} . For every i in $\{1, \dots, n\}$, let ε_i be the element of \mathfrak{t}^* given by $\varepsilon_i(H) = h_i$, with H being the above matrix. The root space of \mathfrak{g} is

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\pm 2\varepsilon_i \mid 1 \leq i \leq n\}.$$

Indeed, for i and j in $\{1, \dots, n\}$, define

$$\begin{aligned} E_{\varepsilon_i + \varepsilon_j} &= E_{i, j+n} + E_{j, i+n}, & E_{\varepsilon_i - \varepsilon_j} &= E_{i, j} + E_{j+n, i+n} \\ E_{-\varepsilon_i - \varepsilon_j} &= E_{i+n, j} - E_{j+n, i} & E_{2\varepsilon_i} &= E_{i, i+n} \\ E_{-2\varepsilon_i} &= E_{i+n, i} \end{aligned}$$

Then, for every H in \mathfrak{t} and every α in Δ , $\text{ad}(H)E_\alpha = \alpha(H)E_\alpha$. The subset

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \mid 1 \leq i \leq n\} \subset \Delta$$

forms a positive root system of Δ . The half sum of the positive roots is

$$\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \dots + \varepsilon_n$$

while

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$$

is the simple system.

$\mathbf{D}_n = \mathfrak{so}(2n, \mathbb{C}), n \geq 4$

Let \mathfrak{g} be the Lie algebra

$$\mathfrak{so}(2n, \mathbb{C}) := \{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid X + X^t = 0\},$$

i.e. the orthogonal Lie algebra of $2n$ -by- $2n$ matrices. Let \mathfrak{t} be the Lie subalgebra of \mathfrak{g} generated by all matrices H of the form

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} \end{pmatrix},$$

where $h_1, \dots, h_n \in \mathbb{C}$. The Lie subalgebra \mathfrak{t} turns out to be a Cartan subalgebra of \mathfrak{g} . As in the case of $\mathfrak{so}(2n+1, \mathbb{C})$, for every i in $\{1, \dots, n\}$, let ε_i be the elements of the dual space \mathfrak{t}^* given by $\varepsilon_i(H) = h_i$, where H is the above matrix. The root space of \mathfrak{g} is

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

Indeed, for every $\alpha \in \Delta$, let E_α be the matrix defined in the case of $\mathfrak{so}(2n+1, \mathbb{C})$ but without the $(2n+1)$ -line and arrow. One can verify that, for every $H \in \mathfrak{t}$ and every $\alpha \in \Delta$, $\text{ad}(H)E_\alpha = \alpha(H)E_\alpha$. The subset

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \subset \Delta$$

forms a positive system of Δ while the half sum of the positive roots is

$$\rho = (n-1)\varepsilon_1 + (n-2)\varepsilon_2 + \dots + \varepsilon_{n-1}.$$

The simple system is

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}.$$

For later use, we give in the following table the first fundamental weight for each classical Lie algebra (see Definition 1.2.19). For more details, one can see [Kna96, Appendice C].

\mathfrak{g}	n	first fundamental weight $\bar{\omega}_1$
$\mathfrak{sl}(n, \mathbb{K})$	$n \geq 2$	$\frac{n-1}{n}\varepsilon_1 - \frac{1}{n}\varepsilon_2 - \dots - \frac{1}{n}\varepsilon_n$
$\mathfrak{so}(2n+1, \mathbb{K})$	$n \geq 2$	ε_1
$\mathfrak{sp}(n, \mathbb{K})$	$n \geq 3$	ε_1
$\mathfrak{so}(2n, \mathbb{K})$	$n \geq 4$	ε_1

Table 1.5: First fundamental weights of classical Lie algebras

1.2.2 Weyl group

We continue with the same notation. Namely \mathfrak{g} is a complex semisimple Lie algebra, \mathfrak{t} is a Cartan subalgebra of \mathfrak{g} and Δ stands for the set of \mathfrak{t} -roots of \mathfrak{g} . For every root α in Δ , we introduce the root reflection

$$s_\alpha(\phi) = \phi - 2\frac{\langle \phi, \alpha \rangle}{\langle \alpha, \alpha \rangle}, \forall \phi \in \mathfrak{t}^*$$

One can check that s_α is -1 on $\mathbb{R}\{\alpha\}$ and 1 on the orthogonal complement of $\mathbb{R}\{\alpha\}$ and so it is an orthogonal transformation with respect to B .

Definition 1.2.20 (Weyl group). *We define the Weyl group $W(\Delta(\mathfrak{g}, \mathfrak{t}))$ (or $W(\mathfrak{g}, \mathfrak{t})$ or just W , if there is no confusion) to be the subgroup of the orthogonal group of \mathfrak{t}^* generated by the root reflections s_α for all $\alpha \in \Delta$.*

Proposition 1.2.21 (See [Kna96, Proposition 2.41]). *For every α and β in Δ , $s_\alpha(\beta)$ belongs to Δ . In other words, W carries Δ to Δ .*

Since W maps the finite set Δ of generators of \mathfrak{t}_0^* to itself, we deduce that W is a finite group.

Definition 1.2.22 (Length of $w \in W$). *Let w be an element of W . The number*

$$\ell(w) := \#\{\alpha \in \Delta^+ \mid w\alpha < 0\}$$

is called the length of w .

For later use, we describe the Weyl group for each classical Lie algebra.

Weyl group of $A_n = \mathfrak{sl}(n+1, \mathbb{C})$, $n \geq 1$

Let $\mathfrak{g} := \mathfrak{sl}(n+1, \mathbb{C})$ and \mathfrak{t} be the Cartan subalgebra of \mathfrak{g} of all traceless diagonal matrices defined for $\mathfrak{sl}(n+1, \mathbb{C})$ in Subsection 1.2.1. The Weyl group $W(\mathfrak{g}, \mathfrak{t})$ is isomorphic to the symmetric group S_{n+1} of $n+1$ objects. More precisely, for the element $s_{\varepsilon_i - \varepsilon_j}$ of W we have

$$s_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) = \begin{cases} \varepsilon_j & \text{if } k = i \\ \varepsilon_i & \text{if } k = j \\ \varepsilon_k & \text{otherwise} \end{cases}$$

and so there is a canonical isomorphism, mapping the element $s_{\varepsilon_i - \varepsilon_j}$ of W to the 2-cycle (ij) of S_{n+1} .

Weyl group of $\mathfrak{B}_n = \mathfrak{so}(2n + 1, \mathbb{C})$, $n \geq 2$

Let $\mathfrak{g} := \mathfrak{so}(2n + 1, \mathbb{C})$ and \mathfrak{t} be the Cartan subalgebra of $\mathfrak{so}(2n + 1, \mathbb{C})$ defined in Subsection 1.2.1. The Weyl group $W(\mathfrak{g}, \mathfrak{t})$ is isomorphic to the semidirect product $S_n \times (\mathbb{Z}_2)^n$. More precisely, for the element $s_{\varepsilon_i - \varepsilon_j}$ of W , we have

$$s_{\varepsilon_i - \varepsilon_j}(\varepsilon_k) = \begin{cases} \varepsilon_j & \text{if } k = i \\ \varepsilon_i & \text{if } k = j \\ \varepsilon_k & \text{otherwise,} \end{cases}$$

for the element $s_{\varepsilon_i + \varepsilon_j}$ we have

$$s_{\varepsilon_i + \varepsilon_j}(\varepsilon_k) = \begin{cases} -\varepsilon_j & \text{if } k = i \\ -\varepsilon_i & \text{if } k = j \\ \varepsilon_k & \text{otherwise,} \end{cases}$$

while

$$s_{\varepsilon_i}(\varepsilon_k) = \begin{cases} -\varepsilon_i & \text{if } k = i \\ \varepsilon_k & \text{otherwise.} \end{cases}$$

Let x_i , $1 \leq i \leq n$, be the element of $(\mathbb{Z}_2)^n$ having all its coordinates $\bar{0}$ except for the i -th one which is $\bar{1}$. Define a map from $W(\mathfrak{g}, \mathfrak{t})$ into $S_n \times (\mathbb{Z}_2)^n$ which maps the reflection $s_{\varepsilon_i - \varepsilon_j}$ to the element $((ij), \bar{0})$ of $S_n \times (\mathbb{Z}_2)^n$, where (ij) stands for the 2-cycle of i and j in S_n , the reflection $s_{\varepsilon_i + \varepsilon_j}$ to the element $((ij), x_i + x_j)$ and s_{ε_i} to $(1, x_i)$. One can check that this map turns out to be a well-defined group isomorphism between $W(\mathfrak{g}, \mathfrak{t})$ and $S_n \times (\mathbb{Z}_2)^n$.

In a similar way, one finds the Weyl groups of $C_n = \mathfrak{sp}(n, \mathbb{C})$, $n \geq 3$, and $D_n := \mathfrak{so}(2n, \mathbb{C})$, $n \geq 4$ (For more details, see [Hum78, Subsection 12.2]) The following table gives the Weyl group for each classical Lie algebra of Table 1.2:

\mathfrak{g}	Weyl group W	$ W $
$\mathfrak{sl}(n + 1, \mathbb{C})$	S_{n+1}	$(n + 1)!$
$\mathfrak{so}(2n + 1, \mathbb{C})$	$S_n \times (\mathbb{Z}_2)^n$	$2^n(n + 1)!$
$\mathfrak{sp}(n, \mathbb{C})$	$S_n \times (\mathbb{Z}_2)^n$	$2^n(n + 1)!$
$\mathfrak{so}(2n, \mathbb{C})$	$S_n \times (\mathbb{Z}_2)^{n-1}$	$2^{n-1}(n + 1)!$

Table 1.6: Weyl groups of classical Lie algebras

Lemma 1.2.23 (Chevalley's Lemma, see [Kna96, Proposition 2.72]). *Using the above notation, let λ be an element of \mathfrak{t}^* and W_λ be the stabilizer of λ in W , i.e.*

$$W_\lambda := \{w \in W \mid w\lambda = \lambda\}.$$

Then W_λ is generated by all elements s_α with $\alpha \in \Delta$ such that $\langle \lambda, \alpha \rangle = 0$.

1.2.3 Dynkin diagrams and Cartan matrices

To each complex simple Lie algebra \mathfrak{g} we attach a diagram known as the Dynkin diagram of \mathfrak{g} . This is a diagram having l vertices, $l \in \mathbb{N}$, each one indexed by a distinct number belonging to $\{1, \dots, l\}$. The i -vertex and the j -vertex are connected by $A_{ij}A_{ji}$ edges, with A_{ij} , $1 \leq i, j \leq l$ being entries of a matrix A , such that

1. $A_{ij} \in \mathbb{Z}$,
2. $A_{ii} = 2$ for all i ,
3. $A_{ij} \leq 0$ for $i \neq j$,
4. $A_{ij} = 0$ if and only $A_{ji} = 0$,
5. there exists a positive diagonal matrix D such that DAD^{-1} is symmetric positive definite.

Definition 1.2.24 (Cartan matrix). *A square matrix $A = (A_{ij})$ having the above properties is called Cartan matrix.*

In the case of a complex simple Lie algebra \mathfrak{g} , using the notation fixed above, we draw a diagram having l vertices, where $l = \text{rk}_{\mathbb{C}}(\mathfrak{g})$, i.e. the complex rank of \mathfrak{g} , each vertex corresponding to an element of the simple system Π of Δ . We connect the vertices corresponding to α_i and α_j of Π by $A_{ij}A_{ji}$ edges, where

$$A_{ij} := 2 \frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}.$$

This diagram turns out to be a Dynkin diagram (see [Kna96, Proposition 2.52]). When we want to indicate that a simple root is greater than another one, we put on the edge(s) connecting them an arrow pointing from the vertex corresponding to the longer root to the vertex corresponding to the shorter one. Using the classification of Dynkin diagrams and the fact that nonisomorphic complex simple Lie algebras correspond to different Dynkin diagrams we obtain a classification up to isomorphism of all complex simple Lie algebras. Namely, there are 8 distinct families of (connected) Dynkin diagrams, given in Tables 1.7 and 1.8.

Therefore, we obtain the following theorem.

Theorem 1.2.25 (Classification of complex simple Lie algebras). *Any complex simple Lie algebra is isomorphic with one of the Lie algebras of Table 1.7. The Lie algebras A_n, B_n, C_n and D_n are the classical Lie algebras while E_6, E_7, E_8, F_4 and G_2 are called exceptional Lie algebras.*

For an exhaustive presentation of the above discussion see [Kna96, Chapter II].

1.2. Complex semisimple Lie algebras and root decomposition

\mathfrak{g}	n	Dynkin Diagram	Cartan matrix
$A_n := \mathfrak{sl}(n+1, \mathbb{C})$	$n \geq 1$		$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$
$B_n := \mathfrak{so}(2n+1, \mathbb{C})$	$n \geq 2$		$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$
$C_n := \mathfrak{sp}(n, \mathbb{C})$	$n \geq 3$		$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$
$D_n := \mathfrak{so}(2n, \mathbb{C})$	$n \geq 4$		$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$

Table 1.7: Dynkin diagrams and Cartan matrices of classical Lie algebras

\mathfrak{g}	Dynkin Diagram	Cartan matrix
E_6		$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$
E_7		$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$
E_8		$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$
F_4		$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$
G_2		$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Table 1.8: Dynkin diagrams and Cartan matrices of exceptional Lie algebras

1.2.4 Reductive Lie algebras

Definition 1.2.26 (Reductive Lie algebra). *A Lie algebra \mathfrak{g} is said to be reductive if, for every ideal \mathfrak{h} of \mathfrak{g} , there is an ideal \mathfrak{h}' of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}'$.*

Definition 1.2.27 (Reductive in \mathfrak{g}). *Let \mathfrak{g} be a complex Lie algebra and \mathfrak{h} be a Lie subalgebra of \mathfrak{g} . The Lie subalgebra \mathfrak{h} is said to be reductive in \mathfrak{g} if the action of \mathfrak{h} on \mathfrak{g} through the adjoint map*

$$\text{ad}_{\mathfrak{g}} : \mathfrak{h} \rightarrow \mathfrak{gl}(\mathfrak{g})$$

of Definition 1.1.18 is completely reducible, i.e. \mathfrak{g} is decomposed into a direct sum of minimal \mathfrak{h} -invariant nonzero subspaces.

Definition 1.2.28 (Derived Lie subalgebra). *The Lie subalgebra*

$$[\mathfrak{g}, \mathfrak{g}] := \{[X, Y] \in \mathfrak{g} \mid X, Y \in \mathfrak{g}\}$$

of \mathfrak{g} is called the derived Lie subalgebra of \mathfrak{g} .

Lemma 1.2.29 (See [Kna96, Corollary 1.56]). *Every reductive Lie algebra \mathfrak{g} can be decomposed as*

$$\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}],$$

where $Z_{\mathfrak{g}}$ stands for its center, i.e. the Lie subalgebra of all elements $X \in \mathfrak{g}$ such that $\text{ad}X = 0$, and $[\mathfrak{g}, \mathfrak{g}]$ is a semisimple subalgebra of \mathfrak{g} . Conversely, any Lie algebra of this type is reductive.

From the above lemma it is clear that reductive Lie algebras constitute a generalization of semisimple Lie algebras. In fact, we have the following inclusion

$$\{\text{simple Lie algebras}\} \subset \{\text{semisimple Lie algebras}\} \subset \{\text{reductive Lie algebras}\}$$

Moreover if \mathfrak{g} is a reductive Lie algebra, we can always find a bilinear form B such that B is symmetric, \mathfrak{g} -invariant and nondegenerate, i.e. having the same "good" properties of the Killing form of a semisimple Lie algebra. To do so, consider the Killing form $B_{[\mathfrak{g}, \mathfrak{g}]}$ of the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of \mathfrak{g} and let $\{X_i\}$ be a basis of $Z_{\mathfrak{g}}$. Define B to be $B_{[\mathfrak{g}, \mathfrak{g}]}$ on $[\mathfrak{g}, \mathfrak{g}]$, with $[\mathfrak{g}, \mathfrak{g}]$ and $Z_{\mathfrak{g}}$ being orthogonal with respect to B , while on $Z_{\mathfrak{g}}$ is such that the basis $\{X_i\}$ is orthogonal. One can check that such a form B is symmetric, \mathfrak{g} -invariant and nondegenerate.

If we suppose that \mathfrak{g} is complex, using the root space decomposition of the complex semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, one obtains a similar decomposition for \mathfrak{g} . More precisely, suppose that \mathfrak{t}' is a Cartan subalgebra of $[\mathfrak{g}, \mathfrak{g}]$ and set

$$\mathfrak{t} := Z_{\mathfrak{g}} \oplus \mathfrak{t}'.$$

One can check that \mathfrak{t} is a maximal abelian Lie subalgebra of \mathfrak{g} while for every H in \mathfrak{t} , $\text{ad}H$ acts by zero if $H \in Z_{\mathfrak{g}}$ and by the roots of $\Delta([\mathfrak{g}, \mathfrak{g}], \mathfrak{t}')$ if $H \in \mathfrak{t}'$. We say that \mathfrak{t} is a Cartan subalgebra of \mathfrak{g} . By extending the roots of $\Delta([\mathfrak{g}, \mathfrak{g}], \mathfrak{t}')$ to be zero on $Z_{\mathfrak{g}}$ and without changing the notation for the extended elements, one defines

$$\mathfrak{g}_{\alpha} := \{X \in \mathfrak{g} \mid \text{ad}(H)X = \alpha(H)X, \forall H \in \mathfrak{t}\}.$$

If $\mathfrak{g}_\alpha \neq \{0\}$, imitating the terminology of complex semisimple Lie algebras, we say that α is a root of \mathfrak{g} while \mathfrak{g}_α is the corresponding root subspace. The set $\Delta(\mathfrak{g}, \mathfrak{t})$ (or just Δ if there is no ambiguity) is called the root system of \mathfrak{g} . It is immediate that \mathfrak{g} has a root decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})} \mathfrak{g}_\alpha.$$

Suppose that $\Delta^+([\mathfrak{g}, \mathfrak{g}], \mathfrak{t}')$ is a positive system of $\Delta([\mathfrak{g}, \mathfrak{g}], \mathfrak{t}')$. The corresponding extended elements in $\Delta(\mathfrak{g}, \mathfrak{t})$ forms a positive system $\Delta^+(\mathfrak{g}, \mathfrak{t})$ for $\Delta(\mathfrak{g}, \mathfrak{t})$. One can extend the other notions introduced for a complex semisimple Lie algebra in the case of a reductive one.

1.3 Real semisimple Lie algebras

Let \mathfrak{g}_0 be a real semisimple Lie algebra and B be its Killing form. Recall from Lemma 1.2.4 that the complexified Lie algebra $\mathfrak{g} := (\mathfrak{g}_0)^\mathbb{C}$ is also semisimple. Let $\theta : \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$ be an involution of \mathfrak{g}_0 , i.e. a Lie algebra morphism such that $\theta^2 = id$.

Definition 1.3.1 (Cartan involution). *An involution θ of \mathfrak{g}_0 is said to be a Cartan involution if the bilinear form*

$$B_\theta : (X, Y) \mapsto -B(X, \theta(Y)), X, Y \in \mathfrak{g}_0$$

is positive definite.

Proposition 1.3.2 (See [Kna96, Theorem 6.16]). *Every real semisimple Lie algebra \mathfrak{g}_0 admits a Cartan involution θ . If θ' is another Cartan involution of \mathfrak{g}_0 , then there is a Lie algebra automorphism φ of \mathfrak{g}_0 such that $\theta' = \varphi \circ \theta \circ \varphi^{-1}$.*

For the rest of this section, we fix \mathfrak{g}_0 to be a real semisimple Lie algebra and θ be a Cartan involution of \mathfrak{g}_0 . Let \mathfrak{k}_0 and \mathfrak{p}_0 be the $+1$ and -1 eigenspace of θ , respectively.

Definition 1.3.3 (Cartan decomposition-Lie algebra case). *The decomposition*

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

of \mathfrak{g}_0 into eigenspaces of θ is called Cartan decomposition of \mathfrak{g}_0 .

Lemma 1.3.4. *Let θ be a Cartan involution of \mathfrak{g}_0 and $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be the corresponding Cartan decomposition. One has that:*

$$(a) [\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0, \quad (b) [\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad (c) [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0.$$

Proof. Let $X_1, X_2 \in \mathfrak{k}_0$ and $Y_1, Y_2 \in \mathfrak{p}_0$. Since θ is a Lie algebra morphism, one has that

$$\begin{aligned} \theta[X_1, X_2] &= [\theta X_1, \theta X_2] \\ &= [X_1, X_2], \end{aligned}$$

$$\begin{aligned} \theta[X_1, Y_1] &= [\theta X_1, \theta Y_1] \\ &= [X_1, -Y_1] \\ &= -[X_1, Y_1] \end{aligned}$$

and

$$\begin{aligned}\theta[Y_1, Y_2] &= [\theta Y_1, \theta Y_2] \\ &= [-Y_1, -Y_2] \\ &= [Y_1, Y_2].\end{aligned}$$

Therefore $[X_1, X_2] \in \mathfrak{k}_0$, $[X_1, Y_1] \in \mathfrak{p}_0$ and $[Y_1, Y_2] \in \mathfrak{k}_0$ and thus $[\mathfrak{k}_0, \mathfrak{k}_0] \subseteq \mathfrak{k}_0$, $[\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0$ and $[\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{k}_0$. □

Lemma 1.3.5. *Let σ be an involution of \mathfrak{g}_0 and $\mathfrak{g}_0 = \mathfrak{r}_0 \oplus \mathfrak{s}_0$ be its decomposition into its ± 1 eigenspaces. Then σ is a Cartan involution if and only if $B|_{\mathfrak{r}_0 \times \mathfrak{r}_0}$ is negative and $B|_{\mathfrak{s}_0 \times \mathfrak{s}_0}$ is positive definite.*

Proof. For $X \in \mathfrak{r}_0$, $B(X, X) = B(X, \sigma(X))$ while for $Y \in \mathfrak{s}_0$, $B(Y, Y) = -B(Y, \sigma(Y))$. It is clear that σ is a Cartan involution if and only if the condition of the statement holds. □

Example 1.3.6 (Cartan decomposition of classical Lie algebras). For the Lie algebra $\mathfrak{gl}(n, \mathbb{R})$ we define the involution $\theta(X) = -X^t$, $X \in \mathfrak{gl}(n, \mathbb{C})$, and the bilinear map tr_θ given by

$$\text{tr}_\theta(X, Y) := -\text{tr}(X\theta(Y)), \quad X, Y \in \mathfrak{gl}(n, \mathbb{R}).$$

One can check that tr_θ is a positive-definite form. Indeed, if E_{ij} stands for the matrix having all its entries zero apart from the (i, j) -entry which is 1, then

$$\begin{aligned}\text{tr}_\theta(E_{ij}, E_{ij}) &= -\text{tr}(E_{ij}(-E_{ij})^t) \\ &= \text{tr}(E_{ij}E_{ji}) \\ &= 1\end{aligned}$$

and thus tr_θ is positive-definite.

Let \mathfrak{g}_0 be the real Lie algebra $\mathfrak{sl}(n, \mathbb{R})$ of Table 1.2. Then, according to Table 1.4, the Killing form B of \mathfrak{g}_0 is given by

$$B(X, Y) = 2n\text{tr}(XY), \quad X, Y \in \mathfrak{g}_0.$$

and thus the map B_θ of Definition 1.3.1 is

$$B_\theta(X, Y) = -2n\text{tr}(X\theta(Y)), \quad \text{for } X, Y \in \mathfrak{g}_0.$$

In other words, B_θ coincides, up to a positive constant, with the restriction of the map tr_θ on \mathfrak{g}_0 . Therefore, B_θ is positive definite on \mathfrak{g}_0 and the restriction of θ on \mathfrak{g}_0 is a Cartan involution. The same argument applies for $\mathfrak{g}_0 = \mathfrak{so}(n, \mathbb{R})$ and $\mathfrak{g}_0 = \mathfrak{sp}(n, \mathbb{R})$. The following table gives the Cartan decomposition of each classical Lie algebra.

\mathfrak{g}_0	\mathfrak{k}_0	\mathfrak{p}_0
$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{so}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R}) \cap \{\text{symmetric matrices}\}$
$\mathfrak{so}(n, \mathbb{R})$	$\mathfrak{so}(n, \mathbb{R})$	$\{0\}$
$\mathfrak{sp}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R}) \cap \mathfrak{so}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R}) \cap \{\text{symmetric matrices}\}$

Table 1.9: Cartan decomposition of classical Lie algebras

Let \mathfrak{g} be the complexification of \mathfrak{g}_0 and note again by θ the complex-linear extension of θ on \mathfrak{g} , by \mathfrak{k} the complexification of \mathfrak{k}_0 and by \mathfrak{p} the complexification of \mathfrak{p}_0 . Define the real subalgebra $\mathfrak{u}_0 := \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ of $\mathfrak{g}^{\mathbb{R}}$. It is clear that \mathfrak{u}_0 is a real form of \mathfrak{g} .

Definition 1.3.7 (Real compact Lie algebra). *A real Lie algebra is said to be compact if its Killing form is negative definite.*

Using Lemma 1.3.5, one deduces that the real Lie subalgebra \mathfrak{u}_0 of $\mathfrak{g}^{\mathbb{R}}$ is a compact real form of \mathfrak{g} . Therefore, as a corollary of Proposition 1.3.2, one obtains that:

Corollary 1.3.8. *Every complex semisimple Lie algebra admits a compact real form.*

Theorem 1.3.9. *A real Lie algebra \mathfrak{u}_0 is compact if and only if there is a real compact semisimple Lie group U such that \mathfrak{u}_0 is its Lie algebra.*

Proof. The "only if" direction is Proposition 4.27 of [Kna96]. The "if" direction is a consequence of the semisimplicity of U and Corollary 4.26 of [Kna96]. \square

One can obtain a similar decomposition on the Lie group level.

Theorem 1.3.10 (Cartan decomposition-Lie group case, see [Kna96, Theorem 6.31]). *Let G be a connected semisimple Lie group of Lie algebra \mathfrak{g}_0 , θ be the a Cartan involution and*

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

be the corresponding Cartan decomposition. Suppose that K is the subgroup of G corresponding to the Lie subalgebra \mathfrak{k}_0 of \mathfrak{g}_0 . Then

(i) *There exists a Lie group automorphism Θ such that $T_e\Theta = \theta$ and $\Theta^2 = id$,*

(ii) *K is the subgroup of G fixed by Θ ,*

(iii) *The map*

$$K \times \mathfrak{p}_0 \ni (k, X) \mapsto k \exp X \in G$$

is a diffeomorphism,

(iv) *K is closed,*

(v) *$Z_G \subseteq K$,*

(vi) *K is compact if and only if Z_G is finite,*

(vii) *If Z_G is finite, K is a maximal compact subgroup of G .*

Using the following proposition, we can assume that G has finite center and thus K is a maximal compact subgroup of G .

Proposition 1.3.11 (See [Kna96, Proposition 6.30]). *If G is a semisimple Lie group, the group G/Z_G has trivial center.*

1.4 Representations of complex Lie algebras

1.4.1 Basic definitions

Definition 1.4.1 (Lie algebra representation). *Let \mathfrak{g} be a real or complex Lie algebra and V be a complex vector space. A representation (ϕ, V) of \mathfrak{g} on V is a Lie algebra morphism*

$$\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V).$$

In other words, (ϕ, V) defines an action of \mathfrak{g} on V in such a way that the Lie algebra structure of \mathfrak{g} is respected. We will use these two equivalent points of view without distinction. Moreover, when there is no ambiguity, we may refer to a representation (ϕ, V) just by ϕ or V while, for X in \mathfrak{g} and v in V , we may omit ϕ and write Xv instead of $\phi(X)v$.

Example 1.4.2 (Trivial representation). Let \mathfrak{g} be a Lie algebra and V be a finite-dimensional vector space. The Lie algebra morphism $\phi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ sending every element of \mathfrak{g} to zero is called the trivial representation of \mathfrak{g} on V .

Example 1.4.3 (Adjoint representation of \mathfrak{g}). Let \mathfrak{g} be a Lie algebra and $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ be its adjoint map defined in 1.1.18. Using the fact that the Lie bracket $[\cdot, \cdot]$ of \mathfrak{g} is by definition skew-symmetric and satisfies the Jacobi identity (see Definition 1.1.10), one has that for every X, Y and Z in \mathfrak{g}

$$\begin{aligned} \text{ad}[X, Y](Z) &= [[X, Y], Z] \\ &= [[X, Z], Y] + [X, [Y, Z]] \\ &= [X, [Y, Z]] - [Y, [X, Z]] \\ &= \text{ad}X \circ \text{ad}Y(Z) - \text{ad}Y \circ \text{ad}X(Z) \\ &= [\text{ad}X, \text{ad}Y](Z). \end{aligned}$$

Therefore

$$\text{ad}[X, Y] = [\text{ad}X, \text{ad}Y]$$

for every X and Y in \mathfrak{g} . In other words, ad is a representation of \mathfrak{g} on itself. This representation is called the adjoint representation of \mathfrak{g} .

The following proposition let us consider a representation of a Lie algebra \mathfrak{g} as a module of its universal enveloping algebra $U(\mathfrak{g})$.

Proposition 1.4.4 (See [Kna96, Corollary 3.6]). *There is an equivalence (of categories) between the representations of \mathfrak{g} and the unital left $U(\mathfrak{g})$ -modules.*

Recall that a module over a unital algebra is said to be unital if the unit of the algebra acts as the identity on the module. The equivalence for the above result is given by the universal property of $U(\mathfrak{g})$. Namely, if ϕ is a representation of \mathfrak{g} , it satisfies the relation (1.4) and so it can be extended to a map $\tilde{\phi} : U(\mathfrak{g}) \rightarrow \text{End}(V)$ which endows V with a unital left $U(\mathfrak{g})$ -module structure. Conversely, if V is a unital left $U(\mathfrak{g})$ -module, we equip V with a vector space structure by setting $\lambda v := \lambda \cdot v$, $\lambda \in \mathbb{C}$, where \cdot stands for the $U(\mathfrak{g})$ -module action on V , while \mathfrak{g} acts on V through its canonical embedding in $U(\mathfrak{g})$. One can check that the two maps are inverses of each other.

Definition 1.4.5 (Subrepresentation of a Lie algebra representation). *If W is a subspace of a representation V of \mathfrak{g} , we will say that W is a subrepresentation of V if W is stable by the action of \mathfrak{g} , i.e. if, for every element X in \mathfrak{g} and every vector w in W , the vector Xw lies in W .*

Definition 1.4.6 (Irreducible representation). *A representation V of \mathfrak{g} that has no proper nonzero subrepresentations is said to be irreducible.*

Definition 1.4.7 (Interwining operator). *Let (ϕ_1, V_1) and (ϕ_2, V_2) be representations of \mathfrak{g} . A linear map $T : V_1 \rightarrow V_2$ is said to be an interwining operator (or \mathfrak{g} -equivariant operator) if*

$$T \circ \phi_1(X) = \phi_2(X) \circ T \text{ for every } X \text{ in } \mathfrak{g}.$$

Definition 1.4.8 (Equivalent representations). *We will say that two representations V_1 and V_2 of \mathfrak{g} are equivalent if there is a bijective interwining operator $\phi : V_1 \rightarrow V_2$.*

Proposition 1.4.9 (Schur's Lemma, see [Kna96, Proposition 4.8]). *Let (ϕ_1, V_1) and (ϕ_2, V_2) be irreducible representations of \mathfrak{g} and $T : V_1 \rightarrow V_2$ be a an interwining operator. Then T is the zero map or an isomorphism.*

Proof. Let (ϕ_1, V_1) , (ϕ_2, V_2) and $T : V_1 \rightarrow V_2$ be as above. If V_1 or V_2 is the zero vector space, we are done. Suppose that both are nontrivial and let v be an element of $\ker T \subseteq V_1$. Then for every X in \mathfrak{g} , $\phi_1(X)v$ belongs to $\ker T$. Indeed, for every $X \in \mathfrak{g}$,

$$T(\phi_1(X)v) = \phi_2(X)T(v) = 0$$

and thus $\phi_1(X) \ker T \subseteq \ker T$. In other words, $\ker T$ is a subrepresentation of V_1 . Using a similar argument, one shows that the image $\text{im}(T)$ of T in V_2 is a subrepresentation of V_2 . Since V_1 is irreducible, either $\ker T$ coincides with V_1 , and thus T is the zero map, or $\ker T$ is the zero subspace and T is a monomorphism. For the same reason, $\text{im}(T)$ is either the zero subspace or V_2 , and thus T is the zero map, or T is an epimorphism. From the above discussion, one deduces that T is either the zero map or an isomorphism. \square

Corollary 1.4.10 (See [Kna96, Corollary 4.9]). *If V is a finite-dimensional irreducible representation of \mathfrak{g} and $T : V \rightarrow V$ is an interwining operator, then T is a scalar operator.*

We have a similar result which generalizes the above corollary including the case of infinite-dimensional irreducible representations.

Proposition 1.4.11 (Dixmier, see [Kna96, Theorem 5.19]). *If V is an irreducible representation of \mathfrak{g} and $T : V \rightarrow V$ is an interwining operator, then T is a scalar operator.*

1.4.2 Representations of reductive Lie algebras

Definition 1.4.12 (Weights of V). *Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{t} be a Cartan subalgebra of \mathfrak{g} and V be a representation of \mathfrak{g} . For every element λ in the dual space \mathfrak{t}^* , we define the vector subspace*

$$V_\lambda := \{v \in V \mid Hv = \lambda(H)v, \forall H \in \mathfrak{t}\}.$$

If $V_\lambda \neq \{0\}$, we say that λ is a weight of V with V_λ being the corresponding weight subspace. The elements of V_λ are called weight vectors of λ . We denote by $\Pi(V)$ the set of all weights of V .

Example 1.4.13. In the case of the adjoint representation ad of a Lie algebra \mathfrak{g} (see Example 1.4.3), the corresponding weights are the roots of \mathfrak{g} .

Example 1.4.14. Let \mathfrak{g} be the special linear Lie algebra $\mathfrak{sl}(n, \mathbb{C})$, \mathfrak{t} be the Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ defined in Subsection 1.2.1, i.e. the subalgebra of traceless diagonal matrices $H = \text{diag}(h_1, \dots, h_n)$. Let ε_i be the linear functional mapping $H = \text{diag}(h_1, \dots, h_n)$ to h_i and (ϕ, V) , where $V = \bigwedge^k \mathbb{C}^n$, $1 \leq k \leq n$, be the representation of \mathfrak{g} given by

$$\phi(X)(v_{j_1} \wedge \dots \wedge v_{j_k}) = \sum_{s=1}^k v_{j_1} \wedge \dots \wedge Xv_{j_s} \wedge \dots \wedge v_{j_k},$$

with X acting on every v_{j_s} , $1 \leq s \leq k$, by matrix multiplication. If $\{v_1, \dots, v_n\}$ is the canonical basis of \mathbb{C}^n , for the general element $H = \text{diag}(h_1, \dots, h_n)$ of \mathfrak{t} , we have

$$\begin{aligned} \phi(H)(v_{j_1} \wedge \dots \wedge v_{j_k}) &= \sum_{s=1}^k v_{j_1} \wedge \dots \wedge Hv_{j_s} \wedge \dots \wedge v_{j_k} \\ &= \sum_{s=1}^k v_{j_1} \wedge \dots \wedge (h_{j_s} v_{j_s}) \wedge \dots \wedge v_{j_k} \\ &= \left(\sum_{s=1}^k h_{j_s} \right) (v_{j_1} \wedge \dots \wedge v_{j_k}) \\ &= \sum_{s=1}^k \varepsilon_{j_s}(H) (v_{j_1} \wedge \dots \wedge v_{j_k}) \end{aligned}$$

Therefore $v_{j_1} \wedge \dots \wedge v_{j_k}$ is a weight vector of weight $\sum_{s=1}^k \varepsilon_{j_s}$ and we can easily deduce that V is the direct sum of its weight subspaces.

Definition 1.4.15 (Algebraically integral weight). *An element λ of \mathfrak{t}^* is said to be algebraically integral if, for every $\alpha \in \Delta$, we have*

$$2 \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

Proposition 1.4.16 (See [Kna96, Proposition 4.62]). *An element λ of \mathfrak{t}^* is algebraically integral if and only if*

$$2 \frac{\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \in \mathbb{Z}$$

for every simple root α_i .

Corollary 1.4.17. *The fundamental weights $\bar{\omega}_i$ of Definition 1.2.19 are all algebraically integral.*

The fact that the representation of Example 1.4.14 can be decomposed into a direct sum of its weight subspaces it is a general fact that holds for every finite-dimensional representation of \mathfrak{g} , as the following Proposition indicates.

Proposition 1.4.18 (See [Kna96, Theorem 5.4]). *Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{t} be a Cartan subalgebra of \mathfrak{g} and Δ be the root system of $(\mathfrak{g}, \mathfrak{t})$. Set \mathfrak{t}_0 to be the real form of \mathfrak{t} defined in (1.6), i.e. the real vector space generated by all H_α , $\alpha \in \Delta$. If (ϕ, V) is a representation of \mathfrak{g} , then*

- (a) $\phi(\mathfrak{t})$ acts diagonally on V ,
- (b) each weight of V is algebraically integral and therefore real valued on \mathfrak{t}_0 ,
- (c) for each α in Δ and every weight μ of V , we have $\phi(\mathfrak{g}_\alpha)V_\mu \subseteq V_{\mu+\alpha}$.

Fix an ordering on \mathfrak{t}_0^* and let Δ^+ be the set of positive roots corresponding to this ordering. Since every element of $\Pi(V)$ is real valued on \mathfrak{t}_0 , according to Lemma 1.2.14, they all belong to \mathfrak{t}_0^* .

Definition 1.4.19 (Highest weight of a representation). *In the above setting, the maximal element of $\Pi(V)$ is called the highest weight of the representation V .*

Definition 1.4.20 (Dominant weight). *An element λ of \mathfrak{t}^* is said to be dominant with respect to $\Delta^+ \subset \Delta$ if, for every $\alpha \in \Delta^+$, we have*

$$\langle \lambda, \alpha \rangle \geq 0.$$

If for every $\alpha \in \Delta^+$ the above inequality is strict, λ is said to be strictly dominant.

Proposition 1.4.21. *The half sum ρ of positive roots of Δ^+ is strictly dominant. In particular, for every simple root $\alpha_i \in \Pi$, one has*

$$2 \frac{\langle \rho, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1.$$

Proof. Since every positive root is a nonnegative sum of simple roots, it suffices to prove the second statement. Let α_i be a simple root and β be a positive root of Δ^+ . Then

$$s_{\alpha_i}(\beta) = \beta - 2 \frac{\langle \beta, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i$$

and, according to Proposition 1.2.21, $s_{\alpha_i}(\beta)$ must be a root. Let

$$\beta = \sum_k n_k \alpha_k$$

be the decomposition of β in terms of simple roots. Then

$$\begin{aligned} s_{\alpha_i}(\beta) &= \sum_k n_k \alpha_k - 2 \frac{\langle \beta, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i \\ &= \sum_{k \neq i} n_k \alpha_k + \left(n_i - 2 \frac{\langle \beta, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \right) \alpha_i. \end{aligned}$$

If there is some coefficient n_k , $k \neq i$, such that $n_k > 0$, then all coefficients must be nonnegative and thus s_{α_i} is a positive root. If $n_k = 0$ for every $k \neq i$, then $\beta = \alpha$ and $s_{\alpha_i}(\beta) = -\alpha_i$. In other words

$$s_{\alpha_i}(\beta) \text{ is } \begin{cases} -\alpha & \text{if } \beta = \alpha_i \\ > 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} 2s_{\alpha_i}(\rho) &= s_{\alpha_i}(2\rho - \alpha_i) + s_{\alpha_i}(\alpha_i) \\ &= 2\rho - \alpha_i - \alpha_i \\ &= 2\rho - 2\alpha_i \end{aligned}$$

and thus $s_{\alpha_i}(\rho) = \rho - \alpha_i$ and

$$2 \frac{\langle \beta, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = 1.$$

□

Theorem 1.4.22 (Theorem of the highest weight, see [Kna96, Theorem 5.5]). *The set of equivalence classes of finite-dimensional irreducible representations of \mathfrak{g} stands to one-to-one correspondence with the set of dominant algebraically integral elements λ of \mathfrak{t}^* , the correspondence being that λ is the highest weight of $(\phi_\lambda, V^\lambda)$. Moreover:*

- (a) *The weight subspace $(V^\lambda)_\lambda$ corresponding to the highest weight λ is 1-dimensional,*
- (b) *For each α in Δ^+ , $\phi_\lambda(\mathfrak{g}_\alpha)(V^\lambda)_\lambda = 0$ and $(V^\lambda)_\lambda$ is the only weight subspace satisfying this property,*
- (c) *The weights of V^λ are of the form $\lambda - \sum n_i \alpha_i$, with $n_i \geq 0$ and α_i being in Δ^+ ,*
- (d) *For each element w in W and each μ in \mathfrak{t}^* , we have $\dim((V^\lambda)_{w\mu}) = \dim((V^\lambda)_\mu)$,*
- (e) *For every μ in $\Pi(V)$, we have $\langle \mu, \mu \rangle \leq \langle \lambda, \lambda \rangle$, with equality holding if and only if μ is in the W -orbit of λ .*

Remarks 1.4.23.

1. Because of (e), a weight of V^λ which belongs to the W -orbit of λ is said to be extremal.
2. A vector v satisfying the condition $\mathfrak{g}_\alpha(\mathbb{C}\{v\}) = \{0\}$ for every $\alpha \in \Delta^+$ is called maximal. From (b), the vector space of the maximal vectors of an irreducible representation is the weight subspace corresponding to the highest weight. According to (a), this space is 1-dimensional.
3. Theorem 1.4.22 ensures that all the weights of V^λ are real valued on the real form \mathfrak{t}_0 .
4. When there is no confusion, we will refer to the highest weight representation $(\phi_\lambda, V^\lambda)$ just by (ϕ, V) , omitting λ .

Example 1.4.24. We consider again the Example 1.4.14 of the representation $V = \bigwedge^k \mathbb{C}^n$, $1 \leq k \leq n$, of $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. We saw that the weight vectors are the vectors $v_{j_1} \wedge \dots \wedge v_{j_k}$, $1 \leq j_1 < \dots < j_k \leq n$, the corresponding weights for every vector of this form being $\sum_{s=1}^k \varepsilon_{j_s}$. Fix the positive system of $\Delta(\mathfrak{g}, \mathfrak{t})$ to be

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

Then the highest weight of V is $\varepsilon_1 + \dots + \varepsilon_k$ and the corresponding highest weight vector is $v_1 \wedge \dots \wedge v_k$.

Example 1.4.25 (Fundamental representations). According to Corollary 1.4.16, the fundamental weights $\bar{\omega}_i$ of a complex simple Lie algebra \mathfrak{g} are all algebraically integral while one can verify that they are dominant, as well. Therefore, for each of them, we define a representation $V^{\bar{\omega}_i}$ of highest weight $\bar{\omega}_i$ called the i -th fundamental representation of \mathfrak{g} .

Definition 1.4.26 (Standard representation). *Let \mathfrak{g} be a complex simple Lie algebra. The finite-dimensional irreducible representation $V^{\bar{\omega}_1}$ of highest weight $\bar{\omega}_1$ is called the standard representation of \mathfrak{g} .*

Theorem 1.4.27 (See [Kna96, Theorem 5.29]). *Let V be a finite-dimensional representation of a complex semisimple Lie algebra \mathfrak{g} . Then V is completely reducible, i.e. there are irreducible subrepresentations W_1, \dots, W_k of V , $k \in \mathbb{N}$, such that $V = W_1 \oplus \dots \oplus W_k$.*

Let V be a finite dimensional representation of a complex semisimple Lie algebra \mathfrak{g} and let

$$V = \bigoplus_i W_i$$

be the decomposition of Theorem 1.4.27 into a direct sum of irreducible subrepresentations of V . In this decomposition of V , a certain isomorphism class might appear more than once.

Definition 1.4.28 (Isotypic component). *The direct sum of all irreducible subrepresentations of V appearing in its decomposition and belonging to the same equivalence class is called the isotypic component of V corresponding to this equivalence class. If, with respect to a positive system Δ^+ of Δ , the highest weight corresponding to this equivalence class is λ , the isotypic component is called the λ -isotypic component of V .*

Suppose, now, that \mathfrak{g} is a complex reductive Lie algebra. Using Lemma 1.2.29, decompose \mathfrak{g} into $\mathfrak{g} = Z_{\mathfrak{g}} \oplus [\mathfrak{g}, \mathfrak{g}]$, with $[\mathfrak{g}, \mathfrak{g}]$ being semisimple, and let \mathfrak{t}' and $\mathfrak{t} := Z_{\mathfrak{g}} \oplus \mathfrak{t}'$ be Cartan subalgebras of $[\mathfrak{g}, \mathfrak{g}]$ and \mathfrak{g} respectively. Suppose that V is an irreducible representation of \mathfrak{g} . The following proposition ensures that $Z_{\mathfrak{g}}$ acts on V by diagonalizable operators.

Proposition 1.4.29 (See [Dix77, Proposition 1.7.9]). *Let \mathfrak{g} be a complex Lie algebra and \mathfrak{h} be a Lie subalgebra of \mathfrak{g} , reductive in \mathfrak{g} . Then, every completely reducible representation V of \mathfrak{g} is also completely reducible as representation of \mathfrak{h} .*

It is clear that $Z_{\mathfrak{g}}$ is reductive in \mathfrak{g} (see Definition 1.2.27) and so, according to Proposition 1.4.29, $Z_{\mathfrak{g}}$ acts by scalars on V . Therefore, V turns out to be an irreducible representation of the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, say of highest weight $\lambda' \in (\mathfrak{t}')^*$ with respect to some positive system Δ^+ of $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$. Let λ_Z be the element of $Z_{\mathfrak{g}}^*$ by which $Z_{\mathfrak{g}}$ acts on V and set λ to be the element of \mathfrak{t}^* determined by λ' and λ_Z . Motivated by the terminology of representation theory of semisimple Lie algebras, we will say that λ is the highest weight of V .

For later use, we give the following propositions, providing us with a necessary and sufficient condition so that a finite-dimensional representation V of \mathfrak{g} is completely reducible.

Proposition 1.4.30 (See [Ser06, page 51]). *Let \mathfrak{g} be a reductive Lie algebra and V be a finite-dimensional representation of \mathfrak{g} . Then, V is completely reducible if and only if $Z_{\mathfrak{g}}$ acts on V by diagonalizable operators.*

1.4.3 Casimir element

For the proof of Theorem 1.4.27, one uses a particular element of $U(\mathfrak{g})$ called the Casimir element of \mathfrak{g} .

Definition 1.4.31 (Casimir element of \mathfrak{g}). *Let $\{X_i\}$ and $\{\tilde{X}_i\}$ be dual bases of \mathfrak{g} with respect to B . We call Casimir element of \mathfrak{g} the element Ω of $U(\mathfrak{g})$ defined by*

$$\Omega = \sum_{i,j} B(\tilde{X}_i, \tilde{X}_j) X_i X_j.$$

Lemma 1.4.32 (See [Kna96, Proposition 5.24]). *The Casimir element Ω is independent of the choice of the bases $\{X_i\}$ and $\{\tilde{X}_i\}$.*

Example 1.4.33. Let \mathfrak{g} be the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ of Table 1.2. The matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a vector space basis for $\mathfrak{sl}(2, \mathbb{C})$. Recall that

$$[H, X] = 2X, \quad [H, Y] = -2Y \quad \text{and} \quad [X, Y] = H.$$

The Killing form of $\mathfrak{sl}(2, \mathbb{C})$ is given by

$$B(L_1, L_2) = 4\text{tr}(L_1 L_2), \quad \forall L_1, L_2 \in \mathfrak{sl}(2, \mathbb{C})$$

and one can verify that the vectors

$$\tilde{H} = \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tilde{X} = \frac{1}{4} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \tilde{Y} = \frac{1}{4} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

form a dual basis of $\{H, X, Y\}$ with respect to B . Therefore

$$\begin{aligned} \Omega &= B(\tilde{H}, \tilde{H})H^2 + B(\tilde{X}, \tilde{Y})XY + B(\tilde{Y}, \tilde{X})YX \\ &= \frac{1}{8}H^2 + \frac{1}{4}XY + \frac{1}{4}YX \end{aligned}$$

The element Ω belongs to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, i.e. it commutes with every element in $\mathfrak{sl}(2, \mathbb{C})$. More precisely, one has that:

$$\begin{aligned}
 4(H\Omega - \Omega H) &= \frac{1}{2}H^3 + HXY + HYX - \frac{1}{2}H^3 - XYH - YXH \\
 &= HXY + HYX - XYH - YXH \\
 &= [H, X]Y + XHY + [H, Y]X + YHX - XYH - YXH \\
 &= 2XY + XHY - 2YX + YHX - XYH - YXH \\
 &= 2XY - 2YX + X(HY - YH) + Y(HX - XH) \\
 &= 2XY - 2YX + X[H, Y] + Y[H, X] \\
 &= 2XY - 2YX - 2XY + 2YX \\
 &= 0.
 \end{aligned}$$

Therefore Ω commutes with the element H and by a similar calculation one can show that it also commutes with X and Y and so is an element of $Z(\mathfrak{g})$.

The fact that the Casimir element of $\mathfrak{sl}(2, \mathbb{C})$ belongs to the center of $U(\mathfrak{sl}(2, \mathbb{C}))$ is not a particularity of $\mathfrak{sl}(2, \mathbb{C})$ but a result of the general properties of Ω , as the following proposition indicates.

Proposition 1.4.34 (See [Kna96, Proposition 5.24 and Proposition 5.28]). *Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{t} be a Cartan subalgebra of \mathfrak{g} , Δ be the corresponding root system and fix Δ^+ to be a positive system for Δ . Moreover, consider the real form \mathfrak{t}_0 of \mathfrak{t} generated by the elements H_α , $\alpha \in \Delta$, and let $\{H_i\}$ be an orthonormal basis of \mathfrak{t}_0 with respect to B . For every $\alpha \in \Delta$, fix an element E_α in \mathfrak{g}_α such that $B(E_\alpha, E_{-\alpha}) = 1$. Then*

$$(a) \quad \Omega = \sum_i H_i^2 + \sum_{\alpha \in \Delta} E_\alpha E_{-\alpha},$$

(b) Ω belongs to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ and acts by a scalar on the irreducible \mathfrak{g} -representation V of highest weight λ , the scalar being $\|\lambda + \rho\|^2 - \|\rho\|^2$, where ρ stands for the half sum of the positive roots,

(c) Ω acts by 0 on the irreducible representation V of highest weight λ if and only if λ is zero. In this case V is the trivial representation of Example 1.4.2.

1.4.4 Weyl dimension formula and Kostant multiplicity formula

As we have already seen, having fixed a positive system Δ^+ for Δ , to each dominant and algebraically integral element λ of \mathfrak{t}^* , corresponds a finite-dimensional irreducible representation. The following theorem computes the dimension of the finite-dimensional irreducible \mathfrak{g} -representation of highest weight λ .

Theorem 1.4.35 (Weyl dimensional formula, see [Kna96, Theorem 5.84]). *Let V be the irreducible representation of \mathfrak{g} with highest weight $\lambda \in \mathfrak{t}^*$. The dimension of V is*

$$\dim(V) = \frac{\prod_{\alpha \in \Delta^+} \langle \lambda + \rho, \alpha \rangle}{\prod_{\alpha \in \Delta^+} \langle \rho, \alpha \rangle}.$$

Example 1.4.36. Let \mathfrak{g} be the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ of Table 1.2 and $V := \mathbb{C}^3$ equipped with the canonical action of \mathfrak{g} . Let v_1, v_2, v_3 be the elements of the canonical basis of \mathbb{C}^3 . We consider the Cartan subalgebra \mathfrak{t} and the elements ε_i of \mathfrak{t}^* , defined in Subsection 1.2.1. The matrices

$$H_{\varepsilon_1 - \varepsilon_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_{\varepsilon_2 - \varepsilon_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

form a basis of \mathfrak{t} while the root system of \mathfrak{g} with respect to \mathfrak{t} is

$$\Delta = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3)\}.$$

We choose

$$\Delta^+ := \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$$

to be the positive root system of Δ and

$$\Pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\}$$

is the corresponding simple system. The matrices

$$E_{\varepsilon_1 - \varepsilon_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\varepsilon_1 - \varepsilon_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{\varepsilon_2 - \varepsilon_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

are elements of $\mathfrak{g}_{\varepsilon_1 - \varepsilon_2}$, $\mathfrak{g}_{\varepsilon_1 - \varepsilon_3}$ and $\mathfrak{g}_{\varepsilon_2 - \varepsilon_3}$ respectively, and one can verify that they act on v_1 by 0. Thus v_1 is a maximal vector of V . Let $\lambda = c_1(\varepsilon_1 - \varepsilon_2) + c_2(\varepsilon_2 - \varepsilon_3)$ be the weight corresponding to v_1 . Then

$$\begin{aligned} 1 \cdot v_1 &= H_{1,2} \cdot v_1 & 0 \cdot v_1 &= H_{2,3} \cdot v_1 \\ &= \lambda(H_{1,2})v_1 & &= \lambda(H_{2,3})v_1 \\ &= (2c_1 - c_2)v_1 & &= (-c_1 + 2c_2)v_1, \end{aligned}$$

from where we obtain the linear system

$$\begin{cases} 2c_1 - c_2 &= 1 \\ -c_1 + 2c_2 &= 0 \end{cases}$$

with solution $(c_1, c_2) = (\frac{2}{3}, \frac{1}{3})$. Thus $\lambda = \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3$ and V contains a copy V' of the irreducible representation of highest weight $\frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3$. Let us calculate the dimension of this representation using the Weyl Dimensional Formula of Theorem 1.4.35. We have $\rho = \varepsilon_1 - \varepsilon_2$ and

$$\begin{aligned}
 \dim(V') &= \frac{\prod_{\alpha \in \Delta^+} \langle \lambda + \rho, \alpha \rangle}{\prod_{\alpha \in \Delta^+} \langle \rho, \alpha \rangle} \\
 &= \frac{\langle \lambda + \rho, \varepsilon_1 - \varepsilon_2 \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_2 \rangle} \cdot \frac{\langle \lambda + \rho, \varepsilon_1 - \varepsilon_3 \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_3 \rangle} \cdot \frac{\langle \lambda + \rho, \varepsilon_2 - \varepsilon_3 \rangle}{\langle \rho, \varepsilon_2 - \varepsilon_3 \rangle} \\
 &= \left(\frac{\langle \lambda, \varepsilon_1 - \varepsilon_2 \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_2 \rangle} + 1 \right) \left(\frac{\langle \lambda, \varepsilon_1 - \varepsilon_3 \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_3 \rangle} + 1 \right) \left(\frac{\langle \lambda, \varepsilon_2 - \varepsilon_3 \rangle}{\langle \rho, \varepsilon_2 - \varepsilon_3 \rangle} + 1 \right) \\
 &= \left(\frac{1}{1} + 1 \right) \left(\frac{1}{2} + 1 \right) \left(\frac{0}{1} + 1 \right) \\
 &= 3
 \end{aligned}$$

Since $\dim V = \dim V'$, V' coincides with V , i.e. V is irreducible and of highest weight $\lambda = \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3$. In particular, V is the standard representation of $\mathfrak{so}(3, \mathbb{C})$ (see Example 1.4.25).

Let

$$Q^+ := \left\{ \sum_i n_i \alpha_i \mid \alpha_i \in \Pi, n_i \in \mathbb{N} \cup \{0\} \right\},$$

i.e. the subset of all elements of \mathfrak{t}^* which can be written as a nonnegative integer sum of simple roots.

Definition 1.4.37 (Kostant partition function). *The function $\mathcal{P} : \mathfrak{t}^* \rightarrow \mathbb{N} \cup \{0\}$ which maps every nonzero element of Q^+ to the number of positive ways, apart from order, that it can be written as a nonnegative integer sum of positive roots while it maps every other element of \mathfrak{t}^* to zero is called the Kostant partition function.*

Theorem 1.4.38 (Kostant multiplicity formula, see [Kna96, Corollary 5.83]). *Let V be the finite-dimensional irreducible representation of \mathfrak{g} with highest weight $\lambda \in \mathfrak{t}^*$ and μ be an element of \mathfrak{t}^* . The multiplicity of μ as a weight of V is given by*

$$\sum_{w \in W} (-1)^{\ell(w)} \mathcal{P}(w(\lambda + \rho) - (\mu + \rho)),$$

where $l(\cdot)$ stands for the length of the indicated element of W (See Definition 1.2.22).

Example 1.4.39. Suppose that $\mathfrak{g} := \mathfrak{sl}(3, \mathbb{C})$ and V is the standard representation of \mathfrak{g} , i.e. the irreducible representation of highest weight $\lambda = \bar{\omega}_1 = \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3$. According to Example 1.4.36, $\dim V = 3$ and thus V has at most 3 distinct weights while the Weyl group of $\mathfrak{sl}(3, \mathbb{C})$ is

$$W = \{1, s_{\varepsilon_1 - \varepsilon_2}, s_{\varepsilon_1 - \varepsilon_3}, s_{\varepsilon_2 - \varepsilon_3}, s_{\varepsilon_1 - \varepsilon_2} s_{\varepsilon_2 - \varepsilon_3}, s_{\varepsilon_1 - \varepsilon_3} s_{\varepsilon_2 - \varepsilon_3}\}.$$

Using Theorem 1.4.38, one finds (after long calculations) that the weights of V are exactly

$$\begin{aligned}
 \bar{\omega}_1 &= \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3 \\
 \bar{\omega}_1 - (\varepsilon_1 + \varepsilon_2) &= -\frac{1}{3}\varepsilon_1 + \frac{2}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3 \\
 \bar{\omega}_1 - (\varepsilon_1 + \varepsilon_3) &= -\frac{1}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 + \frac{2}{3}\varepsilon_3,
 \end{aligned}$$

each of multiplicity one.

Another way to obtain the weights of V is by using the action of W . As we have seen in Theorem 1.4.22, W permutes the weights of V and so the above elements must be weights of V since they all belong to the W -orbit of the highest weight $\bar{\omega}_1$. On the other hand, for dimension reasons, they are the only weights of V , necessarily each of multiplicity one.

1.5 Representations of Lie groups

Definition 1.5.1 (Lie group representation). *Let G be a Lie group and $(V, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. Let $\mathcal{B}(V)$ stand for the group of bounded automorphisms of V with bounded inverse. A Lie group representation (Φ, V) of G on V is a morphism $\Phi : G \rightarrow \mathcal{B}(V)$ such that the map*

$$(g, v) \in G \times V \mapsto \Phi(g)v \in V$$

is continuous and $\Phi(1)$ is the identity operator. If there is no ambiguity, instead of (Φ, V) we write just Φ or V .

Definition 1.5.2. *We will say that (Φ, V) is irreducible if, apart from V and $\{0\}$, there is no closed G -invariant subspace of V . (Φ, V) is said to be unitary if $\Phi(g)$ is a unitary operator for every $g \in G$, i.e.*

$$\langle \Phi(g)v_1, \Phi(g)v_2 \rangle = \langle v_1, v_2 \rangle$$

for every $g \in G$ and $v_1, v_2 \in V$.

We note that if (Φ, V) is a finite-dimensional Lie group representation, according to Proposition 1.1.32, its tangent map $\phi := T_e\Phi$ defines a Lie algebra representation of the Lie algebra \mathfrak{g} of G . The inverse is not always true, unless G is connected and simply connected.

Example 1.5.3 (Adjoint representation of G). The adjoint map Ad of a Lie group G , defined in Example 1.1.9 is a Lie group representation of G on $\mathfrak{g} := \text{Lie}(G)$. Its tangent map, by definition, is the adjoint representation of \mathfrak{g} .

Example 1.5.4 (Dual representation). Let (π, V) be a representation of a Lie group G . We define the dual representation (π^*, V^*) to be the representation of G on the dual space V^* of V given by

$$\pi^* : (g, T) \in G \times V^* \mapsto (\pi(g)^*T : v \mapsto T(\pi(g^{-1})v))$$

for every $v \in V$.

Example 1.5.5 (Regular representation). Let G be a Lie group and set $L^2(G)$ to be the Hilbert space of all square-integrable functions defined on G . Define a representation $l \times r$ of the Lie group $G \times G$ (see Example 1.1.4) on $L^2(G)$ given by

$$[(l \times r)(x, y)f](g) = f(x^{-1}gy)$$

for every $f \in L^2(G)$ and every $x, y, g \in G$. The representation $l \times r$ is called the bi-regular representation of G . The restrictions of $l \times r$ on $G \times \{1\}$ (respectively on $\{1\} \times G$) is called left (respectively right) regular representation of G . For more details, see [Wal88, Subsection 1.3].

Definition 1.5.6 (Interwining operators). *Let (π_1, V_1) and (π_2, V_2) be representations of G and $T : V_1 \rightarrow V_2$ be a bounded linear operator. We say that T is an interwining (or G -equivariant) operator if, for every $g \in G$, one has*

$$T \circ \pi_1(g) = \pi_2(g) \circ T.$$

Definition 1.5.7 (Equivalent representations). *Let (π_1, V_1) and (π_2, V_2) be representations of G . Then (π_1, V_1) and (π_2, V_2) are said to be equivalent if there is an invertible interwining operator $T : V_1 \rightarrow V_2$. The set of equivalence classes of G is denoted \widehat{G} .*

We can adapt the results obtained for complex semisimple Lie algebras, in the case of a real compact Lie group.

Lemma 1.5.8 (See [Kna96, Corollary 4.25]). *The Lie algebra \mathfrak{g}_0 of a compact Lie group G is reductive.*

We have already seen how one can extend the representation theory of complex semisimple Lie algebras in the case of reductive ones.

Proposition 1.5.9 (See [Kna96, Corollary 1.103]). *In a real compact Lie group G every closed abelian Lie subgroup is a torus, i.e. a closed subgroup of G isomorphic with $(S_1)^n$ for some $n \in \mathbb{N}$, where S_1 is the Lie group of the unit circle. We call maximal torus of G a maximal closed abelian Lie subgroup of G .*

Proposition 1.5.10 (See [Kna96, Proposition 4.30]). *The maximal tori of G are exactly the closed abelian Lie subgroups of G corresponding to the maximal abelian subalgebras of $\mathfrak{g} := \text{Lie}(G)$. In particular, the Lie algebra of a maximal torus is a Cartan subalgebra of \mathfrak{g} .*

Let (Φ, V) be a finite-dimensional representation of the real compact Lie group G , not necessarily irreducible. If $\langle \cdot, \cdot \rangle'$ is a Hermitian form on V , we define a Hermitian form $\langle \cdot, \cdot \rangle$ given by

$$\langle u, v \rangle := \int_G \langle \Phi(g)u, \Phi(g)v \rangle' dg, \quad u, v \in V,$$

where dg is a Haar measure on G , i.e. an outer regular Borelian finite measure on G which is invariant by left translations. One can easily verify that $\langle \cdot, \cdot \rangle$ is $\Phi(G)$ -invariant, i.e. for every $g \in G$ and every $u, v \in V$, we have that $\langle \Phi(g)u, \Phi(g)v \rangle = \langle u, v \rangle$. Therefore, we obtain the following Proposition.

Proposition 1.5.11. *Let G be a compact Lie group and V be a finite-dimensional representation of G . Then, V admits a $\Phi(G)$ -invariant Hermitian form.*

Corollary 1.5.12 (See [Kna96, Theorem 9.4]). *Let G be a compact Lie group. Every representation V of G is completely reducible, i.e. V can be decomposed into a direct sum of irreducible subrepresentations.*

Remark 1.5.13. As we did in the case of Lie algebra finite-dimensional representations (see Definition 1.4.28), one can define the isotypic components of a representation of G , in a similar way.

Corollary 1.5.14 (See [Kna96, Proposition 7.15]). *Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{u}_0 be a real compact form of \mathfrak{g} and (ϕ, V) be a finite-dimensional representation of \mathfrak{g} . Then V admits a Hermitian form $\langle \cdot, \cdot \rangle$ such that, for every $X \in \mathfrak{u}_0$ and every $u, v \in V$, one has*

$$\langle \phi(X)u, v \rangle + \langle u, \phi(X)v \rangle = 0.$$

In other words, $\phi(X)^ = -\phi(X)$ for every X in \mathfrak{u}_0 . Consequently, if $\bar{\cdot}$ stands for the conjugation with respect to \mathfrak{u}_0 , then*

$$\phi(X)^* = -\phi(\bar{X}),$$

for every X in \mathfrak{g} .

Let G be a real compact Lie group, T be a maximal torus of G , \mathfrak{t}_0 be the Lie algebra of T and \mathfrak{t} be its complexification. The Lie algebra \mathfrak{t} turns out to be a Cartan subalgebra of $\text{Lie}(G)$. Indeed, Proposition 1.5.10 ensures that \mathfrak{t} is a maximal abelian subalgebra of $\text{Lie}(G)^\mathbb{C}$. On the other hand, for $\mathfrak{u}_0 = \text{Lie}(G)$, $\mathfrak{g} = (\mathfrak{u}_0)^\mathbb{C}$ and ϕ being the adjoint representation of \mathfrak{g} defined in 1.1.18, Proposition 1.5.14 assures that the linear maps $\text{ad}H$ are skew-symmetric and thus diagonalizable for all $H \in \mathfrak{t}$.

Definition 1.5.15 (Analytically integral element). *In the above setting, an element $\lambda \in \mathfrak{t}^*$ is said to be analytically integral if there is a Lie group morphism $\xi_\lambda : T \rightarrow \mathbb{C}$ such that $T_e \xi_\lambda(H) = \lambda(H)$ for every $H \in \mathfrak{t}_0$.*

Proposition 1.5.16 (See [Kna96, Proposition 4.59]). *In the above setting, every analytically integral element $\lambda \in \mathfrak{t}^*$ is algebraically integral.*

Theorem 1.5.17 (Theorem of the highest weight, see [Kna96, Theorem 5.110]). *Let G be a compact connected Lie group, T be a maximal torus of G , \mathfrak{t} be the complexification of its Lie algebra \mathfrak{t}_0 , Δ be the corresponding root system and Δ^+ be a positive system for Δ . Up to isomorphism, the finite-dimensional irreducible representations V of G stand in one-one correspondence with the dominant analytically integral elements λ of \mathfrak{t}^* , the correspondence being that λ is the highest weight of V .*

1.6 (\mathfrak{g}, K) -modules

Let G be a connected semisimple Lie group of finite center and of Lie algebra \mathfrak{g}_0 , \mathfrak{g} be its complexified Lie algebra and K be a maximal compact subgroup of G .

Definition 1.6.1 (K -finite vectors). *A vector v of V is said to be K -finite if*

$$\dim \langle \pi(K)v \rangle < \infty,$$

where $\langle \pi(K)v \rangle$ stands for the vector subspace of V generated by all $\pi(k)v$, $k \in K$.

Set V_K to be the set of all K -finite vectors of V and, for every equivalence class $(\delta, E_\delta) \in \widehat{K}$, where \widehat{K} stands for the set of equivalence classes of all unitary irreducible representations of K , let

$$V_K(\delta) := \sum_{L \in \text{Hom}_K(E_\delta, V)} L(E_\delta)$$

be the δ K -type of V .

Definition 1.6.2. A representation (π, V) is said to be admissible if

$$\dim V_K(\delta) < \infty$$

for every $\delta \in \widehat{K}$.

Theorem 1.6.3 (Harish-Chandra, see [Har53]). Any irreducible unitary representation V of G is admissible.

Definition 1.6.4 (Multiplicity of δ in V). If V is an admissible representation of G , then we call $\dim V_K(\delta)$ the (finite) multiplicity of δ in V .

Theorem 1.6.5 (Harish-Chandra, (see [Vog81, Proposition 0.3.5] or [Har54])). Let (π, V) be an admissible representation of V . For every $v \in V_K$ and every $X \in \mathfrak{g}_0$, the limit

$$\pi(X)v := \lim_{t \rightarrow 0} \frac{\pi(\exp(tX))v - v}{t}$$

exists and $\pi(X)v$ belongs to V_K .

Using the above theorem, one obtains an action of \mathfrak{g}_0 , and thus by complexifying, an action of \mathfrak{g} , on V_K .

Proposition 1.6.6 (See [Vog81, Proposition 0.3.5]). There is a bijection between the closed G -invariant subspaces of V and the K -invariant \mathfrak{g}_0 -invariant subspaces of V_K , the bijection being that if $U \subseteq V$ is closed and G -invariant then $U_K := U \cap V_K$ is the corresponding K -invariant \mathfrak{g} -invariant subspace of V_K .

One can check that the K -action and the \mathfrak{g} -action on V_K are compatible satisfying the following properties:

- (i). Every vector v of V_K is K -finite,
- (ii). The differential of the K -action on V_K coincides with the action of \mathfrak{k}_0 coming from the \mathfrak{g} -action on V_K . In other words,

$$\pi(X)v = \lim_{t \rightarrow 0} \frac{\pi(\exp(tX))v - v}{t}$$

for every $X \in \mathfrak{k}_0$ and $v \in V_K$,

- (iii). For every $X \in \mathfrak{k}_0$ and every $k \in K$,

$$\pi(\text{Ad}(k)X) = \pi(k)\pi(X)\pi(k)^{-1}.$$

Motivated by the above fact, we have the following definition.

Definition 1.6.7. A (\mathfrak{g}, K) -module is a pair (π, X) , with X being a complex vector space and π being a map

$$\pi : \mathfrak{g} \cup K \longrightarrow X$$

such that

- (i) $\pi|_{\mathfrak{g}}$ is a complex Lie algebra representation, $\pi|_K$ is a group representation,
- (ii) the above compatibility conditions (i) – (iii) hold, with X replacing V_K .

As we did for the representations of Lie algebras or Lie groups, in an obvious way, one can define irreducible, unitary or admissible (\mathfrak{g}, K) -modules. For more details see [Vog81, Chapter 0].

Definition 1.6.8 (Equivalent (\mathfrak{g}, K) -modules). *Two (\mathfrak{g}, K) -modules are said to be equivalent if there is an invertible (\mathfrak{g}, K) -module map $T : V_1 \rightarrow V_2$, i.e. an invertible linear map that commutes both with the actions of \mathfrak{g} and K . We denote with \widehat{G} the set of equivalence classes of irreducible (\mathfrak{g}, K) -modules.*

Definition 1.6.9 (Harish-Chandra module). *Let V be a representation of a Lie group G . The (\mathfrak{g}, K) -module V_K of V is called the Harish-Chandra module of V .*

Definition 1.6.10 (Infinitesimally equivalent representations). *Let V_1 and V_2 be representations of a Lie group G . We say that V_1 and V_2 are infinitesimally equivalent if their corresponding Harish-Chandra modules are equivalent as (\mathfrak{g}, K) -modules.*

The following theorem gives a strong motivation for the study of (\mathfrak{g}, K) -modules.

Theorem 1.6.11 (see [Vog81, Theorem 0.3.6 and Theorem 0.3.10]).

1. Every irreducible (\mathfrak{g}, K) -module X is the Harish-Chandra module of some irreducible admissible representation V of G .
2. Let V_1 and V_2 be irreducible unitary representations of G . Then they are admissible. Moreover, they are equivalent if and only if they are infinitesimal equivalent.

1.7 Harish-Chandra isomorphism and infinitesimal characters

Let \mathfrak{g} be a complex semisimple Lie algebra, V be an irreducible representation of \mathfrak{g} (not necessarily finite-dimensional) and $Z(\mathfrak{g})$ be the center of the enveloping algebra $U(\mathfrak{g})$. Note that by contrast with the trivial center $Z_{\mathfrak{g}}$ of \mathfrak{g} , the center $Z(\mathfrak{g})$ of the corresponding universal enveloping algebra not only it is nontrivial but we will see that it has an important role to representation theory of \mathfrak{g} . Using Proposition 1.4.11, one can verify that the elements of $Z(\mathfrak{g})$ act by scalars on V . Therefore, there is a map $\chi_V : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ such that $zv = \chi_V(z)v$, for every z in $Z(\mathfrak{g})$ and every v in V . A representation on which $Z(\mathfrak{g})$ acts by scalars is called quasisimple.

Let

$$\mathcal{P} := \sum_{\alpha \in \Delta^+} U(\mathfrak{g})E_{\alpha} \quad \text{and} \quad \mathcal{N} := \sum_{\alpha \in \Delta^+} E_{-\alpha}U(\mathfrak{g}),$$

where E_{α} is a nonzero element of \mathfrak{g}_{α} , $\alpha \in \Delta$.

Proposition 1.7.1. *With the above notation, $U(\mathfrak{g})$ is decomposed into*

$$U(\mathfrak{g}) = U(\mathfrak{t}) \oplus (\mathcal{P} + \mathcal{N}). \quad (1.7)$$

Moreover, the projection of every element of $Z(\mathfrak{g})$ in the second factor of this decomposition belongs to $\mathcal{P} \cap \mathcal{N}$.

Proof. The fact that $U(\mathfrak{g}) = U(\mathfrak{h}) + (\mathcal{P} + \mathcal{N})$ is a consequence of Poincaré-Birkoff-Witt's Theorem 1.1.44. So it suffices to show that this sum is direct. Again using Theorem 1.1.44, an element in \mathcal{P} can be written as a linear combination of terms of the form

$$E_{-\beta_1}^{q_1} \dots E_{-\beta_k}^{q_k} H_1^{m_1} \dots H_l^{m_l} E_{\beta_1}^{p_1} \dots E_{\beta_k}^{p_k} E_\alpha. \quad (1.8)$$

The elements $E_{\beta_1}^{p_1} \dots E_{\beta_k}^{p_k} E_\alpha$, belonging to $U(\mathfrak{n})$, can be written as linear combinations of terms $E_{\beta_1}^{p'_1} \dots E_{\beta_k}^{p'_k}$ and for each such term we have $p'_1 + \dots + p'_k > 0$. Therefore, every element of the form (1.8) can be written as a linear combination of elements

$$E_{-\beta_1}^{q_1} \dots E_{-\beta_k}^{q_k} H_1^{m_1} \dots H_l^{m_l} E_{\beta_1}^{p'_1} \dots E_{\beta_k}^{p'_k},$$

with $p'_1 + \dots + p'_k > 0$. For the sake of simplicity we use p_i instead of p'_i . Therefore each element in \mathcal{P} can be decomposed as a linear combination of terms

$$E_{-\beta_1}^{q_1} \dots E_{-\beta_k}^{q_k} H_1^{m_1} \dots H_l^{m_l} E_{\beta_1}^{p_1} \dots E_{\beta_k}^{p_k} \quad (1.9)$$

with $p_1 + \dots + p_k > 0$. Similarly, one can show that each element in \mathcal{N} can be written as a linear combination of terms (1.9) with $q_1 + \dots + q_k > 0$. Therefore an element in $\mathcal{P} + \mathcal{N}$ is written as a linear sum of terms

$$E_{-\beta_1}^{q_1} \dots E_{-\beta_k}^{q_k} H_1^{m_1} \dots H_l^{m_l} E_{\beta_1}^{p_1} \dots E_{\beta_k}^{p_k},$$

with $q_1 + \dots + q_k > 0$ or $p_1 + \dots + p_k > 0$ and thus $U(\mathfrak{t}) \cap (\mathcal{P} + \mathcal{N}) = 0$.

If the element (1.9) is taken in $Z(\mathfrak{g}) \cap (\mathcal{P} + \mathcal{N})$, then for every H in \mathfrak{t} we have

$$\begin{aligned} 0 &= \text{ad}(H) \left(E_{-\beta_1}^{q_1} \dots E_{-\beta_k}^{q_k} H_1^{m_1} \dots H_l^{m_l} E_{\beta_1}^{p_1} \dots E_{\beta_k}^{p_k} \right) \\ &= - \sum_{1 \leq i \leq k} q_i \beta_i(H) + \sum_{1 \leq i \leq k} p_i \beta_i(H) \\ &= \sum_{1 \leq i \leq k} (p_i - q_i) \beta_i(H) \end{aligned}$$

and thus $p_1 + \dots + p_k > 0$ if and only if $q_1 + \dots + q_k > 0$. Consequently, if z belongs to $Z(\mathfrak{g})$, its second component with respect to the decomposition (1.7) lies in $\mathcal{P} \cap \mathcal{N}$. \square

Let $\gamma'_n : Z(\mathfrak{g}) \rightarrow U(\mathfrak{t})$ be the projection of $Z(\mathfrak{g})$ in the first factor with respect to the above decomposition $U(\mathfrak{g}) = U(\mathfrak{t}) \oplus (\mathcal{P} + \mathcal{N})$ and $\tau_n : \mathfrak{t} \rightarrow U(\mathfrak{t})$ be the linear map which sends the element H of \mathfrak{t} to the element $H - \rho(H)$ of $U(\mathfrak{t})$. The map τ_n satisfies the relation (1.4) and so it can be extended to a map from $U(\mathfrak{t})$ to $U(\mathfrak{t})$, which we note again by τ_n .

Definition 1.7.2 (Harish-Chandra map). *We define the Harish-Chandra map $\gamma : Z(\mathfrak{g}) \rightarrow U(\mathfrak{t})$ to be the composition map*

$$\gamma := \tau_n \circ \gamma'_n.$$

Using the action of the Weyl group W on \mathfrak{t}^* , we define by duality a W -action on \mathfrak{t} which can be extended to a W -action on $U(\mathfrak{t})$. Let $U(\mathfrak{t})^W$ stand for the space of W -invariants of $U(\mathfrak{t})$, i.e.

$$U(\mathfrak{t})^W := \{u \in U(\mathfrak{t}) \mid wu = u \text{ for every } w \in W\}.$$

Theorem 1.7.3 (Harish-Chandra isomorphism, see [Kna96, Theorem 4.44]). *The map γ does not depend on the choice of the positive system Δ^+ . Moreover it is an algebra isomorphism of $Z(\mathfrak{g})$ onto $U(\mathfrak{t})^W$.*

Remark 1.7.4. Since \mathfrak{t} is abelian, the algebra $U(\mathfrak{t})$ can be regarded as the symmetric algebra $S(\mathfrak{t}^*)$ or as the algebra of polynomial functions on \mathfrak{t}^* , the evaluation being that the element $u \in U(\mathfrak{t})$ evaluated on $\lambda \in \mathfrak{t}^*$ gives the value of the extended $\lambda : U(\mathfrak{t}) \rightarrow \mathbb{C}$ applied on u . In other words, $U(\mathfrak{t}) \simeq \mathbb{C}[\mathfrak{t}] \simeq S(\mathfrak{t})$ and $U(\mathfrak{t})^W \simeq \mathbb{C}[\mathfrak{t}]^W \simeq S(\mathfrak{t})^W$.

Example 1.7.5. In Example 1.4.33, we saw that the Casimir element of $\mathfrak{sl}(2, \mathbb{C})$ is

$$\Omega = \frac{1}{8}H^2 + \frac{1}{4}XY + \frac{1}{4}YX,$$

where H, X and Y are defined in the same example. The matrix X corresponds to the positive root $\varepsilon_1 - \varepsilon_2$, Ω belongs to $Z(\mathfrak{sl}(2, \mathbb{C}))$ and by the relation $[X, Y] = H$, we obtain

$$\begin{aligned} \Omega &= \frac{1}{8}H^2 + \frac{1}{4}XY + \frac{1}{4}YX \\ &= \frac{1}{8}H^2 + \frac{1}{4}[X, Y] + \frac{1}{4}YX + \frac{1}{4}YX \\ &= \frac{1}{8}H^2 + \frac{1}{4}H + \frac{1}{2}YX. \end{aligned}$$

Hence $\gamma'_n(\Omega) = \frac{1}{8}H^2 + \frac{1}{4}H$. On the other hand,

$$\begin{aligned} \tau_n(H) &= H - \rho(H) \\ &= H - \frac{1}{2}(\varepsilon_1 - \varepsilon_2)(H) \\ &= H - 1 \end{aligned}$$

and

$$\begin{aligned} \gamma(\Omega) &= \tau_n(\gamma'_n(\Omega)) \\ &= \frac{1}{8}\tau_n(H^2) + \frac{1}{4}\tau_n(H) \\ &= \frac{1}{8}(H - 1)^2 + \frac{1}{4}(H - 1) \\ &= \frac{1}{8}H^2 - \frac{1}{8}. \end{aligned}$$

The nontrivial element of W sends H to $-H$ and we verify that $\gamma(\Omega)$ is indeed W -invariant. On the other hand, the subalgebra of the W -invariants of $\mathbb{C}[H]$ is exactly $\mathbb{C}[H^2]$, i.e. the subalgebra of $\mathbb{C}[H]$ generated by $\gamma(\Omega)$. Consequently, the center of $U(\mathfrak{sl}(2, \mathbb{C}))$ is exactly $Z(\mathfrak{sl}(2, \mathbb{C})) = \mathbb{C}[\Omega]$.

For λ in \mathfrak{t}^* , let

$$\chi_\lambda : \begin{cases} Z(\mathfrak{g}) & \rightarrow \mathbb{C} \\ z & \mapsto \gamma(z)(\lambda) := \lambda(\gamma(z)) \end{cases}$$

It is clear that χ_λ is an algebra morphism. As a consequence of Harish-Chandra's Theorem 1.7.3, we obtain the following corollary.

Corollary 1.7.6 (See [Kna96, Theorem 5.6.2]). *If $\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$ is an algebra morphism, then there is $\lambda \in \mathfrak{t}^*$ such that $\chi = \chi_\lambda$. If μ is another element of \mathfrak{t}^* , we have $\chi_\lambda = \chi_\mu$ if and only if λ and μ belong to the same W -orbit, i.e. there is an element w of W such that $\lambda = w\mu$.*

Using the above corollary, we can attach to each quasisimple representation V an element λ of \mathfrak{t}^* (or better the W -orbit of λ in \mathfrak{t}^*). More precisely, if χ is the infinitesimal character of V , according to Corollary 1.7.6, there is some λ in \mathfrak{t}^* such that $\chi = \chi_\lambda$.

Definition 1.7.7 (Infinitesimal character). *In the above setting and by abuse of notation we will say that λ is the infinitesimal character of V .*

Example 1.7.8. Let V be the finite-dimensional representation of \mathfrak{g} with highest weight λ . Let z be an element of $Z(\mathfrak{g})$ and v^+ be a maximal vector of V . Then z is decomposed into $z = \gamma'_n(z) + (z - \gamma'_n(z))$, with $\gamma'_n(z)$ being in $U(\mathfrak{t})$ and $(z - \gamma'_n(z))$ being in $\mathcal{P} \cap \mathcal{N}$, while

$$\begin{aligned} z \cdot v^+ &= \gamma'_n(z) \cdot v^+ + (z - \gamma'_n(z)) \cdot v^+ \\ &= \lambda(\gamma'_n(z))v^+ \\ &= (\lambda + \rho)(\gamma(z)) \\ &= \chi_{\lambda+\rho}(z). \end{aligned}$$

Therefore, the infinitesimal character of the irreducible finite-dimensional representation V of highest weight λ is $\lambda + \rho$.

CHAPTER 2

CLIFFORD ALGEBRAS AND SPIN MODULES

In this chapter we introduce the basic notions of Clifford algebras and spaces of spinors. Then we specialize them in the case of complex semisimple Lie algebras in order to define a Dirac operator. The main references for this chapter are the books [HP06] and [GW09].

2.1 Clifford Algebras

Definition 2.1.1 (Clifford algebra). *Let V be a real or complex vector space equipped with a symmetric bilinear form B . We define the Clifford algebra $\mathbf{C}(V)$ of V to be the quotient*

$$\mathbf{C}(V) := T(V)/I,$$

where $I := (u \otimes u - \frac{1}{2}B(u, u))$, i.e. the two-sided ideal of V generated by the indicated element for all $u \in V$.

Remark 2.1.2. Many authors define the ideal I to be $I := (u \otimes u + B(u, u))$ (for example, see [HP06]). Nevertheless, note that in the case where V is complex, which is the case we are interested in, the two definitions turn out to be equivalent, up to multiplication of B by a complex scalar.

An alternative way to define the Clifford algebra of V is by using the following universal property. The Clifford algebra of V is any pair (C, γ) , where C is a unital algebra and $\gamma : V \rightarrow C$ is a linear morphism, such that for each unital algebra A and each vector space morphism $\phi : V \rightarrow A$ with the property $\phi(v)^2 = \frac{1}{2}B(v, v)$ for all $v \in V$, there exists a unique unital algebra morphism $\tilde{\phi} : C \rightarrow A$ with $\tilde{\phi} \circ \gamma = \phi$. In other words, there is a unique $\tilde{\phi} : C \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc}
 V & \xrightarrow{\gamma} & C \\
 & \searrow \phi & \downarrow \tilde{\phi} \\
 & & A
 \end{array} \tag{2.1}$$

If (C', γ') is another Clifford algebra of V , by setting $A = C'$ and $\phi = \gamma'$ in (2.1) and by using the universal property of C and C' , one obtains that there are unique maps $\tilde{\gamma}' : C \rightarrow C'$ and $\tilde{\gamma} : C' \rightarrow C$ such that the following diagram commutes:

$$\begin{array}{ccc}
 V & \xrightarrow{\gamma} & C \\
 & \searrow \gamma' & \uparrow \tilde{\gamma} \\
 & & C'
 \end{array}$$

In other words,

$$\gamma = \tilde{\gamma} \circ \gamma' \tag{2.2}$$

$$\gamma' = \tilde{\gamma}' \circ \gamma \tag{2.3}$$

and the maps $\tilde{\gamma}$ and $\tilde{\gamma}'$ are unique with this property. Using these equalities, one deduces that

$$(\tilde{\gamma} \circ \tilde{\gamma}') \circ \gamma = \tilde{\gamma} \circ \gamma' \tag{2.4}$$

On the other hand, if we set $A = C$ and $\phi = \tilde{\gamma} \circ \gamma'$ in (2.1), there is a unique unital algebra morphism $\tilde{\phi}$ such that $\tilde{\gamma} \circ \gamma' = \tilde{\phi} \circ \gamma$. From (2.4) and (2.2), one deduces that $\tilde{\phi} = \tilde{\gamma} \circ \tilde{\gamma}'$ and $\tilde{\phi} = id_C$ and thus $\tilde{\gamma} \circ \tilde{\gamma}' = id_C$. With a similar argument, one obtains that $\tilde{\gamma}' \circ \tilde{\gamma} = id_{C'}$ and consequently $\tilde{\gamma}$ and $\tilde{\gamma}'$ are unital algebra isomorphisms establishing an isomorphism between C and C' .

Theorem 2.1.3. *If $\{v_i\}_{1 \leq i \leq n}$ is a basis of V , the set*

$$\{\gamma(v_{i_1}) \dots \gamma(v_{i_k}) \mid 0 \leq k \leq n, 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$$

forms a vector space basis of $\mathbf{C}(V)$.

Example 2.1.4 (\mathbb{C} as a Clifford algebra). Let $V = \mathbb{R}$ be equipped with the canonical bilinear form of signature -1 and $\{\varepsilon_1\}$ be the canonical basis of \mathbb{R} . If $v_1 := \sqrt{2}\varepsilon_1$, according to Theorem 2.1.3, the set $\{1, \gamma(v_1)\}$ forms a vector space basis of $\mathbf{C}(\mathbb{R})$. Since $\gamma(v_1)$ squares to -1 , this algebra is canonically isomorphic to \mathbb{C} , the algebra of complex numbers.

Example 2.1.5 (\mathbb{H} as a Clifford algebra). Let $V = \mathbb{R}^2$ be equipped with the canonical bilinear form of signature -2 . Let $\{\varepsilon_1, \varepsilon_2\}$ be its canonical basis and set $v_i := \sqrt{2}\varepsilon_s, s = 1, 2$. According to Theorem 2.1.3, the set $\{1, i := \gamma(v_1), j := \gamma(v_2), k := \gamma(v_1)\gamma(v_2)\}$ forms a vector space basis of the Clifford algebra $\mathbf{C}_{(0,2)}(\mathbb{R}^2)$ of \mathbb{R}^2 where $(0, 2)$ indicates the signature of the bilinear form. The basis elements satisfy the relations:

$$i^2 = j^2 = k^2 = -1 \text{ and } ij = k, jk = i, ki = j$$

which are exactly the defining relations of the quaternion algebra \mathbb{H} . Hence, the Clifford algebra $\mathbf{C}_{(0,2)}(\mathbb{R}^2)$ is isomorphic to the quaternion algebra \mathbb{H} .

Note that if the bilinear form B is the trivial one, the Clifford algebra $\mathbf{C}(V)$ is isomorphic with the exterior algebra $\bigwedge V$ of V . Indeed, the ideal used in Definition 2.1.1 is $(u \otimes u)$, $u \in V$, i.e. the one used in the definition of the exterior algebra. However, if B is nontrivial, instead of an algebra isomorphism, we have a canonical vector space isomorphism given by the following map.

Definition 2.1.6 (Chevalley map). *The Chevalley map is the linear map $j : \bigwedge V \longrightarrow \mathbf{C}(V)$ defined by:*

$$v_1 \wedge \cdots \wedge v_k \in \bigwedge^k V \mapsto \frac{1}{k!} \sum_{\sigma \in S_k} v_{\sigma(1)} \cdots v_{\sigma(k)} \in T(V)/I = \mathbf{C}(V),$$

where S_k is the symmetric group of k objects.

Proposition 2.1.7. *Chevalley map j is a vector space isomorphism. If B is trivial, j is a unital algebra isomorphism.*

Proof. Let $\{Z_s\}_{1 \leq s \leq n}$ be a basis of V such that for every s, t , $1 \leq s \neq t \leq n$, $B(Z_s, Z_t) = 0$ (if B is nondegenerate we can take $\{Z_s\}$ to be an orthonormal basis). Then one can check that the map j maps the element $Z_{s_1} \wedge \cdots \wedge Z_{s_k}$ of $\bigwedge V$ to the element $Z_{s_1} \cdots Z_{s_k}$ of $\mathbf{C}(V)$. In other words, according to Theorem 2.1.3, j sends a basis of $\bigwedge V$ to a basis of $\mathbf{C}(V)$ and thus the result follows. \square

2.2 Spaces of spinors

For the rest of this chapter, V will stand for a finite-dimensional complex vector space equipped with a nondegenerate form B . We would like to classify, up to isomorphism, the irreducible modules of $\mathbf{C}(V)$.

Definition 2.2.1. *An irreducible $\mathbf{C}(V)$ -module is called space of spinors for $\mathbf{C}(V)$.*

We treat separately the 2 cases for V being even or odd dimensional. Let us start with V being an even-dimensional complex vector space, say of dimension $2n$, and U^+ , U^- being two maximal isotropic dual subspaces of V . More precisely, if $\{Z_i\}_{1 \leq i \leq 2n}$ is an orthonormal basis of V and

$$e_j := \frac{Z_{2j-1} + iZ_{2j}}{\sqrt{2}}, \quad e_{-j} := \frac{Z_{2j-1} - iZ_{2j}}{\sqrt{2}}, \quad j = 1, \dots, n, \quad (2.5)$$

the subspaces $U^+ := \langle e_j | 1 \leq j \leq n \rangle$ and $U^- := \langle e_{-j} | 1 \leq j \leq n \rangle$ are maximal isotropic and dual, i.e. for every i and j , $1 \leq i, j \leq n$, one has

$$\begin{aligned} B(e_i, e_j) &= 0, \\ B(e_{-i}, e_{-j}) &= 0, \\ B(e_i, e_{-j}) &= \delta_{ij}. \end{aligned}$$

We set S_V to be the vector space $\bigwedge U^-$ equipped with an action of V on S_V given by

$$\begin{aligned} e_{-i} \cdot (e_{-i_1} \wedge \dots \wedge e_{-i_k}) &:= e_{-i} \wedge e_{-i_1} \wedge \dots \wedge e_{-i_k}, \\ e_i \cdot (e_{-i_1} \wedge \dots \wedge e_{-i_k}) &:= \sum_j (-1)^{j+1} B(e_i, e_{-i_j}) e_{-i_1} \wedge \dots \wedge \widehat{e_{-i_j}} \wedge \dots \wedge e_{-i_k} \end{aligned}$$

for every $e_{-i} \in U^-$ and $e_i \in U^+$. The hat in the second expression indicates that e_{-i_j} is omitted. One can verify that this action of V can be extended to an action of $\mathbf{C}(V)$. Note that:

$$\gamma(e_{-j})(e_{-i_1} \wedge \dots \wedge e_{-i_k}) = \begin{cases} 0 & \text{if } j \in \{i_1, \dots, i_k\} \\ e_{-j} \wedge e_{-i_1} \wedge \dots \wedge e_{-i_k} & \text{if } j \notin \{i_1, \dots, i_k\} \end{cases}$$

$$\gamma(e_j)(e_{-i_1} \wedge \dots \wedge e_{-i_k}) = \begin{cases} (-1)^{j+1} e_{-i_1} \wedge \dots \wedge \widehat{e_{-i_j}} \wedge \dots \wedge e_{-i_k} & \text{if } j \in \{i_1, \dots, i_k\} \\ 0 & \text{if } j \notin \{i_1, \dots, i_k\} \end{cases}$$

Equipped with the above action, one can show that S_V is an irreducible $\mathbf{C}(V)$ -module (see [HP06, Lemma 2.2.3]) and thus it is a space of spinors. Furthermore, using this fact, $\mathbf{C}(V)$ turns out to be a simple algebra (see [HP06, Corollary 2.2.5]). We summarize the above discussion in the following theorem.

Theorem 2.2.2 (See [GW09, Theorem 6.1.3]). *Let V be a $2n$ -dimensional complex vector space. Up to isomorphism, there is exactly one space of spinors S_V , of dimension 2^n , i.e. the one described above. Moreover, the map $\gamma : \mathbf{C}(V) \rightarrow \text{End}(S_V)$ giving the action of $\mathbf{C}(V)$ on S_V is injective.*

Remark 2.2.3. It is useful to say that S_V is a \mathbb{Z}_2 -graded module, the $\bar{0}$ -part being the even subspace of $\bigwedge^k U^-$ and the $\bar{1}$ -part the odd one. On the other hand, from Theorem 2.1.3, we can deduce that $\mathbf{C}(V)$ is a \mathbb{Z}_2 -graded algebra and this grading is compatible with the grading of S_V .

Suppose now that V is an odd-dimensional complex vector space, say of dimension $(2n+1)$. We want to determine up to isomorphism all its spaces of spinors. Let U^+ and U^- be n -dimensional maximal isotropic dual subspaces of V . Set

$$\bar{V} := U^+ \oplus U^-$$

and let $e_0 \in V$ be a non zero vector belonging to the orthogonal complement of \bar{V} with $B(e_0, e_0) = 2$. Clearly $V = \bar{V} \oplus \mathbb{C}\{e_0\}$ and \bar{V} is even dimensional. We consider the space of spinors $S := S_{\bar{V}}$ of $\mathbf{C}(\bar{V})$ and we want to equip it with an action of e_0 so as to be a $\mathbf{C}(V)$ -module, i.e. in such a way that the structure of $\mathbf{C}(V)$ is respected. From the definition of $\mathbf{C}(V)$, for the element $\gamma(e_0)$ we have:

$$\begin{cases} \gamma(e_0)^2 = 1 \\ \gamma(e_0)\gamma(e_i) = -\gamma(e_i)\gamma(e_0) & \text{for every } i=1, \dots, n \\ \gamma(e_0)\gamma(e_{-i}) = -\gamma(e_{-i})\gamma(e_0) & \text{for every } i=1, \dots, n. \end{cases}$$

Here $\{e_i\}$ and $\{e_{-i}\}$ are, as before, bases of U^+ and U^- respectively with $B(e_i, e_{-j}) = \delta_{ij}$ for every i, j , $1 \leq i, j \leq n$. Therefore $\gamma(e_0)$, as an operator acting on S , must have at most two eigenspaces, namely with eigenvalues 1 and -1 , while, for every $i = 1, \dots, n$, $\gamma(e_i)$ and $\gamma(e_{-i})$ must interchange these eigenspaces. A natural choice for the action of e_0 would be taking as the 1-eigenspace the even or the odd subspace with respect to the \mathbb{Z}_2 -gradation of S and as the -1 -eigenspace the other one. They turn out to be the only possibilities for the action of e_0 , giving two nonisomorphic spaces of spinors. The following theorem summarizes the above discussion.

Theorem 2.2.4 (See [GW09, Theorem 6.1.3]). *Let V be an $2n + 1$ -dimensional complex vector space. Up to isomorphism, there are exactly two non isomorphic spaces of spinors S_1 and S_{-1} , each of dimension 2^n , i.e. the ones described above. More precisely, they are isomorphic as $C(\bar{V})$ -modules while e_0 acts by 1 on the $\bar{1}$ -part of S_1 and by -1 on the $\bar{1}$ -part of S_{-1} .*

2.2.1 Hermitian form on S

Let V_0 be a real form of V equipped with a inner product (\cdot, \cdot) and $\{Z_j\}$ be an orthonormal basis of V_0 with respect to (\cdot, \cdot) . The inner product (\cdot, \cdot) is extended uniquely to a Hermitian form on $V := (V_0)^\mathbb{C}$, again denoted by (\cdot, \cdot) . Using (\cdot, \cdot) , we define a complex bilinear form $B(\cdot, \cdot)$ on V , given by $B(X, Y) := -(X, \tilde{Y})$ for $X, Y \in V$, where $\tilde{\cdot}$ stands for the conjugation with respect to V_0 . Then, the set $\{Z_j := iZ'_j\}$ forms an orthonormal basis of V with respect to B . As in (2.5), set

$$e_j := \frac{Z_{2j-1} + iZ_{2j}}{\sqrt{2}}, \quad e_{-j} := \frac{Z_{2j-1} - iZ_{2j}}{\sqrt{2}}, \quad j = 1, \dots, n,$$

$U^+ := \langle e_j | 1 \leq j \leq n \rangle$ and $U^- := \langle e_{-j} | 1 \leq j \leq n \rangle$. The subspaces U^+ and U^- are maximal isotropic dual subspaces with respect to B and define $S := \bigwedge U^-$ to be a space of spinors for $C(V)$. On $\bigwedge U^-$, we define a Hermitian form $\langle \cdot, \cdot \rangle$ as follows: For $e_{-i_1} \wedge \dots \wedge e_{-i_l}$ and $e_{-j_1} \wedge \dots \wedge e_{-j_l}$ in $\bigwedge^l U^-$, we set

$$\langle e_{-i_1} \wedge \dots \wedge e_{-i_l}, e_{-j_1} \wedge \dots \wedge e_{-j_l} \rangle := \det((e_{-i_s}, e_{-j_t})_{s,t}), \quad (2.6)$$

while $\bigwedge^l U^-$ and $\bigwedge^{l'} U^-$ are orthogonal for $l \neq l'$, and $\langle 1, 1 \rangle = 1$. One can verify that $\langle \cdot, \cdot \rangle$ is, indeed, a Hermitian form and the elements of the form $e_{i_1} \wedge \dots \wedge e_{i_l}$, with $1 \leq i_1 < \dots < i_l \leq n$ and $1 \leq l \leq n$, form an orthonormal basis for S .

Proposition 2.2.5. *With the above notation, the adjoint operator of $\gamma(X)$, $X \in V$, is*

$$\gamma(X)^* = -\gamma(\tilde{X}).$$

Proof. For $X = e_{-k}$, $1 \leq k \leq n$, by definition:

$$\langle e_{-k} \wedge e_{-i_1} \wedge \dots \wedge e_{-i_l}, e_{-j_1} \wedge \dots \wedge e_{-j_{l'}} \rangle = \langle e_{-i_1} \wedge \dots \wedge e_{-i_l}, \gamma(e_{-k})^*(e_{-j_1} \wedge \dots \wedge e_{-j_{l'}}) \rangle.$$

The left-hand side of the equality is 0 unless $\{k, i_1, \dots, i_l\} = \{j_1, \dots, j_{l'}\}$, where it is 1. One can verify that

$$\langle e_{-k} \wedge e_{-i_1} \wedge \dots \wedge e_{-i_l}, e_{-j_1} \wedge \dots \wedge e_{-j_{l'}} \rangle = \langle e_{-i_1} \wedge \dots \wedge e_{-i_l}, \gamma(e_k)(e_{-j_1} \wedge \dots \wedge e_{-j_{l'}}) \rangle$$

and thus $\gamma(e_{-k})^* = \gamma(e_k)$, i.e. $\gamma(e_{-k})^* = -\gamma(\widetilde{e_{-k}})$. Using a similar argument, one shows that $\gamma(e_k)^* = -\gamma(\widetilde{e_k})$ and, if necessary, $\gamma(e_0)^* = -\gamma(\widetilde{e_0})$, with e_0 being as above, i.e. a fixed nonzero element of V , orthogonal to $U^+ \oplus U^-$ with $B(e_0, e_0) = 2$. Therefore $\gamma(X)^* = -\gamma(\widetilde{X})$ for every X in V . \square

2.2.2 Spin module of $\mathfrak{so}(V)$

Apart from the $\mathbf{C}(V)$ -module structure, S admits a structure of representation of the Lie algebra

$$\mathfrak{so}(V) := \mathfrak{so}(V, B) = \{T \in \mathfrak{gl}(V) \mid B(Tu, v) + B(u, Tv) = 0, \forall u, v \in V\}.$$

For a and b in V , let $R_{a,b}$ be the element of $\mathfrak{gl}(V)$ defined by

$$R_{a,b}(v) := B(b, v)a - B(a, v)b,$$

for $v \in V$. Then $R_{a,b}$ belongs to $\mathfrak{so}(V)$. Indeed, for every u and v in V , we have:

$$\begin{aligned} B(R_{a,b}(u), v) &= B(b, u)B(a, v) - B(a, u)B(b, v) \\ &= -B(u, R_{a,b}(v)). \end{aligned}$$

Proposition 2.2.6 (See [GW09, Lemma 6.2.1]). *The elements $R_{a,b}$, $a, b \in V$, span $\mathfrak{so}(V)$.*

Let $\mathbf{C}_2(V)$ be the vector subspace of $\mathbf{C}(V)$ given as the image of the subspace $\bigwedge^2 V$ via the Chavalley map (see Definition 2.1.6) and let φ be the linear map defined by

$$\varphi : \begin{cases} \mathfrak{so}(V) & \longrightarrow \mathbf{C}_2(V) \\ R_{a,b} & \mapsto \frac{1}{2}[\gamma(a), \gamma(b)] \end{cases}$$

Let us now equip $\mathbf{C}(V)$ with its canonical Lie algebra structure, i.e. the Lie algebra structure inherited by its associative algebra structure (see Example 1.1.11).

Proposition 2.2.7 (See [GW09, Lemma 6.2.2]). *The map φ is a well-defined injective Lie algebra morphism. Moreover*

$$[\varphi(X), \gamma(v)] = \gamma(Xv) \tag{2.7}$$

for X in $\mathfrak{so}(V)$ and v in V .

Definition 2.2.8 (Spin representation of $\mathfrak{so}(V)$). *Let S be a space of spinors for V . The Lie algebra representation of $\mathfrak{so}(V)$ on S obtained by*

$$\varphi : \mathfrak{so}(V) \longrightarrow \mathbf{C}_2(V) \longrightarrow \text{End}(S) \tag{2.8}$$

is called *spin representation (or spin module) of $\mathfrak{so}(V)$.*

Proposition 2.2.9 (See [GW09, Proposition 6.2.3 and Proposition 6.2.4]). *If V is even-dimensional, say of dimension $2n$, the spin module S of $\mathfrak{so}(V)$ is decomposed into two inequivalent irreducible subrepresentations of highest weights $\frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{n-1} + \varepsilon_n)$ and $\frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_{n-1} - \varepsilon_n)$. If V is odd-dimensional, say of dimension $2n + 1$, the spin module S of $\mathfrak{so}(V)$ is irreducible and of highest weight $\frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_n)$.*

2.3 Spin Groups

Let V be a finite-dimensional complex vector space equipped with a nondegenerate symmetric bilinear form B . Let $\mathbf{C}(V) := \mathbf{C}(V, B)$ be the corresponding Clifford algebra, γ be the canonical projection of V in $\mathbf{C}(V)$ and S be a space of spinors. We consider the spin representation of $\mathfrak{so}(V)$ obtained by the diagram (2.8). For later use, we want to integrate this representation in order to obtain a representation of the Lie group $SO(V)$ on S . Nevertheless, the group $SO(V)$ being non simply connected, this is not possible, as the following example illustrates.

Example 2.3.1. Suppose that $G = SO(3)$. Then the complexified Lie algebra of G is $\mathfrak{g} := \mathfrak{so}(3, \mathbb{C})$, i.e.

$$\mathfrak{g} = \{X \in \mathfrak{gl}(3, \mathbb{C}) \mid X + X^t = 0\}.$$

Then, a Cartan subalgebra of \mathfrak{g} is $\mathfrak{t} := \mathbb{C}\{H\}$, where

$$H = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This Cartan subalgebra corresponds to the maximal torus

$$T := \left\{ \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in [0, 2\pi] \right\}$$

The corresponding root system $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ is $\Delta = \{\pm\varepsilon_1\}$, where ε_1 is the element of \mathfrak{t}^* mapping the above matrix H to 1. Fix $\Delta^+ := \{\varepsilon_1\}$ to be the positive system of Δ . Then, the set Λ of all dominant algebraically integral elements of \mathfrak{t}^* is

$$\Lambda = \frac{1}{2}\mathbb{N}\{\varepsilon_1\}.$$

Consider the spin representation S of $\mathfrak{so}(3, \mathbb{C})$ (see Definition 2.2.8), which, according to Proposition 2.2.9, is of highest weight $\lambda := \frac{1}{2}\varepsilon_1$. Suppose that λ is analytically integral, i.e. there is a group morphism

$$\xi_\lambda : T \rightarrow \mathbb{C}$$

such that $T_e \xi_\lambda = \lambda$. Moreover, consider the matrix

$$X := \begin{pmatrix} 0 & 2\pi & 0 \\ -2\pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

of \mathfrak{t} . According to Example 1.1.37, the exponential of matrix Lie groups coincides with the standard exponential map of matrices. Thus, one can check that $\exp X = 1$. On the other hand, $\lambda(X) = -i\pi$ and

$$\begin{aligned} 1 &= \xi_\lambda(\exp X) \\ &= e^{T_e \xi_\lambda(X)} \\ &= e^{\lambda(X)} \\ &= e^{-i\pi} \\ &= -1 \end{aligned}$$

which is a contradiction. Therefore, λ is not analytically integral and so the spin representation S of $\mathfrak{so}(3, \mathbb{C})$ does not integrate to a representation of $SO(3)$.

In order to overcome this problem, we have to consider a double cover of $SO(V)$, denoted $\text{Spin}(V)$, instead of $SO(V)$. In what follows, we construct this double cover $\text{Spin}(V)$ in terms of the Clifford algebra $\mathbf{C}(V)$ of V .

Set α to be the linear map defined by

$$\alpha : V \ni v \mapsto -\gamma(v) \in \mathbf{C}(V).$$

Then α is annihilated on the ideal I used in the definition of $\mathbf{C}(V)$ and thus can be uniquely extended on an endomorphism of $\mathbf{C}(V)$, denoted again by α . Then, α is an involution of $\mathbf{C}(V)$, i.e. an automorphism such that $\alpha^2 = id$, given by

$$\alpha(\gamma(v_1) \dots \gamma(v_p)) = (-1)^p \gamma(v_1) \dots \gamma(v_p)$$

for every element v_1, \dots, v_p of V .

Definition 2.3.2 (Main involution of $\mathbf{C}(V)$). *The automorphism $\alpha : \mathbf{C}(V) \rightarrow \mathbf{C}(V)$ is called the main involution of $\mathbf{C}(V)$.*

Set $\mathbf{C}^+(V)$ (respectively $\mathbf{C}^-(V)$) to be the $+1$ -eigenspace (respectively -1 -eigenspace) of α , i.e. the subspace of $\mathbf{C}(V)$ generated by products of an even (respectively odd) number of elements $\gamma(v)$, $v \in V$. Then, $\mathbf{C}(V)$ is decomposed into

$$\mathbf{C}(V) = \mathbf{C}^+(V) \oplus \mathbf{C}^-(V).$$

On the other hand, let $\tau : \mathbf{C}(V) \rightarrow \mathbf{C}(V)$ be the unique antiautomorphism of $\mathbf{C}(V)$ defined by

$$V \ni v \mapsto \gamma(v) \in \mathbf{C}(V).$$

In other words, τ is the antiautomorphism of $\mathbf{C}(V)$ mapping the element

$$\gamma(v_1) \dots \gamma(v_p), \quad v_1, \dots, v_p \in V$$

to

$$\gamma(v_p) \dots \gamma(v_1).$$

Definition 2.3.3 (Main antiautomorphism of $\mathbf{C}(V)$). *The map $\tau : \mathbf{C}(V) \rightarrow \mathbf{C}(V)$ is called the main antiautomorphism of $\mathbf{C}(V)$.*

Let us now define a conjugation \cdot^* on $\mathbf{C}(V)$ given by

$$\mathbf{C}(V) \ni \gamma(v) \mapsto \gamma(v)^* := \tau(\alpha(v)) \in \mathbf{C}(V), \quad v \in V.$$

Then

$$\gamma(v)\gamma(v)^* = \gamma(v)^*\gamma(v) = -\frac{1}{2}B(v, v)$$

and, if $B(v, v) = -2$,

$$\gamma(v)\gamma(v)^* = 1 = \gamma(v)^*\gamma(v).$$

Thus $\gamma(v)$ is invertible, the inverse being $\gamma(v)^*$. Moreover, for every $y \in V$, one can check that

$$\alpha(\gamma(v))\gamma(y)\gamma(v)^* = \gamma(s_v y),$$

where $s_v y := y + B(v, y)y$, i.e. the orthogonal reflection of y with respect to the hyperplane defined by v . In other words, the $*$ -conjugation by $\gamma(v)$ on $\mathbf{C}(V)$ corresponds to s_v on V . Set

$$\text{Pin}(V, B) = \{x \in \mathbf{C}(V) \mid xx^* = 1 \text{ and } \alpha(x)\gamma(V)\gamma(v)^* = \gamma(V)\}.$$

The following proposition ensures that $\text{Pin}(V) := \text{Pin}(V, B)$ is a subgroup of the group $\mathbf{C}(V)^\times$ of all invertible elements of $\mathbf{C}(V)$.

Proposition 2.3.4. *In a finite dimensional \mathbb{K} -algebra A , an element $a \in A$ is invertible if and only if it is not a zero divisor.*

Proof. If $a \in A$ is invertible, it is obvious that a is not a zero divisor. On the other hand, suppose that a is an element of A such that a is not a zero divisor and let N be the maximal integer such that

$$1, a, \dots, a^N$$

are linearly independent. Then, there are $c_0, c_1, \dots, c_{N+1} \in \mathbb{K}$, not all zero, such that

$$\sum_{i=0}^{N+1} c_i a^i = 0.$$

Moreover,

$$a\left(\sum_{i=1}^{N+1} c_i a^{i-1}\right) = \left(\sum_{i=1}^{N+1} c_i a^{i-1}\right)a = -c_0,$$

with c_0 being nonzero since a is not a zero divisor. Therefore, a is an invertible element of A , the inverse being

$$-\frac{1}{c_0} \left(\sum_{i=1}^{N+1} c_i a^{i-1}\right).$$

□

Theorem 2.3.5 (See [GW09, Theorem 6.3.2]). *There is a continuous topological group homomorphism*

$$\pi : \text{Pin}(V) \rightarrow O(V)$$

such that, for every $x \in \text{Pin}(V)$ and every $v \in V$,

$$\alpha(x)\gamma(v)x^* = \gamma(\pi(x)v).$$

Moreover, π is surjective and $\ker \pi = \{\pm 1\}$.

The topological group $\text{Pin}(V)$ turns out to be a Lie group. Indeed, Theorem 2.3.5 indicates that $\text{Pin}(V)$ is a double cover of the compact group $O(V)$ and thus it is compact as well. Consequently, $\text{Pin}(V)$ is a closed subgroup of $\mathbf{C}(V)^\times$, which, as an open set of the finite-dimensional vector space $\mathbf{C}(V)$, is a manifold with its group multiplication and inversion being smooth maps. In other words, $\mathbf{C}(V)^\times$ is a Lie group and $\text{Pin}(V)$ is a closed subgroup of it. Hence, $\text{Pin}(V)$ is a Lie group as well.

Set

$$\text{Spin}(V, B) := \text{Pin}(V) \cap \mathbf{C}^+(V),$$

i.e. $\text{Spin}(V, B)$ is the closed Lie subgroup of $\text{Pin}(V)$ generated by products of an even number of elements $\gamma(v)$, $v \in V$. As a consequence, the image of π restricted to $\text{Spin}(V) := \text{Spin}(V, B)$ is generated by an even number of reflections and thus it lies in $SO(V)$. In particular, π maps $\text{Spin}(V)$ onto $SO(V)$ with its kernel being $\{\pm 1\}$. In other words, $\text{Spin}(V)$ is a double cover of the Lie group $SO(V)$.

Proposition 2.3.6 (See [HP06, Subsection 2.1.14]). *The Lie group $\text{Spin}(V)$ is connected.*

Moreover, one can integrate the spin representation (2.8) of $\mathfrak{so}(V)$ to obtain a representation

$$\Phi : \text{Spin}(V) \rightarrow \mathbf{C}(V) \rightarrow GL(S) \quad (2.9)$$

of $\text{Spin}(V)$ on S , called spin representation of $\text{Spin}(V)$. Clearly, the tangent map of Φ is the map φ of Definition 2.2.8.

2.4 Clifford algebra and spinors for Lie algebras

Let \mathfrak{g} be a complex semisimple Lie algebra, B its Killing form, θ a Cartan involution of a real form \mathfrak{g}_0 of \mathfrak{g} such that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is the corresponding Cartan decomposition and $\mathfrak{g}_c = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ the corresponding compact real form of \mathfrak{g} . Extend θ to a \mathbf{C} -linear map again denoted by θ and let $\bar{\cdot}$ be the conjugation with respect to \mathfrak{g}_0 . Then $\theta(\bar{\cdot})$ is the conjugation with respect to \mathfrak{g}_c and let \mathfrak{h} be a

$$\theta(\bar{\cdot})\text{-stable Lie subalgebra of } \mathfrak{g}. \quad (2.10)$$

Then, \mathfrak{h} is reductive and the restriction $B|_{\mathfrak{h} \times \mathfrak{h}}$ of the Killing form B to \mathfrak{h} is nondegenerate. Moreover \mathfrak{g} is decomposed into

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{q} := \mathfrak{h}^\perp$ with respect to B , while

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p})$$

and

$$\mathfrak{q} = (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p}).$$

Furthermore, the restriction of the adjoint representation $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ of \mathfrak{g} to \mathfrak{h} induces a linear map

$$\text{ad}_{\mathfrak{q}} : \mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{q}) := \mathfrak{so}(\mathfrak{q}, B_{\mathfrak{q}}),$$

where $B_{\mathfrak{q}} := B|_{\mathfrak{q} \times \mathfrak{q}}$. Indeed, for X_1, X_2 in \mathfrak{h} and Y_1, Y_2 in \mathfrak{q} , from the \mathfrak{g} -invariance of the Killing form (see Lemma 1.1.40), we obtain

$$B([X_1, Y_1], X_2) = -B(Y_1, [X_1, X_2]) = 0$$

which ensures that $[\mathfrak{h}, \mathfrak{q}] \subseteq \mathfrak{h}^\perp = \mathfrak{q}$. Thus, the image of $\text{ad}_{\mathfrak{q}}$ lies in $\text{End}(\mathfrak{q})$ while, again from the \mathfrak{g} -invariance of B , one obtains that:

$$B(\text{ad}_{\mathfrak{q}}(X_1)Y_1, Y_2) + B(Y_1, \text{ad}_{\mathfrak{q}}(X_1)Y_2) = 0,$$

which is exactly the defining property of $\mathfrak{so}(\mathfrak{q})$. Hence, as expected, the image of $\text{ad}_{\mathfrak{q}}$ lies in $\mathfrak{so}(\mathfrak{q})$. Moreover, note that, according to Proposition 1.7.6 of [Dix77], the Lie subalgebra \mathfrak{h} is reductive in \mathfrak{g} .

Consider the Clifford algebra $\mathbf{C}(\mathfrak{q}) := \mathbf{C}(\mathfrak{q}, B_{\mathfrak{q}})$ and let $S := S_{\mathfrak{q}}$ be a space of spinors for $\mathbf{C}(\mathfrak{q})$, while $\varphi : \mathfrak{so}(\mathfrak{q}) \rightarrow \text{End}(S)$ is the spin representation 2.8. Then the composition map

$$\mathfrak{h} \xrightarrow{\text{ad}} \mathfrak{so}(\mathfrak{q}) \xrightarrow{\varphi} \mathbf{C}(\mathfrak{q}) \xrightarrow{\gamma} \text{End}(S) \quad (2.11)$$

equips S with a structure of Lie algebra representation of \mathfrak{h} . In fact, all involved maps are Lie algebras morphisms and so is the case for their composition.

Definition 2.4.1 (Spin representation of \mathfrak{h}). *The Lie algebra representation of \mathfrak{h} on S defined above is called spin representation of \mathfrak{h} .*

Let $\mathfrak{q}^+, \mathfrak{q}^-$ be two maximal dual isotropic subspaces of \mathfrak{q} and $\{e_i\}, \{e_{-i}\}$ be bases for \mathfrak{q}^+ and \mathfrak{q}^- , respectively. If \mathfrak{q} is even-dimensional, say of dimension $2n$, \mathfrak{q} is decomposed as $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^-$ while if \mathfrak{q} is odd-dimensional, say of dimension $2n + 1$, we consider a vector e_0 of \mathfrak{q} , orthogonal to $\mathfrak{q}^+ \oplus \mathfrak{q}^-$, such that $B(e_0, e_0) = 2$, and \mathfrak{q} is decomposed as $\mathfrak{q} = \mathfrak{q}^+ \oplus \mathfrak{q}^- \oplus \mathbb{C}\{e_0\}$. Thus the set $\{e_{-n}, \dots, e_{-1}, e_1, \dots, e_n\}$ in the even case and the set $\{e_{-n}, \dots, e_{-1}, e_0, e_1, \dots, e_n\}$ in the odd case form vector space bases for \mathfrak{q} . For i and j in $\{-n, \dots, n\}$, let $\Phi_{i,j}$ be the operator sending e_j to e_i and all the other basis vectors to 0. From now on, we will use the notation $\{e_{-n}, \dots, e_n\}$ or $\{-n, \dots, n\}$ and we will include or exclude e_0 and 0 depending on the parity of $\dim(\mathfrak{q})$.

Proposition 2.4.2. *For each X in \mathfrak{h} ,*

$$\text{ad}X = \frac{1}{2} \sum_{i,j} B(X, [e_j, e_{-i}]) R_{e_i, e_{-j}},$$

as an element of $\mathfrak{so}(\mathfrak{q})$.

Proof. Let X be an element of \mathfrak{h} and, for i, j in $\{-n, \dots, n\}$, let $\{(\text{ad}X)_{ij}\}$ be the entries of the matrix corresponding to $\text{ad}X$ with respect to the base $\{e_{-n}, \dots, e_n\}$, the (i, j) -entry being the one corresponding to the elements e_i and e_j . Then

$$\begin{aligned} (\text{ad}X)_{ij} &= B(\text{ad}X(e_j), e_{-i}) \\ &= B([X, e_j], e_{-i}) \\ &= B(X, [e_j, e_{-i}]) \end{aligned}$$

and

$$\begin{aligned}
 2\text{ad}X &= 2 \sum_{i,j} (\text{ad}X)_{ij} \Phi_{i,j} \\
 &= \sum_{i,j} B(X, [e_j, e_{-i}]) \Phi_{i,j} \\
 &\quad + \sum_{i,j} B(X, [e_j, e_{-i}]) \Phi_{i,j} \\
 &= \sum_{i,j} B(X, [e_j, e_{-i}]) \Phi_{i,j} \\
 &\quad + \sum_{i,j} B(X, [e_{-i}, e_j]) \Phi_{-j,-i} \\
 &= \sum_{i,j} B(X, [e_j, e_{-i}]) \Phi_{i,j} \\
 &\quad - \sum_{i,j} B(X, [e_j, e_{-i}]) \Phi_{-j,-i} \\
 &= \sum_{i,j} B(X, [e_j, e_{-i}]) \{ \Phi_{i,j} - \Phi_{-j,-i} \}.
 \end{aligned}$$

Hence

$$\text{ad}X = \frac{1}{2} \sum_{i,j} B(X, [e_j, e_{-i}]) \{ \Phi_{i,j} - \Phi_{-j,-i} \}.$$

On the other hand, one can check that $R_{e_i, e_{-j}} = \Phi_{i,j} - \Phi_{-j,-i}$ and so

$$\text{ad}X = \frac{1}{2} \sum_{i,j} B(X, [e_j, e_{-i}]) R_{e_i, e_{-j}},$$

as wanted. □

Corollary 2.4.3. *For each X in \mathfrak{h} ,*

$$\text{ad}X = \frac{1}{2} \sum_j R_{[X, e_j], e_{-j}}$$

as an element of $\mathfrak{so}(\mathfrak{q})$.

Proof. Let X be in \mathfrak{h} . Using Proposition 2.4.2, we obtain

$$\begin{aligned}
 \text{ad}X &= \frac{1}{2} \sum_{i,j} B(X, [e_j, e_{-i}]) R_{e_i, e_{-j}} \\
 &= \frac{1}{2} \sum_{i,j} B([X, e_j], e_{-i}) R_{e_i, e_{-j}} \\
 &= \frac{1}{2} \sum_j \sum_i B([X, e_j], e_{-i}) R_{e_i, e_{-j}} \\
 &= \frac{1}{2} \sum_j R_{\sum_i B([X, e_j], e_{-i}) e_i, e_{-j}} \\
 &= \frac{1}{2} \sum_j R_{[X, e_j], e_{-j}}.
 \end{aligned}$$

□

Corollary 2.4.4. *Let ϕ be the map 2.8. For each X in \mathfrak{h} ,*

$$\varphi(\text{ad}X) = \frac{1}{4} \sum_j [\gamma([X, e_j]), \gamma(e_{-j})].$$

Proof. It suffices to apply φ to the equality of the Corollary 2.4.3. □

2.5 Weights of spin modules

Let $\mathfrak{t}_{\mathfrak{h}}$ be a Cartan subalgebra of \mathfrak{h} and $\mathfrak{t}_{\mathfrak{q}}$ be a subspace of \mathfrak{q} such that $\mathfrak{t} := \mathfrak{t}_{\mathfrak{h}} + \mathfrak{t}_{\mathfrak{q}}$ is a Cartan subalgebra of \mathfrak{g} . This is always possible thanks to Proposition 1.7.6 of [Dix77] which ensures that \mathfrak{h} is reductive in \mathfrak{g} . Then $\mathfrak{t}_{\mathfrak{h}}$ is abelian and acts by (simultaneously) diagonalizable operators on \mathfrak{g} . To extend $\mathfrak{t}_{\mathfrak{h}}$ to a Cartan subalgebra of \mathfrak{g} , it suffices to choose $\mathfrak{t} \supseteq \mathfrak{t}_{\mathfrak{h}}$ to be a subalgebra of \mathfrak{g} maximal with respect to these properties. Then, if $H = H_1 + H_2$, $H_1 \in \mathfrak{h}$, $H_2 \in \mathfrak{q}$, is an element of the Cartan subalgebra \mathfrak{t} , one can verify that H_1 belongs to $\mathfrak{t}_{\mathfrak{h}}$. In other words, \mathfrak{t} is of the form $\mathfrak{t}_{\mathfrak{h}} + \mathfrak{t}_{\mathfrak{q}}$ with $\mathfrak{t}_{\mathfrak{q}}$ being a subspace of \mathfrak{q} .

Let $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ be the root system of \mathfrak{g} with respect to \mathfrak{t} and set

$$\Delta_0 := \{\alpha \in \Delta \mid \alpha|_{\mathfrak{t}_{\mathfrak{h}}} = 0\},$$

i.e. the set of the roots that vanish on $\mathfrak{t}_{\mathfrak{h}}$. Note that Δ_0 can be empty (see [MZ14b, Lemma 2.4]). Let H_0 be a vector of $\mathfrak{t}_{\mathfrak{q}}$ such that $\alpha(H_0) \neq 0$ for every α in $\Delta \setminus \Delta_0$. The remark (3) following Theorem 1.4.22 ensures that we can take H_0 in such a way that $\text{ad}H_0$ has only real eigenvalues on \mathfrak{q} .

Definition 2.5.1. *A $\mathfrak{t}_{\mathfrak{h}}$ -weight β of \mathfrak{q} is said to be positive (respectively negative) if $\beta(H_0) > 0$ (respectively $\beta(H_0) < 0$) while we write $\beta > 0$ (respectively $\beta < 0$).*

Let β_1, \dots, β_k be the positive weights of \mathfrak{q} , repeated according to their multiplicities. From the \mathfrak{g} -invariance of B and since $B_{\mathfrak{q}} := B|_{\mathfrak{q} \times \mathfrak{q}}$ is nondegenerate, we deduce that the negative $\mathfrak{t}_{\mathfrak{h}}$ -weights are exactly $-\beta_1, \dots, -\beta_k$. In fact, β_j and $-\beta_j$ must be nondegenerately paired and thus their multiplicities must be equal. Let $(\mathfrak{q}^1)^+$ and $(\mathfrak{q}^1)^-$ be the subspaces of \mathfrak{q} corresponding to the positive and negative $\mathfrak{t}_{\mathfrak{h}}$ -weight subspaces respectively. Let $\{e_1, \dots, e_k\}$ and $\{e_{-1}, \dots, e_{-k}\}$ be bases of $(\mathfrak{q}^1)^+$ and $(\mathfrak{q}^1)^-$ respectively, such that the vector $e_{\pm i}$ corresponds to $\pm\beta_i$ and $B(e_i, e_{-j}) = \delta_{ij}$ for every i and j in $\{1, \dots, k\}$. Let

$$\mathfrak{q}^0 := \{X \in \mathfrak{q} \mid [\mathfrak{t}_{\mathfrak{h}}, X] = 0\},$$

i.e. the subspace of \mathfrak{q} consisting of the vectors that commute with $\mathfrak{t}_{\mathfrak{h}}$, and set $(\mathfrak{q}^0)^+$ and $(\mathfrak{q}^0)^-$ to be maximal isotropic dual subspaces of \mathfrak{q}^0 . For $(\mathfrak{q}^0)^+$ and $(\mathfrak{q}^0)^-$, consider bases $\{f_1, \dots, f_l\}$ and $\{f_{-1}, \dots, f_{-l}\}$ respectively, such that $B(f_i, f_{-j}) = \delta_{ij}$ for i and j in $\{1, \dots, l\}$. If necessary, consider an extra vector f_0 in \mathfrak{q}^0 , orthogonal to $(\mathfrak{q}^0)^+ \oplus (\mathfrak{q}^0)^-$. Then $\mathfrak{q}^0 = (\mathfrak{q}^0)^+ \oplus (\mathfrak{q}^0)^-$ or $\mathfrak{q}^0 = (\mathfrak{q}^0)^+ \oplus (\mathfrak{q}^0)^- \oplus \mathbb{C}\{f_0\}$, depending on $\dim \mathfrak{g}^0$. Set

$$\mathfrak{q}^+ := (\mathfrak{q}^0)^+ \oplus (\mathfrak{q}^1)^+$$

and

$$\mathfrak{q}^- := (\mathfrak{q}^0)^- \oplus (\mathfrak{q}^1)^-.$$

One can verify that \mathfrak{q}^+ and \mathfrak{q}^- are maximal isotropic dual subspaces of \mathfrak{q} while $\{f_1, \dots, f_l, e_1, \dots, e_k\}$ and $\{f_{-1}, \dots, f_{-l}, e_{-1}, \dots, e_{-k}\}$ are dual bases of \mathfrak{q}^+ and \mathfrak{q}^- , respectively.

Let H be a vector of $\mathfrak{t}_{\mathfrak{h}}$. According to Lemma 2.4.4 and the above discussion, one has

$$\begin{aligned} \varphi(\text{ad}H) &= \frac{1}{4} \sum_j [\gamma([H, e_j]), \gamma(e_{-j})] + \frac{1}{4} \sum_j [\gamma([H, f_j]), \gamma(f_{-j})] \\ &= \frac{1}{4} \sum_j [\gamma([H, e_j]), \gamma(e_{-j})] && \text{(since } [\mathfrak{t}_{\mathfrak{h}}, \mathfrak{q}^0] = 0 \text{)} \\ &= \frac{1}{4} \sum_{j>0} [\gamma([H, e_j]), \gamma(e_{-j})] + \frac{1}{4} \sum_{j<0} [\gamma([H, e_j]), \gamma(e_{-j})] \\ &= \frac{1}{4} \sum_{j>0} \beta_j(H) [\gamma(e_j), \gamma(e_{-j})] - \frac{1}{4} \sum_{j<0} \beta_j(H) [\gamma(e_j), \gamma(e_{-j})] \\ &= \frac{1}{4} \sum_{j>0} \beta_j(H) [\gamma(e_j), \gamma(e_{-j})] + \frac{1}{4} \sum_{j>0} \beta_j(H) [\gamma(e_j), \gamma(e_{-j})] \\ &= \frac{1}{2} \sum_{j>0} \beta_j(H) [\gamma(e_j), \gamma(e_{-j})] \\ &= \frac{1}{2} \sum_{j>0} \beta_j(H) \{ \gamma(e_j) \gamma(e_{-j}) - \gamma(e_{-j}) \gamma(e_j) \} \\ &= \sum_{j>0} \beta_j(H) \gamma(e_j) \gamma(e_{-j}) - \frac{1}{2} \sum_{j>0} \beta_j(H) \\ &= \sum_{j>0} \beta_j(H) \{ \gamma(e_j) \gamma(e_{-j}) - \frac{1}{2} \} \end{aligned}$$

Definition 2.5.2. Let $I = (i_1, \dots, i_s)$ and $J = (j_1, \dots, j_t)$, with $1 \leq s \leq k$, $1 \leq t \leq l$, $1 \leq i_1 < \dots < i_s \leq k$ and $1 \leq j_1 < \dots < j_t \leq l$. We define $u_{I,J}$ to be the element

$$u_{I,J} := e_{-i_1} \wedge \dots \wedge e_{-i_s} \wedge f_{-j_1} \wedge \dots \wedge f_{-j_t}$$

of S .

Proposition 2.5.3. Let ρ and $\rho_{\mathfrak{h}}$ be the half sum of positive roots of Δ^+ and $\Delta_{\mathfrak{h}}^+$ respectively. With the above notation, $u_{I,J}$ is a $\mathfrak{t}_{\mathfrak{h}}$ -weight vector of weight

$$\rho - \rho_{\mathfrak{h}} - \sum_{i \in I} \beta_i = \frac{1}{2} \left\{ \sum_{i \notin I} \beta_i - \sum_{i \in I} \beta_i \right\},$$

where, by abuse of notation, we write $i \in I$ if $i \in \{i_1, \dots, i_s\}$.

Proof. Let $u_{I,J}$ be an element of S as above. For each H in $\mathfrak{t}_{\mathfrak{h}}$,

$$\varphi(\text{ad}H)(u_{I,J}) = \sum_{i>0} \beta_i(H) \left\{ \gamma(e_i) \gamma(e_{-i}) - \frac{1}{2} \right\} u_{I,J},$$

One can verify that for each i , $\{\gamma(e_i) \gamma(e_{-i}) - \frac{1}{2}\}$ acts either by $\frac{1}{2}$, if i does not belong to I , or by $-\frac{1}{2}$ otherwise. Hence

$$\begin{aligned} \varphi(\text{ad}H)(u_{I,J}) &= \sum_{i \notin I} \beta_i(H) \left\{ \gamma(e_i) \gamma(e_{-i}) - \frac{1}{2} \right\} u_{I,J} + \sum_{i \in I} \beta_i(H) \left\{ \gamma(e_i) \gamma(e_{-i}) - \frac{1}{2} \right\} u_{I,J} \\ &= \frac{1}{2} \sum_{i \notin I} \beta_i(H) u_{I,J} - \frac{1}{2} \sum_{i \in I} \beta_i(H) u_{I,J} \\ &= \frac{1}{2} \left\{ \sum_{i \notin I} \beta_i - \sum_{i \in I} \beta_i \right\} (H) u_{I,J} \\ &= \left(\rho - \rho_{\mathfrak{h}} - \sum_{i \in I} \beta_i \right) (H) u_{I,J} \end{aligned}$$

and $u_{I,J}$ is a weight vector of weight $\rho - \rho_{\mathfrak{h}} - \sum_{i \in I} \beta_i = \frac{1}{2} \left\{ \sum_{i \notin I} \beta_i - \sum_{i \in I} \beta_i \right\}$. □

CHAPTER 3

KOSTANT'S CUBIC DIRAC OPERATOR

In this chapter we discuss the basic theory of cubic Dirac operators. More precisely, in Section 3.1, we discuss the basic properties of cubic Dirac operators, while in Sections 3.2 and 3.3 we develop the main results for cubic Dirac operators acting on spin modules twisted by finite-dimensional modules and (\mathfrak{g}, K) -modules respectively.

3.1 Definition and basic properties

We continue with the same notation. Namely, \mathfrak{g} is a complex semisimple Lie algebra, θ is a Cartan involution of \mathfrak{g} and \mathfrak{h} is a Lie subalgebra of \mathfrak{g} such that condition (2.10) holds, i.e. \mathfrak{h} is reductive, θ -stable and the restriction $B|_{\mathfrak{h} \times \mathfrak{h}}$ of the Killing form B of \mathfrak{g} to \mathfrak{h} is nondegenerate. Then, as before, \mathfrak{g} can be decomposed into $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, where $\mathfrak{q} := \mathfrak{h}^\perp$ with respect to the Killing form B . We consider the Clifford algebra $\mathbf{C}(\mathfrak{q})$ of \mathfrak{q} and a space of spinors $S := S_{\mathfrak{q}}$ for $\mathbf{C}(\mathfrak{q})$ equipped with the $U(\mathfrak{h})$ -module structure induced by the map (2.11).

Definition 3.1.1 (Kostant's Cubic Dirac operator, see [Kos99]). *We define Kostant's cubic Dirac operator D to be the element of $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$ given by:*

$$D := \sqrt{2} \left\{ \sum_{i=1}^l B(\tilde{e}_i, \tilde{e}_j) e_i \otimes \gamma(e_j) - 1 \otimes \gamma(c) \right\} \in U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q}),$$

where $\{\tilde{e}_i\}$ and $\{e_i\}$ are dual bases of \mathfrak{q} and $\gamma(c)$ is the image of the 3-exterior form

$$(X, Y, Z) \in \mathfrak{q} \times \mathfrak{q} \times \mathfrak{q} \mapsto B(X, [Y, Z])$$

via the Chevalley map (2.1.6). In other words, if $\{e_i\}$ and $\{\tilde{e}_i\}$ are dual spaces of \mathfrak{q} with respect to B , then

$$\gamma(c) = \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) \gamma(e_i) \gamma(e_j) \gamma(e_k). \quad (3.1)$$

Remark 3.1.2. The constant $\sqrt{2}$ in Definition 3.1.1 is related to the convention that we made in order to define Clifford algebras (see Remark 2.1.2) and is used to simplify the formula for D^2 .

Definition 3.1.3 (Cubic term). *The element $1 \otimes \gamma(c)$ (or by abuse of notation the element $\gamma(c)$) is called the cubic term of D .*

Lemma 3.1.4. *The operators D and $1 \otimes \gamma(c)$ do not depend on the choice of the bases $\{\tilde{e}_i\}$ and $\{e_i\}$.*

Proof. Let $\{\tilde{e}'_s\}$ and $\{e'_t\}$ be dual bases of \mathfrak{q} and let $A = (a_{ij})$ and $B = (b_{ij})$ be the transition matrices from $\{\tilde{e}'_s\}$ to $\{\tilde{e}_s\}$ and $\{e'_t\}$ to $\{e_t\}$, respectively. Then, due to the duality of the bases involved, A and B^t are inverses of each other. Thus

$$\begin{aligned} \sum_{s,t} B(\tilde{e}'_s, \tilde{e}'_t) e'_s \otimes \gamma(e'_t) &= \sum_{s,t,i,j,k,l} a_{si} a_{tj} b_{sk} b_{tl} B(\tilde{e}_i, \tilde{e}_j) e_k \otimes \gamma(e_l) \\ &= \sum_{i,j,k,l} \left(\sum_s a_{si} b_{sk} \right) \left(\sum_t a_{tj} b_{tl} \right) B(\tilde{e}_i, \tilde{e}_j) e_k \otimes \gamma(e_l) \\ &= \sum_{i,j,k,l} \delta_{ik} \delta_{jl} B(\tilde{e}_i, \tilde{e}_j) e_k \otimes \gamma(e_l) \\ &= \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) e_i \otimes \gamma(e_j), \end{aligned}$$

In the same way, one shows that

$$\begin{aligned} \gamma(c') &:= \frac{1}{6} \sum_{i,j,k} B(\tilde{e}'_s, [\tilde{e}'_t, \tilde{e}'_l]) \gamma(e'_s) \gamma(e'_t) \gamma(e'_l) \\ &= \gamma(c), \end{aligned}$$

and thus $1 \otimes \gamma(c)$ does not depend on the choice of the bases. Adding the two equalities, one obtains that

$$D' := \sum_{s,t} B(\tilde{e}'_s, \tilde{e}'_t) e'_s \otimes \gamma(e'_t) - 1 \otimes \sum_{i < j < k} B(\tilde{e}'_s, [\tilde{e}'_t, \tilde{e}'_l]) \gamma(e'_s) \gamma(e'_t) \gamma(e'_l) = D.$$

□

We need to embed the Lie algebra \mathfrak{h} into $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$. To do so, we use the Lie algebra morphism (2.11) to define the following map

Definition 3.1.5 (Diagonal embedding of \mathfrak{h}). *The map $(\cdot)_\Delta : \mathfrak{h} \rightarrow U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$ given by the formula*

$$\mathfrak{h} \ni X \mapsto X_\Delta := X \otimes 1 + 1 \otimes \varphi(\text{ad}X) \in U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$$

is called diagonal embedding of \mathfrak{h} in $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$.

Indeed, the map $(\cdot)_\Delta$, as its name indicates, embeds diagonally the Lie algebra \mathfrak{h} into $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$.

Proposition 3.1.6. *The map $(\cdot)_\Delta : \mathfrak{h} \rightarrow U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$ is a Lie algebra morphism.*

Proof. Let X_1 and X_2 be vectors of \mathfrak{h} . Then

$$\begin{aligned}
 [X_1, X_2]_\Delta &= [X_1, X_2] \otimes 1 + 1 \otimes (\varphi \circ \text{ad})[X_1, X_2] \\
 &= [X_1, X_2] \otimes 1 + 1 \otimes [(\varphi \circ \text{ad})X_1, (\varphi \circ \text{ad})X_2] \\
 &= X_1 X_2 \otimes 1 + 1 \otimes ((\varphi \circ \text{ad})X_1)((\varphi \circ \text{ad})X_2) \\
 &\quad - X_2 X_1 \otimes 1 - 1 \otimes ((\varphi \circ \text{ad})X_2)((\varphi \circ \text{ad})X_1) \\
 &= (X_1 \otimes 1 + 1 \otimes (\varphi \circ \text{ad})X_1)(X_2 \otimes 1 + 1 \otimes (\varphi \circ \text{ad})X_2) \\
 &\quad - (X_2 \otimes 1 + 1 \otimes (\varphi \circ \text{ad})X_2)(X_1 \otimes 1 + 1 \otimes (\varphi \circ \text{ad})X_1) \\
 &= (X_1)_\Delta (X_2)_\Delta - (X_2)_\Delta (X_1)_\Delta \\
 &= [(X_1)_\Delta, (X_2)_\Delta]
 \end{aligned}$$

and thus the map $(\cdot)_\Delta$ is a Lie algebra morphism. \square

As a consequence of the above lemma, $(\cdot)_\Delta$ can be extended to a unital Lie algebra morphism

$$(\cdot)_\Delta : U(\mathfrak{h}) \rightarrow U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q}).$$

Proposition 3.1.7. *The Lie algebra morphism $(\cdot)_\Delta : U(\mathfrak{h}) \rightarrow U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$ is injective.*

Proof. For $p \in \mathbb{N}$, let

$$(U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q}))_p := (U(\mathfrak{g}))_p \otimes \mathbf{C}(\mathfrak{q}),$$

where $(U(\mathfrak{g}))_p$ is the standard filtration of $U(\mathfrak{g})$, induced by Theorem 1.1.44. In the same way, we define a filtration for $U(\mathfrak{h})$. Then the map $(\cdot)_\Delta$ is compatible with these filtrations. If u is a monomial of the form (1.5) and of degree p , then

$$u_\Delta = u \otimes 1 + \text{terms of lower filtration degrees},$$

and the result follows. \square

3.2 Cubic Dirac operators for finite dimensional modules

Let (π, V) be a Lie algebra representation of \mathfrak{g} . Using the morphism $(\cdot)_\Delta$, $V \otimes S$ turns out to be a Lie algebra representation of \mathfrak{h} . Namely, for $u \in V$ and $v \in S$, we define the action to be

$$X(u \otimes v) := X_\Delta \cdot (u \otimes v) = (\pi(X)u) \otimes v + u \otimes (\varphi \circ \text{ad}(X)v), \quad X \in \mathfrak{h}. \quad (3.2)$$

Note that under this action, $V \otimes S$ is the tensor product representation defined by the representation V of \mathfrak{g} , which by restriction is a representation of \mathfrak{h} , and by the representation S of \mathfrak{h} . Moreover, the operator D acts on $V \otimes S$, with the first factor acting on V via π and the second one on S via the Clifford multiplication γ .

Proposition 3.2.1. *The operators D and $1 \otimes \gamma(c)$ of Definition 3.1.1 are \mathfrak{h} -equivariant.*

Proof. Let H be a connected, simply connected Lie group with Lie algebra \mathfrak{h} (see Theorem 1.1.22) and $\text{Ad}_q : \mathfrak{h} \rightarrow \text{Aut}(\mathfrak{q})$ be the restriction of the adjoint representation of H on \mathfrak{q} . Let X be an element of \mathfrak{h} , t a real number and $\{\tilde{e}_i\}$ and $\{e_i\}$ dual bases of \mathfrak{q} . Then

$$\begin{aligned} \gamma_t &:= \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) \gamma(\text{Ad}_q(\exp(tX))e_i) \gamma(\text{Ad}_q(\exp(tX))e_j) \gamma(\text{Ad}_q(\exp(tX))e_k) \\ &= \frac{1}{6} \sum_{i,j,k} B(\text{Ad}_q(\exp(tX))\tilde{e}_i, [\text{Ad}_q(\exp(tX))\tilde{e}_j, \text{Ad}_q(\exp(tX))\tilde{e}_k]) \\ &\quad \gamma(\text{Ad}_q(\exp(tX))e_i) \gamma(\text{Ad}_q(\exp(tX))e_j) \gamma(\text{Ad}_q(\exp(tX))e_k). \end{aligned}$$

and

$$\begin{aligned} D_t &:= \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) \pi(\text{Ad}_q(\exp(tX))e_i) \otimes \gamma(\text{Ad}_q(\exp(tX))e_j) - 1 \otimes \gamma_t \\ &= \sum_{i,j} B(\text{Ad}_q(\exp(tX))\tilde{e}_i, \text{Ad}_q(\exp(tX))\tilde{e}_j) \pi(\text{Ad}_q(\exp(tX))e_i) \otimes \gamma(\text{Ad}_q(\exp(tX))e_j) - 1 \otimes \gamma_t. \end{aligned}$$

$\{\text{Ad}_q(\exp(tX))\tilde{e}_i\}$ and $\{\text{Ad}_q(\exp(tX))e_i\}$ are dual bases of \mathfrak{q} and, according to Lemma 3.1.4, γ_t coincides with $\gamma(c) = \gamma_0$ and D_t coincides with $D = D_0$.

Hence

$$\begin{aligned} 0 &= \frac{d}{dt} \Big|_{t=0} \gamma_t \\ &= \frac{d}{dt} \Big|_{t=0} \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) \gamma(\text{Ad}_q(\exp(tX))e_i) \gamma(\text{Ad}_q(\exp(tX))e_j) \gamma(\text{Ad}_q(\exp(tX))e_k) \\ &= \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) \gamma([X, e_i]) \gamma(e_j) \gamma(e_k) \\ &\quad + \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) \gamma(e_i) \gamma([X, e_j]) \gamma(e_k) \\ &\quad + \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) \gamma(e_i) \gamma(e_j) \gamma([X, e_k]). \end{aligned}$$

On the other hand, Lemma 2.2.7 ensures that

$$\gamma([X, e_j]) = [\varphi(\text{ad}X), \gamma(e_j)] = \varphi(\text{ad}X)\gamma(e_j) - \gamma(e_j)\varphi(\text{ad}X) \quad (3.4)$$

and so

$$\begin{aligned}
0 &= \sum_{i < j < k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) (\varphi(\text{ad}X)\gamma(e_i) - \gamma(e_i)\varphi(\text{ad}X))\gamma(e_j)\gamma(e_k) \\
&\quad + \sum_{i < j < k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k])\gamma(e_i) (\varphi(\text{ad}X)\gamma(e_j) - \gamma(e_j)\varphi(\text{ad}X))\gamma(e_k) \\
&\quad + \sum_{i < j < k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k])\gamma(e_i)\gamma(e_j) (\varphi(\text{ad}X)\gamma(e_k) - \gamma(e_k)\varphi(\text{ad}X)) \\
&= \varphi(\text{ad}X) \circ \gamma(c) - \gamma(c) \circ \varphi(\text{ad}X).
\end{aligned}$$

Next

$$(\pi(X) \otimes 1) \circ (1 \otimes \gamma(c)) - (1 \otimes \gamma(c)) \circ (\pi(X) \otimes 1) = 0$$

and thus

$$(1 \otimes \gamma(c)) \circ X_\Delta - X_\Delta \circ (1 \otimes \gamma(c)) = 0, \quad \forall X \in \mathfrak{h}. \quad (3.4)$$

In the same way, one obtains that

$$\begin{aligned}
0 &= \frac{d}{dt}\Big|_{t=0} (D_t + \gamma_t) \\
&= \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) \pi\left(\frac{d}{dt}\Big|_{t=0} \text{Ad}_q(\exp(tX))e_i\right) \otimes \gamma(e_j) \\
&\quad + \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) \pi(e_i) \otimes \gamma\left(\frac{d}{dt}\Big|_{t=0} \text{Ad}_q(\exp(tX))e_j\right) \\
&= \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) \pi([X, e_i]) \otimes \gamma(e_j) + \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) \pi(e_i) \otimes \gamma([X, e_j]).
\end{aligned}$$

Replacing again by (3.3) and using the fact that

$$\pi([X, e_i]) = \pi(X)\pi(e_i) - \pi(e_i)\pi(X), \quad \forall X \in \mathfrak{h},$$

gives that

$$X_\Delta \circ (D + 1 \otimes \gamma(c)) - (D + 1 \otimes \gamma(c)) \circ X_\Delta = 0, \quad \forall X \in \mathfrak{h},$$

which, in combination with (3.4), gives:

$$X_\Delta \circ D - D \circ X_\Delta = 0, \quad \forall X \in \mathfrak{h}.$$

Thus, both D and $1 \otimes \gamma(c)$ are \mathfrak{h} -equivariant. \square

Corollary 3.2.2. *Both $\ker(D)$ and $\ker(1 \otimes \gamma(c))$ are representations of the Lie algebra \mathfrak{h} .*

Proof. As a consequence of Proposition 3.2.1, the kernel of the operators D and $1 \otimes \gamma(c)$ are stable by the action of \mathfrak{h} . Indeed, if x is a vector of $\ker(D)$, then

$$\begin{aligned}
\{D \circ X_\Delta\}(x) &= \{X_\Delta \circ D\}(x) \\
&= X_\Delta(Dx) \\
&= 0.
\end{aligned}$$

Therefore $\ker D$ is \mathfrak{h} -stable, i.e. a representation of \mathfrak{h} . The same argument applies for $1 \otimes \gamma(c)$. \square

Hence $\ker(D)$ and $\ker(1 \otimes \gamma(c))$ are representations of \mathfrak{h} and, \mathfrak{h} being reductive, Theorem 1.4.27 ensures that each can be decomposed into a direct sum of irreducible subrepresentations.

Let $\mathfrak{t}_{\mathfrak{h}}$ be a Cartan subalgebra of \mathfrak{h} . We extend $\mathfrak{t}_{\mathfrak{h}}$ to a Cartan subalgebra $\mathfrak{t} := \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$ of \mathfrak{g} , where $\mathfrak{t}_{\mathfrak{q}} \subseteq \mathfrak{q}$ (see Section 2.5), while we fix a positive system Δ^+ for $\Delta := \Delta(\mathfrak{g}, \mathfrak{h})$, choosing a lexicographic ordering considering $\mathfrak{t}_{\mathfrak{h}}$ before $\mathfrak{t}_{\mathfrak{q}}$. In other words, we fix a regular element $\xi \in \mathfrak{t}^*$, i.e. an element such that $\langle \xi, \alpha \rangle \neq 0$ for every $\alpha \in \Delta$, so that if $\alpha \in \Delta^+$ and $\alpha|_{\mathfrak{t}_{\mathfrak{h}}} \neq 0$, then $\langle \xi|_{\mathfrak{t}_{\mathfrak{h}}}, \alpha|_{\mathfrak{t}_{\mathfrak{h}}} \rangle \neq 0$.

Definition 3.2.3. *When a positive system $\Delta^+ \subseteq \Delta$ has been chosen with respect to a regular element $\xi \in \mathfrak{t}^*$ so that if $\alpha \in \Delta^+$ and $\alpha|_{\mathfrak{t}_{\mathfrak{h}}} \neq 0$, then $\langle \xi|_{\mathfrak{t}_{\mathfrak{h}}}, \alpha|_{\mathfrak{t}_{\mathfrak{h}}} \rangle \neq 0$, we will say that Δ^+ satisfies the $\mathfrak{t}_{\mathfrak{h}}$ -property.*

For more details, see [MZ14b]. The above ordering for Δ determines a positive system $\Delta_{\mathfrak{h}}^+$ for $\Delta_{\mathfrak{h}} := \Delta(\mathfrak{h}, \mathfrak{t}_{\mathfrak{h}})$ (see [MZ14b, Lemma 1.1]). More precisely, an element $\beta \in \Delta_{\mathfrak{h}}$ belongs to $\Delta_{\mathfrak{h}}^+$ if $\langle \beta, \xi|_{\mathfrak{t}_{\mathfrak{h}}} \rangle > 0$. Set $\rho_{\mathfrak{h}}$ to be the half sum of the positive $\mathfrak{t}_{\mathfrak{h}}$ -roots of $\Delta_{\mathfrak{h}}^+$. Let λ be a dominant integral element of $(\mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}})^*$ and set

$$W_{\lambda}^1 := \{w \in W \mid w(\lambda + \rho)|_{\mathfrak{t}_{\mathfrak{q}}} = 0 \text{ and } w(\lambda + \rho)|_{\mathfrak{t}_{\mathfrak{h}}} \text{ is } \Delta_{\mathfrak{h}}^+ \text{-dominant}\}. \quad (3.5)$$

Theorem 3.2.4 (See [HKP09, Theorem 4.2], [Kos99, Theorem 4.24] and [MZ14b, Theorem 2.5]). *Let V be the finite-dimensional irreducible representation of \mathfrak{g} of highest weight λ . The kernel $\ker(D)$ of D is decomposed into a sum of irreducible subrepresentations of \mathfrak{h} :*

$$\ker(D) = \bigoplus_{w \in W_{\lambda}^1} 2^d F_{w(\lambda + \rho) - \rho_{\mathfrak{h}}},$$

where $d := \lfloor \frac{\dim \mathfrak{t}_{\mathfrak{q}}}{2} \rfloor$ and F_{μ} is the irreducible representation of \mathfrak{h} of highest weight $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$.

Remark 3.2.5. One can verify that $F_{w(\lambda + \rho) - \rho_{\mathfrak{h}}}$ appears with multiplicity 2^d in $V \otimes S$. Thus the whole isotypic component belongs to $\ker(D)$.

The proof of the above theorem is based, as many other results for the cubic Dirac operator D , on the following crucial property of the square D^2 of D . Let $\Omega_{\mathfrak{g}}$ and $\Omega_{\mathfrak{h}}$ be the Casimir elements of \mathfrak{g} and \mathfrak{h} respectively (see Subsection 1.4.3). Set $\Omega_{\mathfrak{h}_{\Delta}} := (\Omega_{\mathfrak{h}})_{\Delta}$, i.e. the image of $\Omega_{\mathfrak{h}}$ via $(\cdot)_{\Delta}$.

Proposition 3.2.6 (See [Kos99, Theorem 2.16]). *The square of the cubic Dirac operator D is*

$$D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{h}_{\Delta}} + \|\rho\|^2 - \|\rho_{\mathfrak{h}}\|^2,$$

where ρ and $\rho_{\mathfrak{h}}$ stand for the half-sum of all positive roots of \mathfrak{g} and \mathfrak{h} , respectively, with respect to the positive root system Δ^+ .

Corollary 3.2.7. *Let $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$ be a weight of $V \otimes S$. The operator D^2 acts on the \mathfrak{h} -isotypic component $(V \otimes S)_{\mu}$ by the scalar*

$$\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2.$$

Proof. According to Theorem 3.2.6,

$$D^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{h}_\Delta} + \|\rho\|^2 - \|\rho_{\mathfrak{h}}\|^2.$$

Clearly the term $\Omega_{\mathfrak{g}} \otimes 1$ acts on the first factor of $V \otimes S$. According to Theorem 1.4.34, $\Omega_{\mathfrak{g}}$ acts on the irreducible representation V of highest weight λ by the scalar

$$\|\lambda + \rho\|^2 - \|\rho\|^2.$$

Since $\Omega_{\mathfrak{h}}$ is the Casimir element of \mathfrak{h} , again using Theorem 1.4.34, one has that $\Omega_{\mathfrak{h}}$ acts via $(\cdot)_{\Delta}$ on the isotypic component $(V \otimes S)_{\mu}$ by the scalar

$$\|\mu + \rho_{\mathfrak{h}}\|^2 - \|\rho_{\mathfrak{h}}\|^2.$$

Thus D^2 acts on $(V \otimes S)_{\mu}$ by the scalar of the statement. \square

Remark 3.2.8. Note that Corollary 3.2.7 is still valid in the case of an irreducible infinite-dimensional representation. Namely, if V is such a representation, say of infinitesimal character $\lambda + \rho \in \mathfrak{t}^*$, then the above argument applies in the same way and D^2 acts on the isotypic component of weight $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$ of $V \otimes S$ by the same scalar.

Let us suppose now that \mathfrak{g}_0 is a real form of \mathfrak{g} and

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

is its Cartan decomposition with respect to the Cartan involution θ . According to Lemma 3.1.4, the real subalgebra $\mathfrak{u}_0 := \mathfrak{k}_0 \oplus i\mathfrak{p}_0$ of $\mathfrak{g}^{\mathbb{R}}$ is a compact real form of \mathfrak{g} and $(\cdot, \cdot) := -B_{\mathfrak{u}_0}(\cdot, \cdot) = -B(\cdot, \cdot)|_{\mathfrak{u}_0 \times \mathfrak{u}_0}$, where $B_{\mathfrak{u}_0}$ is the Killing form of \mathfrak{u}_0 while $B|_{\mathfrak{u}_0 \times \mathfrak{u}_0}$ stands for the restriction of the Killing form B of \mathfrak{g} to \mathfrak{u}_0 , defines a inner product on \mathfrak{u}_0 . We extend (\cdot, \cdot) to be a Hermitian form on \mathfrak{g} , denoted again by (\cdot, \cdot) . One can check that

$$B(X, Y) = -(X, \theta(\bar{Y})), \forall X, Y \in \mathfrak{g},$$

where $\bar{\cdot}$ stands for conjugation with respect to \mathfrak{g}_0 . We note that $\theta(\bar{\cdot})$ is the conjugation with respect to the real form \mathfrak{u}_0 . We restrict these forms on \mathfrak{q} and we preserve the same notation for the restrictions. For $V_0 := \mathfrak{q} \cap \mathfrak{u}_0$, $V := \mathfrak{q}$ and $(\bar{\cdot}) := \theta(\bar{\cdot})$, we define $\{Z_j\}$, $\{u_{\pm j}\}$, U^{\pm} and S , as we did before Lemma 2.2.5. Then, according to Lemma 2.2.5, one has:

$$\gamma(X)^* = -\gamma(\theta(\bar{X})), \forall X \in \mathfrak{q}. \quad (3.6)$$

On the other hand, since V is finite dimensional, according to 1.5.14, we can find a Hermitian form $\langle \cdot, \cdot \rangle$ on V such that

$$(\pi(X))^* = -\pi(X), \forall X \in \mathfrak{u}_0.$$

In other words,

$$(\pi(X))^* = -\pi(\theta(\bar{X})), \forall X \in \mathfrak{g}.$$

Let $\langle \cdot, \cdot \rangle$ be the Hermitian form defined on $V \otimes S$ using the Hermitian forms of V and S .

Proposition 3.2.9. *The operators D and $1 \otimes \gamma(c)$ are both self-adjoint with respect to the Hermitian form $\langle \cdot, \cdot \rangle$.*

Proof. Let $\{W_j\}$ and $\{W'_j\}$ be orthonormal bases of $\mathfrak{q} \cap \mathfrak{k}_0$ and $\mathfrak{q} \cap i\mathfrak{p}_0$ respectively, with respect to (\cdot, \cdot) . Then, $\{Z_j := iW_j\}$ and $\{Z'_j := iW'_j\}$ are orthonormal basis of $\mathfrak{q} \cap \mathfrak{k}$ and $\mathfrak{q} \cap \mathfrak{p}$ respectively, with respect to the Killing form B of \mathfrak{g} . Then

$$D = \sqrt{2} \left\{ \sum \pi(Z_j) \otimes \gamma(Z_j) + \sum \pi(Z'_j) \otimes \gamma(Z'_j) - 1 \otimes \gamma(c) \right\},$$

where

$$\gamma(c) = \sum_{i < j < k} B(Z_i, [Z_j, Z_k]) \gamma(Z_i) \gamma(Z_j) \gamma(Z_k) + \sum_{j < k}^i B(Z_i, [Z'_j, Z'_k]) \gamma(Z_i) \gamma(Z'_j) \gamma(Z'_k).$$

We have

$$\begin{aligned} (\pi(Z_j) \otimes \gamma(Z_j))^* &= -\pi(\theta(\overline{Z_j})) \otimes \{-\gamma(\theta(\overline{Z_j}))\} \\ &= \pi(Z_j) \otimes \gamma(Z_j) \end{aligned}$$

and

$$\begin{aligned} (\pi(Z'_j) \otimes \gamma(Z'_j))^* &= -\pi(\theta(\overline{Z'_j})) \otimes \{-\gamma(\theta(\overline{Z'_j}))\} \\ &= \pi(Z'_j) \otimes \gamma(Z'_j). \end{aligned}$$

On the other hand,

$$\begin{aligned} (B(Z_i, [Z_j, Z_k]) \gamma(Z_i) \gamma(Z_j) \gamma(Z_k))^* &= (B(iW_i, [iW_j, iW_k]) \gamma(iW_i) \gamma(iW_j) \gamma(iW_k))^* \\ &= -\overline{B(W_i, [W_j, W_k])} \gamma(W_k)^* \gamma(W_j)^* \gamma(W_i)^* \\ &= -B(W_i, [W_j, W_k]) (-\gamma(\theta(\overline{W_k}))) (-\gamma(\theta(\overline{W_j}))) (-\gamma(\theta(\overline{W_i}))) \\ &= B(W_i, [W_j, W_k]) \gamma(W_k) \gamma(W_j) \gamma(W_i) \\ &= -B(W_i, [W_j, W_k]) \gamma(W_i) \gamma(W_j) \gamma(W_k) \\ &= B(Z_i, [Z_j, Z_k]) \gamma(Z_i) \gamma(Z_j) \gamma(Z_k) \end{aligned}$$

and

$$\begin{aligned} (B(Z_i, [Z'_j, Z'_k]) \gamma(Z_i) \gamma(Z'_j) \gamma(Z'_k))^* &= (B(iW_i, [iW'_j, iW'_k]) \gamma(iW_i) \gamma(iW'_j) \gamma(iW'_k))^* \\ &= -\overline{B(W_i, [W'_j, W'_k])} \gamma(W'_k)^* \gamma(W'_j)^* \gamma(W_i)^* \\ &= -B(W_i, [W'_j, W'_k]) (-\gamma(\theta(\overline{W'_k}))) (-\gamma(\theta(\overline{W'_j}))) (-\gamma(\theta(\overline{W_i}))) \\ &= B(W_i, [W'_j, W'_k]) \gamma(W'_k) \gamma(W'_j) \gamma(W_i) \\ &= -B(W_i, [W'_j, W'_k]) \gamma(W_i) \gamma(W'_j) \gamma(W'_k) \\ &= B(Z_i, [Z'_j, Z'_k]) \gamma(Z_i) \gamma(Z'_j) \gamma(Z'_k) \end{aligned}$$

Therefore

$$\begin{aligned}
 D^* &= \sum (\pi(Z_j) \otimes \gamma(Z_j))^* + \sum (\pi(Z'_j) \otimes \gamma(Z'_j))^* \\
 &\quad - \sum_{i < j < k} 1 \otimes (B(Z_i, [Z_j, Z_k])\gamma(Z_i)\gamma(Z_j)\gamma(Z_k))^* \\
 &\quad - \sum_{i, j < k} 1 \otimes (B(Z_i, [Z'_j, Z'_k])\gamma(Z_i)\gamma(Z'_j)\gamma(Z'_k))^* \\
 &= \sum \pi(Z_j) \otimes \gamma(Z_j) + \sum \pi(Z'_j) \otimes \gamma(Z'_j) \\
 &\quad - \sum_{i < j < k} 1 \otimes B(Z_i, [Z_j, Z_k])(1 \otimes \gamma(Z_i)\gamma(Z_j)\gamma(Z_k)) \\
 &\quad - \sum_{i, j < k} 1 \otimes B(Z_i, [Z'_j, Z'_k])(1 \otimes \gamma(Z_i)\gamma(Z'_j)\gamma(Z'_k)) \\
 &= D.
 \end{aligned}$$

Thus $D^* = D$, i.e. the operator D is self-adjoint. In the same way, we prove the result for $1 \otimes \gamma(c)$. \square

Corollary 3.2.10. *Since D and $\gamma(c)$ are self-adjoint, one has*

$$\begin{aligned}
 \ker(D) &= \ker(D^2), \\
 \ker(\gamma(c)) &= \ker(\gamma(c)^2).
 \end{aligned}$$

Proof. This is a standard result for self-adjoint operators. For the sake of completeness, we prove the statement. The inclusion $\ker(D) \subseteq \ker(D^2)$ is obvious; for the other one suppose that $x \in V \otimes S$ belongs to $\ker(D^2)$. Then

$$\begin{aligned}
 \|Dx\|^2 &= \langle Dx, Dx \rangle \\
 &= \langle x, D^*Dx \rangle \\
 &= \langle x, D^2x \rangle \\
 &= 0
 \end{aligned}$$

and thus $Dx = 0$. The same argument applies for $\gamma(c)$. \square

3.3 Cubic Dirac operators for (\mathfrak{g}, K) -modules

Let G be a Lie group of finite center and \mathfrak{g} be its complexified Lie algebra. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be a Cartan decomposition of \mathfrak{g} and K be the corresponding maximal compact subgroup of G with $\text{Lie}(K)^\mathbb{C} = \mathfrak{k}$ (see Theorem 1.3.10). Note that in the case of a Cartan decomposition, the condition (2.10) is always satisfied. Indeed, according to Lemma 1.5.8, \mathfrak{k} , being the Lie algebra of a compact Lie group, is reductive while, according to Lemma 1.3.5, the restriction of B on

\mathfrak{k} is nondegenerate. Consider the Clifford algebra $\mathbf{C}(\mathfrak{p})$ of \mathfrak{p} and let S be a space of spinors for $\mathbf{C}(\mathfrak{p})$.

Suppose that X is a (\mathfrak{g}, K) -module and consider the tensor product $X \otimes S$. Then, $X \otimes S$ admits an action of $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{p})$, given by:

$$(u \otimes e)(x \otimes s) := ux \otimes es \quad (3.7)$$

for every $u \in U(\mathfrak{g})$, $e \in \mathbf{C}(\mathfrak{p})$, $x \in X$ and $s \in S$. Moreover, we would like to have an action of K on $X \otimes S$. To do so, it suffices to define an action of K on S . Imitating what we did in the Lie algebra context (see Section 2.4), a first idea would be to use the adjoint map

$$\text{Ad} : K \rightarrow SO(\mathfrak{p}).$$

Nevertheless, it is the double cover $\text{Spin}(\mathfrak{p})$ of $SO(\mathfrak{p})$ rather than $SO(\mathfrak{p})$ that acts on S (see Section 2.3). Hence, instead of K , we have to consider a double cover \tilde{K} of K defined by

$$\tilde{K} := \{(k, s) \in K \times \text{Spin}(\mathfrak{p}) \mid p(s) = \text{Ad}(k)\},$$

where $p : \text{Spin}(\mathfrak{p}) \rightarrow SO(\mathfrak{p})$ stands for the covering map. In other words, \tilde{K} is the group defined by the commutative diagram

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{\text{pr}_2} & \text{Spin}(\mathfrak{p}) \\ \text{pr}_1 \downarrow & & \downarrow p \\ K & \xrightarrow{\text{Ad}} & SO(\mathfrak{p}) \end{array}$$

where pr_1 and pr_2 are the projections to the first and second, respectively, factor of \tilde{K} . Then \tilde{K} acts on $X \otimes S$, the action being through K on X and through $\text{Spin}(\mathfrak{p})$ on S . Furthermore, one can easily verify that the differential of this action is the diagonal action \cdot_{Δ} of \mathfrak{k}_0 on $X \otimes S$ (see Definition 3.1.5).

Moreover, $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{p})$ admits an action of \tilde{K} through the adjoint representation of K . Namely, if $\tilde{k} = (k, s) \in \tilde{K}$, then, for every $u \in U(\mathfrak{g})$ and $v \in V$,

$$\tilde{k}(u \otimes \gamma(v)) = (kuk^{-1}) \otimes (\Phi(\tilde{k})\gamma(v)\Phi(\tilde{k})^{-1}), \quad (3.8)$$

where $\Phi : \text{Spin}(\mathfrak{p}) \rightarrow \mathbf{C}(\mathfrak{p})$ stands for the representation (2.9) of $\text{Spin}(\mathfrak{p})$ on S .

Proposition 3.3.1. *For every $\tilde{k} = (k, s) \in \tilde{K}$ and every $v \in V$, one has*

$$\Phi(\tilde{k})\gamma(v)\Phi(\tilde{k})^{-1} = \gamma(\text{Ad}(k)v).$$

Proof. The equality is obtained by integrating the equality (2.7) of Proposition 2.2.7. □

Consider, now, the Dirac operator D of Definition 3.1.1, given by

$$D := \sqrt{2} \sum_{i=1}^l B(\tilde{e}_i, \tilde{e}_j) e_i \otimes \gamma(e_j) \in U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{p}),$$

where $\{\tilde{e}_i\}$ and $\{e_i\}$ are dual bases of \mathfrak{p} . Note that, since $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$ (see Lemma 1.3.4), the cubic term $\gamma(c)$ of (3.1) vanishes. Indeed, one can verify that the coefficients of $\gamma(c)$ are all zero.

Proposition 3.3.2. *The element D of $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{p})$ is \tilde{K} -invariant, i.e. it is fixed by the action (3.8) of \tilde{K} .*

Proof. Let $\{Z_i\}$ be an orthonormal basis of \mathfrak{p} , $\tilde{k} := (k, s)$ be an element of \tilde{K} and set $Z'_i := \text{Ad}(k)Z_i$ to be the elements of the orthogonal basis $\{Z'_i\}$ of \mathfrak{p} . Then, according to Lemma 3.1.4,

$$\begin{aligned} D &= \sum_i Z'_i \otimes \gamma(Z'_i) \\ &= \sum_i \text{Ad}(k)Z_i \otimes \gamma(\text{Ad}(k)Z_i) \end{aligned}$$

and, using Proposition 3.3.1, we obtain

$$\begin{aligned} D &= \sum_i \text{Ad}(k)Z_i \otimes (\Phi(\tilde{k})\gamma(Z_i)\Phi(\tilde{k})^{-1}) \\ &= \sum_i \tilde{k}(Z_i \otimes \gamma(Z_i)) \\ &= \tilde{k}D. \end{aligned}$$

□

Moreover, being an element of $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{p})$, D acts on $X \otimes S$ by the action (3.7) while, according to Proposition 3.3.2, as linear operator acting on $X \otimes S$, is \tilde{K} -equivariant. Therefore one obtains the following proposition.

Proposition 3.3.3. *Let D be the cubic Dirac operator acting on $X \otimes S$. Then both $\ker(D)$ and $\text{im}(D)$ are representations of \tilde{K} .*

Proof. It suffices to prove that $\ker(D)$ and $\text{im}(D)$ are \tilde{K} -stable. Let y be an element of $\ker(D)$ and \tilde{k} be an element of \tilde{K} . Then

$$D(\tilde{k}y) = \tilde{k}\{D(y)\} = 0$$

and thus $\tilde{k}y$ belongs to $\ker D$. On the other hand, if y belongs to $\text{im}(D)$, suppose that z is an element of $X \otimes S$ such that $D(z) = y$. Then

$$\tilde{k}y = \tilde{k}\{D(z)\} = D(\tilde{k}z)$$

and thus $\tilde{k}y$ belongs to $\text{im}(D)$, as well. □

Definition 3.3.4 (Dirac cohomology). *Let X be a (\mathfrak{g}, K) -module. We define the Dirac cohomology $H_D(X)$ of X to be the \tilde{K} -module*

$$H_D(X) = \ker(D) / \ker(D) \cap \text{im}(D).$$

Suppose, now, that X is an irreducible (\mathfrak{g}, K) -module. Then, according to Proposition 1.4.11, X admits an infinitesimal character. In the fall of 1997, in his series of talks [Vog97], David Vogan conjectured the following theorem, proved by J.-S. Huang and P. Pandžić in [HP02].

Theorem 3.3.5 (Vogan's Conjecture, Huang-Pandzic (see [HP02])). *Let X be an irreducible (\mathfrak{g}, K) -module. Assume that $H_D(X)$ is nonzero and let E_μ be a \tilde{K} -type of $H_D(X)$ with highest weight $\mu \in \mathfrak{t}_\mathfrak{k}^* \subseteq \mathfrak{t}^*$, i.e. a copy of the irreducible \tilde{K} -representation of highest weight μ , embedded in $H_D(X)$. Then, the infinitesimal character λ of X is W -conjugate to $\mu + \rho_\mathfrak{k}$, where $\rho_\mathfrak{k}$ stands for the half sum of the positive roots of \mathfrak{k} .*

Moreover, Dirac operators provides us with a test of nonunitarizability of (\mathfrak{g}, K) -modules.

Theorem 3.3.6 (Parthasarathy's Dirac inequality, see [HP06, Proposition 3.5.1] or [Par80b]). *Let X be an irreducible unitary (\mathfrak{g}, K) -module with infinitesimal character λ . Fix a Cartan subalgebra $\mathfrak{t}_\mathfrak{k}$ of \mathfrak{k} and let $\Delta_\mathfrak{k} := \Delta(\mathfrak{k}, \mathfrak{t}_\mathfrak{k})$ be the corresponding root system, $\Delta_\mathfrak{k}^+$ be a positive system of $\Delta_\mathfrak{k}$ and $\rho_\mathfrak{k}$ be the half sum of the positive roots of $\Delta_\mathfrak{k}^+$. If $\mu \in \mathfrak{t}_\mathfrak{k}^*$ is the highest weight of an irreducible representation of \tilde{K} such that the μ -isotypic component of $X \otimes S$ is not trivial, then*

$$\|\lambda\| \leq \|\mu + \rho_\mathfrak{k}\|.$$

The equality holds if and only if $\mu + \rho_\mathfrak{k}$ is W -conjugate to λ .

The proof of Theorem 3.3.6 is based on the following Proposition.

Proposition 3.3.7. *If X is an irreducible unitary (\mathfrak{g}, K) -module, then there is a Hermitian form on $X \otimes S$ such that the Dirac operator*

$$D : X \otimes S \rightarrow X \otimes S$$

is skew-adjoint.

Proof of Proposition 3.3.7. Let (π, X) be an irreducible unitary (\mathfrak{g}, K) -module. Then, there is a Hermitian form $\langle \cdot, \cdot \rangle_X$ on X such that

$$\pi(X)^* = -\pi(X)$$

for every $X \in \mathfrak{g}_0$, where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is a real form of \mathfrak{g} containing the Lie algebra \mathfrak{k}_0 of K and $\mathfrak{k}_0 \oplus \mathfrak{p}_0$ stands for the Cartan decomposition of \mathfrak{g}_0 . On the other hand, consider the Hermitian form $\langle \cdot, \cdot \rangle_S$ of S , constructed in (2.6), and define a Hermitian form (\cdot, \cdot) on $X \otimes S$ as the product of the Hermitian forms of X and S . Namely,

$$(x_1 \otimes s_1, x_2 \otimes s_2) := \langle x_1, x_2 \rangle_X \langle s_1, s_2 \rangle_S$$

for every $x_1, x_2 \in X$ and every $s_1, s_2 \in S$. Note that, according to (3.6), for every $X \in \mathfrak{p}$, one has

$$\gamma(X)^* = -\gamma(\theta(\bar{X})) = \gamma(\bar{X}).$$

Let $\{Z_i\}$ be an orthonormal basis of \mathfrak{p} with respect to the Killing form B of \mathfrak{g} and consider the Dirac operator

$$D = \sum_i \pi(Z_i) \otimes \gamma(Z_i).$$

Since B is positive definite on \mathfrak{p}_0 (see Lemma 1.3.5), we can suppose that the basis $\{Z_i\}$ has been chosen to be a subset of \mathfrak{p}_0 . Then, for every basis element Z_i , one has

$$\begin{aligned} (\pi(Z_i) \otimes \gamma(Z_i))^* &= \pi(Z_i)^* \otimes \gamma(Z_i)^* \\ &= -\pi(Z_i) \otimes \gamma(\overline{Z_i}) \\ &= -\pi(Z_i) \otimes \gamma(Z_i) \end{aligned}$$

and thus

$$\begin{aligned} D^* &= \sum_i (\pi(Z_i) \otimes \gamma(Z_i))^* \\ &= -\sum_i \pi(Z_i) \otimes \gamma(Z_i) \\ &= -D. \end{aligned}$$

□

Now, we can turn to the proof of Theorem 3.3.6.

Proof of Theorem 3.3.6. Let X be an irreducible unitary (\mathfrak{g}, K) -module. Consider the space $X \otimes S$ equipped with the Hermitian form of Proposition 3.3.7. Then, the operator

$$D : X \otimes S \rightarrow X \otimes S$$

is skew-adjoint with respect to this form and its eigenvalues are imaginary. Therefore, D^2 is a semi-negative definite operator, i.e. its eigenvalues are nonpositive. On the other hand, according to Corollary 3.2.7 and Remark 3.2.8, the eigenvalues of D^2 are of the form

$$\|\lambda\|^2 - \|\mu + \rho_{\mathfrak{k}}\|^2,$$

where $\mu \in \mathfrak{k}^*$ is such that the corresponding isotypic component of $X \otimes S$ is nontrivial. Therefore,

$$\|\lambda\|^2 \leq \|\mu + \rho_{\mathfrak{k}}\|^2$$

and the inequality follows.

The equality holds if and only if the μ -isotypic component of $X \otimes S$ belongs to the Dirac cohomology $H_D(X)$. According to Theorem 3.3.5, this is the case if and only if $\mu + \rho_{\mathfrak{k}}$ is W -conjugate to λ .

□

CHAPTER 4

NONCUBIC DIRAC OPERATORS

In this chapter we introduce and study the noncubic Dirac operator \widehat{D} . More precisely, we start by defining \widehat{D} and discussing its basic properties in Sections 4.1, 4.2 and 4.3. Then, in Section 4.4 we find a relation between the kernels of the operators D and \widehat{D} while in Section 4.5 we introduce Weyl's inequalities. Section 4.6 is devoted to studying the kernel $\ker(\widehat{D})$ of the operator \widehat{D} . More precisely, we start by developing a necessary condition so that an isotypic component of $V \otimes S$ belongs to $\ker(\widehat{D})$ which, when \mathfrak{h} is chosen to be a Cartan subalgebra of \mathfrak{g} , turns out to be sufficient as well. This gives us a complete description for $\ker(\widehat{D})$ in this case. In Section 4.7, we study a set $A(\lambda)$, describing $\ker(\widehat{D})$. Finally, in Section 4.8, we calculate $\ker(D^t)$ for t -cubic Dirac operators.

4.1 Definition and basic properties of noncubic Dirac operators

We continue using the same notation. Namely, \mathfrak{g} is a complex semisimple Lie algebra, B is the Killing form of \mathfrak{g} , θ is a Cartan involution of \mathfrak{g} , and Δ is the root system of \mathfrak{g} with respect to a Cartan subalgebra \mathfrak{t} of \mathfrak{g} . Suppose that \mathfrak{h} is a Lie subalgebra of \mathfrak{g} , satisfying (2.10), set $\mathfrak{q} := \mathfrak{h}^\perp$, i.e. the orthogonal complement of \mathfrak{h} with respect to B , and consider the Clifford algebra $\mathbf{C}(\mathfrak{q})$ of \mathfrak{q} and a space of spinors S for $\mathbf{C}(\mathfrak{q})$. In this context, we defined a cubic Dirac operator D (see Definition 3.1.1). Recall that D is the element of $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$ given by

$$D = \sqrt{2} \left\{ \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) e_i \otimes \gamma(e_j) - 1 \otimes \gamma(c) \right\}$$

with $\{\tilde{e}_i\}$ and $\{e_i\}$ being dual bases of \mathfrak{q} with respect to B and $\gamma(c)$ being the cubic term

$$\gamma(c) = \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) \gamma(e_i) \gamma(e_j) \gamma(e_k) \tag{4.1}$$

of D defined in (3.1).

Definition 4.1.1 (Noncubic Dirac operator). We define the noncubic Dirac operator \widehat{D} to be the element of $U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q})$ given by:

$$\widehat{D} = D + \sqrt{2}(1 \otimes \gamma(c)).$$

In other words, if $\{\tilde{e}_i\}$ and $\{e_i\}$ are dual bases of \mathfrak{q} , \widehat{D} is defined to be

$$\widehat{D} = \sqrt{2} \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) e_i \otimes \gamma(e_j).$$

Remark 4.1.2. Note that in the case where \mathfrak{h} is a symmetric Lie subalgebra, i.e. \mathfrak{h} is the set of fixed elements by an involution σ of \mathfrak{g} , then the cubic term $\gamma(c)$ is zero so that cubic and noncubic Dirac operators coincide. Indeed, in this case one can check that $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$ and thus all coefficients in (4.1) vanish.

As we did in the case of the cubic Dirac operator D , we can consider a finite-dimensional irreducible representation (π, V) of \mathfrak{g} and a space of spinors S for $\mathbf{C}(\mathfrak{q})$. Then \widehat{D} acts on $V \otimes S$ with the first factor acting on V and the second one acting on S .

Proposition 4.1.3. The operator \widehat{D} is independent of the choice of the bases, \mathfrak{h} -equivariant and self-adjoint with respect to the Hermitian form of Lemma 3.2.9.

Proof. According to Lemma 3.1.4 and Propositions 3.2.1 and 3.2.9, D and $1 \otimes \gamma(c)$ are respectively independent of the choice of the bases, \mathfrak{h} -equivariant and self-adjoint. Therefore this is the case for $\widehat{D} = D + 1 \otimes \gamma(c)$. \square

Corollary 4.1.4. The vector space $\ker(\widehat{D})$ is a representation of the Lie algebra \mathfrak{h} .

Proof. The proof of Corollary 3.2.2 for $\ker(D)$ and $\ker(1 \otimes \gamma(c))$ applies in the same way for $\ker(\widehat{D})$. \square

4.2 The square of \widehat{D}

For the rest of this chapter, we are interested in the kernel of the operator \widehat{D} . However, in contrast to D , \widehat{D}^2 has not the good property of D^2 , which is crucial for Theorem 3.2.4, describing the kernel of D . As a consequence, calculating the kernel of \widehat{D} , as we did in the case of D with Theorem 3.2.4, turns out to be much more difficult since we can not anymore take advantage of the "scalar" form of D^2 .

We continue with the notation fixed in Chapter 3. More precisely, let $\mathfrak{t}_{\mathfrak{h}}$ be a Cartan subalgebra of \mathfrak{h} , $\mathfrak{t}_{\mathfrak{q}}$ be a vector subspace of \mathfrak{q} such that $\mathfrak{t} := \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$ is a Cartan subalgebra of \mathfrak{g} (see Section 2.5), Δ^+ is a positive system for $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ satisfying the $\mathfrak{t}_{\mathfrak{h}}$ -property of Definition 3.2.3, and $\Delta_{\mathfrak{h}}^+$ is the corresponding positive system for $\Delta_{\mathfrak{h}} := \Delta(\mathfrak{h}, \mathfrak{t}_{\mathfrak{h}})$.

Proposition 4.2.1. Let $\{Z_i\}$ be an orthonormal basis of \mathfrak{q} , with respect to B . Then

$$\widehat{D}^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{h}_{\Delta}} + \|\rho\|^2 - \|\rho_{\mathfrak{h}}\|^2 + 2 \sum_{i < j} [Z_i, Z_j]_{\mathfrak{q}} \otimes \gamma(Z_i)\gamma(Z_j),$$

where $[\cdot, \cdot]_{\mathfrak{q}}$ stands for the orthogonal projection of the bracket onto \mathfrak{q} .

Proof. Let $\{Z_i\}$ be an orthonormal basis of \mathfrak{q} . Then

$$\widehat{D} = \sqrt{2} \sum_i Z_i \otimes \gamma(Z_i)$$

and

$$\begin{aligned} \widehat{D}^2 &= 2 \sum_{i,j} Z_i Z_j \otimes \gamma(Z_i) \gamma(Z_j) \\ &= 2 \sum_i Z_i^2 \otimes \gamma(Z_i)^2 + 2 \sum_{i \neq j} Z_i Z_j \otimes \gamma(Z_i) \gamma(Z_j) \\ &= \sum_i Z_i^2 \otimes 1 + 2 \sum_{i < j} Z_i Z_j \otimes \gamma(Z_i) \gamma(Z_j) - 2 \sum_{i < j} Z_j Z_i \otimes \gamma(Z_i) \gamma(Z_j) \\ &= \sum_i Z_i^2 \otimes 1 + 2 \sum_{i < j} [Z_i, Z_j] \otimes \gamma(Z_i) \gamma(Z_j). \end{aligned}$$

Moreover, if $\{W_k\}$ is an orthonormal basis of \mathfrak{h} with respect to B , then

$$\Omega_{\mathfrak{g}} = \sum_k W_k^2 + \sum_i Z_i^2$$

and

$$\Omega_{\mathfrak{h}_\Delta} = \sum_k [(W_k)_\Delta]^2 = \sum_k W_k^2 \otimes 1 + \sum_k 1 \otimes \varphi(\text{ad}W_k)^2 + 2 \sum_k W_k \otimes \varphi(\text{ad}W_k).$$

while, according to [HP06, Proposition 2.3.7],

$$\sum_k \varphi(\text{ad}W_k)^2 = \|\rho\|^2 - \|\rho_{\mathfrak{h}}\|^2.$$

Therefore

$$\widehat{D}^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{h}_\Delta} + \|\rho_{\mathfrak{g}}\|^2 - \|\rho_{\mathfrak{h}}\|^2 + 2 \sum_k W_k \otimes \varphi(\text{ad}W_k) + 2 \sum_{i < j} [Z_i, Z_j] \otimes \gamma(Z_i) \gamma(Z_j).$$

On the other hand, using Proposition 2.4.2, one obtains that

$$\begin{aligned} 2 \sum_k W_k \otimes \varphi(\text{ad}W_k) &= - \sum_{i,j,k} B(W_k, [Z_i, Z_j]) W_k \otimes \varphi(R_{Z_i, Z_j}) \\ &= - \frac{1}{2} \sum_{i,j,k} B(W_k, [Z_i, Z_j]) W_k \otimes [\gamma(Z_i), \gamma(Z_j)] \\ &= - \frac{1}{2} \sum_{i,j} [Z_i, Z_j]_{\mathfrak{h}} \otimes [\gamma(Z_i), \gamma(Z_j)] \\ &= - \sum_{i,j} [Z_i, Z_j]_{\mathfrak{h}} \otimes \gamma(Z_i) \gamma(Z_j) \\ &= - 2 \sum_{i < j} [Z_i, Z_j]_{\mathfrak{h}} \otimes \gamma(Z_i) \gamma(Z_j). \end{aligned}$$

Here $[\cdot, \cdot]_{\mathfrak{h}}$ stands for the orthogonal projection of the bracket $[\cdot, \cdot]$ onto \mathfrak{h} . Hence

$$\widehat{D}^2 = \Omega_{\mathfrak{g}} \otimes 1 - \Omega_{\mathfrak{h}_{\Delta}} + \|\rho\|^2 - \|\rho_{\mathfrak{h}}\|^2 + 2 \sum_{i < j} [Z_i, Z_j]_{\mathfrak{q}} \otimes \gamma(Z_i)\gamma(Z_j).$$

□

Remark 4.2.2. Note that if \mathfrak{h} is a symmetric subalgebra of \mathfrak{g} , the terms $[Z_i, Z_j]$ belong to \mathfrak{h} . Thus $[Z_i, Z_j]_{\mathfrak{q}} = 0$ and $\widehat{D}^2 = D^2$. Due to Remark 4.1.2, this is what one would expect.

Let us note that the square of the noncubic Dirac operator was calculated by Agricola who also established precise links between this noncubic operator and string theory in physics.

4.3 Some examples

In this section, we calculate the noncubic Dirac operator and its kernel for various cases. Let us start with some examples for the classical Lie algebra $\mathfrak{g} := \mathfrak{sl}(3, \mathbb{C})$ of Table 1.2. Choose

$$\mathfrak{t} := \langle H_{\varepsilon_1 - \varepsilon_2} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_{\varepsilon_2 - \varepsilon_3} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle$$

to be the Cartan subalgebra of \mathfrak{g} defined in Subsection 1.2.1. Then, if ε_i , $1 \leq i \leq 3$, is the linear functional mapping the element

$$H := \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix}$$

to h_i , the set of roots is

$$\Delta = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3)\}$$

with

$$\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$$

being a positive system. Moreover, according to Table 1.4, the Killing form of \mathfrak{g} is given by

$$B(X, Y) = 6\text{tr}(XY), \quad X, Y \in \mathfrak{g}.$$

For every $\alpha \in \Delta$, set

$$h_{\alpha} := \frac{\sqrt{3}}{6}H_{\alpha} \text{ and } e_{\alpha} := \frac{\sqrt{6}}{6}E_{\alpha},$$

with H_{α} and E_{α} being as in Subsection 1.2.1. Then, one can check that $B(h_{\alpha}, h_{\alpha}) = 1$ while

$$B(e_{\alpha}, e_{\beta}) = \begin{cases} 1 & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.3.1. Suppose that

$$\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2}.$$

Then

$$\mathfrak{q} := \mathfrak{h}^\perp = \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} \oplus \mathfrak{g}_{\varepsilon_2 - \varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2 + \varepsilon_3}$$

and one can check that $[\mathfrak{q}, \mathfrak{q}] \subseteq \mathfrak{h}$. As a consequence, $\gamma(c) = 0$, $D = \widehat{D}$ and thus $\ker(D) = \ker(\widehat{D})$ for every representation V of \mathfrak{g} .

Example 4.3.2. Suppose that $\mathfrak{h} := \mathfrak{t}$. Then

$$\mathfrak{q} := \mathfrak{h}^\perp = \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} \oplus \mathfrak{g}_{\varepsilon_2 - \varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2 + \varepsilon_3}$$

while the cubic term is

$$\gamma(c) = \frac{\sqrt{6}}{6} \gamma(e_{\varepsilon_1 - \varepsilon_3}) \gamma(e_{-\varepsilon_1 + \varepsilon_2}) \gamma(e_{-\varepsilon_2 + \varepsilon_3}) - \frac{\sqrt{6}}{6} \gamma(e_{-\varepsilon_1 + \varepsilon_3}) \gamma(e_{\varepsilon_1 - \varepsilon_2}) \gamma(e_{\varepsilon_2 - \varepsilon_3}).$$

Suppose that V is the standard representation of \mathfrak{g} , i.e. the irreducible representation of highest weight $\lambda = \bar{\omega}_1 = \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3$. According to Example 1.4.39, the weights of V are

$$\begin{aligned} \bar{\omega}_1 &= \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3 \\ \bar{\omega}_1 - (\varepsilon_1 - \varepsilon_2) &= -\frac{1}{3}\varepsilon_1 + \frac{2}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3 \\ \bar{\omega}_1 - (\varepsilon_1 - \varepsilon_3) &= -\frac{1}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 + \frac{2}{3}\varepsilon_3, \end{aligned}$$

each of multiplicity 1, and say v_1, v_2 and v_3 are nonzero weight vectors for each of these weights, respectively. Hence, as stated by Theorem 3.2.4, the kernel of the cubic Dirac operator D is

$$\begin{aligned} \ker(D) &= \langle v_1 \otimes 1, v_1 \otimes e_{-\varepsilon_2 + \varepsilon_3}, v_2 \otimes e_{-\varepsilon_1 + \varepsilon_2}, v_2 \otimes e_{-\varepsilon_1 + \varepsilon_2} \wedge e_{-\varepsilon_1 + \varepsilon_3}, \\ &\quad v_3 \otimes e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_2 + \varepsilon_3}, v_3 \otimes e_{-\varepsilon_1 + \varepsilon_2} \wedge e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_2 + \varepsilon_3} \rangle. \end{aligned}$$

On the other hand, the kernel of the noncubic Dirac operator coincides with $\ker(D)$ (see Proposition 4.7.5). Namely,

$$\begin{aligned} \ker(\widehat{D}) &= \langle v_1 \otimes 1, v_1 \otimes e_{-\varepsilon_2 + \varepsilon_3}, v_2 \otimes e_{-\varepsilon_1 + \varepsilon_2}, v_2 \otimes e_{-\varepsilon_1 + \varepsilon_2} \wedge e_{-\varepsilon_1 + \varepsilon_3}, \\ &\quad v_3 \otimes e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_2 + \varepsilon_3}, v_3 \otimes e_{-\varepsilon_1 + \varepsilon_2} \wedge e_{-\varepsilon_1 + \varepsilon_3} \wedge e_{-\varepsilon_2 + \varepsilon_3} \rangle \end{aligned}$$

Suppose now that \mathfrak{g} is the Lie algebra $\mathfrak{sp}(3, \mathbb{C})$ defined in Table 1.2. Let

$$\mathfrak{t} := \langle H_{2\varepsilon_1}, H_{2\varepsilon_2}, H_{2\varepsilon_3} \rangle,$$

be the Cartan subalgebra of \mathfrak{g} defined in Subsection 1.2.1, where

$$H_{2\varepsilon_1} := \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H_{2\varepsilon_2} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $H_{2\varepsilon_3} := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 \end{pmatrix}.$

If ε_i is the linear functional defined in Subsection 1.2.1, mapping the matrix $H_{2\varepsilon_j}$ to δ_{ij} , then the root system of \mathfrak{g} is

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq 3\} \cup \{\pm 2\varepsilon_i\}$$

and

$$\Delta^+ := \{(\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq 3\} \cup \{2\varepsilon_i\}$$

is chosen to be the positive system. Moreover, according to Table 1.4, the Killing form of \mathfrak{g} is

$$B(X, Y) = 8\text{tr}(XY), \quad X, Y \in \mathfrak{g}.$$

For i and j , $1 \leq i < j \leq 3$, set

$$h_{2\varepsilon_i} := \frac{1}{8}H_{2\varepsilon_i} \quad h_{\varepsilon_i \pm \varepsilon_j} := \frac{1}{4}H_{\varepsilon_i \pm \varepsilon_j}$$

$$e_{\pm 2\varepsilon_i} := \frac{\sqrt{2}}{4}E_{\pm 2\varepsilon_i} \quad e_{\varepsilon_i \pm \varepsilon_j} := \frac{1}{4}E_{\varepsilon_i \pm \varepsilon_j},$$

with H_α and E_α being as in Subsection 1.2.1. Then, one can easily check that, for every $\alpha, \beta \in \Delta^+$, $B(h_\alpha, h_\alpha) = 1$ while

$$B(e_\alpha, e_\beta) = \begin{cases} 1 & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.3.3. Let

$$\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{g}_{2\varepsilon_1} \oplus \mathfrak{g}_{2\varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_1 + \varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{-2\varepsilon_1} \oplus \mathfrak{g}_{-2\varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2}$$

and set

$$\begin{aligned} \mathfrak{q} := \mathfrak{h}^\perp = & \mathfrak{g}_{2\varepsilon_3} \oplus \mathfrak{g}_{\varepsilon_1+\varepsilon_3} \oplus \mathfrak{g}_{\varepsilon_1-\varepsilon_3} \oplus \mathfrak{g}_{\varepsilon_2+\varepsilon_3} \oplus \mathfrak{g}_{\varepsilon_2-\varepsilon_3} \\ & \oplus \mathfrak{g}_{-2\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1-\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1+\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2-\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2+\varepsilon_3}. \end{aligned}$$

Then, the cubic term is

$$\begin{aligned} \gamma(c) = & \frac{\sqrt{2}}{4} \gamma(e_{\varepsilon_1+\varepsilon_3}) \gamma(e_{-\varepsilon_1+\varepsilon_3}) \gamma(e_{-2\varepsilon_3}) + \frac{\sqrt{2}}{4} \gamma(e_{\varepsilon_2+\varepsilon_3}) \gamma(e_{-\varepsilon_2+\varepsilon_3}) \gamma(e_{-2\varepsilon_3}) \\ & - \frac{\sqrt{2}}{4} \gamma(e_{-\varepsilon_1-\varepsilon_3}) \gamma(e_{\varepsilon_1-\varepsilon_3}) \gamma(e_{2\varepsilon_3}) - \frac{\sqrt{2}}{4} \gamma(e_{-\varepsilon_2-\varepsilon_3}) \gamma(e_{\varepsilon_2-\varepsilon_3}) \gamma(e_{2\varepsilon_3}). \end{aligned}$$

Set

$$S := \bigwedge (\mathfrak{g}_{-2\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1-\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1+\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2-\varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_2+\varepsilon_3})$$

to be a space of spinors for $\mathbf{C}(\mathfrak{q})$ and suppose that V stands for the standard representation of \mathfrak{g} , i.e. the irreducible representation of highest weight $\lambda := \bar{\omega}_1 = \varepsilon_1$. Using 1.4.35, one can check that the dimension of V is 6, while, on the other hand the W -orbit of ε_1 is $\{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_3\}$. Since, according to Theorem 1.4.22, these elements are weights of V , we deduce that they are the only ones, each of multiplicity 1. Let $v_{\pm i}$, $1 \leq i \leq 3$, be a nontrivial weight vector of $\pm\varepsilon_i$. Then, using Theorem 3.2.4, one obtains

$$\begin{aligned} \ker(D) = & \langle v_1 \otimes 1, v_1 \otimes e_{-\varepsilon_2+\varepsilon_3}, v_1 \otimes e_{-2\varepsilon_3}, v_1 \otimes e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3}, \\ & v_1 \otimes e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-2\varepsilon_3}, v_1 \otimes e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-2\varepsilon_3}, \\ & v_2 \otimes 1, v_2 \otimes e_{-\varepsilon_1+\varepsilon_3}, v_2 \otimes e_{-2\varepsilon_3}, v_2 \otimes e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_3}, \\ & v_2 \otimes e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-2\varepsilon_3}, v_2 \otimes e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-2\varepsilon_3}, \\ & v_{-1} \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3}, v_{-1} \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3}, \\ & v_{-1} \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-2\varepsilon_3}, v_{-1} \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3}, \\ & v_{-1} \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-2\varepsilon_3}, \\ & v_{-1} \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-2\varepsilon_3}, \\ & v_{-2} \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3}, v_{-2} \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_3}, \\ & v_{-2} \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-2\varepsilon_3}, v_{-2} \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_3}, \\ & v_{-2} \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-2\varepsilon_3}, \\ & v_{-2} \otimes e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-2\varepsilon_3}, \\ & v_3 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3}, v_3 \otimes e_{-\varepsilon_1-\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-2\varepsilon_3} \rangle. \end{aligned}$$

One can check that the cubic term $1 \otimes \gamma(c)$ acts trivially on each vector of $\ker(D)$ and so this is the case for the operator $\widehat{D} = D + 1 \otimes \gamma(c)$. In other words, $\ker(D)$ is a subset of $\ker(\widehat{D})$. In particular, this inclusion is strict. Indeed, consider the vector $v_1 \otimes e_{-\varepsilon_2-\varepsilon_3}$ of $V \otimes S$. One can check that $\widehat{D}(v_1 \otimes e_{-\varepsilon_2-\varepsilon_3}) = 0$ while $1 \otimes \gamma(c)(v_1 \otimes e_{-\varepsilon_2-\varepsilon_3}) \neq 0$. Thus, $v_1 \otimes e_{-\varepsilon_2-\varepsilon_3}$ belongs to $\ker(\widehat{D})$ but not to $\ker(D)$.

4.4 Inclusion of the kernels

We continue with the same notation. As we have seen in the previous section, there are many possibilities for the relation of $\ker(\widehat{D})$ and $\ker(D)$. For the case where $\text{rank}_{\mathbb{C}}(\mathfrak{g}) = \text{rank}_{\mathbb{C}}(\mathfrak{h})$, one has the following proposition.

Theorem 4.4.1. *Let \mathfrak{g} be a complex semisimple Lie algebra and \mathfrak{h} a Lie subalgebra of \mathfrak{g} satisfying (2.10). Consider an irreducible finite-dimensional representation V of \mathfrak{g} and the corresponding cubic and noncubic Dirac operators*

$$\begin{aligned} D &: V \otimes S \rightarrow V \otimes S \\ \widehat{D} &: V \otimes S \rightarrow V \otimes S, \end{aligned}$$

respectively. If $\text{rank}_{\mathbb{C}}(\mathfrak{g}) = \text{rank}_{\mathbb{C}}(\mathfrak{h})$, then $\ker(D) \subseteq \ker(\widehat{D})$.

Proof. Let \mathfrak{t} be a common Cartan subalgebra of \mathfrak{g} and \mathfrak{h} , and let Δ and $\Delta_{\mathfrak{h}}$ be the root systems of \mathfrak{g} and \mathfrak{h} , respectively, with respect to \mathfrak{t} . Note that any positive system Δ^+ of Δ satisfies trivially the $\mathfrak{t}_{\mathfrak{h}}$ -property of Definition 3.2.3. Fix such a positive system and let $\Delta_{\mathfrak{h}}^+$ be a positive system for $\Delta_{\mathfrak{h}}$ such that $\Delta_{\mathfrak{h}}^+ \subset \Delta^+$. Let V be the finite-dimensional irreducible \mathfrak{g} -representation of highest weight λ and v_{λ} be the highest weight vector of V . Moreover, for every algebraically integral $\Delta_{\mathfrak{h}}^+$ -dominant element $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$, let F_{μ} be the corresponding finite-dimensional irreducible \mathfrak{h} -representation of highest weight μ . According to Theorem 3.2.4, we will have finished once we show that, for every w of W_{λ}^1 , the representation $F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}}$ belongs to $\ker(\widehat{D})$. Note that in the case of equal ranks, the set W_{λ}^1 defined in (3.5), coincides with

$$W^1 = \{w \in W \mid \Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+\}$$

and does not depend on λ .

First, we consider $F_{\lambda+\rho-\rho_{\mathfrak{h}}}$. Since $[\mathfrak{t}, \mathfrak{q}] \subset \mathfrak{q}$, \mathfrak{q} is decomposed into a direct sum of root spaces and set \mathfrak{q}^+ and \mathfrak{q}^- to be the direct sums corresponding to positive and negative roots respectively. Since the restriction of B on \mathfrak{q} is nondegenerate, \mathfrak{q}^+ and \mathfrak{q}^- are nondegenerately paired, i.e. if $\Delta^+(\mathfrak{q})$ and $\Delta^-(\mathfrak{q})$ are the sets of positive and negative respectively roots of \mathfrak{q} , then $\Delta^-(\mathfrak{q}) = -\Delta^+(\mathfrak{q})$. Moreover, let S be the spin module for $\mathbb{C}(\mathfrak{q})$, obtained by the negative roots of $\Delta^-(\mathfrak{q})$. Note that, according to Proposition 2.5.3, 1 is a maximal weight vector of S . Let $\{e_{\alpha}\}$ and $\{e_{-\alpha}\}$ be root vector bases of \mathfrak{q}^+ and \mathfrak{q}^- respectively such that $B(e_{\alpha}, e_{-\alpha}) = 1$. Then the noncubic Dirac operator is given by

$$\widehat{D} = \sqrt{2} \sum_{\alpha \in \Delta^+(\mathfrak{q})} e_{\alpha} \otimes \gamma(e_{-\alpha}) + \sqrt{2} \sum_{\alpha \in \Delta^+(\mathfrak{q})} e_{-\alpha} \otimes \gamma(e_{\alpha}).$$

Set

$$\widehat{D}_1 := \sqrt{2} \sum_{\alpha \in \Delta^+(\mathfrak{q})} e_{\alpha} \otimes \gamma(e_{-\alpha})$$

and

$$\widehat{D}_2 := \sqrt{2} \sum_{\alpha \in \Delta^+(\mathfrak{q})} e_{-\alpha} \otimes \gamma(e_{\alpha}).$$

Then \widehat{D}_1 acts trivially on the first factor of $v_\lambda \otimes 1$, \widehat{D}_2 acts trivially on the second one and thus \widehat{D} acts trivially on $v_\lambda \otimes 1$. Using the fact that $F_{\lambda+\rho-\rho_{\mathfrak{h}}}$ is irreducible and \widehat{D} is \mathfrak{h} -equivariant, one deduces that the whole representation $F_{\lambda+\rho-\rho_{\mathfrak{h}}}$, generated by $v_\lambda \otimes 1$ as $U(\mathfrak{h})$ -module, belongs to $\ker(\widehat{D})$.

To treat the general case of a subrepresentation $F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}}$ of $\ker(D)$, with w being always in W^1 , instead of Δ^+ , we choose $w\Delta^+$ to be the positive system of Δ . Note that by the definition of W^1 , $\Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+$. Then $w\lambda$ is the highest weight of V with respect to this choice while $w(\rho - \rho_{\mathfrak{h}}) = w\rho - \rho_{\mathfrak{h}}$ is a maximal weight of S with respect to $\Delta_{\mathfrak{h}}^+$. The last statement can be seen by using $w\Delta^+$ instead of Δ^+ in the choice of the space of spinors for $\mathbf{C}(\mathfrak{q})$. More precisely, we define S using again the negative roots of \mathfrak{q} but with respect to $w\Delta^+$ this time. Note that, due to Proposition 2.2.2, the space of spinors S does not depend on the choice of the positive system of Δ . Then, according to Proposition 2.5.3, $w\rho - \rho_{\mathfrak{h}}$ is a maximal weight. Therefore, if $v_{w\lambda}$ and $u_{w\rho-\rho_{\mathfrak{h}}}$ are the corresponding weight vectors of V and S respectively, for every $\alpha \in w\Delta^+(\mathfrak{q})$,

$$\begin{aligned}\pi(e_\alpha)v_{w\lambda} &= 0, \\ \gamma(e_\alpha)u_{w\rho-\rho_{\mathfrak{h}}} &= 0.\end{aligned}$$

Arguing as above, one deduces that $F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}}$ is contained in $\ker(\widehat{D})$. Since $w \in W_\lambda^1$ was arbitrary, from Theorem 3.2.4, we conclude that $\ker(D) \subseteq \ker(\widehat{D})$. \square

Corollary 4.4.2. *In the above setting, if $\text{rank}_{\mathbf{C}}(\mathfrak{g}) = \text{rank}_{\mathbf{C}}(\mathfrak{h})$, the cubic term $1 \otimes \gamma(c)$ acts by zero on $\ker(D)$.*

Proof. Let $x \in \ker(D)$. Then, according to Theorem 4.4.1, x belongs to $\ker(\widehat{D})$ and so $(1 \otimes \gamma(c))x = (\widehat{D} - D)x = 0$. \square

Corollary 4.4.3. *In the above setting, we have*

$$\ker(\widehat{D}) = \ker(D) \oplus \left[\ker(\widehat{D}) \cap \text{im}(D) \right].$$

Proof. Recall that, according to Lemma 3.2.9, D is a self-adjoint operator and thus $V \otimes S$ can be decomposed into

$$V \otimes S = \ker(D) \oplus \text{im}(D). \quad (4.2)$$

Thus the sum on the right-hand side of the statement is, indeed, direct. It suffices to show that

$$\ker(\widehat{D}) \subseteq \ker(D) \oplus \left[\ker(\widehat{D}) \cap \text{im}(D) \right],$$

the other inclusion being obvious. Let x be an element of $\ker(\widehat{D})$. From decomposition (4.2), there are $x_1 \in \ker(D)$ and $x_2 \in \text{im}(D)$ such that $x = x_1 + x_2$. From Theorem 4.4.1, x_1 belongs to $\ker(\widehat{D})$ and so this must be the case for x_2 . Therefore x belongs to $\ker(D) \oplus \left[\ker(\widehat{D}) \cap \text{im}(D) \right]$ and, since x is arbitrary, we obtain the desired inclusion. \square

The following example shows that the condition $\text{rank}_{\mathbf{C}}(\mathfrak{g}) = \text{rank}_{\mathbf{C}}(\mathfrak{h})$ in Theorem 4.4.1 can not be omitted.

Example 4.4.4. Let $\mathfrak{g} := \mathfrak{sl}(3, \mathbb{C})$ of Table 1.2. Choose

$$\mathfrak{t} := \langle H_{\varepsilon_1 - \varepsilon_2} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, H_{\varepsilon_2 - \varepsilon_3} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \rangle$$

to be the Cartan subalgebra of \mathfrak{g} of Subsection 1.2.1. Then, if ε_i is the linear functional defined in Subsection 1.2.1, the set of roots is

$$\Delta = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 - \varepsilon_3), \pm(\varepsilon_2 - \varepsilon_3)\},$$

with

$$\Delta^+ = \{\varepsilon_1 - \varepsilon_2, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3\}$$

being a positive system and $\rho = \varepsilon_1 - \varepsilon_3$ being the half sum of positive roots. Moreover, according to Table 1.4, the Killing form of \mathfrak{g} is given by

$$B(X, Y) = 6\text{tr}(XY), \quad X, Y \in \mathfrak{g}.$$

For every $\alpha \in \Delta$, set

$$h_\alpha := \frac{\sqrt{3}}{6}H_\alpha \text{ and } e_\alpha := \frac{\sqrt{6}}{6}E_\alpha,$$

with E_α being as in Subsection 1.2.1. Then, one can check that $B(h_\alpha, h_\alpha) = 1$ while

$$B(e_\alpha, e_\beta) = \begin{cases} 1 & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases}$$

Suppose that

$$\mathfrak{h} = \mathfrak{t}_\mathfrak{h} := \mathbb{C}\{5H_{\varepsilon_1 - \varepsilon_2} + 4H_{\varepsilon_2 - \varepsilon_3}\}.$$

Then

$$\mathfrak{q} := \mathfrak{h}^\perp = \mathbb{C}\{h_0\} \oplus \mathfrak{n} \oplus \mathfrak{n}^-,$$

where \mathfrak{n} (respectively \mathfrak{n}^-) stands for the sum of the positive (respectively negative) root spaces of \mathfrak{g} , while

$$h_0 := \frac{\sqrt{21}}{42}\{H_{\varepsilon_1 - \varepsilon_2} - 2H_{\varepsilon_2 - \varepsilon_3}\}.$$

Set $\mathfrak{t}_\mathfrak{q} := \mathbb{C}\{h_0\}$ and note that $\mathfrak{t}_\mathfrak{h} \oplus \mathfrak{t}_\mathfrak{q} = \mathfrak{t}$ and Δ^+ satisfies the $\mathfrak{t}_\mathfrak{h}$ -property. Let $S := \bigwedge \mathfrak{n}^-$ be a space of spinors for $\mathbb{C}(\mathfrak{q})$. Note that $B(h_0, h_0) = 1$ and $\gamma(h_0)$ acts either by $\frac{\sqrt{2}}{2}$ on S^+ and $-\frac{\sqrt{2}}{2}$ on S^- either by $-\frac{\sqrt{2}}{2}$ on S^+ and $\frac{\sqrt{2}}{2}$ on S^- (see Theorem 2.2.4). Let (π, V) be the standard representation of \mathfrak{g} , i.e. the irreducible representation of highest weight

$$\lambda := \frac{2}{3}\varepsilon_1 - \frac{1}{3}\varepsilon_2 - \frac{1}{3}\varepsilon_3.$$

Then, $(\lambda + \rho)(h_0) = 0$ and thus the identity element 1 of W belongs to the set W_λ^1 , defined in (3.5)). In other words, according to Theorem 3.2.4, the (1-dimensional) irreducible representation $F_{\lambda+\rho}$ of \mathfrak{h} of highest weight $\lambda + \rho$ belongs to $\ker D$. Moreover, if v_λ^+ stands for the highest

weight vector of V_λ , then $v_\lambda \otimes 1$ belongs to $F_{\lambda+\rho} \subseteq \ker(D)$. Nevertheless,

$$\begin{aligned} \widehat{D}[v_\lambda \otimes 1] &= \pi(h_0)v_\lambda^+ \otimes \gamma(h_0)1 + \sum_{\alpha \in \Delta} \pi(e_\alpha)v_\lambda^+ \otimes \gamma(e_\alpha)1 \\ &= \pi(h_0)v_\lambda^+ \otimes \gamma(h_0)1 \\ &= \pm \frac{\sqrt{2}}{2} \lambda(h_0)v_\lambda^+ \otimes 1 \\ &= \pm \frac{\sqrt{42}}{84} v_\lambda^+ \otimes 1 \neq 0. \end{aligned}$$

One can equivalently show that $\gamma(c)1 \neq 0$. Therefore, $v_\lambda^+ \otimes 1 \notin \ker \widehat{D}$ and thus $\ker D \not\subseteq \ker \widehat{D}$. In other words, if $\text{rank}_{\mathbb{C}}(\mathfrak{g}) \neq \text{rank}_{\mathbb{C}}(\mathfrak{h})$, the statement of Theorem 4.4.1 is not in general true.

The following example shows that the condition $\text{rank}_{\mathbb{C}}(\mathfrak{g}) = \text{rank}_{\mathbb{C}}(\mathfrak{h})$ in Theorem 4.4.1 is not necessary.

Example 4.4.5. Using the same notation as in the previous example, set

$$\mathfrak{h} := \mathbb{C}\{H_{\varepsilon_1 - \varepsilon_3}\} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_3}.$$

Then

$$\mathfrak{q} := \mathfrak{h}^\perp = \mathbb{C}\{h_0\} \oplus \mathfrak{g}_{\varepsilon_1 - \varepsilon_2} \oplus \mathfrak{g}_{\varepsilon_2 - \varepsilon_3} \oplus \mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_2 + \varepsilon_3},$$

where $h_0 := \frac{1}{6}(H_{\varepsilon_1 - \varepsilon_2} - H_{\varepsilon_2 - \varepsilon_3})$. Furthermore, $\mathfrak{t}_{\mathfrak{h}} := \mathbb{C}\{H_{\varepsilon_1 - \varepsilon_3}\}$ is a Cartan subalgebra of \mathfrak{h} and $\mathfrak{t}_{\mathfrak{q}} := \mathbb{C}\{h_0\}$ is such that $\mathfrak{t} = \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$. One can check that the choice of Δ^+ satisfies the $\mathfrak{t}_{\mathfrak{h}}$ -property of Definition 3.2.3, the corresponding positive system for $\Delta_{\mathfrak{h}}$ being $\Delta_{\mathfrak{h}}^+ := \{\varepsilon_1 - \varepsilon_3\}$. Moreover, let $S := \bigwedge(\mathfrak{g}_{-\varepsilon_1 + \varepsilon_2} \oplus \mathfrak{g}_{-\varepsilon_2 + \varepsilon_3})$ be a space of spinors for $\mathbb{C}(\mathfrak{q})$ and V be the adjoint representation of \mathfrak{g} , i.e. the irreducible representation of highest weight $\lambda := \varepsilon_1 - \varepsilon_3$. One can check that $W_\lambda^1 = \{1\}$ and thus, according to Theorem 3.2.4,

$$\ker(D) = F_{\lambda+\rho-\rho_{\mathfrak{h}}} = F_{\frac{3}{2}\varepsilon_1 - \frac{1}{3}\varepsilon_3},$$

i.e. $\ker(D)$ is isomorphic with the irreducible \mathfrak{h} -representation of highest weight $\frac{3}{2}\varepsilon_1 - \frac{1}{3}\varepsilon_3$, the highest weight vector of $\ker(D)$ being $E_{\varepsilon_1 - \varepsilon_3} \otimes 1$, where $E_{\varepsilon_1 - \varepsilon_3}$ is as in Subsection 1.2.1. On the other hand, the cubic term is

$$\begin{aligned} \gamma(c) &= -\frac{1}{2}\gamma(h_0)\gamma(e_{\varepsilon_1 - \varepsilon_2})\gamma(e_{-\varepsilon_1 + \varepsilon_2}) + \frac{1}{2}\gamma(h_0)\gamma(e_{-\varepsilon_1 + \varepsilon_2})\gamma(e_{\varepsilon_1 - \varepsilon_2}) \\ &\quad + \frac{1}{2}\gamma(h_0)\gamma(e_{\varepsilon_2 - \varepsilon_3})\gamma(e_{-\varepsilon_2 + \varepsilon_3}) - \frac{1}{2}\gamma(h_0)\gamma(e_{-\varepsilon_1 + \varepsilon_2})\gamma(e_{\varepsilon_2 - \varepsilon_3}) \end{aligned}$$

and

$$\gamma(c) \cdot 1 = -\frac{1}{2}\gamma(h_0) \cdot 1 + \frac{1}{2}\gamma(h_0) \cdot 1 = 0.$$

Hence

$$1 \otimes \gamma(c)(E_{\varepsilon_1 - \varepsilon_3} \otimes 1) = 0$$

and $E_{\varepsilon_1 - \varepsilon_3} \otimes 1 \in \ker(1 \otimes \gamma(c)) \cap \ker D \subseteq \ker \widehat{D}$. On the other hand, since \widehat{D} is \mathfrak{h} -equivariant, the whole representation $F_{\lambda+\rho-\rho_{\mathfrak{h}}}$ is included in $\ker \widehat{D}$ and thus $\ker D \subseteq \ker \widehat{D}$.

4.5 Weyl's inequalities

In this section, we introduce Weyl's inequalities which will be useful in order to overcome the "bad" form of \widehat{D}^2 and calculate $\ker(\widehat{D})$. The main references of this section are Tao's electronic notes [Tao10] and Brion's article [Bri15].

Let A and B be two Hermitian operators acting on a n -dimensional Hilbert space H . Then A and B have all their eigenvalues real. Arrange them in descending ordering $\alpha_1 \geq \dots \geq \alpha_n$ and $\beta_1 \geq \dots \geq \beta_n$, for A and B , respectively. A natural (but still open) question is to describe the eigenvalues of the operator sum $A + B$. Weyl's inequalities relates the eigenvalues $\gamma_1 \geq \dots \geq \gamma_n$ of $A + B$ to the eigenvalues α_i and β_i .

Proposition 4.5.1. *With A, B, α_i, β_i and γ_i as above, one has:*

$$\gamma_{i+j-1} \leq \alpha_i + \beta_j, \quad (4.3)$$

whenever the indices make sense.

Before the proof of the above proposition, we remark that if x is an element of H such that $\|x\| = 1$, then $\langle x, Ax \rangle \in [\alpha_n, \alpha_1]$. More precisely, using elementary multivariable calculus one can show that

$$\alpha_1 = \max_{\|x\|=1} \langle x, Ax \rangle \quad \text{and} \quad \alpha_n = \min_{\|x\|=1} \langle x, Ax \rangle. \quad (4.4)$$

These relations are special cases of the following lemma, known as minmax principle.

Lemma 4.5.2. *For A and $\alpha_1 \geq \dots \geq \alpha_n$ as above and for j in $\{1, \dots, n\}$,*

$$\alpha_j = \max_{\substack{V \subset H \\ \dim V = j}} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle = \min_{\substack{V \subset H \\ \dim V = n-j+1}} \max_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle.$$

Proof. Let u_1, \dots, u_n be eigenvectors of A corresponding to $\alpha_1 \geq \dots \geq \alpha_n$, respectively. For fixed $j \in \{1, \dots, n\}$, set W to be the $(n - j + 1)$ -dimensional vector subspace of H generated by $\{u_j, \dots, u_n\}$. If V is a j -dimensional vector subspace of H , the dimensions of V and W add up to $n + 1$ and so they have a nonzero intersection. Let y be a vector in this intersection with $\|y\| = 1$. Then $\langle y, Ay \rangle \in [\alpha_1, \alpha_j]$ and thus $\langle y, Ay \rangle \leq \alpha_j$ so that

$$\min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle \leq \alpha_j.$$

For V being the subspace generated by $\{u_1, \dots, u_j\}$, the above inequality holds as equality and thus

$$\max_{\substack{V \subset H \\ \dim V = j}} \min_{\substack{x \in V \\ \|x\|=1}} \langle x, Ax \rangle = \alpha_j.$$

The second equality is proved in a similar way. □

Now we are ready to prove Proposition 4.5.1.

Proof of Proposition 4.5.1. Let $\{u_s\}_{1 \leq s \leq n}$, $\{v_s\}_{1 \leq s \leq n}$ and $\{w_s\}_{1 \leq s \leq n}$ be eigenvectors of A , B and $A + B$ corresponding to the eigenvalues $\{\alpha_s\}_{1 \leq s \leq n}$, $\{\beta_s\}_{1 \leq s \leq n}$ and $\{\gamma_s\}_{1 \leq s \leq n}$ respectively. Let i and j be elements in $\{1, \dots, n\}$ such that $k := i + j - 1$ is also in $\{1, \dots, n\}$ and consider the subspaces spanned by $\{u_i, \dots, u_n\}$, $\{v_j, \dots, v_n\}$ and $\{w_1, \dots, w_k\}$. The dimensions of these subspaces add up to $2n + 1$ and thus there is a nonzero x in the intersection of these spaces. We can suppose that $\|x\| = 1$. Then, according to the above discussion,

$$\begin{aligned} \langle x, Ax \rangle &\in [\alpha_n, \alpha_i], \\ \langle x, Bx \rangle &\in [\beta_n, \beta_j], \\ \langle x, (A + B)x \rangle &\in [\gamma_k, \gamma_1]. \end{aligned} \tag{4.5}$$

Thus $\gamma_k \leq \langle x, (A + B)x \rangle = \langle x, Ax \rangle + \langle x, Bx \rangle \leq \alpha_i + \beta_j$ and the result follows. \square

4.6 Kernel of noncubic Dirac operators

Let \mathfrak{g}_0 be a real semisimple Lie algebra, \mathfrak{h}_0 be a subalgebra of \mathfrak{g}_0 satisfying conditions (2.10) and such that $\text{rank}_{\mathbb{C}}(\mathfrak{g}) = \text{rank}_{\mathbb{C}}(\mathfrak{h})$, and let \mathfrak{t}_0 be a common Cartan subalgebra of \mathfrak{h}_0 and \mathfrak{g}_0 . We omit the zero-indices to indicate complexification. Let λ be a dominant algebraically integral element of \mathfrak{t}^* , with respect to some fixed positive root system $\Delta^+ \subseteq \Delta := \Delta(\mathfrak{g}, \mathfrak{t})$, V be the corresponding finite-dimensional irreducible \mathfrak{g} -representation of highest weight λ and $\Pi(V)$ be the set of weights of V . Moreover, let $\Delta_{\mathfrak{h}} := \Delta(\mathfrak{h}, \mathfrak{t})$ be the roots system of \mathfrak{h} with respect to \mathfrak{t} while fix $\Delta_{\mathfrak{h}}^+$ to be a positive system such that $\Delta_{\mathfrak{h}}^+ \subseteq \Delta^+$. We consider the cubic and the noncubic Dirac operators D and \widehat{D} , respectively, associated to the above data. We are interested in finding a complete description of $\ker(\widehat{D})$.

4.6.1 A necessary condition

In this subsection the basic result is Theorem 4.6.8 which gives a necessary condition so as an \mathfrak{h} -isotypic component $(V \otimes S)_{\mu}$, $\mu \in \mathfrak{t}^*$ belongs to $\ker(\widehat{D})$. To do so, we use Weyl's inequalities introduced in Section 4.5. Recall that according to Propositions 3.2.9 and 4.1.3, the operators D , \widehat{D} and $1 \otimes \gamma(c)$ all are self-adjoint. For a Hermitian operator A acting on an n -dimensional Hilbert space, denote by $\lambda_i(A)$ the i -th eigenvalue of A in descending ordering and counting multiplicities. In other words, for A and B Hermitian, the inequality in Proposition 4.5.1 becomes

$$\lambda_{i+j-1}(A + B) \leq \lambda_i(A) + \lambda_j(B), \tag{4.6}$$

whenever the indices make sense.

Lemma 4.6.1. *For $i \in \{1, \dots, n\}$ with $n = \dim(V \otimes S)$, one has*

$$\lambda_i(D) - \|1 \otimes \gamma(c)\| \leq \lambda_i(\widehat{D}) \leq \lambda_i(D) + \|1 \otimes \gamma(c)\|. \tag{4.7}$$

Remark 4.6.2. The norm $\|\cdot\|$ stands for the operator norm.

Proof. In (4.6), set $A = D$, $B = 1 \otimes \gamma(c)$ and $j = 1$. Then

$$\begin{aligned} \lambda_i(\widehat{D}) &\leq \lambda_i(D) + \lambda_1(1 \otimes \gamma(c)) \\ \Rightarrow \lambda_i(\widehat{D}) &\leq \lambda_i(D) + \|1 \otimes \gamma(c)\|. \end{aligned}$$

The second inequality is obtained by exchanging the roles of D and \widehat{D} . Combining the two inequalities we obtain (4.7). □

Decompose $V \otimes S$ into \mathfrak{h} -isotypic components

$$V \otimes S = \bigoplus_{\mu \in \mathfrak{t}^*} (V \otimes S)_\mu.$$

The operators D , \widehat{D} and $1 \otimes \gamma(c)$, being \mathfrak{h} -equivariant, preserve every isotypic component $(V \otimes S)_\mu$. Let $\lambda_i^{(\mu)}(\cdot)$ be the i -th eigenvalue of the indicated Dirac operator (cubic or noncubic) on the component $(V \otimes S)_\mu$ and let $\|1 \otimes \gamma(c)\|^{(\mu)}$ be the operator norm of $1 \otimes \gamma(c)$ in the same component. Then the proof of Lemma 4.6.1 applies exactly in the same way for $(V \otimes S)_\mu$ and the inequality (4.7) becomes

$$\lambda_i^{(\mu)}(D) - \|1 \otimes \gamma(c)\|^{(\mu)} \leq \lambda_i^{(\mu)}(\widehat{D}) \leq \lambda_i^{(\mu)}(D) + \|1 \otimes \gamma(c)\|^{(\mu)}. \quad (4.8)$$

According to Corollary 3.2.7, on the μ -isotypic component of $V \otimes S$, D^2 acts by the scalar

$$\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2$$

with ρ being the half-sum of the positive roots in Δ^+ . Note that, according to Proposition 3.2.9, D is self-adjoint. Thus, its eigenvalues are real and the eigenvalues of D^2 are nonnegative. As a consequence, on the μ -isotypic component of $V \otimes S$, the eigenvalues $\lambda_i^{(\mu)}(D)$ of D are exactly

$$\pm \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2}.$$

Let us see what happens with the eigenvalues of $1 \otimes \gamma(c)$ on such an isotypic component. One can verify that the actions of $1 \otimes \gamma(c)$ and $-D$ coincide on the trivial representation of \mathfrak{g} tensorised with S . Consequently, using Corollary 3.2.7, we deduce that the eigenvalues of $\gamma(c)$ on the μ_1 -isotypic component S_{μ_1} of S are exactly

$$\pm \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2}.$$

Therefore the eigenvalues of $1 \otimes \gamma(c)$ on $(V \otimes S)_\mu$ are of this form, for μ_1 being a weight of S such that S_{μ_1} is involved in $(V \otimes S)_\mu$. In other words, μ_1 is such that $\mu - \mu_1$ is a weight of V .

Definition 4.6.3. [*Related weight*] When the weight $\mu_1 \in \mathfrak{t}^*$ of S satisfies the above condition, i.e. it is such that $\mu - \mu_1$ is a weight of V , we say that μ_1 is related to μ .

Theorem 4.6.4. *If $\mu \in \mathfrak{t}^*$ is a weight of $V \otimes S$ and $\mu_1 \in \mathfrak{t}^*$ is related to μ , then*

$$\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2 \geq \|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2. \quad (4.9)$$

The equality holds if and only if $\mu - \mu_1$ is an extremal weight of V and $\rho - w(\mu_1 + \rho_{\mathfrak{h}})$ is orthogonal to λ , with $w \in W$ being such that $w(\mu - \mu_1)$ is Δ^+ -dominant. In this case $\rho - w(\mu_1 + \rho_{\mathfrak{h}})$ can be written as a sum $\sum_{\alpha \in A} \alpha$, with $A \subseteq \Delta^+$ and $\langle \alpha, \lambda \rangle = 0$ for every α in A .

For the proof of Theorem 4.6.4, we will need the following lemma.

Lemma 4.6.5. *Let B be a subset of Δ^+ and w be an element of the Weyl group W . Then there is a subset A of Δ^+ such that*

$$w(\rho - \sum_{\beta \in B} \beta) = \rho - \sum_{\alpha \in A} \alpha.$$

Proof of Lemma 4.6.5. Let $\mathfrak{h}' := \mathfrak{t}$ and $\mathfrak{q}' := (\mathfrak{h}')^\perp$. Then $\mathfrak{q}' = \mathfrak{n} \oplus \mathfrak{n}^-$, where

$$\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha},$$

and let $S' := \bigwedge \mathfrak{n}^-$ be a space of spinors for $\mathbf{C}(\mathfrak{q})$. Moreover, let $\Delta_w^+ := w\Delta^+$ and \mathfrak{n}_w (respectively \mathfrak{n}_w^-) be the direct sum of positive (respectively negative) root spaces with respect to the positive system Δ_w^+ . Let $S'_w := \bigwedge \mathfrak{n}_w^-$ be the space of spinors of $\mathbf{C}(\mathfrak{q})$. Then $w\rho$ is the half sum of positive roots of Δ_w^+ , $w\rho_{\mathfrak{h}'} = 0$ and, according to Proposition 2.5.3, the element

$$w\rho - \sum_{\beta \in B} w\beta = w(\rho - \sum_{\beta \in B} \beta)$$

is a weight of S'_w . On the other hand, according to Theorem 2.2.2, S'_w and S' are isomorphic as $\mathbf{C}(\mathfrak{q})$ -modules and thus they have the same weights. As a consequence, using once more Proposition 2.5.3, we deduce that there is a subset A of Δ^+ such that

$$w(\rho - \sum_{\beta \in B} \beta) = \rho - \sum_{\alpha \in A} \alpha.$$

□

Proof of Theorem 4.6.4. Let μ and μ_1 as above. Then

$$\begin{aligned} \|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2 &\geq \|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2 \\ \Leftrightarrow \|\lambda + \rho\|^2 - \|\rho\|^2 &\geq \|\mu + \rho_{\mathfrak{h}}\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2 \\ \Leftrightarrow \|\lambda\|^2 + 2\langle \lambda, \rho \rangle &\geq \|\mu - \mu_1\|^2 + 2\langle \mu - \mu_1, \mu_1 + \rho_{\mathfrak{h}} \rangle \\ \Leftrightarrow (\|\lambda\|^2 - \|\mu - \mu_1\|^2) + 2\langle \lambda, \rho \rangle &\geq 2\langle \mu - \mu_1, \mu_1 + \rho_{\mathfrak{h}} \rangle. \end{aligned}$$

According to Proposition 2.5.3, there is a subset B of Δ^+ (more precisely of $\Delta^+ \setminus \Delta_{\mathfrak{h}}^+$) such that

$$\mu_1 = \rho - \rho_{\mathfrak{h}} - \sum_{\beta \in B} \beta$$

and thus

$$\mu_1 + \rho_{\mathfrak{h}} = \rho - \sum_{\beta \in B} \beta.$$

Let w be an element of W such that $w(\mu - \mu_1)$ is dominant. Using Lemma 4.6.5, one can find a subset A of Δ^+ such that

$$w(\mu_1 + \rho_{\mathfrak{h}}) = \rho - \sum_{\alpha \in A} \alpha.$$

Consequently

$$\begin{aligned} \langle \mu - \mu_1, \mu_1 + \rho_{\mathfrak{h}} \rangle &= \langle w(\mu - \mu_1), w(\mu_1 + \rho_{\mathfrak{h}}) \rangle \\ &= \langle w(\mu - \mu_1), \rho - \sum_{\alpha \in A} \alpha \rangle \\ &= \langle w(\mu - \mu_1), \rho \rangle - \sum_{\alpha \in A} \langle w(\mu - \mu_1), \alpha \rangle. \end{aligned}$$

Replacing in the above inequality to obtain

$$\begin{aligned} \Leftrightarrow \quad & (\|\lambda\|^2 - \|\mu - \mu_1\|^2) + 2\langle \lambda, \rho \rangle \geq 2\langle \mu - \mu_1, \mu_1 + \rho_{\mathfrak{h}} \rangle \\ & (\|\lambda\|^2 - \|\mu - \mu_1\|^2) + 2\langle \lambda, \rho \rangle \geq 2\langle w(\mu - \mu_1), \rho \rangle - 2 \sum_{\alpha \in A} \langle w(\mu - \mu_1), \alpha \rangle \\ \Leftrightarrow \quad & (\|\lambda\|^2 - \|\mu - \mu_1\|^2) + 2\langle \lambda - w(\mu - \mu_1), \rho \rangle \geq -2 \sum_{\alpha \in A} \langle w(\mu - \mu_1), \alpha \rangle. \end{aligned}$$

This is always true. More precisely, since $\mu - \mu_1$ is a weight of V and λ is extremal, we always have $\|\lambda\|^2 \geq \|\mu - \mu_1\|^2$. In addition, $w(\mu - \mu_1)$ is a weight of V and so $\lambda - w(\mu - \mu_1)$ can be written as a positive sum of positive roots. According to Proposition 1.4.21, ρ is dominant and so the left-hand side of the last inequality is always nonnegative. On the other hand, $w(\mu - \mu_1)$ is also dominant and so the right-hand side is always nonpositive.

The equality holds if and only if

$$(\|\lambda\|^2 - \|\mu - \mu_1\|^2) + 2\langle \lambda - w(\mu - \mu_1), \rho \rangle = 0,$$

which means that $\lambda = w(\mu - \mu_1)$ and thus $\mu - \mu_1$ is extremal, and

$$\langle w(\mu - \mu_1), \alpha \rangle = 0 \text{ for every } \alpha \text{ in } A,$$

which is equivalent to $\langle \lambda, \alpha \rangle = 0$ for every α in A . □

Remark 4.6.6. In (4.9) both sides are nonnegative. More precisely, if S' is the space of spinors of $\mathbf{C}(\mathfrak{n} \oplus \mathfrak{n}^-)$ defined in the proof of Lemma 4.6.5, then ρ and $\mu_1 + \rho_{\mathfrak{h}}$ are weights of S' with ρ being extremal. Thus $\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2 \geq 0$ and from Theorem 4.6.4 we obtain

$$\sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2} \geq \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2}, \quad (4.10)$$

with the equality holding exactly when the equality in (4.9) holds.

Definition 4.6.7. We say that the equality in (4.9) holds for a weight $\mu \in \mathfrak{t}^*$ of $V \otimes S$, if there is $\mu_1 \in \mathfrak{t}^*$ related to μ such that the equality in (4.9) holds. In this case, we will say that μ_1 is strongly related to μ .

Theorem 4.6.8. Let $\mu \in \mathfrak{t}^*$ be a weight of $V \otimes S$. If the equality in (4.9) does not hold for μ , then $(V \otimes S)_\mu$ does not contribute to $\ker(\widehat{D})$. In other words, the isotypic component $(V \otimes S)_\mu$ has nontrivial intersection with $\ker(\widehat{D})$ only if the equality in (4.9) holds for $(V \otimes S)_\mu$.

For the proof, we will use the following lemma.

Lemma 4.6.9. Let $\mu \in \mathfrak{t}^*$ be a weight of $V \otimes S$. If the equality in (4.9) does not hold for μ , then $(V \otimes S)_\mu$ does not contribute to $\ker(D)$.

Proof of Lemma 4.6.9. Suppose that $(V \otimes S)_\mu$ does contribute to $\ker(D)$ and thus there is a nonzero vector v of $(V \otimes S)_\mu$ such that $Dv = 0$. Then

$$\begin{aligned} D^2v &= 0 \\ \Rightarrow \{ \|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2 \} v &= 0 \\ \Rightarrow \|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2 &= 0. \end{aligned}$$

Since D^2 acts by the scalar $\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2$, i.e. trivially, on $(V \otimes S)_\mu$, the whole isotypic component lies in $\ker(D)$. On the other hand, from Theorem 4.4.1, $1 \otimes \gamma(c)$ also acts trivially on this component. In other words, $\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2 = 0$ for every μ_1 related to μ . Therefore, for such a μ both sides in (4.9) are zero and the equality holds, which is a contradiction. Consequently $(V \otimes S)_\mu$ does not contribute to $\ker(D)$. \square

Now, we can turn to the proof of Theorem 4.6.8.

Proof of Theorem 4.6.8. Let μ be a weight of $V \otimes S$ for which (4.9) is strict. Set n to be the dimension of the isotypic component $(V \otimes S)_\mu$ and let i be an element of $\{1, \dots, n\}$. We are interested in $\lambda_i^{(\mu)}(\widehat{D})$, the i -th eigenvalue of \widehat{D} on $(V \otimes S)_\mu$. According to Lemma 4.6.9, the i -th eigenvalue $\lambda_i^{(\mu)}(D)$ of D is nonzero and suppose that it is positive. Then $\lambda_i^{(\mu)}(D) = \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2}$ and using (4.8) and Remark 4.6.6, one obtains

$$\begin{aligned} \lambda_i^{(\mu)}(\widehat{D}) &\geq \lambda_i^{(\mu)}(D) - \|1 \otimes \gamma(c)\|^{(\mu)} \\ &= \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2} - \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2} \\ &> 0, \end{aligned}$$

where μ_1 is a weight of S related to μ such that $\sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2} = \|1 \otimes \gamma(c)\|^{(\mu)}$.

If $\lambda_i^{(\mu)}(D)$ is negative, i.e. $\lambda_i^{(\mu)}(D) = -\sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2}$, then with a similar argument

$$\begin{aligned} \lambda_i^{(\mu)}(\widehat{D}) &\leq \lambda_i^{(\mu)}(D) + \|1 \otimes \gamma(c)\|^{(\mu)} \\ &= -\sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2} + \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2} \\ &< 0. \end{aligned}$$

Hence, in both cases $\lambda_i^{(\mu)}(\widehat{D})$ is nonzero and since i was arbitrary, all the eigenvalues of \widehat{D} in $(V \otimes S)_\mu$ are nonzero. Therefore, this component does not contribute to $\ker(\widehat{D})$. \square

As a consequence, in order to study $\ker(\widehat{D})$ we are interested only in $(V \otimes S)_\mu$ for which the equality in (4.9) holds.

Lemma 4.6.10. *If μ and μ_1 are strongly related, then the elements*

$$\begin{aligned} w \star \mu &:= w(\mu + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}}, \\ w \star \mu_1 &:= w(\mu_1 + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}} \end{aligned}$$

of \mathfrak{t}^* satisfy the equality in (4.9).

Proof. We have that

$$\begin{aligned} & \|\lambda + \rho\|^2 - \|w \star \mu + \rho_{\mathfrak{h}}\|^2 = \|\rho\|^2 - \|w \star \mu_1 + \rho_{\mathfrak{h}}\|^2 \\ \Leftrightarrow & \|\lambda + \rho\|^2 - \|w(\mu + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}} + \rho_{\mathfrak{h}}\|^2 = \|\rho\|^2 - \|w(\mu_1 + \rho_{\mathfrak{h}}) - \rho_{\mathfrak{h}} + \rho_{\mathfrak{h}}\|^2 \\ \Leftrightarrow & \|\lambda + \rho\|^2 - \|w(\mu + \rho_{\mathfrak{h}})\|^2 = \|\rho\|^2 - \|w(\mu_1 + \rho_{\mathfrak{h}})\|^2 \\ \Leftrightarrow & \|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2 = \|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2 \end{aligned}$$

□

Remark 4.6.11. Note that $w \star \mu$ and $w \star \mu_1$ in general fail to be weights of $V \otimes S$ and S respectively.

4.6.2 A sufficient condition

In this subsection, we suppose that $\mathfrak{h} = \mathfrak{t}$, i.e. \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} . Under this condition, we will prove a sufficient condition so as an \mathfrak{h} -isotypic component $(V \otimes S)_\mu$ of $V \otimes S$ contributes to $\ker(\widehat{D})$. Combining this condition with Theorem 4.6.8 we obtain a complete description for $\ker(\widehat{D})$ in Theorem 4.6.13 which is the main result of this subsection.

Let $\mu \in \mathfrak{t}^*$ be a weight of $V \otimes S$ and $\mu_1 \in \mathfrak{t}^*$ be a weight of S related to μ . According to Lemma 4.6.10, if μ and μ_1 are strongly related then it will also be the case for $w \star \mu = w\mu$ and $w \star \mu_1 = w\mu_1$ for every w in W . Note that $w\mu$ and $w\mu_1$ turn out to be weights of $V \otimes S$ and S respectively. More precisely, the fact that $w\mu_1$ is a weight of S is a consequence of Proposition 2.5.3 with $\rho_{\mathfrak{h}} = 0$ and with $w\Delta^+$ being the positive system of Δ . Then $w\mu = w(\mu - \mu_1) + w\mu_1$ is a weight of $V \otimes S$ since $\mu - \mu_1$, and thus $w(\mu - \mu_1)$, is a weight of V .

Let μ be a weight of $V \otimes S$ and set

$$(V \otimes S)^{[\mu]} := \bigoplus_{w \in W/W_\mu} (V \otimes S)_{w\mu},$$

where W_μ stands for the stabilizer of μ in W . We consider the quotient W/W_μ instead of W in order to avoid having twice the same isotypic component in the direct sum. From the previous discussion, if the equality in (4.9) holds for μ , then it will also hold for any isotypic component appearing in $(V \otimes S)^{[\mu]}$ and so, using the action of W/W_μ , we can suppose that μ is such that if μ_1 is strongly related to μ , then $\mu - \mu_1$ is dominant. In this case, according to Theorem 4.6.4, $\mu - \mu_1$ is an extremal weight of V and, since it is dominant, it must be equal

to λ . Indeed, according to Theorem 1.4.22, a finite-dimensional irreducible representation has a unique extremal dominant weight. Then μ_1 can be written as

$$\mu_1 = \rho - \sum_{\alpha \in A} \alpha$$

for some $A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+$ and so

$$\mu = \lambda + \rho - \sum_{\alpha \in A} \alpha.$$

Consequently, if

$$\tilde{A}(\lambda) := \{\lambda + \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+\}$$

and

$$A(\lambda) := \tilde{A}(\lambda)/W, \quad (4.11)$$

i.e. $A(\lambda)$ is the set of equivalence classes of $\tilde{A}(\lambda)$ with respect to W , the equality in (4.9) holds exactly for the isotypic components appearing in

$$\bigoplus_{\mu \in A(\lambda)} (V \otimes S)^{[\mu]}.$$

Note that for distinct μ and μ' in $\tilde{A}(\lambda)$ belonging to the same W -orbit, we have $(V \otimes S)^{[\mu]} = (V \otimes S)^{[\mu']}$. In order to avoid having twice the same block in the above direct sum, we have to consider the quotient $A(\lambda)$ instead of $\tilde{A}(\lambda)$.

Lemma 4.6.12. *If the equality in (4.9) holds for μ , then $(V \otimes S)_\mu$ is 1-dimensional. In other words, there is only one weight μ_1 of S related to μ .*

Proof. Assume that $(V \otimes S)_\mu$ contains a nontrivial vector of the form $v_\lambda \otimes e$, with v_λ being a highest weight vector in V and e being a weight vector of S . Suppose that there is a nontrivial weight vector $v_{\lambda'}$ of weight λ' in V and a nontrivial weight vector e' in S such that $v_{\lambda'} \otimes e'$ also belongs to $(V \otimes S)_\mu$. Let A and A' be subspaces of Δ^+ such that the weights of S corresponding to e and e' are

$$\rho - \sum_{\alpha \in A} \alpha \quad \text{and} \quad \rho - \sum_{\alpha' \in A'} \alpha',$$

respectively. We remark that, according to Theorem 4.6.4, $\langle \alpha, \lambda \rangle = 0$, for every α in A . Therefore

$$\begin{aligned} \lambda + \rho - \sum_{\alpha \in A} \alpha &= \lambda' + \rho - \sum_{\alpha' \in A'} \alpha' \\ \Rightarrow \lambda - \lambda' &= \sum_{\alpha \in A} \alpha - \sum_{\alpha' \in A'} \alpha' \\ \Rightarrow \langle \lambda - \lambda', \lambda \rangle &= - \sum_{\alpha' \in A'} \langle \alpha', \lambda \rangle \end{aligned}$$

The left side of the above equality is nonnegative while the right one is nonpositive and thus both must be equal to 0. Therefore $\langle \lambda - \lambda', \lambda \rangle = 0$ and

$$\begin{aligned} \|\lambda'\|^2 &= \|(\lambda' - \lambda) + \lambda\|^2 \\ &= \|\lambda' - \lambda\|^2 + \|\lambda\|^2 + 2\langle \lambda' - \lambda, \lambda \rangle \\ &= \|\lambda' - \lambda\|^2 + \|\lambda\|^2 \\ &\geq \|\lambda\|^2. \end{aligned}$$

Since λ is an extremal weight of V , the equality must hold. Therefore $\|\lambda' - \lambda\| = 0$ or equivalently $\lambda' = \lambda$. \square

Theorem 4.6.13. *Let \mathfrak{g} be a complex semisimple Lie algebra, $\mathfrak{h} := \mathfrak{t}$ be a Cartan subalgebra of \mathfrak{g} , Δ be the corresponding root system and set Δ^+ to be a positive system of Δ . For a Δ^+ -dominant algebraically integral element $\lambda \in \mathfrak{t}^*$, let V be the corresponding finite-dimensional irreducible representation of highest weight λ and $A(\lambda)$ be the set defined in (4.11). Then the kernel of the noncubic Dirac operator \widehat{D} acting on $V \otimes S$ is*

$$\ker(\widehat{D}) = \bigoplus_{\mu \in A(\lambda)} (V \otimes S)^{[\mu]}. \quad (4.12)$$

Proof. In view of Theorem 4.6.8, it suffices to prove that each isotypic component of the sum on the right-hand side of the above relation lies in $\ker(\widehat{D})$. Let $(V \otimes S)_\mu$ be such a component, i.e. μ is a weight of $V \otimes S$ such that (4.9) holds as an equality. According to Lemma 4.6.12, there is a weight μ_0 of V such that $(V \otimes S)_\mu \subseteq V_{\mu_0} \otimes S$, where V_{μ_0} is the weight subspace of V corresponding to μ_0 . Then

$$\widehat{D}[(V \otimes S)_\mu] \subseteq (V \otimes S)_\mu \subseteq V_{\mu_0} \otimes S. \quad (4.13)$$

On the other hand, each additive term of \widehat{D} acts on an element of $V_{\mu_0} \otimes S$ giving an element in $V_{\mu_0+\alpha} \otimes S$ for some α in Δ , where $V_{\mu_0+\alpha}$ is the weight subspace of V corresponding to $\mu_0 + \alpha$ (it may be trivial if $\mu_0 + \alpha$ is not a weight of V). Thus

$$\widehat{D}[(V \otimes S)_\mu] \subseteq \widehat{D}[V_{\mu_0} \otimes S] \subseteq \bigoplus_{\alpha \in \Delta} V_{\mu_0+\alpha} \otimes S.$$

Moreover,

$$\bigoplus_{\alpha \in \Delta} V_{\mu_0+\alpha} \otimes S \subseteq [V_{\mu_0} \otimes S]^\perp,$$

where the orthogonal complement is taken with respect to the Hermitian product $\langle \cdot, \cdot \rangle$ of $V \otimes S$, defined before Proposition 3.2.9. Indeed, for distinct weights ν and ν' of V , the corresponding weight subspaces V_ν and $V_{\nu'}$ are orthogonal and in consequence, this is the case for V_{μ_0} and $V_{\mu_0+\alpha}$ for $\alpha \in \Delta$. Since $\langle \cdot, \cdot \rangle$ is defined as the tensor product of the Hermitian products on V and on S , we conclude that $V_{\mu_0} \otimes S$ is orthogonal to $V_{\mu_0+\alpha} \otimes S$ and

$$\widehat{D}[(V \otimes S)_\mu] \subseteq \bigoplus_{\alpha \in \Delta} V_{\mu_0+\alpha} \otimes S \subseteq [V_{\mu_0} \otimes S]^\perp. \quad (4.14)$$

Combining (4.13) and (4.14), one obtains that:

$$\widehat{D}[(V \otimes S)_\mu] \subseteq [V_{\mu_0} \otimes S] \cap [V_{\mu_0} \otimes S]^\perp = \{0\}$$

and thus $(V \otimes S)_\mu$ lies in $\ker(\widehat{D})$. \square

Corollary 4.6.14. *Let V be the finite-dimensional irreducible representation of highest weight λ and $A(\lambda)$ be the set defined in (4.11). Then*

$$\ker(\widehat{D}) / \ker(D) = \bigoplus_{\mu \in A(\lambda) \setminus \{\lambda + \rho\}} (V \otimes S)^{[\mu]}.$$

Proof. From Theorem 3.2.4, one has

$$\ker(D) = \bigoplus_{w \in W} F_{w(\lambda + \rho)}, \quad (4.15)$$

where $F_{w(\lambda + \rho)}$ is the 1-dimensional irreducible representation of \mathfrak{h} associated to the weight $w(\lambda + \rho)$. Indeed, the set W'_λ in Theorem 3.2.4 coincides with W , while $\rho_{\mathfrak{h}}$ is clearly zero. On the other hand, the isotypic component $(V \otimes S)_{w(\lambda + \rho)}$ of $V \otimes S$ is 1-dimensional, since it consists of the 1-dimensional weight subspace $V_{w\lambda}$ tensorized with the 1-dimensional weight subspace $S_{w\rho}$. These weight subspaces are both 1-dimensional because they are the highest weight subspaces of V and S , respectively, with respect to the positive root system $w\Delta^+$. Therefore, $F_{w(\lambda + \rho)}$ coincides with $(V \otimes S)_{w(\lambda + \rho)}$ and (4.15) can be written as

$$\ker(D) = \bigoplus_{w \in W} (V \otimes S)_{w(\lambda + \rho)}. \quad (4.16)$$

Furthermore, since $\lambda + \rho$ is strictly dominant, from Lemma 1.2.23, $W_{\lambda + \rho}$ is trivial and

$$(V \otimes S)^{[\lambda + \rho]} = \bigoplus_{w \in W} (V \otimes S)_{w(\lambda + \rho)}. \quad (4.17)$$

Combining (4.16) and (4.17), one obtains that $\ker(D)$ coincides with $(V \otimes S)^{[\lambda + \rho]}$ and thus

$$\ker(\widehat{D}) / \ker(D) = \bigoplus_{\mu \in A(\lambda) \setminus \{\lambda + \rho\}} (V \otimes S)^{[\mu]}.$$

\square

Lemma 4.6.15. *Let $\{v_\nu \mid \nu \in \Pi(V)\}$ be a weight vector basis of V , with v_ν being a weight vector corresponding to the weight $\nu \in \mathfrak{t}^*$. The kernel $\ker(\widehat{D})$ of \widehat{D} admits a vector space basis consisting of elements of the form $v_\nu \otimes e_I$, with ν being a weight of V and I a subset of Δ^+ .*

Remark 4.6.16. For the element e_I , we use the notation introduced before Lemma 2.5.3. More precisely, if $\{e_\alpha\}_{\alpha \in \Delta}$ is a basis of \mathfrak{q} such that

$$\langle e_\alpha, e_\beta \rangle = \begin{cases} 1 & \text{if } \alpha + \beta = 0 \\ 0 & \text{otherwise} \end{cases},$$

and I is a subset of Δ^+ , we set e_I to be the element of S defined by

$$e_I := \bigwedge_{\alpha \in I} e_{-\alpha}.$$

Proof of Lemma 4.6.15. From Theorem 4.6.13, it is clear that for a weight μ of $V \otimes S$, if

$$(V \otimes S)_\mu \cap \ker(\widehat{D}) \neq \{0\},$$

then

$$(V \otimes S)_\mu \subseteq \ker(\widehat{D}).$$

Since every $(V \otimes S)_\mu$ admits a basis of the above form, it will be the case for $\ker(\widehat{D})$. \square

Recall that, if $\{e_\alpha\}_{\alpha \in \Delta}$ is the above basis of \mathfrak{q} , then \widehat{D} is given by

$$\widehat{D} = \sum_{\alpha \in \Delta} e_\alpha \otimes \gamma(e_{-\alpha}).$$

Proposition 4.6.17. *For $\{e_\alpha\}_{\alpha \in \Delta}$ as above,*

$$\ker(\widehat{D}) = \bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha})).$$

In other words, if an element x of $V \otimes S$ is annihilated by \widehat{D} , then it is annihilated by every additive term $\pi(e_\alpha) \otimes \gamma(e_{-\alpha})$, $\alpha \in \Delta$, of \widehat{D} .

Proof. One inclusion is obvious. For the other one, suppose x is an element in $\ker(\widehat{D})$. From Lemma 4.6.15, we can suppose that x is of the form $v_\nu \otimes e_I$ for some weight ν of V and for I being a subset of Δ^+ . Let α be a root of Δ . One has

$$\pi(e_\alpha) \otimes \gamma(e_{-\alpha})(v_\nu \otimes e_I) \in V_{\nu+\alpha} \otimes S,$$

while, on the other hand, using an argument like the one used in the proof of Theorem 4.6.13, one obtains that

$$(\widehat{D} - \pi(e_\alpha) \otimes \gamma(e_{-\alpha}))(v_\nu \otimes e_I) \in (V_{\nu+\alpha} \otimes S)^\perp.$$

Moreover,

$$\begin{aligned} 0 &= \widehat{D}(v_\nu \otimes e_I) \\ &= (\pi(e_\alpha) \otimes \gamma(e_{-\alpha}))(v_\nu \otimes e_I) + (\widehat{D} - \pi(e_\alpha) \otimes \gamma(e_{-\alpha}))(v_\nu \otimes e_I). \end{aligned}$$

Since $(\pi(e_\alpha) \otimes \gamma(e_{-\alpha}))(v_\nu \otimes e_I)$ and $(\widehat{D} - \pi(e_\alpha) \otimes \gamma(e_{-\alpha}))(v_\nu \otimes e_I)$ are orthogonal and add to 0, both must be 0. Consequently, x belongs to $\ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha}))$ and

$$\ker(\widehat{D}) \subseteq \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha})).$$

The choice of α was arbitrary and, thus, the inclusion remains true for any root α of Δ . Hence

$$\ker(\widehat{D}) = \bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha})).$$

\square

4.6.3 Isotypic components

From Corollary 3.2.7, one deduces that the cubic Dirac operator D acts in the same way on every irreducible representation of an isotypic component. Namely, on the isotypic component corresponding to an extremal weight $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$, it acts by the scalar

$$\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2.$$

As a consequence, either the whole isotypic component belongs to $\ker(D)$ or its intersection with $\ker(D)$ is trivial. On the other hand, according to Theorem 4.6.13, this turns out to be the case for the noncubic Dirac operator \widehat{D} in the case where $\mathfrak{h} = \mathfrak{t}$. Note that, since each isotypic component of $V \otimes S$ is generated by "monomials", i.e. by elements of the form $v \otimes u$ with v and u being weight vectors of V and S respectively, one deduces that this is the case for $\ker(D)$ and, in the case where $\mathfrak{h} = \mathfrak{t}$, for $\ker(\widehat{D})$. Nevertheless, as the following example implies, this property for \widehat{D} is not true if $\mathfrak{h} \neq \mathfrak{t}$.

Example 4.6.18. Let $\mathfrak{g} := \mathfrak{sl}(4, \mathbb{C})$ and \mathfrak{t} be the Cartan subalgebra of \mathfrak{g} consisted of all traceless diagonal matrices

$$H := \begin{pmatrix} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & h_4 \end{pmatrix}$$

in \mathfrak{g} . Set ε_i , $1 \leq i \leq 4$, to be the linear functional mapping the matrix H to h_i . Then, the root system of \mathfrak{g} is

$$\Delta := \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq 4\}$$

while

$$\Delta^+ := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq 4\}$$

is a positive system of Δ . Let

$$\mathfrak{h} := \mathfrak{t} \oplus \mathfrak{g}_{\varepsilon_3 - \varepsilon_4} \oplus \mathfrak{g}_{-\varepsilon_3 + \varepsilon_4}$$

and $\mathfrak{q} := \mathfrak{h}^\perp$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}.$$

Consider the Clifford algebra $\mathbf{C}(\mathfrak{q})$ of \mathfrak{q} and the space of spinors S for $\mathbf{C}(\mathfrak{q})$, determined by the negative root spaces of \mathfrak{q} . For every $\alpha \in \Delta$, set

$$e_\alpha := \frac{\sqrt{2}}{4} E_\alpha,$$

where E_α stands for the root vector of \mathfrak{g}_α defined in Subsection 1.2.1. More precisely, for $\alpha = \varepsilon_i - \varepsilon_j$, E_α is the matrix E_{ij} having all its entries 0 except for its (i, j) -entry which is 1. Then, for every $\alpha, \beta \in \Delta$, we have

$$B(e_\alpha, e_\beta) = \delta_{\alpha, \beta}.$$

Furthermore, let V be the standard representation of \mathfrak{g} , i.e. the finite-dimensional irreducible representation of highest weight

$$\lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4.$$

In particular, V is isomorphic with the representation obtained by the standard action of \mathfrak{g} on \mathbb{C}^4 , its weights being

$$\begin{aligned}\mu_1 &:= \lambda = \frac{3}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_2 &:= \lambda - \varepsilon_1 + \varepsilon_2 = -\frac{1}{4}\varepsilon_1 + \frac{3}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_3 &:= \lambda - \varepsilon_1 + \varepsilon_3 = -\frac{1}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 + \frac{3}{4}\varepsilon_3 - \frac{1}{4}\varepsilon_4 \\ \mu_4 &:= \lambda - \varepsilon_1 + \varepsilon_4 = -\frac{1}{4}\varepsilon_1 - \frac{1}{4}\varepsilon_2 - \frac{1}{4}\varepsilon_3 + \frac{3}{4}\varepsilon_4\end{aligned}$$

and, if v_1, v_2, v_3 and v_4 are the standard basis of \mathbb{C}^4 , then v_i is the weight vector corresponding to μ_i , $1 \leq i \leq 4$.

For the pair $(\mathfrak{g}, \mathfrak{h})$, consider the noncubic Dirac operator \widehat{D} given by

$$\begin{aligned}\widehat{D} &= \frac{1}{2}E_{\varepsilon_1-\varepsilon_2} \otimes \gamma(e_{-\varepsilon_1+\varepsilon_2}) + \frac{1}{2}E_{\varepsilon_1-\varepsilon_3} \otimes \gamma(e_{-\varepsilon_1+\varepsilon_3}) + \frac{1}{2}E_{\varepsilon_1-\varepsilon_4} \otimes \gamma(e_{-\varepsilon_1+\varepsilon_4}) \\ &\quad + \frac{1}{2}E_{\varepsilon_2-\varepsilon_3} \otimes \gamma(e_{-\varepsilon_2+\varepsilon_3}) + \frac{1}{2}E_{\varepsilon_2-\varepsilon_4} \otimes \gamma(e_{-\varepsilon_2+\varepsilon_4}) \\ &\quad + \frac{1}{2}E_{-\varepsilon_1+\varepsilon_2} \otimes \gamma(e_{\varepsilon_1-\varepsilon_2}) + \frac{1}{2}E_{-\varepsilon_1+\varepsilon_3} \otimes \gamma(e_{\varepsilon_1-\varepsilon_3}) + \frac{1}{2}E_{-\varepsilon_1+\varepsilon_4} \otimes \gamma(e_{\varepsilon_1-\varepsilon_4}) \\ &\quad + \frac{1}{2}E_{-\varepsilon_2+\varepsilon_3} \otimes \gamma(e_{\varepsilon_2-\varepsilon_3}) + \frac{1}{2}E_{-\varepsilon_2+\varepsilon_4} \otimes \gamma(e_{\varepsilon_2-\varepsilon_4}).\end{aligned}$$

and the vectors

$$\begin{aligned}x_1 &:= v_3 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4} \\ x_2 &:= v_4 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3}.\end{aligned}$$

Then

$$\begin{aligned}\widehat{D}x_1 &= v_2 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \neq 0 \\ \widehat{D}x_2 &= -v_2 \otimes e_{-\varepsilon_1+\varepsilon_3} \wedge e_{-\varepsilon_1+\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \neq 0\end{aligned}$$

and

$$\widehat{D}(x_1 + x_2) = 0.$$

Therefore, the elements x_1, x_2 and $x_1 + x_2$ belong to the same isotypic component of $V \otimes S$, the element $x_1 + x_2$ belonging to $\ker(\widehat{D})$, while x_1 and x_2 do not. In other words, only a part of this isotypic component belongs to $\ker(\widehat{D})$.

4.7 Explicit description of the set $A(\lambda)$

In order to make more explicit our description (4.12) of the kernel of the noncubic Dirac operator \widehat{D} , we study the set $A(\lambda)$ involved in the formula. We continue here with the assumption that \mathfrak{h} coincides with a Cartan subalgebra \mathfrak{t} of \mathfrak{g} . The main result of this paragraph is the following proposition.

Proposition 4.7.1. *The set $\{\mu \in \widetilde{A}(\lambda) \mid \mu - \lambda \text{ is } \Delta^+\text{-dominant}\}$ is a complete family of representatives for $A(\lambda)$.*

In other words, by taking all elements

$$\lambda + \rho - \sum_{\alpha \in A} \alpha \in \widetilde{A}(\lambda),$$

with $\rho - \sum_{\alpha \in A} \alpha$ being Δ^+ -dominant, we obtain a complete family of representatives for $A(\lambda)$.

Recall that

$$\lambda + \rho - \sum_{\alpha \in A} \alpha$$

belongs to $\widetilde{A}(\lambda)$ if A is a subset of $(\mathbb{R}\{\lambda\})^\perp \cap \Delta^+$. The rest of this section is devoted to the proof of Proposition 4.7.1.

Lemma 4.7.2. *Let A and A' be subsets of $(\mathbb{R}\{\lambda\})^\perp \cap \Delta^+$ and set*

$$\begin{aligned} \mu &= \lambda + \rho - \sum_{\alpha \in A} \alpha \\ \text{and } \mu' &= \lambda + \rho - \sum_{\alpha' \in A'} \alpha' \end{aligned}$$

to be the corresponding elements of $\widetilde{A}(\lambda)$. Then μ and μ' are W -conjugate if and only if they are W_λ -conjugate, where W_λ stands for the stabilizer of λ in W .

Proof. The "if" direction is obvious. For the "only if" direction, let us suppose that w is an element of W such that $\mu = w\mu'$. Then

$$\begin{aligned} \mu &= w\mu' \\ \implies \lambda + \rho - \sum_{\alpha \in A} \alpha &= w(\lambda + \rho - \sum_{\alpha' \in A'} \alpha') \\ \implies \lambda - w\lambda &= w(\rho - \sum_{\alpha' \in A'} \alpha') - (\rho - \sum_{\alpha \in A} \alpha) \\ \implies \langle \lambda - w\lambda, \lambda \rangle &= -\langle (\rho - \sum_{\alpha \in A} \alpha) - w(\rho - \sum_{\alpha' \in A'} \alpha'), \lambda \rangle \\ \implies \langle \lambda - w\lambda, \lambda \rangle &= -\langle \rho - w(\rho - \sum_{\alpha' \in A'} \alpha'), \lambda \rangle. \end{aligned}$$

Since $w\lambda$ is a weight of V , $\lambda - w\lambda$ is a nonnegative sum of positive roots, while λ is dominant. Thus, the left-hand side of the above expression is nonnegative. On the other hand, using Lemma 4.6.5, one can see that the right-hand side is nonpositive and so both sides must be 0. Then $w\lambda = \lambda$, since

$$\begin{aligned} \|w\lambda\|^2 &= \|(w\lambda - \lambda) + \lambda\|^2 \\ &= \|w\lambda - \lambda\|^2 + \|\lambda\|^2 \\ &\geq \|\lambda\|^2. \end{aligned}$$

But $\|w\lambda\|^2 = \|\lambda\|^2$, $\|w\lambda - \lambda\|^2 = 0$ and thus $w\lambda = \lambda$. Consequently, w belongs to W_λ . \square

Corollary 4.7.3. *The set $A(\lambda)$ coincides with $\tilde{A}(\lambda)/W_\lambda$.*

As a consequence, if $\{\nu_i\}_i$ is a complete family of representatives for the W_λ -equivalence classes of the set

$$\{\rho - \sum_{\alpha \in A} \alpha \mid A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+\},$$

then the set $\{\lambda + \nu_i\}_i$ is a complete family of representatives for $A(\lambda)$. The following lemma ensures that we can choose ν_i to be Δ^+ -dominant and so it completes the proof of Proposition 4.7.1.

Lemma 4.7.4. *Any element of the form*

$$\rho - \sum_{\alpha \in A} \alpha$$

with $A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+$ is W_λ -conjugate to a Δ^+ -dominant element.

Proof. Let $\Delta_1^+ := (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+$, i.e. the set of the positive roots orthogonal to λ , and set $\Delta_2^+ := \Delta^+ \setminus \Delta_1^+$. The set $\Delta_1 := \Delta_1^+ \cup (-\Delta_1^+)$ is a root system. Indeed, if α and β are elements of Δ_1^+ , then, from the W -invariance of the Killing form (see Lemma 1.1.41),

$$\begin{aligned} \langle s_\alpha(\beta), \lambda \rangle &= \langle \beta, s_\alpha(\lambda) \rangle \\ &= \langle \beta, \lambda \rangle \\ &= 0, \end{aligned}$$

and thus Δ_1 is stable with respect to the reflections generated by its elements. The other properties for Δ_1 being a root system are obviously satisfied. Let W_1 be the Weyl group defined by Δ_1 and ρ_1 be the half-sum of its positive roots. Then, W_1 stabilizes the element $\rho - \rho_1$, i.e. the half-sum of the positive roots included in $\Delta_2 := \Delta_2^+ \cup (-\Delta_2^+)$. Note that Δ_2 is not, in general, a root system. Indeed, if β is a root of Δ_1 and γ is a root in Δ_2^+ , then

$$\begin{aligned} \langle s_\beta(\gamma), \lambda \rangle &= \langle \gamma, s_\beta(\lambda) \rangle \\ &= \langle \gamma, \lambda \rangle \\ &> 0. \end{aligned}$$

Thus $W_1\Delta_2^+ = \Delta_2^+$, W_1 does stabilize $\rho - \rho_1$ and $\langle \rho - \rho_1, \beta \rangle = 0$ for any $\beta \in \Delta_1^+$. Since the Weyl group of a root system acts transitively on its Weyl chambers, if $A \subseteq \Delta_1^+$, then there is some w_0 in W_1 such that

$$w_0(\rho_1 - \sum_{\alpha \in A} \alpha)$$

is Δ_1^+ -dominant. In this case, for every $\beta \in \Delta_1^+$,

$$\begin{aligned}
 \langle w_0(\rho - \sum_{\alpha \in A} \alpha), \beta \rangle &= \langle \rho - \rho_1 + w_0(\rho_1 - \sum_{\alpha \in A} \alpha), \beta \rangle \\
 &= \langle \rho - \rho_1, \beta \rangle + \langle w_0(\rho_1 - \sum_{\alpha \in A} \alpha), \beta \rangle \\
 &= \langle w_0(\rho_1 - \sum_{\alpha \in A} \alpha), \beta \rangle \\
 &\geq 0.
 \end{aligned}$$

Therefore $w_0(\rho - \sum_{\alpha \in A} \alpha)$ is Δ_1^+ -dominant.

We will be finished once we show that this is also the case for Δ_2^+ . More precisely, we want to show that

$$\langle w_0(\rho - \sum_{\alpha \in A} \alpha), \gamma \rangle \geq 0,$$

for any γ in Δ_2^+ . Since $w_0^{-1}\Delta_2^+ = \Delta_2^+$, it suffices to show that

$$\langle \rho - \sum_{\alpha \in A} \alpha, \gamma \rangle \geq 0,$$

for any γ in Δ_2^+ . Suppose that for some γ in Δ_2^+ ,

$$\langle \rho - \sum_{\alpha \in A} \alpha, \gamma \rangle < 0.$$

Then one has that

$$s_\gamma(\rho - \sum_{\alpha \in A} \alpha) = \rho - \sum_{\alpha \in A} \alpha + n\gamma \tag{4.18}$$

with $n := -\frac{2}{\|\gamma\|^2} \langle \rho - \sum_{\alpha \in A} \alpha, \gamma \rangle$ being positive.

On the other hand, according to Lemma 4.6.5, the set of weights of S is W -stable, and so the element

$$s_\gamma(\rho - \sum_{\alpha \in A} \alpha)$$

must be a weight of S . Thus, from Proposition 2.5.3, it must be of the form

$$s_\gamma(\rho - \sum_{\alpha \in A} \alpha) = \rho - \sum_{\beta \in B} \beta \tag{4.19}$$

for some subset B of Δ^+ .

Combining the equalities (4.18) and (4.19), one obtains that

$$\begin{aligned}
 \rho - \sum_{\alpha \in A} \alpha + n\gamma &= \rho - \sum_{\beta \in B} \beta \\
 \implies n\gamma &= \sum_{\alpha \in A} \alpha - \sum_{\beta \in B} \beta \\
 \implies n\langle \gamma, \lambda \rangle &= - \sum_{\beta \in B} \langle \beta, \lambda \rangle.
 \end{aligned}$$

The left-hand side of the last equality is positive, since γ has been chosen to be an element of Δ_2^+ , while the right-hand side is non positive, since B consists of positive roots and λ is dominant. This is a contradiction. Consequently

$$\langle \rho - \sum_{\alpha \in A} \alpha, \gamma \rangle \geq 0$$

for any γ in Δ_2^+ and thus the element $w_0(\rho - \sum_{\alpha \in A} \alpha)$ is Δ^+ -dominant.

From Lemma 1.2.23, W_1 coincides with the stabilizer W_λ of λ in W . Hence the element $\rho - \sum_{\alpha \in A} \alpha$ turns out to be W_λ -conjugate with a dominant one. \square

This completes Proposition 4.7.1. The fact that $\ker(\widehat{D})$ and $\ker(D)$ may coincide can be characterized as follows.

Proposition 4.7.5. *For \mathfrak{g} , \mathfrak{h} , λ and Δ_1^+ as above, one has $\ker(\widehat{D}) = \ker(D)$ if and only if W_λ acts transitively on the set*

$$\{\rho - \sum_{\alpha \in A} \alpha \mid A \subseteq \Delta_1^+\},$$

i.e. there is only one W_λ -orbit in this set.

Proof. According to Corollary 4.6.14, $\ker(\widehat{D})$ coincides with $\ker(D)$ if and only if $A(\lambda) \setminus \{\lambda + \rho\} = \emptyset$, i.e. $A(\lambda) = \{\lambda + \rho\}$. In other words, if and only if there is only one W -orbit in $\widetilde{A}(\lambda)$, or, using Corollary 4.7.3, if and only if there is only one W_λ -orbit in $\widetilde{A}(\lambda)$. From the discussion following Corollary 4.7.3, this is equivalent to saying that there is only one W_λ -orbit in the set

$$\{\rho - \sum_{\alpha \in A} \alpha \mid A \subseteq \Delta_1^+\}.$$

\square

In Section 5.1 of Chapter 5, we will give an explicit more practical description for $\ker(\widehat{D})$ which will permit to study the noncubic Dirac operator in the case of the standard representations of classical Lie algebras.

4.8 t -cubic Dirac operators

We continue with the same notation. Namely, \mathfrak{g} is a complex semisimple Lie algebra, \mathfrak{h} is a Lie subalgebra of \mathfrak{g} satisfying the condition (2.10) and such that $\text{rank}_{\mathbb{C}}(\mathfrak{g}) = \text{rank}_{\mathbb{C}}(\mathfrak{h})$, and \mathfrak{t} is a common Cartan subalgebra of \mathfrak{g} and \mathfrak{h} . Let $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ and $\Delta_{\mathfrak{h}} := \Delta(\mathfrak{h}, \mathfrak{t})$ be the corresponding root systems of \mathfrak{g} and \mathfrak{h} respectively. In $\Delta_{\mathfrak{h}}$ choose a positive system $\Delta_{\mathfrak{h}}^+$ and extend it to a positive system Δ^+ for Δ . Decompose \mathfrak{g} into

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

where $\mathfrak{q} := \mathfrak{h}^\perp$, and consider the Clifford algebra $\mathbf{C}(\mathfrak{q})$ of \mathfrak{q} and a space of spinors S for $\mathbf{C}(\mathfrak{q})$. Let V be an irreducible finite-dimensional representation of \mathfrak{g} of highest weight $\lambda \in \mathfrak{t}^*$.

Definition 4.8.1. In the above context, for $t \in [0, \frac{1}{2}]$, we define D^t to be the operator

$$D^t : V \otimes S \rightarrow V \otimes S$$

given by

$$D^t := \sqrt{2} \left\{ \sum_{i=1}^l B(\tilde{e}_i, \tilde{e}_j) e_i \otimes \gamma(e_j) - 3t(1 \otimes \gamma(c)) \right\} \in U(\mathfrak{g}) \otimes \mathbf{C}(\mathfrak{q}),$$

where $\{\tilde{e}_i\}$ and $\{e_i\}$ are dual bases of \mathfrak{q} and $\gamma(c)$ is the cubic term (4.1). Note that, for $t = \frac{1}{3}$, one obtains the cubic Dirac operator D of Definition 3.1.1 while, for $t = 0$, one obtains the noncubic Dirac operator \widehat{D} of Definition 4.1.1. For $t \in (0, \frac{1}{3}]$, we call D^t the t -cubic Dirac operator.

The main result of this section is the following theorem.

Theorem 4.8.2. Let V be an irreducible finite-dimensional representation of \mathfrak{g} of highest weight $\lambda \in \mathfrak{t}^*$. For $t \in (0, \frac{1}{3}]$, consider the t -cubic Dirac operator

$$D^t : V \otimes S \rightarrow V \otimes S$$

and the cubic Dirac operator

$$D^{1/3} = D : V \otimes S \rightarrow V \otimes S$$

of Definition 3.1.1. Then, for every $t \in (0, \frac{1}{3}]$,

$$\ker(D^t) = \ker(D).$$

Proof. Let \mathfrak{g} , \mathfrak{h} and V be as above. The inclusion $\ker(D) \subseteq \ker(D^t)$ is a consequence of Corollary 4.4.2. More precisely, if $x \in \ker(D)$, then, according to this corollary, $1 \otimes \gamma(c)x = 0$. Therefore,

$$\begin{aligned} D^t(x) &= (D + (1 - 3t)1 \otimes \gamma(c))(x) \\ &= D(x) - (1 - 3t)(1 \otimes \gamma(c))(x) \\ &= 0 \end{aligned}$$

and so $\ker(D) \subseteq \ker(D^t)$.

For the other inclusion, following what we did in Subsection 4.6.1, we will use once more Weyl's inequalities of Section 4.5. Let

$$\begin{aligned} \lambda_1(D) &\geq \dots \geq \lambda_n(D), \\ \lambda_1(D) &\geq \dots \geq \lambda_n(D), \\ \lambda_1(1 \otimes \gamma(c)) &\geq \dots \geq \lambda_n(1 \otimes \gamma(c)) \end{aligned}$$

be the eigenvalues of D , D^t and $1 \otimes \gamma(c)$ respectively, in descending ordering and counting multiplicities. For $A := D$, $B := (1 - 3t)(1 \otimes \gamma(c))$ and $j = 1$, the inequality (4.6) gives

$$\begin{aligned} \lambda_i(D^t) &\leq \lambda_i(D) + (1 - 3t)\lambda_1(1 \otimes \gamma(c)) \\ \Rightarrow \lambda_i(D^t) &\leq \lambda_i(D) + (1 - 3t)\|1 \otimes \gamma(c)\|, \end{aligned}$$

where $\|1 \otimes \gamma(c)\|$ stands for the operator norm of $1 \otimes \gamma(c)$. Exchanging the roles of D and D^t gives

$$\begin{aligned} \lambda_i(D) + (1 - 3t)\lambda_n(1 \otimes \gamma(c)) &\leq \lambda(D^t) \\ \Rightarrow \lambda_i(D) - (1 - 3t)\|1 \otimes \gamma(c)\| &\leq \lambda(D^t). \end{aligned}$$

Therefore, in analogy with inequality (4.7), one obtains

$$\lambda_i(D) - (1 - 3t)\|1 \otimes \gamma(c)\| \leq \lambda_i(D^t) \leq \lambda_i(D) + (1 - 3t)\|1 \otimes \gamma(c)\|.$$

Let us restrict our study to an isotypic component $(V \otimes S)_\mu$, $\mu \in \mathfrak{t}^*$, of $V \otimes S$. If $\lambda_i^{(\mu)}(\cdot)$ corresponds to the i -th eigenvalue of the corresponding operator acting on $(V \otimes S)_\mu$ and $\|1 \otimes \gamma(c)\|^{(\mu)}$ stands for the operator norm of $1 \otimes \gamma(c)$ when it acts on $(V \otimes S)_\mu$, using a similar argument as above, one obtains

$$\lambda_i^{(\mu)}(D) - (1 - 3t)\|1 \otimes \gamma(c)\|^{(\mu)} \leq \lambda_i^{(\mu)}(D^t) \leq \lambda_i^{(\mu)}(D) + (1 - 3t)\|1 \otimes \gamma(c)\|^{(\mu)}. \quad (4.20)$$

We have already seen in the proof of Lemma 4.6.1 that

$$\begin{aligned} \lambda_i^{(\mu)}(D) &= \pm \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2}, \\ \|1 \otimes \gamma(c)\|^{(\mu)} &= \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2}, \end{aligned}$$

where $\mu_1 \in \mathfrak{t}^*$ is a weight of S related to μ (see Definition 4.6.3). Therefore, replacing in (4.20), one obtains

$$\pm \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2} - (1 - 3t)\sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2} \leq \lambda_i^{(\mu)}(D^t) \quad (4.21)$$

and

$$\lambda_i^{(\mu)}(D^t) \leq \pm \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2} + (1 - 3t)\sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2}. \quad (4.22)$$

We are interested in the i -th eigenvalue $\lambda_i^{(\mu)}(D^t)$ of D^t . Suppose that the i -th eigenvalue $\lambda_i^{(\mu)}(D)$ of D is nonnegative, i.e.

$$\lambda_i^{(\mu)}(D) = \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2}.$$

Then, using (4.21), for $t \in (0, \frac{1}{3}]$, one obtains

$$\begin{aligned} \lambda_i^{(\mu)}(D^t) &\geq \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2} - (1 - 3t)\sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2} \\ &\geq \sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_{\mathfrak{h}}\|^2} - \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_{\mathfrak{h}}\|^2} \\ &\geq 0. \end{aligned}$$

For the last inequality, see Theorem 4.6.4 and Remark 4.6.6. If $\lambda_i^{(\mu)}(D^t) = 0$, then the equalities in the above inequalities must hold and one obtains

$$\sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_6\|^2} - (1 - 3t)\sqrt{\|\rho\|^2 - \|\mu_1 + \rho_6\|^2} = 0,$$

$$\sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_6\|^2} - \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_6\|^2} = 0.$$

Since t is nonzero, subtracting the above equalities gives

$$\sqrt{\|\lambda + \rho\|^2 - \|\mu + \rho_6\|^2} = \sqrt{\|\rho\|^2 - \|\mu_1 + \rho_6\|^2} = 0$$

and so $\lambda_i^{(\mu)}(D) = 0$.

The same argument applies in the case where the i -th eigenvalue $\lambda_i^{(\mu)}(D)$ of D is nonpositive. In other words, the eigenvalues of D in $(V \otimes S)_\mu$ all are zero and so $(V \otimes S)_\mu \subset \ker(D)$. Hence, if there is $x \in (V \otimes S)_\mu$ such that $D^t(x) = 0$, i.e. D^t has a zero eigenvalue in $(V \otimes S)_\mu$, then the whole component $(V \otimes S)_\mu$ is contained in $\ker(D)$ and so $D(x) = 0$ and $\ker(D_t) \subseteq \ker(D)$, as we wanted. \square

CHAPTER 5

KERNEL OF NONCUBIC DIRAC OPERATORS FOR THE CLASSICAL LIE ALGEBRAS

In this paragraph we apply the results of the previous chapter in the case of the classical Lie algebras and for V being the standard representation of Example 1.4.25, i.e. the irreducible representation having as highest weight the first fundamental weight $\bar{\omega}_1$ of the corresponding Lie algebra \mathfrak{g} . Recall that $\bar{\omega}_1$ is defined as the unique element of the dual space \mathfrak{t}^* of the Cartan subalgebra \mathfrak{t} of \mathfrak{g} such that

$$2 \frac{\langle \bar{\omega}_1, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_1, \dots, \alpha_l$ are the simple roots for a positive system Δ^+ of $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$.

5.1 Preliminaries

Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{t} be a Cartan subalgebra of \mathfrak{g} and set $\mathfrak{h} := \mathfrak{t}$. Let V be a finite-dimensional irreducible representation of \mathfrak{g} and set $\Pi(V)$ to be the set of all weights of V . For every weight $\nu \in \Pi(V)$, set

$$\begin{aligned} {}^1\Delta^{(\nu)} &:= \{\alpha \in \Delta^+ \mid \nu + \alpha \text{ is a weight of } V\}, \\ {}^2\Delta^{(\nu)} &:= \{\alpha \in \Delta^+ \mid \nu - \alpha \text{ is a weight of } V\}, \\ \text{and } \mathcal{P}^{(\nu)} &:= \{I \in \mathcal{P}(\Delta^+) \mid {}^1\Delta^{(\nu)} \subseteq I \text{ and } {}^2\Delta^{(\nu)} \cap I = \emptyset\}, \end{aligned}$$

where $\mathcal{P}(\Delta^+)$ stands for the power set of Δ^+ .

Definition 5.1.1. *We say that a representation V of \mathfrak{g} satisfies property $(*)$, if for every weight ν of V and every root α of ${}^1\Delta^{(\nu)}$ (respectively of ${}^2\Delta^{(\nu)}$), we have $\pi(e_\alpha)v_\nu \neq 0$ (respectively*

$\pi(e_{-\alpha})v_\nu \neq 0$) for every nonzero weight vector v_ν of V_ν . In other words, for ν and $\nu + \alpha$ (respectively $\nu - \alpha$) being in $\Pi(V)$, the linear map

$$\begin{aligned} \pi(e_\alpha) &: V_\nu \longrightarrow V_{\nu+\alpha} \\ (\text{respectively } \pi(e_{-\alpha}) &: V_\nu \longrightarrow V_{\nu-\alpha}) \end{aligned}$$

is injective.

Theorem 5.1.2. Suppose ν , ${}^1\Delta^{(\nu)}$ and ${}^2\Delta^{(\nu)}$ are as above and V satisfies property (*). Then, one has

$$\ker(\widehat{D}) = \bigoplus_{\nu \in \Pi(V)} \langle v_\nu \otimes e_I \mid I \in \mathcal{P}^{(\nu)} \rangle.$$

Proof. Let $v_\nu \otimes e_I$ be a nonzero element of $V \otimes S$ and α be a positive root of Δ^+ . If $\nu + \alpha$ is not a weight of V , i.e. $\alpha \notin {}^1\Delta^{(\nu)}$, then $\pi(e_\alpha)v_\nu = 0$ and so $\pi(e_\alpha) \otimes \gamma(e_{-\alpha})(v_\nu \otimes e_I) = 0$. On the other hand, if $\nu + \alpha$ is a weight of V , i.e. $\alpha \in {}^1\Delta^{(\nu)}$, and since V satisfies property (*), $\pi(e_\alpha)v_\nu \neq 0$. So $\pi(e_\alpha) \otimes \gamma(e_{-\alpha})(v_\nu \otimes e_I) = 0$ if and only if $\gamma(e_{-\alpha})e_I = 0$ or equivalently if and only if I contains α . Summarizing, we have

$$\pi(e_\alpha) \otimes \gamma(e_{-\alpha})(v_\nu \otimes e_I) \begin{cases} = 0 & \text{if } \alpha \notin {}^1\Delta^{(\nu)} \text{ or } \alpha \in I \\ \neq 0 & \text{if } \alpha \in {}^1\Delta^{(\nu)} \text{ and } \alpha \notin I \end{cases} \quad (5.1)$$

Using a similar argument, we obtain

$$\pi(e_{-\alpha}) \otimes \gamma(e_\alpha)(v_\nu \otimes e_I) \begin{cases} = 0 & \text{if } \alpha \notin {}^2\Delta^{(\nu)} \text{ or } \alpha \notin I \\ \neq 0 & \text{if } \alpha \in {}^2\Delta^{(\nu)} \text{ and } \alpha \in I \end{cases} \quad (5.2)$$

Therefore, the element $v_\nu \otimes e_I$ belongs to

$$\bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha}))$$

if and only if for every α in ${}^1\Delta^{(\nu)}$ we have α belonging to I and for every α in ${}^2\Delta^{(\nu)}$ we have α not belonging to I . This is equivalent to say that ${}^1\Delta^{(\nu)} \subseteq I$ and ${}^2\Delta^{(\nu)} \cap I = \emptyset$, i.e. $I \in \mathcal{P}^{(\nu)}$. Hence,

$$\bigcap_{\alpha \in \Delta} \ker(\pi(e_\alpha) \otimes \gamma(e_{-\alpha})) = \bigoplus_{\nu \in \Pi(V)} \langle v_\nu \otimes e_I \mid I \in \mathcal{P}^{(\nu)} \rangle.$$

Using Proposition 4.6.17, we obtain that

$$\ker(\widehat{D}) = \bigoplus_{\nu \in \Pi(V)} \langle v_\nu \otimes e_I \mid I \in \mathcal{P}^{(\nu)} \rangle.$$

□

The following table contains the dimensions of the kernels $\ker(D)$ and $\ker(\widehat{D})$ for \mathfrak{g} being a classical Lie algebra, \mathfrak{h} a Cartan subalgebra of \mathfrak{g} and V the standard representation of \mathfrak{g} .

\mathfrak{g}	$\dim \ker(D) = W $	$\dim \ker(\widehat{D})$
$\mathfrak{sl}(n+1, \mathbb{C})$	$(n+1)!$	$(n+1) \times 2^{\frac{n(n-1)}{2}}$
$\mathfrak{so}(2n+1, \mathbb{C})$	$2^n n!$	$2n \times 2^{(n-1)^2}$
$\mathfrak{sp}(n, \mathbb{C})$	$2^n n!$	$2n \times 2^{(n-1)^2}$
$\mathfrak{so}(2n, \mathbb{C})$	$2^{n-1} n!$	$2n \times 2^{(n-1)(n-2)}$

Table 5.1: Dimensions of the kernels of cubic and noncubic Dirac operators

5.2 The Lie Algebra $A_n = \mathfrak{sl}(n+1, \mathbb{C})$

Recall that

$$\mathfrak{g} := \mathfrak{sl}(n+1, \mathbb{C}) = \{X \in \mathfrak{gl}(n+1, \mathbb{C}) \mid \operatorname{tr}(X) = 0\}$$

while the subspace

$$\mathfrak{t} = \langle H_{\varepsilon_i - \varepsilon_j} := E_{ii} - E_{jj} \mid 1 \leq i < j \leq n+1 \rangle,$$

where E_{ij} is the matrix having all its entries 0 except for the (i, j) -entry which is 1, forms a Cartan subalgebra of \mathfrak{g} . If ε_i is the linear functional given by

$$\varepsilon_i(E_{jj}) = \delta_{ij},$$

then the root system $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ of \mathfrak{g} is

$$\Delta = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n+1\},$$

with the matrix $E_{\varepsilon_i - \varepsilon_j} := E_{ij} - E_{ji}$ being the root vector corresponding to $\varepsilon_i - \varepsilon_j$. We choose

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n+1\}$$

to be the set of positive roots of Δ . According to Table 1.5, the highest weight λ of the fundamental representation V of \mathfrak{g} is

$$\lambda = \frac{n}{n+1}\varepsilon_1 - \frac{1}{n+1}\varepsilon_2 - \dots - \frac{1}{n+1}\varepsilon_{n+1}.$$

Lemma 5.2.1. *The fundamental representation V of $\mathfrak{sl}(n+1, \mathbb{C})$ is $(n+1)$ -dimensional.*

Proof. Using Weyl dimensional formula (see Proposition 1.4.35), one obtains that

$$\begin{aligned}
 \dim(V) &= \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \\
 &= \prod_{j>1} \left(1 + \frac{\langle \lambda, \varepsilon_1 - \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_j \rangle}\right) \times \prod_{1 < i < j} \left(1 + \frac{\langle \lambda, \varepsilon_i - \varepsilon_j \rangle}{\langle \rho, \varepsilon_i - \varepsilon_j \rangle}\right) \\
 &= \prod_{j>1} \left(1 + \frac{\langle \lambda, \varepsilon_1 - \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_j \rangle}\right) \\
 &= \prod_{j>1} \left(1 + \frac{1}{j-1}\right) \\
 &= n + 1.
 \end{aligned}$$

□

On the other hand, the W -orbit of λ contains exactly $(n + 1)$ -many elements. Indeed, W acts on Δ by permutations, the element $s_{\varepsilon_i - \varepsilon_j}$ permuting the roots ε_i and ε_j (see Subsection 1.2.2). Therefore, one can verify that $W\{\lambda\} = \{\mu_j := \lambda - \varepsilon_1 + \varepsilon_j \mid 1 \leq j \leq n + 1\}$ and thus $|W\{\lambda\}| = n + 1$. Since the set $\Pi(V)$ of all weights of V is W -stable, by using Lemma 5.2.1, one deduces that the set of weights $\Pi(V)$ of V coincides with the W -orbit of λ , with each weight being of multiplicity 1.

We can verify all the above facts using an explicit construction of V . The representation V can be realized as the complex vector space \mathbb{C}^{n+1} endowed with the standard action of $M_{n+1}(\mathbb{C})$ on \mathbb{C}^{n+1} , i.e. matrix multiplication. If w_j stands for the j -th element of the standard basis of \mathbb{C}^{n+1} , w_j is the weight vector corresponding to the weight $\mu_j = \lambda - \varepsilon_1 + \varepsilon_j$ of V . Indeed, for $i, j, k \in \{1, \dots, n + 1\}$, $i < j$, one has

$$H_{\varepsilon_i - \varepsilon_j} w_k = (E_{ii} - E_{jj}) w_k = \begin{cases} w_k & \text{if } i = k \\ -w_k & \text{if } j = k \\ 0 & \text{otherwise,} \end{cases}$$

i.e. $H_{\varepsilon_i - \varepsilon_j} w_k = \mu_k(H_{\varepsilon_i - \varepsilon_j}) w_k$. Due to weight restrictions, the only roots that may act nontrivially on w_k are either $\varepsilon_i - \varepsilon_k$, with $i < k$, or $-\varepsilon_k + \varepsilon_l$, with $k < l$. In other words,

$$\begin{aligned}
 {}^1\Delta^{(\mu_k)} &= \{\varepsilon_i - \varepsilon_k \mid 1 \leq i < k\}, \\
 \text{and } {}^2\Delta^{(\mu_k)} &= \{\varepsilon_k - \varepsilon_l \mid k < l \leq n + 1\}.
 \end{aligned}$$

Moreover, for $i, j, k \in \{1, \dots, n + 1\}$, one has

$$E_{\varepsilon_i - \varepsilon_j} w_k = (E_{ij} - E_{ji}) w_k = \begin{cases} w_i & \text{if } j = k \\ 0 & \text{otherwise.} \end{cases}$$

Therefore V satisfies the property (*) and we can use Theorem 5.1.2.

Proposition 5.2.2. *For the fundamental representation V of $\mathfrak{sl}(n + 1, \mathbb{C})$, one has*

$$\ker(\widehat{D}) = \bigoplus_{1 \leq k \leq n+1} \langle w_k \otimes e_{-\varepsilon_1 + \varepsilon_k} \wedge \dots \wedge e_{-\varepsilon_{k-1} + \varepsilon_k} \wedge e_J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\mu_k)} \cup {}^2\Delta^{(\mu_k)}) \rangle.$$

Proof. The proposition is a reformulation of Theorem 5.1.2. □

Corollary 5.2.3. For the fundamental representation V of $\mathfrak{sl}(n + 1, \mathbb{C})$, one has

$$\begin{aligned}\dim(\ker(\widehat{D})) &= (n + 1) \times 2^{\frac{n(n-1)}{2}} \\ \dim(\ker(D)) &= (n + 1)!\end{aligned}$$

Proof. Using Proposition 5.2.2, one has

$$\begin{aligned}\dim(\ker(\widehat{D})) &= \sum_k \dim(\langle w_k \otimes e_{-\varepsilon_1 + \varepsilon_k} \wedge \dots \wedge e_{-\varepsilon_{k-1} + \varepsilon_k} \wedge e_J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\mu_k)} \cup {}^2\Delta^{(\mu_k)}) \rangle) \\ &= \sum_k \#\{J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\mu_k)} \cup {}^2\Delta^{(\mu_k)})\} \\ &= \sum_k 2^{\frac{n(n-1)}{2}} \\ &= (n + 1) \times 2^{\frac{n(n-1)}{2}}.\end{aligned}$$

On the other hand, according to Theorem 3.2.4 and Table 1.6, one has

$$\dim \ker(D) = |W| = (n + 1)!$$

□

5.3 The Lie Algebra $B_n = \mathfrak{so}(2n + 1, \mathbb{C})$

Recall that

$$\mathfrak{g} := \mathfrak{so}(2n + 1, \mathbb{C}) = \{X \in \mathfrak{gl}(2n + 1, \mathbb{C}) \mid X + X^t = 0\}$$

while the subspace \mathfrak{t} generated by all the matrices H of the form

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} & \\ & & & & 0 \end{pmatrix}, \quad (5.3)$$

where $h_1, \dots, h_n \in \mathbb{C}$, is a Cartan subalgebra of \mathfrak{g} . For $i \in \{1, \dots, n\}$, let ε_i be the element of the dual space \mathfrak{t}^* given by $\varepsilon_i(H) = h_i$, where H is the above element of \mathfrak{t} . Then, the root system $\Delta(\mathfrak{g}, \mathfrak{t})$ of \mathfrak{g} is

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\pm\varepsilon_i \mid 1 \leq i \leq n\}.$$

Indeed, for $\alpha = \pm\varepsilon_i \pm \varepsilon_j$, with $1 \leq i < j \leq n$, let E_α be the matrix being everywhere 0 except for the entries corresponding to the i^{th} and j^{th} pairs which are given by

$$E_\alpha = \begin{pmatrix} & i & & j \\ 0 & & X_\alpha & \\ -X_\alpha^t & & & 0 \end{pmatrix} \begin{matrix} i \\ \\ j \\ \end{matrix},$$

with X_α given by

$$\begin{aligned} X_{\varepsilon_i - \varepsilon_j} &= \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, & X_{\varepsilon_i + \varepsilon_j} &= \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \\ X_{-\varepsilon_i + \varepsilon_j} &= \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, & X_{-\varepsilon_i - \varepsilon_j} &= \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \end{aligned}$$

If $\alpha = \pm\varepsilon_i$, define E_α to be the matrix having all its entries 0 except for the entries corresponding to the i^{th} pair and the $(2n+1)^{\text{th}}$ entry which are given by

$$E_\alpha = \begin{pmatrix} & i^{\text{th}} \text{ pair} & & (2n+1)^{\text{th}} \text{ entry} \\ 0 & & X_\alpha & \\ -X_\alpha^t & & & 0 \end{pmatrix}$$

where

$$X_{\varepsilon_i} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad X_{-\varepsilon_i} = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

One can easily verify that E_α is a root vector corresponding to the root α . We fix

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{\varepsilon_i \mid 1 \leq i \leq n\}$$

to be the set of positive roots of Δ . Then

$$\rho = (n - \frac{1}{2})\varepsilon_1 + (n - \frac{3}{2})\varepsilon_2 + \dots + \frac{1}{2}\varepsilon_n$$

is the half sum of the positive roots of Δ^+ , while, according to Table 1.5, $\lambda = \varepsilon_1$ is the highest weight of the fundamental representation V of \mathfrak{g} .

Lemma 5.3.1. *The fundamental representation V of $\mathfrak{so}(2n+1, \mathbb{C})$ is $(2n+1)$ -dimensional.*

Proof. Using Weyl dimensional formula of (see Proposition 1.4.35), one obtains

$$\begin{aligned} \dim(V) &= \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \\ &= \prod_{i < j} \left(1 + \frac{\langle \lambda, \varepsilon_i - \varepsilon_j \rangle}{\langle \rho, \varepsilon_i - \varepsilon_j \rangle}\right) \times \prod_{i < j} \left(1 + \frac{\langle \lambda, \varepsilon_i + \varepsilon_j \rangle}{\langle \rho, \varepsilon_i + \varepsilon_j \rangle}\right) \times \prod_i \left(1 + \frac{\langle \lambda, \varepsilon_i \rangle}{\langle \rho, \varepsilon_i \rangle}\right) \\ &= \prod_j \left(1 + \frac{\langle \lambda, \varepsilon_1 - \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_j \rangle}\right) \times \prod_j \left(1 + \frac{\langle \lambda, \varepsilon_1 + \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 + \varepsilon_j \rangle}\right) \times \left(1 + \frac{\langle \lambda, \varepsilon_1 \rangle}{\langle \rho, \varepsilon_1 \rangle}\right) \\ &= n \times \frac{2n-1}{n} \times \frac{2n+1}{2n-1} \\ &= 2n + 1. \end{aligned}$$

□

A construction of V is obtained by considering the vector space \mathbb{C}^{2n+1} endowed with the canonical action of $M_n(\mathbb{C})$, i.e. the matrix multiplication. Consider the vector space basis of \mathbb{C}^{2n+1} consisting of the elements

$$w_{\pm 1} = \begin{pmatrix} 1 \\ \mp i \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, w_{\pm 2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mp i \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, w_{\pm n} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \mp i \\ 0 \end{pmatrix} \quad \text{and} \quad w_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^{2n+1}.$$

The matrix H of (5.3) acts on the vector $w_{\pm i}$ by $\pm \varepsilon_i$ and on the vector w_0 by $\varepsilon_0 := 0$. Therefore, w_1 is a weight vector of the element λ and, since, using Lemma 5.3.1, one obtains that $\dim(V) = \dim(\mathbb{C}^{2n+1})$, we deduce that the constructed representation of \mathfrak{g} on \mathbb{C}^{2n+1} is equivalent to V . Consequently, the set of the weights $\Pi(V)$ of V is $\Pi(V) = \{\pm \varepsilon_i \mid 1 \leq i \leq n\} \cup \{\varepsilon_0\}$. Due to weight restrictions, the only roots that may act nontrivially on the w_k -weight space are either $\varepsilon_i - \varepsilon_k$, with $i < k$, either $-\varepsilon_k + \varepsilon_l$, with $k < l$, either $-\varepsilon_k - \varepsilon_l$, with $1 \leq l \leq n$, or $-\varepsilon_k$. In other words,

$$\begin{aligned} {}^1\Delta^{(\varepsilon_k)} &= \{\varepsilon_i - \varepsilon_k \mid 1 \leq i < k\}, \\ \text{and } {}^2\Delta^{(\varepsilon_k)} &= \{\varepsilon_k - \varepsilon_l \mid k < l \leq n\} \cup \{\varepsilon_k + \varepsilon_l \mid 1 \leq l \leq n\} \cup \{\varepsilon_k\} \end{aligned}$$

In the same way, for $-\varepsilon_k$, $k \neq 0$, one obtains that

$$\begin{aligned} {}^1\Delta^{(-\varepsilon_k)} &= {}^2\Delta^{(\varepsilon_k)}, \\ \text{and } {}^2\Delta^{(-\varepsilon_k)} &= {}^1\Delta^{(\varepsilon_k)}, \end{aligned}$$

while for ε_0 , one has

$${}^1\Delta^{(\varepsilon_0)} = \{\varepsilon_i \mid 1 \leq i \leq n\} = {}^2\Delta^{(\varepsilon_0)}.$$

Furthermore, V satisfies property (*). Indeed, for the weight vectors w_k and w_{-k} , we have $E_{\pm \varepsilon_i - \varepsilon_k} w_k = w_{\pm i}$ for every $i \neq k$, $E_{\pm \varepsilon_i + \varepsilon_k} w_{-k} = w_{\pm i}$, for every $i \neq k$, while for w_0 , $E_{\pm \varepsilon_i} w_0 = w_{\pm i}$. Therefore, we can use Theorem 5.1.2.

Proposition 5.3.2. *For the fundamental representation V of $\mathfrak{so}(2n + 1, \mathbb{C})$, one has*

$$\ker(\widehat{D}) = \bigoplus_{0 < k \leq n} \langle w_k \otimes e_{I^{(k)}} \wedge e_J, w_{-k} \otimes e_{I^{(-k)}} \wedge e_J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\varepsilon_k)} \cup {}^2\Delta^{(\varepsilon_k)}) \rangle,$$

where $e_{I^{(k)}}$ (respectively $e_{I^{(-k)}}$) stands for the monomial of $S = \bigwedge \mathfrak{n}^-$ containing all the root vectors $e_{-\alpha}$, with α being in ${}^1\Delta^{(\varepsilon_k)}$ (respectively for α being in ${}^1\Delta^{(-\varepsilon_k)}$).

Proof. The proposition is a reformulation of Theorem 5.1.2. □

Corollary 5.3.3. *For the fundamental representation V of $\mathfrak{so}(2n + 1, \mathbb{C})$, one has*

$$\begin{aligned} \dim(\ker(\widehat{D})) &= 2n \times 2^{(n-1)^2} \\ \dim(\ker(D)) &= 2^n n! \end{aligned}$$

where E_{ij} is the matrix having all its entries zero apart from the (i, j) -place where the entry is 1. One can easily verify that, for an element α of Δ , E_α corresponds to a root vector of the indicated root. We fix

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\} \cup \{2\varepsilon_i \mid 1 \leq i \leq n\} \subset \Delta$$

to be the set of positive roots of Δ . Then, the half sum ρ of the positive roots is

$$\rho = n\varepsilon_1 + (n-1)\varepsilon_2 + \dots + \varepsilon_n,$$

while, according to Table 1.5, the first fundamental weight λ of \mathfrak{g} is ε_1 .

Lemma 5.4.1. *The fundamental representation V of $\mathfrak{sp}(n, \mathbb{C})$ is $2n$ -dimensional.*

Proof. Using Weyl dimensional formula (see Proposition 1.4.35), one obtains that

$$\begin{aligned} \dim(V) &= \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \\ &= \prod_{i < j} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_i - \varepsilon_j \rangle}{\langle \rho, \varepsilon_i - \varepsilon_j \rangle}\right) \times \prod_{i < j} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_i + \varepsilon_j \rangle}{\langle \rho, \varepsilon_i + \varepsilon_j \rangle}\right) \times \prod_i \left(1 + \frac{\langle \varepsilon_1, 2\varepsilon_i \rangle}{\langle \rho, 2\varepsilon_i \rangle}\right) \\ &= \prod_{j > 1} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_1 - \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_j \rangle}\right) \times \prod_{j > 1} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_1 + \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 + \varepsilon_j \rangle}\right) \times \left(1 + \frac{\langle \varepsilon_1, 2\varepsilon_1 \rangle}{\langle \rho, 2\varepsilon_1 \rangle}\right) \\ &= n \times \frac{2n}{n+1} \times \frac{2(n+1)}{2n} \\ &= 2n. \end{aligned}$$

□

The standard representation V can be obtained as the vector space \mathbb{C}^{2n} endowed with the standard matrix multiplication action of $M_{2n}(\mathbb{C})$ on it. Indeed, consider the vector space basis of \mathbb{C}^{2n} consisting of the elements

$$w_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, w_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad w_{-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, w_{-n} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{C}^{2n}.$$

In other words, for i , $1 \leq i \leq n$, the vectors w_i and w_{-i} are respectively the i -th and the $n+i$ -th elements of the standard basis of \mathbb{C}^{2n} . Then, one can verify that the matrix H of (5.4) acts on $w_{\pm i}$ by $\pm\varepsilon_i$ and so w_1 is a weight vector of λ . Using Lemma 5.4.1, one obtains that $\dim(V) = \dim(\mathbb{C}^{2n})$, and so the constructed representation of \mathfrak{g} on \mathbb{C}^{2n} is equivalent to V . Consequently, the set of the weights $\Pi(V)$ of V is $\Pi(V) = \{\pm\varepsilon_i \mid 1 \leq i \leq n\}$. Due to weight restrictions, the only roots that may act nontrivially on the ε_k -weight space are either $\varepsilon_i - \varepsilon_k$, with $i < k$, either $-\varepsilon_k + \varepsilon_l$, with $k < l$, or $-\varepsilon_k - \varepsilon_l$, with $1 \leq l \leq n$. In other words,

$$\begin{aligned} {}^1\Delta^{(\varepsilon_k)} &= \{\varepsilon_i - \varepsilon_k \mid 1 \leq i < k\}, \\ \text{and } {}^2\Delta^{(\varepsilon_k)} &= \{\varepsilon_k - \varepsilon_l \mid k < l \leq n\} \cup \{\varepsilon_k + \varepsilon_l \mid 1 \leq l \leq n\}. \end{aligned}$$

In the same way, for $-\varepsilon_k$, one obtains that

$$\begin{aligned} {}^1\Delta^{(-\varepsilon_k)} &= {}^2\Delta^{(\varepsilon_k)}, \\ \text{and } {}^2\Delta^{(-\varepsilon_k)} &= {}^1\Delta^{(\varepsilon_k)}. \end{aligned}$$

We have that V satisfies property $(*)$. Indeed, for the weight vectors w_k and w_{-k} and for i , $1 \leq i \leq n$, we have $E_{\pm\varepsilon_i - \varepsilon_k} w_k = w_{\pm i}$ and $E_{\pm\varepsilon_i + \varepsilon_k} w_{-k} = w_{\pm i}$. Therefore, we can use Theorem 5.1.2.

Proposition 5.4.2. *For the fundamental representation V of $\mathfrak{sp}(2n, \mathbb{C})$, one has*

$$\ker(\widehat{D}) = \bigoplus_{0 < k \leq n} \langle w_k \otimes e_{I^{(k)}} \wedge e_J, w_{-k} \otimes e_{I^{(-k)}} \wedge e_J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\varepsilon_k)} \cup {}^2\Delta^{(\varepsilon_k)}) \rangle,$$

where $e_{I^{(k)}}$ (respectively $e_{I^{(-k)}}$) stands for the monomial of $S = \bigwedge \mathfrak{n}^-$ containing all the root vectors $e_{-\alpha}$, with α being in ${}^1\Delta^{(\varepsilon_k)}$ (respectively for α being in ${}^1\Delta^{(-\varepsilon_k)}$).

Proof. The proposition is a reformulation of Theorem 5.1.2. □

Corollary 5.4.3. *For the fundamental representation V of $\mathfrak{sp}(2n, \mathbb{C})$, one has*

$$\begin{aligned} \dim(\ker(\widehat{D})) &= 2n \times 2^{(n-1)^2} \\ \dim(\ker(D)) &= 2^n n! \end{aligned}$$

Proof. Using Proposition 5.3.2, one obtains that

$$\begin{aligned} \dim(\ker(\widehat{D})) &= 2 \sum_{0 < k \leq n} \dim(\langle w_k \otimes e_{I^{(k)}} \wedge e_J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\varepsilon_k)} \cup {}^2\Delta^{(\varepsilon_k)}) \rangle) \\ &= 2 \sum_{0 < k \leq n} \#\{J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\varepsilon_k)} \cup {}^2\Delta^{(\varepsilon_k)})\} \\ &= 2 \sum_{0 < k \leq n} 2^{(n-1)^2} \\ &= 2n \times 2^{(n-1)^2}. \end{aligned}$$

On the other hand, according to Theorem 3.2.4 and Table 1.6, one has

$$\dim \ker(D) = |W| = 2^n n!$$

□

5.5 The Lie Algebra $D_n = \mathfrak{so}(2n, \mathbb{C})$

This case is very similar with the case of $\mathfrak{so}(2n + 1, \mathbb{C})$. Let

$$\mathfrak{g} := \mathfrak{so}(2n, \mathbb{C}) = \{X \in \mathfrak{gl}(2n, \mathbb{C}) \mid X + X^t = 0\}.$$

The subalgebra \mathfrak{t} generated by all the matrices H of the form

$$H = \begin{pmatrix} \begin{pmatrix} 0 & ih_1 \\ -ih_1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & ih_2 \\ -ih_2 & 0 \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix} \end{pmatrix}, \quad (5.5)$$

where $h_1, \dots, h_n \in \mathbb{C}$, is a Cartan subalgebra of \mathfrak{g} . For $i \in \{1, \dots, n\}$, let ε_i be the element of the dual space \mathfrak{t}^* given by $\varepsilon_i(H) = h_i$, where H is the above element of \mathfrak{t} . Then, the root system $\Delta(\mathfrak{g}, \mathfrak{t})$ of \mathfrak{g} is

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}.$$

Indeed, for $\alpha = \pm\varepsilon_i \pm \varepsilon_j$, with $1 \leq i < j \leq n$, if E_α is the matrix obtained by deleting the $(2n + 1)$ -th row and column of the corresponding matrix defined for $\mathfrak{so}(2n + 1, \mathbb{C})$, E_α is a root vector corresponding to α . Fix

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}$$

to be the set of positive roots of Δ . Then

$$\rho = (n - 1)\varepsilon_1 + (n - 2)\varepsilon_2 + \dots + \varepsilon_{n-1}$$

is the half sum of the positive roots, while, according to Table 1.5, $\lambda = \varepsilon_1$ is the highest weight of the fundamental representation V of \mathfrak{g} .

Lemma 5.5.1. *The fundamental representation V of $\mathfrak{so}(2n, \mathbb{C})$ is $2n$ -dimensional.*

Proof. Using Weyl dimensional formula of (see Proposition 1.4.35), one obtains

$$\begin{aligned} \dim(V) &= \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \rho, \alpha \rangle}{\langle \rho, \alpha \rangle} \\ &= \prod_{i < j} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_i - \varepsilon_j \rangle}{\langle \rho, \varepsilon_i - \varepsilon_j \rangle}\right) \times \prod_{i < j} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_i + \varepsilon_j \rangle}{\langle \rho, \varepsilon_i + \varepsilon_j \rangle}\right) \\ &= \prod_{j > 1} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_1 - \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 - \varepsilon_j \rangle}\right) \times \prod_{j > 1} \left(1 + \frac{\langle \varepsilon_1, \varepsilon_1 + \varepsilon_j \rangle}{\langle \rho, \varepsilon_1 + \varepsilon_j \rangle}\right) \\ &= n \times 2 \\ &= 2n. \end{aligned}$$

□

A construction of V is given by considering the vector space \mathbb{C}^{2n} endowed with the canonical action of $M_{2n}(\mathbb{C})$, i.e. the matrix multiplication. Consider the vector space basis of \mathbb{C}^{2n+1} consisting of the elements

$$w_{\pm 1} = \begin{pmatrix} 1 \\ \mp i \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, w_{\pm 2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \mp i \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \dots, w_{\pm n} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ \mp i \end{pmatrix} \in \mathbb{C}^{2n}.$$

One can easily verify that $w_{\pm k}$ is a weight vector of $\pm \varepsilon_k$. Therefore, w_1 is a weight vector of λ and, since, using Lemma 5.5.1 one obtains $\dim(V) = \dim(\mathbb{C}^{2n})$, we deduce that the constructed representation of \mathfrak{g} on \mathbb{C}^{2n} is equivalent to V . Consequently, the set of the weights $\Pi(V)$ of V is $\Pi(V) = \{\pm \varepsilon_i \mid 1 \leq i \leq n\}$. Due to weight restrictions, the only roots that may act nontrivially on the w_k -weight space are either $\varepsilon_i - \varepsilon_k$, with $i < k$, either $-\varepsilon_k + \varepsilon_l$, with $k < l$, or $-\varepsilon_k - \varepsilon_l$, with $1 \leq l \leq n$. In other words,

$$\begin{aligned} {}^1\Delta^{(\varepsilon_k)} &= \{\varepsilon_i - \varepsilon_k \mid 1 \leq i < k\}, \\ \text{and } {}^2\Delta^{(\varepsilon_k)} &= \{\varepsilon_k - \varepsilon_l \mid k < l \leq n\} \cup \{\varepsilon_k + \varepsilon_l \mid 1 \leq l \leq n\}. \end{aligned}$$

In the same way, for $-\varepsilon_k$, one obtains that

$$\begin{aligned} {}^1\Delta^{(-\varepsilon_k)} &= {}^2\Delta^{(\varepsilon_k)}, \\ \text{and } {}^2\Delta^{(-\varepsilon_k)} &= {}^1\Delta^{(\varepsilon_k)}. \end{aligned}$$

Moreover, V satisfies property (*). Indeed, for the weight vectors w_k and w_{-k} , and for $i \neq k$, we have $E_{\pm \varepsilon_i - \varepsilon_k} w_k = w_{\pm i}$ and $E_{\pm \varepsilon_i + \varepsilon_k} w_{-k} = w_{\pm i}$. Therefore, we can use Theorem 5.1.2.

Proposition 5.5.2. *For the fundamental representation V of $\mathfrak{so}(2n, \mathbb{C})$, one has*

$$\ker(\widehat{D}) = \bigoplus_{0 < k \leq n} \langle w_k \otimes e_{I^{(k)}} \wedge e_J, w_{-k} \otimes e_{I^{(-k)}} \wedge e_J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\varepsilon_k)} \cup {}^2\Delta^{(\varepsilon_k)}) \rangle,$$

where $e_{I^{(k)}}$ (respectively $e_{I^{(-k)}}$) stands for the monomial of $S = \bigwedge \mathfrak{n}^-$ containing all the root vectors $e_{-\alpha}$, with α being in ${}^1\Delta^{(\varepsilon_k)}$ (respectively for α being in ${}^1\Delta^{(-\varepsilon_k)}$).

Proof. The proposition is a reformulation of Theorem 5.1.2. □

Corollary 5.5.3. *For the fundamental representation V of $\mathfrak{so}(2n, \mathbb{C})$, one has*

$$\begin{aligned} \dim(\ker(\widehat{D})) &= 2n \times 2^{(n-1)(n-2)} \\ \dim(\ker(D)) &= 2^{n-1}n! \end{aligned}$$

Proof. Using Proposition 5.3.2, one obtains that

$$\begin{aligned}
 \dim(\ker(\widehat{D})) &= 2 \sum_{0 < k \leq n} \dim(\langle w_k \otimes e_{I^{(k)}} \wedge e_J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\varepsilon_k)} \cup {}^2\Delta^{(\varepsilon_k)}) \rangle) \\
 &= 2 \sum_{0 < k \leq n} \#\{J \mid J \subseteq \Delta^+ \setminus ({}^1\Delta^{(\varepsilon_k)} \cup {}^2\Delta^{(\varepsilon_k)})\} \\
 &= 2 \sum_{0 < k \leq n} 2^{(n-1)(n-2)} \\
 &= 2n \times 2^{(n-1)(n-2)}.
 \end{aligned}$$

On the other hand, according to Theorem 3.2.4 and Table 1.6, one has

$$\dim \ker(D) = |W| = 2^{n-1}n!$$

□

CHAPTER 6

KERNEL OF NONCUBIC DIRAC OPERATORS FOR THE EXCEPTIONAL LIE ALGEBRAS

In this chapter, we apply the results of Chapter 4 in the case of exceptional Lie algebras when V is the standard representation of Example 1.4.25, i.e. the irreducible representation having as highest weight the first fundamental weight $\bar{\omega}_1$ of the corresponding Lie algebra \mathfrak{g} . The necessary information about exceptional Lie algebras and their root decompositions can be found in [FH91], [Jac71] or [Kna96].

6.1 Preliminaries

In this chapter, \mathfrak{g} will stand for an exceptional Lie algebra, Δ will be the root system of \mathfrak{g} , $\Pi := \{\alpha_1, \dots, \alpha_l\}$ will be the set of simple roots of Δ and Δ^+ will be the set of positive roots determined by Π . We are always interested in calculating the kernel of the noncubic Dirac operator \widehat{D} in the case where the highest weight λ of the \mathfrak{g} -representation V is the first fundamental weight $\bar{\omega}_1$ of \mathfrak{g} . We use the notation introduced in the proof of Lemma 4.7.4. Namely,

$$\begin{aligned}\Delta_1 &:= \{\alpha \in \Delta \mid \langle \lambda, \alpha \rangle = 0\}, \\ \Delta_2 &:= \Delta \setminus \Delta_1, \\ \Delta_i^+ &:= \Delta_i \cap \Delta^+, \quad i = 1, 2.\end{aligned}$$

Let ρ_1 be the half-sum of the positive roots of Δ_1^+ . As we have already shown in the proof of Lemma 4.7.4, Δ_1 is a root system with Δ_1^+ being the set of positive roots. Moreover, one can verify that $\Delta_1^+ = \Delta^+ \cap \langle \alpha_2, \dots, \alpha_l \rangle$ and thus $\Pi_1 := \{\alpha_2, \dots, \alpha_l\}$ is the set of simple roots for Δ_1^+ . Consequently, the Dynkin diagram of Δ_1 is obtained from the Dynkin diagram of Δ by deleting the edge corresponding to the simple root α_1 .

To calculate $\ker(\widehat{D})$, according to Theorem 4.6.13, it suffices to determine the set $A(\lambda)$. In

other words, using Proposition 4.7.1, it suffices to find all dominant weights of the form

$$\rho - \sum_{\alpha \in A} \alpha, \quad A \subseteq \Delta_1^+.$$

To find these weights, we will need the following lemma.

Lemma 6.1.1. *With the above notation, let A be a subset of Δ_1^+ . The weight $\rho - \sum_{\alpha \in A} \alpha$ is Δ^+ -dominant if and only if the weight $\rho_1 - \sum_{\alpha \in A} \alpha$ is Δ_1^+ -dominant.*

Proof. If $\rho - \sum_{\alpha \in A} \alpha$ is Δ^+ -dominant, obviously it is Δ_1^+ -dominant. Moreover, $\rho_1 - \sum_{\alpha \in A} \alpha$ is Δ_1^+ -dominant. Indeed, as we have already seen in the proof of Proposition 4.7.1, $\langle \rho - \rho_1, \beta \rangle = 0$ for every β in Δ_1^+ and so

$$\begin{aligned} \langle \rho_1 - \sum_{\alpha \in A} \alpha, \beta \rangle &= \langle (\rho - \rho_1) + \rho - \sum_{\alpha \in A} \alpha, \beta \rangle \\ &= \langle \rho - \sum_{\alpha \in A} \alpha, \beta \rangle. \end{aligned} \tag{6.1}$$

Therefore, $\rho_1 - \sum_{\alpha \in A} \alpha$ is Δ_1^+ -dominant if and only if $\rho - \sum_{\alpha \in A} \alpha$ is Δ_1^+ -dominant, which is the case for $\rho - \sum_{\alpha \in A} \alpha$ being Δ^+ -dominant.

On the other hand, if $\rho_1 - \sum_{\alpha \in A} \alpha$ is Δ_1^+ -dominant, then, as we have seen in the proof of Proposition 4.7.1, $\rho - \sum_{\alpha \in A} \alpha$ is Δ^+ -dominant. For the sake of completeness, we repeat the argument. Suppose that

$$\langle \rho - \sum_{\alpha \in A} \alpha, \beta \rangle < 0,$$

for some β in Δ_2^+ . Then, the element $s_\beta(\rho - \sum_{\alpha \in A} \alpha) = \rho - \sum_{\alpha \in A} \alpha + n\beta$, with n being some positive integer, is a weight of S . Thus, there is some subset Γ of Δ^+ such that

$$\rho - \sum_{\alpha \in A} \alpha + n\beta = \rho - \sum_{\gamma \in \Gamma} \gamma.$$

Hence

$$\begin{aligned} \langle \rho - \sum_{\alpha \in A} \alpha + n\beta, \lambda \rangle &= \langle \rho - \sum_{\gamma \in \Gamma} \gamma, \lambda \rangle \\ \Rightarrow n\langle \beta, \lambda \rangle &= - \sum_{\gamma \in \Gamma} \langle \gamma, \lambda \rangle. \end{aligned}$$

Since the right-hand side is positive while the left-hand side is nonpositive, this is a contradiction. Therefore, $\langle \rho - \sum_{\alpha \in A} \alpha, \beta \rangle > 0$, for every β in Δ_2^+ , while from the relation (6.1), $\rho - \sum_{\alpha \in A} \alpha$ is Δ_1^+ -dominant. This proves the statement. \square

Using the above lemma, in order to determine $A(\lambda)$, it suffices to find all the Δ_1^+ -dominant weights of the form

$$\rho_1 - \sum_{\alpha \in A} \alpha. \tag{6.2}$$

On the other hand, Δ_1 is the root system that one obtains by deleting by the Dynkin diagram of \mathfrak{g} the edge corresponding to the first simple root α_1 . Let \mathfrak{g}_1 be the Lie algebra corresponding to Δ_1 and choose \mathfrak{h}_1 to be a Cartan subalgebra of \mathfrak{g}_1 . Then $\mathfrak{q}_1 := \mathfrak{h}_1^\perp = \mathfrak{n}_1 \oplus \mathfrak{n}_1^-$, where \mathfrak{n}_1 (respectively \mathfrak{n}_1^-) is the sum of all positive (respectively negative) root spaces of \mathfrak{g}_1 . Then the Δ_1^+ -dominant weights of the form (6.2) are exactly the Δ_1^+ -dominant weights of the space of spinors S_1 of the Clifford algebra $\mathbf{C}(\mathfrak{n}_1 \oplus \mathfrak{n}_1^-)$.

6.2 The Lie Algebra E_6

Consider the vector space \mathbb{R}^8 , let $\{\varepsilon_i\}$ be its standard basis and set

$$V := \{v \in \mathbb{R}^8 \mid \langle \varepsilon_6 - \varepsilon_7, v \rangle = \langle \varepsilon_7 + \varepsilon_8, v \rangle = 0\}.$$

The root space of E_6 is the subset

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j \leq 5\} \cup \left\{ \frac{1}{2} \sum_{i=1}^8 (-1)^{n(i)} \varepsilon_i \in V \mid \sum_{i=1}^8 n(i) \text{ even} \right\}$$

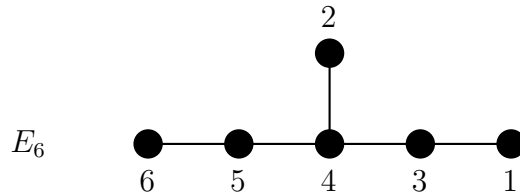
of V . Then

$$\begin{aligned} \Pi := \{ & \alpha_1 := \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5 - \varepsilon_4 - \varepsilon_3 - \varepsilon_2 + \varepsilon_1), \alpha_2 := \varepsilon_2 + \varepsilon_1, \\ & \alpha_3 := \varepsilon_2 - \varepsilon_1, \alpha_4 := \varepsilon_3 - \varepsilon_2, \alpha_5 := \varepsilon_4 - \varepsilon_3, \alpha_6 := \varepsilon_5 - \varepsilon_4 \} \end{aligned}$$

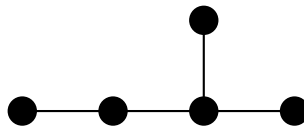
is a simple system for Δ and

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid i > j\} \cup \left\{ \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 + \sum_{i=1}^5 (-1)^{n(i)} \varepsilon_i) \mid \sum_{i=1}^5 n(i) \text{ even} \right\}$$

is the corresponding positive system determined by Π . Moreover, the Dynkin diagram of E_6 is



and by deleting the edge corresponding to α_1 , one obtains the diagram



which, according to Table 1.7 is the Dynkin diagram corresponding to the classical Lie algebra $D_5 = \mathfrak{so}(10, \mathbb{C})$. In other words, Δ_1 is the root system of the Lie algebra $\mathfrak{so}(10, \mathbb{C})$ and there is a canonical embedding of $\mathfrak{so}(10, \mathbb{C})$ into E_6 . Consider the space of spinors S_1 of the Clifford

algebra $\mathbb{C}(\mathfrak{n}_1 \oplus \mathfrak{n}_1^-)$, where \mathfrak{n}_1 and \mathfrak{n}_1^- are the sum of the positive and negative respectively root spaces of $\mathfrak{so}(10, \mathbb{C})$. Then, the weights (6.2) we are looking for are the Δ_1^+ -dominant weights of S_1 and the set $A(\lambda)$ for E_6 is

$$A(\lambda) = \{\lambda + \nu_1 \mid \nu_1 \text{ is } \Delta_1^+ \text{-dominant weight of } S_1\}.$$

Hence

$$\ker(\widehat{D}) = \bigoplus_{\mu \in A(\lambda)} (V \otimes S)^{[\mu]}.$$

6.3 The Lie Algebra E_7

Consider the vector space \mathbb{R}^8 , let $\{\varepsilon_i\}$ be its standard basis and set

$$V := \{v \in \mathbb{R}^8 \mid \langle \varepsilon_7 + \varepsilon_8, v \rangle = 0\}.$$

The root space of E_7 is the subset

$$\Delta := \{\pm \varepsilon_i \pm \varepsilon_j \mid i < j \leq 6\} \cup \{\pm(\varepsilon_7 - \varepsilon_8)\} \cup \left\{ \frac{1}{2} \sum_{i=1}^8 (-1)^{n(i)} \varepsilon_i \in V \mid \sum_{i=1}^8 n(i) \text{ even} \right\}.$$

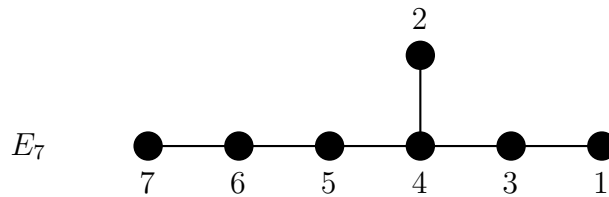
of V . Then

$$\begin{aligned} \Pi := \{ & \alpha_1 := \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5 - \varepsilon_4 - \varepsilon_3 - \varepsilon_2 + \varepsilon_1), \alpha_2 := \varepsilon_2 + \varepsilon_1, \\ & \alpha_3 := \varepsilon_2 - \varepsilon_1, \alpha_4 := \varepsilon_3 - \varepsilon_2, \alpha_5 := \varepsilon_4 - \varepsilon_3, \alpha_6 := \varepsilon_5 - \varepsilon_4, \alpha_7 := \varepsilon_6 - \varepsilon_5 \} \end{aligned}$$

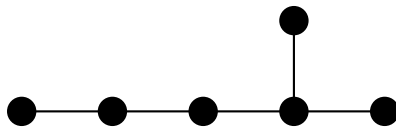
is a simple system of Δ and

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid i > j\} \cup \left\{ \frac{1}{2}(\varepsilon_8 - \varepsilon_7 + \sum_{i=1}^6 (-1)^{n(i)} \varepsilon_i) \mid \sum_{i=1}^6 n(i) \text{ odd} \right\}$$

is the corresponding positive system determined by Π . Moreover, the Dynkin diagram of E_7 is



and by deleting the edge corresponding to α_1 , one obtains the diagram



which, according to Table 1.7 is the Dynkin diagram corresponding to the classical Lie algebra $D_6 = \mathfrak{so}(12, \mathbb{C})$. In other words, Δ_1 is the root system of the Lie algebra $\mathfrak{so}(12, \mathbb{C})$ and there

is a canonical embedding of $\mathfrak{so}(12, \mathbb{C})$ into E_7 . Consider the space of spinors S_1 of the Clifford algebra $\mathbb{C}(\mathfrak{n}_1 \oplus \mathfrak{n}_1^-)$, where \mathfrak{n}_1 and \mathfrak{n}_1^- are the sum of the positive and negative respectively root spaces of $\mathfrak{so}(12, \mathbb{C})$. Then, the weights (6.2) we are looking for are the Δ_1^+ -dominant weights of S_1 and the set $A(\lambda)$ for E_7 is

$$A(\lambda) = \{\lambda + \nu_1 \mid \nu_1 \text{ is } \Delta_1^+ \text{-dominant weight of } S_1\}.$$

Hence

$$\ker(\widehat{D}) = \bigoplus_{\mu \in A(\lambda)} (V \otimes S)^{[\mu]}.$$

6.4 The Lie Algebra E_8

Consider the vector space $V := \mathbb{R}^8$ and let $\{\varepsilon_i\}$ be its standard basis. The root space of E_8 is

$$\Delta := \{\pm\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \left\{ \frac{1}{2} \sum_{i=1}^8 (-1)^{n(i)} \varepsilon_i \in V \mid \sum_{i=1}^8 n(i) \text{ even} \right\}$$

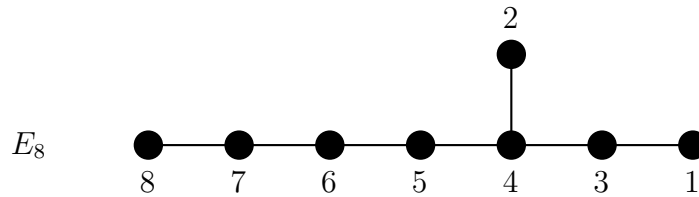
Then

$$\begin{aligned} \Pi := \{ & \alpha_1 := \frac{1}{2}(\varepsilon_8 - \varepsilon_7 - \varepsilon_6 - \varepsilon_5 - \varepsilon_4 - \varepsilon_3 - \varepsilon_2 + \varepsilon_1), \alpha_2 := \varepsilon_2 + \varepsilon_1, \alpha_3 := \varepsilon_2 - \varepsilon_1, \\ & \alpha_4 := \varepsilon_3 - \varepsilon_2, \alpha_5 := \varepsilon_4 - \varepsilon_3, \alpha_6 := \varepsilon_5 - \varepsilon_4, \alpha_7 := \varepsilon_6 - \varepsilon_5, \alpha_8 := \varepsilon_7 - \varepsilon_6 \} \end{aligned}$$

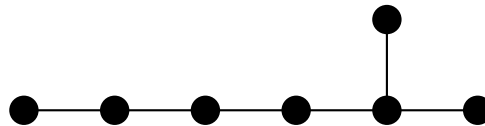
is a simple system of Δ and

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid i > j\} \cup \left\{ \frac{1}{2}(\varepsilon_8 + \sum_{i=1}^7 (-1)^{n(i)} \varepsilon_i) \mid \sum_{i=1}^7 n(i) \text{ even} \right\}$$

is the corresponding positive system determined by Π . Moreover, the Dynkin diagram of E_8 is



and by deleting the edge corresponding to α_1 , one obtains the diagram



which, according to Table 1.7 is the Dynkin diagram corresponding to the classical Lie algebra $D_4 = \mathfrak{so}(14, \mathbb{C})$. In other words, Δ_1 is the root system of the Lie algebra $\mathfrak{so}(14, \mathbb{C})$ and there is a canonical embedding of $\mathfrak{so}(14, \mathbb{C})$ into E_8 . Consider the space of spinors S_1 of the Clifford

algebra $\mathbb{C}(\mathfrak{n}_1 \oplus \mathfrak{n}_1^-)$, where \mathfrak{n}_1 and \mathfrak{n}_1^- are the sum of the positive and negative respectively root spaces of $\mathfrak{so}(14, \mathbb{C})$. Then, the weights (6.2) we are looking for are the Δ_1^+ -dominant weights of S_1 and the set $A(\lambda)$ for E_8 is

$$A(\lambda) = \{\lambda + \nu_1 \mid \nu_1 \text{ is } \Delta_1^+ \text{-dominant weight of } S_1\}.$$

Hence

$$\ker(\widehat{D}) = \bigoplus_{\mu \in A(\lambda)} (V \otimes S)^{[\mu]}.$$

6.5 The Lie Algebra F_4

Let $V := \mathbb{R}^4$ and

$$\Delta := \{\pm \varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{\pm \varepsilon_i\} \cup \left\{ \frac{1}{2}(\pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \right\},$$

where ε_i is the element of the dual space of \mathbb{R}^4 sending the i -th element of the standard basis to 1 and all the others to 0. The set Δ is the root system of F_4 and we fix

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j \mid i < j\} \cup \{\varepsilon_i\} \cup \left\{ \frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \right\}$$

to be the set of positive roots of Δ . Then,

$$\Pi := \left\{ \alpha_1 := \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4), \alpha_2 := \varepsilon_4, \alpha_3 := \varepsilon_3 - \varepsilon_4, \alpha_4 := \varepsilon_2 - \varepsilon_3 \right\}$$

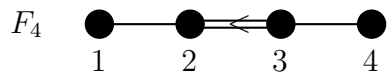
is the set of simple roots of Δ . The highest weight λ of the standard representation is $\lambda = \bar{\omega}_1 = \varepsilon_1$ and the set of roots being orthogonal with λ is

$$\Delta_1 = \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 < i < j\} \cup \{\pm \varepsilon_i \mid 1 < i\}.$$

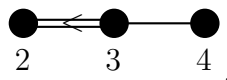
Then, the half sum of the positive roots of Δ_1^+ is $\rho_1 = \frac{5}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$ and

$$\Pi_1 := \{\alpha_2, \alpha_3, \alpha_4\}$$

is the set of simple roots of Δ_1 . The Dynkin diagram of F_4 is



By deleting the edge corresponding to α_1 , we obtain the Dynkin diagram of Δ_1 :



This is exactly the Dynkin diagram of the Lie algebra B_3 . With the previous notation, the Δ_1^+ -dominant weights of the spin module S_1 for B_3 , are given in Table 6.1.

Weight	Vectors
$\frac{5}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	1
$\frac{5}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_3}$ $e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{3}{2}\varepsilon_4$	$e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{3}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2}$ $e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3}$ $e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_4}$
$\frac{3}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_4}$ $e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2-\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}$
$\frac{1}{2}\varepsilon_2 + \frac{1}{2}\varepsilon_3 + \frac{1}{2}\varepsilon_4$	$e_{-\varepsilon_2} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_2-\varepsilon_3}$ $e_{-\varepsilon_2} \wedge e_{-\varepsilon_3} \wedge e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_3+\varepsilon_4}$ $e_{-\varepsilon_4} \wedge e_{-\varepsilon_2+\varepsilon_3} \wedge e_{-\varepsilon_2+\varepsilon_4} \wedge e_{-\varepsilon_3+\varepsilon_4} \wedge e_{-\varepsilon_3-\varepsilon_4}$

Table 6.1: Δ_1^+ -dominant weights of S_1

According to the discussion in the beginning of this chapter, the set $A(\lambda)$ consists of all elements of the form $\rho - \rho_1 + \mu_1$, with μ_1 being a weight of the above matrix. Using Theorem 4.6.13, the kernel of the noncubic Dirac operator \widehat{D} is

$$\ker(\widehat{D}) = \bigoplus_{\mu_1} (V \otimes S)^{[(\rho - \rho_1) + \mu_1]},$$

where μ_1 runs over the set of the weights of the above table. The dimension of each isotypic component $(V \otimes S)^{[(\rho - \rho_1) + \mu_1]}$ is given by Table 6.1.

6.6 The Lie Algebra G_2

Let $V := \{v \in \mathbb{R}^3 \mid \langle \varepsilon_1 + \varepsilon_2 + \varepsilon_3, v \rangle = 0\}$, where $\{\varepsilon_i\}$ stands for the standard basis of \mathbb{R}^3 . The root system of G_2 is

$$\begin{aligned} \Delta := & \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_2 - \varepsilon_3), \pm(\varepsilon_1 - \varepsilon_3)\} \\ & \cup \{\pm(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3), \pm(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3), \pm(2\varepsilon_3 - \varepsilon_1 - \varepsilon_2)\} \end{aligned}$$

while we fix

$$\begin{aligned} \Delta^+ := & \{\varepsilon_1 - \varepsilon_2, -\varepsilon_2 + \varepsilon_3, -\varepsilon_1 + \varepsilon_3\} \\ & \cup \{-2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, -2\varepsilon_2 + \varepsilon_1 + \varepsilon_3, 2\varepsilon_3 - \varepsilon_1 - \varepsilon_2\} \end{aligned}$$

to be the set of positive roots of Δ . The set of simple roots is

$$\Pi := \{\alpha_1 := \varepsilon_1 - \varepsilon_2, \alpha_2 := -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$$

and the highest weight λ of the standard representation is $\lambda = \bar{\omega}_1 = -\varepsilon_2 + \varepsilon_3$. Then, the set of roots annihilated by λ is $\Delta_1 := \{\alpha_2\}$ and the stabilizer of λ in W is $W_\lambda = W(\Delta_1) = \{1, s_{\alpha_2}\}$, i.e. the Weyl group generated by Δ_1 . One can verify that $s_{\alpha_2}(\rho) = \rho - \alpha_2$ and thus W_λ acts transitively on the set $\{\rho, \rho - \alpha_2\}$. Using Proposition 4.7.5, one deduces that $\ker(\widehat{D}) = \ker(D)$. Therefore, according to Theorem 3.2.4,

$$\ker(\widehat{D}) = \bigoplus_{w \in W} (V \otimes S)_{w(\lambda + \rho)},$$

with $(V \otimes S)_{w(\lambda + \rho)}$ being the 1-dimensional isotypic component corresponding to the weight $w(\lambda + \rho)$.

For the sake of completeness, let us say that the set Δ_1 of all roots orthogonal to λ is $\Delta_1 = \{\alpha_2 = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3\}$ while the corresponding Dynkin diagram has only one edge, i.e., according to Table 1.7, it is the Dynkin diagram of the classical Lie algebra $A_1 = \mathfrak{sl}(2, \mathbb{C})$.

CHAPTER 7

GEOMETRIC DIRAC OPERATORS

In this chapter we introduce and study the cubic and noncubic geometric Dirac operator \mathcal{D} and $\widehat{\mathcal{D}}$, respectively. More precisely, we start by developing the basic notions of homogeneous spaces and homogeneous vector bundles in Section 7.1. Next, in Section 7.2 we present Peter-Weyl Theorem for compact Lie groups and we discuss the decomposition of the space of sections of homogeneous vector bundles into a sum of finite-dimensional irreducible subrepresentations. In Section 7.3, we consider a particular homogeneous vector bundle and we study the cubic geometric Dirac operator, while in Sections 7.4 and 7.5 we deal with the corresponding noncubic and t -cubic geometric Dirac operators. Finally, in Section 7.6, we compute the Dirac indices of both algebraic and geometric cubic or noncubic Dirac operators.

7.1 Homogeneous spaces and homogeneous vector bundles

Let G be a compact connected Lie group, H be a Lie subgroup of G and \mathfrak{g}_0 and \mathfrak{h}_0 be respectively the Lie algebras of G and H while \mathfrak{g} and \mathfrak{h} are their complexifications. Consider the quotient G/H .

Proposition 7.1.1. *The quotient G/H admits the structure of a smooth manifold. Its tangent space $T_{eG}G/H$ based at the point eG coincides with the vector space $\mathfrak{g}_0/\mathfrak{h}_0$.*

Proof. The fact that G/H is a smooth manifold is an application of the Quotient Manifold Theorem (see [Lee13, Theorem 7.10]). For the second statement see [Pau, Theorem 1.64]. \square

Definition 7.1.2 (Homogeneous space). *The quotient G/H is called homogeneous space for G .*

Example 7.1.3. Let $G := SO(n+1)$ and $H := SO(n)$, with H being embedded in G by the map

$$x \in SO(n) \mapsto \tilde{x} := \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \in SO(n+1).$$

Then $X := SO(n+1)/SO(n)$ is a homogeneous space. It turns out to be diffeomorphic with the n -dimensional sphere \mathbb{S}_n (see [Pau, Subsection 1.5.5]).

Note that the homogeneous space G/H admits a canonical smooth left G -action by left multiplication. Suppose, now, that (π, E) is a (complex) finite dimensional representation of H equipped with a norm $\|\cdot\|_E$. Set

$$\mathcal{E} := G \times E / \sim,$$

where \sim is the equivalence relation identifying $(gh, h^{-1}v) \sim (g, v)$ for every $g \in G$, $h \in H$ and $v \in E$, and consider the canonical projection

$$p : \begin{cases} \mathcal{E} & \rightarrow G/H \\ [g, v] & \mapsto gH. \end{cases}$$

Proposition 7.1.4. *The pair (p, \mathcal{E}) is a vector bundle over G/H , the fiber being $p^{-1}(gH) \simeq E$.*

We equip \mathcal{E} with the smooth left G -action given by

$$x[g, v] := [xg, v]$$

for every $x, g \in G$ and every $v \in E$. This action is compatible with the map p , i.e., for every $x, g \in G$ and every $v \in E$,

$$p(x[g, v]) = xp([g, v]).$$

Definition 7.1.5 (Homogeneous vector bundle). *The vector bundle (p, \mathcal{E}) equipped with the above action of G is called the homogeneous vector bundle over G/H induced by E .*

Let $\Gamma(G/H, \mathcal{E})$ be the space of smooth sections of (p, \mathcal{E}) , i.e. the space of all smooth maps $\tilde{f} : G/H \rightarrow \mathcal{E}$ such that $p \circ \tilde{f} = id_{G/H}$. We equip $\Gamma(G/H, \mathcal{E})$ with a smooth left G -action given by

$$x \cdot \tilde{f} : g \mapsto x\{\tilde{f}(x^{-1}g)\}. \quad (7.1)$$

Moreover, consider the vector space $\mathcal{C}^\infty(G)$ of all smooth functions of G . Recall that G (and thus H) acts by right translations on $\mathcal{C}^\infty(G)$ via the map

$$r(x) : f \mapsto (r(x)f : g \mapsto f(gx)) \quad (7.2)$$

for every $g, x \in G$. Then H acts on $\mathcal{C}^\infty(G) \otimes E$ and set

$$[\mathcal{C}^\infty(G) \otimes E]^H$$

to be the set of all H -stable vector of $\mathcal{C}^\infty(G) \otimes E$. In other words,

$$[\mathcal{C}^\infty(G) \otimes E]^H = \{f : G \rightarrow E \mid f \text{ is smooth, } f(gh^{-1}) = hf(g), \forall g \in G, \forall h \in H\}. \quad (7.3)$$

Moreover, equip $[\mathcal{C}^\infty(G) \otimes E]^H$ with the $L^2(G, E)$ -norm, i.e. the norm given by

$$\|f\|^2 := \int_G \|f(x)\|_E^2 dx$$

for every $f \in [\mathcal{C}^\infty(G) \otimes E]^H$, where dx is a bi-invariant Haar measure of G , i.e. an outer regular Borelian finite measure on G which is invariant by left and right translations (for more details see [Kna96, Chapter 4]). Define a smooth left G -action on $[\mathcal{C}^\infty(G) \otimes E]^H$, given by left translations:

$$x \cdot f := l(x)f : g \mapsto f(x^{-1}g) \quad (7.4)$$

for every f in $[\mathcal{C}^\infty(G) \otimes E]^H$ and every g, x in G . One can check that this action is well-defined.

Definition 7.1.6. The representation $[\mathcal{C}^\infty(G) \otimes E]^H$ of G is called the smooth representation induced by E (or the induced representation of E) and it is denoted by $\text{Ind}_H^G(E)$.

Theorem 7.1.7 (Frobenius reciprocity, see [GK94, Subsection 3.3]). Let V be a representation of G and E be a representation of H . If $\text{Res}_H^G(V)$ stands for the restriction of V to a representation of H , then

$$\text{Hom}_G(V, \text{Ind}_H^G(E)) \simeq \text{Hom}_H(\text{Res}_H^G(V), E).$$

Theorem 7.1.8 (See [Bot57]). As representations of G , $\Gamma(G/H, \mathcal{E})$ and $\text{Ind}_H^G(E)$ are equivalent, the isomorphism being

$$[\mathcal{C}^\infty(G) \otimes E]^H \ni f \mapsto (\tilde{f} : gH \mapsto [g, f(g)]) \in \Gamma(G/H, \mathcal{E}).$$

7.2 Peter-Weyl Theorem for homogeneous vector bundles

Let G be a compact semisimple Lie group, $L^2(G)$ be the Hilbert space of square-integrable functions defined on G and $\mathcal{C}^\infty(G)$ be the subspace of $L^2(G)$ of all smooth functions defined on G .

Proposition 7.2.1 (See [Kna96, Theorem 4.20]). The subspace $\mathcal{C}^\infty(G)$ is dense in the Hilbert space $L^2(G)$.

On $L^2(G)$, consider the representation $l \times r$ of the Lie group $G \times G$ by left and right translations constructed in Example 1.5.5. More precisely, for every $f \in L^2(G)$ and every $g, x, y \in G$, $l \times r$ is given by

$$(l \times r)(x, y)f(g) = f(x^{-1}gy).$$

Theorem 7.2.2 (Peter-Weyl, see [Dei05, Theorem 11.4.1] or [PW27]). Let G be a compact Lie group and \widehat{G} be the set of equivalence classes of all irreducible unitary representations of G . Suppose that for every $\pi \in \widehat{G}$, V_π is a representative for π and V_π^* is the dual representation of V_π (see Example 1.5.4). Then, as representations of $G \times G$,

$$L^2(G) \simeq \bigoplus_{\pi \in \widehat{G}} V_\pi \otimes V_\pi^*, \tag{7.5}$$

with the action of $G \times G$ on each component $V_\pi \otimes V_\pi^*$ being given as follows: the subgroup $G \times \{1\} \simeq G$ of $G \times G$ acts on the first factor V_π via the representation π and trivially on the second factor V_π^* , while the subgroup $\{1\} \times G \simeq G$ acts trivially on the first factor V_π and via the representation π^* on the second factor V_π^* .

Let H be a closed subgroup of G , E be an irreducible representation of H and \mathcal{E} be the homogeneous vector bundle over G/H induced by E . Then, according to Theorem 7.1.8, the space of smooth sections $\Gamma(G/H, \mathcal{E})$ of \mathcal{E} coincides, as representation of G , with $[\mathcal{C}^\infty(G) \otimes E]^H$. Using Proposition 7.2.1, one can show that $[\mathcal{C}^\infty(G) \otimes E]^H$ is dense in $[L^2(G) \otimes E]^H$. We are interested in a decomposition of $[L^2(G) \otimes E]^H$ similar to (7.5). Consider the space $L^2(G) \otimes E$.

Then, $L^2(G) \otimes E$ admits an action of $G \times H$, $G \times \{1\}$ acting only on the first factor by left translations and $\{1\} \times H$ acting on $L^2(G)$ by right translations and on E by the given action of H on E . Moreover, the isomorphism establishing (7.5) can be canonically extended to an isomorphism

$$\Phi : L^2(G) \otimes E \xrightarrow{\sim} \widehat{\bigoplus_{\pi \in \widehat{G}} V_\pi \otimes V_\pi^* \otimes E}. \quad (7.6)$$

For each component $V_\pi \otimes V_\pi^* \otimes E$, $G \times \{1\}$ acts only on the first factor by π while $\{1\} \times H$ acts only on the second factor by π^* and on the third factor by the given representation of H on E . Then, the map Φ is $G \times H$ -equivariant and thus the representations of $G \times H$ involved in (7.6) are equivalent. Furthermore, Φ establishes an isomorphism of the $G \times \{1\}$ -representations $[L^2(G) \otimes E]^H$ and $\widehat{\bigoplus_{\pi \in \widehat{G}} V_\pi \otimes [V_\pi^* \otimes E]^H}$ of $(\{1\} \times H)$ -stable vectors. In other words, we obtain the following theorem.

Theorem 7.2.3. *Let G be a compact connected semisimple Lie group, H be a closed subgroup of G and E be a representation of H . Then, as representations of $G \simeq G \times \{1\}$, one has*

$$\Phi : [L^2(G) \otimes E]^H \xrightarrow{\sim} \widehat{\bigoplus_{\pi \in \widehat{G}} V_\pi \otimes [V_\pi^* \otimes E]^H}. \quad (7.7)$$

7.3 Twisted spin bundles and geometric cubic Dirac operators

Let G be a real compact connected semisimple Lie group, H be a closed subgroup of G and let \mathfrak{g} and \mathfrak{h} be respectively the complexified Lie algebras of G and H . Moreover, suppose that \mathfrak{h} satisfies the condition (2.10) and let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q},$$

where $\mathfrak{q} = \mathfrak{h}^\perp$. Consider the Clifford algebra $\mathbf{C}(\mathfrak{q})$ of \mathfrak{q} and let S be a space of spinors for it. Then \mathfrak{h} acts on S by the spin representation

$$\mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{q}) \rightarrow \mathbf{C}(\mathfrak{q}) \rightarrow \text{End } S$$

of Definition 2.4.1. Furthermore, consider a finite dimensional representation E of \mathfrak{h} such that $S \otimes E$ integrates to a representation of H and let $\mathcal{S} \otimes \mathcal{E}$ stand for the corresponding induced homogeneous bundle over G/H . Then, the space of smooth sections $\Gamma(G/H, \mathcal{S} \otimes \mathcal{E})$ of $\mathcal{S} \otimes \mathcal{E}$ admits the action (7.1) of G and, according to Theorem 7.1.8, as representation of G , it is equivalent with the induced representation $[\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$. Therefore, the smooth sections of $\Gamma(G/H, \mathcal{S} \otimes \mathcal{E})$ can be seen as smooth functions $f : G \rightarrow S \otimes E$ satisfying the following equivariance condition:

$$f(gh^{-1}) = hf(g), \quad \forall g \in G, \forall h \in H.$$

We still denote by r the map

$$r : \mathfrak{g} \times \mathcal{C}^\infty(G) \rightarrow \mathcal{C}^\infty(G),$$

obtained by differentiating the map (7.2) of right translations. More precisely, r is the map given by

$$r(X) : f \mapsto (r(X)f : g \mapsto \frac{d}{dt}|_{t=0} f(g \exp tX)) \quad (7.8)$$

for every $X \in \mathfrak{g}$ and every $g \in G$. Define $\mathcal{D}' : \mathcal{C}^\infty(G) \otimes (S \otimes E) \rightarrow \mathcal{C}^\infty(G) \otimes (S \otimes E)$ to be the operator given by

$$\mathcal{D}' = \sqrt{2} \left\{ \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) r(e_i) \otimes \gamma(e_j) \otimes 1 - 1 \otimes \gamma(c) \otimes 1 \right\},$$

where $\gamma(c)$ is the cubic term (3.1) of Definition 3.1.1, and with the first factor acting on $\mathcal{C}^\infty(G)$, the second one on S while on E the action is trivial.

Lemma 7.3.1. *The operators \mathcal{D}' and $1 \otimes \gamma(c) \otimes 1$ do not depend on the choice of the bases $\{e_i\}$ and $\{\tilde{e}_i\}$.*

Proof. The proof is identical with the proof of Lemma 3.1.4. \square

Proposition 7.3.2. *The operators \mathcal{D}' and $1 \otimes \gamma(c) \otimes 1$ map $[\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$ to itself.*

Proof. Let f be a function of $\mathcal{C}^\infty(G)$, u be an element of S , v be a vector of E , h be an element of H and X be an element of \mathfrak{q} . Then

$$(r(\text{Ad}(h)X) \otimes \gamma(\text{Ad}(h)X) \otimes 1)(f \otimes u \otimes v) = (r(\text{Ad}(h)X)f) \otimes (\gamma(\text{Ad}(h)X)u) \otimes v \quad (7.9)$$

Let $\widetilde{\text{Ad}(h)}$ be an element of the double cover $\text{Spin}(\mathfrak{q})$ of $SO(\mathfrak{q})$ corresponding to $\text{Ad}(h)$ via the covering map

$$p : \text{Spin}(\mathfrak{q}) \rightarrow SO(\mathfrak{q}).$$

Then, according to Proposition 3.3.1,

$$\gamma(\text{Ad}(h)X) = \Phi(\widetilde{\text{Ad}(h)})\gamma(X)\Phi(\widetilde{\text{Ad}(h)})^{-1}$$

and one can easily verify that, if π stands for the representation of H on $S \otimes E$, then

$$(\Phi(\widetilde{\text{Ad}(h)})\gamma(X)\Phi(\widetilde{\text{Ad}(h)})^{-1} \otimes 1) = \pi(h) \circ (\gamma(X) \otimes 1) \circ \pi(h)^{-1}.$$

Hence

$$1 \otimes \gamma(\text{Ad}(h)X) \otimes 1 = (r(h) \otimes \pi(h))(1 \otimes \gamma(X) \otimes 1)(r(h) \otimes \pi(h))^{-1}.$$

Therefore, using Lemma 7.3.1 and the $\text{Ad}(G)$ -invariance of the Killing form B (see Lemma 1.1.41), one obtains that

$$\begin{aligned} 1 \otimes \gamma(c) \otimes 1 &= \frac{1}{6} \sum_{i,j,k} B(\text{Ad}(h)\tilde{e}_i, [\text{Ad}(h)\tilde{e}_j, \text{Ad}(h)\tilde{e}_k]) 1 \otimes \gamma(\text{Ad}(h)e_i)\gamma(\text{Ad}(h)e_j)\gamma(\text{Ad}(h)e_k) \otimes 1 \\ &= \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) 1 \otimes \gamma(\text{Ad}(h)e_i)\gamma(\text{Ad}(h)e_j)\gamma(\text{Ad}(h)e_k) \otimes 1 \\ &= \frac{1}{6} \sum_{i,j,k} B(\tilde{e}_i, [\tilde{e}_j, \tilde{e}_k]) 1 \otimes (r(h) \otimes \pi(h)) \circ \gamma(e_i)\gamma(e_j)\gamma(e_k) \circ (r(h) \otimes \pi(h))^{-1} \otimes 1 \\ &= (r(h) \otimes \pi(h)) \circ (1 \otimes \gamma(c) \otimes 1) \circ (r(h) \otimes \pi(h))^{-1}. \end{aligned}$$

and thus the statement is true for $1 \otimes \gamma(c) \otimes 1$.

On the other hand,

$$\begin{aligned} r(\text{Ad}(h)X) \otimes \gamma(\text{Ad}(h)X) \otimes 1 &= (r(h)r(X)r(h)^{-1}) \otimes (\widetilde{\Phi(\text{Ad}(h))\gamma(X)\Phi(\text{Ad}(h))^{-1}}) \otimes 1 \\ &= (r(h)r(X)r(h)^{-1}) \otimes (\pi(h) \circ [\gamma(X) \otimes 1] \circ \pi(h)^{-1}) \\ &= (r(h) \otimes \pi(h))(r(X) \otimes \gamma(X) \otimes 1)(r(h)^{-1} \otimes \pi(h)^{-1}). \end{aligned}$$

Suppose $\{Z_i\}$ is an orthonormal basis of \mathfrak{q} . By using Lemma 7.3.1, the invariance of $1 \otimes \gamma(c) \otimes 1$ proven above and the last relation, one obtains that:

$$\begin{aligned} \mathcal{D}' &= \sum_i r(\text{Ad}(h)Z_i) \otimes \gamma(\text{Ad}(h)Z_i) \otimes 1 - 1 \otimes \gamma(c) \otimes 1 \\ &= \sum_i (r(h) \otimes \pi(h))(r(Z_i) \otimes \gamma(Z_i) \otimes 1)(r(h)^{-1} \otimes \pi(h)^{-1}) \\ &\quad - (r(h) \otimes \pi(h)) \circ (1 \otimes \gamma(c) \otimes 1) \circ (r(h) \otimes \pi(h))^{-1} \\ &= (r(h) \otimes \pi(h)) \left(\sum_i r(Z_i) \otimes \gamma(Z_i) \otimes 1 - 1 \otimes \gamma(c) \otimes 1 \right) (r(h)^{-1} \otimes \pi(h)^{-1}) \\ &= (r(h) \otimes \pi(h)) \circ \mathcal{D}' \circ (r(h)^{-1} \otimes \pi(h)^{-1}). \end{aligned}$$

Consequently

$$\mathcal{D}' = (r(h) \otimes \pi(h)) \circ \mathcal{D}' \circ (r(h)^{-1} \otimes \pi(h)^{-1})$$

and thus \mathcal{D}' maps $[\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$ to itself. \square

Definition 7.3.3 (Geometric cubic Dirac operator). *Let $[\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$ be the vector space (7.3), r be the map (7.8) and $\{\tilde{e}_i\}$ and $\{e_i\}$ be dual bases of \mathfrak{q} . We call geometric cubic Dirac operator the operator*

$$\mathcal{D} : [\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H \rightarrow [\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$$

given by

$$\mathcal{D} = \sqrt{2} \left\{ \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) r(e_i) \otimes \gamma(e_j) \otimes 1 - 1 \otimes \gamma(c) \otimes 1 \right\},$$

where $\gamma(c)$ is the cubic term (3.1) of Definition 3.1.1, with the first factor acting on $\mathcal{C}^\infty(G)$, the second one on S while on E the action is trivial.

Remark 7.3.4. Note that the constant $\sqrt{2}$ of the above definition, as in the case of algebraic cubic Dirac operators (see Definition 3.1.1), is motivated by a simpler formula for \mathcal{D}^2 . In particular, it is easy to see that, in analogy with Proposition 3.2.6, one has:

$$\mathcal{D}^2 = r(\Omega_{\mathfrak{g}}) \otimes 1 \otimes 1 - (r \otimes \gamma)(\Omega_{\mathfrak{h}_\Delta}) \otimes 1 + (\|\rho\|^2 - \|\rho_{\mathfrak{h}}\|^2).$$

Let us equip $[\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$ with the action (7.4) of G , i.e. with the action by left translations.

Proposition 7.3.5. *The operator $\mathcal{D} : [\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H \rightarrow [\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$ is G -equivariant.*

Proof. The G -equivariance of \mathcal{D} is a direct result of the commutativity of left and right translations. Namely, for a function h of $\mathcal{C}^\infty(G)$ and for $r(x)$, $x \in G$, (respectively $l(y)$, $y \in G$) being the right translation of h by x (respectively the left translation of h by y), one has that

$$(r(x) \circ l(y)(h))(g) = h(y^{-1}gx) = (l(y) \circ r(x)(h))(g)$$

for every $g \in G$. Therefore, by taking the differential of r , one obtains

$$(r(X) \circ l(y)(f))(g) = (l(y) \circ r(X)(f))(g)$$

for every $X \in \mathfrak{g}$ and every $g, y \in G$.

Consequently, for the action of G on $[\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$, we have

$$\begin{aligned} l(x) \circ \mathcal{D} &= \sqrt{2}(l(x) \otimes 1 \otimes 1) \circ \left\{ \sum_i r(Z_i) \otimes \gamma(Z_i) \otimes 1 - 1 \otimes \gamma(c) \otimes 1 \right\} \\ &= \sqrt{2} \left\{ \sum_i l(x)r(Z_i) \otimes \gamma(Z_i) \otimes 1 - l(x) \otimes \gamma(c) \otimes 1 \right\} \\ &= \sqrt{2} \left\{ \sum_i r(Z_i)l(x) \otimes \gamma(Z_i) \otimes 1 - l(x) \otimes \gamma(c) \otimes 1 \right\} \\ &= \sqrt{2} \left\{ \sum_i r(Z_i) \otimes \gamma(Z_i) \otimes 1 - 1 \otimes \gamma(c) \otimes 1 \right\} \circ (l(x) \otimes 1 \otimes 1) \\ &= \mathcal{D} \circ l(x), \end{aligned}$$

for every x in G . Hence \mathcal{D} is a G -equivariant operator. \square

Corollary 7.3.6. *The kernel of \mathcal{D} is a representation of G .*

Proof. Let f be a function of $\ker \mathcal{D}$ and x be an element of G . It suffices to show that $l(x)f$ lies in $\ker \mathcal{D}$, as well. Indeed,

$$\mathcal{D} \circ (l(x)f) = l(x) \circ (\mathcal{D}f) = 0$$

and thus $\ker \mathcal{D}$ is a representation of G . \square

Using the fact that $[\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$ is dense in $[L^2(G) \otimes (S \otimes E)]^H$ (see Proposition 7.2.1), we extend continuously the operator \mathcal{D} on $[L^2(G) \otimes (S \otimes E)]^H$ which, as representation of G , according to Theorem 7.2.3, is equivalent to

$$\widehat{\bigoplus_{\pi \in \widehat{G}} V_\pi} \otimes [V_\pi^* \otimes (S \otimes E)]^H.$$

We are interested in the kernel of \mathcal{D} . We follow Landweber who, in [Lan00], calculates this kernel in the case where $\text{rk}_{\mathbb{C}}(\mathfrak{g}) = \text{rk}_{\mathbb{C}}(\mathfrak{h})$.

Theorem 7.3.7 (Landweber (see [Lan00, Theorem 4])). *Let G be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H be a closed subgroup of G such that its complexified Lie algebra \mathfrak{h} satisfies condition (2.10). Moreover, suppose that there is a common Cartan subalgebra \mathfrak{t} of \mathfrak{g} and \mathfrak{h} , and E is an irreducible finite dimensional representation of \mathfrak{h} of highest weight $\mu \in \mathfrak{t}^*$ with respect to a fixed positive root system $\Delta_{\mathfrak{h}}^+$ such that $S \otimes E$ lifts*

to a representation of H . Then, if there is $w \in W$ such that $w(\mu + \rho_{\mathfrak{h}})$ is analytically integral and dominant for a positive system $\Delta^+ \supseteq \Delta_{\mathfrak{h}}^+$ of Δ , the kernel $\ker(\mathcal{D})$ of the geometric Dirac operator

$$\mathcal{D} : [L^2(G) \otimes (S \otimes E)]^H \rightarrow [L^2(G) \otimes (S \otimes E)]^H$$

is equivalent with the irreducible representation $V_{w(\mu + \rho_{\mathfrak{h}}) - \rho}$ of highest weight $w(\mu + \rho_{\mathfrak{h}}) - \rho$ of G . Otherwise, $\ker(\mathcal{D})$ is trivial otherwise.

Let $\mathfrak{t}_{\mathfrak{h}}$ be a Cartan subalgebra of \mathfrak{h} and $\mathfrak{t}_{\mathfrak{q}}$ be a subspace of \mathfrak{q} such that $\mathfrak{t} := \mathfrak{t}_{\mathfrak{h}} \oplus \mathfrak{t}_{\mathfrak{q}}$ is a Cartan subalgebra of \mathfrak{g} (see Section 2.5). Furthermore, fix a positive system Δ^+ for Δ satisfying the $\mathfrak{t}_{\mathfrak{h}}$ -property and let $\Delta_{\mathfrak{h}}^+$ be the corresponding positive system for $\Delta_{\mathfrak{h}}$. Then the set \widehat{G} of equivalence classes of all unitary irreducible representations of G , stands to one-by-one correspondence with the set Λ of all dominant analytically integral elements λ of \mathfrak{t}^* (see Theorem 1.5.17). In other words, if V_{λ} stands for the irreducible representation of highest weight λ of G , then the equivalence (7.7) is written as

$$\Phi : [L^2(G) \otimes (S \otimes E)]^H \xrightarrow{\sim} \widehat{\bigoplus_{\lambda \in \Lambda} V_{\lambda}} \otimes [V_{\lambda}^* \otimes (S \otimes E)]^H. \quad (7.10)$$

Moreover, according to Corollary 1.5.12, $[L^2(G) \otimes (S \otimes E)]^H$ is decomposed into a direct sum of irreducible subrepresentations and it is clear that Φ maps the λ -isotypic component

$$[L^2(G) \otimes (S \otimes E)]_{\lambda}^H$$

of $[L^2(G) \otimes (S \otimes E)]^H$ to $V_{\lambda} \otimes [V_{\lambda}^* \otimes E]^H$. Let

$$\mathcal{D}_{\lambda} : [L^2(G) \otimes (S \otimes E)]_{\lambda}^H \rightarrow [L^2(G) \otimes (S \otimes E)]_{\lambda}^H$$

be the restriction of \mathcal{D} on this λ -isotypic component. We would like to see how \mathcal{D}_{λ} acts on

$$V_{\lambda} \otimes [V_{\lambda}^* \otimes E]^H$$

via the isomorphism Φ of (7.10). Recall that \mathcal{D}_{λ} is the operator

$$\mathcal{D}_{\lambda} = \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) r(e_i) \otimes \gamma(e_j) \otimes 1 - 1 \otimes \gamma(c) \otimes 1,$$

where $\{\tilde{e}_i\}$ and $\{e_i\}$ are dual bases of \mathfrak{q} and $\gamma(c)$ is the cubic term (3.1). The terms $r(\cdot)$ act on the space $[L^2(G) \otimes (S \otimes E)]_{\lambda}^H$ by the differential of the map (7.2) of right translations on $L^2(G)$ and thus, according to Theorem 7.2.2, the corresponding action on $V_{\lambda} \otimes [V_{\lambda}^* \otimes (S \otimes E)]^H$ is on V_{λ}^* . On the other hand, the terms $\gamma(\cdot)$ act on S both for $[L^2(G) \otimes (S \otimes E)]_{\lambda}^H$ and $V_{\lambda} \otimes [V_{\lambda}^* \otimes (S \otimes E)]^H$. Hence the operator

$$\mathcal{D}_{\lambda} : [L^2(G) \otimes (S \otimes E)]_{\lambda}^H \rightarrow [L^2(G) \otimes (S \otimes E)]_{\lambda}^H,$$

via Φ , corresponds to the operator

$$1 \otimes D_{V_\lambda^*} \otimes 1 : V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H \rightarrow V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H,$$

where $D_{V_\lambda^*}$ is the algebraic Dirac operator acting on $V_\lambda^* \otimes S$ (see Definition 3.1.1). Note that the group representation action of G on $V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H$ occurs on the first factor and thus the operator $1 \otimes D_{V_\lambda^*} \otimes 1$ is clearly G -equivariant. In other words, we conclude that:

Proposition 7.3.8. *As above, let*

$$\mathcal{D}_\lambda : [L^2(G) \otimes (S \otimes E)]_\lambda^H \rightarrow [L^2(G) \otimes (S \otimes E)]_\lambda^H$$

be the geometric Dirac operator restricted to the λ -isotypic component of $[L^2(G) \otimes (S \otimes E)]^H$,

$$D_{V_\lambda^*} : V_\lambda^* \otimes S \rightarrow V_\lambda^* \otimes S$$

be the algebraic Dirac operator and

$$\Phi : [L^2(G) \otimes (S \otimes E)]_\lambda^H \xrightarrow{\sim} V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H \quad (7.11)$$

be the isomorphism (7.10), restricted to the λ -isotypic component. Then the diagram

$$\begin{array}{ccc} [L^2(G) \otimes (S \otimes E)]_\lambda^H & \xrightarrow{\mathcal{D}_\lambda} & [L^2(G) \otimes (S \otimes E)]_\lambda^H \\ \downarrow \Phi & & \downarrow \Phi \\ V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H & \xrightarrow{1 \otimes D_{V_\lambda^*} \otimes 1} & V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H \end{array}$$

is commutative, i.e.

$$\Phi \circ \mathcal{D}_\lambda = (1 \otimes D_{V_\lambda^*} \otimes 1) \circ \Phi.$$

Therefore, the isomorphism

$$\Phi : [L^2(G) \otimes (S \otimes E)]_\lambda^H \xrightarrow{\sim} V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H$$

of (7.11) establishes an equivalence

$$\ker \mathcal{D}_\lambda \simeq V_\lambda \otimes [\ker D_{V_\lambda^*} \otimes E]^H \quad (7.12)$$

of representations of G . Since G acts trivially on the component $[V_\lambda^* \otimes (S \otimes E)]^H$ of $V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H$, $V_\lambda \otimes [\ker D_{V_\lambda^*} \otimes E]^H$ turns out to be equivalent to a direct sum of $\dim[\ker D_{V_\lambda^*} \otimes E]^H$ copies of V_λ which, for the sake of simplicity, we denote

$$\dim [\ker D_{V_\lambda^*} \otimes E]^H V_\lambda.$$

Corollary 7.3.9. *As representations of G ,*

$$\ker \mathcal{D}_\lambda \simeq \dim [\ker D_{V_\lambda^*} \otimes E]^H V_\lambda.$$

The following theorem generalizes Landweber's result [Lan00, Theorem 4] in the case of non-equal complex ranks of \mathfrak{g} and \mathfrak{h} .

Theorem 7.3.10. *Let G be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H be a closed subgroup of G such that its complexified Lie algebra \mathfrak{h} satisfies condition (2.10). Moreover, suppose that $\mathfrak{t}_{\mathfrak{h}}$ is a Cartan subalgebra of \mathfrak{h} and $\mathfrak{t}_{\mathfrak{q}} \subseteq \mathfrak{q}$ is such that $\mathfrak{t} := \mathfrak{t}_{\mathfrak{h}} + \mathfrak{t}_{\mathfrak{q}}$ is a Cartan subalgebra of \mathfrak{g} . Let $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ be the corresponding root system of \mathfrak{g} , fix Δ^+ to be a positive system for Δ satisfying the $\mathfrak{t}_{\mathfrak{h}}$ -property of Definition 3.2.3, and $\Delta_{\mathfrak{h}}^+$ to be the corresponding positive system for $\Delta_{\mathfrak{h}}$. Fix E to be an irreducible finite-dimensional representation of \mathfrak{h} of highest weight $\mu \in \mathfrak{t}_{\mathfrak{h}}^*$ with respect to $\Delta_{\mathfrak{h}}^+$ such that $S \otimes E$ lifts to a representation of H . Set Λ to be the set of all analytically integral and Δ^+ -dominant elements of \mathfrak{t}^* and, for every $\lambda \in \Lambda$, set*

$$W_{\lambda}^1 := \{w \in W \mid w(\lambda + \rho)|_{\mathfrak{t}_{\mathfrak{q}}} = 0 \text{ and } w(\lambda + \rho)|_{\mathfrak{t}_{\mathfrak{h}}} \text{ is } \Delta_{\mathfrak{h}}^+ \text{-dominant}\}$$

If there are $\lambda \in \Lambda$ and $w \in W_{\lambda}^1$ such that $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$ is analytically integral and Δ^+ -dominant, the kernel of the extended on $[L^2(G) \otimes (S \otimes E)]^H$ geometric Dirac operator

$$\mathcal{D} : [L^2(G) \otimes (S \otimes E)]^H \rightarrow [L^2(G) \otimes (S \otimes E)]^H$$

is equivalent with 2^d copies of the G -representation $V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho}$ of highest weight $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$, where $d := \lfloor \frac{\dim \mathfrak{t}_{\mathfrak{q}}}{2} \rfloor$. Otherwise, $\ker \mathcal{D} = 0$.

Proof. Fix Δ^+ to be a positive system for Δ satisfying the $\mathfrak{t}_{\mathfrak{h}}$ -property and let $\Delta_{\mathfrak{h}}^+$ be the corresponding positive system for $\Delta_{\mathfrak{h}}$. Let Λ be the set of all dominant analytically integral elements of \mathfrak{t}^* with respect to Δ^+ . Decompose $[L^2(G) \otimes (S \otimes E)]^H$ into a Hilbert sum of its isotypic components and, for every $\lambda \in \Lambda$, let \mathcal{D}_{λ} be the restriction of \mathcal{D} on the λ -isotypic component. It is clear that

$$\ker \mathcal{D} = \widehat{\bigoplus_{\lambda \in \Lambda} \ker \mathcal{D}_{\lambda}}. \quad (7.13)$$

According to Corollary 7.3.9,

$$\ker \mathcal{D}_{\lambda} \simeq \dim [\ker D_{V_{\lambda}^*} \otimes E]^H V_{\lambda}$$

and thus

$$\ker \mathcal{D}_{\lambda} \simeq \dim \operatorname{Hom}_H(E^*, \ker D_{V_{\lambda}^*}) V_{\lambda}.$$

Moreover, the spin module S is self-dual, i.e. it coincides with its dual (see [Del99]), and thus $D_{V_{\lambda}^*}$ can be seen as the dual operator $D_{V_{\lambda}^*}^*$ of $D_{V_{\lambda}}$ acting on $V_{\lambda}^* \otimes S^*$. Then

$$\begin{aligned} \operatorname{Hom}_H(E^*, \ker D_{V_{\lambda}^*}) &\simeq \operatorname{Hom}_H(E^*, \ker D_{V_{\lambda}}^*) \\ &= \operatorname{Hom}_H(E^*, (\ker D_{V_{\lambda}})^*) \\ &= \operatorname{Hom}_H(\ker D_{V_{\lambda}}, E) \end{aligned}$$

and thus

$$\ker \mathcal{D}_{\lambda} \simeq \dim \operatorname{Hom}_H(\ker D_{V_{\lambda}}, E) V_{\lambda}. \quad (7.14)$$

For more details about the relation between algebraic and geometric Dirac operators, see [MZ14a].

According to Theorem 3.2.4,

$$\ker(D_{V_\lambda}) = \bigoplus_{w \in W_\lambda^1} 2^d F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}},$$

where $F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}}$ is the irreducible representation of H of highest weight $w(\lambda + \rho) - \rho_{\mathfrak{h}} \in \mathfrak{t}_{\mathfrak{h}}^*$ and $d := \lceil \frac{\dim \mathfrak{t}_{\mathfrak{g}}}{2} \rceil$. Hence

$$\mathrm{Hom}_H(\ker D_{V_\lambda}, E) \simeq \bigoplus_{w \in W_\lambda^1} 2^d \mathrm{Hom}_H(F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}}, E).$$

On the other hand, according to Proposition 1.4.9,

$$\mathrm{Hom}_H(F_{w(\lambda+\rho)-\rho_{\mathfrak{h}}}, E) \simeq \begin{cases} \mathbb{C} & \text{if } \mu = w(\lambda + \rho) - \rho_{\mathfrak{h}} \\ 0 & \text{otherwise} \end{cases}$$

and so

$$\dim \mathrm{Hom}_H(\ker D_{V_\lambda}, E) = \sum_{w \in W_\lambda^1} 2^d \delta_{\mu, w(\lambda+\rho)-\rho_{\mathfrak{h}}}.$$

By replacing in (7.14), we obtain

$$\ker \mathcal{D}_\lambda \simeq \sum_{w \in W_\lambda^1} 2^d \delta_{\mu, w(\lambda+\rho)-\rho_{\mathfrak{h}}} V_\lambda$$

while (7.13) gives

$$\begin{aligned} \ker \mathcal{D} &\simeq \sum_{\substack{w \in W_\lambda^1 \\ \lambda \in \Lambda}} 2^d \delta_{\mu, w(\lambda+\rho)-\rho_{\mathfrak{h}}} V_\lambda \\ &= \sum_{\substack{w \in W_\lambda^1 \\ \lambda \in \Lambda}} 2^d \delta_{w^{-1}(\mu+\rho_{\mathfrak{h}})-\rho, \lambda} V_{w^{-1}(\mu+\rho_{\mathfrak{h}})-\rho} \end{aligned}$$

Note that if there are $\lambda \in \Lambda$ and $w \in W_\lambda^1$ such that $\mu = w(\lambda + \rho) - \rho_{\mathfrak{h}}$, then they will be unique. More precisely, if $w' \in W_{\lambda'}^1$ and $\lambda' \in \Lambda$ is another pair such that $\mu = w'(\lambda' + \rho) - \rho_{\mathfrak{h}}$, then

$$\begin{aligned} &w'(\lambda' + \rho) - \rho_{\mathfrak{h}} = w(\lambda + \rho) - \rho_{\mathfrak{h}} \\ \Rightarrow &w'(\lambda' + \rho) = w(\lambda + \rho) \\ \Rightarrow &w^{-1}w'(\lambda' + \rho) = \lambda + \rho. \end{aligned}$$

Since $\lambda + \rho$ and $\lambda' + \rho'$ both are strictly dominant with respect to Δ^+ , we deduce that $w^{-1}w'\Delta^+ = \Delta^+$ and so $w^{-1}w' = 1$. Consequently, $w' = w$ and $\lambda' = \lambda$.

Therefore, if there are $\lambda \in \Lambda$ and $w \in W_\lambda^1$ such that $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$ is Δ^+ -dominant and analytically integral, then

$$\ker \mathcal{D} = 2^d V_{w^{-1}(\mu+\rho_{\mathfrak{h}})-\rho}.$$

Otherwise, $\ker \mathcal{D} = 0$. □

7.4 Geometric noncubic Dirac operators

We continue with the same notation. Namely, G is a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H is a closed subgroup with complexified Lie algebra \mathfrak{h} , $\mathfrak{t}_{\mathfrak{h}}$ is a Cartan subalgebra of \mathfrak{h} and $\mathfrak{t}_{\mathfrak{q}}$ is a subspace of $\mathfrak{q} := \mathfrak{h}^{\perp}$ such that $\mathfrak{t} := \mathfrak{t}_{\mathfrak{h}} + \mathfrak{t}_{\mathfrak{q}}$ is a Cartan subalgebra of \mathfrak{g} . Then \mathfrak{g} is decomposed into

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$$

and let $\mathbf{C}(\mathfrak{q})$ be the Clifford algebra of \mathfrak{q} . Moreover, let S be a space of spinors of $\mathbf{C}(\mathfrak{q})$,

$$\mathfrak{h} \rightarrow \mathfrak{so}(\mathfrak{q}) \rightarrow \mathbf{C}(\mathfrak{q}) \rightarrow \text{End } S$$

be the spin representation of Definition 2.8 and E be a finite dimensional irreducible representation of \mathfrak{h} such that $S \otimes E$ integrates to a representation of H . Let $\mathcal{S} \otimes \mathcal{E}$ be the corresponding homogeneous vector bundle over G/H induced by $S \otimes E$. According to Theorem 7.1.8, as representation of G , the space of smooth sections $\Gamma(G/H, \mathcal{S} \otimes \mathcal{E})$ of $\mathcal{S} \otimes \mathcal{E}$ is equivalent with $[\mathcal{C}^{\infty}(G) \otimes (S \otimes E)]^H$.

Definition 7.4.1 (Geometric noncubic Dirac operator). *We define the geometric noncubic Dirac operator $\widehat{\mathcal{D}}$ to be the operator*

$$\widehat{\mathcal{D}} : [\mathcal{C}^{\infty}(G) \otimes (S \otimes E)]^H \rightarrow [\mathcal{C}^{\infty}(G) \otimes (S \otimes E)]^H$$

given by

$$\widehat{\mathcal{D}} = \sqrt{2} \left\{ \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) r(e_i) \otimes \gamma(e_j) \otimes 1 \right\},$$

where $\{e_i\}$ and $\{\tilde{e}_i\}$ are dual bases of \mathfrak{q} . In other words, if \mathcal{D} is the geometric Dirac operator of Definition 7.3.3 and $\gamma(c)$ is the cubic term (3.1) of Definition 3.1.1, then

$$\widehat{\mathcal{D}} = \mathcal{D} + 1 \otimes \gamma(c) \otimes 1.$$

According to Lemma 7.3.1, \mathcal{D} , $1 \otimes \gamma(c) \otimes 1$ and thus $\widehat{\mathcal{D}}$ do not depend on the choice of the bases $\{e_i\}$ and $\{\tilde{e}_i\}$. On the other hand, according to Proposition 7.3.5, \mathcal{D} is G -equivariant while one can verify that this is also the case for $1 \otimes \gamma(c) \otimes 1$. Therefore, $\widehat{\mathcal{D}}$, being the sum of \mathcal{D} and $1 \otimes \gamma(c) \otimes 1$, is G -equivariant and so its kernel is a representation of G .

Fix a positive system Δ^+ of Δ and let Λ stand for the set of all Δ^+ -dominant analytically integral elements of \mathfrak{t}^* . Using Proposition 7.2.1, extend the operator $\widehat{\mathcal{D}}$ to $[L^2(G) \otimes (S \otimes E)]^H$ which is, via the isomorphism S of (7.10), equivalent as a G -representation to the Hilbert space sum

$$\widehat{\bigoplus_{\lambda \in \Lambda} V_{\lambda} \otimes [V_{\lambda}^* \otimes (S \otimes E)]^H}. \quad (7.15)$$

Let $\widehat{\mathcal{D}}_{\lambda}$ be the restriction of $\widehat{\mathcal{D}}$ to the λ -isotypic component

$$[L^2(G) \otimes (S \otimes E)]_{\lambda}^H$$

of $[L^2(G) \otimes (S \otimes E)]^H$ and denote again by Φ the restriction of the operator (7.10) to this isotypic component. Then, one obtains the analog of Proposition 7.3.8 in the case of noncubic Dirac operators.

Proposition 7.4.2. *As above, let*

$$\widehat{\mathcal{D}}_\lambda : [L^2(G) \otimes (S \otimes E)]_\lambda^H \rightarrow [L^2(G) \otimes (S \otimes E)]_\lambda^H$$

be the geometric Dirac operator restricted to the λ -isotypic component of $[L^2(G) \otimes (S \otimes E)]^H$,

$$\widehat{D}_{V_\lambda^*} : V_\lambda^* \otimes S \rightarrow V_\lambda^* \otimes S$$

be the algebraic noncubic Dirac operator corresponding to V_λ^* and

$$\Phi : [L^2(G) \otimes (S \otimes E)]_\lambda^H \xrightarrow{\sim} V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H \quad (7.16)$$

be the isomorphism (7.10), restricted to the λ -isotypic component. Then the diagram

$$\begin{array}{ccc} [L^2(G) \otimes (S \otimes E)]_\lambda^H & \xrightarrow{\widehat{\mathcal{D}}_\lambda} & [L^2(G) \otimes (S \otimes E)]_\lambda^H \\ \downarrow \Phi & & \downarrow \Phi \\ V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H & \xrightarrow{1 \otimes \widehat{D}_{V_\lambda^*} \otimes 1} & V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H \end{array}$$

is commutative, i.e.

$$\Phi \circ \widehat{\mathcal{D}}_\lambda = (1 \otimes \widehat{D}_{V_\lambda^*} \otimes 1) \circ \Phi.$$

In particular, as representations of G , one has

$$\ker \widehat{\mathcal{D}}_\lambda \simeq \dim \left[\ker \widehat{D}_{V_\lambda^*} \otimes E \right]^H V_\lambda.$$

Proposition 7.4.3. *Suppose that G is a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H a closed subgroup of G with complexified Lie algebra \mathfrak{h} such that $\text{rank}_{\mathbb{C}}(\mathfrak{h}) = \text{rank}_{\mathbb{C}}(\mathfrak{g})$, and E a finite-dimensional irreducible representation of \mathfrak{h} such that $S \otimes E$ integrates to a representation of H . Let \mathcal{D} and $\widehat{\mathcal{D}}$ be the geometric cubic and noncubic, respectively, Dirac operators defined on $[L^2(G) \otimes (S \otimes E)]^H$. Then*

$$\ker \mathcal{D} \subseteq \ker \widehat{\mathcal{D}}.$$

Proof. In the above setting, let f be an element of $[L^2(G) \otimes (S \otimes E)]^H$ such that $\mathcal{D}f = 0$. According to Theorem 7.3.10, $\Phi f \in \ker (1 \otimes D_{V_\lambda^*} \otimes 1)$ and since, by Theorem 4.4.1, $\ker D_{V_\lambda^*} \subseteq \ker \widehat{D}_{V_\lambda^*}$, we deduce that $\Phi f \in \ker (1 \otimes \widehat{D}_{V_\lambda^*} \otimes 1)$. Consequently,

$$\Phi \circ \widehat{\mathcal{D}}f = (1 \otimes \widehat{D}_{V_\lambda^*} \otimes 1) \circ \Phi f = 0$$

and, because Φ is injective,

$$\widehat{\mathcal{D}}f = 0.$$

In other words, $\ker \mathcal{D} \subseteq \ker \widehat{\mathcal{D}}$. □

Suppose now that H coincides with a maximal torus T of G (see Proposition 1.5.9) with complexified Lie algebra \mathfrak{t} . Recall that in this case, according to Proposition 1.5.10, \mathfrak{t} is a Cartan subalgebra of \mathfrak{g} . Let $\mu \in \mathfrak{t}^*$ be Δ^+ -dominant and algebraically integral, and E be the corresponding 1-dimensional representation of \mathfrak{t} , i.e. the representation given by

$$Xv = \mu(X)v, \quad \forall X \in \mathfrak{t}, \forall v \in E.$$

Moreover, suppose that μ is such that the representation $S \otimes E$ of \mathfrak{t}^* integrates to a representation of T . In analogy with Theorem 7.3.10, we obtain the following theorem for the kernel of the geometric noncubic Dirac operator.

Theorem 7.4.4. *Let G be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , T be a maximal torus of G such that its complexified Lie algebra \mathfrak{t} is θ -stable. Fix E to be a 1-dimensional representation of \mathfrak{t} determined by a Δ^+ -dominant element μ of \mathfrak{t}^* with respect to a fixed positive root system Δ^+ for $\Delta := \Delta(\mathfrak{g}, \mathfrak{t})$ and such that $S \otimes E$ lifts to a representation of T . For every element λ of the set Λ of all Δ^+ -dominant and analytically integral elements of \mathfrak{t}^* , set*

$$K_\lambda := \left\{ \rho - \sum_{\alpha \in A} \alpha \mid A \subseteq (\mathbb{R}\{\lambda\})^\perp \cap \Delta^+ \text{ and } \rho - \sum_{\alpha \in A} \alpha \text{ is } \Delta^+ \text{-dominant} \right\}$$

and

$$\mathcal{A}_\mu := \{(\lambda, \kappa) \in \Lambda \times K_\lambda \mid \lambda + \kappa = \mu\}$$

Then, the kernel of the noncubic Dirac operator

$$\widehat{\mathcal{D}} : [L^2(G) \otimes (S \otimes E)]^T \rightarrow [L^2(G) \otimes (S \otimes E)]^T$$

is

$$\ker \widehat{\mathcal{D}} = \sum_{(\lambda, \kappa) \in \mathcal{A}_\mu} \dim(V \otimes S)_\mu V_\lambda$$

where V_λ is the finite-dimensional irreducible G -representation of highest weight λ .

Proof. For every $\lambda \in \Lambda$, let $\widehat{\mathcal{D}}_\lambda$ be the restriction of $\widehat{\mathcal{D}}$ on the λ -isotypic component of $[L^2(G) \otimes (S \otimes E)]^H$. It is clear that

$$\ker \widehat{\mathcal{D}} = \bigoplus_{\lambda \in \Lambda} \ker \widehat{\mathcal{D}}_\lambda \tag{7.17}$$

and thus it suffices to determine $\ker \widehat{\mathcal{D}}_\lambda$ for every $\lambda \in \Lambda$. According to Proposition 7.4.2,

$$\ker \widehat{\mathcal{D}}_\lambda \simeq \dim \left[\ker \widehat{D}_{V_\lambda^*} \otimes E \right]^H V_\lambda \tag{7.18}$$

and arguing as in the case of geometric cubic Dirac operator, we obtain that

$$\ker \widehat{\mathcal{D}}_\lambda = \dim \operatorname{Hom}_H(\ker \widehat{D}_{V_\lambda}, E) V_\lambda.$$

Recall that, according to Theorem 4.6.13,

$$\begin{aligned} \ker \widehat{D}_{V_\lambda} &= \bigoplus_{\nu \in A(\lambda)} (V_\lambda \otimes S)^{[\nu]} \\ &= \bigoplus_{\kappa \in K_\lambda} (V_\lambda \otimes S)^{[\lambda+\kappa]} \\ &= \bigoplus_{\substack{\kappa \in K_\lambda \\ w \in W/W_{\lambda+\kappa}}} (V_\lambda \otimes S)_{w(\lambda+\kappa)}. \end{aligned}$$

Therefore,

$$\begin{aligned} \ker \widehat{\mathcal{D}}_\lambda &= \dim \operatorname{Hom}_H(\ker D_{V_\lambda}, E)V_\lambda \\ &= \dim \operatorname{Hom}_H\left(\bigoplus_{\substack{\kappa \in K_\lambda \\ w \in W/W_{\lambda+\kappa}}} (V_\lambda \otimes S)_{w(\lambda+\kappa)}, E\right)V_\lambda \\ &= \sum_{\substack{\kappa \in K_\lambda \\ w \in W/W_{\lambda+\kappa}}} \dim \operatorname{Hom}_H((V_\lambda \otimes S)_{w(\lambda+\kappa)}, E)V_\lambda. \end{aligned}$$

Since μ has been chosen to be Δ^+ -dominant,

$$\begin{aligned} \ker \widehat{\mathcal{D}}_\lambda &= \sum_{\kappa \in K_\lambda} \dim \operatorname{Hom}_H((V_\lambda \otimes S)_{\lambda+\kappa}, E)V_\lambda \\ &= \sum_{\kappa \in K_\lambda} \dim(V_\lambda \otimes S)_{\lambda+\kappa} \cdot \delta_{\mu, \lambda+\kappa} \cdot V_\lambda \\ &= \sum_{\kappa \in K_\lambda} \dim(V_\lambda \otimes S)_\mu \cdot \delta_{\mu-\kappa, \lambda} \cdot V_\lambda \end{aligned}$$

Hence, by replacing in (7.17), we obtain

$$\begin{aligned} \ker \widehat{\mathcal{D}} &= \sum_{\substack{\lambda \in \Lambda \\ \kappa \in K_\lambda}} \dim(V \otimes S)_\mu \cdot \delta_{\mu-\kappa, \lambda} \cdot V_{\mu-\kappa} \\ &= \sum_{(\lambda, \kappa) \in \mathcal{A}_\mu} \dim(V \otimes S)_\mu V_\lambda. \end{aligned}$$

□

7.5 Geometric t -cubic Dirac operators

In view of [Agr03], [Sle87a], [Sle87b] and [Sle88], one can consider geometric analogs for the t -cubic Dirac operators of Section 4.8.

Definition 7.5.1 (Geometric noncubic Dirac operator). *For $t \in [0, \frac{1}{3}]$, we define \mathcal{D}^t to be the operator*

$$\mathcal{D}^t : [\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H \rightarrow [\mathcal{C}^\infty(G) \otimes (S \otimes E)]^H$$

given by

$$\mathcal{D}^t = \sqrt{2} \left\{ \sum_{i,j} B(\tilde{e}_i, \tilde{e}_j) r(e_i) \otimes \gamma(e_j) \otimes 1 - 3t(1 \otimes \gamma(c) \otimes 1) \right\},$$

where $\{\tilde{e}_i\}$ and $\{e_i\}$ are dual bases of \mathfrak{g} and $\gamma(c)$ is the cubic term (4.1). Note that, for $t = \frac{1}{3}$, one obtains the geometric cubic Dirac operator \mathcal{D} of Definition 7.3.3 while, for $t = 0$, one obtains the geometric noncubic Dirac operator $\widehat{\mathcal{D}}$ of Definition 7.4.1. For $t \in (0, 1]$, we call \mathcal{D}^t the geometric t -cubic Dirac operator.

As in the case of \mathcal{D} and $\widehat{\mathcal{D}}$, one can verify that the operators \mathcal{D}^t are well-defined, do not depend on the choice of the bases $\{e_i\}$ and $\{\tilde{e}_i\}$, and are G -equivariant so that their kernels are G -representations. We are interested in describe these kernels. By analogy with Theorem 4.8.2, we have the following theorem.

Theorem 7.5.2. *Let G be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H be a closed subgroup of G such that its complexified Lie algebra \mathfrak{h} satisfies condition (2.10). Moreover, suppose that there is a common Cartan subalgebra \mathfrak{t} of \mathfrak{g} and \mathfrak{h} , and E is an irreducible finite-dimensional \mathfrak{h} -representation of highest weight $\mu \in \mathfrak{t}^*$ with respect to a fixed positive root system $\Delta_{\mathfrak{h}}^+$ such that $S \otimes E$ lifts to a representation of H . Then, for $t \in (0, \frac{1}{3}]$, the kernel $\ker(\mathcal{D}^t)$ of the geometric t -cubic Dirac operator*

$$\mathcal{D}^t : [L^2(G) \otimes (S \otimes E)]^H \rightarrow [L^2(G) \otimes (S \otimes E)]^H$$

coincides with the kernel $\ker(\mathcal{D})$ of the geometric cubic Dirac operator \mathcal{D} . In particular, if there is $w \in W$ such that $w(\mu + \rho_{\mathfrak{h}}) - \rho$ is analytically integral and dominant with respect to a positive system $\Delta^+ \supseteq \Delta_{\mathfrak{h}}^+$ of Δ , $\ker(\mathcal{D}^t)$ is equivalent with the irreducible representation $V_{w(\mu + \rho_{\mathfrak{h}}) - \rho}$ of highest weight $w(\mu + \rho_{\mathfrak{h}}) - \rho$ of G . Otherwise, $\ker(\mathcal{D}^t)$ is trivial.

Proof. The proof is a "translation" of Theorem 4.8.2, expressed for algebraic t -cubic Dirac operators D^t , in the case of geometric t -cubic Dirac operators \mathcal{D}^t . We follow [Lan00] or what we did for Theorem 7.3.10. More precisely, consider the isomorphism (7.7)

$$\Phi : [L^2(G) \otimes E]^H \xrightarrow{\sim} \widehat{\bigoplus_{\pi \in \widehat{G}} V_{\pi} \otimes [V_{\pi}^* \otimes E]^H}. \quad (7.19)$$

of Theorem 7.2.3. As in the case of geometric cubic Dirac operator \mathcal{D} , decompose $[L^2(G) \otimes E]^H$ into isotypic components, let

$$[L^2(G) \otimes E]_{\lambda}^H$$

be the isotypic component corresponding to the dominant analytically integral weight $\lambda \in \mathfrak{t}^*$, and

$$\mathcal{D}_{\lambda}^t : [L^2(G) \otimes (S \otimes E)]_{\lambda}^H \rightarrow [L^2(G) \otimes (S \otimes E)]_{\lambda}^H$$

the restriction of \mathcal{D}^t on this λ -isotypic component. It is clear that

$$\ker \mathcal{D}^t = \widehat{\bigoplus_{\lambda \in \Lambda} \ker \mathcal{D}_{\lambda}^t} \quad (7.20)$$

and so it suffices to prove the statement for \mathcal{D}_{λ}^t and \mathcal{D}_{λ} . Using an argument as the one used for Proposition 7.3.8, we obtain the commutative diagram

$$\begin{array}{ccc}
[L^2(G) \otimes (S \otimes E)]_\lambda^H & \xrightarrow{\mathcal{D}_\lambda^t} & [L^2(G) \otimes (S \otimes E)]_\lambda^H \\
\downarrow \Phi & & \downarrow \Phi \\
V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H & \xrightarrow{1 \otimes D_{V_\lambda^*}^t \otimes 1} & V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H.
\end{array}$$

Therefore, the isomorphism Φ , restricted to $[L^2(G) \otimes (S \otimes E)]_\lambda^H$, establishes an equivalence

$$\ker(\mathcal{D}_\lambda^t) \simeq V_\lambda \otimes [\ker D_{V_\lambda^*}^t \otimes E]^H$$

of representations of G . Since G acts trivially on the component $[V_\lambda^* \otimes (S \otimes E)]^H$ of $V_\lambda \otimes [V_\lambda^* \otimes (S \otimes E)]^H$, $V_\lambda \otimes [\ker D_{V_\lambda^*}^t \otimes E]^H$ turns out to be equivalent to a direct sum of $\dim[\ker D_{V_\lambda^*}^t \otimes E]^H$ copies of V_λ which, for the sake of simplicity, we denote

$$\dim [\ker D_{V_\lambda^*}^t \otimes E]^H V_\lambda.$$

According to Theorem 4.8.2, for $t \in (0, \frac{1}{3}]$, $\ker(D_{V_\lambda^*}^t) = \ker(D_{V_\lambda^*})$ and thus

$$\dim [\ker D_{V_\lambda^*}^t \otimes E]^H = \dim [\ker D_{V_\lambda^*} \otimes E]^H.$$

Therefore,

$$\ker(\mathcal{D}_\lambda^t) = \dim [\ker D_{V_\lambda^*} \otimes E]^H V_\lambda$$

and, using Corollary 7.3.9, one deduces that

$$\ker(\mathcal{D}_\lambda^t) = \ker(\mathcal{D}_\lambda),$$

as we wanted. □

7.6 Dirac indices

7.6.1 Algebraic Dirac index

Let \mathfrak{g} be a complex semisimple Lie algebra and \mathfrak{h} be a subalgebra of \mathfrak{g} such that the condition (2.10) is satisfied and $\text{rank}_{\mathbb{C}}(\mathfrak{h}) = \text{rank}_{\mathbb{C}}(\mathfrak{g})$. Consider the Clifford algebra $\mathbf{C}(\mathfrak{q})$ of $\mathfrak{q} := \mathfrak{h}^\perp$ and a space of spinors S for $\mathbf{C}(\mathfrak{q})$. Then S has a natural Z_2 -grading $S = S^+ \oplus S^-$, with $\gamma(v)$, $v \in V$, interchanging S^+ and S^- (see Remark 2.2.3). For every representation V of \mathfrak{g} (not necessarily finite dimensional), define an algebraic cubic Dirac operator D and let

$$D^\pm : V \otimes S^\pm \rightarrow V \otimes S^\mp$$

be the corresponding restricted Dirac operators on $V \otimes S^\pm$.

Definition 7.6.1 (Representation of finite length). *The representation V of \mathfrak{g} is said to have finite length if V has a finite composition series, i.e. if there is a finite series of subrepresentations*

$$0 = V_0 \subsetneq V_1 \subsetneq \dots \subsetneq V_n = V$$

of V such that every quotient V_i/V_{i-1} is an irreducible representation of \mathfrak{g} . In this case, we say that V has (finite) length n . If there is no finite composition series for V , we say that V has infinite length.

Note that every irreducible representation of \mathfrak{g} has length 1. Denote by $\mathcal{F}(\mathfrak{g})$ the set of equivalence classes of all \mathfrak{g} -representations of finite length.

In the same way as above, define $\mathcal{F}(\mathfrak{h})$ to be the set of equivalence classes of all \mathfrak{h} -representations of finite length. Let E_1, E_2 and E_3 be representations of finite length of \mathfrak{h} and $\phi : E_1 \rightarrow E_2$ and $\psi : E_2 \rightarrow E_3$ be \mathfrak{h} -equivariant morphism. We will say that a sequence of morphisms

$$0 \longrightarrow E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3 \longrightarrow 0$$

is a short exact sequence if the image of each morphism is the kernel of the next. For each short exact sequence of the above form, a relation

$$(E_2) = (E_1) + (E_3)$$

yields in \mathcal{F} .

Definition 7.6.2 (Groethendieck group). *The Groethendieck group $K(\mathfrak{h})$ of \mathfrak{h} -representations of finite length is the abelian group obtained as the quotient of $\mathcal{F}(\mathfrak{h})$ by the group generated by all such relations. The elements of $K(\mathfrak{h})$ are called virtual modules or virtual representations of \mathfrak{h} .*

Remark 7.6.3. In an obvious way, one can define the Groethendieck group $K(G)$ for the representations of finite length of a compact Lie group G .

Definition 7.6.4 (Algebraic Dirac index). *For every \mathfrak{g} -representation of finite length V , we define the Dirac index of V to be the element*

$$I_D(V) := \ker(D_V^+) - \ker(D_V^-)$$

of $K(\mathfrak{h})$.

Lemma 7.6.5. *If V is finite-dimensional, then*

$$I_D(V) = V \otimes S^+ - V \otimes S^-.$$

Proof. According to Proposition 3.2.9, the operator D is self-adjoint and so $\ker(D) \cap \text{im}(D) = 0$ and

$$V \otimes S = \ker(D) \oplus \text{im}(D).$$

Moreover, D maps isomorphically $\text{im}(D^+)$ onto D^- and thus, in the Groethendieck group $K(\mathfrak{h})$, one has

$$V \otimes S^+ - V \otimes S^- = \ker(D^+) - \ker(D^-) = I_D(V).$$

□

The above definitions can be extended in the case of irreducible (\mathfrak{g}, K) -modules. Mehdi, Pandžić and Vogan proved in [MPV17b] that if X is a unitary irreducible (\mathfrak{g}, K) -module, one obtains again that

$$I_D(X) = X \otimes S^+ - X \otimes S^-.$$

On the other hand, one can define the noncubic Dirac index of $V \in \mathcal{F}(\mathfrak{h})$ as the element

$$I_{\widehat{D}}(V) := \ker(\widehat{D}^+) - \ker(\widehat{D}^-)$$

of $K(\mathfrak{h})$. Note that, as we have already seen, $\ker(\widehat{D}^\pm)$ and $\ker(D^\pm)$ in general do not coincide. Nevertheless, by using a similar argument as in the proof above, one obtains the following proposition.

Proposition 7.6.6. *Let V be a virtual module of $K(\mathfrak{h})$. Then*

$$I_{\widehat{D}}(V) = V \otimes S^+ - V \otimes S^-.$$

In particular,

$$I_{\widehat{D}}(V) = I_D(V).$$

7.6.2 Geometric Dirac index

Let G be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H be a closed subgroup of G with complexified Lie algebra \mathfrak{h} such that \mathfrak{h} satisfies condition (2.10) and $\text{rank}_{\mathbb{C}}(\mathfrak{g}) = \text{rank}_{\mathbb{C}}(\mathfrak{h})$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}, \quad \mathfrak{q} := \mathfrak{h}^\perp$$

be the corresponding decomposition of \mathfrak{g} , S be a spin module for the Clifford algebra $\mathbb{C}(\mathfrak{q})$ of \mathfrak{q} and E be a finite-dimensional irreducible representation of \mathfrak{h} such that the representation $S \otimes E$ of \mathfrak{h} integrates to a representation of H . Consider the (extended) geometric cubic Dirac operator

$$\mathcal{D} : [L^2(G) \otimes (S \otimes E)]^H \rightarrow [L^2(G) \otimes (S \otimes E)]^H,$$

the \mathbb{Z}_2 -grading $S = S^+ \oplus S^-$ of Remark 2.2.3, and the corresponding restrictions

$$\mathcal{D}^\pm : [L^2(G) \otimes (S^\pm \otimes E)]^H \rightarrow [L^2(G) \otimes (S^\mp \otimes E)]^H \quad (7.21)$$

of \mathcal{D} .

Definition 7.6.7 (Geometric Dirac index). *Let E be a finite-dimensional irreducible representation of \mathfrak{h} such that $S \otimes E$ integrates to a representation of H and*

$$\mathcal{D}^\pm : [L^2(G) \otimes (S^\pm \otimes E)]^H \rightarrow [L^2(G) \otimes (S^\mp \otimes E)]^H$$

be the restrictions of the geometric cubic Dirac operator \mathcal{D} on the indicated spaces. The element

$$\text{Index}_G \mathcal{D} := \ker(\mathcal{D}^+) - \ker(\mathcal{D}^-)$$

of $K(G)$ is called the geometric Dirac index corresponding to E .

More generally, one can define the G -index of an elliptic differential operator P . Recall that, if E and F are vector bundles over a manifold M , an operator

$$P : \Gamma(E) \rightarrow \Gamma(F)$$

is said to be a differential operator of order l if, for every chart (U_i, φ_i) of M , P has the following expression in local coordinates:

$$P|_{U_i} = \sum_{|\alpha| \leq l} a_i^\alpha \otimes \frac{\partial^\alpha}{\partial x^{i,\alpha}},$$

where α are multi-indices with $a_i^\alpha \neq 0$ for some α with $|\alpha| = l$ and a_i^α are elements of $\Gamma(\text{Hom}(E, F))|_{U_i}$. To such an operator P , one attaches a section $\sigma(P)$ of $\Gamma(TM \otimes \text{Hom}(E, F))$ which, in the local coordinates of (U_i, φ_i) , maps the differential form $(y; \xi) = \xi_j(dx^j)_y$ of T_y^*M to

$$\sigma(P)(y; \xi) = \sum_{|\alpha|=l} \xi^\alpha a_i^\alpha(y) \in \text{Hom}(E_y, F_y)$$

The element $\sigma(P)$ is called the principal symbol of the operator P while P is said to be elliptic if, for every element $(y; \xi)$ of $T_y^*M \setminus \{0\}$, its principal symbol evaluated at $(y; \xi)$ is an isomorphism of the corresponding vector bundle fibers E_y and F_y .

One can check that both \mathcal{D} and $\widehat{\mathcal{D}}$ are elliptic differential operators over the homogeneous space G/H . More precisely, the principal symbol of each is the Clifford multiplication, which is invertible, so the operators are elliptic.

Theorem 7.6.8 (See [Lan00, Theorem 3]). *Let G be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H be a closed subgroup of G with complexified Lie algebra \mathfrak{h} such that \mathfrak{h} satisfies condition (2.10). Suppose that there is a common Cartan subalgebra \mathfrak{t} of \mathfrak{g} and \mathfrak{h} , and fix E to be an irreducible finite dimensional \mathfrak{h} -representation of highest weight $\mu \in \mathfrak{t}^*$ with respect to a fixed positive root system $\Delta_{\mathfrak{h}}^+$. Extend $\Delta_{\mathfrak{h}}^+$ to a positive system Δ^+ of Δ and set*

$$W^1 := \{w \in W \mid \Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+\}.$$

If there is $w \in W^1$ such that $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$ is analytically integral and Δ^+ -dominant, then

$$\text{Index}_G \mathcal{D} = (-1)^{\ell(w)} V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho},$$

where $V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho}$ is the finite-dimensional irreducible representation of G of highest weight $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$. Otherwise, $\text{Index}_G \mathcal{D} = 0$.

The proof of the above theorem is based on the following theorem, due to R. Bott (see [Bot65]).

Theorem 7.6.9 (See [Lan00, Theorem 2]). *Let M and N be a finite-dimensional representation of H and consider the corresponding representations $[L^2(G) \otimes M]^H$ and $[L^2(G) \otimes N]^H$ of G . If*

$$P : [L^2(G) \otimes M]^H \rightarrow [L^2(G) \otimes N]^H$$

is a G -equivariant elliptic differential operator, then its G -index is the element of $K(G)$ given by

$$\text{Index}_G P = ([L^2(G) \otimes M]^H) - ([L^2(G) \otimes N]^H)$$

In other words, the G -index of P does not depend on the operator itself but only on its domain and its range.

In an obvious way, one can define the index of the noncubic geometric Dirac operator $\widehat{\mathcal{D}}$. In this case, the proof of Theorem 7.6.8 is still valid for $\widehat{\mathcal{D}}$ and one obtains the analogue of this theorem in the case of $\widehat{\mathcal{D}}$.

Theorem 7.6.10. *Let G be a compact connected semisimple Lie group with complexified Lie algebra \mathfrak{g} , H be a closed subgroup of G with complexified Lie algebra \mathfrak{h} such that \mathfrak{h} satisfies condition (2.10). Suppose that there is a common Cartan subalgebra \mathfrak{t} of \mathfrak{g} and \mathfrak{h} and fix E to be an irreducible finite dimensional \mathfrak{h} -representation of highest weight $\mu \in \mathfrak{t}^*$ with respect to a fixed positive root system $\Delta_{\mathfrak{h}}^+$. Extend $\Delta_{\mathfrak{h}}^+$ to a positive system Δ^+ of Δ and set*

$$W^1 := \{w \in W \mid \Delta_{\mathfrak{h}}^+ \subseteq w\Delta^+\}$$

If there is $w \in W^1$ such that $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$ is analytically integral and Δ^+ -dominant, then

$$\text{Index}_G \widehat{\mathcal{D}} = (-1)^{\ell(w)} V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho},$$

where $V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho}$ is the finite-dimensional irreducible representation of G of highest weight $w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho$. Otherwise, $\text{Index}_G \widehat{\mathcal{D}} = 0$. In particular,

$$\text{Index}_G \widehat{\mathcal{D}} = \text{Index}_G \mathcal{D}.$$

For the sake of completeness, we repeat the proof of Theorem 7.6.8 in the case of $\widehat{\mathcal{D}}$.

Proof of Theorem 7.6.10. Let

$$\widehat{\mathcal{D}}^+ : [L^2(G) \otimes (S^+ \otimes E)]^H \rightarrow [L^2(G) \otimes (S^- \otimes E)]^H$$

be the operator (7.21), i.e. the restriction of $\widehat{\mathcal{D}}$ on the corresponding subspace. According to Theorem 7.6.9, the index of $\widehat{\mathcal{D}}$ is

$$\text{Index}_G(\widehat{\mathcal{D}}) = ([L^2(G) \otimes (S^+ \otimes E)]^H) - ([L^2(G) \otimes (S^- \otimes E)]^H). \quad (7.22)$$

On the other hand, according to Proposition 1.4.11,

$$\begin{aligned} ([L^2(G) \otimes (S^{\pm} \otimes E)]^H) &= \sum_{\lambda \in \Lambda} (V_{\lambda}) \dim \text{Hom}_G(V_{\lambda}, [L^2(G) \otimes (S^{\pm} \otimes E)]^H) \\ &= \sum_{\lambda \in \Lambda} (V_{\lambda}) \dim \text{Hom}_H(V_{\lambda}, S^{\pm} \otimes E), \end{aligned}$$

where the last equality comes from Theorem 7.1.7. By replacing in (7.22), we obtain

$$\begin{aligned}
 \text{Index}_G(\widehat{\mathcal{D}}) &= \sum_{\lambda \in \Lambda} (V_\lambda) (\dim \text{Hom}_H(V_\lambda, S^+ \otimes E) - \dim \text{Hom}_H(V_\lambda, S^- \otimes E)) \\
 &= \sum_{\lambda \in \Lambda} (V_\lambda) (\dim \text{Hom}_H(V_\lambda, S^+ \otimes E) - \dim \text{Hom}_H(V_\lambda, S^- \otimes E)) \\
 &= \sum_{\lambda \in \Lambda} (V_\lambda) (\dim \text{Hom}_H(V_\lambda \otimes S^+, E) - \dim \text{Hom}_H(V_\lambda \otimes S^-, E)) \\
 &= \sum_{\substack{\lambda \in \Lambda \\ w \in W^1}} (-1)^{\ell(w)} (V_\lambda) \delta_{w(\lambda+\rho) - \rho_{\mathfrak{h}}, \mu}
 \end{aligned}$$

Therefore, if there is no w in W such that $w(\mu + \rho_{\mathfrak{h}}) - \rho$ is Δ^+ -dominant and analytically integral, then $\text{Index}_G(\widehat{\mathcal{D}}) = 0$. Otherwise, if there is w in W and $\lambda \in \Lambda$ such that $w(\mu + \rho_{\mathfrak{h}}) - \rho = \lambda$, then one can verify that w^{-1} must belong to W^1 and $w^{-1}(\lambda + \rho) - \rho_{\mathfrak{h}} = \mu$. Therefore,

$$\text{Index}_G \widehat{\mathcal{D}} = (-1)^{\ell(w)} V_{w^{-1}(\mu + \rho_{\mathfrak{h}}) - \rho}.$$

□

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Résumé

Mots-clefs: théorie de représentations, groupes de Lie, algèbres de Lie, opérateur de Dirac.

Cette thèse porte sur l'étude des opérateurs de Dirac non-cubiques dans le cadre de la théorie des représentations des groupes de Lie. Après avoir présenté des notions de la théorie de Lie et des algèbres de Clifford, nous rappelons les propriétés principales des opérateurs de Dirac cubiques D introduits par Kostant en 1999. Ces résultats ont rapidement suscité un vif intérêt. En particulier, à la fin des années 2000, Vogan introduit une cohomologie définie par l'opérateur de Kostant et suggère une classification cohomologique des représentations. La cohomologie de Dirac a été calculée pour diverses familles de représentations, telles que les séries discrètes, les modules $A_q(\lambda)$ ou les modules de dimension finie. Pour les modules de dimension finie, la cohomologie de Dirac coïncide avec le noyau de D . Il apparaît que l'opérateur de Dirac de Kostant est une version algébrique d'un opérateur différentiel issu d'une famille continue d'opérateurs de Dirac géométriques introduits par Slebarski dans les années 1980 dans le cadre de fibrés au-dessus d'espaces homogènes G/H de groupes compacts. Ce qui distingue l'opérateur de Dirac de Kostant est qu'il est le seul membre de cette famille dont le carré, généralisant une formule de Parthasarathy, diffère de l'opérateur de Casimir à un scalaire près. Cette propriété a des applications importantes en théorie des représentations des groupes de Lie. Le carré des opérateurs de Dirac non-cubiques, i.e des autres membres de la famille d'opérateurs de Slebarski, a été calculé par Agricola qui a également établi des liens précis entre ces opérateurs non-cubiques et la théorie des cordes en physique. Par ailleurs, les opérateurs de Dirac non-cubiques sont des opérateurs différentiels invariants, et donc leur noyau est le siège de représentations (de dimension finie) de groupes compacts. Dans cette thèse nous étudions le noyau des opérateurs de Dirac non-cubiques, et nous montrons, sous certaines conditions sur les espaces homogènes G/H , que ce noyau contient le noyau de l'opérateur de Dirac cubique. Nous obtenons en fait une formule explicite pour le noyau que nous appliquons aux cas des algèbres de Lie classiques et des algèbres de Lie exceptionnelles. Nous constatons que certaines propriétés des opérateurs non-cubiques sont analogues à celles de l'opérateur de Dirac de Kostant, tel que l'indice. Nous déduisons également quelques observations sur les opérateurs de Dirac géométrique non-cubiques.

Abstract

Keywords: representation theory, Lie groups, Lie algebras, Dirac operators.

This thesis focuses on the study of noncubic Dirac operators within the framework of representation theory of Lie groups. After recalling basic notions of Lie theory and Clifford algebras, we present the main properties of cubic Dirac operators D introduced by Kostant in 1999. These results quickly aroused great interest. In particular, in the late 1990's, Vogan introduced a cohomology defined by Kostant operator D and suggested a cohomological classification of

representations. Dirac cohomology was computed for various families of representations, such as the discrete series, $A_q(\lambda)$ modules or finite dimensional representations. It turns out that for finite dimensional modules, Dirac cohomology coincides with the kernel of D . It appears that Kostant's Dirac operator is an algebraic version of a specific member of a continuous family of geometric Dirac operators introduced by Slebarski in the mid 1980's in the context of bundles over homogeneous spaces G/H of compact groups. What distinguishes the cubic Dirac operator is that it is the only member of this family whose square, generalizing Parthasarathy's formula, differs from the Casimir operator up to a scalar. This property has important applications in representation theory of Lie groups. The square of the noncubic Dirac operators, i.e. of the other members of Slebarski's family, was calculated by Agricola who also established precise links between these noncubic operators and string theory in physics. Actually, noncubic Dirac operators are invariant differential operators, and therefore their kernels define (finite-dimensional) representations of compact groups. In this thesis we study the kernel of noncubic Dirac operators, and we show that, under certain conditions on the homogeneous spaces G/H , the kernel contains the kernel of the cubic Dirac operator. We obtain an explicit formula for the kernel which we apply to the case of classical Lie algebras and of exceptional Lie algebras. We remark that some properties of noncubic operators are analogous to those of Kostant cubic Dirac operator, such as the index. We also deduce some observations on noncubic geometric Dirac operators.

INDEX

- (\mathfrak{g}, K) -module, 82
- $\mathfrak{t}_\mathfrak{h}$ -property, 108

- Cartan involution, 66
- Cartan matrix, 62
- Casimir element, 75
- Chevalley map, 89
- Clifford algebra, 87
 - main antiautomorphism of, 94
 - main involution of, 94
 - space of spinors of, 89

- Dirac cohomology, 113
- Dirac index
 - algebraic, 188
 - geometric, 189
- Dirac operator
 - t -noncubic, 145
 - cubic, 103
 - cubic term of, 104
 - geometric t -noncubic, 185
 - geometric cubic, 176
 - geometric noncubic, 182
 - noncubic, 118

- equivalent representations, 70, 80
 - infinitesimally, 83

- Frobenius reciprocity, 173
- fundamental weights, 56

- Groethendieck group, 188

- Harish-Chandra isomorphism, 85

- homogeneous space, 171
- homogeneous vector bundle, 172

- induced representation, 173
- interwining operator, 70

- Kostant
 - multiplicity formula, 78
 - partition function, 78

- lexicographic ordering, 55
- Lie algebra, 43
 - (complex) rank of, 54
 - of Lie group, 46
 - semisimple, 53
 - abelian, 43
 - adjoint map of, 44
 - adjoint representation of, 69
 - Cartan involution of, 66
 - Cartan subalgebra of, 54
 - compact, 68
 - ideal of, 53
 - Killing form of, 50
 - Lie subalgebra of, 43
 - derived, 65
 - real form of complex, 44
 - reductive, 65
 - root decomposition of, 54
 - simple, 53
 - universal enveloping algebra of, 51
- Lie group, 41
 - Cartan decomposition of, 68
 - exponential map of, 47

- semisimple, 53
- Parthasarathy's Dirac inequality, 114
- Peter-Weyl Theorem, 173
- property (*), 149
- representation of Lie algebra, 69
 - infinitesimal character of, 86
 - irreducible, 70
 - isotypic component of, 74
 - of finite length, 188
 - subrepresentation of, 70
 - weight of, 70
 - algebraically integral, 71
 - analytically integral, 81
 - dominant, 72
 - highest, 72
- representation of Lie group, 79
 - K -finite vectors of, 81
 - admissible, 82
 - Harish-Chandra module of, 83
 - irreducible, 79
- root system, 54
 - Dynkin diagram of, 62
 - positive system of, 55
 - simple system of, 56
 - Weyl group of, 60
- Schur's lemma, 70
- spin representation, 92
- standard representation, 74
- Vogan's conjecture, 114
- Weyl dimensional formula, 76
- Weyl's inequalities, 128