



HAL
open science

Geometry and prequantisation of 2-plectic manifolds

Gabriel Sevestre

► **To cite this version:**

Gabriel Sevestre. Geometry and prequantisation of 2-plectic manifolds. Differential Geometry [math.DG]. Université de Lorraine, 2021. English. NNT : 2021LORR0142 . tel-03378218

HAL Id: tel-03378218

<https://hal.univ-lorraine.fr/tel-03378218v1>

Submitted on 14 Oct 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



AVERTISSEMENT

Ce document est le fruit d'un long travail approuvé par le jury de soutenance et mis à disposition de l'ensemble de la communauté universitaire élargie.

Il est soumis à la propriété intellectuelle de l'auteur. Ceci implique une obligation de citation et de référencement lors de l'utilisation de ce document.

D'autre part, toute contrefaçon, plagiat, reproduction illicite encourt une poursuite pénale.

Contact : ddoc-theses-contact@univ-lorraine.fr

LIENS

Code de la Propriété Intellectuelle. articles L 122. 4

Code de la Propriété Intellectuelle. articles L 335.2- L 335.10

http://www.cfcopies.com/V2/leg/leg_droi.php

<http://www.culture.gouv.fr/culture/infos-pratiques/droits/protection.htm>

Géométrie et préquantification des variétés 2-plectiques**Thèse****Présentée et soutenue publiquement pour l'obtention du titre de****DOCTEUR DE L'UNIVERSITE DE LORRAINE****Mention : Mathématiques****par Gabriel Sevestre****Sous la direction de Tilmann Wurzbacher****24 Juin 2021****Membres du jury :**

Directeur de thèse :	Tilmann WURZBACHER	Professeur à l'Université de Lorraine
Rapporteurs et examineurs :	Konrad WALDORF	Professeur à l'Université de Greifswald
	Marco ZAMBON	Professeur à KU Leuven
Examineurs :	Honglei LANG	Professeure à l'Université agronomique de Chine
	Camille LAURENT-GENGOUX	Professeur à l'Université de Lorraine
Examinatrice et présidente du jury :	Simone GUTT	Professeure à l'Université libre de Bruxelles

Remerciements

Je veux exprimer tout d'abord mes profonds remerciements à mon directeur de thèse Tilmann Wurzbacher. L'aide qu'il a apportée dans tous les aspects de ma vie de doctorant fut précieuse au cours de ces quatre années. L'étendue impressionnante de son savoir mathématique et son intérêt tout particulier pour la physique ont enrichi ma réflexion. Grâce à lui, j'ai mieux compris le sens de ce qu'est la recherche en mathématiques.

Je veux le remercier pour son enthousiasme et sa patience, qui m'ont donné les connaissances et la confiance nécessaires à l'élaboration de ce travail. Nos discussions furent un réel bonheur, et j'espère qu'elles se poursuivront à l'avenir.

Je veux ensuite remercier Camille Laurent-Gengoux qui m'a lui aussi aidé, et avec qui j'ai beaucoup appris sur ces objets intrigants que sont les groupoïdes. Je reste souvent fasciné par la profusion de ses idées, dont certaines m'ont guidé dans la résolution des problèmes que j'ai eu à rencontrer.

À Laure, qui est à mes côtés depuis le début de cette thèse, je veux aussi exprimer ma gratitude pour son intérêt toujours sincère pour mon travail, et pour le soutien qu'elle m'a apporté dans des moments parfois difficiles. Je doute que tu lises ce manuscrit, mais tu peux maintenant retenir son titre !

Merci aux doctorants que j'ai pu rencontrer à l'IECL où règne une ambiance chaleureuse, loin de l'image poussiéreuse du mathématicien misanthrope. Merci à mes amis et à ma famille pour leur soutien, mais surtout pour m'avoir parfois distrait de la thèse et permis de penser à autre chose qu'aux mathématiques.

Je remercie enfin la région Grand-Est pour le soutien financier qui a permis ce travail.

Contents

Introduction et résumé de la thèse	4
1 Variétés multisymplectiques	11
1.1 Généralités	11
1.2 Observables multisymplectiques	14
1.3 Premières approches de la préquantification 2-plectique	16
2 Geometrisation of $H^3(M, \mathbb{Z})$	22
2.1 Principal $PU(H)$ -bundles	23
2.2 Bundle gerbes	25
2.3 Groupoid S^1 -central extensions	28
2.4 Principal 2-group bundles	32
3 A prequantisation map for 2-plectic manifolds	38
Introduction	38
3.1 Global geometric prequantisation of integral pre-2-plectic manifolds	42
3.2 Infinitesimal symmetries of bundle gerbes	49
3.3 A prequantisation map for integral 2-plectic manifolds	58
3.4 The case of exact 2-plectic manifolds	67
4 Weak infinitesimal symmetries of local bundle gerbes	71
5 Lagrangian submanifolds of standard multisymplectic manifold	74
Introduction	74
5.1 Multisymplectic vector spaces and manifolds, and Lagrangian submanifolds	77
5.2 Standard multisymplectic vector spaces and manifolds	80
5.3 Normal forms of Lagrangian submanifolds of standard multisymplectic manifolds	85
5.4 Appendix: Two results in foliated differential topology	87

6 Appendices	91
6.1 Lie groupoids and strict Lie 2-groups	91
6.2 Lie 2-algebras	100
Bibliography	108

Introduction et résumé de la thèse

Cette thèse est dédiée à l'étude des variétés multisymplectiques et plus particulièrement 2-plectiques, avec un accent mis sur la préquantification de ces derniers objets. La géométrie multisymplectique est l'étude des variétés différentiables M munies d'une $k+1$ forme différentielle fermée et non dégénérée. Le couple (M, ω) est alors appelé variété multisymplectique. Il s'agit bien d'une généralisation de la géométrie symplectique dans le sens où l'on retrouve cette dernière en considérant le cas $k = 1$.

Comme la géométrie symplectique trouve ses origines dans la formulation abstraite de la mécanique Hamiltonienne, la géométrie multisymplectique apparut en premier lieu dans des formulations de théorie des champs classique. Le modèle "habituel" en physique théorique est donné par un sous-fibré du fibré multicotangent d'une variété, appelé parfois *espace des phases étendu*, qui est alors canoniquement équipé d'une forme multisymplectique. Citons ici quelques références utiles pour ces points de vue : [Kij73], [H12], [RR09], [CnCI91], [Got91].

La première différence notable entre les géométries symplectiques et multisymplectiques est, outre la disparition sur la restriction de la dimension d'une telle variété, l'absence en général d'un théorème de forme normale ou d'un théorème de Darboux. Pour $k \geq 2$, il existe plus d'un type linéaire pour une k -forme linéaire sur un espace vectoriel (i.e. ces formes ne sont pas reliées par un isomorphisme linéaire). Par exemple en dimension 6, il existe 3 types linéaire pour des 3-formes non-dégénérées, et il est possible alors de construire une variété 2-plectique qui possède localement ces trois types simultanément ([Ryv]). Cependant, il est possible pour des variétés multisymplectiques particulières, ou en imposant de fortes hypothèses sur une variété multisymplectique générale, de montrer des théorèmes à la Darboux. Un de ces théorèmes fera l'objet d'un chapitre de cette thèse.

La géométrie multisymplectique admet par ailleurs des *observables* qui sont alors des formes différentielles, en lieu et place des fonctions pour la géométrie symplectique. En effet, étant donnée une variété k -plectique (M, ω) , on définit les $k-1$ -formes Hamiltoniennes $\alpha \in \Omega_{Ham}^{k-1}(M, \omega)$ avec l'équation dite de "Hamilton-De Donder-Weyl":

$$d\alpha = \iota_{X_\alpha} \omega,$$

pour un champ de vecteurs $X_\alpha \in \mathfrak{X}(M)$, appelé alors *champ de vecteurs Hamiltonien associé à α* . Cependant, la généralisation naïve de la structure d’algèbre de Lie des observables dans le cas symplectique est insuffisante ici car, si on définit le *crochet de Poisson* de deux formes Hamiltoniennes $\alpha, \beta \in \Omega_{Ham}^{k-1}(M, \omega)$ avec

$$\{\alpha, \beta\} = \iota_{X_\alpha} \iota_{X_\beta} \omega,$$

l’équation de Jacobi n’est vérifiée qu’à une $k-2$ forme exacte près.

Si on peut outrepasser cette difficulté en considérant le quotient des formes Hamiltoniennes par les formes exactes (voir par exemple [CI96]), l’approche la plus récente, initiée dans [BHR10], est de faire usage des structures algébriques *supérieures*, et en particulier des ∞ -algèbres de Lie. On considère alors le complexe de de Rham jusqu’aux $k-1$ -formes Hamiltoniennes, et des crochets jusqu’à l’ordre $k-1$ sont définis. Notons qu’il existe aussi des approches de ces observables en considérant des multichamps de vecteurs (voir par exemple [FPR05]).

Si le point de vue des ∞ -algèbres de Lie sur les observables d’une variété multisymplectique est généralement bien accepté et a conduit à de nouvelles définitions liées par exemple à la géométrie des applications *co-moment* ([CFRZ16]), un produit entre ces observables qui serait compatible en un certain sens avec les crochets est encore absent, et donc l’algèbre de Poisson de la géométrie symplectique n’est pas tout à fait généralisée. Le produit usuel des formes différentielles est trop naïf, et on peut trouver des contres exemples simples dans ce sens.

La quantification des variétés multisymplectiques est aussi un sujet d’étude qui est encore assez peu compris. Une première approche dans [Rog13] prouve un théorème d’isomorphisme entre la Lie 2-algèbre des observables d’une variété 2-plectique, et la Lie 2-algèbre associée à l’algébroïde de Courant construite à partir d’une trivialisatoin locale de la forme 2-plectique dans un bon recouvrement ouvert de la variété. Ce résultat sera ensuite généralisé dans [FRS14], qui montre un théorème donnant cette fois un quasi-isomorphisme entre la Lie n -algèbre d’une variété n -plectique, et la Lie n -algèbre (en fait une *algèbre de Lie différentielle graduée*) des *quantomorphismes infinitésimaux*, construite elle aussi à partir de données locales sur la variété n -plectique. Ces deux résultats généralisent bien sûr l’isomorphisme bien connu de la géométrie symplectique, entre les observables et les quantomorphismes, sections de l’algébroïde d’Atiyah respectant certaines conditions.

Pour le point de vue plus global en géométrie 2-plectique, il semble maintenant plus communément accepté de choisir comme objet préquantifiant une gerbe ou extension S^1 -centrale de groupoïdes (qui sont classifiés par $H^3(M, \mathbb{Z})$), dans des formulations plus modernes que, par exemple, avec des *faisceaux de groupoïdes* comme dans [Bry08]. Cependant, les sections d'un éventuel objet linéaire associé aux gerbes ou extensions S^1 -centrales de groupoïdes sont encore mal comprises, compliquant ainsi l'accès à la généralisation de la représentation de l'algèbre des observables sur les sections du fibré préquantifiant en géométrie symplectique. Des essais pour des définitions ont été faits dans [Rog11] et [BSS18] par exemple, mais ces définitions donnent des objets non-triviaux uniquement dans le cas où la classe de Dixmier-Douady de la gerbe est de torsion, et donc où la variété 2-plectique est exacte.

Les précédents aspects ne concernent par ailleurs uniquement que la préquantification. La quantification dans son ensemble semble à ce jour encore plus éloignée, et aucune tentative n'a encore été faite pour définir ce que devrait être une polarisation en géométrie multisymplectique.

Nous allons maintenant donner un résumé de cette thèse, en détaillant grossièrement le contenu de chaque chapitre.

Chapitre 1. Ce chapitre donne dans les deux premières sections les définitions générales des variétés multisymplectiques et des observables associées. On rappelle en section 1.2 la structure de Lie 2-algèbre sur les observables d'une variété 2-plectique, le cas général n'étant pas utile dans cette thèse. Enfin en section 1.3, on rappelle deux résultats de préquantification multisymplectique, qui servent d'introduction aux résultats du chapitre 3.

Chapitre 2. Ce chapitre dresse une liste des objets géométriques associés à $H^3(M, \mathbb{Z})$ pour une variété différentiable M , ainsi que des théorèmes donnant la bijection avec la bonne classe d'isomorphismes de ces objets. On y étudie dans l'ordre les $PU(H)$ -fibrés principaux, les gerbes, les extensions S^1 -centrales de groupoïdes, les BS^1 -2-fibrés principaux et les $\Gamma_{U(H) \rtimes PU(H)}$ -2-fibrés principaux. Les théorèmes mentionnés sont tous connus mais on y montre le résultat nouveau suivant.

Théorème 2.4.6. *Il y a une bijection*

$$H^1(M, \Gamma_{U(H) \rtimes PU(H)}) \cong H^3(M, \mathbb{Z}).$$

Ce résultat montre alors une bijection entre $H^3(M, \mathbb{Z})$, et les classes d'isomorphismes des $\Gamma_{U(H) \times PU(H)}$ -2-fibrés principaux. Il découle de la littérature existante sur le sujet, mais n'a pas été observé à ce jour, et offre donc un potentiel nouvel objet d'étude pour considérer la préquantification des variétés 2-plectiques, principale motivation de ce chapitre.

Chapitre 3. Ce chapitre est l'adaptation quasi identique d'un article écrit en collaboration avec Tilmann Wurzbacher, appelé "A prequantisation map for 2-plectic manifolds". Seule une annexe de l'article a été retirée, car une version plus détaillée sert d'annexe pour cette thèse.

La section 3.1 donne les résultats préliminaires nécessaires à la définition de la préquantification d'une variété 2-plectique considérée dans ce chapitre et s'achève par la définition suivante.

Définition 3.1.16. Soit une variété 2-plectique (M, ω) , avec $[\omega] \in H^3(M, \mathbb{Z})$. Une *préquantification géométrique globale* de (M, ω) est un $PU(H)$ -fibré principal $\Pi : Y \rightarrow M$, telle que la gerbe associée ("lifting bundle gerbe") possède une structure connective ayant comme 3-courbure $(-2i\pi)\omega$.

On étudie ensuite en section 3.2 les champs de vecteurs multiplicatifs sur les gerbes, et on donne la définition des *symétries infinitésimales faibles*.

Définition 3.2.9. Soit (P, Y) une gerbe sur une variété M , de structure connective (A, θ) . On appelle *symétrie infinitésimale faible* un triplet (ξ, X, B) où $B \in \Omega^1(Y)$ et (ξ, X) est un champ de vecteurs multiplicatif sur le groupoïde $P \rightrightarrows Y$ sujet aux conditions

- $\mathcal{L}_\xi A = (t^* - s^*)B$
- $\mathcal{L}_X \theta = dB$.

Cette définition permet alors de montrer le résultat suivant.

Proposition 3.2.11. Soit (P, Y) une gerbe sur une variété M de structure connective (A, θ) ayant comme 3-courbure $(-2i\pi) \cdot \omega \in \Omega^3(M)$. Alors un champ de vecteurs multiplicatif (ξ, X) sur le groupoïde $P \rightrightarrows Y$ est une symétrie infinitésimale faible si et seulement si X se projette sur M en un champ de vecteurs Hamiltonien pour ω .

La fin de cette section est dédiée à décrire la structure de Lie 2-algèbre sur les symétries infinitésimales faibles, qui découle directement de la structure

de Lie 2-algèbre sur les champ de vecteurs multiplicatifs. Grâce à la forme particulière de l'algèbroïde associée à $P \rightrightarrows Y$, on obtient donc une structure de Lie 2-algèbre sur le complexe

$$\mathfrak{X}_V(Y) \oplus C^\infty(Y) \rightarrow \mathcal{V}_{(P,Y)}$$

où $\Gamma(Y, \text{Lie}(P)) \cong \mathfrak{X}_V(Y) \oplus C^\infty(Y)$, avec $\mathfrak{X}_V(Y)$ l'espace des champs de vecteurs verticaux par rapport à $\pi : Y \rightarrow M$, et $\mathcal{V}_{(P,Y)}$ est l'espace des symétries infinitésimales faibles.

On montre ensuite le résultat suivant en section 3.3.

Théorème 3.3.5. *Soit (M, ω) une variété 2-plectique avec $[\omega] \in H^3(M, \mathbb{Z})$, et (P, Y) une préquantification géométrique globale de (M, ω) . Il y a alors un quasi-isomorphisme de Lie 2-algèbres*

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \Omega_{Ham}^1(M, \omega) \\ \Phi_1 \downarrow & \swarrow \Phi_2 & \downarrow \Phi_1 \\ \mathfrak{X}_V(Y) \oplus C^\infty(Y) & \longrightarrow & \mathcal{V}_{(P,Y)} \end{array}$$

entre la Lie 2-algèbre des observables et la Lie 2-algèbre formée par les symétries infinitésimales faibles.

La section 3.4 est finalement dédiée à préciser ce quasi-isomorphisme dans le cas où la variété 2-plectique est exacte.

Chapitre 4. Ce chapitre, plus court que les autres, peut être compris comme un ajout au chapitre précédent. Il détaille les symétries infinitésimales faibles dans le cas où la gerbe préquantifiante est construite à partir de données locales. On montre qu'on retrouve alors exactement la définition des *quantomorphisms infinitésimaux* de [FRS14].

Chapitre 5. Ce chapitre est tiré d'un article écrit en collaboration avec Tilmann Wurzbacher, intitulé "Lagrangian submanifolds of standard multisymplectic manifold", et publié dans *Geometric and Harmonic Analysis on Homogeneous Spaces. TJC 2017. Springer Proceedings in Mathematics and Statistics, vol 290*. Il vise à donner une preuve détaillée d'un théorème de forme normale de George Martin, qui généralise dans le cas où la variété k -plectique M est *standard*, un théorème d'Alan Weinstein qui donne une

forme normale d'une forme symplectique autour d'une sous-variété Lagrangienne.

Afin de montrer ce théorème, on commence en section 4.1 par définir les sous variétés Lagrangiennes des variétés multisymplectiques, en s'aidant des espaces vectoriels multisymplectiques.

Définition 5.1.2. Soit (V, ω) un espace vectoriel k -plectique, $U \subset V$ un sous-espace vectoriel et $1 \leq j \leq k$. On définit le j -ème complément orthogonal de U comme:

$$U^{\perp, j} := \{v \in V \mid \iota_v \wedge u_1 \wedge \dots \wedge u_j \omega = 0, \forall u_1, \dots, u_j \in U\}.$$

On dit alors que $U \subset V$ est un *sous-espace vectoriel j -Lagrangien* si $U = U^{\perp, j}$. Une sous variété L d'une variété k -plectique M est alors appelée *j -Lagrangienne* si en tout point $p \in L$, $T_p L$ est un sous-espace vectoriel j -Lagrangien de $T_p M$.

On définit ensuite les variétés multisymplectiques standards.

Définitions 5.2.1 et 5.2.4. Soit V un espace vectoriel et $k > 1$. On dit que V est un *espace vectoriel k -plectique standard* si (V, ω) est un espace vectoriel k -plectique et s'il existe un sous-espace $W \subset V$ tel que:

- (1) $\forall u, v \in W, \iota_u \wedge v \omega = 0$
- (2) $\dim(W) = \dim(\Lambda^k((V/W)^*))$
- (3) $\text{codim}(W) > k$.

On dira alors qu'une variété k -plectique (M, ω) est *standard* si pour tout point $p \in M$, l'espace vectoriel $(T_p M, \omega_p)$ est standard.

Ceci permet alors d'énoncer le théorème principal.

Théorème 5.3.1. Soit (M, ω) une variété k -plectique standard. Soit la distribution $W \subset TM$ définie comme $W := \cup_{p \in M} W_\omega(p)$ et soit L une sous variété k -Lagrangienne de M complémentaire à W (c'est à dire $T_p M = T_p L \oplus W|_p, \forall p \in L$). Si W est une distribution intégrable, alors il existe des voisinages ouverts U et V de L dans respectivement M et $\Lambda^k(T^*L)$, et un difféomorphisme $\phi : U \rightarrow V$ tel que

$$\phi|_L = id_L \text{ et } \phi^*(\omega^{\Lambda^k(T^*L)}) = \omega \text{ sur } U.$$

La preuve est grossièrement une adaptation de "l'astuce de Moser" utilisé pour la preuve du théorème de Weinstein, rendu possible grâce à l'existence de la distribution $W \subset TM$. On y utilise aussi deux résultats de topologie différentielle montrés en annexe de ce chapitre.

Annexes. Le Chapitre 6 est constitué de deux annexes, traitant respectivement des groupoïdes de Lie ainsi que des 2-groupes de Lie stricts, et des Lie 2-algèbres. On y énonce les définitions et résultats principaux qui sont utilisés au cours de cette thèse, principalement dans les chapitres 2 et 3.

Chapter 1

Variétés multisymplectiques

1.1 Généralités

Dans cette section, nous allons donner les définitions et concepts généraux concernant les variétés multisymplectiques. Nous nous intéresserons plus particulièrement au cas des variétés 2-plectiques, objets centraux de cette étude.

Definition 1.1.1. Soit M une variété différentiable, et $\omega \in \Omega^{k+1}(M)$, $k \geq 1$. On dit que ω est une *forme multisymplectique*, ou *k-plectique*, sur M si ω est fermée et non-dégénérée dans le sens suivant:

$$\iota_v \omega = 0 \text{ si et seulement si } v = 0$$

pour tout $v \in TM$. Nous dirons alors que le couple (M, ω) est une *variété multisymplectique*, ou *k-plectique*.

On retrouve bien les variétés symplectiques avec $k = 1$. Notons que la condition de non-dégénérescence est plus faible pour une variété multisymplectique générale, car alors l'application

$$\omega^\# : TM \rightarrow \Lambda^k T^*M; \quad v \mapsto \iota_v \omega$$

est seulement une injection.

Example 1.1.2. Soit M une variété différentiable de dimension $n \geq 2$, orientable. Alors si ω est une forme volume sur M , le couple (M, ω) est une variété $(n-1)$ -plectique.

Example 1.1.3. Soit Q une variété différentiable. On note $M = \Lambda^k T^*Q$, avec $1 \leq k \leq \dim(M)$. Soit $\theta \in \Omega^k(M)$ définie par

$$\theta_{\alpha_q}(v_1, \dots, v_k) = \alpha_q(\pi_{*\alpha_q}v_1, \dots, \pi_{*\alpha_q}v_k)$$

pour tout $\alpha_q \in \Lambda^k T_q^*Q$ avec $q \in Q$, $v_1, \dots, v_k \in T_{\alpha_q}(M)$, et $\pi : M \rightarrow Q$ est la projection canonique. On note $\omega = d\theta$. Alors (M, ω) est une variété k -plectique. Pour voir la non-dégénérescence de ω , on peut par exemple considérer des coordonnées q^i et p_I , avec $I = (i_1, \dots, i_k)$, sur respectivement un ouvert $U \subset Q$ et $\Lambda^k T^*U$; un calcul direct montre qu'alors

$$\omega|_U = \sum_{i_1, \dots, i_k} dp_{i_1, \dots, i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}$$

ce dont on peut aisément tirer la non-dégénérescence de ω . Notons qu'il s'agit d'une généralisation directe de la forme symplectique canonique sur le fibré cotangent d'une variété.

Example 1.1.4. Soit $\pi : E \rightarrow \Sigma$ un fibré de fibre typique Q , avec $\dim_{\mathbb{R}}(\Sigma) = n$. Soit ω la forme n -plectique de l'exemple précédent sur $\Lambda^n T^*E$. On pose, pour $0 \leq k \leq n$

$$\Lambda_k^n T^*E := \{\alpha \in \Lambda^n T^*E \mid \iota_{u_1} \dots \iota_{u_k} \alpha = 0 \text{ dès que } u_1, \dots, u_k \in \ker(\pi_*)\}.$$

Alors $\Lambda_k^n T^*E \subset \Lambda^n T^*E$ est un sous-fibré, et le pullback de ω est non-dégénéré. Donc pour tout $0 \leq k \leq n$, $\Lambda_k^n T^*E$ est une variété n -plectique.

Example 1.1.5. Soit M une variété complexe munie d'une forme volume holomorphe Ω . Alors la partie réelle de Ω donne une structure multisymplectique à M .

Example 1.1.6. Soit G un groupe de Lie réel, semi-simple et compact. On considère la forme $\omega \in \Omega^3(G)$ bi-invariante donnée en l'élément neutre e par

$$\omega_e(X, Y, Z) = \kappa([X, Y], Z)$$

où $X, Y, Z \in \text{Lie}(G)$ et κ la forme de Killing. Alors (G, ω) est une variété 2-plectique.

Les variétés symplectiques possèdent de forts invariants qui disparaissent lorsque l'on considère la généralisation multisymplectique. Nous allons voir maintenant les deux différences majeures entre symplectique et multisymplectique.

Une première différence notable est qu'une variété multisymplectique peut potentiellement exister en n'importe quelle dimension égale ou plus grande que 2, comme le montre le premier exemple. La condition de

non-dégénérescence étant plus faible que dans le cas symplectique, aucune restriction n'existe sur la dimension, en particulier il existe des variétés multisymplectique de dimension impaire.

La deuxième différence est la disparition en général d'un résultat de forme locale pour les variétés multisymplectiques. Le théorème de Darboux assure que, étant donné une variété symplectique (M, ω) de dimension $2n$ et un point $x \in M$, il existe un voisinage $U \subset M$ de x et des coordonnées locales (q_i, p^i) tels que

$$\omega|_U = \sum_i dq_i \wedge dp^i.$$

En particulier, en un point $x \in M$ n'importe quel forme symplectique possède le *type linéaire* de la forme bilinéaire canonique sur $V \oplus V^*$ pour un espace vectoriel V

$$((v, \varphi), (w, \psi)) \mapsto \varphi(w) - \psi(v)$$

avec $(v, \varphi), (w, \psi) \in V \oplus V^*$. C'est à dire qu'il existe un isomorphisme $T_x M \cong V \oplus V^*$ qui tire la forme bilinéaire canonique sur $V \oplus V^*$ sur la forme symplectique au point x . Ces deux observations sont fausses en général en géométrie symplectique : non seulement il existe maintenant plusieurs type linéaires pour des formes multilinéaires alternées et non-dégénérées, mais ce type linéaire peut aussi varier au sein d'une même variété multisymplectique.

En 2-plectique et dimension 6 par exemple, il existe 3 types linéaires. Pour un espace vectoriel V muni d'une base $\{e_1, \dots, e_6\}$, ces types linéaires sont

$$(i) \quad e_1^* \wedge e_2^* \wedge e_3^* + e_4^* \wedge e_5^* \wedge e_6^*$$

$$(ii) \quad e_1^* \wedge e_3^* \wedge e_5^* - e_1^* \wedge e_4^* \wedge e_6^* - e_2^* \wedge e_3^* \wedge e_6^* - e_2^* \wedge e_4^* \wedge e_5^*$$

$$(iii) \quad e_1^* \wedge e_5^* \wedge e_6^* - e_2^* \wedge e_4^* \wedge e_6^* + e_3^* \wedge e_4^* \wedge e_5^*.$$

Le type linéaire (i) correspond à des sommes de deux formes volumes, (ii) correspond aux groupes de Lie semi-simples compacts, et (iii) correspond au multicotangents. Le résultat suivant donne un fort contre-exemple à l'existence de théorèmes de Darboux en géométrie multisymplectique

Lemma 1.1.7 ([Ryv]). *On considère sur \mathbb{R}^6 la 3-forme*

$$\omega^f = dx_1 \wedge dx_3 \wedge dx_5 - dx_1 \wedge dx_4 \wedge dx_6 - dx_2 \wedge dx_3 \wedge dx_6 + f \cdot dx_2 \wedge dx_4 \wedge dx_5$$

où $f \in C^\infty(\mathbb{R}^6)$ ne dépend que de x_2, x_4 et x_5 . Alors (\mathbb{R}^6, ω^f) est une variété 2-plectique et on a pour $x \in \mathbb{R}^6$

- $\omega^f|_x$ est du type linéaire (i) si $f(x) > 0$
- $\omega^f|_x$ est du type linéaire (ii) si $f(x) < 0$
- $\omega^f|_x$ est du type linéaire (iii) si $f(x) = 0$.

Donc, en prenant par exemple $f(x) = x_2$ dans le lemme précédent, la forme ω^f possède les trois types linéaires dans n'importe quel voisinage ouvert de $0 \in \mathbb{R}^6$.

Des théorèmes de Darboux peuvent cependant être montrés sous des hypothèses supplémentaires assez fortes sur la variété multisymplectique en question. Un théorème de ce genre sera considéré en chapitre 5.

1.2 Observables multisymplectiques

Definition 1.2.1. Soit (M, ω) une variété k -plectique, et $\alpha \in \Omega^{k-1}(M)$. On dit que α est une *forme Hamiltonienne* s'il existe $X_\alpha \in \mathfrak{X}(M)$ tel que

$$d\alpha = \iota_{X_\alpha} \omega.$$

Le champ de vecteur X_α est alors appelé *champ Hamiltonien* (associé à la forme Hamiltonienne α). L'ensemble des formes Hamiltoniennes sera noté $\Omega_{Ham}^{k-1}(M, \omega)$, et l'ensemble des champs Hamiltoniens sera noté $\mathfrak{X}_{Ham}(M)$.

Notons que la non-dégénérescence de la forme multisymplectique assure pour une forme Hamiltonienne l'unicité du champ de vecteurs Hamiltonien associé. Cependant l'existence n'est pas assurée. Si l'on considère par exemple $M = \mathbb{R}^6$ muni de la forme 2-plectique

$$dx_1 \wedge dx_5 \wedge dx_6 - dx_2 \wedge dx_4 \wedge dx_6 + dx_3 \wedge dx_4 \wedge dx_5$$

alors la forme $\alpha = x_3 dx_4$ n'est pas Hamiltonienne. En général on a donc $\Omega_{Ham}^{n-1}(M, \omega) \subsetneq \Omega^{n-1}(M)$. Notons cependant que l'égalité a lieu si l'on considère une forme volume comme forme multisymplectique, par exemple.

La structure algébrique sur les observables d'une variété multisymplectique requiert l'usage de algèbres de Lie *supérieures*, ou ∞ -algèbres de Lie. Nous n'étudierons ici que le cas 2-plectique.

Proposition 1.2.2 ([Rog13]). *Soit (M, ω) une variété 2-plectique. Alors*

$$C^\infty(M) \xrightarrow{d} \Omega_{Ham}^1(M, \omega)$$

possède une structure de Lie 2-algèbre, avec les crochets suivants

- $l_1 = d$
- $l_2(\alpha, \beta) = \iota_{X_\alpha} \iota_{X_\beta} \omega$
- $l_2(\alpha, f) = 0$
- $l_3(\alpha, \beta, \gamma) = \omega(X_\alpha, X_\beta, X_\gamma)$

pour tout $\alpha, \beta, \gamma \in \Omega_{Ham}^1(M, \omega)$ et $f \in C^\infty(M)$.

Remark 1.2.3. Pour une variété n -plectique, la structure algébrique associée est une Lie n -algèbre ([Rog12],[Ryv16]), définie avec des crochets similaires.

Remarquons tout d'abord que cette Lie 2-algèbre n'est pas stricte. Pour cause, le crochet l_2 ne vérifie pas l'identité de Jacobi et $\Omega_{Ham}^1(M, \omega)$ n'est pas une algèbre de Lie. Par contre, le Jacobiateur est bien égal à la différentielle de la fonction donnée par le crochet l_3 . Certains textes de la littérature considèrent donc l'espace $\Omega^1(M, \omega)/d(C^\infty(M))$, qui possède alors une structure d'algèbre de Lie. Notons de même que d'autres structures ont été considérées, en rajoutant cependant des hypothèses sur la variété 2-plectique, et/ou en étendant les formes Hamiltoniennes à tout degré, considérant ainsi des multichamps de vecteurs (voir par exemple [FPR03], [CI96]).

On a un morphisme de Lie 2-algèbres

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \Omega_{Ham}^1(M, \omega) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{X}_{Ham}(M) \end{array}$$

où la seule flèche non triviale envoie une forme Hamiltonienne sur son unique champ de vecteurs Hamiltonien associé. Ce morphisme n'est jamais un quasi-isomorphisme.

En géométrie symplectique, l'algèbre des observables est pourvue d'une structure d'algèbre de Poisson : un produit existe sur les observables qui *dérive* le crochet de Lie donné par la forme symplectique. En géométrie 2-plectique, et même n -plectique générale, cette structure est pour le moment absente.

Remarquons par exemple qu'en 2-plectique, le produit d'une 1-forme Hamiltonienne avec une fonction n'est en général pas une 1-forme Hamiltonienne. Si on reprend par exemple la forme 2-plectique sur \mathbb{R}^6

$$dx_1 \wedge dx_5 \wedge dx_6 - dx_2 \wedge dx_4 \wedge dx_6 + dx_3 \wedge dx_4 \wedge dx_5$$

Alors par exemple $\alpha = x_1 dx_6$ est une 1-forme Hamiltonienne avec champ Hamiltonien associé $X_\alpha = -\partial_{x_5}$, et son produit avec la fonction $f(x) = x_2$ n'est pas une 1-forme Hamiltonienne.

1.3 Premières approches de la préquantification 2-plectique

Nous allons énoncer dans cette section deux résultats concernant la préquantification 2-plectique. Ceci sert d'introduction aux chapitres suivants, qui considéreront le problème d'un point de vue plus global.

Rappelons d'abord quelques faits bien connus de la quantification symplectique ([Kos70], [Sou66], [GS77], [Woo92]).

Etant donnée une variété symplectique (M, ω) , avec $[\omega] \in H^2(M, \mathbb{Z})$, il existe un fibré S^1 -principal $\tau : P \rightarrow M$ de courbure $(-2i\pi) \cdot \omega$. Alors on a un morphisme d'algèbres de Lie

$$C^\infty(M) \rightarrow \mathfrak{X}^{S^1}(P) \quad f \mapsto \widetilde{X}_f + \xi_f$$

où \widetilde{X}_f est le lift horizontal du champ Hamiltonien associé à f , et $\xi_f(p) = \left. \frac{d}{dt} \right|_0 (p \cdot \exp(2i\pi f(\tau(p))))$ pour $p \in P$. Les champs de vecteurs invariants par l'action de S^1 sont les sections de l'algébroïde d'Atiyah $TP/S^1 \rightarrow M$. De plus, ce morphisme devient un isomorphisme si l'on restreint l'image aux champs de vecteurs qui respectent la connexion, c'est à dire les champs de vecteurs $X \in \mathfrak{X}^{S^1}(P)$ tels que $\mathcal{L}_X A = 0$, où A est la connexion sur P de courbure $(-2i\pi) \cdot \omega$. Les deux résultats que nous allons voir maintenant considèrent cet aspect de la préquantification en géométrie 2-plectique.

L'algébroïde d'Atiyah est alors remplacée par une *algébroïde de Courant*.

Definition 1.3.1. Une *algébroïde de Courant* sur M est un fibré vectoriel $C \rightarrow M$, muni d'une forme bilinéaire $\langle \cdot, \cdot \rangle$ symétrique, et d'un crochet $[\cdot, \cdot]_C$ sur les section $\Gamma(C)$ et d'un morphisme de fibré vectoriels $\rho : C \rightarrow TM$, tels que les conditions suivantes sont réunies

- $[e_1, [e_2, e_3]_C]_C = [[e_1, e_2]_C, e_3]_C + [e_2, [e_1, e_3]_C]_C$
- $\rho([e_1, e_2]_C) = [\rho(e_1), \rho(e_2)]$
- $[e_1, fe_2]_C = f[e_1, e_2]_C + \rho(e_1)(f)e_2$
- $[e_1, e_1]_C = \frac{1}{2}D \langle e_1, e_1 \rangle$
- $\rho(e_1)(\langle e_2, e_3 \rangle) = \langle [e_1, e_2]_C, e_3 \rangle + \langle e_2, [e_1, e_3]_C \rangle$

avec $e_1, e_2, e_3 \in \Gamma(C)$, $f \in C^\infty(M)$ et $D : C^\infty(M) \rightarrow \Gamma(C)$ est définie par $\langle Df, e \rangle = \rho(e)(f)$ pour $e \in \Gamma(C)$ et $f \in C^\infty(M)$.

Remark 1.3.2. Nous n'énoncerons pas ici les propriétés fines des algébroïdes de Courant. Pour un exposé plus complet, voir [Roy99].

L'exemple fondamental d'une algébroïde de Courant est le fibré $C = TM \oplus T^*M \rightarrow M$, avec la structure

- $[(v, \alpha), (v', \alpha')]_C = ([v, v'], \mathcal{L}_v \alpha' - \mathcal{L}_{v'} \alpha - \frac{1}{2}d(\iota_v \alpha' - \iota_{v'} \alpha))$
- $\langle (v, \alpha), (v', \alpha') \rangle = \iota_v \alpha' + \iota_{v'} \alpha$

pour $(v, \alpha), (v', \alpha') \in \mathfrak{X}(M) \oplus \Omega^1(M) = \Gamma(C)$.

Soit (M, ω) une variété 2-plectique, avec $[\omega] \in H^3(M, \mathbb{Z})$. Soit $\mathcal{U} = \{U_i\}_{i \in I}$ un bon recouvrement ouvert de la variété M . Notons $U_{i_1, \dots, i_k} = U_{i_1} \cap \dots \cap U_{i_k}$. On considère une *trivialisat*ion de ω dans ce recouvrement

- $\omega|_{U_i} = d\theta_i$ avec $\theta_i \in \Omega^2(U_i)$
- $\theta_j - \theta_i = dA_{ij}$ avec $A_{ij} \in \Omega^1(U_{ij})$
- $A_{ij} + A_{jk} + d\log h_{ijk} = A_{ik}$ avec $h_{ijk} \in C^\infty(U_{ijk}, S^1)$.

Soit alors le fibré vectoriel sur M

$$C = \left(\coprod_i TU_i \oplus T^*U_i \right) / \sim$$

où $(i, v, \alpha) \sim (j, w, \beta)$ s'il existe $x \in U_{ij}$ tel que $(v, \alpha) \in T_x U_i \oplus T_x^* U_i$, $(w, \beta) \in T_x U_j \oplus T_x^* U_j$, $v = w$ et $\beta = \alpha + (\iota_v dA_{ij})|_x$. On vérifie facilement que \sim est une relation d'équivalence et

$$C \rightarrow M; [i, v, \alpha] \mapsto x$$

si $x \in U_i$ et $(v, \alpha) \in T_x U_i \oplus T_x^* U_i$ définit un fibré vectoriel lisse sur M . De plus pour $x \in U_i$ et $v \in T_x U_i$ on pose $s_i(v) := (i, v, -\iota_v \theta_i)$. Si $x \in U_{ij}$, on observe que $s_i(v) \sim s_j(v)$ et donc $s(v) := [s_i(v)]$ définit un homomorphisme de fibré vectoriel sur M

$$\begin{array}{ccc} TM & \xrightarrow{s} & C \\ \downarrow & & \downarrow \\ M & \xlongequal{\quad} & M. \end{array}$$

Donc toute section $e \in \Gamma(C)$ peut s'écrire de façon unique $e = s(X) + \alpha$, pour $X \in \mathfrak{X}(M)$ et $\alpha \in \Omega^1(M)$ (ici α définit une section globale de C via $[0, \alpha]$). On peut alors définir un morphisme $\rho : C \rightarrow TM$

$$\rho(s(v) + \alpha) = v.$$

On définit encore pour $X \in \mathfrak{X}(M)$ et $\alpha \in \Omega^1(M)$

- $\langle s(X) + \alpha, s(X') + \alpha' \rangle = \iota_X \alpha' + \iota_{X'} \alpha$
- $[s(X) + \alpha, s(X') + \alpha']_C = ([X, X'], \mathcal{L}_X \alpha' - \mathcal{L}_{X'} \alpha - \frac{1}{2} d(\iota_X \alpha' - \iota_{X'} \alpha))$.

Alors $C \rightarrow M$ est une algébroïde de Courant. On construit ensuite une structure de Lie 2-algèbre sur

$$C^\infty(M) \rightarrow \Gamma(C)$$

avec

- $l_1(f) = (0, df)$
- $l_2(e_1, e_2) = [e_1, e_2]_C$
- $l_2(e, f) = \frac{1}{2} \langle e, df \rangle$
- $l_3(e_1, e_2, e_3) = -\frac{1}{6} T(e_1, e_2, e_3)$

où $f \in C^\infty(M)$, $e, e_1, e_2, e_3 \in \Gamma(C)$ et

$$T(e_1, e_2, e_3) = \langle [e_1, e_2]_C, e_3 \rangle + \langle [e_3, e_1]_C, e_2 \rangle + \langle [e_2, e_3]_C, e_1 \rangle.$$

Theorem 1.3.3 ([Rog13]). *Soit (M, ω) une variété 2-plectique avec $[\omega] \in H^3(M, \mathbb{Z})$. Il y a un morphisme de Lie 2-algèbres*

$$\begin{array}{ccc}
C^\infty(M) & \longrightarrow & \Omega_{Ham}^1(M, \omega) \\
\downarrow id & \swarrow \Phi_2 & \downarrow \Phi_1 \\
C^\infty(M) & \longrightarrow & \Gamma(C)
\end{array}$$

entre la Lie 2-algèbre des observables et la Lie 2-algèbre donnée par l'algébroïde de Courant construite ci-dessus. Ce morphisme est donné par

- $\Phi_1(\alpha) = s(X_\alpha) - \alpha$
- $\Phi_2(\alpha, \beta) = -\frac{1}{2}(\beta(X_\alpha) - \alpha(X_\beta))$.

De plus ce morphisme est un isomorphisme de Lie 2-algèbre si on restreint $\Gamma(C)$ aux sections $e = s(X) + \alpha$ telles que

$$[e, s(X')]_C = s([X, X'])$$

pour tout $X' \in \mathfrak{X}(M)$.

Remark 1.3.4. La restriction dans le théorème ci-dessus donne une Lie 2-algèbre avec la même structure.

Notons que dans ce théorème, nous obtenons un isomorphisme de Lie 2-algèbre, ce qui est plus fort qu'un quasi-isomorphisme. Ceci généralise donc le morphisme d'algèbre de Lie entre les observables et les sections de l'algébroïde d'Atiyah en géométrie symplectique. On peut aussi construire cette algébroïde directement en "twistant" le crochet des champs de vecteurs avec la forme symplectique, dans une démarche similaire à la construction de l'algébroïde de Courant ci-dessus.

Nous allons maintenant énoncer un deuxième résultat similaire. On considère encore une variété 2-plectique (M, ω) avec $[\omega] \in H^3(M, \mathbb{Z})$, et une trivialisations de ω dans un bon recouvrement ouvert de la variété M , en reprenant les notations de la construction précédente.

On définit alors L_0 comme l'ensemble des $(X, \{f_{ij}\}, \{\alpha_i\})$, où $X \in \mathfrak{X}(M)$, $f_{ij} \in C^\infty(U_{ij})$ et $\alpha_i \in \Omega^1(U_i)$ sujets aux conditions

- $\mathcal{L}_X \theta_i = d\alpha_i$
- $\mathcal{L}_X A_{ij} = \alpha_j - \alpha_i + df_{ij}$.

Alors le complexe

$$\oplus C^\infty(U_i) \xrightarrow{l_1} L_0$$

est une Lie 2-algèbre avec la structure suivante

- $l_1(\{f_i\}) = (0, \{f_i - f_j\}, \{df_i\})$
- $l_2((X, \{f_{ij}\}, \{\alpha_i\}), (X', \{f'_{ij}\}, \{\alpha'_i\})) = ([X, X'], \{df'_{ij}(X) - df_{ij}(X')\}, \{\mathcal{L}_X \alpha'_i - \mathcal{L}_{X'} \alpha_i\})$
- $l_2((X, \{f_{ij}\}, \{\alpha_i\}), \{g_i\}) = dg_i(X)$
- $l_3 = 0$.

On a alors le résultat suivant

Theorem 1.3.5. *[FRS14] Soit (M, ω) une variété 2-plectique. On trivialisé ω dans un bon recouvrement ouvert en reprenant les notation de la construction précédente. Il y a alors un morphisme de Lie 2-algèbres*

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \Omega_{Ham}^1(M, \omega) \\ \Phi_1 \downarrow & \swarrow \Phi_2 & \downarrow \Phi_1 \\ \oplus C^\infty(U_i) & \xrightarrow{l'_1} & L_0 \end{array}$$

donné par les équations suivantes pour $f \in C^\infty(M)$ et $\alpha, \beta \in \Omega_{Ham}^1(M, \omega)$

- $\Phi_1(f) = \{f|_{U_i}\}$
- $\Phi_1(\alpha) = (X_\alpha, \{\iota_{X_\alpha} A_{ij}\}, \{\iota_{X_\alpha} \theta_i + \alpha|_{U_i}\})$
- $\Phi_2(\alpha, \beta) = \{(\beta(X_\alpha) - \alpha(X_\beta))|_{U_i} + \iota_{X_\alpha} \iota_{X_\beta} \theta_i\}$.

De plus, ce morphisme est un quasi-isomorphisme en imposant la condition supplémentaire suivante sur les éléments $(X, \{f_{ij}\}, \{\alpha_i\}) \in L_0$

$$f_{ij} + f_{jk} + d\log h_{ijk}(X) = f_{ik}.$$

Dans le théorème précédent, le complexe

$$\oplus C^\infty(U_i) \xrightarrow{l'_1} L'_0$$

avec L'_0 l'ensemble des $(X, \{f_{ij}\}, \{\alpha_i\})$ où $X \in \mathfrak{X}(M)$, $f_{ij} \in C^\infty(U_{ij})$ et $\alpha_i \in \Omega^1(U_i)$ sujets aux conditions

- $\mathcal{L}_X \theta_i = d\alpha_i$
- $\mathcal{L}_X A_{ij} = \alpha_j - \alpha_i + df_{ij}$
- $f_{ij} + f_{jk} + \text{dlog} h_{ijk}(X) = f_{ik}$

reste bien sûr une Lie 2-algèbre, avec les mêmes crochets de structure que ceux définis pour la Lie 2-algèbre $\oplus C^\infty(U_i) \xrightarrow{l_1} L_0$.

Remark 1.3.6. La condition

$$f_{ij} + f_{jk} + \text{dlog} h_{ijk}(X) = f_{ik}$$

sur le couple $(X, \{f_{ij}\})$ avec $X \in \mathfrak{X}(M)$ et $f_{ij} \in C^\infty(U_{ij})$ exprime les données locales d'un champ de vecteurs *multiplicatif* sur une *gerbe locale* de classe Dixmier-Douady $[h_{ijk}] \in H^3(M, \mathbb{Z})$. Rajouter ensuite les conditions

- $\mathcal{L}_X \theta_i = d\alpha_i$
- $\mathcal{L}_X A_{ij} = \alpha_j - \alpha_i + df_{ij}$

fait alors du triplet $(X, \{f_{ij}\}, \{\alpha_i\})$ les données d'une *symétrie infinitésimale faible*, sur une gerbe locale de classe Dixmier-Douady $[h_{ijk}] \in H^3(M, \mathbb{Z})$ et structure connective donnée par (A_{ij}, θ_i) , appelées *quantomorphismes infinitésimaux* dans [FRS14]. Ceci sera détaillé et globalisé dans les chapitres ultérieurs.

Chapter 2

Geometrisation of $H^3(M, \mathbb{Z})$

In this section, we will study the various geometric objects associated to an element of $H^3(M, \mathbb{Z})$, for M a finite-dimensional differentiable manifold. For every notion, we will give several geometric and topological constructions, and give results linking $H^3(M, \mathbb{Z})$ to the right definition of isomorphism class of the object that we consider. We will eventually highlight the similarities between these objects. In this whole section M will denote a finite dimensional differentiable manifold. Since the presented constructions do not depend on the choice of a good open cover of the manifold M , we allow ourselves to fix such a covering $\mathcal{U} = \{U_i\}_{i \in I}$. We will use the notation $U_{i_1 \dots i_k} = U_{i_1} \cap \dots \cap U_{i_k}$.

For every geometrisation of $H^3(M, \mathbb{Z})$ studied here, with the exception of principal $PU(H)$ -bundles, the adequate general theory needed to understand their general structure involves the use of (at least) 2-categories. Since we only want to understand the link these objects have with $H^3(M, \mathbb{Z})$, these aspects will not be mentioned here.

We will also disregard the various notions of connective structures that exist on these objects. We will however give geometric constructions whenever possible.

The main motivation in studying the geometrisation of $H^3(M, \mathbb{Z})$ is, of course, to arrive at a better understanding of the objects that should appear upon developing 2-plectic quantisation.

2.1 Principal $PU(H)$ -bundles

We begin with the oldest geometrisation of $H^3(M, \mathbb{Z})$, which goes back to ([DD63]).

Let H be a separable, infinite-dimensional Hilbert space. We have the well-known exact sequence of Lie groups

$$1 \rightarrow S^1 \rightarrow U(H) \rightarrow PU(H) \rightarrow 1$$

where $U(H)$ is the space of unitary operators on H , and $PU(H) = U(H)/S^1$. We will denote the natural projection by $q : U(H) \rightarrow PU(H)$.

Now let $P \rightarrow M$ be a principal $PU(H)$ -bundle. Denote by $g_{ij} : U_{ij} \rightarrow PU(H)$ the transition functions of the bundle $P \rightarrow M$. We use conventions such that:

$$g_{jk}g_{ij} = g_{ik}.$$

Upon choosing lifts :

$$\begin{array}{ccc} & & U(H) \\ & \nearrow \widetilde{g}_{ij} & \downarrow \\ U_{ij} & \xrightarrow{g_{ij}} & PU(H) \end{array}$$

one observes that the \widetilde{g}_{ij} 's do not define a cocycle but

$$a_{ijk} := \widetilde{g}_{ik}^{-1} \widetilde{g}_{jk} \widetilde{g}_{ij}$$

lies in S^1 because $q(a_{ijk}) = 1$. It is easily verified that $\{a_{ijk}\}$ defines an element $[a]$ of $H^2(M, S^1) = H^3(M, \mathbb{Z})$. The class $[a]$ does not depend on the choice of the open cover nor the choice of lifts. This class is called the *Dixmier-Douady class of the bundle P* . Note that this class being torsion means that the bundle P lifts to a principal $U(H)$ -bundle $\widetilde{P} \rightarrow M$ such that $\widetilde{P}/S^1 \cong P$. So the Dixmier-Douady class can be seen as an obstruction to lifting the bundle P to a principal $U(H)$ -bundle over M .

Constructing a map $H^3(M, \mathbb{Z}) \rightarrow H^1(M, PU(H))$ is more complicated and will not be detailed here. However we have the following result, whose proof can be found, e.g., in ([Bry08]).

Theorem 2.1.1. *There is an isomorphism*

$$H^3(M, \mathbb{Z}) \cong H^1(M, PU(H))$$

with $H^1(M, PU(H))$ describing isomorphism classes of principal $PU(H)$ -bundles over M .

Remark 2.1.2. The isomorphism of Theorem 2.1.1 can also be understood in terms of classifying and Eilenberg-MacLane spaces (see, e.g., [Spa81] for a detailed discussion). Let indeed G be a Lie group. There is a universal principal G -bundle $EG \rightarrow BG$ which gives an isomorphism

$$H^1(M, G) = [M, BG]$$

between the equivalence classes of principal G -bundles over M and homotopy classes of smooth maps $M \rightarrow BG$. It is well known that finding a principal G -bundle $E \rightarrow B$ with E being contractible, means that this bundle is universal in the above sense.

Moreover, we have the Eilenberg-MacLane space $K(G, n)$ for $n \in \mathbb{N}$, coming from ordinary cohomology, that satisfy

$$K(G, n) = B(K(G, n-1)).$$

We interpret here $K(G, 0)$ as G . Recall that for $n \geq 2$, G must be abelian. Therefore we have isomorphisms

$$H^n(M, G) = [M, K(G, n)] = [M, B(K(G, n-1))]$$

between the n^{th} ordinary cohomology of M with values in G , and isomorphism classes of principal $K(G, n-1)$ -principal bundles.

The principal S^1 -bundle $U(H) \rightarrow PU(H)$ is universal since $U(H)$ is contractible by Kuiper's theorem. Therefore, taking $G = \mathbb{Z}$ in the preceding isomorphism yields

$$H^3(M, \mathbb{Z}) = [M, K(\mathbb{Z}, 3)] = [M, B(K(\mathbb{Z}, 2))] = [M, BPU(H)] = H^1(M, PU(H))$$

giving the isomorphism of Theorem 2.1.1.

This gives the first geometric realization of a class in $H^3(M, \mathbb{Z})$. However, given such a class, it is quite complicated to *directly* construct a $PU(H)$ -principal bundle with the right Dixmier-Douady class. We will now introduce geometric objects that solve this problem.

2.2 Bundle gerbes

We consider in this section bundle gerbes, first introduced by Murray in ([Mur96]). We will first give the general definitions concerning the objects themselves as well as isomorphisms between them, and then state the result linking these objects to $H^3(M, \mathbb{Z})$.

Definition 2.2.1. Let M be a smooth manifold. A S^1 -bundle gerbe (P, Y) over M consists of:

- a smooth manifold Y with a surjective submersion $Y \xrightarrow{\pi} M$,
- a principal S^1 -bundle P over $Y^{[2]}$,
- a smooth isomorphism $\mu : \pi_3^* P \otimes \pi_1^* P \rightarrow \pi_2^* P$ of principal S^1 -bundles over $Y^{[3]}$ (where $\pi_i : Y^{[3]} \rightarrow Y^{[2]}$ denotes the projection that forgets the i^{th} factor).

Furthermore, the isomorphism μ has to verify an associativity condition, which can be expressed by the commutativity of the following diagram of S^1 -bundles over $Y^{[4]}$:

$$\begin{array}{ccc} \pi_{34}^* P \otimes \pi_{14}^* P \otimes \pi_{12}^* P & \xrightarrow{\mu \otimes id} & \pi_{24}^* P \otimes \pi_{12}^* P \\ \downarrow id \otimes \mu & & \downarrow \mu \\ \pi_{34}^* P \otimes \pi_{13}^* P & \xrightarrow{\mu} & \pi_{23}^* P \end{array}$$

Here $\pi_{ij} : Y^{[4]} \rightarrow Y^{[2]}$ denotes the projection which forgets the i^{th} and j^{th} factors.

The map μ in the preceding definition will be referred to as a *bundle gerbe product*. Note that for $(y_1, y_2, y_3) \in Y^{[3]}$, this map gives an S^1 -equivariant isomorphism

$$P_{y_1, y_2} \otimes P_{y_2, y_3} \cong P_{y_1, y_3}$$

where P_{y_i, y_j} denotes the fiber above an element $(y_i, y_j) \in Y^{[2]}$.

We will now describe how an element of $H^3(M, \mathbb{Z})$ is associated to a bundle gerbe.

Let (P, Y) be a bundle gerbe over M . Choose local sections $\sigma_i : U_i \rightarrow Y$. Then for $x \in U_{ij}$, $(\sigma_i(x), \sigma_j(x)) \in Y^{[2]}$ so we denote P_{ij} the pullback of P by

the map $(\sigma_i, \sigma_j) : U_{ij} \rightarrow Y^{[2]}$. Choosing sections $s_{ij} : U_{ij} \rightarrow P_{ij}$, there is a unique $g_{ijk} : U_{ijk} \rightarrow S^1$ such that for every $x \in U_{ijk}$:

$$\mu(s_{jk}(x), s_{ij}(x)) = g_{ijk}(x) \cdot s_{ik}(x).$$

Using the associativity of the bundle gerbe multiplication, we can show that $[g] \in H^2(M, S^1) = H^3(M, \mathbb{Z})$. Furthermore $[g]$ does not depend on the choice of the cover nor the choices of sections σ_i and s_{ij} .

Definition 2.2.2. The class $[g] \in H^3(M, \mathbb{Z})$ constructed above is called the *Dixmier-Douady class of the bundle gerbe* (P, Y) .

Before giving the right notion of morphisms between bundle gerbes which respects the Dixmier-Douady class, we have to consider several constructions with bundle gerbes.

We can construct the *dual* (P^*, Y) of a bundle gerbe (P, Y) by simply taking the dual P^* of the principal S^1 -bundle $P \rightarrow Y^{[2]}$. Since $P \otimes P^* \cong Y^{[2]} \times S^1$, we can construct a bundle gerbe product on (P^*, Y) .

We can also construct the *product* of two bundle gerbes (P_1, Y_1) and (P_2, Y_2) over M . Denoting the projections $Y_1 \xrightarrow{\pi_1} M$ and $Y_2 \xrightarrow{\pi_2} M$, consider the fibered product $Y_{1\pi_1} \times_{\pi_2} Y_2$ with projections $p_1 : Y_{1\pi_1} \times_{\pi_1} Y_2 \rightarrow Y_1$ and $p_2 : Y_{1\pi_1} \times_{\pi_2} Y_2 \rightarrow Y_2$. Thus we have maps $p_1^{[2]} : (Y_{1\pi_1} \times_{\pi_2} Y_2)^{[2]} \rightarrow Y_1^{[2]}$ and $p_2^{[2]} : (Y_{1\pi_1} \times_{\pi_2} Y_2)^{[2]} \rightarrow (Y_2)^{[2]}$, which we can use to consider the pullback S^1 -bundle

$$P_1 \otimes P_2 := (p_1^{[2]})^* P_1 \otimes (p_2^{[2]})^* P_2$$

over $Y_{1\pi_1} \times_{\pi_2} Y_2$. A bundle gerbe product can be easily defined using the products coming from the two bundle gerbes (P_1, Y_1) and (P_2, Y_2) . Then $(P_1 \otimes P_2, Y_{1\pi_1} \times_{\pi_2} Y_2)$ is a bundle gerbe over M .

Finally let $Y \xrightarrow{\pi} M$ be a surjective submersion, and $Q \rightarrow Y$ be a principal S^1 -bundle. Denoting the projections $q_i : Y^{[2]} \rightarrow Y$, for $i = 1, 2$ we can form the S^1 -bundle

$$\delta(Q) := (q_1^* Q)^* \otimes q_2^* Q$$

over $Y^{[2]}$. If $\pi_i : Y^{[3]} \rightarrow Y^{[2]}$, for $i = 1, 2, 3$, is the projection that forgets the i^{th} component, then $\pi_1^* \delta(Q) \otimes \pi_3^* \delta(Q) = (\kappa_{13}^* Q)^* \otimes \kappa_{12}^* Q \otimes (\kappa_{23}^* Q)^* \otimes \kappa_{13}^* Q$ (where $\kappa_{ij} : Y^{[3]} \rightarrow Y$ is the projection that forgets the i^{th} and j^{th} component), then it is naturally isomorphic to $(\kappa_{23}^* Q)^* \otimes \kappa_{12}^* Q = \pi_2^* \delta(Q)$, giving a

bundle gerbe product. Thus $(\delta(Q), Y)$ is a bundle gerbe over M .

We are now ready to define the notion of "stable" isomorphism between bundle gerbes.

Definition 2.2.3. Let (P_1, Y_1) and (P_2, Y_2) be two bundle gerbes over M . We denote the projections $\pi_1 : Y_1 \rightarrow M$ and $\pi_2 : Y_2 \rightarrow M$. We say that (P_1, Y_1) and (P_2, Y_2) are *stably isomorphic* if there exists principal S^1 -bundles Q_1, Q_2 over $Y_{1\pi_1} \times_{\pi_2} Y_2$ with an isomorphism of principal S^1 -bundles

$$(p_1^{[2]})^* P_1 \otimes \delta(Q_1) \cong \delta(Q_2) \otimes (p_2^{[2]})^* P_2$$

over $(Y_{1\pi_1} \times_{\pi_2} Y_2)^{[2]}$, which commutes with the bundle gerbe multiplications.

This defines an equivalence relation between bundle gerbes, and it turns out that this is the right notion of isomorphism that preserves the Dixmier-Douady class.

Theorem 2.2.4. [MS00] *Two bundle gerbes are stably isomorphic if and only if they have the same Dixmier-Douady class.*

Remark 2.2.5. In other words, we have a bijection between $H^3(M, \mathbb{Z})$ and the stable isomorphism classes of bundle gerbes over M . Moreover, a bundle gerbe has torsion Dixmier-Douady class if and only if it is stably isomorphic to a bundle gerbe of the form $(\delta(Q), Y)$. It is in fact quite easy, given a class $[\lambda] = [\lambda_{ijk}] \in H^3(M, \mathbb{Z})$, to construct a bundle gerbe over M with Dixmier-Douady class $[\lambda]$. Consider the projection $Y := \coprod U_i \rightarrow M$. Then $Y^{[2]} = \coprod U_{ij}$, and we consider the trivial S^1 -bundle $P := \coprod U_{ij} \times S^1 \rightarrow \coprod U_{ij}$. Then the map

$$\mu : \pi_3^* P \otimes \pi_1^* P \rightarrow \pi_2^* P, \quad (x_{jk}, g) \otimes (x_{ij}, g') \mapsto (x_{ik}, gg' \lambda_{ijk}(x))$$

where $x \in U_{ijk}$, defines a bundle gerbe multiplication, and the Dixmier-Douady class of (P, Y) is $[\lambda]$ (here x_{jk} denote x but with its membership to U_{jk} being underlined). Such a bundle gerbe is called *local*.

We finish this section by linking principal $PU(H)$ -bundles to bundle gerbes.

Let $Y \rightarrow M$ be a principal $PU(H)$ -bundle. Then $Y^{[2]} \cong Y \times PU(H)$ and we have a well defined map $\psi : Y^{[2]} \rightarrow PU(H)$ such that for $(y, y') \in Y^{[2]}$

$$y' = y \cdot \psi(y, y').$$

Thus we can pullback the S^1 -bundle $U(H) \rightarrow PU(H)$ via the map ψ to obtain a principal S^1 -bundle $P \rightarrow Y$. A bundle gerbe multiplication can be defined by

$$\mu((y, y', u), (y', y'', u')) = (y, y'', uu'),$$

for $y, y', y'' \in Y^{[3]}$ and $u, u' \in U(H)$. This gives a bundle gerbe (P, Y) over M , called the *lifting bundle gerbe associated to $Y \rightarrow M$* .

Remark 2.2.6. This construction works for every principal G -bundle, with G sitting in an exact sequence

$$1 \rightarrow S^1 \rightarrow \hat{G} \rightarrow G \rightarrow 1.$$

Consider a principal $PU(H)$ -bundle $Y \rightarrow M$, with Dixmier-Douady class $[\lambda] \in H^3(M, \mathbb{Z})$. Then choosing sections $\sigma_i : U_i \rightarrow Y$, we obtain transition functions $g_{ij} : U_{ij} \rightarrow PU(H)$ and for $x \in U_{ij}$, $\psi(\sigma_i(x), \sigma_j(x)) = g_{ij}(x)$. Then sections $s_{ij} : U_{ij} \rightarrow \psi^*(U(H))$ are defined by $s_{ij}(x) = (\sigma_i(x), \sigma_j(x), \widetilde{g}_{ij}(x))$ where $\widetilde{g}_{ij} : U_{ij} \rightarrow U(H)$ are lifts of the transition functions g_{ij} . We obtain

$$\begin{aligned} \mu(s_{jk}(x), s_{ij}(x)) &= (\sigma_i(x), \sigma_k(x), \widetilde{g}_{jk}(x)\widetilde{g}_{ij}(x)) \\ &= (\sigma_i(x), \sigma_k(x), \widetilde{g}_{ik}(x)\lambda_{ijk}(x)) \\ &= \lambda_{ijk}(x)s_{ik}(x). \end{aligned}$$

This shows that the Dixmier-Douady class of the lifting bundle gerbe associated to the principal $PU(H)$ -bundle $Y \rightarrow M$ is equal to the Dixmier-Douady class of the bundle itself. Combining this construction with the fact that $H^3(M, \mathbb{Z})$ is isomorphic to both the isomorphism classes of principal $PU(H)$ -bundles, and the stable isomorphism classes of bundle gerbes, we obtain the following

Theorem 2.2.7. *Every bundle gerbe is stably isomorphic to a lifting bundle gerbe of some principal $PU(H)$ -bundle.*

2.3 Groupoid S^1 -central extensions

The next class of geometric objects that we investigate here are groupoid S^1 -central extensions, introduced in [BX11].

Definition 2.3.1. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid. A Lie groupoid $Q_1 \rightrightarrows G_0$ is called an *S^1 -central extension of $G_1 \rightrightarrows G_0$* if there exists an exact sequence of Lie groupoids

$$1 \rightarrow G_0 \times S^1 \xrightarrow{\chi} Q_1 \xrightarrow{\rho} G_1 \rightarrow 1$$

over id_{G_0} such that the image of χ is central in Q_1 . We will denote such objects by $Q_1 \rightarrow G_1 \rightrightarrows G_0$.

It is immediate to see that, given such a sequence, the manifold Q_1 is equipped with an action of S^1 , namely for $q \in Q_1$ and $\lambda \in S^1$

$$q \cdot \lambda := q \circ \chi(s(q), \lambda).$$

Moreover this action makes $Q_1 \xrightarrow{\rho} G_1$ into a principal S^1 -bundle since, for $q, q' \in Q_1$ such that $\rho(q) = \rho(q')$, we have that $s(q) = s(q')$ and $t(q) = t(q')$, so we can consider the product $q^{-1} \circ q'$ which is in the kernel of ρ , so there is a unique $\lambda \in S^1$ such that

$$q^{-1} \circ q' = \chi(s(q), \lambda) \text{ and therefore } q' = q \cdot \lambda.$$

Note that this action is compatible with the groupoid multiplication in the sense that

$$(q \cdot \lambda) \circ q' = q \circ (q' \cdot \lambda) = (q \circ q') \cdot \lambda$$

for $q, q' \in Q_1$ and $\lambda \in S^1$ such that the above equation makes sense.

We will only consider S^1 -central extensions of certain groupoids, in order to give a description in term of $H^3(M, \mathbb{Z})$. Let $Y \xrightarrow{\pi} M$ be a surjective submersion. Then we consider the S^1 -central extensions of the groupoid $Y^{[2]} \rightrightarrows Y$, that is, sequences

$$1 \rightarrow Y \times S^1 \rightarrow P \rightarrow Y^{[2]} \rightarrow 1.$$

Such sequences are in fact in one to one correspondence with bundle gerbes: in the above sequence, the pair (P, Y) is a bundle gerbe over M . The groupoid multiplication gives a bundle gerbe multiplication, and the fact that the groupoid multiplication is compatible with the S^1 -action and associative imply that the axioms in the definition of a bundle gerbe are satisfied.

Definition 2.3.2. Let $\pi_i : Y_i \rightarrow M$ ($i = 1, 2$) be two surjective submersions. An *isomorphism between two S^1 -central extensions* $Q_1 \rightarrow Y_1^{[2]} \rightrightarrows Y_1$ and $Q_2 \rightarrow Y_2^{[2]} \rightrightarrows Y_2$ is a Morita equivalence $Q_1 \xleftarrow{\tau_1} Z \xrightarrow{\tau_2} Q_2$ such that $\pi_1 \circ \tau_1 = \pi_2 \circ \tau_2$, with a right action of S^1 on Z which is compatible with the Morita equivalence in the sense that

$$(q_1 \cdot \lambda) \cdot z \cdot q_2 = q_1 \cdot (z \cdot \lambda) \cdot q_2 = q_1 \cdot z \cdot (q_2 \cdot \lambda)$$

for all $\lambda \in S^1$, $(q_1, q_2) \in Q_1 \times Q_2$ and $z \in Z$ such that the above formula makes sense.

We will show that this definition is equivalent to the associated bundle gerbes being stably isomorphic.

Let then $Q_1 \xleftarrow{\tau_1} Z \xrightarrow{\tau_2} Q_2$ be an isomorphism between the two S^1 -central extensions $Q_1 \rightarrow Y_1^{[2]} \rightrightarrows Y_1$ and $Q_2 \rightarrow Y_2^{[2]} \rightrightarrows Y_2$. Because the S^1 -action is free on both Q_1 and Q_2 it is straightforward to verify that it is also free on Z . Then the map

$$Z \xrightarrow{(\tau_1, \tau_2)} Y_{1\pi_1} \times_{\pi_2} Y_2$$

is well-defined and is a principal S^1 -bundle since for $z, z' \in Z$ with $\tau_1(z) = \tau_1(z')$, $\tau_2(z) = \tau_2(z')$, we have unique $q_1 \in Q_1$ and $q_2 \in Q_2$ such that $q_1 \cdot z = z'$ and $z \cdot q_2 = z'$. Taking the image by τ_1 of both these equations shows that $s(q_1) = t(q_1)$ so q_1 lies in the image of $Y_1 \times S^1$ and then there is a unique $\lambda \in S^1$ such that $z' = z \cdot \lambda$. Then consider the S^1 -bundle $\delta(Z) \rightarrow (Y_{1\pi_1} \times_{\pi_2} Y_2)^{[2]}$ as in the previous section. We have an isomorphism of principal S^1 -bundles:

$$(p_1^{[2]})^* Q_1 \otimes \delta(Z) \cong \delta(Z) \otimes (p_2^{[2]})^* Q_2.$$

Let indeed be $((y_1, y_2), (y'_1, y'_2)), q_1, z, z' \in (p_1^{[2]})^* Q_1 \otimes \delta(Z)$. It is such that $(\tau_1, \tau_2)(z) = (y_1, y_2)$, $(\tau_1, \tau_2)(z') = (y'_1, y'_2)$ and $(s, t)(q_1) = (y_1, y'_1)$, with $(y_1, y'_1) \in Y_1^{[2]}$ and $(y_2, y'_2) \in Y_2^{[2]}$. Then $\tau_1(q_1 \cdot z') = t(q_1) = y_1 = \tau_1(z)$, so there is a unique $q_2 \in Q_2$, $(s, t)(q_2) = (y_2, y'_2)$ such that $q_1 \cdot z' = z \cdot q_2$. Thus the correspondence

$$[q_1, z, z'] \mapsto [z, z', q_2]$$

gives the desired isomorphism. Moreover, it can be shown using the properties of Morita equivalences that the isomorphism indeed respects the bundle gerbes products. Thus, the bundle gerbes (Q_1, Y_1) and (Q_2, Y_2) are stably isomorphic.

Now consider two stably isomorphic bundle gerbes (P_1, Y_1) and (P_2, Y_2) over M , with surjective submersions $\pi_1 : Y_1 \rightarrow M$ and $\pi_2 : Y_2 \rightarrow M$. Then we have principal S^1 -bundles $Q_1 \xrightarrow{\tau_1} Y_1$ and $Q_2 \xrightarrow{\tau_2} Y_2$ with an isomorphism

$$(p_1^{[2]})^* P_1 \otimes \delta(Q_1) \cong \delta(Q_2) \otimes (p_2^{[2]})^* P_2$$

of S^1 -bundles over $(Y_{1\pi_1} \times_{\pi_2} Y_2)^{[2]}$. The fact that this isomorphism respects the bundle gerbe multiplications means there is an isomorphism of the associated Lie groupoids $(p_1^{[2]})^* P_1 \otimes \delta(Q_1) \rightrightarrows Y_{1\pi_1} \times_{\pi_2} Y_2$ and $\delta(Q_2) \otimes (p_2^{[2]})^* P_2 \rightrightarrows Y_{1\pi_1} \times_{\pi_2} Y_2$. The Lie groupoid $(p_1^{[2]})^* P_1 \rightrightarrows Y_{1\pi_1} \times_{\pi_2} Y_2$ is the pullback groupoid of $P_1 \rightrightarrows Y_1$ by the projection $Y_{1\pi_1} \times_{\pi_2} Y_2 \rightarrow Y_1$, therefore these two groupoids are Morita equivalent. Moreover, consider the pullback groupoid of $\delta(Q_1) \rightrightarrows Y_1$ by $Q_1 \xrightarrow{\tau_1} Y_1$, which we will denote by $\widetilde{Q}_1 \rightrightarrows Q_1$; as well as the pullback groupoid of $Y_1^{[2]} \times S^1 \rightrightarrows Y_1$ by $Q_1 \xrightarrow{\tau_1} Y_1$, which we will denote by $\widetilde{Y}_1 \rightrightarrows Q_1$.

Elements of \widetilde{Q}_1 are triples $(q, [\widetilde{q}, \widetilde{q}'], q')$, with $\tau_1(\widetilde{q}) = \tau_1(q)$, $\tau_1(\widetilde{q}') = \tau_1(q')$, and the class $[\widetilde{q}, \widetilde{q}']$ taken with respect to the S^1 -action $(\widetilde{q}, \widetilde{q}') \cdot \lambda = (\widetilde{q} \cdot \lambda, \widetilde{q}' \cdot \lambda)$. Thus given such a triple, the quantity $\psi(q, \widetilde{q}) \cdot \psi(q', \widetilde{q}')^{-1} \in S^1$ is well-defined. Thus the map

$$(q, [\widetilde{q}, \widetilde{q}'], q') \mapsto (q, (\tau_1(q), \tau_1(q')), \psi(q, \widetilde{q}) \cdot \psi(q', \widetilde{q}')^{-1}), q')$$

defines an isomorphism between the Lie groupoids $\widetilde{Q}_1 \rightrightarrows Q_1$ and $\widetilde{Y}_1 \rightrightarrows Q_1$. This shows that the Lie groupoids $\delta(Q_1) \rightrightarrows Y_1$ and $Y_1^{[2]} \times S^1 \rightrightarrows Y_1$ are Morita equivalent.

Therefore, $(p_1^{[2]})^* P_1 \otimes \delta(Q_1) \rightrightarrows Y_{1\pi_1} \times_{\pi_2} Y_2$ is Morita equivalent to $P_1 \rightrightarrows Y_1$, and $\delta(Q_2) \otimes (p_2^{[2]})^* P_2 \rightrightarrows Y_{1\pi_1} \times_{\pi_2} Y_2$ is Morita equivalent to $P_2 \rightrightarrows Y_2$. Combining these facts with the isomorphism given by the stable isomorphism between (P_1, Y_1) and (P_1, Y_2) shows the Lie groupoids $P_1 \rightrightarrows Y_1$ and $P_2 \rightrightarrows Y_2$ are Morita equivalent. Showing that this Morita equivalence has the desired properties is tedious but straightforward. Then the S^1 -central extensions $P_1 \rightarrow Y_1^{[2]} \rightrightarrows Y_1$ and $P_2 \rightarrow Y_2^{[2]} \rightrightarrows Y_2$ are isomorphic.

Thus two bundle gerbes are stably isomorphic if and only if the associated S^1 -central extensions of Lie groupoids are isomorphic. Because stable isomorphisms are classified by $H^3(M, \mathbb{Z})$, we obtain the following:

Theorem 2.3.3. *There is a bijection between $H^3(M, \mathbb{Z})$ and the isomorphism classes of S^1 -central extensions of Lie groupoids $Y^{[2]} \rightrightarrows Y$, for a surjective submersion $Y \xrightarrow{\pi} M$.*

Remark 2.3.4. This theorem can be shown directly using an exact sequence of cohomology groups as in [BX11].

2.4 Principal 2-group bundles

Here we will consider principal 2-group bundle, focusing in particular on the cases where the 2-group is BS^1 or $\Gamma_{U(H) \rtimes PU(H)} = (U(H) \rtimes PU(H) \rightrightarrows PU(H))$. There exists several definitions of Lie 2-group bundles (compare, e.g., [GS15],[BS07]), but we follow the point of view of [Wal18].

For a Lie groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ with a surjective submersion $\pi : G_0 \rightarrow M$, the Lie groupoid $\mathcal{G} \times_\pi \mathcal{G}$ is defined to be the full sub-Lie groupoid of $\mathcal{G} \times \mathcal{G}$ over $G_0 \times_\pi G_0$. This means that the space of morphisms of the Lie groupoid $\mathcal{G} \times_\pi \mathcal{G}$ is the space $\{g, g' \in G_1 \mid \pi \circ s(g) = \pi \circ s(g') \text{ and } \pi \circ t(g) = \pi \circ t(g')\}$.

Definition 2.4.1. Let $\mathcal{G} = G_1 \rightrightarrows G_0$ be a Lie groupoid, and $\Gamma = \Gamma_1 \rightrightarrows \Gamma_0$ be a Lie 2-group. We say that \mathcal{G} is a *principal Γ -2-bundle over M* if there is a Lie groupoid morphism

$$\begin{array}{ccc} G_1 & \longrightarrow & M \\ \Downarrow & & \Downarrow \\ G_0 & \xrightarrow{\pi} & M \end{array}$$

with π being a surjective submersion, together with an action $R : \mathcal{G} \times \Gamma \rightarrow \mathcal{G}$ such that the functor

$$\mathcal{G} \times \Gamma \xrightarrow{(id \times R)} \mathcal{G} \times_\pi \mathcal{G}$$

is a Morita morphism.

The general definition of isomorphisms between these objects is as follows

Definition 2.4.2. Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ and $\mathcal{G}' = (G'_1 \rightrightarrows G'_0)$ be two principal Γ 2-bundles over M . Note the projections $\pi : G_0 \rightarrow M$ and $\pi' : G'_0 \rightarrow M$. An *isomorphism between \mathcal{G} and \mathcal{G}'* is a Morita equivalence $G_1 \xleftarrow{\tau} Z \xrightarrow{\tau'} G'_1$, with $\pi \circ \tau = \pi' \circ \tau'$, equipped with a right action of the Lie group Γ_1 , $Z \times \Gamma_1 \rightarrow Z$ such that

$$R(\tau_1(z), t(\gamma)) = \tau_1(R(z, \gamma)) \text{ and } R(\tau_2(z), s(\gamma)) = \tau_2(R(z, \gamma))$$

for all $z \in Z$ and $\gamma \in \Gamma_1$, and compatible with the groupoid multiplications in the sense that

$$R(g_1 \cdot z \cdot g_2, \gamma_1 \circ \gamma_Z \circ \gamma_2) = R(g_1, \gamma_1) \cdot R(z, \gamma_Z) \cdot R(g_2, \gamma_2)$$

for all appropriate $z \in Z$, $g_1 \in G_1$, $g_2 \in G_2$ and $\gamma_1, \gamma_Z, \gamma_2 \in \Gamma_1$.

We now state the main classification result about these objects, beginning with a definition of cocycles with values in a 2-group.

Definition 2.4.3. Let Γ be a Lie 2-group, given by a crossed module of Lie groups $H \rtimes G \rightrightarrows G$. Denote the structure morphisms $r : H \rightarrow G$ and $\alpha : G \rightarrow \text{Aut}(H)$. Let furthermore $\mathcal{U} = \{U_i\}_{i \in I}$ be a good open cover of the manifold M . A Γ -cocycle on M is a collection of smooth maps $g_{ij} : U_{ij} \rightarrow G$ and $a_{ijk} : U_{ijk} \rightarrow H$ such that

- $g_{ik} = r(a_{ijk})g_{jk}g_{ij}$
- $a_{ikl}\alpha(g_{kl})(a_{ijk}) = a_{ijl}a_{jkl}$.

Two such cocycles (a_{ijk}, g_{ij}) , (a'_{ijk}, g'_{ij}) are equivalent if there exist smooth maps $h_i : U_i \rightarrow G$ and $e_{ij} : U_{ij} \rightarrow H$ such that

- $g'_{ij}h_i = r(e_{ij})h_i g_{ij}$
- $a'_{ijk}\alpha(g_{jk})(e_{ij})e_{jk} = e_{ik}\alpha(h_k)(a_{ijk})$.

We will denote the set of equivalence classes of such cocycles by $H^1(M, \Gamma)$.

Remark 2.4.4. Such cocycles are in fact morphisms of simplicial manifolds between the Čech simplex associated to a good open cover of the manifold M , and the Lie 2-group Γ , both considered as simplicial manifolds.

These cocycles classify the principal 2-group bundles.

Theorem 2.4.5 ([Wal18]). *Let Γ be a Lie 2-group. There is a bijection between the isomorphism classes of principal Γ -2-bundles over M , and $H^1(M, \Gamma)$.*

An element of $H^1(M, BS^1)$ is then a collection of maps $\lambda_{ijk} : U_{ijk} \rightarrow S^1$ such that

$$\lambda_{ijk}\lambda_{ikl} = \lambda_{jkl}\lambda_{ijl}$$

Thus we have $H^1(M, BS^1) = H^2(M, S^1) = H^3(M, \mathbb{Z})$, and $H^3(M, \mathbb{Z})$ classifies the principal BS^1 -2-bundles over M . We will however link these principal BS^1 -2-bundles with the objects considered in the previous sections more precisely. Consider for this a principal BS^1 -2-bundle over M . It is a Lie groupoid $P \rightrightarrows Y$ with a functor

$$\begin{array}{ccc} P & \longrightarrow & M \\ \Downarrow & & \Downarrow \\ Y & \xrightarrow{\pi} & M \end{array}$$

where $\pi : Y \rightarrow M$ is a surjective submersion, and $P \times S^1 \rightarrow P$ an action preserving the source and target maps of the Lie groupoid $P \rightrightarrows Y$. Moreover, the fact that $P \times S^1 \xrightarrow{id, R} P \times_{\pi} P$ is a Morita morphism translates to the diagram

$$\begin{array}{ccc} P \times S^1 & \longrightarrow & P \times_{\pi} P \\ \downarrow & & \downarrow \\ Y \times Y & \longrightarrow & Y^{[2]} \times Y^{[2]} \end{array}$$

being a pullback diagram, where $P \times_{\pi} P = \{(p, p') \in P \times P \mid (s(p), s(p')) \in Y^{[2]} \text{ and } (t(p), t(p')) \in Y^{[2]}\}$. This means that for all $(p, p') \in P \times_{\pi} P$, there exists a unique $\lambda \in S^1$ such that $p' = R(p, \lambda)$. Therefore $P \xrightarrow{(s, t)} Y^{[2]}$ is a principal S^1 -bundle, and, denoting $\varepsilon : Y \rightarrow P$ the unit map, we have an S^1 -central extension

$$1 \rightarrow Y \times S^1 \xrightarrow{R \circ (\varepsilon, id)} P \xrightarrow{(s, t)} Y^{[2]} \rightarrow 1$$

It is also straightforward to verify that a S^1 -central extension gives a BS^1 -principal 2-bundle. Therefore there is a one-to-one correspondence between bundle gerbes, S^1 -central extensions of groupoids of the form $P \rightarrow Y^{[2]} \rightrightarrows Y$ for a surjective submersion $Y \xrightarrow{\pi} M$, and BS^1 principal 2-bundles. Moreover, an isomorphism between two principal BS^1 -2-bundles, being a Morita equivalence satisfying extra conditions, corresponds exactly to an isomorphism between the associated S^1 -central extensions of groupoids.

Now consider a bundle gerbe (\tilde{P}, \tilde{Y}) over M . We have seen that it is stably isomorphic to a lifting bundle gerbe of some principal $PU(H)$ -bundle $Y \xrightarrow{\pi} M$. Note $P \rightrightarrows Y$ the associated groupoid. Now recall that the S^1 -bundle $P \rightarrow Y^{[2]}$ is the pullback of $U(H) \rightarrow PU(H)$ via the map $\psi : Y^{[2]} \rightarrow PU(H)$. Consider the Lie groupoid $Y \times U(H) \rightrightarrows Y$ with

$$s(y, u) = y \cdot q(u) \text{ and } t(y, u) = y,$$

multiplication given by $(y, u) \circ (y \cdot q(u), u') = (y, uu')$ and unit $\varepsilon(y) = (y, 1)$. Then this Lie groupoid is isomorphic to $P \rightrightarrows Y$ via the map $f : Y \times U(H) \rightarrow P$,

$$f(y, u) = (y, y \cdot q(u), u).$$

Moreover we have a diagram

$$\begin{array}{ccc}
Y \times U(H) & \xrightarrow{\pi \circ pr_Y} & M \\
\Downarrow & & \Downarrow \\
Y & \xrightarrow{\pi} & M
\end{array}$$

And there is a right action

$$\begin{array}{ccc}
(Y \times U(H)) \times (U(H) \rtimes PU(H)) & \xrightarrow{R} & Y \times U(H) \\
\Downarrow & & \Downarrow \\
Y \times PU(H) & \longrightarrow & Y
\end{array}$$

given by

$$R((y, \tilde{u}), (u, g)) = (y \cdot g, c_{g^{-1}}(\tilde{u})u),$$

where $c_g(u) = gu g^{-1}$ for $g \in PU(H)$ and $u \in U(H)$. It is straightforward to verify that this indeed defines a 2-group action of $\Gamma_{U(H) \rtimes PU(H)}$ on the Lie groupoid $Y \times U(H) \rightrightarrows Y$ and that the functor

$$(Y \times U(H)) \times (U(H) \rtimes PU(H)) \xrightarrow{(id \times R)} (Y \times U(H)) \times_{\pi} (Y \times U(H))$$

is a Morita morphism. Therefore the Lie groupoid $Y \times U(H) \rightrightarrows Y$ has the structure of a $\Gamma_{U(H) \rtimes PU(H)}$ principal 2-bundle over M .

We will in fact show that principal $\Gamma_{U(H) \rtimes PU(H)}$ -2-bundles are classified by $H^3(M, \mathbb{Z})$. Recall that an element of $H^1(M, \Gamma_{U(H) \rtimes PU(H)})$ is a collection of maps $g_{ij} : U_{ij} \rightarrow PU(H)$ and $a_{ijk} : U_{ijk} \rightarrow U(H)$ such that

- $g_{ik} = q(a_{ijk})g_{jk}g_{ij}$
- $a_{ikl}c_{g_{kl}}(a_{ijk}) = a_{ijl}a_{jkl}$

with $q : U(H) \rightarrow PU(H)$ being the projection. Equivalence between two such cocycles (a_{ijk}, g_{ij}) and (a'_{ijk}, g'_{ij}) is given by a collection of maps $h_i : U_i \rightarrow PU(H)$ and $e_{ij} : U_{ij} \rightarrow U(H)$ such that

- $g'_{ij}h_i = q(e_{ij})h_i g_{ij}$
- $a'_{ijk}c_{g_{jk}}(e_{ij})e_{jk} = e_{ik}c_{h_k}(a_{ijk})$.

Theorem 2.4.6. *There is a bijection*

$$H^1(M, \Gamma_{U(H) \rtimes PU(H)}) \cong H^3(M, \mathbb{Z}).$$

Proof. We will show that $\Gamma_{U(H) \times PU(H)}$ -cocycles correspond bijectively to S^1 -cocycles of degree 2. Let $[(a_{ijk}, g_{ij})]$ be an element of $H^1(M, \Gamma_{U(H) \times PU(H)})$. Then choose lifts $\widetilde{g}_{ij} : U_{ij} \rightarrow U(H)$ of the maps g_{ij} . Then

$$h_{ijk} = \widetilde{g}_{ik}^{-1} a_{ijk} \widetilde{g}_{jk} \widetilde{g}_{ij}$$

takes values in S^1 because $q(h_{ijk}) = 1$ and thus $[h_{ijk}] \in H^2(M, S^1) = H^3(M, \mathbb{Z})$.

If $\{(a_{ijk}, g_{ij})\}, \{(a'_{ijk}, g'_{ij})\}$ are two equivalent cocycles with equivalence given by maps $h_i : U_i \rightarrow PU(H)$ and $e_{ij} : U_{ij} \rightarrow U(H)$ then

$$b_{ij} = \widetilde{h}_i^{-1} \widetilde{g'_{ij}}^{-1} e_{ij} \widetilde{h}_j \widetilde{g}_{ij}$$

give an equivalence between h_{ijk} and h'_{ijk} . Here $\widetilde{h}_i : U_i \rightarrow U(H)$ are lifts of the maps $h_i : U_i \rightarrow PU(H)$. So we have a well defined map $H^1(M, \Gamma_{U(H) \times PU(H)}) \rightarrow H^3(M, \mathbb{Z})$.

Now consider $[\lambda_{ijk}] \in H^3(M, \mathbb{Z})$. Then we obtain maps $g_{ij} : U_{ij} \rightarrow PU(H)$ such that $[g_{ij}] \in H^1(M, PU(H))$ and lifts $\widetilde{g}_{ij} : U_{ij} \rightarrow U(H)$ that verify

$$\lambda_{ijk} = \widetilde{g}_{ik}^{-1} \widetilde{g}_{jk} \widetilde{g}_{ij}$$

Then $[(1, g_{ij})]$ defines an element of $H^1(M, \Gamma_{U(H) \times PU(H)})$. For $\{\lambda_{ijk}\}$ and $\{\lambda'_{ijk}\}$ two equivalent cocycles in $H^3(M, \mathbb{Z})$, we obtain equivalent cocycles $\{g_{ij}\}$ and $\{g'_{ij}\}$ with values in $PU(H)$, and equivalence means that there are maps $\tau_{ij} : U_{ij} \rightarrow S^1$ and $k_i : U_i \rightarrow PU(H)$ such that

$$\lambda'_{ijk} \tau_{jk} \tau_{ij} = \tau_{ik} \lambda_{ijk}$$

and

$$g'_{ij} = k_i^{-1} g_{ij} k_j$$

Then $\{(k_i, 1)\}$ gives an equivalence between $\{(1, g_{ij})\}$ and $\{(1, g'_{ij})\}$ in $H^1(M, \Gamma_{U(H) \times PU(H)})$. So we also have a well defined map $H^3(M, \mathbb{Z}) \rightarrow H^1(M, \Gamma_{U(H) \times PU(H)})$.

Finally we have to verify that these two maps are mutually inverse. Starting with $[(a_{ijk}, g_{ij})] \in H^1(M, \Gamma_{U(H) \times PU(H)})$ we obtain $[\lambda_{ijk}] \in H^3(M, \mathbb{Z})$ and therefore $[g'_{ij}] \in H^1(M, PU(H))$ such that

$$\begin{aligned}\lambda_{ijk} &= \widetilde{g'_{ik}}^{-1} \widetilde{g'_{jk}} \widetilde{g'_{ij}} \\ &= \widetilde{g_{ik}}^{-1} a_{ijk} \widetilde{g_{jk}} \widetilde{g_{ij}}\end{aligned}$$

Then $\{(1, \widetilde{g'_{ij}} \widetilde{g'_{ij}}^{-1})\}$ gives an equivalence in $H^1(M, \Gamma_{U(H) \times PU(H)})$ between $\{(a_{ijk}, g_{ij})\}$ and $\{(1, g'_{ij})\}$.

Starting with $[\lambda_{ijk}] \in H^3(M, \mathbb{Z})$, we have $[g_{ij}] \in H^1(M, PU(H))$ such that

$$\lambda_{ijk} = \widetilde{g'_{ik}}^{-1} \widetilde{g'_{jk}} \widetilde{g'_{ij}}$$

Then we obtain the cocycle $[(1, g_{ij})] \in H^1(M, \Gamma_{U(H) \times PU(H)})$, and going back to $H^3(M, \mathbb{Z})$, we obtain again $[\lambda_{ijk}]$. This finishes the proof. \square

Remark 2.4.7. The 2-groups BS^1 and $\Gamma_{U(H) \times PU(H)}$ are in fact equivalent Lie 2-groups. It is easy to see that they are Morita equivalent since the isotropy groups of the Lie groupoid $U(H) \times PU(H) \rightrightarrows PU(H)$ are isomorphic to S^1 . The Morita equivalence is given by the following diagram

$$\begin{array}{ccc} S^1 & U(H) & U(H) \times PU(H) \\ \Downarrow & \swarrow & \searrow q \\ * & & PU(H) \end{array}$$

with the obvious action of S^1 on $U(H)$ and the other action given by

$$(\tilde{u}, (u, q(\tilde{u}))) \mapsto \tilde{u}u.$$

It is straightforward to verify that this gives an equivalence between $\Gamma_{U(H) \times PU(H)}$ and BS^1 . Thus, using the fact that Γ -cocycles for a Lie 2-group Γ are just certain morphisms, we obtain that every $\Gamma_{U(H) \times PU(H)}$ -cocycle is equivalent to a BS^1 -cocycle, and thus the theorem. We also mention the result in [NW13], that gives an equivalence between the 2-categories formed by principal 2-bundles given by equivalent Lie 2-groups.

Chapter 3

A prequantisation map for 2-plectic manifolds

This chapter is a quasi identical adaptation of an article written in collaboration with Tilmann Wurzbacher and titled : "A prequantisation map for 2-plectic manifolds". Only one appendix is missing from the original paper, to avoid the duplication of the appendices of this thesis.

Introduction

Geometric quantisation of symplectic manifolds gives an excellent guideline and many rigorous results on how to go from classical to quantum mechanics. Its procedure for a manifold equipped with an integral symplectic form ω can be divided in four steps. First a geometric set-up is established from the data: a principal S^1 -bundle $q : P \rightarrow M$ with connection 1-form A whose curvature $F^A = dA$ equals $(-i2\pi)\omega$. Then a Lie algebra isomorphism, the "prequantisation map" from $C^\infty(M)$, the observables of (M, ω) , to the vector fields on P that preserve A , i.e., the symmetries of (P, A) , is constructed:

$$(K) \quad f \mapsto \widetilde{X}_f + (q^* f) \cdot \mathfrak{z},$$

where, for $f \in C^\infty(M)$, X_f denotes the Hamiltonian vector field fulfilling $df = \iota_{X_f}\omega$, \widetilde{X}_f its horizontal lift to P and \mathfrak{z} the fundamental vector field of the S^1 -action on P , defined by $\mathfrak{z}_p = \left. \frac{d}{d\tau} \right|_{\tau=0} p \cdot \exp(i2\pi \tau)$. (See the fundamental articles of Kostant and Souriau: [Kos70] resp. [Sou66].)

The last step of geometric prequantisation is the linear representation of the above symmetries on the vector space of S^1 -equivariant \mathbb{C} -valued functions on P , whereas the fourth step concerns the passage to quantisation

and involves more deliberate and delicate choices, such as polarisations and metaplectic structures (compare, e.g., [Woo92]). Though completely general uniqueness statements for the outcome of geometric quantisation are lacking (due essentially to the above mentioned choices in the fourth step), crucial classes of symplectic manifolds, as cotangent bundles, Kähler manifolds and coadjoint orbits of several types of Lie groups are “quantised” without ambiguities. Moreover, geometric quantisation is widely accepted in mathematical and theoretical physics as a very useful source of intuition when it comes to quantisation of point particles.

In contrast, modelling and, especially, quantising classical field theories beyond point particles, i.e., with sources of dimension bigger than one, is a highly active research area. Progress here has most certainly a strong impact on mathematics and physics. The last years have seen a new wave of interest in formulating classical field theories in terms of multisymplectic geometry, i.e., via manifolds provided with non-generate closed $(n+1)$ -forms for $n \geq 1$ replacing the symplectic (a.k.a. 1-plectic) manifolds used in mechanics. (Such couples (M, ω) are also called “ n -plectic manifolds”; compare [RW19] for a recent discussion of their properties.) This approach to classical field theory was strongly advocated and developed by the Polish school around Kijowski and Tulczyjew in the seventies of the last century (see, e.g., [Kij73]), and later by the Spanish school (see, e.g., [RR09]), but can be traced back at least to work of Volterra in the 19th century. One main reason for the revival of multisymplectic geometry is certainly the increased acceptance of higher geometric and algebraic structures, as the Lie ∞ -algebra of observables on a multisymplectic manifold, formulated in its modern form by Stasheff and co-workers, and Baez and Rogers (see [Zam12] for a very general construction of observable algebras).

In this chapter, we develop the geometric part of prequantisation for 2-plectic manifolds.

Our first important result is the existence of a global geometrisation of the datum of an integral pre-2-plectic form ω on a manifold M . More precisely we prove:

Theorem (Global two-plectic geometric prequantisation). *Let ω be an integral, closed three-form on a finite-dimensional manifold M . Then there exists a smooth principal $PU(H)$ -bundle $\Pi : Y \rightarrow M$, such that the associated lifting bundle gerbe (P, Y) has a connective structure (A, θ) whose 3-curvature equals $(-i2\pi)\omega$, and a fortiori the bundle Π has Dixmier-Douady*

class $[\omega]$.

Since the observables of a 2-plectic manifolds (M, ω) do not form a Lie algebra but a Lie 2-algebra $L_\bullet(M, \omega)$, given by the two-term complex $C^\infty(M) \xrightarrow{d} \Omega_{Ham}^1(M, \omega)$ together with brackets l_2 and l_3 , the natural analogon of (K) is a Lie 2-algebra morphism whose target are “vector fields on the total space P of the gerbe (or equivalently on the associated Lie groupoid $P \rightrightarrows Y$)”. We construct a natural Lie 2-algebra $WSym(P, Y)$ of weak symmetries out of the multiplicative vector fields on P that preserve the chosen connective structure up to a one-form on Y , and $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \cong \Gamma(Y, Lie(P))$, the sections of the Lie algebroid of P . Then we show that there is a natural, explicit Lie 2-algebra morphism from the observables to the symmetries, in close analogy to (K):

Theorem (Two-plectic prequantisation map). *Let (M, ω) be a 2-plectic manifold with ω an integral form. Then there is a natural Lie 2-algebra quasi-isomorphism $\Phi = (\Phi_1, \Phi_2) : L_\bullet(M, \omega) \rightarrow WSym(P, Y)$:*

$$\begin{array}{ccc} C^\infty(M) & \longrightarrow & \Omega_{Ham}^1(M, \omega) \\ \Phi_1 \downarrow & \swarrow \Phi_2 & \downarrow \Phi_1 \\ \mathfrak{X}_V(Y) \oplus C^\infty(Y) & \longrightarrow & \mathcal{V}_{(P, Y)} \quad . \end{array}$$

Here $\Omega_{Ham}^1(M, \omega)$ denotes the space of one-forms α defining a Hamiltonian vector field X_α via $d\alpha = \iota_{X_\alpha}\omega$, $L_\bullet(M, \omega)$ the Lie 2-algebra of observables of (M, ω) , $\mathfrak{X}_V(Y)$ the vertical vector fields on the total space of the bundle $\Pi : Y \rightarrow M$ of the preceding theorem, and $\mathcal{V}_{(P, Y)}$ the degree-0 term of $WSym(P, Y)$.

Let us briefly describe the organisation of this chapter. In Section 3.1, we collect some “folklore” about principal bundles, lifting bundle gerbes and central S^1 -extensions of Lie groupoids, together with the appropriate type of connections and connective structures. In case we could not point to a proof in the literature, we furnished it for completeness. The main contribution is here the above theorem on global two-plectic geometric prequantisation. Section 3.2 relies on the dictionary from lifting bundle gerbes (P, Y) to a certain type of Lie groupoids $P \rightrightarrows Y$, namely central S^1 -extensions of $Y^{[2]} \rightrightarrows Y$, where $Y^{[2]} = \{(y_1, y_2) \in Y^2 \mid \Pi(y_1) = \Pi(y_2)\}$ and $\Pi : Y \rightarrow M$ is the $PU(H)$ -bundle constructed in Section 3.1. We analyse the multiplicative vector fields on $P \rightrightarrows Y$ in thorough detail, explaining notably their S^1 -invariance and their relation to objects on the base Y . Then we explain the Lie 2-algebra structure

on $\text{WSym}(P, Y)$, the above mentioned weak symmetries of $P \rightrightarrows Y$, together with a fixed multiplicative connection A and a curving θ . Finally, we unravel here the crucial relation for multiplicative vector fields between weak preservation of the connective structure on P and the existence of a Hamiltonian generator on M . In Section 3.3, we give the analogon of (K), namely the Lie 2-algebra quasi-isomorphism of the theorem on the two-plectic prequantisation map. To achieve this goal, we first observe that there is a natural Lie 2-algebra morphism from $L_{\bullet}(M, \omega)$ to the Lie 2-algebra formed by the sections of $\text{Lie}(P)$, identified with vertical vector fields plus functions on Y , and the space of all multiplicative vector fields on P , and then refine it to the morphism Φ of the theorem.

Finally, in Section 3.4 we expose our constructions and results in detail in the case of exact 2-plectic manifolds, i.e., for manifolds M with a non-degenerate 3-form ω possessing a potential $\chi \in \Omega^2(M)$ such that $d\chi = \omega$. (The exact case is important for the application of our constructions to many of the classical field theories considered in physics.)

The results and proofs of this chapter are independent and different from the research of Krepski and Vaughan in [KV], and of Fiorenza, Rogers and Schreiber (see, e.g., [FRS14]). The main difference stems from our objective to always give a global differential-geometric picture (including a $PU(H)$ -bundle geometrising the cohomology class of the 2-plectic form) and explicit formulae, notably for the arising Lie 2-algebra morphisms. For the reader interested in comparing our work to these approaches to prequantisation of 2-plectic manifolds, we insert precise remarks at several places in the body of the chapter.

Another approach to 2-plectic prequantization is given by Rogers in [Rog13] (compare also [Rog11]), where the Lie 2-algebra of observables of a 2-plectic manifold (M, ω) is embedded into a Lie 2-algebra associated to the sections of a Courant algebroid, that in turn is associated to (M, ω) . This procedure parallels the interpretation of the symplectic prequantization map (K) above as realising the symplectic observables, $C^{\infty}(M)$, inside the sections of the Atiyah algebroid associated to an integral symplectic manifold. This procedure differs from ours since we stress the construction of a global geometric prequantization, and the realisation of the observables as “vector fields on this global geometric object”, refined to weak infinitesimal symmetries. Nevertheless, combining Rogers’ above ideas with unpublished work of Collier ([Col]) should allow for a construction analogous to our two-plectic prequantization map, but in Brylinski’s more algebraic (Čech) framework.

Finally, a natural idea is to use transgression of differential forms to pass from a finite-dimensional integral 2-plectic manifold (M, ω) to the induced integral two-form $\hat{\omega}$ on LM , the loop space of M , and to try quantising the latter (infinite-dimensional) (pre-)symplectic manifold. Prominent examples are here the standard Cartan 3-form on a compact semi-simple Lie group, leading to a derivation of many properties of Wess-Zumino-Witten models (compare, e.g., [Koh02]), and volume forms on compact three-manifolds, leading to interesting geometry on knot spaces (cf. [Bry08]). More recently, Sämann, Szabo and collaborators combined the idea of transgression with symplectic groupoids and higher algebraic structure on the observables of a 2-plectic manifold to analyse properties of a tentative quantisation of integral 2-plectic manifolds (see, e.g., [SS13]). We do not use transgression to loop spaces here since we are aiming to stay in the 2-plectic framework.

(Pre-)quantization of manifolds with integral three-forms is also surveyed in a recent preprint of Bunk ([Bun]).

3.1 Global geometric prequantisation of integral pre-2-plectic manifolds

In this section we construct a global geometric prequantisation of a pre-2-plectic manifold. Our approach naturally relies on principal $PU(H)$ -bundles, bundle gerbes and central extensions of groupoids. As a preparation, we show several technical results needed to ensure the existence of connections on certain principal bundles in infinite dimensions. With the exception of Theorem 3.1.15 all results are explicitly known or folkloristic. We give proofs only in case we could not find a reference covering the exact statement.

Definition 3.1.1. Let M be a manifold and $\omega \in \Omega^3(M)$. We say that (M, ω) is a *pre-2-plectic manifold* if ω is closed.

It is well-known that $H^3(M, \mathbb{Z})$ for a finite-dimensional manifold M , or for a suitable infinite-dimensional manifold, parametrizes the equivalence classes of smooth principal $PU(H)$ -bundles over M (here H is a separable complex Hilbert space and $PU(H)$ the Banach Lie group of projective unitary isomorphisms of H), cf., e.g., Section 18.4 in [HJJS08]. Thus, given an integral, closed 3-form ω on M , we have via the canonical surjection $H^3(M, \mathbb{Z}) \rightarrow H_{dR}^3(M, \mathbb{Z})$ a class $c \in H^3(M, \mathbb{Z})$ and an associated principal

$PU(H)$ -bundle $Y \rightarrow M$ with characteristic class equal to c . The class c is called the *Dixmier-Douady class* of $Y \rightarrow M$, and projects to $[\omega]$. Non-uniqueness arises from torsion in $H^3(M, \mathbb{Z})$, exactly as in the case of geometric prequantisation of symplectic manifolds. Our first technical goal is to assure us of the existence of principal connections on these bundles. We treat a slightly more general situation than strictly needed here.

Definition 3.1.2. A possibly infinite-dimensional manifold M is called *smoothly paracompact* if for every open covering $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ of M there exists a smooth partition of unity $\{\chi_\alpha\}_{\alpha \in \Lambda}$ subordinate to \mathcal{U} .

Remark 3.1.3. The preceding notion is studied in detail in Section III.16 of [KM97].

Proposition 3.1.4. *Let G be a Banach Lie group with Lie algebra \mathfrak{g} , M a smoothly paracompact manifold and $\pi : Q \rightarrow M$ a principal G -bundle. Then there exists a smooth principal connection on Q .*

Proof. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering of M such that there are principal G -bundle isomorphisms $\psi_\alpha : Q|_{U_\alpha} = \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times G$ over id_{U_α} , and let A_α be the pullback via ψ_α of the connection 1-form of the canonical flat connection on $U_\alpha \times G$. A direct verification shows (as in the finite-dimensional case) that the \mathfrak{g} -valued 1-form $A = \sum_\alpha (\chi_\alpha \circ \pi) \cdot A_\alpha$ is a connection 1-form on the G -bundle π . \square

Corollary 3.1.5. *Let M be a finite-dimensional manifold and ω a closed, integral 3-form on M . Then there exists a smooth principal $PU(H)$ -bundle (with H a separable complex Hilbert space), such that its Dixmier-Douady class equals (up to torsion) the class of ω in $H^3(M, \mathbb{Z})$, and all such bundles possess smooth principal connections.*

Before going to the gerbe interpretation of 3-forms, we need some more information concerning the existence of connections. We start with the following:

Lemma 3.1.6. *Let H be a separable complex Hilbert space, $U(H)$ the unitary group of H with the norm topology and $PU(H) = U(H)/S^1$ the projective unitary group of H with the ensuing quotient topology. Then:*

- (i) $S^1 \rightarrow U(H) \rightarrow PU(H)$ is a short exact sequence of smooth Banach Lie groups,
- (ii) the preceding sequence induces a smooth principal S^1 -bundle $U(H) \rightarrow PU(H)$,

(iii) the principal S^1 -bundle $U(H) \rightarrow PU(H)$ possesses a smooth principal connection.

Proof. (i) is well-known.

Let us recall that the Lie algebra of $U(H)$ is given as $\mathfrak{u}(H) = T_{\text{id}_H} U(H) = \{\xi \in \mathcal{B}(H) \mid \xi^* = -\xi\}$, where $\mathcal{B}(H)$ is the space of bounded linear endomorphisms of H . We denote the Lie algebra of $S^1 = \{\exp(it) \cdot \text{id}_H \mid t \in \mathbb{R}\}$ by $\mathfrak{z} = \{it \cdot \text{id}_H \mid t \in \mathbb{R}\}$ and of $PU(H)$ by $\mathfrak{pu}(H) = \mathfrak{u}(H)/\mathfrak{z}$.

(ii) Observing that $\mathfrak{z} \subset \mathfrak{u}(H)$ is 1-dimensional, we construct a closed complement of this subspace as follows. First let us fix the convention that the scalar product \langle, \rangle of H is complex-linear in its second entry. Upon choosing an element v of norm one in H , we define a continuous linear functional φ on $\mathcal{B}(H)$ by $\varphi(\xi) = -i \langle v, \xi(v) \rangle$. It is clear that φ takes real values on $\mathfrak{u}(H)$ and that $\varphi(i \cdot \text{id}_H) = 1$. The kernel of φ will be denoted by \mathfrak{m} and it is a closed linear complement of \mathfrak{z} , i.e.,

$$(*) \quad \mathfrak{u}(H) = \mathfrak{z} \oplus \mathfrak{m} \quad \text{as real Banach spaces.}$$

Since the exponential map $\exp^{U(H)} : \mathfrak{u}(H) \rightarrow U(H)$ is given by the exponential series, $\exp^{U(H)}$ is a local diffeomorphism and it follows as in the finite-dimensional case that $U(H) \rightarrow PU(H)$ is a smooth principal S^1 -bundle (notably being locally trivial!).

(iii) Since S^1 is a central subgroup of $U(H)$, the splitting $(*)$ is invariant under the adjoint action of $U(H)$ restricted to S^1 . Thus the canonical $\mathfrak{u}(H)$ -valued 1-form on $U(H)$, followed by the projection onto \mathfrak{z} defined by this splitting, yields a connection 1-form on $U(H) \rightarrow PU(H)$, that is furthermore invariant under the action of $U(H)$ on itself by left multiplication. (Compare, e.g., Theorem 11.1 in Chapter II of [KN96] for a proof in a finite-dimensional situation, that immediately generalizes to our setup.) \square

Remark 3.1.7. In fact, all maps and constructions in the preceding lemma are real-analytic.

Definition 3.1.8. Let $\pi : Y \rightarrow M$ be a surjective submersion of smooth Banach manifolds. Then we set $Y^{[0]} := M$, $Y^{[1]} := Y$ and for a natural number $k \geq 2$, $Y^{[k]} := Y \times_{\pi} Y \times_{\pi} \dots \times_{\pi} Y = \{(y_1, \dots, y_k) \in Y \mid \pi(y_1) = \dots = \pi(y_k)\}$. We call $Y^{[k]}$ the k -fold fiber product of Y with itself (over M).

Remark 3.1.9. All fiber products $Y^{[k]}$ for $k \geq 0$ are smooth since π is assumed to be a surjective submersion.

Lemma 3.1.10. *Let G be a Banach Lie group and $Y \xrightarrow{\pi} M$ a principal G -bundle in the category of smooth Banach manifolds. Then there is a unique, smooth map $\psi : Y^{[2]} \rightarrow G$ such that for all $(y, y') \in Y^{[2]}$, $y' = y \cdot \psi(y, y')$.*

Proof. The pair $(y, y') \in Y^2$ is in $Y^{[2]}$ if and only if $\pi(y) = \pi(y')$, that is if and only if y and y' are in the same orbit of the free and transitive G -action on π -fibers. Thus it exists a unique $g = \psi(y, y') \in G$ such that $y' = y \cdot g = y \cdot \psi(y, y')$, yielding $\psi : Y^{[2]} \rightarrow G$. The map ψ is smooth by the implicit function theorem for differentiable maps between Banach manifolds. \square

We now recall the definition of S^1 -bundle gerbes that goes back to [Mur96].

Definition 3.1.11. Let M be a smooth manifold. A S^1 -bundle gerbe (P, Y) over M consists of:

- a smooth manifold Y with a surjective submersion $Y \xrightarrow{\pi} M$,
- a principal S^1 -bundle P over $Y^{[2]}$,
- a smooth isomorphism $\mu : \pi_3^*P \otimes \pi_1^*P \rightarrow \pi_2^*P$ of principal S^1 -bundles over $Y^{[3]}$ (where $\pi_i : Y^{[3]} \rightarrow Y^{[2]}$ denotes the projection that forgets the i^{th} factor).

Furthermore, the isomorphism μ has to verify an associativity condition, which can be expressed by the commutativity of the following diagram of S^1 -bundles over $Y^{[4]}$:

$$\begin{array}{ccc} \pi_{34}^*P \otimes \pi_{14}^*P \otimes \pi_{12}^*P & \xrightarrow{\mu \otimes id} & \pi_{24}^*P \otimes \pi_{12}^*P \\ \downarrow id \otimes \mu & & \downarrow \mu \\ \pi_{34}^*P \otimes \pi_{13}^*P & \xrightarrow{\mu} & \pi_{23}^*P. \end{array}$$

Here $\pi_{ij} : Y^{[4]} \rightarrow Y^{[2]}$ denotes the projection which forgets the i^{th} and j^{th} factors.

Remark 3.1.12. Since we are only dealing with S^1 -bundle gerbes in this article, we allow ourselves to use the shorter term *bundle gerbes* for them as well.

Suppose we are given a principal G -bundle $Y \rightarrow M$ with a Lie group G that possesses a central S^1 -extension :

$$1 \rightarrow S^1 \rightarrow \widehat{G} \xrightarrow{q} G \rightarrow 1,$$

and consider the map $\psi : Y^{[2]} \rightarrow G$ (as in Lemma 1.10). Then we can pull back the principal S^1 -bundle $\widehat{G} \rightarrow G$ via ψ to obtain a principal S^1 -bundle $P \rightarrow Y^{[2]}$. We may then define a map μ via the multiplication on \widehat{G} , namely:

$$\mu((y, y', \hat{g}), (y', y'', \hat{g}')) = (y, y'', \hat{g} \cdot \hat{g}'),$$

for $y, y', y'' \in Y^{[3]}$ and $\hat{g}, \hat{g}' \in \widehat{G}$. This gives a bundle gerbe (P, Y) over M , called the *lifting bundle gerbe associated to $Y \rightarrow M$* .

There is an equivalent description of bundle gerbes, via extensions of groupoids. Since the below result will be crucial for us and it is only hinted at in the original source [BX11], we give the idea of its proof.

First we need the

Definition 3.1.13. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid. A Lie groupoid $Q_1 \rightrightarrows G_0$ is called an *S^1 -central extension of $G_1 \rightrightarrows G_0$* if there exists an exact sequence of Lie groupoids

$$1 \rightarrow G_0 \times S^1 \xrightarrow{\chi} Q_1 \xrightarrow{\rho} G_1 \rightarrow 1$$

over id_{G_0} such that the image of χ is central in Q_1 .

Of course, source and target map of the groupoid $G_0 \times S^1 \rightrightarrows G_0$ are given by the projection on G_0 and its multiplication is defined by the multiplication of S^1 .

Proposition 3.1.14. *Given a surjective submersion $Y \xrightarrow{\pi} M$, the S^1 -bundle gerbes (P, Y) over M (with the given π part of the data) are in bijection with the central S^1 -extensions of the Lie groupoid $Y^{[2]} \rightrightarrows Y$ (with $Y^{[2]}$ defined by the given π).*

Proof. Fix $\pi : Y \rightarrow M$ during this proof.

Given a bundle gerbe in the above sense, we have notably a principal S^1 -bundle $P \rightarrow Y^{[2]}$. Denoting its projection by ρ , we have the second morphism of a sequence

$$(*) \quad 1 \rightarrow Y \times S^1 \rightarrow P \rightarrow Y^{[2]} \rightarrow 1.$$

Furthermore, if $\Delta : Y \rightarrow Y^{[2]}$ is the diagonal embedding, the isomorphism μ induces a canonical section of the pullback bundle $\Delta^*P \rightarrow Y$ and an “inverse” $P_{(y_1, y_2)} \rightarrow P_{(y_2, y_1)}, p \mapsto p^{-1}$ (cf., e.g., [Mur96] or [Mur10] for these facts). The map ρ now yields a source $s = \pi_2 \circ \rho$ and a target $t = \pi_1 \circ \rho$, as well as a multiplication map $m : P_s \times_t P \rightarrow P$, defined in terms of μ , and the above

canonical section yields a unit map $\varepsilon : Y \rightarrow P, y \mapsto \varepsilon(y) \in P_{(y,y)}$. Obviously ρ is then a surjective Lie groupoid morphism. The kernel of ρ is easily seen to be the image of the injective morphism $\chi : Y \times S^1 \rightarrow P, \chi(y, z) = \varepsilon(y) \cdot z$.

On the other hand, given a central S^1 -extension as described by (*), we define a right-action of S^1 on P by $p \cdot z := m(p, \chi(y, z))$ for $z \in S^1$ and p in P lying over (y', y) in $Y^{[2]}$, and m the multiplication of the groupoid $P \rightrightarrows Y$. Finally, we define μ via the groupoid multiplication $m : P_{(y_1, y_2)} \times P_{(y_2, y_3)} \rightarrow P_{(y_1, y_3)}$. The associativity of the multiplication m implies the associativity condition necessary for μ to be a “bundle gerbe multiplication”. \square

We now briefly recall connective structures on bundle gerbes whose detailed study can be found in the literature (see, e.g., [Mur96], [Ste00]). A *bundle gerbe connection* on a bundle gerbe (P, Y) over M is given by a connection one-form A on the total space of the principal S^1 -bundle $P \rightarrow Y^{[2]}$, such that A is *multiplicative* in the following sense:

$$m^*A = \text{proj}_1^*A + \text{proj}_2^*A,$$

where $m, \text{proj}_1, \text{proj}_2 : P_s \times_t P \rightarrow P$ are respectively the partial multiplication and the projections onto the first and second factor (the maps s and t denote the source and target maps of the groupoid). In fact, given any connection on the principal S^1 -bundle $P \rightarrow Y^{[2]}$, it is always possible to modify it to obtain a multiplicative connection. For this and the sequel up to the theorem, compare Section 4.4 of [Mur10].

The curvature of such a connection descends to a 2-form θ on Y , more precisely, there exists a $\theta \in i\Omega^2(Y)$ fulfilling:

$$dA = (t^* - s^*)\theta.$$

We call such a 2-form θ a *curving*. Finally, since $d(t^* - s^*)\theta = 0$, there exists $\kappa \in i\Omega^3(M)$ such that $d\theta = \Pi^*\kappa$. (We denote from now the map $Y \rightarrow M$ by Π to distinguish it from the number π .) We call κ the *3-curvature* and observe that $(-2\pi i)\kappa$ is integral. Note that the curving θ is not unique, in fact, if $\theta' = \theta + \Pi^*\eta$, where η is a closed, $i\mathbb{R}$ -valued 2-form on M , we still have $(t^* - s^*)\theta' = dA$ and $d\theta' = \Pi^*\kappa$. Finally, $\frac{i}{2\pi}\kappa$ represents the image of the Dixmier-Douady class c of the bundle gerbe (P, Y) over M in the de Rham cohomology of M .

In the sequel, we will often call (A, θ) a *connective structure* on (P, Y) .

Now consider a class $c \in H^3(M, \mathbb{Z})$, and $Y \rightarrow M$ a principal $PU(H)$ -bundle with Dixmier-Douady class equal to c . Then the lifting bundle gerbe associated to the exact sequence $S^1 \rightarrow U(H) \rightarrow PU(H)$ also has Dixmier-Douady class c (see e.g. [Mur10]). This will be useful in the proof of the next theorem, which will allow us to define a global geometric prequantisation of a pre-2-plectic manifold.

Theorem 3.1.15. *Let ω be an integral, closed 3-form on a finite-dimensional manifold M . Then there exists a smooth principal $PU(H)$ -bundle $\Pi : Y \rightarrow M$, such that the associated lifting bundle gerbe (P, Y) has a connective structure (A, θ) whose 3-curvature equals $(-2\pi i)\omega$ and, a fortiori, the bundle Π has Dixmier-Douady class $[\omega]$.*

Proof. We know that there exists a principal $PU(H)$ -bundle $\Pi : Y \rightarrow M$ with Dixmier-Douady class projecting to $[\omega]$. Consider the lifting bundle gerbe (P, Y) associated to this bundle. We know from Lemma 3.1.6 that there exists a principal connection on $U(H) \rightarrow PU(H)$, which we pullback to $P \rightarrow Y^{[2]}$ via ψ . (Recall that the principal bundle $P \rightarrow Y^{[2]}$ can be realized as the pullback of $U(H) \rightarrow PU(H)$ via ψ .) If necessary, we modify this connection and we can assume it to be multiplicative. Let us denote the resulting connection 1-form by A . Then the above considerations imply that there exists $\theta' \in \Omega^2(Y)$ such that $(t^* - s^*)\theta' = dA$ and $\omega' \in \Omega^3(M)$ such that $d\theta' = \Pi^*\omega'$. Since $[\omega - \omega'] = 0$ there exists a 2-form $\kappa \in \Omega^2(M)$ such that $\omega = \omega' + d\kappa$. Consider $\theta = \theta' + \Pi^*\kappa$. Replacing θ' by θ in the connective structure of the lifting bundle gerbe does not change the connection of the principal S^1 -bundle over $Y^{[2]}$, and now we have $d\theta = d\theta' + \Pi^*d\kappa = \Pi^*(\omega' + d\kappa) = \Pi^*(\omega)$. \square

Definition 3.1.16. A *global geometric prequantisation* of an integral pre-2-plectic manifold (M, ω) is given by a principal $PU(H)$ -bundle $\Pi : Y \rightarrow M$, and a bundle gerbe (P, Y) over M with connective structure (A, θ) whose 3-curvature is equal to $(-2\pi i)\omega$.

Remark 3.1.17. In Definition 2.2.4 of [FRS14] a *prequantisation* is essentially a connective structure with 3-curvature equal to $(-2\pi i)\omega$. Our definition asks for a more complete set-up for the connection to arise, notably a principal $PU(H)$ -bundle and a gerbe (or equivalently, a central S^1 -extension of groupoids) on its total space having the “right” 3-curvature. In [KV], the notion of a quantisation used is closer to ours, but the analogon of our $PU(H)$ -bundle projection Π is there only a surjective submersion.

3.2 Infinitesimal symmetries of bundle gerbes

In this section we turn our attention to *multiplicative* vector fields on groupoids, intensely studied by Kirill Mackenzie (see [MX98] or [BD19] for a more complete treatment of multiplicative tensors on Lie groupoids).

Definition 3.2.1. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid. A *multiplicative vector field* (on $G_1 \rightrightarrows G_0$) is given by a couple (ξ, X) in $\mathfrak{X}(G_1) \oplus \mathfrak{X}(G_0)$ such that the following diagram is a Lie groupoid morphism:

$$\begin{array}{ccc} G_1 & \xrightarrow{\xi} & TG_1 \\ \Downarrow & & \Downarrow \\ G_0 & \xrightarrow{X} & TG_0 \end{array}$$

The Lie algebra of multiplicative vector fields is denoted by $\mathfrak{X}_{mult}(G_1)$.

For the rest of the section, we will denote by M a manifold, by (P, Y) a bundle gerbe over M , by A a multiplicative connection on $P \rightarrow Y^{[2]}$, and by θ resp. ω a curving resp. the 3-curvature of (A, θ) . As discussed earlier, we have a natural Lie groupoid structure $P \rightrightarrows Y$, centrally extending $Y^{[2]} \rightrightarrows Y$ by S^1 . We denote by s, t the source resp. target maps of this groupoid, m its partial multiplication and ε its unit map. We denote the principal S^1 -bundle projection $P \rightarrow Y^{[2]}$ again by ρ . Observe that $\rho = (t, s)$.

We will consider multiplicative vector fields on the Lie groupoid $P \rightrightarrows Y$ in more detail now, starting with two important lemmas.

Lemma 3.2.2. *Let (ξ, X) be a multiplicative vector field on the Lie groupoid $P \rightrightarrows Y$. Then ξ is S^1 -invariant.*

Proof. In this proof, we identify the Lie algebra of S^1 with \mathbb{R} . We begin with a computation, for $y \in Y$, $z, z' \in S^1$:

$$\begin{aligned} A_{\varepsilon(y) \cdot zz'}(\xi_{\varepsilon(y) \cdot zz'}) &= A_{m(\varepsilon(y) \cdot z, \varepsilon(y) \cdot z')}(\xi_{m(\varepsilon(y) \cdot z, \varepsilon(y) \cdot z')}) \\ &= A_{m(\varepsilon(y) \cdot z, \varepsilon(y) \cdot z')}(m_*(\xi_{\varepsilon(y) \cdot z}, \xi_{\varepsilon(y) \cdot z'})) \\ &= A_{\varepsilon(y) \cdot z}(\xi_{\varepsilon(y) \cdot z}) + A_{\varepsilon(y) \cdot z'}(\xi_{\varepsilon(y) \cdot z'}). \end{aligned}$$

Thus we obtain, for a fixed $y \in Y$, a smooth map $F : S^1 \rightarrow \mathbb{R}$, $F(z) = A_{\varepsilon(y) \cdot z}(\xi_{\varepsilon(y) \cdot z})$, such that $F(zz') = F(z) + F(z')$. This shows that F must be zero, and therefore that for every $y \in Y$, $z \in S^1$, $\xi_{\varepsilon(y) \cdot z}$ is horizontal. Since the horizontal subbundle $H \subset TP$ is invariant under S^1 , this also shows that there exists a horizontal vector $u(y, z) \in H_{\varepsilon(y)}$ such that $\xi_{\varepsilon(y) \cdot z} = z_*(u(y, z))$

(where z also denotes the right-action of an element $z \in S^1$). For $z = 1$ we obtain $u(y, 1) = \xi_{\varepsilon(y)} = \varepsilon_*(X_y)$. Moreover we have, using the multiplicativity of ξ :

$$\begin{aligned}\rho_*(u(y, z)) &= \rho_*(\xi_{\varepsilon(y) \cdot z}) \\ &= (s, t)_*(\xi_{\varepsilon(y) \cdot z}) \\ &= (X_y, X_y) \\ &= \rho_*(u(y, 1)).\end{aligned}$$

So the vector $u(y, z) - u(y, 1)$ is both horizontal and vertical, and therefore must be zero. Thus we obtain that $\xi_{\varepsilon(y) \cdot z} = z_*(\xi_{\varepsilon(y)})$.

Finally we compute, for $(y, y') \in Y^{[2]}$, $p \in P_{(y, y')}$ and $z \in S^1$ (using again the multiplicativity of ξ):

$$\begin{aligned}\xi_{p \cdot z} &= \xi_{m(p, \varepsilon(y') \cdot z)} \\ &= m_*(\xi_p, \xi_{\varepsilon(y') \cdot z}) \\ &= m_*(\xi_p, z_*\xi_{\varepsilon(y')}) \\ &= z_*(m_*(\xi_p, \xi_{\varepsilon(y')})) \\ &= z_*\xi_p.\end{aligned}$$

This concludes the proof. □

Remark 3.2.3. Observe that, if (ξ, X) is a multiplicative vector field on $P \rightrightarrows Y$, then we have $s_*\xi = X \circ s$, $t_*\xi = X \circ t$, and, since $\Pi \circ s = \Pi \circ t$, we obtain for all $(y, y') \in Y^{[2]}$:

$$\Pi_{*y}X_y = \Pi_{*y'}X_{y'}.$$

This shows that the vector field X is *projectable along* Π , i.e., there exists $\bar{X} \in \mathfrak{X}(M)$ such that $\Pi_*X = \bar{X} \circ \Pi$. Thus, for $(y, y') \in Y^{[2]}$ the map:

$$(y, y') \mapsto (X_y, X_{y'})$$

defines a vector field on $Y^{[2]}$. We will denote this vector field by X_2 .

These considerations allow us to give a general formula for multiplicative vector fields on $P \rightrightarrows Y$.

Lemma 3.2.4. *Let (ξ, X) be a multiplicative vector field on the Lie groupoid $P \rightrightarrows Y$. Then ξ may be written as:*

$$\widetilde{X}_2 + (t^* - s^*)g \cdot \mathfrak{z},$$

where \widetilde{X}_2 is the horizontal lift of the vector field X_2 with respect to the multiplicative connection A , $g \in C^\infty(Y)$ and \mathfrak{z} is the fundamental vector field associated to the S^1 -action on P . The function g is unique up to addition of $\Pi^* f$ for a $f \in C^\infty(M)$.

Proof. Since ξ is S^1 -invariant, it can be written as

$$\widetilde{Z} + \rho^* h \cdot \mathfrak{z},$$

where $Z \in \mathfrak{X}(Y^{[2]})$, \widetilde{Z} is its horizontal lift with respect to A and $h \in C^\infty(Y^{[2]})$. We have for $p \in P_{(y,y')}$

$$\rho_{*p}(\widetilde{Z}_p + (\rho^* h \cdot \mathfrak{z})_p) = Z_{(y,y')}$$

because \widetilde{Z} is ρ -related to Z . On the other hand

$$\rho_{*p}(\widetilde{Z}_p + (\rho^* h \cdot \mathfrak{z})_p) = (t, s)_{*p}(\widetilde{Z}_p) = (X_y, X_{y'})$$

because $(\xi, X) = (\widetilde{Z} + \rho^* h \cdot \mathfrak{z}, X)$ is multiplicative. This shows that $Z = X_2$. Thus (\widetilde{X}_2, X) is multiplicative, and the vector field $(\rho^* h \cdot \mathfrak{z}, 0)$ must be multiplicative as well.

We will now show the following identity for all $y, y', y'' \in Y^{[3]}$:

$$h(y, y') + h(y', y'') = h(y, y'')$$

implying in turn that the function h descends to a function on Y . We have, for all $p \in P_{(y,y')}$, $p' \in P_{(y',y'')}$

$$\begin{aligned} h(y, y') + h(y', y'') &= A_p((\rho^* h \cdot \mathfrak{z})_p) + A_{p'}((\rho^* h \cdot \mathfrak{z})_{p'}) \\ &= (m^* A)_{(p,p')}((\rho^* h \cdot \mathfrak{z})_p, (\rho^* h \cdot \mathfrak{z})_{p'}) \\ &= A_{m(p,p')} (m_{*(p,p')}((\rho^* h \cdot \mathfrak{z})_p, (\rho^* h \cdot \mathfrak{z})_{p'})) \\ &= A_{m(p,p')}((\rho^* h \cdot \mathfrak{z})_{m(p,p')}) \\ &= h(y, y''), \end{aligned}$$

where we used the multiplicativity of $\rho^* h \cdot \mathfrak{z}$ to obtain the fourth equality. Thus the function $h \in C^\infty(Y^{[2]})$ descends to a function $g \in C^\infty(Y)$. Finally, since $(t^* - s^*) \circ \Pi^* = 0$, the function $g \in C^\infty(Y)$ is unique only up to addition of $\Pi^* f$ with f a smooth function on M . \square

In [BEL20] the authors construct a strict Lie 2-algebra structure on multiplicative vector fields. In fact, this construction generalizes to any \mathcal{LA} -groupoid ([OW19]). See the appendices of this thesis for the relevant definitions and notations concerning Lie 2-algebras and crossed modules of Lie algebras, as well as this construction. We will here, however, describe the Lie 2-algebra of multiplicative vector fields more explicitly, in a fashion tailored to our needs.

Consider the extension of groupoids

$$\begin{array}{ccccc} Y \times S^1 & \longrightarrow & P & \longrightarrow & Y^{[2]} \\ \Downarrow & & \Downarrow & & \Downarrow \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y \end{array}$$

We obtain an exact sequence of vector bundles over Y

$$\begin{array}{ccccc} Y \times \mathbb{R} & \longrightarrow & \text{Lie}(P) & \longrightarrow & T_V Y \\ \downarrow & & \downarrow & & \downarrow \\ Y & \xlongequal{\quad} & Y & \xlongequal{\quad} & Y \end{array}$$

where the vector bundles considered here are the Lie algebroids of the respective Lie groupoids. Here we have denoted by $\text{Lie}(P) = (\ker s_*)|_{\varepsilon(Y)} \subset TP|_{\varepsilon(Y)}$ the Lie algebroid of the Lie groupoid $P \rightrightarrows Y$, and $T_V Y := \ker(\Pi_*)$, where Π is the projection from Y to M . The sections of the vector bundle $T_V Y$ form the space of vertical vector fields, denoted by $\mathfrak{X}_V(Y)$. The map $t_* : \text{Lie}(P) \rightarrow T_V Y$ is the anchor of the Lie algebroid $\text{Lie}(P)$.

The multiplicative connection A defines an explicit splitting $\sigma : \mathfrak{X}_V(Y) \rightarrow \Gamma(Y, \text{Lie}(P))$. First let $Z \in \mathfrak{X}_V(Y)$. Then the map:

$$(y, y') \mapsto (Z_y, 0)$$

for $(y, y') \in Y^{[2]}$, defines a vector field on $Y^{[2]}$, since Z is vertical. We denote this vector field by $(Z, 0)$. We also set $(0, Z) : (y, y') \mapsto (0, Z_{y'})$. Then we define:

$$\sigma(Z)(y) := \widetilde{(Z, 0)}_{\varepsilon(y)}.$$

This map σ is indeed a splitting of the above sequence of vector bundles (but not of algebroids!) since $t_*(\sigma(Z)(y)) = t_*\left(\widetilde{(Z, 0)}_{\varepsilon(y)}\right) = Z_y$. Thus we have an

isomorphism of $C^\infty(Y)$ -modules $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \rightarrow \Gamma(Y, \text{Lie}(P))$, $(Z, h) \mapsto \sigma(Z) + \kappa(h)$, where $\kappa(h)(y) = h(y) \cdot \mathfrak{z}|_{\varepsilon(y)}$. This isomorphism allows us to describe the strict Lie 2-algebra structure on $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \rightarrow \mathfrak{X}_{mult}(P)$ in useful detail. (Compare the below proposition.)

Remark 3.2.5. (Compare [Mac05] and [MX98]). We recall several facts concerning Lie groupoids and algebroids, which will be useful in the following. Let $G_1 \rightrightarrows G_0$ be a Lie groupoid and $\text{Lie}(G_1) \rightarrow G_0$ the associated Lie algebroid.

- (1) The sections of $\text{Lie}(G_1)$ are isomorphic as a $C^\infty(G_0)$ -module to the right-invariant resp. left-invariant vector fields on G_1 via standard identifications $a \mapsto \overrightarrow{a}$ resp. $a \mapsto \overleftarrow{a}$.
- (2) The Lie bracket on $\Gamma(G_0, \text{Lie}(G_1))$ is inferred from the right-invariant vector fields on G_1 .
- (3) The map $\Gamma(G_0, \text{Lie}(G_1)) \rightarrow \mathfrak{X}_{mult}(G_1)$, $a \mapsto (\overrightarrow{a} + \overleftarrow{a}, X(a))$, where $X(a)$ is the vector field on G_0 associated to a via the anchor map of $\text{Lie}(G_1)$, is a Lie algebra morphism.

We can now give explicit formulae for the crossed module $\Gamma(Y, \text{Lie}(P)) \rightarrow \mathfrak{X}_{mult}(P)$, relying on the general construction recalled in the appendices of this thesis.

Proposition 3.2.6. *Let $Y \xrightarrow{\Pi} M$, $P \xrightarrow{\rho} Y^{[2]}$ and (A, θ) be as throughout this section. Then:*

- (i) $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \rightarrow \Gamma(Y, \text{Lie}(P))$, $(Z, h) \mapsto \sigma(Z) + \kappa(h)$, is a $C^\infty(Y)$ -module isomorphism,
- (ii) the Lie bracket on $\Gamma(Y, \text{Lie}(P))$ induces the following bracket on $\mathfrak{X}_V(Y) \oplus C^\infty(Y)$:

$$[(Z, h), (Z', h')] = ([Z, Z'], Z(h') - Z'(h) - \theta(Z, Z')),$$
- (iii) the Lie algebra morphism $\Gamma(Y, \text{Lie}(P)) \rightarrow \mathfrak{X}_{mult}(P)$, $a \mapsto (\overrightarrow{a} + \overleftarrow{a}, X(a))$ translates to $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \xrightarrow{\eta} \mathfrak{X}_{mult}(P)$,

$$(Z, h) \mapsto (\widetilde{Z}_2 + (t^* - s^*)h \cdot \mathfrak{z}, Z),$$
- (iv) the action of $\mathfrak{X}_{mult}(P)$ on $\Gamma(Y, \text{Lie}(P))$ given by $(\xi, X).a = b$ with $\overrightarrow{b} = [\xi, \overrightarrow{a}]$ induces the following action on $\mathfrak{X}_V(Y) \oplus C^\infty(Y)$:

$$(\xi, X).(Z, h) = ([X, Z], X(h) - Z(g) - \theta(X, Z)),$$
 where $\xi = \widetilde{X}_2 + (t^* - s^*)g \cdot \mathfrak{z}$ as in Lemma 3.2.4.

Proof. The first statement has already been proven. Let now (Z, h) be in $\mathfrak{X}_V(Y) \oplus C^\infty(Y)$.

The vector field $\overrightarrow{\sigma(Z)}$ is horizontal and:

$$\begin{aligned} \rho_* \left(\overrightarrow{\sigma(Z)} \right) (p) &= \rho_* \circ (R_p)_* \left(\widetilde{((Z, 0))}_{\varepsilon(t(p))} \right) \\ &= (t_* \left(\widetilde{((Z, 0))}_{\varepsilon(t(p))} \right), 0) \\ &= (Z_{t(p)}, 0). \end{aligned}$$

Thus $\overrightarrow{\sigma(Z)} = \widetilde{(Z, 0)}$. We continue with the computation

$$\begin{aligned} \overrightarrow{\kappa(h)}(p) &= (R_p)_* \left(\left. \frac{d}{d\tau} \right|_{\tau=0} \varepsilon(t(p)) \cdot \exp(i2\pi\tau h(t(p))) \right) \\ &= \left. \frac{d}{d\tau} \right|_{\tau=0} (p \cdot \exp(i2\pi\tau h(t(p)))) \\ &= (t^*h \cdot \mathfrak{z})|_p. \end{aligned}$$

In view of the \mathbb{R} -linear isomorphism from part (i), the Lie bracket on $\mathfrak{X}_V(Y) \oplus C^\infty(Y)$ can be determined from the below commutator of right-invariant vector fields:

$$\begin{aligned} &[\overrightarrow{\sigma(Z) + \kappa(h)}, \overrightarrow{\sigma(Z') + \kappa(h')}] \\ &= [\overrightarrow{\sigma(Z)}, \overrightarrow{\sigma(Z')}] - [\overrightarrow{\sigma(Z')}, \overrightarrow{\kappa(h)}] + [\overrightarrow{\sigma(Z)}, \overrightarrow{\kappa(h')}] + [\overrightarrow{\kappa(h)}, \overrightarrow{\kappa(h')}] \\ &= [\widetilde{((Z, 0))}, \widetilde{((Z', 0))}] - [\widetilde{((Z', 0))}, t^*h \cdot \mathfrak{z}] + [\widetilde{((Z, 0))}, t^*h' \cdot \mathfrak{z}] \\ &= ([\widetilde{Z}, \widetilde{Z'}], 0) - t^*\theta(Z, Z') \cdot \mathfrak{z} - t^*(Z'(h)) \cdot \mathfrak{z} + t^*(Z(h')) \cdot \mathfrak{z} \\ &= \sigma([\widetilde{Z}, \widetilde{Z'}] + \kappa(Z(h') - Z'(h) - \theta(Z, Z'))). \end{aligned}$$

We deduce that the Lie bracket is given by

$$[(Z, h), (Z', h')] = ([Z, Z'], Z(h') - Z'(h) - \theta(Z, Z')).$$

Similarly to the above calculation, one obtains $\overleftarrow{\sigma(Z)} = \widetilde{(0, Z)}$, and thus $\overrightarrow{\sigma(Z)} + \overleftarrow{\sigma(Z)} = \widetilde{Z_2}$. Also, $\overleftarrow{\kappa(h)} = -s^*h \cdot \mathfrak{z}$ and $\overrightarrow{\kappa(h)} + \overleftarrow{\kappa(h)} = (t^* - s^*)h \cdot \mathfrak{z}$. Therefore the Lie algebra morphism $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \xrightarrow{\mathcal{L}} \mathfrak{X}_{mult}(P)$ is explicitly described by

$$\overset{\circ}{\eta}(Z, h) = (\widetilde{Z_2} + (t^* - s^*)h \cdot \mathfrak{z}, Z).$$

Finally, the action of a multiplicative vector field (ξ, X) on (Z, h) reads

$$\begin{aligned} [\xi, \overrightarrow{\sigma(Z) + \kappa(h)}] &= [\widetilde{X}_2 + (t^* - s^*)g \cdot \mathfrak{z}, \widetilde{(Z, 0)} + t^*h \cdot \mathfrak{z}] \\ &= ([\widetilde{X}, \widetilde{Z}], 0) - t^*\theta(X, Z) \cdot \mathfrak{z} + t^*X(h) \cdot \mathfrak{z} - t^*Z(g) \cdot \mathfrak{z} \\ &= \sigma([X, Z]) + \kappa(X(h) - Z(g) - \theta(X, Z)). \end{aligned}$$

Thus the action is given by $[(\xi, X), (Z, h)] = ([X, Z], X(h) - Z(g) - \theta(X, Z))$.

Since Z is vertical, $Z(\Pi^*f) = 0$ for all $f \in C^\infty(M)$. Accordingly, adding Π^*f to g does not change the right hand side in the preceding equation. With Lemma 3.2.4, it follows that the action in (iv) is well-defined. \square

Going back to the Lie 2-algebra structure, we have obtained the following

Proposition 3.2.7. *The Lie algebras $\mathfrak{X}_V(Y) \oplus C^\infty(Y) (\cong \Gamma(Y, \text{Lie}(P)))$ and $\mathfrak{X}_{mult}(P)$, together with the map $\overset{\circ}{\eta}$ and the action defined above, form a Lie algebra crossed module, hence a strict Lie 2-algebra. We will denote this Lie 2-algebra by $\mathfrak{X}_{mult}(P, \Pi) := \mathfrak{X}_{mult}(P, \Pi, A)$.*

Remark 3.2.8. As mentioned above, the crossed module structure is abstractly known to exist on the complex $\Gamma(Y, \text{Lie}(P)) \rightarrow \mathfrak{X}_{mult}(P)$ (cf. [BEL20] and [OW19]). Here, we simply expressed the section space of the Lie algebroid and the bracket resp. the action of $\mathfrak{X}_{mult}(P)$ on it in a convenient form.

In symplectic prequantisation, there is the important notion of vector fields that preserve the connection (“infinitesimal quantomorphisms”). It turns out that the direct generalisation of this notion is too naive but the following definition gives a very useful 2-plectic analogue.

Definition 3.2.9. Let (ξ, X) be a multiplicative vector field on the Lie groupoid $P \rightrightarrows Y$ with connective structure (A, θ) , and B a 1-form on Y . We say that (ξ, X, B) is a *weak infinitesimal symmetry* if

- $\mathcal{L}_\xi A = (t^* - s^*)B$
- $\mathcal{L}_X \theta = dB$.

We denote the space of weak infinitesimal symmetries by $\mathcal{V}_{(P,Y)} := \mathcal{V}_{(P,Y;A,\theta)} := \{(\xi, X, B) \in \mathfrak{X}_{mult}(P) \oplus \Omega^1(Y) \mid \mathcal{L}_\xi A = (t^* - s^*)B, \mathcal{L}_X \theta = dB\}$. By abuse of language, we will call a multiplicative vector field (ξ, X) a weak infinitesimal symmetry, if there exists a one-form B such that (ξ, X, B) is such a symmetry.

Remark 3.2.10. Given a weak infinitesimal symmetry, the 1-form B is unique up to addition of the pullback under Π of a closed 1-form on M .

We first characterize the weak infinitesimal symmetries in terms of the multi-symplectic notion of ‘‘Hamiltonian vector fields’’, which will be considered in more detail in Section 3.3.

Proposition 3.2.11. *Let (ξ, X) be a multiplicative vector field on the Lie groupoid $P \rightrightarrows Y$. Then (ξ, X) is a weak infinitesimal symmetry if and only if X projects to a ‘‘Hamiltonian vector field’’, i.e., there exists $\alpha \in \Omega^1(M)$ such that $d\alpha = \iota_{\Pi_* X} \omega$.*

Proof. Recall that ξ can be written as $\widetilde{X}_2 + (t^* - s^*)f \cdot \mathfrak{z}$. Denote by $\bar{X} \in \mathfrak{X}(M)$ the vector field upon which X projects. We have:

$$\begin{aligned} \mathcal{L}_\xi A &= d(\iota_\xi A) + \iota_\xi dA \\ &= d((t^* - s^*)f) + \iota_{\widetilde{X}_2} (t^* - s^*)\theta \\ &= (t^* - s^*)(df + \iota_X \theta) \end{aligned}$$

therefore, the form $B \in \Omega^1(Y)$ such that $\mathcal{L}_\xi A = (t^* - s^*)B$ must take the form:

$$B = df + \iota_X \theta + \Pi^* \alpha$$

with $\alpha \in \Omega^1(M)$. Next we have on one hand $\mathcal{L}_X \theta = d(\iota_X \theta) + \Pi^* \iota_{\bar{X}} \omega$ and on the other hand $\mathcal{L}_X \theta = dB$ so:

$$\begin{aligned} \Pi^* \iota_{\bar{X}} \omega &= dB - d(\iota_X \theta) \\ &= d(\iota_X \theta + \Pi^* \alpha + df) - d(\iota_X \theta) \\ &= \Pi^* d\alpha \end{aligned}$$

therefore \bar{X} is Hamiltonian. The converse is easily seen to be true, finishing the proof. \square

From the preceding proof we easily obtain the following.

Corollary 3.2.12. *Let (ξ, X, B) be a weak infinitesimal symmetry. Write $\xi = \widetilde{X}_2 + (t^* - s^*)g \cdot \mathfrak{z}$, for a certain $g \in C^\infty(Y)$. Then there exists $\alpha \in \Omega^1(M)$ such that:*

$$d\alpha = \iota_{\Pi_* X} \omega \quad \text{and} \quad B = \iota_X \theta + dg + \Pi^* \alpha.$$

Remark 3.2.13. The space of weak infinitesimal symmetries is also considered in [KV], where they are referred to as *vector fields that preserve* (A, θ) . Our Proposition 3.2.14 below corresponds to Proposition 3.17 and Corollary 3.18 in [KV].

As in the case of multiplicative vector fields, the space of weak infinitesimal symmetries can be viewed as the degree-0 term of a strict Lie 2-algebra. Consider the two-term complex of vector spaces given by

$$\mathfrak{X}_V(Y) \oplus C^\infty(Y) \xrightarrow{\eta} \mathcal{V}_{(P,Y)}$$

with $\eta(Z, h) = (\widetilde{Z}_2 + (t^* - s^*)h \cdot \mathfrak{z}, Z, \iota_Z \theta + dh) = (\overset{\circ}{\eta}(Z, h), \iota_Z \theta + dh)$. We define a bracket on $\mathcal{V}_{(P,Y)}$ by $[(\xi, X), B], (\xi', X'), B'] = (([\xi, \xi'], [X, X']), \mathcal{L}_X B' - \mathcal{L}_{X'} B)$, and the natural induced action of weak infinitesimal symmetries on $\mathfrak{X}_V(Y) \oplus C^\infty(Y)$ (i.e. the forms B do not act). We then have the following:

Proposition 3.2.14. *The weak infinitesimal symmetries*

$$\mathfrak{X}_V(Y) \oplus C^\infty(Y) \xrightarrow{\eta} \mathcal{V}_{(P,Y)}$$

form a Lie algebra crossed module, hence a strict Lie 2-algebra. Moreover, this Lie 2-algebra maps naturally to the Lie 2-algebra of multiplicative vector fields (upon forgetting the 1-form component of a weak symmetry).

Proof. We first show that $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \xrightarrow{\eta} \mathcal{V}_{(P,Y)}$ is a Lie algebra crossed module. It then follows that this two-term complex forms indeed a strict Lie 2-algebra.

For $(Z, h) \in \mathfrak{X}_V(Y) \oplus C^\infty(Y)$, we show that the conditions of Definition 3.2.9 are satisfied for $\overset{\circ}{\eta}(Z, h)$ and $B = \iota_Z \theta + dh$. In fact, we have:

$$\begin{aligned} \mathcal{L}_{\widetilde{Z}_2 + (t^* - s^*)h \cdot \mathfrak{z}} A &= d(\iota_{(t^* - s^*)h \cdot \mathfrak{z}} A) + \iota_{\widetilde{Z}_2} dA \\ &= (t^* - s^*)(dh + \iota_Z \theta) = (t^* - s^*)B. \end{aligned}$$

Moreover, since Z is a vertical vector field, $\iota_Z d\theta = \iota_Z \Pi^* \omega = 0$. Thus we have:

$$\begin{aligned} \mathcal{L}_Z \theta &= d(\iota_Z \theta) + \iota_Z d\theta \\ &= dB. \end{aligned}$$

In view of Proposition 3.2.7, the only additional equation we have to verify is the ‘‘form part’’ of the second equation of Definition 6.2.6 in the appendices.

This boils down to checking that, for $(\xi, X, B) \in \mathcal{V}_{(P,Y)}$ and $(Z, h) \in \mathfrak{X}_V(Y) \oplus C^\infty(Y)$

$$\iota_{[X,Z]}\theta + d(X(h) - Z(g) - \theta(X, Z)) = \mathcal{L}_X(\iota_Z\theta + dh) - \mathcal{L}_Z(\iota_X\theta + dg + \Pi^*\alpha).$$

The last equation follows easily upon using the general formulae $\xi = \widetilde{X}_2 + (t^* - s^*)g$ and $B = \iota_X\theta + dg + \Pi^*\alpha$.

We conclude with

$$\begin{aligned} \mathcal{L}_X(\iota_Z\theta + dh) - \mathcal{L}_Z(B) &= d(X(h) - Z(g) - \theta(X, Z)) + \iota_X d\iota_Z\theta - \mathcal{L}_Z\iota_X\theta \\ &= d(X(h) - Z(g) - \theta(X, Z)) + \iota_X\mathcal{L}_Z\theta - \mathcal{L}_Z\iota_X\theta \\ &= \iota_{[X,Z]}\theta + d(X(h) - Z(g) - \theta(X, Z)). \end{aligned}$$

Finally, the Lie 2-algebra $\mathfrak{X}_V(Y) \oplus C^\infty(Y) \xrightarrow{\eta} \mathcal{V}_{(P,Y)}$ is mapped to the Lie 2-algebra of multiplicative vector fields on $P \rightrightarrows Y$ by the natural morphism $\Psi = (\Psi_1, \Psi_2)$ defined as follows (here (ξ, X, B) is an element of $\mathcal{V}_{(P,Y)}$):

$$\Psi_1|_{\mathfrak{X}_V(Y) \oplus C^\infty(Y)} = \text{id}_{\mathfrak{X}_V(Y) \oplus C^\infty(Y)}, \quad \Psi_1(\xi, X, B) = (\xi, X) \text{ and } \Psi_2 = 0.$$

This finishes the proof. □

Definition 3.2.15. We denote the above *Lie 2-algebra of weak infinitesimal symmetries* by $\text{WSym}(P, Y) := \text{WSym}(P, Y; A, \theta)$, suppressing often the dependence on the connective structure (A, θ) notationally.

3.3 A prequantisation map for integral 2-plectic manifolds

Let us recall Kostant's prequantisation map in symplectic geometry (compare e.g. [Kos70]). For an integral symplectic manifold (M, ω) , let $q : P \rightarrow M$ be a principal S^1 -bundle with connection whose curvature equals the pullback of $-2\pi i\omega$. For $f \in C^\infty(M)$, X_f denotes the Hamiltonian vector field fulfilling $df = \iota_{X_f}\omega$, \widetilde{X}_f its horizontal lift to P and \mathfrak{z} the fundamental vector field of the S^1 -action on P , given by $\mathfrak{z}_p = \left. \frac{d}{d\tau} \right|_{\tau=0} p \cdot \exp(i2\pi\tau)$. Then the Lie algebra morphism $C^\infty(M) \rightarrow \mathfrak{X}^{S^1}(P)$, the S^1 -invariant vector fields on P , given by $f \mapsto \widetilde{X}_f + q^*f \cdot \mathfrak{z}$ is the prequantisation map.

Its 2-plectic analogue will be developed in this section as a Lie 2-algebra morphism from the Lie 2-algebra of observables associated to a 2-plectic manifold (see Proposition 3.3.3 below) to a certain Lie 2-algebra of multiplicative vector fields on the global geometric prequantisation of an integral 2-plectic manifold (M, ω) . Recall that this prequantisation is given by a lifting bundle gerbe associated to a principal $PU(H)$ -bundle $\Pi : Y \rightarrow M$ with connective structure (A, θ) such that its 3-curvature equals $(-2\pi i)\omega$.

We also fix a principal connection on Π and denote, for a vector field $X \in \mathfrak{X}(M)$, the ensuing horizontal lift to Y by X^h . Its curvature will be written Ω , a $\mathfrak{pu}(H)$ -valued 2-form on Y . Given two vector fields Z and Z' on Y , we have a vertical vector field $\Omega(Z, Z')$ that is in $y \in Y$ defined by the following formula:

$$\Omega(Z, Z')|_y := \left. \frac{d}{dt} \right|_{t=0} y \cdot \exp(t\Omega_y(Z_y, Z'_y)),$$

where $y \cdot g$ denotes the action of $g \in PU(H)$ on $y \in Y$. Notably, we have for $X, X' \in \mathfrak{X}(M)$, the vertical vector field $\Omega(X^h, X'^h)$ on Y fulfilling

$$[X, X']^h = [X^h, (X')^h] + \Omega(X^h, (X')^h)$$

(see, e.g., Section 19 of [Mic08], for this approach to the curvature of a principal bundle).

Finally, we recall that given a Π -projectable vector field Z on Y , we have introduced in Section 3.2 the notation Z_2 for the vector field on $Y^{[2]}$ that is given by $(y, y') \mapsto (Z_y, Z_{y'})$.

We now give the fundamental definitions regarding 2-plectic manifolds and the associated Lie 2-algebra of observables. Of course, Definitions 3.3.1 and 3.3.2, as well as Proposition 3.3.3 generalize immediately to the multisymplectic (or n -plectic) case (compare [Rog11]), but we stick here to the 2-plectic case.

Definition 3.3.1. Let M be a manifold and $\omega \in \Omega^3(M)$. We say that (M, ω) is a *2-plectic manifold* if ω is closed and non-degenerate in the following sense:

$$\iota_X \omega = 0 \quad \text{if and only if} \quad X = 0$$

for every vector field $X \in \mathfrak{X}(M)$.

Definition 3.3.2. Let (M, ω) be a 2-plectic manifold, and $\alpha \in \Omega^1(M)$. We say that α is a *Hamiltonian form* if there exists a vector field X_α such that:

$$d\alpha = \iota_{X_\alpha} \omega.$$

We also say that X_α is the *Hamiltonian vector field associated to α* . We denote the space of Hamiltonian forms by $\Omega_{Ham}^1(M, \omega)$.

Note that, given $\alpha \in \Omega_{Ham}^1(M, \omega)$, the associated Hamiltonian vector field is unique, by the non-degeneracy of ω . The converse is not true: the Hamiltonian form, given a Hamiltonian vector field, is only unique up to a closed 1-form.

The following natural algebraic structure on the space of Hamiltonian forms seems to be explicitly stated for the first time in [BHR10] (compare also [Rog11]). We refer to the appendix (more precisely Section 6.2) for relevant definitions and notations.

Proposition 3.3.3. *Let (M, ω) be a 2-plectic manifold. The complex:*

$$C^\infty(M) \xrightarrow{d} \Omega_{Ham}^1(M, \omega)$$

can be given the structure of a Lie-2 algebra by setting:

- $l_1(f) = df$ and $l_1(\alpha) = 0$
- $l_2(\alpha, \beta) = \iota_{X_\alpha} \iota_{X_\beta} \omega$
- $l_2(\alpha, f) = 0$
- $l_3(\alpha, \beta, \gamma) = \omega(X_\alpha, X_\beta, X_\gamma)$

for all $\alpha, \beta, \gamma \in \Omega_{Ham}^1(M, \omega)$ and $f \in C^\infty(M)$.

We will denote this Lie 2-algebra by $L_\bullet(M, \omega)$, and call it the *Lie 2-algebra (of observables) of (M, ω)* . We will also use the following notation for the bracket of forms: $\{\alpha, \beta\} := l_2(\alpha, \beta)$.

We are now ready to describe the 2-plectic analogue of Kostant's prequantisation map in symplectic geometry.

Proposition 3.3.4. *Let (M, ω) be a 2-plectic manifold with integral ω , and (P, Y) a global geometric prequantisation of (M, ω) with connective structure (A, θ) . Then there is a Lie 2-algebra morphism $\Phi = (\Phi_1, \Phi_2) : L_\bullet(M, \omega) \rightarrow \mathfrak{X}_{mult}(P, \Pi)$:*

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{d} & \Omega_{Ham}^1(M, \omega) \\ \Phi_1 \downarrow & \swarrow \Phi_2 & \downarrow \Phi_1 \\ \mathfrak{X}_V(Y) \oplus C^\infty(Y) & \xrightarrow{\mathfrak{h}} & \mathfrak{X}_{mult}(P) \end{array}$$

where:

- $\Phi_1(f) = (0, \Pi^* f)$
- $\Phi_1(\alpha) = \left(\widetilde{(X_\alpha^h)_2}, X_\alpha^h \right)$
- $\Phi_2(\alpha, \beta) = (\Omega(X_\alpha^h, X_\beta^h), \theta(X_\alpha^h, X_\beta^h) + \Pi^*(\alpha(X_\beta) - \beta(X_\alpha)))$.

Proof. The map Φ_1 is a chain map because $X_{df} = 0$ and $\mathring{\eta}((0, \Pi^* f)) = 0$, because $s^* \Pi^* f = t^* \Pi^* f$. Now we have to verify

$$(*) \quad \Phi_1(\{\alpha, \beta\}) = [\Phi_1(\alpha), \Phi_1(\beta)] + \mathring{\eta}(\Phi_2(\alpha, \beta))$$

and

$$(**) \quad -\Phi_2(df, \alpha) = [\Phi_1(f), \Phi_1(\alpha)].$$

We recall that $X_{\{\alpha, \beta\}} = [X_\alpha, X_\beta]$ and compute

$$\begin{aligned} \Phi_1(\{\alpha, \beta\}) &= (([X_\alpha, X_\beta]^h)_2, [X_\alpha, X_\beta]^h) \\ &= (([X_\alpha^h, X_\beta^h] + \Omega(X_\alpha^h, X_\beta^h))_2, [X_\alpha^h, X_\beta^h] + \Omega(X_\alpha^h, X_\beta^h)) \\ &= ((\widetilde{(X_\alpha^h)_2}, \widetilde{(X_\beta^h)_2}) + \Omega(\widetilde{(X_\alpha^h)_2}, \widetilde{(X_\beta^h)_2}) + (t^* - s^*)\theta(X_\alpha^h, X_\beta^h) \cdot \mathfrak{z}, [X_\alpha^h, X_\beta^h] \\ &\quad + \Omega(X_\alpha^h, X_\beta^h)) \end{aligned}$$

where the second equality follows from the formula $dA = (t^* - s^*)\theta$ for a connective structure, recalled in Section 1. On the other hand,

$$\begin{aligned} [\Phi_1(\alpha), \Phi_1(\beta)] + \mathring{\eta}(\Phi_2(\alpha, \beta)) &= ((\widetilde{(X_\alpha^h)_2}, \widetilde{(X_\beta^h)_2}), [X_\alpha^h, X_\beta^h]) \\ &\quad + (\Omega(\widetilde{(X_\alpha^h)_2}, \widetilde{(X_\beta^h)_2}) + (t^* - s^*)\theta(X_\alpha^h, X_\beta^h) \cdot \mathfrak{z}, \Omega(X_\alpha^h, X_\beta^h)). \end{aligned}$$

Thus equation (*) is shown. For (**) we have

$$\Phi_2(df, \alpha) = (0, \Pi^*(df(X_\alpha)))$$

and

$$\begin{aligned} [\Phi_1(f), \Phi_1(\alpha)] &= [(0, \Pi^* f), (\widetilde{(X_\alpha^h)_2}, X_\alpha^h)] \\ &= -(0, X_\alpha^h(\Pi^* f)) \\ &= -(0, \Pi^*(df(X_\alpha))). \end{aligned}$$

The verification of the fourth equation of Definition 6.2.15 is a long, but straightforward computation, whose crucial points we will explain in the sequel. First of all, we have terms of the type:

$$\begin{aligned}\Phi_2(\{\alpha, \beta\}, \gamma) &= (\Omega([X_\alpha, X_\beta]^h, X_\gamma^h), \theta([X_\alpha, X_\beta]^h, X_\gamma^h)) \\ &\quad + \Pi^*(\{\alpha, \beta\}(X_\gamma) - \gamma([X_\alpha, X_\beta])) \\ &= (\Omega([X_\alpha, X_\beta]^h, X_\gamma^h), \theta([X_\alpha^h, X_\beta^h], X_\gamma^h)) \\ &\quad + \theta(\Omega(X_\alpha, X_\beta), X_\gamma^h) + g(\alpha, \beta, \gamma)\end{aligned}$$

where we set

$$g(\alpha, \beta, \gamma) := \Pi^*(\{\alpha, \beta\}(X_\gamma) - \gamma([X_\alpha, X_\beta])).$$

Furthermore, we have

$$\begin{aligned}-\gamma([X_\alpha, X_\beta]) &= -X_\alpha(\gamma(X_\beta)) + X_\beta(\gamma(X_\alpha)) + d\gamma(X_\alpha, X_\beta) \\ &= -X_\alpha(\gamma(X_\beta)) + X_\beta(\gamma(X_\alpha)) + \omega(X_\alpha, X_\beta, X_\gamma)\end{aligned}$$

and

$$\{\alpha, \beta\}(X_\gamma) = -\omega(X_\alpha, X_\beta, X_\gamma).$$

Thus we obtain

$$g(\alpha, \beta, \gamma) = \Pi^*(-X_\alpha(\gamma(X_\beta)) + X_\beta(\gamma(X_\alpha))).$$

Finally, we have on the RHS of the fourth equation of Definition 6.2.15 terms of the type

$$\begin{aligned}[\Phi_1(\alpha), \Phi_2(\beta, \gamma)] &= [\widetilde{((X_\alpha^h)_2, X_\alpha^h)}, (\Omega(X_\beta^h, X_\gamma^h), \theta(X_\beta^h, X_\gamma^h) + \Pi^*(\beta(X_\gamma) - \gamma(X_\beta)))] \\ &= (\widetilde{(X_\alpha^h)_2}, X_\alpha^h) \cdot (\Omega(X_\beta^h, X_\gamma^h), \theta(X_\beta^h, X_\gamma^h) + \Pi^*(\beta(X_\gamma) - \gamma(X_\beta))) \\ &= ([X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h)], X_\alpha^h(\theta(X_\beta^h, X_\gamma^h) + \Pi^*(\beta(X_\gamma) - \gamma(X_\beta)))) \\ &\quad - \theta(X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h)).\end{aligned}$$

Let us now explain why the LHS and the RHS of the fourth equation of Definition 6.2.15 coincide here. Firstly, the terms with $g(\alpha, \beta, \gamma)$ cancel out with the terms of the type $X_\alpha^h(\Pi^*(\beta(X_\gamma) - \gamma(X_\beta)))$. Secondly, $l'_3(\Phi_1(\alpha), \Phi_1(\beta), \Phi_1(\gamma)) = 0$ since $l'_3 = 0$, and

$$\Phi_1(l'_3(\alpha, \beta, \gamma)) = (0, \Pi^*(\omega(X_\alpha, X_\beta, X_\gamma))) = (0, (d\theta)(X_\alpha^h, X_\beta^h, X_\gamma^h)).$$

Developping the $d\theta$ -term and comparing with the terms of the type $\theta([X_\alpha, X_\beta]^h, X_\gamma^h)$ resp. $X_\alpha^h(\theta(X_\beta^h, X_\gamma^h))$ show that with respect to the discussed contributions LHS and RHS are equal. We next observe that the term $\theta(X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h))$ and its signed cyclic permutations appear identically on both sides of the equation.

To conclude, we start by observing that

$$[X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h)] = X_\alpha^h(\Omega(X_\beta^h, X_\gamma^h)),$$

where the last term is interpreted as the vertical vector field associated to the $\mathfrak{pu}(H)$ -valued map $X_\alpha^h(\Omega(X_\beta^h, X_\gamma^h))$. To show this, denote by $C \in \Omega^1(Y) \otimes \mathfrak{pu}(H)$ a principal connection on $\Pi : Y \rightarrow M$ with fixed curvature Ω . Then $C(X_\alpha^h) = 0$ because X_α^h is horizontal and $dC(X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h)) = 0$ because $dC = -\frac{1}{2}[C, C] + \Omega$ and Ω is horizontal. So we obtain

$$\begin{aligned} C([X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h)]) &= -dC(X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h)) \\ &\quad + X_\alpha^h(C(\Omega(X_\beta^h, X_\gamma^h)) - \Omega(X_\beta^h, X_\gamma^h)(C(X_\alpha^h))) \\ &= X_\alpha^h(\Omega(X_\beta^h, X_\gamma^h)). \end{aligned}$$

(Note that $\Omega(X_\beta^h, X_\gamma^h)$ is here interpreted both as a vertical vector field and a $\mathfrak{pu}(H)$ -valued map on Y). Observing that $[X_\alpha^h, \Omega(X_\beta^h, X_\gamma^h)]$ is vertical thus yields the formula.

Since the curvature Ω is a horizontal form, we have $\Omega([X_\alpha, X_\beta]^h, X_\gamma^h) = \Omega([X_\alpha^h, X_\beta^h], X_\gamma^h)$ and, by the Bianchi identity, we have $d\Omega(X_\alpha^h, X_\beta^h, X_\gamma^h) = 0$. It follows that the remaining terms on both sides of the fourth equation of Definition 6.2.15 indeed coincide. \square

Remark 3.3.5. Since in the preceding proposition the first Lie 2-algebra $L_\bullet(M, \omega)$ is non-strict, whereas $\mathfrak{X}_{mult}(P, \Pi)$, the target of the morphism Φ , is strict, it is not surprising that we need the “correcting component” Φ_2 .

In symplectic geometry, restricting the target space of Kostant’s prequantisation map to the vector fields that preserve the connection (which are generally referred to as infinitesimal quantomorphisms), gives a Lie algebra isomorphism. We now show the 2-plectic analogue: since the map Φ_1 in Proposition 3.3.4 actually lies in $\mathcal{V}_{(P, Y)}$, we can “restrict” this Lie 2-algebra morphism to the weak infinitesimal symmetries, which then gives a quasi-isomorphism.

Theorem 3.3.6. *The Lie 2-algebra morphism of Proposition 3.3.4 refines to a quasi-isomorphism of Lie 2-algebras $\Phi = (\Phi_1, \Phi_2) : L_\bullet(M, \omega) \rightarrow WSym(P, Y)$ as follows*

$$\begin{array}{ccc}
C^\infty(M) & \xrightarrow{d} & \Omega_{Ham}^1(M, \omega) \\
\Phi_1 \downarrow & \swarrow \Phi_2 & \downarrow \Phi_1 \\
\mathfrak{X}_V(Y) \oplus C^\infty(Y) & \xrightarrow{\eta} & \mathcal{V}_{(P,Y)}
\end{array}$$

where

- $\Phi_1(f) = (0, \Pi^* f)$
- $\Phi_1(\alpha) = \left(\widetilde{(X_\alpha^h)_2}, X_\alpha^h, \iota_{X_\alpha^h} \theta + \Pi^* \alpha \right)$
- $\Phi_2(\alpha, \beta) = \left(\Omega(X_\alpha^h, X_\beta^h), \theta(X_\alpha^h, X_\beta^h) + \Pi^*(\alpha(X_\beta) - \beta(X_\alpha)) \right)$.

Proof. First we show the Φ_1 map indeed takes values in $\mathcal{V}_{(P,Y)}$. Denote $B_\alpha = \iota_{X_\alpha^h} \theta + \Pi^* \alpha$ for $\alpha \in \Omega_{Ham}^1(M, \omega)$. Because $(t^* - s^*)\Pi^* \alpha = 0$ it is clear that $\mathcal{L}_{\widetilde{(X_\alpha^h)_2}} A = (t^* - s^*)B_\alpha$. Moreover:

$$\begin{aligned}
\mathcal{L}_{X_\alpha^h} \theta &= d\iota_{X_\alpha^h} \theta + \iota_{X_\alpha^h} d\theta \\
&= d(B_\alpha - \Pi^* \alpha) + \Pi^* \iota_{X_\alpha} \omega \\
&= dB_\alpha - \Pi^*(d\alpha - \iota_{X_\alpha} \omega) \\
&= dB_\alpha.
\end{aligned}$$

We obviously have $\Phi_1(df) = (0, 0, \Pi^* df) = \eta(0, \Pi^* f) = \eta(\Phi_1(f))$.

To verify that Φ defines a Lie 2-algebra morphism, the only additional equality to be satisfied is the ‘‘form part’’ of the second equation in Definition 6.2.15. Concretely, we have to check the following equation:

$$(*) \quad \iota_{X_{\{\alpha, \beta\}}^h} \theta + \Pi^* \{\alpha, \beta\} = \mathcal{L}_{X_\alpha^h} B_\beta - \mathcal{L}_{X_\beta^h} B_\alpha + \iota_{\Omega(X_\alpha^h, X_\beta^h)} \theta + d(\theta(X_\alpha^h, X_\beta^h) + \Pi^*(\alpha(X_\beta) - \beta(X_\alpha))).$$

We start with the first term of the LHS of (*):

$$(I) \quad \iota_{X_{\{\alpha, \beta\}}^h} \theta = \iota_{[X_\alpha^h, X_\beta^h]} \theta + \iota_{\Omega(X_\alpha^h, X_\beta^h)} \theta.$$

We rewrite now the first two terms of the RHS of (*), via

$$\mathcal{L}_{X_\alpha^h} B_\beta - \mathcal{L}_{X_\beta^h} B_\alpha = \mathcal{L}_{X_\alpha^h} (\iota_{X_\beta^h} \theta + \Pi^* \beta) - \mathcal{L}_{X_\beta^h} (\iota_{X_\alpha^h} \theta + \Pi^* \alpha)$$

and

$$\begin{aligned}
\mathcal{L}_{X_\alpha^h} \iota_{X_\beta^h} \theta - \mathcal{L}_{X_\beta^h} \iota_{X_\alpha^h} \theta &= \mathcal{L}_{X_\alpha^h} \iota_{X_\beta^h} \theta - d(\iota_{X_\beta^h} \iota_{X_\alpha^h} \theta) - \iota_{X_\beta^h} d\iota_{X_\alpha^h} \theta \\
&= \mathcal{L}_{X_\alpha^h} \iota_{X_\beta^h} \theta - d(\theta(X_\alpha^h, X_\beta^h)) - \iota_{X_\beta^h} \mathcal{L}_{X_\alpha^h} \theta + \iota_{X_\beta^h} \iota_{X_\alpha^h} d\theta \\
&= \iota_{[X_\alpha^h, X_\beta^h]} \theta - d(\theta(X_\alpha^h, X_\beta^h)) - \Pi^* \iota_{X_\alpha} \iota_{X_\beta} \omega.
\end{aligned}$$

Furthermore, we calculate

$$\begin{aligned}
\mathcal{L}_{X_\alpha^h} \Pi^* \beta - \mathcal{L}_{X_\beta^h} \Pi^* \alpha &= d\iota_{X_\alpha^h} \Pi^* \beta + \iota_{X_\alpha^h} d\Pi^* \beta - d\iota_{X_\beta^h} \Pi^* \alpha - \iota_{X_\beta^h} d\Pi^* \alpha \\
&= d\Pi^*(\beta(X_\alpha) - \alpha(X_\beta)) + 2\Pi^* \iota_{X_\alpha} \iota_{X_\beta} \omega.
\end{aligned}$$

We thus arrive at

$$\mathcal{L}_{X_\alpha^h} B_\beta - \mathcal{L}_{X_\beta^h} B_\alpha = \iota_{[X_\alpha^h, X_\beta^h]} \theta - d(\theta(X_\alpha^h, X_\beta^h)) + \Pi^* \{\alpha, \beta\} + d\Pi^*(\beta(X_\alpha) - \alpha(X_\beta)).$$

Reordering the terms in the preceding equation and employing equation (I) immediately gives equation (*).

Finally we show that this morphism defines a quasi-isomorphism.

We have that $\ker(\eta) = \{(Z, g) \in \mathfrak{X}_V(Y) \oplus C^\infty(Y) \mid Z = 0, (t^* - s^*)g = 0, dg = 0\} = \{(0, \Pi^* f) \mid f \in C^\infty(M) \text{ and } df = 0\}$ implying that the map Φ_1 is an isomorphism between $\ker(d)$ and $\ker(\eta)$.

We turn to proving that Φ_1 defines an isomorphism from $\Omega_{Ham}^1(M, \omega) / \text{im}(d)$ to $\mathcal{V}_{(P, Y)} / \text{im}(\eta)$. Since a multiplicative vector field (ξ, X) is a weak infinitesimal symmetry if and only if X projects to a Hamiltonian vector field, it is clear that there is an $\alpha \in \Omega_{Ham}^1(M, \omega)$ and a $Z \in \mathfrak{X}_V(Y)$ such that $X - Z = X_\alpha^h$. Since $\xi = \widetilde{X}_2 + (t^* - s^*)g \cdot \mathfrak{z}$ for a certain $g \in C^\infty(Y)$, we have for all $f \in C^\infty(M)$ that $\xi - \widetilde{Z}_2 - (t^* - s^*)(g + \Pi^* f) \cdot \mathfrak{z} = \widetilde{X}_\alpha^h$. Now, if $B \in \Omega^1(Y)$ such that $\mathcal{L}_X \theta = dB$, it can be written as $B = \iota_X \theta + dg + \Pi^*(\alpha + df)$, and then $B - \iota_Z \theta - dg = \iota_X \theta + \Pi^*(\alpha + df)$. Thus we have $(\xi, X, B) - \eta(Z, g + \Pi^* f) = \Phi_1(\alpha + df)$. Unicity of α (up to differentials of functions) now yields the desired isomorphism. \square

Remark 3.3.7. We observe that the quasi-isomorphism Φ of the preceding theorem is injective as a morphism of cochain complexes of vector spaces but, of course, is far from being surjective.

Remark 3.3.8. Theorem 5.1 of [KV] shows that $L_\bullet(M, \omega)$ and $WSym(P, Y)$ are quasi-isomorphic via the existence of an “invertible butterfly”. Our techniques are completely different and yield an explicit map of Lie 2-algebras that turns out to be a quasi-isomorphism. From the point of view of [FRS14], the preceding theorem is a “globalisation” of the 2-plectic specialisation of Theorem 4.2.2 there concerning prequantisation in n-plectic geometry. There the authors give a quasi-isomorphism between the Lie n-algebra of observables of a n-plectic manifold, and what they call the *infinitesimal quantomorphisms*, defined via a Čech-Deligne complex. We do not rely on open coverings as used there but formulate the algebraic objects in a global way by first associating a global geometric prequantisation to an integral 2-plectic manifold.

Remark 3.3.9. Proposition 3.3.4 and Theorem 3.3.6 above both apply to integral pre-2-plectic manifolds, i.e., manifolds equipped with an integral, closed 3-form ω , not necessarily non-degenerate. However, since we do not have unicity of Hamiltonian vector fields associated to Hamiltonian 1-forms anymore, we have to modify the Lie 2-algebra of observables (compare [Zam12]). Given an integral pre-2-plectic manifold (M, ω) , we set $\tilde{L}_0 = \{(\alpha, X) \in \Omega^1(M) \oplus \mathfrak{X}(M) \mid d\alpha = \iota_X \omega\}$. Then

$$\tilde{L}_{-1} = C^\infty(M) \xrightarrow{l_1} \tilde{L}_0$$

is a Lie 2-algebra with brackets

- $l_1(f) = (df, 0)$
- $l_2((\alpha, X), (\alpha', X')) = (\iota_X \iota_{X'} \omega, [X, X'])$
- $l_2((\alpha, X), f) = 0$
- $l_3((\alpha, X), (\alpha', X'), (\alpha'', X'')) = \omega(X, X', X'')$

for $f \in C^\infty(M)$ and $(\alpha, X), (\alpha', X'), (\alpha'', X'') \in \tilde{L}_0$. Denoting this Lie 2-algebra by $\tilde{L}_\bullet(M, \omega)$, we have Lie 2-algebra morphisms

$$\tilde{L}_\bullet(M, \omega) \rightarrow \mathfrak{X}_{mult}(P, \Pi) \text{ and } \tilde{L}_\bullet(M, \omega) \rightarrow WSym(P, Y)$$

where the bundle gerbe (P, Y) , together with the principal $PU(H)$ -bundle $Y \xrightarrow{\Pi} M$, is a global geometric prequantisation of the pre-2-plectic manifold (M, ω) . The formulae giving the two Lie 2-algebra morphisms (abusively both called Φ) are easily adapted from those given in Proposition 3.3.4 and Theorem 3.3.6. A direct check shows that the proofs of both results adapt as well to this case.

Definition 3.3.10. Let (M, ω) be a (pre-)2-plectic manifold with integral ω , and (P, Y) a global geometric prequantisation of (M, ω) . The Lie 2-algebra morphism $\Phi = (\Phi_1, \Phi_2)$ from $L_\bullet(M, \omega)$ to $\text{WSym}(P, Y; A, \theta)$ of the above theorem is called the *prequantisation map* of (M, ω) .

3.4 The case of exact 2-plectic manifolds

In this section, we expose the case of 2-plectic manifolds (M, ω) with a potential $\chi \in \Omega^2(M)$ for the 3-form ω , i.e., $d\chi = \omega$. This case is important for applications in physics, compare, e.g., [Kij73], [RR09] and [RW19].

Recall that for $P \xrightarrow{r} B$ and $P' \xrightarrow{r'} B$ two principal S^1 -bundles over B we may form a new bundle $P \otimes P' \rightarrow B$ defined by

$$P \otimes P' := (P_r \times_{r'} P') / S^1,$$

where the quotient is taken so that $(p, p') \sim (pz, p'z^{-1})$ for $p \in P$, $p' \in P'$ and $z \in S^1$. We may also form the dual bundle P^* defined by the same fiber bundle but provided with the action $p \cdot z = pz^{-1}$.

Let $Y \xrightarrow{\Pi} M$ be a surjective submersion and consider $Q \xrightarrow{r} Y$ a principal S^1 -bundle, with connection 1-form $\tilde{A} \in i\Omega^1(Q)$. We denote $F^{\tilde{A}} \in i\Omega^2(Y)$ the curvature of this principal connection, so that $d\tilde{A} = r^*F^{\tilde{A}}$. Furthermore, we put $\pi_k : Y^{[2]} \rightarrow Y$ for $k = 1, 2$, where π_k is the projection that omits the k^{th} factor. Then

$$1 \rightarrow Y \times S^1 \rightarrow (\pi_1^*Q)^* \otimes \pi_2^*Q \rightarrow Y^{[2]} \rightarrow 1$$

is a S^1 -central extension of the Lie groupoid $Y^{[2]} \rightrightarrows Y$, and therefore $\delta(Q) := (\pi_1^*Q)^* \otimes \pi_2^*Q$ yields a bundle gerbe. In fact (compare [MS00]), a bundle gerbe has vanishing Dixmier-Douady class precisely when it is isomorphic (in an appropriate sense) to a bundle gerbe of the form $(\delta(Q), Y)$, and a choice of an isomorphism $P \rightarrow \delta(Q)$ is called a *trivialisation* of the bundle gerbe (P, Y) .

Let us recall several observations concerning the Lie groupoid $\delta(Q) \rightrightarrows Y$. An element of $\delta(Q)$ is written as a quadruple $([q_1, q_2], y_1, y_2)$ where $q_i \in Q$, $(y_1, y_2) \in Y^{[2]}$ such that $r(q_1) = y_1$, $r(q_2) = y_2$, and $[q_1, q_2]$ is the class stemming from the equivalence relation: $(q_1, q_2) \sim (q_1z, q_2z)$ for all $z \in S^1$. In the sequel, we denote such a quadruple by the equivalence classe $[q_1, q_2]$. Then the structure maps of the Lie groupoid $\delta(Q) \rightrightarrows Y$ are given by

- $s([q_1, q_2]) = r(q_2)$
- $t([q_1, q_2]) = r(q_1)$.

Note that for $[q_1, q_2], [q_3, q_4]$ in $\delta(Q)$, such that $s([q_1, q_2]) = r(q_2) = r(q_3) = t([q_3, q_4])$, we may assume without loss of generality that $q_2 = q_3$, since the equivalence classes are defined by the orbits of the S^1 -action. Therefore the groupoid multiplication can be written as

$$m([q_1, q_2], [q_2, q_3]) = [q_1, q_3].$$

For $y \in Y$ we have furthermore

$$\delta(Q)_{(y,y)} = ((\pi_1^*Q)^* \otimes \pi_2^*Q)_{(y,y)} = Q_y^* \otimes Q_y.$$

Thus denoting by $\Delta : Y \rightarrow Y^{[2]}$ the diagonal inclusion, the bundle $\Delta^*(\delta(Q))$ is canonically trivialised. Taking for $y \in Y$ any $q \in Q_y$, the canonical section $\epsilon : Y \rightarrow \Delta^*(\delta(Q))$ is given by $\epsilon(y) = [q, q]$ and ϵ is taken as the unit map of the Lie groupoid $\delta(Q) \rightrightarrows Y$. Finally the inverse map is given as

$$[q_1, q_2]^{-1} = [q_2, q_1].$$

We now describe a connective structure on $(\delta(Q), Y)$. Consider the projections $(\pi_1^*Q)^* \times_{Y^{[2]}} \pi_2^*Q \xrightarrow{p_k} Q$, where p_k projects to the k^{th} factor ($k = 1, 2$), and the 1-form $A = p_1^*\tilde{A} - p_2^*\tilde{A}$. Then A defines a principal connection on $(\pi_1^*Q)^* \otimes \pi_2^*Q$. More precisely, we have:

Lemma 3.4.1. *We keep the notations from the previous considerations. Then*

(i) *the connection A on $\delta(Q) \rightarrow Y^{[2]}$ is multiplicative,*

(ii) *a curving of this connection is given by the curvature $F^{\tilde{A}}$ of \tilde{A} .*

Therefore $(A, F^{\tilde{A}})$ is a connective structure on the bundle gerbe $(\delta(Q), Y)$. Moreover, the 3-curvature is identically zero and thus $(\delta(Q), Y)$ has vanishing Dixmier-Douady class.

Proof. For $q_1, q_2, q_3 \in Q$ and $v_{q_1}, v_{q_2}, v_{q_3}$ tangent vectors at the respective points, we compute

$$\begin{aligned} (m^*A)_{[q_1, q_2], [q_2, q_3]}((v_{q_1}, v_{q_2}), (v_{q_2}, v_{q_3})) &= A_{[q_1, q_3]}(v_{q_1}, v_{q_3}) = \tilde{A}_{q_1}(v_{q_1}) - \tilde{A}_{q_3}(v_{q_3}) = \\ \tilde{A}_{q_1}(v_{q_1}) - \tilde{A}_{q_2}(v_{q_2}) + \tilde{A}_{q_2}(v_{q_2}) - \tilde{A}_{q_3}(v_{q_3}) &= A_{[q_1, q_2]}(v_{q_1}, v_{q_2}) + A_{[q_2, q_3]}(v_{q_2}, v_{q_3}) = \\ (\text{proj}_1^*A + \text{proj}_2^*A)_{[q_1, q_2], [q_2, q_3]}((v_{q_1}, v_{q_2}), (v_{q_2}, v_{q_3})), \end{aligned}$$

where $\text{proj}_k : \delta(Q)_s \times_t \delta(Q) \rightarrow \delta(Q)$ is the projection onto the k^{th} factor, for $k = 1, 2$. Thus A is multiplicative.

Moreover, we have

$$dA = p_1^* d\tilde{A} - p_2^* d\tilde{A} = p_1^* r^* F^{\tilde{A}} - p_2^* r^* F^{\tilde{A}} = (t^* - s^*) F^{\tilde{A}}.$$

This shows that $F^{\tilde{A}}$ provides a curving for A , and that $(A, F^{\tilde{A}})$ is indeed a connective structure for $(\delta(Q), Y)$. Since $dF^{\tilde{A}} = 0$, the 3-curvature equals zero, and therefore $(\delta(Q), Y)$ has vanishing Dixmier-Douady class. \square

We now construct the prequantisation map for an exact 2-plectic manifold. Let (M, ω) be a 2-plectic manifold with $\omega = d\chi$ an exact 3-form. Consider the trivial bundle $\Pi : Y := M \times PU(H) \rightarrow M$, equipped with the trivial connection, that we denote here by $C \in \Omega^1(M \times PU(H)) \otimes \mathfrak{pu}(H)$. Recall that C is given by

$$C_{(m,g)}(u_m, v_g) = (L_{g^{-1}})_* v_g,$$

i.e., C is the pullback of the Maurer-Cartan form on $PU(H)$ via the projection $Y \rightarrow PU(H)$.

Observe that $Y^{[2]} = M \times PU(H) \times PU(H)$. We then have $\psi : M \times PU(H) \times PU(H) \rightarrow PU(H)$, $\psi(m, g, g') = g^{-1}g'$ (compare Lemma 1.10). We also define $Q := M \times U(H) \rightarrow Y$. Then $\delta(Q) := (\pi_1^* Q)^* \otimes \pi_2^* Q$ is isomorphic to $\psi^* U(H)$, the pullback of the principal S^1 -bundle $U(H) \rightarrow PU(H)$ by the map ψ . To see this, note that the elements of $\delta(Q)$ are triples $(m, [u, u'])$, where $m \in M$, $u, u' \in U(H)$, and the class $[u, u']$ is taken with respect to $(u, u') \sim (uz, u'z)$ for all $z \in S^1$. The elements of $\psi^* U(H)$ are triples (m, g, u) , where $m \in M$, $g \in PU(H)$ and $u \in U(H)$. Then the isomorphism $\delta(Q) \rightarrow \psi^* U(H)$ is given by

$$(*) \quad (m, [u, u']) \mapsto (m, q(u), u^{-1}u'),$$

with $U(H) \xrightarrow{q} PU(H)$ being the canonical projection. The map $(*)$ is well-defined and equivariant, and therefore an isomorphism of principal S^1 -bundles.

Let \tilde{A} be the principal connection on Q defined by the Maurer-Cartan form on $U(H)$, projected onto $i\mathbb{R} = \text{Lie}(S^1)$ via a splitting $\mathfrak{pu}(H) \rightarrow \mathfrak{u}(H)$ (see Lemma 1.6). Let $F^{\tilde{A}}$ be the curvature of this connection. On the bundle $\delta(Q) \rightarrow Y^{[2]}$, we consider as above the connection $A = p_1^* \tilde{A} - p_2^* \tilde{A}$ (recall

that A is multiplicative by the preceding lemma). We set $\theta = \Pi^*\chi + F^{\tilde{A}}$. Then (A, θ) is a connective structure on the bundle gerbe $(\delta(Q), Y)$, with 3-curvature $(-i2\pi)\omega$. For a vector field $X \in \mathfrak{X}(M)$, the horizontal lift X^h to Y with respect to the connection C is simply $(X, 0)$, which we denote again by X . Then we have $X_2 = (X, 0, 0)$ and we continue to denote this vector field by X . The horizontal lift of a vector field $Z \in \mathfrak{X}(Y^{[2]})$ with respect to the connection A will be denoted by \tilde{Z} .

We conclude with an explicit description of the components of the Lie 2-algebra morphism of Theorem 3.3.6 in the exact case. For the sake of better readability, we omit the symbol Π^* for pullbacks of functions and differential forms with respect to the projection $\Pi : Y \rightarrow M$. Furthermore, given a vector field V on a factor of a product $A \times B$, we denote its trivial extension to this product again by V . With these conventions, we obtain for $f \in C^\infty(M)$ and $\alpha, \beta \in \Omega_{Ham}^1(M, d\chi)$:

- $\Phi_1(f) = (0, f)$
- $\Phi_1(\alpha) = (\tilde{X}_\alpha, X_\alpha, \iota_{X_\alpha}\chi + \alpha)$
- $\Phi_2(\alpha, \beta) = (0, \chi(X_\alpha, X_\beta) + \alpha(X_\beta) - \beta(X_\alpha)).$

Chapter 4

Weak infinitesimal symmetries of local bundle gerbes

This short chapter is an addendum to the precedent one, dedicated to computing the data of a weak infinitesimal symmetry of a local bundle gerbe (see Remark 2.2.5), and showing that we recover the *infinitesimal quantomorphisms* of [FRS14], already mentioned in Section 1.3.

Let (M, ω) be a 2-plectic manifold. We trivialize ω on the elements of a good open cover $\mathcal{U} = \{U_i\}_{i \in I}$ of M

- $\omega|_{U_i} = d\theta_i$ with $\theta_i \in \Omega^2(U_i)$
- $\theta_j - \theta_i = dA_{ij}$ with $A_{ij} \in \Omega^1(U_{ij})$
- $A_{ij} + A_{jk} + \text{dlog}(h_{ijk}) = A_{ik}$ with $h_{ijk} \in C^\infty(U_{ijk}, S^1)$.

Consider the local bundle gerbe $(\coprod U_{ij} \times S^1, \coprod U_i)$ over M , with bundle gerbe product

$$m((x_{jk}, z), (x_{ij}, z')) = (x_{ik}, zz'h_{ijk}(x)),$$

with $x = x_{ij} = x_{jk} = x_{ik} \in U_{ijk}$ and $z, z' \in S^1$. Note that we have

$$m = (m^{\check{C}} \circ (p_1, p'_1), m^{S^1}(p_2, m^{S^1} \circ (p'_2, h_{ijk} \circ \pi \circ p_1)))$$

where $m^{\check{C}}$ is the multiplication of the Čech groupoid $\coprod U_{ij} \rightrightarrows \coprod U_i$, $\pi : \coprod U_{ij} \rightarrow M$ is the natural projection, m^{S^1} is the multiplication of S^1 , and $p_1, p'_1 : (\coprod U_{ij} \times S^1) \times (\coprod U_{ij} \times S^1) \rightarrow \coprod U_{ij}$ and $p_2, p'_2 : (\coprod U_{ij} \times S^1) \times (\coprod U_{ij} \times S^1) \rightarrow S^1$ are projections on the respective first and second factors of $\coprod U_{ij}$ and S^1 . A tangent vector $\eta_z \in T_z S^1$, with $z \in S^1$, is always of the

form $\eta_z = c \cdot \mathfrak{z}|_z$ with $c \in \mathbb{R}$ and \mathfrak{z} the rotation vector field on S^1 . Therefore tangent vectors $(\xi_{jk}, c \cdot \mathfrak{z}|_z) \in T_x U_{jk} \times T_z S^1$ and $(\eta_{ij}, c' \cdot \mathfrak{z}|_{z'}) \in T_x U_{ij} \times T_{z'} S^1$ are composable if $\xi = \eta$ (as elements of $T_x M$) and then we have

$$\begin{aligned} m_*((\xi_{jk}, c \cdot \mathfrak{z}|_z), (\xi_{ij}, c' \cdot \mathfrak{z}|_{z'})) &= (\xi_{ik}, m_*^{S^1}(c \cdot \mathfrak{z}|_z, m_*^{S^1}(c' \cdot \mathfrak{z}|_{z'}, (h_{ijk})_*(\xi))) \\ &= (\xi_{ik}, m_*^{S^1}(c \cdot \mathfrak{z}|_z, c' \cdot \mathfrak{z}|_{z' h_{ijk}(x)} + (L_{z'})_*(h_{ijk})_*(\xi))) \\ &= (\xi_{ik}, m_*^{S^1}(c \cdot \mathfrak{z}|_z, c' \cdot \mathfrak{z}|_{z' h_{ijk}(x)} + \text{dlog} h_{ijk}(\xi) \cdot \mathfrak{z}|_{z' h_{ijk}(x)})) \\ &= (\xi_{ik}, (c + c' + \text{dlog} h_{ijk}(\xi)) \cdot \mathfrak{z}|_{zz' h_{ijk}(x)}). \end{aligned}$$

Consider now the connection 1-form C on $\coprod U_{ij} \times S^1$, given by $C_{ij} = A_{ij} - \Theta$, where Θ is the Maurer-Cartan form on S^1 . We compute

$$m^*(C)((\xi_{jk}, c \cdot \mathfrak{z}|_z), (\xi_{ij}, c' \cdot \mathfrak{z}|_{z'})) = A_{ik}(\xi_{ik}) - (c + c' + \text{dlog} h_{ijk}(\xi))$$

and moreover

$$(pr_1^*(C) + pr_2^*(C))((\xi_{jk}, c \cdot \mathfrak{z}|_z), (\xi_{ij}, c' \cdot \mathfrak{z}|_{z'})) = A_{ij}(\xi_{ij}) + A_{jk}(\xi_{jk}) - (c + c'),$$

where $pr_i : (\coprod U_{ij} \times S^1)_s \times_t (\coprod U_{ij} \times S^1) \rightarrow \coprod U_{ij} \times S^1$, $i = 1, 2$ are the projections on the i^{th} factor. We obtain that $m^*C = pr_1^*C + pr_2^*C$ and therefore C defines a multiplicative connection. Moreover $dC = (t^* - s^*)\theta$, because $d\Theta = 0$ and $\theta_j - \theta_i = dA_{ij}$ on U_{ij} . Therefore (C, θ) gives a connective structure on $(\coprod U_{ij} \times S^1, \coprod U_i)$, with 3-curvature ω .

Now consider a multiplicative vector field (ξ, X) on the Lie groupoid $\coprod U_{ij} \times S^1 \rightrightarrows \coprod U_i$. Then ξ is a collection of $\xi_{ij} = (\xi_{ij}^1, \xi_{ij}^2)$ with $\xi_{ij}^1(x, z) \in T_x U_{ij}$ and $\xi_{ij}^2(x, z) \in T_z S^1$, for all $x \in U_{ij}$ and $z \in S^1$, and X is a collection of $X_i \in \mathfrak{X}(U_i)$. Since (ξ, X) is multiplicative, we have

$$s_*(\xi_{ij}(x, z)) = X_j(x) \text{ and } t_*(\xi_{ij}(x, z)) = X_i(x).$$

Therefore considering $\pi \circ s = \pi \circ t$ for $\pi : \coprod U_i \rightarrow M$ the natural projection, we obtain $X_i = X_j$ on U_{ij} , and therefore X is a globally defined vector field on M , and moreover $\xi_{ij}^1(x, z) = X(x)$ for all $x \in U_{ij}$ and $z \in S^1$.

Now, considering that ξ is S^1 -invariant, and upon trivializing the bundle TS^1 , we obtain the description $\xi_{ij} = (X, f_{ij} \cdot \mathfrak{z})$ with $f_{ij} \in C^\infty(U_{ij})$. The multiplicativity condition gives

$$m_*((X(x), f_{jk}(x) \cdot \mathfrak{z}|_z), (X(x), f_{ij}(x) \cdot \mathfrak{z}|_{z'})) = (X(x), f_{ik}(x) \cdot \mathfrak{z}|_{zz' h_{ijk}(x)})$$

for all $x \in U_{ijk}$. But we know that

$$\begin{aligned} & m_*((X(x), f_{jk}(x) \cdot \mathfrak{z}|_z), (X(x), f_{ij}(x) \cdot \mathfrak{z}|_{z'})) \\ &= (X(x), (f_{jk}(x) + f_{ij}(x) + \mathrm{d}\log h_{ijk}(X)) \cdot \mathfrak{z}|_{zz'h_{ijk}(x)}) \end{aligned}$$

and therefore the f_{ij} 's must satisfy the condition

$$f_{ij} + f_{jk} + \mathrm{d}\log h_{ijk}(X) = f_{ik}.$$

Moreover, if (ξ, X, B) is a weak infinitesimal symmetry, then B is a collection of $B_i \in \Omega^1(U_i)$ such that

- $\mathcal{L}_\xi(A_{ij} - \Theta) = B_j - B_i$
- $\mathcal{L}_X \theta_i = dB_i$.

We compute

$$\begin{aligned} \mathcal{L}_\xi(A_{ij} - \Theta) &= \mathcal{L}_{(X, f_{ij}, \mathfrak{z})}(A_{ij} - \Theta) \\ &= \mathcal{L}_X A_{ij} - df_{ij}. \end{aligned}$$

Therefore the conditions above may be rewritten as

- $\mathcal{L}_X A_{ij} = B_j - B_i + df_{ij}$
- $\mathcal{L}_X \theta_i = dB_i$.

This shows that *infinitesimal quantomorphisms* in [FRS14] correspond to the data of weak infinitesimal symmetries of the prequantizing local bundle gerbe (of course, our B_i are called α_i there and in Theorem 1.3.5).

Chapter 5

Lagrangian submanifolds of standard multisymplectic manifold

This chapter is taken from an article written in collaboration with Tilmann Wurzbacher : "Lagrangian submanifolds of standard multisymplectic manifolds". This article appeared in : *Geometric and Harmonic Analysis on Homogeneous Spaces. TJC 2017. Springer Proceedings in Mathematics and Statistics, vol 290.*

Introduction

It is well-known that Lagrangian submanifolds play a central role in symplectic geometry. This can easily be traced back to the search for so-called "generating functions" of (local) symplectomorphisms in the framework of the Hamilton-Jacobi method for integrating Hamilton's equation (see the classical reference [Arn89], Sections 47-48). This method is closely connected to the observations that the graph of a diffeomorphism between two symplectic manifolds is Lagrangian if and only if the diffeomorphism is symplectic and that the image of a one-form is Lagrangian (inside the cotangent bundle) if and only if the form is closed. Alan Weinstein deduced from such classical facts his famous symplectic creed: "Everything is a Lagrangian submanifold". Of course, from a modern perspective the main argument for this creed is ... Weinstein's fundamental result from 1971 (see [Wei71]):

Weinstein's normal form theorem. *Let L be a Lagrangian submanifold of a symplectic manifold (M, ω) . Then there exist open neighborhoods U and*

V of L in M respectively T^*L , and a diffeomorphism $\phi : U \rightarrow V$ such that $\phi|_L = id_L$ and $\phi^*(\omega^{T^*L}) = \omega$ on U .

Classical mechanics is geometrized by the Hamiltonian approach on cotangent bundles and more generally on symplectic manifolds, whereas its higher dimensional analogue, classical field theory, can be formulated in a Hamiltonian way on multicotangent or jet bundles, and leads more generally to multisymplectic manifolds (cf. [RW19], Section 2 for a recent account of this). A multisymplectic manifold is a manifold together with a nondegenerate, closed $(k+1)$ -form ω with $k \in \mathbb{N}$; $k = 1$ being the symplectic case.

In a 1988 article ([Mar88]) Geoffrey Martin extended Weinstein's result to an important class of multisymplectic manifolds including multicotangent bundles. (Note that he reserves the term "multisymplectic" for the class of multisymplectic manifolds where his theorem applies.) The proof of his main result (Lemma 2.1) being rather cryptic, and in parts being reduced to mere hints for the reader, his precocious results fell into oblivion, not receiving the deserved attention.

The spanish school on differential-geometric methods in mathematical physics revived multisymplectic geometry (in its modern definition) at the end of the last century, and Manuel de León, David Martín de Diego and Aitor Santamaría-Merino gave in [dds03] a rather detailed framework for Martin's normal form theorem. Unfortunately, the necessary condition that a certain naturally associated subbundle of the tangent bundle of the ambient manifold should be integrable is not emphasised in their proof of Martin's main result (see the proof of Lemma 3.24 in the cited article).

Since multisymplectic geometry is by now emerging fast as the "right" (higher) geometric formulation of classical field theory, thanks to the advent of rather well-suited homotopical and homological methods, the interest in Martin's result is growing and we felt compelled to give a self-contained, detailed account of his result and techniques. It turns out that one crucially needs "folkloristic" extensions of two standard theorems in differential topology to a foliated setting (these being of independent interest, in fact). Once established, Martin's ingenious idea that the path method of Jürgen Moser (see [Mos65]) applies though a multisymplectic form of degree $k+1$ does not yield an isomorphism between the tangent bundle and the bundle of k -forms, goes through and yields the following result:

Martin's normal form theorem (Theorem 1 below). *Let (M, ω) be*

a standard k -plectic manifold, with $k > 1$. Let the distribution $W \subset TM$ be defined as $W := \cup_{p \in M} W_\omega(p)$ and let L be a k -Lagrangian submanifold of M complementary to W (that is $T_p M = T_p L \oplus W|_p$, $\forall p \in L$). If W is integrable, there exist open neighborhoods U and V of L in M and $\Lambda^k(T^*L)$, and a diffeomorphism $\phi : U \rightarrow V$ such that:

$$\phi|_L = id_L \text{ and } \phi^*(\omega^{\Lambda^k(T^*L)}) = \omega \text{ on } U.$$

Results of a related but more global nature were obtained by Frans Cantrijn, Alberto Ibort and Manuel de León in 1999 (see Theorem 7.3 in [CId99]) and Michael Forger and Sandra Z. Yepes in 2013 (see Theorem 7 in [FY13]). In both cases the focus is shifted from the local situation near a Lagrangian submanifold to the foliation associated to an involutive Lagrangian distribution and its leaf space, implying an important role for regularity assumptions on the foliation, and for connections on the leaves.

We conclude the introduction by summarising the chapter's content. In Section 5.1 we give the basic definitions, as multisymplectic vector spaces and manifolds and their isotropic and Lagrangian subspaces respectively submanifolds. We also give here some examples of isotropic and Lagrangian submanifolds of multisymplectic manifolds. Section 5.2 introduces the notions of "standard" multisymplectic vector spaces and manifolds, central for this article. We prove the fundamental properties of a standard multisymplectic vector space (V, ω) (with ω a $(k+1)$ -linear form and $k > 1$), notably the existence of a unique subspace $W \subset V$ that is isomorphic to $\Lambda^k(V/W)^*$ via the natural contraction map (compare Lemma 5.2.2 and Proposition 5.2.4). On the level of manifolds, we explain why multicotangent bundles are standard multisymplectic manifolds. In Section 5.3 we give a detailed proof of Martin's normal form theorem (see above), expanding and explaining Martin's extremely brief original proof. In an Appendix of this chapter we give complete proofs for the extension of two classical differential-topological results to foliated manifolds, more precisely, we show a foliated tubular neighborhood theorem and a foliated relative Poincaré lemma (see Theorems 5.4.1 and 5.4.3).

5.1 Multisymplectic vector spaces and manifolds, and Lagrangian submanifolds

In this section, we give the basic definitions used in the chapter, together with some examples. We will work over the real numbers and all manifolds will be smooth. The algebraic considerations for vector spaces hold true over fields of characteristic zero instead of the reals.

Definition 5.1.1. Let V be a vector space, $k \geq 1$ and $\omega \in \Lambda^{k+1}(V^*)$. We say that (V, ω) is a k -plectic vector space (or simply a multisymplectic vector space) if ω is nondegenerate, in the sense that :

$$\omega^\sharp : V \rightarrow \Lambda^k(V^*); v \mapsto \iota_v \omega$$

is injective.

As in the symplectic case, we can define orthogonal subspaces with respect to ω , but in this setting we have more than just one " ω -orthogonal complement" for a given subspace of V :

Definition 5.1.2. Let (V, ω) be a k -plectic vector space, $U \subset V$ a subspace and $1 \leq j \leq k$. We define the j -th orthogonal complement of U with respect to ω as follows:

$$U^{\perp, j} := \{v \in V \mid \iota_v \wedge u_1 \wedge \dots \wedge u_j \omega = 0, \forall u_1, \dots, u_j \in U\}.$$

We say that $U \subset V$ is a j -isotropic subspace (respectively, a j -Lagrangian subspace) if $U \subset U^{\perp, j}$ (respectively if $U = U^{\perp, j}$).

Going to manifolds we have:

Definition 5.1.3. Let M be a manifold and $\omega \in \Lambda^{k+1}T^*M$. We say that (M, ω) is a k -plectic manifold, or simply a multisymplectic manifold, if the form ω is closed and nondegenerate, in the sense that for all $q \in M$, the map :

$$\omega_q^\sharp : T_q M \rightarrow \Lambda^k(T_q^* M); v_q \mapsto \iota_{v_q} \omega_q$$

is injective.

Analogously to the linear case, we will say that a regular submanifold L is a j -isotropic respectively j -Lagrangian submanifold of M , if, for each $p \in L$, $T_p L$ is a j -isotropic respectively j -Lagrangian subspace of $T_p M$.

Before studying a special class of multisymplectic manifolds in Sections 5.2 and 5.3, we will give general examples of multisymplectic manifolds and isotropic submanifolds. Note that if N is a submanifold of M of dimension n , then N is j -isotropic for all $j \geq n$ in a trivial way. Thus in the following examples, we will only consider "interesting" isotropic and Lagrangian submanifolds, where this is not the case.

Example 5.1.4. Let M be an orientable manifold of dimension m and ω a volume form on M . Then (M, ω) is a $(m-1)$ -plectic manifold. In this case there are no non-trivial examples (in the sense stated above) of isotropic submanifolds of M .

Example 5.1.5. Let Q be a manifold, $k \geq 1$ and the dimension of Q being greater or equal to $k+1$. Then the manifold $M := \Lambda^k(T^*Q)$ is naturally equipped with a k -plectic form. Indeed let $\theta \in \Omega^k(M)$ be defined by :

$$\theta_{\alpha_p}(v_1, \dots, v_k) := \alpha_p(\pi_{*\alpha_p}(v_1), \dots, \pi_{*\alpha_p}(v_k)),$$

where $\alpha_p \in M$, $v_j \in T_{\alpha_p}(M)$, and $\pi : M \rightarrow Q$ is the canonical projection. Then $\omega := -d\theta$ is a k -plectic form on M . This construction is the generalization of the symplectic form on a cotangent bundle. The zero-section of $\Lambda^k(T^*M)$ is a k -Lagrangian manifold, and the fibers of π are 1-Lagrangian. To see this, we can work in local coordinates. A direct computation shows then that if (q^i) are coordinates on an open subset $U \subset Q$ and (p_i) are coordinates on the fibers of $\Lambda^k(T^*U)$, we have :

$$\omega|_U = - \sum_{i_1, \dots, i_k} dp_{i_1, \dots, i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k}.$$

Using this local description it is easy to see that Q is k -Lagrangian and the fibers are 1-Lagrangian. More generally, for $\alpha \in \Omega^k(Q)$, we have that $im(\alpha) \subset M$ is a k -Lagrangian manifold if and only if α is closed. This follows from $\alpha^*\theta = \alpha$ (where, on the left-side, α is regarded as a map $\alpha : Q \rightarrow M$), implying $\alpha^*\omega = -d\alpha$.

Example 5.1.6. Let (M, η) be a k -plectic manifold and $\omega \in \Omega^{k+1}(M \times M)$ the form given by:

$$\omega = p_1^*\eta - p_2^*\eta,$$

where for $i = 1, 2$, the map p_i is the projection $p_i : M \times M \rightarrow M$ on the i -th factor. Then $(M \times M, \omega)$ is a k -plectic manifold. Considering a diffeomorphism $\phi : M \rightarrow M$, we claim that Γ_ϕ , the graph of ϕ , is k -Lagrangian

if and only if ϕ is a symplectomorphism in the sense that $\phi^*\eta = \eta$. Indeed $T_{(q,\phi(q))}(M \times M) = \{(u_q, \phi_{*q}(u_q)) \mid u_q \in T_q M\}$. Then for $(u_i, \phi_*(u_i)) \in T_{(q,\phi(q))}(M \times M)$ ($1 \leq i \leq k$) we obtain:

$$\begin{aligned} \omega_{(q,\phi(q))}((u_1, \phi_*(u_1)), \dots, (u_k, \phi_*(u_k))) &= \eta_q(u_1, \dots, u_k) - \eta_{\phi(q)}(\phi_*(u_1), \dots, \phi_*(u_k)) \\ &= \eta_q(u_1, \dots, u_k) - (\phi^*\eta)_q(u_1, \dots, u_k), \end{aligned}$$

showing the claim.

Example 5.1.7. Let M be a complex manifold with a holomorphic volume form Ω . Then setting $\omega = \Re(\Omega)$, the real part of Ω , turns (M, ω) into a multisymplectic manifold. To get a feeling of how Lagrangian submanifolds may look in this case, we consider $M = \mathbb{C}^3 = \mathbb{R}^6$ and $\Omega = dz^1 \wedge dz^2 \wedge dz^3 = dz^{123}$. We find:

$$\omega = dx^{123} - dx^{156} - dx^{246} - dx^{345},$$

where we have omitted wedge products and x^i are coordinates in \mathbb{R}^6 . Then the manifold $\{x^1 = x^2 = x^3 = 0\}$ is 2-Lagrangian, and the manifold $\{x^2 = x^3 = x^5 = x^6 = 0\}$ is 1-Lagrangian.

Example 5.1.8. (Compare [CI99], Section 3). Let $M = \mathbb{R}^6$ and

$$\omega = dx^{145} + dx^{246} + dx^{356} + dx^{456}.$$

Then ω is a 2-plectic form, and $L_2 = \{x^1 = x^3 = x^4 = x^6 = 0\}$ and $L_3 = \{x^4 = x^5 = x^6 = 0\}$ are (linear) 1-Lagrangian submanifolds of (M, ω) of different dimensions. (Note that (\mathbb{R}^6, ω) is symplectomorphic to the multicotangent bundle $\Lambda^2(T^*\mathbb{R}^3)$ with the multisymplectic form defined in Example 2 above.)

Example 5.1.9. Let G be a real semi-simple, compact Lie group. Consider the Cartan form $\omega \in \Omega^3(G)$, which is the bi-invariant form defined at the neutral element e by :

$$\omega_e(\xi, \eta, \zeta) := \langle [\xi, \eta], \zeta \rangle,$$

where $\xi, \eta, \zeta \in \mathfrak{g}$ (the Lie algebra of G) and $\langle \cdot, \cdot \rangle$ is the Killing form. The form ω is closed because it is bi-invariant, and it is nondegenerate because the Killing form is nondegenerate and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Consider $T \subset G$, a torus. Its Lie algebra \mathfrak{t} is abelian and thus T is 1-isotropic. Thus if T is a maximal torus then it is 1-Lagrangian.

5.2 Standard multisymplectic vector spaces and manifolds

In this section, we will be interested in a special class of multisymplectic vector spaces and manifolds, important in applications of multisymplectic geometry to classical field theories.

Definition 5.2.1. Let V be a vector space and $k > 1$. We say that V is a standard k -plectic vector space, if (V, ω) is a k -plectic vector space and there exists a subspace $W \subset V$ such that:

- (1) $\forall u, v \in W, \iota_{u \wedge v} \omega = 0$
- (2) $\dim(W) = \dim(\Lambda^k((V/W)^*))$
- (3) $\text{codim}(W) > k$.

Let us also consider the following condition :

$$\dim(W) \geq \text{codim}(W). \quad (3')$$

Concentrating on the higher degree cases ($k > 1$) we then have the following relations between these conditions:

Lemma 5.2.2. *Let (V, ω) be a k -plectic vector space with $k > 1$. Then :*

- (i) *conditions (1) and (2) imply that the map $\omega^\sharp|_W : W \rightarrow \Lambda^k V^*$ induces a linear isomorphism :*

$$\chi : W \rightarrow \Lambda^k(V/W)^*,$$

- (ii) *if conditions (1) and (2) are satisfied, then condition (3) is equivalent to condition (3'),*

- (iii) *if (V, ω) is standard, then $\dim(W) \geq 2$.*

Remark 5.2.3. In reference [ddS03] these multisymplectic vector spaces are called of type $(k+1, 0)$.

Proof. We denote the projection $V \rightarrow V/W$ by π . Then the subspace $\pi^*(\Lambda^k(V/W)^*) \subset \Lambda^k V^*$ is given by :

$$\pi^*(\Lambda^k(V/W)^*) = \{\eta \in \Lambda^k V^* \mid \iota_v \eta = 0, \forall v \in W\}.$$

By condition (1), $\omega^\sharp(w) \in \pi^*(\Lambda^k(V/W)^*)$ whenever $w \in W$; thus ω^\sharp induces a unique injective linear map $\chi : W \rightarrow \Lambda^k(V/W)^*$ such that $\pi^* \circ \chi = \omega^\sharp|_W$.

Moreover, χ is a linear isomorphism by condition (2) ; thus proving the first assertion.

Now put $d = \dim(W)$, $c = \text{codim}(W)$. Then $\dim(V) = c + d$. Assume conditions (1) – (3) to hold, and $d < c$. By condition (2), $d = \binom{c}{k}$, thus $d = \frac{c(c-1)\dots(c-(k-1))}{k(k-1)\dots 1} \geq c$. This contradiction shows that conditions (1) – (3) imply condition (3').

Now assume conditions (1), (2), (3') to hold and $c \leq k$. Since $c < k$ is easily seen to contradict (2), then $c = k$ and $d = 1$. By (3'), $\dim(V) = 2$ and thus $1 = \dim(\Lambda^k(V/W)^*)$, implying $k = 1$, contradicting the assumptions. Thus the conditions (1), (2), (3') imply conditions (1) – (3), and the second assertion is proven.

Now assume that $d \leq 1$. If $d = 0$, then $\Lambda^k V^* = \{0\}$, contradicting the fact that ω is nondegenerate. Now if $d = 1$, we have $1 = \binom{\dim(V)-1}{k}$ and therefore $k+1 = \dim(V)$, implying $c = k$; this contradiction proves the last assertion. \square

If (V, ω) is a standard k -plectic vector space, with $k > 1$, then the subspace W satisfying Definition 5.2.1 is unique as shows the following :

Proposition 5.2.4. *Let (V, ω) be a standard k -plectic vector space, with $k > 1$, and W, \widetilde{W} two subspaces satisfying Definition 5.2.1. Then $W = \widetilde{W}$.*

Proof. First we show that $W \cap \widetilde{W}$ has codimension at most 1 in \widetilde{W} . To do this, assume the opposite : $\text{codim}_{\widetilde{W}}(W \cap \widetilde{W}) > 1$. Then, there exists linearly independent vectors u, v of \widetilde{W} such that $\text{span}(u, v) \cap W = \{0\}$; thus we can find $\eta \in \Lambda^k(V/W)^*$ such that $\iota_{u \wedge v} \eta \neq 0$. But, for all $w \in W$, $\iota_w \iota_{u \wedge v} \omega = 0$, so there cannot exist a $w \in W$ such that $\eta = \iota_w \omega$, and this contradicts the fact that the map χ is an isomorphism.

Now suppose $W \neq \widetilde{W}$. Then there exists a non-zero vector $z \in \widetilde{W}$ such that $\text{span}(z) \cap W = \{0\}$. For all $w \in W \cap \widetilde{W}$, $\eta \in \Lambda^{k-1}(V/W)$:

$$\chi^*(\pi(z) \wedge \eta)(w) = (\pi(z) \wedge \eta)(\chi(w)) = \omega(w, z, \eta) = 0,$$

where χ^* denotes the dual of the map χ , and $\pi : V \rightarrow V/W$ is the canonical projection. The above equation is well-defined because for $w \in W$, $\iota_w \omega$ depends only on its evaluation on element of $\Lambda^k(V/W)$, because of condition (1) in Definition 5.2.1. Denote $Z = \text{span}(z)$. The above computation shows that:

$$\chi^*(\pi(Z) \wedge \Lambda^{k-1}(V/W)) \subset \text{ann}_{W^*}(W \cap \widetilde{W}),$$

where $\text{ann}_{W^*}(W \cap \widetilde{W}) = \{\eta \in W^* \mid \eta(w) = 0, \forall w \in W \cap \widetilde{W}\}$. This implies, together with $\text{codim}_{\widetilde{W}}(W \cap \widetilde{W}) \leq 1$, that $\dim(\chi^*(\pi(Z) \wedge \Lambda^{k-1}(V/W))) \leq 1$. Furthermore :

$$\dim(\pi(Z) \wedge \Lambda^{k-1}(V/W)) = \dim(\Lambda^{k-1}(V/W)) > 1,$$

because $\text{codim}(W) > k$. This shows a contradiction, and thus the Proposition.

□

The preceding proposition allows to denote such a subspace by W_ω and motivates the next definition :

Definition 5.2.5. Let M be a manifold, $k > 1$, and $\omega \in \Lambda^{k+1}(T^*M)$. We say that (M, ω) is a standard k -plectic manifold if (M, ω) is a k -plectic manifold and if for each $p \in M$, $(T_p M, \omega_p)$ is a standard k -plectic vector space. For all $p \in M$ the unique subspace of $T_p M$ satisfying Definition 5.2.1 is denoted by $W_\omega(p)$ or simply $W(p)$.

The remainder of this section is dedicated to showing that standard multi-symplectic vector spaces are in fact symplectomorphic to a canonical k -plectic model that we will describe now.

Proposition 5.2.6. *Let (V, ω) be a standard k -plectic vector space. Then the subspace W_ω is 1-Lagrangian. Moreover, there exists a k -Lagrangian vector space $L \subset V$ complementary to W_ω and the map χ induces (for all choices of such L) an isomorphism:*

$$W_\omega \cong \Lambda^k(L^*).$$

Proof. Condition (1) in Definition 5.2.1 implies that W_ω is 1-isotropic. Now if $w \in W_\omega^{\perp, 1}$ but $w \notin W_\omega$, then we can find $\eta \in \Lambda^k(V/W_\omega)^*$ such that $\iota_w \eta \neq 0$. But, for all $u \in W_\omega$, $\iota_u \iota_w \omega = 0$ and thus there cannot exist a $u \in W_\omega$ such that $\eta = \iota_u \omega$. This property contradicts the fact that the map χ is an isomorphism, and therefore proves that W_ω is 1-Lagrangian.

Now let \widetilde{L} be any subspace complementary to W_ω . We may canonically identify V/W_ω and \widetilde{L} since the restriction to \widetilde{L} of the projection $\pi : V \rightarrow V/W_\omega$ is an isomorphism. Thus we have a canonical isomorphism $\chi : W_\omega \xrightarrow{\cong} \Lambda^k(\widetilde{L}^*)$. We will search for a k -Lagrangian complement of the form $L = \{v + Av \mid v \in \widetilde{L}\}$ for some linear map $A : \widetilde{L} \rightarrow W_\omega$. For L to be k -Lagrangian, it has to verify that $L \subset L^{\perp, k}$, i.e., for all $v_j \in \widetilde{L}$, $j = 1, \dots, k+1$:

$$\omega(v_1 + Av_1, \dots, v_{k+1} + Av_{k+1}) = 0.$$

This condition suffices here because assuming that there exists an element $u \in L^{\perp,k} \setminus L$, we may write $u = v + w$, with $v \in L$ and $w \in W_\omega$. For $u_1, \dots, u_k \in L$ we compute then:

$$\begin{aligned}\omega(u, u_1, \dots, u_k) &= \omega(v, u_1, \dots, u_k) + \omega(w, u_1, \dots, u_k) \\ &= \omega(w, u_1, \dots, u_k) = 0\end{aligned}$$

because $v \in L \subset L^{\perp,k}$, and $u \in L^{\perp,k}$. Thus we obtain $w \in W_\omega \cap L^{\perp,k}$. But then for all $v_j = x_j + y_j$, $v_j \in V$, $x_j \in L$ and $y_j \in W_\omega$:

$$\begin{aligned}\omega(w, v_1, \dots, v_k) &= \omega(v, u_1, \dots, u_k) + \omega(w, u_1, \dots, u_k) \\ &= 0.\end{aligned}$$

By the nondegeneracy of ω , we obtain $w = 0$, thus $u = v \in L$ which is a contradiction to $u \in L^{\perp,k} \setminus L$.

Now we return to the construction of the linear map A . We have:

$$\begin{aligned}\omega(v_1 + Av_1, \dots, v_{k+1} + Av_{k+1}) &= \omega(v_1, \dots, v_{k+1}) \\ &\quad + \sum_{j=1, \dots, k+1} (-1)^{j+1} \omega(Av_j, v_1, \dots, \widehat{v}_j, \dots, v_{k+1}).\end{aligned}$$

Let $\Phi := \chi \circ A$, then the Lagrangian condition reads as follows:

$$\omega(v_1, \dots, v_{k+1}) = - \sum_{j=1, \dots, k+1} (-1)^{j+1} \Phi(v_j)(v_1, \dots, \widehat{v}_j, \dots, v_{k+1}).$$

We denote by T the application $T : \widetilde{L} \rightarrow \Lambda^k(\widetilde{L}^*)$, $v \mapsto \iota_v \omega$. If $\Phi = -\frac{1}{k+1} T$, then the above condition is verified. Thus the map $A := (\chi)^{-1} \circ \Phi$ has the property that its graph $L = \Gamma_\Phi$ is a k -Lagrangian space, complementary to W_ω . \square

Definition 5.2.7. Let V be a vector space and ω_{can} the canonical $(k+1)$ -form on the space $\mathcal{V} := V \oplus \Lambda^k(V^*)$ given by:

$$\omega_{can}(v_1 \oplus \alpha_1, \dots, v_{k+1} \oplus \alpha_{k+1}) = \sum_{i=1}^k (-1)^{i+1} \alpha_i(v_1, \dots, v_{i-1}, \widehat{v}_i, v_{i+1}, \dots, v_{k+1}),$$

for all $v_j \in V$, and $\alpha_j \in \Lambda^k(V^*)$. Then ω_{can} is a k -plectic form. We call $(\mathcal{V}, \omega_{can})$ a canonical k -plectic vector space.

Lemma 5.2.8. *Let (V, ω) be a k -plectic vector space with $k > 1$. Then (V, ω) is isomorphic to a canonical k -plectic vector space $(L \oplus \Lambda^k(L^*), \omega_{can})$ if and only if (V, ω) is standard.*

Proof. Let (V, ω) be a standard k -plectic vector space, and L a k -Lagrangian subspace complementary to W_ω . We define ω_{can} as above on the space $L \oplus \Lambda^k(L^*)$. Let $\gamma := id_L \oplus \chi : L \oplus W_\omega \rightarrow L \oplus \Lambda^k(L^*)$, where we again canonically identified V/W_ω and L . Then γ is a linear isomorphism and furthermore we find:

$$\begin{aligned} \omega_{can}(\gamma(u_1 \oplus w_1), \dots, \gamma(u_{k+1} \oplus w_{k+1})) &= \omega_{can}(u_1 \oplus \iota_{w_1}\omega, \dots, u_{k+1} \oplus \iota_{w_{k+1}}\omega) \\ &= \sum_{j=1 \dots k+1} (-1)^{j+1} \iota_{w_j}\omega(u_1, \dots, \widehat{u_j}, \dots, u_{k+1}) \\ &= \omega(u_1 \oplus w_1, \dots, u_{k+1} \oplus w_{k+1}), \end{aligned}$$

i.e. $\gamma^*\omega_{can} = \omega$. Therefore, the k -plectic space (V, ω) is symplectomorphic to $(L \oplus \Lambda^k(L^*), \omega_{can})$.

To show the converse, first note that if $(L \oplus \Lambda^k(L^*), \omega_{can})$ is a canonical k -plectic vector space, then the space L identified with $L \times \{0\} \subset L \oplus \Lambda^k(L^*)$, is k -Lagrangian, and the space $\Lambda^k(L^*)$, identified with $\{0\} \times \Lambda^k(L^*) \subset L \oplus \Lambda^k(L^*)$ is 1-Lagrangian. Indeed :

$$\omega_{can}((v_1, 0), \dots, (v_{k+1}, 0)) = 0,$$

and if $\omega_{can}((v, \alpha), (v_1, 0), \dots, (v_k, 0)) = 0$ for all $v_j \in L$, then $\alpha(v_1, \dots, v_k) = 0$ and then $\alpha = 0$. This shows that L is k -Lagrangian. Moreover :

$$\omega_{can}((0, \alpha), (0, \beta), (v_1, \gamma_1), \dots, (v_{k-2}, \gamma_{k-2})) = 0,$$

and if $\omega_{can}((v, \alpha), (0, \beta), (v_1, \gamma_1), \dots, (v_{k-2}, \gamma_{k-2})) = 0$ for all $v_j \in V$ and $\gamma_j \in \Lambda^k(L^*)$, then $\iota_v\beta = 0$ and thus $v = 0$. This shows that $\Lambda^k(L^*)$ is 1-Lagrangian. Thus, if a k -plectic linear space (V, ω) is symplectomorphic to a space $(L \oplus \Lambda^k(L^*), \omega_{can})$, then, pulling back the 1-Lagrangian space $\Lambda^k(L^*)$ to V with this symplectomorphism gives a linear subspace $W \subset V$ satisfying Definition 5.2.1. \square

Remark. Consider now a manifold Q , $k > 1$ and $(\Lambda^k(T^*Q), \omega^{\Lambda^k(T^*Q)})$ the k -plectic structure exposed in Example 5.1.5. We have :

$$T(\Lambda^k(T^*Q))|_Q = TQ \oplus \Lambda^k(T^*Q).$$

Recall that each fiber of $\Lambda^k(T^*Q)$ is 1-Lagrangian, and Q is k -Lagrangian. Thus, at each point $p \in Q$, the form $\omega^{\Lambda^k(T^*Q)}$ evaluated at the point p is in fact the canonical form on the space $T_pQ \oplus \Lambda^k(T_p^*Q)$. Using the coordinate expression of the form :

$$\omega^{\Lambda^k(T^*Q)}|_U = - \sum_{i_1, \dots, i_k} dp_{i_1, \dots, i_k} \wedge dq^{i_1} \wedge \dots \wedge dq^{i_k},$$

with $U \subset Q$ an open set, (q^i) coordinates on U and (p_i) coordinates on the fibers of T^*U , we see that at any point α_q of $\Lambda^k(T^*Q)$, the fiber $\Lambda^k(T_q^*Q)$ is 1-Lagrangian and the tangent space at α_q satisfies the conditions of Definition 5.2.1. This implies that $(\Lambda^k(T^*Q), \omega^{\Lambda^k(T^*Q)})$ is a standard k -plectic manifold.

5.3 Normal forms of Lagrangian submanifolds of standard multisymplectic manifolds

In this section we give a proof of the main result, that first appeared in equivalent form as Lemma 2.1 in Geoffrey Martin's 1988 article [Mar88]:

Theorem 5.3.1. *Let (M, ω) be a standard k -plectic manifold with $k > 1$. Let the distribution $W \subset TM$ be defined as $W := \cup_{p \in M} W_\omega(p)$ and let L be a k -Lagrangian submanifold of M complementary to W (that is $T_p M = T_p L \oplus W_p$ for all p in L). If W is integrable, there exist open neighborhoods U and V of L in M and $\Lambda^k(T^*L)$, and a diffeomorphism $\phi : U \rightarrow V$ such that:*

$$\phi|_L = id_L \text{ and } \phi^*(\omega^{\Lambda^k(T^*L)}) = \omega \text{ on } U.$$

Proof. It follows from the definition of a standard k -plectic manifold and the results found in the linear case, that if L is complementary to W we have an isomorphism of vector bundles :

$$\chi : W|_L \rightarrow \Lambda^k(T^*L),$$

which is given by contraction of vectors in W with ω , using the identification $TM|_L/W|_L \cong TL$. (Note that we write in the sequel often W_x for the fiber of the vector bundle $W \rightarrow M$ over x in M .)

Using this map we construct the following vector bundle isomorphism :

$$\Psi : TM|_L = TL \oplus W|_L \cong TL \oplus \Lambda^k(T^*L) = T(\Lambda^k(T^*L))|_L,$$

acting as the identity on vectors of TL , and transforming vectors of W via χ (i.e., $\Psi = id_{TL} \oplus \chi$). Furthermore, this map is for each $x \in L$ an isomorphism between the multisymplectic vector spaces $(T_x M, \omega_x)$ and $(T_x(\Lambda^k T^*L), \omega_x^{\Lambda^k T^*L})$. We now wish to find a diffeomorphism $f : U_1 \rightarrow U_2$, where U_1, U_2 are neighborhoods of L respectively in M and in $\Lambda^k(T^*L)$, such that $f|_L = id_L$ and for every $x \in L$, $T_x f = \Psi_x$. Such a map f then fulfills $(f^* \omega^{\Lambda^k(T^*L)})|_L = \omega|_L$.

By the foliated tubular neighborhood theorem, we may find a neighborhood U of L in M , a neighborhood V of the zero-section in $W|_L$, and a diffeomorphism ϕ , which is the identity along L , maps each leaf of the foliation

to a fiber of $W|_L \rightarrow L$, and has as its differential at any point of L the identity. Let $f := \chi \circ \phi$. Then f maps L to the zero section of $\Lambda^k(T^*L)$, and is a diffeomorphism onto an open subset $U' \subset \Lambda^k(T^*L)$ which contains L (as the zero section). Furthermore we have for $x \in L$, $T_x f|_{W_x} = \chi$ and $T_x f|_{T_x L} = id_{T_x L}$ (where we have identified $T_x(W_x)$ with W_x). Thus we obtain for $x \in L$, $T_x f = \Psi_x$, and upon putting $\tilde{\omega} := f^* \omega^{\Lambda^k(T^*L)}$, we arrive, by the above said, at $\tilde{\omega}|_L = \omega|_L$.

We now want to show that for any vector fields X, Y (defined in U) and tangent to W , $\iota_{X \wedge Y} \tilde{\omega} = 0$. Let $p \in U$ and F_p be the leaf of the foliation defined by W , which passes through p . This leaf also passes through a point of L , say x . Then ϕ maps this leaf to the space W_x , and thus f maps F_p to $\Lambda^k(T_x^*L)$. Thus if, X_p, Y_p are vectors in W_p , we may consider that $T_p f(X_p), T_p f(Y_p)$ are vectors of $\Lambda^k(T_x^*L)$. Since this space is 1-Lagrangian with respect to $\omega^{\Lambda^k(T^*L)}$, we find $\iota_{X \wedge Y} \tilde{\omega} = 0$.

Working on an open neighborhood U of L in M , we adapt now the well-known Moser path method (see [Mos65]) to our situation. Let $\omega_t = \omega + t(\tilde{\omega} - \omega)$ for $t \in [0, 1]$. Then we have $\omega_t|_L = \omega|_L$ and $\frac{\partial}{\partial t} \omega_t = \tilde{\omega} - \omega =: \omega'$ so $\omega'|_L = 0$. Thus for $x \in L$, $\omega_t(x)$ is nondegenerate for all $t \in [0, 1]$ and the set of points (t, x) such that $\omega_t(x)$ is nondegenerate is an open subset of $\mathbb{R} \times M$. So, shrinking U if necessary, we may suppose that ω_t is nondegenerate in U for all $t \in [0, 1]$. We also have that $d\omega_t = d\omega' = 0$. By the relative Poincaré lemma, there exist a neighborhood U of L in M and $\mu \in \Omega^k(U)$ with $d\mu = \omega'$ and $\mu|_L = 0$. Moreover -upon using Theorem 5.4.3- we can choose μ such that $\iota_v \mu = 0$ whenever $v \in W$, because ω' vanishes when contracted with two vectors of W (because both ω and $\tilde{\omega}$ have this property). Therefore μ may be interpreted as a section $U \rightarrow \Lambda^k(TM/W)^*$. Let us now take a look at the map :

$$\omega_t^\sharp : W \rightarrow \Lambda^k(T^*M),$$

given by contraction of ω_t with vectors of W . For $u, v \in W$, lying over the same point, $\iota_{u \wedge v} \omega_t = 0$; so ω_t^\sharp may be seen as a map :

$$\omega_t^\sharp : W \rightarrow \Lambda^k(TM/W)^* ; u \mapsto \iota_u \omega_t,$$

and, by the nondegeneration of ω_t , this map is injective, and thus is an isomorphism for dimensional reasons. Then for each $t \in [0, 1]$, there exists a unique vector field X_t (which take values in W) such that :

$$\iota_{X_t} \omega_t + \mu = 0$$

The association $(t, x) \rightarrow X_t(x)$ thus gives a (time-dependent) vector field tangent to W . But $\mu|_L = 0$ so we deduce that for $x \in L$, for all $t \in$

$[0, 1]$, $X_t(x) = 0$ by the nondegeneration of ω_t . Let ϕ_t be the curve of local diffeomorphisms tangent to X_t . We have $\phi_t|_L = id|_L$. So $\forall t \in [0, 1]$ $\phi_t|_L$ is defined. But if $D(\phi) \subset [0, 1] \times U$ is the domain of ϕ , then $[0, 1] \times L \subset D(\phi)$ so, by the openness of $D(\phi)$, we may suppose (again shrinking the domain U if necessary), that ϕ_t is defined in U for all $t \in [0, 1]$. Now we compute :

$$\begin{aligned} \frac{\partial}{\partial t} (\phi_t^* \omega_t) &= \phi_t^* \left(\frac{\partial}{\partial t} \omega_t + \mathcal{L}_{X_t} \omega_t \right) \\ &= \phi_t^* (\omega' - d\mu) = 0. \end{aligned}$$

Let $g := \phi_1$. Then $g^* \tilde{\omega} = \omega$, so if $\phi = g \circ f$ (where f is defined above), we obtain $\phi^* (\omega^{\wedge^k (T^*L)}) = \omega$, and maintain $\phi|_L = id_L$, concluding the proof. \square

5.4 Appendix: Two results in foliated differential topology

In this appendix we give proofs for two "folkloristic" but subtle (and useful) extensions of well-known results in differential topology. Both are used in [Mar88] but ask for a detailed proof. A brief sketch of a proof of the first result is given on the pages 88-89 in [CC00].

We begin with the tubular neighborhood theorem, in the presence of a foliation:

Theorem 5.4.1. *[Foliated tubular neighborhood theorem] Let M be a manifold, $W \subset TM$ an integrable distribution, and N a submanifold complementary to W in the sense that $W|_N \oplus TN = TM|_N$. Then there exist an open neighborhood U of N in M , and a diffeomorphism ϕ from U onto an open subset of $W|_N$ containing the zero section, such that $\phi|_N = id_N$, the differential of ϕ at any point of N is the identity, and ϕ maps for all p in N the leaf of the foliation defined by W passing through it to the fiber $\phi(U) \cap (W|_p)$ of $W|_N \rightarrow N$, intersected with $\phi(U)$.*

Proof. Let g be a fixed (auxiliary) Riemannian metric on the manifold M .

Given $q \in M$, the leaf \mathcal{W}_q of the foliation \mathcal{W} defined by the distribution W and containing q is given as an injectively immersed submanifold $j_q : F_q \rightarrow M$ (with image $j_q(F_q) = \mathcal{W}_q$). The induced Riemannian metric $j_q^*(g)$ defines an exponential map $exp^{\mathcal{W}}$, notably one has $exp_q^{\mathcal{W}} : T_q(F_q) \rightarrow F_q$, defined on an open neighborhood of 0. Since $T_q(F_q)$ is canonically identified with $W_q = T_q(\mathcal{W}_q)$ via the differential of j_q , and j_q is smooth, $exp_q^{\mathcal{W}}$ is a smooth

map from an open neighborhood of 0 in W_q to M , having values in \mathscr{W}_q . Restricting q to be an element of N we obtain a map $exp^{\mathscr{W},N}$ from a subset of $W|_N$ containing N to M .

Let us now show that $exp^{\mathscr{W},N}$ is, in fact, smooth on an open neighborhood of N in $W|_N$. Fix q in N and a coordinate chart $M \supset U \xrightarrow{\varphi} V_1 \times V_2 \subset \mathbb{R}^{m-d} \times \mathbb{R}^d$, such that the fibers of $\pi : V_1 \times V_2 \rightarrow V_1$ are the leaves of the foliation \mathscr{W} (d is the rank of this foliation). Furthermore, we can assume that $\varphi(q) = 0$ and denote the elements of \mathbb{R}^{m-d} resp. \mathbb{R}^d by x resp. z .

We denote $\varphi(U \cap N)$ by N and $T\varphi(W)$ by W if no ambiguities are possible. By the assumption $TM|_N = TN \oplus W|_N$ we have $\forall q \in N \subset V_1 \times V_2$ that $\mathbb{R}^m = T_q(V_1 \times V_2) = T_qN \oplus W_q = T_qN \oplus \mathbb{R}^d$ and thus the natural projection $\pi_q : T_qN \rightarrow \mathbb{R}^{m-d}$ is a linear isomorphism. Thus $\pi|_N : N \rightarrow V_1$ has everywhere maximal rank equal to the dimension of V_1 . Shrinking V_1 and V_2 if necessary, we can assume that $\pi|_N : N \rightarrow V_1$ is a diffeomorphism whose inverse is described by $(id_{V_1}, f) : V_1 \rightarrow V_1 \times V_2$, where $f : V_1 \rightarrow V_2$ is smooth and $N = \Gamma_f$, the graph of f . The map ψ given by $\psi(x, z) = (x, z - f(x)) =: (x, y)$ is a diffeomorphism of $V_1 \times V_2$ to an open subset of \mathbb{R}^m . Restricting ψ to an appropriate open neighborhood of 0, the image of ψ is a product of open subsets of \mathbb{R}^{m-d} and \mathbb{R}^d . Furthermore, $\psi(0) = 0$, ψ preserves the leaves of \mathscr{W} , and maps $N = \Gamma_f$ to $\{y = 0\}$.

Post-composing φ with ψ yields a chart of M near q compatible with the foliation \mathscr{W} and "adapted" to N . Obviously, we can construct a locally finite covering of N by open subsets of M that are domains of such charts, again denoted by $\varphi : U \rightarrow V_1 \times V_2$ for simplicity.

In these coordinates $exp^{\mathscr{W},N}_{(x,0)}$ is given as the time-one value of the solution of the following differential equation :

$$\frac{d^2 y^k}{dt^2} + \sum_{i,j} \Gamma_{i,j}^k(x, y) \frac{dy^i}{dt} \frac{dy^j}{dt} = 0 \quad \text{for } 1 \leq k \leq d,$$

subject to the initial condition that $x \in V_1$, $y(0) = 0$ and $\frac{dy}{dt}(0) \in W_{(x,0)}$. Standard results on smooth ordinary differential equations depending smoothly on parameters imply that there exists an open subset $\tilde{O} \subset W|_N$ containing N , where $exp^{\mathscr{W},N}$ is uniquely defined and smooth.

Upon identifying, for $q \in N$, $T_q\tilde{O} = T_qN \oplus W_q = T_qM$, we obtain that $D(exp^{\mathscr{W},N}_q) = id_{T_qM}$. By the below cited Proposition 5.4.2, it follows that there exists an open neighborhood O of N in $\tilde{O} \subset W|_N$ such that $exp^{\mathscr{W},N}|_O$ is a diffeomorphism onto its image U , an open neighborhood of N in M . Calling its inverse ϕ , this latter map fulfills the conditions stated in Theorem 5.4.1. □

The last argument relies on a standard result in differential topology (cf., e.g., Proposition 7.3 in [GG73]):

Proposition 5.4.2. *Let Y and Y' be two manifolds, and $X \subset Y$, $X' \subset Y'$ two regular submanifolds. Let $f : Y \rightarrow Y'$ be a smooth map satisfying :*

- $f|_X : X \rightarrow X'$ is a diffeomorphism
- $T_x f : T_x Y \rightarrow T_{f(x)} Y'$ is an isomorphism for all $x \in X$

Then there exists an open neighborhood V of X in Y such that $f(V)$ is open in Y' , and $f|_V$ is a diffeomorphism.

Now we show the relative Poincaré lemma, again in the presence of a foliation:

Theorem 5.4.3. [*Foliated relative Poincaré lemma*] *Let M be a smooth manifold and $N \subset M$ a submanifold. Let ω be a closed $(k+1)$ -form on M which vanishes when pulled back to N . Then there exists a neighborhood U of N in M , and a k -form μ defined on U , such that $d\mu = \omega|_U$ and $\mu|_N = 0$. Moreover, if there exists an integrable distribution $W \subset TM$ complementary to N , and such that $\iota_{u \wedge v} \omega = 0$ whenever x is in M and u and v are in $W_x \subset T_x M$, we may choose μ such that $\iota_X \mu = 0$, for all vector fields X taking value in W and defined on an open subset of U .*

Proof. By the (standard) tubular neighborhood theorem, there exist U and V neighborhoods of N in M respectively E (where $E \rightarrow N$ can be chosen to be any vector bundle such that $E \oplus TN = TM|_N$), and a diffeomorphism $\phi : U \rightarrow V$ fixing N pointwise. Thus in what follows, we can and will assume to be in an open neighborhood U of N in M , which is also a vector bundle $\pi : E = U \rightarrow N$ over N . Let us consider the map:

$$H : [0, 1] \times U \rightarrow U, (t, x) \mapsto t \cdot x = tx.$$

If we denote $H_t(x) := H(t, x)$ then $H_0 = \iota \circ \pi$ (where $\iota : N \rightarrow U = E$ is the inclusion of N as the zero-section of E), and $H_1 = id_E = id_U$. Let $Y_t(x) := \frac{d}{dt} H_t(x)$. The smooth map Y_t is not a vector field since H is not a flow, but the following formula still holds:

$$\frac{d}{dt}(H_t^* \omega) = d(H_t^* \iota_{Y_t} \omega) + H_t^* \iota_{Y_t} d\omega, \quad (*)$$

where, for α a $(k+1)$ -form, $H_t^* \iota_{Y_t} \alpha$ is the following (well-defined!) k -form:

$$(H_t^* \iota_{Y_t} \alpha)_x(v_1, \dots, v_k) = \alpha_{tx}(Y_t(x), T_x H_t(v_1), \dots, T_x H_t(v_k)),$$

for $x \in U$ and $v_j \in T_x U$. For a proof of (*) see [GS77].
 Since $\iota^* \omega = 0$ and ω is closed we obtain:

$$\begin{aligned}
 \omega|_U &= H_1^* \omega - H_0^* \omega \\
 &= \int_{[0,1]} \left(\frac{d}{dt} H_t^* \omega \right) dt \\
 &= \int_{[0,1]} (d(H_t^* \iota_{Y_t} \omega)) dt \\
 &= d \int_{[0,1]} (H_t^* \iota_{Y_t} \omega) dt \\
 &= d\mu,
 \end{aligned}$$

where we set $\mu := \int_{[0,1]} (H_t^* \iota_{Y_t} \omega) dt$. Moreover $\mu|_N = 0$ because $Y_t|_N = 0$.
 To prove the last part of the theorem, we apply Theorem 5.4.1 in order to choose a foliated tubular neighborhood U of N with respect to W . We can thus assume that the fibers of $U \rightarrow N$ are the fibers of $W|_N \rightarrow N$. Then for $x \in U$, $Y_t(x) \in W_{tx}$, implying for X a vector field tangent to W :

$$\begin{aligned}
 (\iota_X H_t^* \iota_{Y_t} \omega)_x(v_1, \dots, v_{k-1}) &= (H_t^* \iota_{Y_t} \omega)_x(X(x), v_1, \dots, v_{k-1}) \\
 &= \omega_{tx}(Y_t(x), T_x H_t(X(x)), T_x H_t(v_1), \dots, T_x H_t(v_{k-1})) \\
 &= 0
 \end{aligned}$$

since $Y_t(x)$ and $T_x H_t(X(x))$ are both in W_{tx} . □

Chapter 6

Appendices

6.1 Lie groupoids and strict Lie 2-groups

In this section, we give a short presentation of Lie groupoids and strict Lie 2-groups, together with the associated concepts used throughout this thesis. For more Lie groupoids, see for example [Mac05], and for general Lie 2-groups, see [BL04]. Certain concepts recalled here are well-known, and we give the definitions only in order to fix the notations.

We begin with the general definitions of Lie groupoids, as well as morphisms between them.

Definition 6.1.1. A *Lie groupoid* consists of two smooth manifolds G_1 and G_0 , with *source* and *target* maps $s, t : G_1 \rightarrow G_0$ which are surjective submersions, a smooth *unit* map $\varepsilon : G_0 \rightarrow G_1$, a smooth *partial multiplication* $(g, g') \mapsto g \circ g'$ defined on the set $G_{1s} \times_t G_1 := \{(g, g') \in G_1 \times G_1 \mid s(g) = t(g')\}$ and an smooth *inversion* map $\iota : G_1 \rightarrow G_1$, such that

- $s(g \circ g') = s(g')$ and $t(g \circ g') = t(g)$
- $g \circ (g' \circ g'') = (g \circ g') \circ g''$
- $s(\varepsilon(x)) = t(\varepsilon(x)) = x$
- $g \circ \varepsilon(s(g)) = g$ and $\varepsilon(t(g)) \circ g = g$
- $s(\iota(g)) = t(g)$, $t(\iota(g)) = s(g)$ and $g \circ \iota(g) = \varepsilon(t(g))$, $\iota(g) \circ g = \varepsilon(s(g))$

for all $g, g', g'' \in G_1$ with $s(g) = t(g')$ and $s(g') = t(g'')$, and all $x \in G_0$. We will use the notation $g^{-1} = \iota(g)$ for $g \in G_1$. The partial multiplication of a Lie groupoid will often be referred to just as the *multiplication*. Finally, a Lie groupoid will be denoted as $\mathcal{G} = (G_1 \rightrightarrows G_0)$, or simply \mathcal{G} .

Remark 6.1.2. A Lie groupoid could also be defined as a category internal to the category of smooth manifold, with all morphisms being invertible.

Before defining morphisms between Lie groupoids, we exhibit some examples used in this thesis.

Example 6.1.3. Let M be a smooth manifold. Then $M \rightrightarrows M$ is a Lie groupoid, with source, target and unit maps being the identity of M . Therefore an element $x \in M$ is only composable with itself, and we have $x \circ x = x$. In other words, this groupoid only has identity morphisms. We can also construct the *pair groupoid* $M \times M \rightrightarrows M$ with structure maps given by

- $s(x, x') = x'$
- $t(x, x') = x$
- $(x, x') \circ (x', x'') = (x, x'')$
- $\varepsilon(x) = (x, x)$
- $(x, x')^{-1} = (x', x)$.

for all $x, x', x'' \in M$. In fact these two Lie groupoids are not equivalent, in a sense we will precise later.

Example 6.1.4. Let Y and M be smooth manifolds, and $\pi : Y \rightarrow M$ be a surjective submersion. Then the space $Y^{[2]} := \{(y, y') \in Y \times Y \mid \pi(y) = \pi(y')\}$ is a smooth manifold and $Y^{[2]} \rightrightarrows Y$ is a Lie groupoid with the following structure maps:

- $s(y, y') = y'$
- $t(y, y') = y$
- $(y, y') \circ (y', y'') = (y, y'')$
- $\varepsilon(y) = (y, y)$
- $(y, y')^{-1} = (y', y)$

for all $y, y', y'' \in Y$ with $\pi(y) = \pi(y') = \pi(y'')$.

Considering the surjective submersion $\coprod U_i \rightarrow M$ for $\mathcal{U} = \{U_i\}_{i \in I}$ a good open cover of the manifold M gives a Lie groupoid structure on $\coprod U_{ij} \rightrightarrows \coprod U_i$, traditionally referred to as the *Čech groupoid* of M (associated to the cover \mathcal{U}).

Example 6.1.5. Let M be a smooth manifold equipped with a smooth right action of a Lie group $M \times G \rightarrow M$. Then $M \times G \rightrightarrows M$ is a Lie groupoid, with target map the projection onto M , and source map the action itself. Then the multiplication is given by

$$(x, g) \circ (x \cdot g, g') = (x, gg')$$

for all $x \in M$ and $g, g' \in G$. We also have $\varepsilon(x) = (x, 1)$ and $(x, g)^{-1} = (x \cdot g, g^{-1})$. This groupoid is called the *action groupoid* (associated to the action $M \times G \rightarrow M$).

Let X be a smooth manifold, $G_1 \rightrightarrows G_0$ be a Lie groupoid, and $f : X \rightarrow G_0$ be a smooth map, such that the space $f^*G_1 := X_f \times_t G_{1s} \times_f X$ is a smooth manifold. This is the case for example if f is a surjective submersion. We can define the *pullback* Lie groupoid $f^*G_1 \rightrightarrows X$ with structure maps

- $s(x, g, x') = x'$
- $t(x, g, x') = x$
- $(x, g, x') \circ (x', g', x'') = (x, g \circ g', x'')$
- $\varepsilon(x) = (x, \varepsilon(f(x)), x)$
- $(x, g, x')^{-1} = (x', g^{-1}, x)$

for all $x, x', x'' \in X$ and $g, g' \in G_1$. This construction is important in order to prove *Morita equivalence* of certain Lie groupoids, as we will see later.

We continue with the definition of morphisms between Lie groupoids.

Definition 6.1.6. A *Lie groupoid morphism* $\varphi = (\varphi_1, \varphi_0)$ between $G_1 \rightrightarrows G_0$ (with source and target maps s, t) and $G'_1 \rightrightarrows G'_0$ (with source and target maps s', t') consists in smooth maps $\varphi_1 : G_1 \rightarrow G'_1$ and $\varphi_0 : G_0 \rightarrow G'_0$ such that

- $s' \circ \varphi_1 = \varphi_0 \circ s$ and $t' \circ \varphi_1 = \varphi_0 \circ t$
- $\varphi_1(g_1 \circ g_2) = \varphi_1(g_1) \circ \varphi_1(g_2)$ for all $g_1, g_2 \in G_1$ such that $s(g_1) = t(g_2)$.

We say that φ_1 is a morphism of groupoids *over* φ_0 . If φ_1 and φ_0 are diffeomorphisms, we say that this defines an isomorphism of Lie groupoids.

Example 6.1.7. Let Y and M be smooth manifolds and $\pi : Y \rightarrow M$ be a surjective submersion. Consider the pair Lie groupoid $Y \times Y \rightrightarrows Y$ and the Lie groupoid $Y^{[2]} \rightrightarrows Y$. Then the inclusion $Y^{[2]} \rightarrow Y \times Y$ is a Lie groupoid morphism over the identity of Y . This means that $Y^{[2]} \rightrightarrows Y$ is a *subgroupoid* of the pair groupoid. Suppose now that $\pi : Y \rightarrow M$ has the structure of a principal G -bundle. Then $Y^{[2]} \rightrightarrows Y$ is isomorphic to the action groupoid $Y \times G \rightrightarrows Y$.

Remark 6.1.8. A Lie groupoid morphism is a smooth functor between the underlying categories of the Lie groupoids, viewed as categories internal to the category of smooth manifolds. It is immediate to verify that, given a morphism (φ_1, φ_0) between two Lie groupoids $G_1 \rightrightarrows G_0$ and $G'_1 \rightrightarrows G'_0$ we have

- $\varphi_1(\varepsilon(x)) = \varepsilon'(\varphi_0(x))$
- $\varphi_1(g^{-1}) = \varphi_1(g)^{-1}$

where $x \in G_0$, $g \in G_1$, and $\varepsilon, \varepsilon'$ denoting the unit maps of $G_1 \rightrightarrows G_0$ respectively $G'_1 \rightrightarrows G'_0$.

In view of the preceding remark, it appears that an *isomorphism* of Lie groupoids gives a stronger relation than the equivalence of the underlying categories, because it asks for both the spaces of objects and morphisms to be diffeomorphic. We will now define other classes of morphisms between Lie groupoids, which will reflect these insights.

Definition 6.1.9. We say a Lie groupoid morphism (φ_1, φ_0) between $G_1 \rightrightarrows G_0$ and $G'_1 \rightrightarrows G'_0$ is a *Morita morphism* if the map

$$s' \circ pr_2 : G_0 \times_{\varphi_0} G'_1 \rightarrow G'_0$$

is a surjective submersion and

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & G'_1 \\ \downarrow s \times t & & \downarrow s' \times t' \\ G_0 \times G_0 & \xrightarrow{\varphi_0 \times \varphi_0} & G'_0 \times G'_0 \end{array}$$

is a pullback diagram.

Remark 6.1.10. The axioms of a Morita morphism assure that (φ_1, φ_0) is an essentially surjective and fully faithful functor from $G_1 \rightrightarrows G_0$ to $G'_1 \rightrightarrows G'_0$. Therefore if a Morita morphism exists between two Lie groupoids, the underlying categories are equivalent.

This allows to define more general morphisms between Lie groupoids.

Definition 6.1.11. A *generalized homomorphism*, or *anafunctor*, between two Lie groupoids $\mathcal{G} = (G_1 \rightrightarrows G_0)$ and $\mathcal{G}' = (G'_1 \rightrightarrows G'_0)$ is a third Lie groupoid $\mathcal{Z} = (Z_1 \rightrightarrows Z_0)$ with Lie groupoid morphisms $\mathcal{G} \leftarrow \mathcal{Z} \rightarrow \mathcal{G}'$ such that the left arrow is a Morita morphism.

In the preceding definition, if the right arrow is also a Morita morphism, we say that \mathcal{Z} together with the morphisms is a *Morita equivalence* between \mathcal{G} and \mathcal{G}' , and that these two Lie groupoids are *Morita equivalent*.

Observe that, if there is a Morita morphism $\mathcal{G} \xrightarrow{\phi} \mathcal{G}'$, then the span $\mathcal{G} \xleftarrow{id_{\mathcal{G}}} \mathcal{G} \xrightarrow{\phi} \mathcal{G}'$ gives a Morita equivalence between \mathcal{G} and \mathcal{G}' .

We will now give equivalent descriptions of generalized homomorphisms and Morita equivalences of Lie groupoids, which are useful in practice.

Definition 6.1.12. Let $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid. A *right action* of \mathcal{G} on a manifold Z consists in a smooth map $\tau : Z \rightarrow G_0$ together with a smooth map $R : Z_{\tau} \times_t G_1 \rightarrow Z$ satisfying the following conditions

- $\tau(R(z, g)) = s(g)$
- $R(R(z, g), g') = R(z, g \circ g')$
- $R(z, \varepsilon(\tau(z))) = z$

for all $z \in Z$ and $g, g' \in G_1$ such that $s(g) = t(g')$. Moreover, if there is a R -invariant map $Z \xrightarrow{\kappa} M$ to a manifold M (i.e. $\kappa(R(z, g)) = \kappa(z)$ for all $(z, g) \in Z_{\tau} \times_t G_1$), we say that Z is a *right principal \mathcal{G} -bundle* if κ is a surjective submersion and if the map

$$(pr_1, R) : Z_{\tau} \times_t G_1 \rightarrow Z_{\kappa} \times_{\kappa} Z$$

is a diffeomorphism.

Remark 6.1.13. Left groupoid actions are defined similarly. In fact, a left action of a groupoid is a right action of the *opposite* groupoid.

Proposition 6.1.14. A *generalized homomorphism* between two Lie groupoids $\mathcal{G} = (G_1 \rightrightarrows G_0)$ and $\mathcal{G}' = (G'_1 \rightrightarrows G'_0)$ is equivalently given by a diagram

$$\begin{array}{ccc} G_1 & & Z & & G'_1 \\ \Downarrow & \swarrow \tau & & \searrow \tau' & \Downarrow \\ G_0 & & & & G'_0 \end{array}$$

where \mathcal{G} respectively \mathcal{G}' acts on Z via τ respectively τ' , such that τ has the structure of a right principal \mathcal{G}' -bundle, and such that the actions of the respective groupoids commute.

In the preceding definition, if τ' also has the structure of a left principal \mathcal{G} -bundle, then \mathcal{G} and \mathcal{G}' are Morita equivalent.

Remark 6.1.15. The proof of the last proposition can be found, e.g., in [BX11].

Remark 6.1.16. In what follows, and in the whole thesis, we will exclusively use this description of generalized homomorphisms and Morita equivalences. Furthermore we will use the notation $G_1 \leftarrow Z \rightarrow G'_1$ for a generalized homomorphism.

Let us now describe some Morita equivalences between the Lie groupoids given as examples.

Example 6.1.17. Let Y and M be smooth manifolds and $\pi : Y \rightarrow M$ be a surjective submersion. Then the following diagram

$$\begin{array}{ccccc} Y^{[2]} & & Y & & M \\ \Downarrow & \swarrow & & \searrow & \Downarrow \\ Y & & id_Y & & M \end{array}$$

with trivial right action of $M \rightrightarrows M$ on Y , and left action

$$((y, y'), y') \mapsto y$$

for $(y, y') \in Y^{[2]}$, gives a Morita equivalence between $Y^{[2]} \rightrightarrows Y$ and $M \rightrightarrows M$. In fact, every Lie groupoid Morita equivalent to $M \rightrightarrows M$ arises in this way.

Example 6.1.18. Let M be a manifold. Then the following diagram

$$\begin{array}{ccccc} M \times M & & M & & * \\ \Downarrow & \swarrow & & \searrow & \Downarrow \\ M & & id_M & & * \end{array}$$

with trivial right action and left action given by

$$((x, x'), x') \mapsto x$$

for all $x, x' \in M$, defines a Morita equivalence between the pair groupoid of M , and the trivial Lie groupoid $* \rightrightarrows *$.

Example 6.1.19. Let X be a smooth manifold, $G_1 \rightrightarrows G_0$ be a Lie groupoid, and $f : X \rightarrow G_0$ a surjective submersion. Then

$$\begin{array}{ccc} f^*G_1 & \xrightarrow{p_{G_1}} & G_1 \\ \Downarrow & & \Downarrow \\ X & \xrightarrow{f} & G_0 \end{array}$$

is a Morita morphism. Thus $f^*G_1 \rightrightarrows X$ and $G_1 \rightrightarrows G_0$ are Morita equivalent Lie groupoids.

Generalized homomorphisms can be obtained via simple morphisms. Let indeed (φ_1, φ_0) be a morphism $\mathcal{G} \rightarrow \mathcal{G}'$. Then the following diagram

$$\begin{array}{ccccc} G_1 & & G_{0\varphi_0} \times_t G'_1 & & G'_1 \\ \Downarrow & & \swarrow pr_1 & & \searrow s' \\ \Downarrow & & & & \Downarrow \\ G_0 & & & & G'_0 \end{array}$$

with right action given by the multiplication of \mathcal{G}' and left action given by

$$g_1 \cdot (g_0, g'_1) \mapsto (t(g_1), \varphi_1(g_1) \circ g'_1)$$

defines a generalized homomorphism between \mathcal{G} and \mathcal{G}' .

We can also compose generalized homomorphisms. Let $G_1 \xleftarrow{\tau_1} Z \xrightarrow{\tau'_1} G'_1$ and $G'_1 \xleftarrow{\tau'_2} Z' \xrightarrow{\tau_2} G''_1$ be two generalized homomorphisms. The composition is then defined as $G_1 \xleftarrow{\tilde{\tau}_1} \tilde{Z} \xrightarrow{\tilde{\tau}'_2} G''_1$ with

$$\tilde{Z} = (Z_{\tau'_1} \times_{\tau_2} Z') / G'_1$$

where the quotient identifies $(z \cdot g'_1, z') \sim (z, g'_1 \cdot z')$ for all $(z, z') \in Z_{\tau'_1} \times_{\tau_2} Z'$ and $g'_1 \in G'_1$ with $t(g'_1) = \tau'_1(z)$ and $s(g'_1) = \tau_2(z')$. The actions are defined using the actions of \mathcal{G} and \mathcal{G}' , and we obtain a generalized homomorphism between \mathcal{G} and \mathcal{G}'' . If we initially had Morita equivalences, then the composition also yields a Morita equivalence.

We give a final definition regarding Lie groupoids.

Definition 6.1.20. Let $G_1 \xleftarrow{\tau_1} Z_1 \xrightarrow{\tau'_1} G'_1$ and $G_1 \xleftarrow{\tau_2} Z_2 \xrightarrow{\tau'_2} G'_1$ be two generalized homomorphisms. A *transformation* between Z_1 and Z_2 is a smooth map $\phi : Z_1 \rightarrow Z_2$ which is equivariant with respect to both groupoid actions and such that $\tau_2 \circ \phi = \tau_1$ and $\tau'_2 \circ \phi = \tau'_1$.

Remark 6.1.21. Lie groupoids, generalized homomorphisms and transformations form a *bigroupoid*, a 2-category where all the morphisms are invertible up to a 2-morphism, and all the 2-morphisms are isomorphisms. We will not make much use of this structure in this thesis, but the definition of a transformation between generalized homomorphisms will be useful to define equivalence of Lie 2-groups.

We now consider Lie 2-groups. There are various definitions of such objects, but we will mostly take the point of view of [Wal18]. Note that we only consider *strict* Lie 2-groups here.

Definition 6.1.22. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ be a Lie groupoid. We say that Γ is a (*strict*) *Lie 2-group* if there are morphisms of groupoids $m : \Gamma \times \Gamma \rightarrow \Gamma$ and $i : \Gamma \rightarrow \Gamma$ satisfying the usual axioms of a group.

Remark 6.1.23. The preceding definition is in fact the definition of a groupoid object in the category of Lie groups. Note that this gives Lie group structures on both Γ_1 and Γ_0 .

In fact, strict Lie 2-groups are in one-to-one correspondence with Lie group crossed modules, which we will now define.

Definition 6.1.24. A *Lie group crossed module* consists of two Lie groups H and G , with homomorphisms $r : H \rightarrow G$ and $\alpha : G \rightarrow \text{Aut}(H)$ subject to the following conditions

- $\alpha(r(h_1))(h_2) = h_1 h_2 h_1^{-1}$
- $r(\alpha(g)(h)) = gr(h)g^{-1}$

Given a Lie group crossed module, we obtain a Lie groupoid $H \times G \rightrightarrows G$ (the semi-direct product being taken with respect to α). Source and target maps are $s(g, h) = gr(h)$, $t(h, g) = g$, partial multiplication $(h, g) \circ (h', gr(h)) = (hh', g)$, unit $\varepsilon(g) = (1, g)$ and inverse $(h, g)^{-1} = (h^{-1}, gr(h))$. The Lie 2-group structure is then given by multiplication $m((h_1, g_1), (h_2, g_2)) = (\alpha(g_2^{-1}, h_1)h_2, g_1g_2)$ and inverse $(h, g)^{-1} = (\alpha(g^{-1}, h)^{-1}, g^{-1})$. We will not prove the other correspondence in full generality, but note that given a Lie 2-group $\Gamma_1 \rightrightarrows \Gamma_0$, it is possible to give the pair $(H = \ker(s) \subset \Gamma_1, G = \Gamma_0)$ the structure of a Lie group crossed module.

We give here two examples that will be used throughout this thesis.

Example 6.1.25. First consider the Lie group crossed module given by $S^1 \rightarrow \{e\}$ and $id_{S^1} : \{e\} \rightarrow Aut(S^1)$. We obtain a Lie groupoid $S^1 \rightrightarrows \{e\}$ with all structure maps (for both the Lie groupoid and Lie 2-group structure) given by the structure maps of the Lie group S^1 . We will denote this Lie 2-group by BS^1 . Note that this construction is possible for all abelian Lie groups.

Example 6.1.26. Consider, for H an infinite dimensional separable Hilbert space, the Lie group $U(H)$ of unitary operators on H , and $PU(H) = U(H)/S^1$. Denote the projection by $q : U(H) \rightarrow PU(H)$. The map $c : U(H) \rightarrow Aut(U(H)); u \mapsto (v \mapsto uvu^{-1})$ factors via a map $c : PU(H) \rightarrow Aut(U(H))$ because S^1 is central in $U(H)$. Verifying the axioms for a Lie crossed module is straightforward. Thus we obtain a Lie 2-group that we will denote by $\Gamma_{U(H) \rtimes PU(H)}$.

We now give the definition of an equivalence of Lie 2-groups.

Definition 6.1.27 ([NW13]). Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ and $\Gamma' = (\Gamma'_1 \rightrightarrows \Gamma'_0)$ be two Lie 2-groups. Note $m_\Gamma : \Gamma \times \Gamma \rightarrow \Gamma$ and $m_{\Gamma'} : \Gamma' \times \Gamma' \rightarrow \Gamma'$ the structure morphisms of the Lie 2-groups. We say that Γ and Γ' are *equivalent* if there is a Morita equivalence $\Gamma_1 \leftarrow Z \rightarrow \Gamma'_1$, together with a transformation between

$$Z \circ m_\Gamma \text{ and } m_{\Gamma'} \circ (Z \times Z)$$

viewed as generalized homomorphisms between $\Gamma \times \Gamma$ and Γ' .

Finally, we give the definition of an action of a Lie 2-group on a Lie groupoid, which will be useful when considering principal 2-group bundles.

Definition 6.1.28. Let $\Gamma = (\Gamma_1 \rightrightarrows \Gamma_0)$ be a Lie 2-group and $\mathcal{G} = (G_1 \rightrightarrows G_0)$ be a Lie groupoid. A right action of Γ on \mathcal{G} is the datum of a Lie groupoid morphism $R = (R_1, R_0) : \mathcal{G} \times \Gamma \rightarrow \mathcal{G}$ such that $R_0(g_0, 1_{\Gamma_0}) = g_0$ and $R_1(g_1, 1_{\Gamma_1}) = g_1$ for all $g_0 \in G_0$ and $g_1 \in G_1$, and such that the following diagram

$$\begin{array}{ccc} \mathcal{G} \times \Gamma \times \Gamma & \xrightarrow{id \times m} & \mathcal{G} \times \Gamma \\ \downarrow R \times id & & \downarrow R \\ \mathcal{G} \times \Gamma & \xrightarrow{R} & \mathcal{G} \end{array}$$

of Lie groupoid morphisms is commutative.

6.2 Lie 2-algebras

In this section, we will give a short presentation of Lie 2-algebras. See [BC04] for a more exhaustive insight on these objects.

Definition 6.2.1. Let $L_\bullet = (L_{-1} \rightarrow L_0)$ be a 2-term complex of vector spaces. A *Lie 2-algebra structure on L_\bullet* consists in (multi)linear graded antisymmetric maps $\{l_k \mid 1 \leq k \leq 3\}$ with the degree of l_k being equal to $2-k$:

- $l_1 : L_{-1} \rightarrow L_0$
- $l_2 : \Lambda^2 L_\bullet \rightarrow L_\bullet$
- $l_3 : \Lambda^3 L_0 \rightarrow L_{-1}$

such that the following equations hold, for $x, y, z, t \in L_0$ and $u, v \in L_{-1}$:

$$l_1(l_2(x, u)) = l_2(x, l_1(u)) \text{ and } l_2(l_1(u), v) = l_2(u, l_1(v))$$

$$l_1(l_3(x, y, z)) + l_2(l_2(x, y), z) - l_2(l_2(x, z), y) + l_2(l_2(y, z), x) = 0$$

$$l_3(l_1(u), x, y) + l_2(l_2(x, y), u) - l_2(l_2(x, u), y) + l_2(l_2(y, u), x) = 0$$

$$\begin{aligned} & l_3(l_2(x, y), z, t) - l_3(l_2(x, z), y, t) + l_3(l_2(x, t), y, z) \\ & + l_3(l_2(y, z), x, t) - l_3(l_2(y, t), x, z) + l_3(l_2(z, t), x, y) \\ & = l_2(l_3(x, y, z), t) - l_2(l_3(x, y, t), z) + l_2(l_3(x, z, t), y) - l_2(l_3(y, z, t), x). \end{aligned}$$

Remark 6.2.2. We make several comments on the equations involved in the preceding definition. The first equations mean that l_1 *derives* the bracket l_2 . The second and third equations show that l_2 satisfies a sort of Jacobi identity, up to a "boundary term" involving the map l_3 . This implies that the quotient $L_0/l_1(L_{-1})$ equipped with the bracket l_2 is a Lie algebra. The fourth equation is however less obvious to interpret in terms of the "classical" Lie theory.

Example 6.2.3. Let \mathfrak{g} be a Lie algebra. Then the complex $0 \rightarrow \mathfrak{g}$ is canonically equipped with a Lie 2-algebra structure. Both maps l_1 and l_3 are zero, and the binary bracket l_2 is the Lie bracket of \mathfrak{g} .

Example 6.2.4. Let \mathfrak{g} be a Lie algebra, and $\mathfrak{g} \xrightarrow{\rho} \text{End}(V)$ be a representation on a vector space V . Let c be a degree-three cocycle of the Chevalley-Eilenberg complex of this representation, that is a closed element $c \in \text{Hom}(\Lambda^3 \mathfrak{g}^*, V)$. Then the complex $V \rightarrow \mathfrak{g}$ with maps

- $l_1 = 0$
- $l_2(\xi, \xi') = [\xi, \xi']$
- $l_2(\xi, v) = \rho(\xi)(v)$
- $l_3 = c$

is a Lie 2-algebra. In fact, Lie 2-algebras with unary map l_1 being identically zero are often referred to as *skeletal* or *minimal* Lie 2-algebras. Moreover, every skeletal Lie 2-algebra arises from a similar construction.

Example 6.2.5. Let $V_\bullet = (V_{-1} \xrightarrow{d} V_0)$ be a 2-term complex of vector spaces. Let $E_{-1} = \text{Hom}(V_0, V_{-1})$ and $E_0 = \{(\varphi, \psi) \in \text{End}(V_{-1}) \times \text{End}(V_0) \mid d \circ \varphi = \psi \circ d\}$. Then the complex

$$E_\bullet = E_{-1} \xrightarrow{\delta} E_0$$

with $\delta(f) = (f \circ d, d \circ f)$ for $f \in E_{-1}$, bracket l_2 defined by

- $l_2((\varphi, \psi), (\varphi', \psi')) = ([\varphi, \varphi'], [\psi, \psi'])$
- $l_2((\varphi, \psi), f) = \varphi \circ f - f \circ \psi$

for $(\varphi, \psi), (\varphi', \psi') \in E_0$, $f \in E_{-1}$, and $[\cdot, \cdot]$ denoting the usual Lie bracket on $\text{End}(V_i)$ for $i = -1, 0$, and with third bracket l_3 being identically zero is a Lie 2-algebra. It is in fact a truncation of the *differential graded Lie algebra* given by the total complex $\text{End}(V_\bullet)$, which is made of three vector spaces (see [SZ12]).

An important class of Lie 2-algebras is given by the following definition.

Definition 6.2.6. A *Lie algebra crossed module* is given by two Lie algebras \mathfrak{h} and \mathfrak{g} , together with a Lie algebra morphism $r : \mathfrak{h} \rightarrow \mathfrak{g}$ and $\alpha : \mathfrak{g} \rightarrow \mathfrak{Der}(\mathfrak{h})$ subject to the following conditions

- $\alpha(r(X))(X') = [X, X']_{\mathfrak{h}}$
- $r(\alpha(\xi)(X)) = [\xi, r(x)]_{\mathfrak{g}}$

for all $X, X' \in \mathfrak{h}$ and $\xi \in \mathfrak{g}$.

In fact, Lie algebra crossed modules are in one-to-one correspondence with *strict* Lie 2-algebras, which means the third bracket l_3 is identically zero. Consider indeed a Lie algebra crossed module given by Lie algebras \mathfrak{h} , \mathfrak{g} and Lie algebra morphisms $r : \mathfrak{h} \rightarrow \mathfrak{g}$ and $\alpha : \mathfrak{g} \rightarrow \mathfrak{Der}(\mathfrak{h})$. Then the complex $\mathfrak{h} \rightarrow \mathfrak{g}$ is a Lie 2-algebra, with brackets given by

- $l_1 = r$
- $l_2(\xi, \xi') = [\xi, \xi']_{\mathfrak{g}}$
- $l_2(\xi, X) = \alpha(\xi)(X)$
- $l_3 = 0$

for all $\xi, \xi' \in \mathfrak{g}$ and $X \in \mathfrak{h}$. To see the inverse correspondence, observe that given a strict Lie 2-algebra $L_{\bullet} = (L_{-1} \rightarrow L_0)$, then Lie algebra structures can be defined on L_{-1} via $[X, X'] = l_2(l_1(X), X')$ for $X, X' \in L_{-1}$ and on L_0 via $[\xi, \xi'] = l_2(\xi, \xi')$. Then the required Lie algebra morphisms are $r = l_1$ and $\alpha(\xi)(X) = l_2(\xi, X)$ for all $\xi \in L_0$ and $X \in L_{-1}$.

Example 6.2.7. Let \mathfrak{h} be a Lie algebra. Consider the Lie algebra $\mathfrak{g} = \mathfrak{Der}(\mathfrak{h})$, with Lie algebra morphisms $ad : \mathfrak{h} \rightarrow \mathfrak{g}$ and the canonical action $\mathfrak{g} \rightarrow \mathfrak{Der}(\mathfrak{h})$. This defines a Lie algebra crossed module, and therefore a strict Lie 2-algebra, often referred to as the *derivation Lie 2-algebra* (of the Lie algebra \mathfrak{h}).

Example 6.2.8. Let P, M be smooth manifold, and $\pi : P \rightarrow M$ be a principal G -bundle. Let $C^{\infty}(P, \mathfrak{g})^G = \{f \in C^{\infty}(P, \mathfrak{g}) \mid f(p \cdot g) = Ad_{g^{-1}}(f(p))\}$, and denote, for a function $f \in C^{\infty}(P, \mathfrak{g})^G$, the G -invariant vector field $\xi_f \in \mathfrak{X}^G(P)$ given by $p \mapsto \left. \frac{d}{dt} \right|_{t=0} p \cdot \exp(tf(p))$. Then the map $\xi : C^{\infty}(P, \mathfrak{g})^G \rightarrow \mathfrak{X}^G(P)$ is a Lie algebra morphism, and together with the canonical action of vector fields on functions, this defines a Lie algebra crossed module.

Lie algebra crossed modules are indeed the infinitesimal version of Lie group crossed modules. Let H, G be two Lie groups, with morphisms $r : H \rightarrow G$ and $\alpha : G \rightarrow Aut(H)$ which form a Lie group crossed module. Then, taking the Lie algebras $Lie(H)$ and $Lie(G)$, as well as the derivatives of the maps r and α gives a Lie algebra crossed module, hence a strict Lie 2-algebra. It is not surprising that we only recover strict Lie 2-algebras, because Lie group crossed modules are in one-to-one correspondence with strict Lie 2-groups.

Lie algebra crossed modules can also be obtained directly from a Lie groupoid. First recall the following

Definition 6.2.9. A *Lie algebroid* (over a manifold M) is a vector bundle $A \rightarrow M$, equipped with a Lie bracket $[\cdot, \cdot]_A : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$ and an *anchor* map $\rho : A \rightarrow TM$, which is a vector bundle morphism satisfying the following conditions

- $\rho([X, Y]_A) = [\rho(X), \rho(Y)]$
- $[X, f \cdot Y]_A = \rho(X)(f) \cdot Y + f \cdot [X, Y]_A$

A Lie algebroid $A_{\mathcal{G}}$ can be associated to a Lie groupoid $\mathcal{G} = (G_1 \rightrightarrows G_0)$ via the following pullback

$$\begin{array}{ccc} A_{\mathcal{G}} & \longrightarrow & \ker(s_*) \\ \downarrow & & \downarrow \\ G_0 & \xrightarrow{\varepsilon} & G_1 \end{array}$$

where ε is the unit map of the Lie groupoid $G_1 \rightrightarrows G_0$, and $\ker(s_*) \subset TG_1$. Given $\sigma \in \Gamma(A_{\mathcal{G}})$, the formula

$$\overrightarrow{\sigma}(g) = (R_g)_*(\sigma(t(g)))$$

for $g \in G_1$, defines a vector field on G_1 , which is right-invariant in the sense that $s_*(\overrightarrow{\sigma}) = 0$ and $\overrightarrow{\sigma}(h \circ g) = (R_g)_*(\overrightarrow{\sigma}(h))$. Moreover, it can be shown that the space of sections of the Lie algebroid $A_{\mathcal{G}}$ is diffeomorphic to the space of right-invariant vector fields on G_1 , which allows to define a bracket on $\Gamma(A_{\mathcal{G}})$. Finally, the composition $\rho_A : A_{\mathcal{G}} \rightarrow \ker(s_*) \xrightarrow{t_*} TG_0$ of vector bundle morphisms gives an anchor map for $A_{\mathcal{G}}$.

Let us now consider, for $\sigma \in \Gamma(A_{\mathcal{G}})$

$$\overleftarrow{\sigma}(g) = (L_g)_*\iota_*(\sigma(s(g)))$$

for $g \in G_1$, and where $\iota : G_0 \rightarrow G_1$ is the inversion map. Then $X_{\sigma} = \overrightarrow{\sigma} + \overleftarrow{\sigma}$ is a *multiplicative vector field* on the Lie groupoid $G_1 \rightrightarrows G_0$, in the sense that the following diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{X_{\sigma}} & TG_1 \\ \Downarrow & & \Downarrow \\ G_0 & \xrightarrow{\rho_A(\sigma)} & TG_0 \end{array}$$

is a Lie groupoid morphism. Denote by $\mathfrak{X}_{mult}(\mathcal{G})$ the Lie algebra of multiplicative vector fields on the Lie groupoid \mathcal{G} . Then $X_\cdot : \Gamma(A_{\mathcal{G}}) \rightarrow \mathfrak{X}_{mult}(\mathcal{G})$; $\sigma \mapsto X_\sigma$ is a Lie algebra morphism, and for $\xi \in \mathfrak{X}_{mult}(\mathcal{G})$ and $\sigma \in \Gamma(A_{\mathcal{G}})$, the equation

$$\xi \cdot \sigma = [\xi, \overrightarrow{\sigma}]$$

defines a right-invariant vector field on G_1 , and therefore a section of the Lie algebroid $A_{\mathcal{G}}$. We obtain a Lie algebra crossed module, and a Lie 2-algebra $\Gamma(A_{\mathcal{G}}) \rightarrow \mathfrak{X}_{mult}(\mathcal{G})$.

Remark 6.2.10. See [Mac05] for the proof of these facts, and [MX98] for more informations on multiplicative vector fields on Lie groupoids. The construction of the above Lie algebra crossed module can be found in [BEL20]. This construction can be generalized to any $\mathcal{L}\mathcal{A}$ -groupoid ([OW19]).

Example 6.2.11. Let M be a manifold, and consider the Lie groupoid $M \rightrightarrows M$, with only identity morphisms. Then a multiplicative vector field is just a vector field on M , and the Lie algebroid is $M \times \{0\}$. Then we obtain the Lie 2-algebra

$$0 \rightarrow \mathfrak{X}(M)$$

which is only the Lie algebra of vector fields, seen as a Lie 2-algebra.

Example 6.2.12. Consider for a manifold M the pair groupoid $M \times M \rightrightarrows M$. A simple computation shows that multiplicative vector fields on $M \times M$ are those of the form (X, X) , for $X \in \mathfrak{X}(M)$. Moreover, the Lie algebroid of the pair Lie groupoid is simply the tangent bundle $TM \rightarrow M$. Therefore, the Lie 2-algebra obtained is

$$\mathfrak{X}(M) \xrightarrow{id} \mathfrak{X}(M)$$

Example 6.2.13. Let Y and M be two smooth manifolds, and $\pi : Y \rightarrow M$ be a surjective submersion. We consider the Lie groupoid $Y^{[2]} \rightrightarrows Y$. It is easily seen that multiplicative vector fields are in fact π -projectable vector fields. Moreover, the Lie algebroid is $\ker(\pi_*) \rightarrow Y$. Therefore we obtain the Lie 2-algebra

$$\mathfrak{X}_{vert}(Y) \rightarrow \mathfrak{X}^\pi(Y)$$

where $\mathfrak{X}_{vert}(Y)$ denotes vertical vector fields on Y (with respect to π), and $\mathfrak{X}^\pi(Y)$ denotes π -projectable vector fields. The arrow is the inclusion, and brackets are given by brackets of vector fields.

Example 6.2.14. Consider a smooth, free and transitive right action of a compact Lie group G on a manifold P , and consider the action groupoid $P \times G \rightrightarrows P$. Then, a result of [Hep09] shows that the category of vector fields on the *differentiable stack* given by this Lie groupoid, is equivalent to the Lie 2-algebra

$$C^\infty(P, \mathfrak{g})^G \rightarrow \mathfrak{X}^G(P)$$

already described in Example 5.2.8. Note that in this example, we did not suppose the compactness of the Lie group G .

We will now define morphisms of Lie 2-algebras.

Definition 6.2.15. Let $(L_\bullet = (L_{-1} \rightarrow L_0), \{l_k\})$ and $(L'_\bullet = (L'_{-1} \rightarrow L'_0), \{l'_k\})$ be two Lie 2-algebras. A *Lie ∞ -algebra morphism* $L_\bullet \xrightarrow{\Phi} L'_\bullet$ is given by linear maps :

- $\Phi_1 : L_\bullet \rightarrow L'_\bullet$
- $\Phi_2 : \Lambda^2 L_\bullet \rightarrow L'_\bullet$

with Φ_1 of degree 0 and Φ_2 of degree -1 , such that for every $x, y, z \in L_0$ and $u \in L_{-1}$:

$$\Phi_1(l_1(u)) = l'_1(\Phi_1(u))$$

$$\Phi_1(l_2(x, y)) = l'_2(\Phi_1(x), \Phi_1(y)) + l'_1(\Phi_2(x, y))$$

$$\Phi_1(l_2(u, x)) = l'_2(\Phi_1(u), \Phi_1(x)) + \Phi_2(l_1(u), x)$$

$$\begin{aligned} & \Phi_2(l_2(x, y), z) - \Phi_2(l_2(x, z), y) + \Phi_2(l_2(y, z), x) + \Phi_1(l_3(x, y, z)) \\ &= l'_2(\Phi_1(x), \Phi_2(y, z)) - l'_2(\Phi_1(y), \Phi_2(x, z)) + l'_2(\Phi_1(z), \Phi_2(x, y)) \\ &+ l'_3(\Phi_1(x), \Phi_1(y), \Phi_1(z)). \end{aligned}$$

Following a suggestion of Camille Laurent-Gengoux, we visualize a Lie 2-algebra morphism $L_\bullet \xrightarrow{\Phi} L'_\bullet$ as below:

$$\begin{array}{ccc} L_{-1} & \xrightarrow{l_1} & L_0 \\ \Phi_1 \downarrow & \swarrow \Phi_2 & \downarrow \Phi_1 \\ L'_{-1} & \xrightarrow{l'_1} & L'_0 \end{array}$$

If a Lie 2-algebra morphism is also a quasi-isomorphism between the underlying complexes, we say that we have a Lie 2-algebra quasi-isomorphism.

Let $(L_\bullet = L_{-1} \rightarrow L_0, \{l_k\})$, $(L'_\bullet = L'_{-1} \rightarrow L'_0, \{l'_k\})$ and $(L''_\bullet = L''_{-1} \rightarrow L''_0, \{l''_k\})$ be three Lie 2-algebras, with morphisms $\Phi : L_\bullet \rightarrow L'_\bullet$ and $\Phi' : L'_\bullet \rightarrow L''_\bullet$. We can define a composition $\Phi'' = \Phi' \circ \Phi$ by the following formulas

- $\Phi''_1 = \Phi'_1 \circ \Phi_1$
- $\Phi''_2 = \Phi'_2 \circ (\Phi_1 \times \Phi_1) + \Phi'_1 \circ \Phi_2$.

Example 6.2.16. Let Y and M be smooth manifolds and $\pi : Y \rightarrow M$ be a surjective submersion. We have a Lie 2-algebra morphism

$$\begin{array}{ccc} \mathfrak{X}_{vert}(Y) & \longrightarrow & \mathfrak{X}^\pi(Y) \\ \downarrow & \nearrow \Phi_2 & \downarrow \Phi_1 \\ 0 & \longrightarrow & \mathfrak{X}(M) \end{array}$$

between the Lie 2-algebra formed by multiplicative vector fields on respectively $Y^{[2]} \rightrightarrows Y$ and $M \rightrightarrows M$. Obviously $\Phi_2 = 0$ and Φ_1 is the projection onto M of a π -projectable vector field on Y . Moreover, it is easily seen that this morphism is a quasi-isomorphism.

Example 6.2.17. Let P and M be smooth manifolds, and $\pi : P \rightarrow M$ be a principal G -bundle. Choose a connection $A \in \Omega^1(P) \otimes \mathfrak{g}$ with curvature $F_A \in \Omega^2(P) \otimes \mathfrak{g}$, and denote, for a vector field $X \in \mathfrak{X}(M)$, its horizontal lift by $\tilde{X} \in \mathfrak{X}^G(P)$. The map $X \mapsto \tilde{X}$ does not define a Lie algebra morphism, however the following diagram

$$\begin{array}{ccc} 0 & \longrightarrow & \mathfrak{X}(M) \\ \downarrow & \nearrow \Phi_2 & \downarrow \Phi_1 \\ C^\infty(P, \mathfrak{g})^G & \longrightarrow & \mathfrak{X}^G(P) \end{array}$$

with $\Phi_1(X) = \tilde{X}$ and $\Phi_2(X, X') = F_A(\tilde{X}, \tilde{X}')$ for $X, X' \in \mathfrak{X}(M)$, defines a Lie 2-algebra morphism. Moreover, it is a quasi-isomorphism of Lie 2-algebras, which is another way of saying that the Atiyah sequence

$$\frac{P \times \mathfrak{g}}{G} \rightarrow \frac{TP}{G} \rightarrow TM$$

of vector bundles over M is exact, and that choosing a connection for the bundle P splits this sequence.

We mention here a result of [BEL20], which sheds a different light on the last two examples.

Theorem 6.2.18. *Let \mathcal{G} and \mathcal{G}' be two Morita equivalent Lie groupoids. Then the Lie 2-algebras formed by the multiplicative vector fields on \mathcal{G} respectively \mathcal{G}' are quasi-isomorphic.*

Finally, we mention homotopies between Lie 2-algebra morphisms. The following definition can be found, e.g., in [Noo13].

Definition 6.2.19. Let Φ, Ψ be two Lie 2-algebra morphisms $L_\bullet \rightarrow L'_\bullet$. A *homotopy from Φ to Ψ* is a linear map $h : L_0 \rightarrow L'_{-1}$ such that h is a usual homotopy between the chain maps Φ_1 and Ψ_1 , i.e.:

$$\Psi_1(x) = \Phi_1(x) + l'_1(h(x)), \quad \Psi_1(u) = \Phi_1(u) + h(l_1(u))$$

for all $x \in L_0$ and $u \in L_{-1}$ and such that furthermore the following equation is verified for all $x, y \in L_0$:

$$\Psi_2(x, y) = \Phi_2(x, y) + h(l_2(x, y)) + l'_2(\Phi_1(x), h(y)) + l'_2(h(x), \Psi_1(y)).$$

It is possible to define composition of such homotopies, and then Lie 2-algebras together with their morphisms and homotopies form a 2-category, in which equivalences are precisely the quasi-isomorphisms.

Remark 6.2.20. Other classes of morphisms of Lie 2-algebras can be defined, namely *butterflies* of Lie 2-algebras (see [Noo13]). Also transformations between butterflies can be defined, which leads to 2-category equivalent to the one mentioned above. In particular, a butterfly between Lie 2-algebras is an equivalence if and only if it induces isomorphisms between the cohomology of the underlying chain complexes of the Lie 2-algebras.

Bibliography

- [Arn89] Vladimir Arnold. *Mathematical methods of classical mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer, 1989.
- [BC04] John C. Baez and Alissa S. Crans. Higher-dimensional algebra. VI. Lie 2-algebras. *Theory Appl. Categ.*, 12:492–538, 2004.
- [BD19] Henrique Bursztyn and Thiago Drummond. Lie theory of multiplicative tensors. *Math. Ann.*, 375(3-4):1489–1554, 2019.
- [BEL20] Daniel Berwick-Evans and Eugene Lerman. Lie 2-algebras of vector fields. *Pacific J. Math.*, 309(1):1–34, 2020.
- [BHR10] John C. Baez, Alexander E. Hoffnung, and Christopher L. Rogers. Categorified symplectic geometry and the classical string. *Comm. Math. Phys.*, 293(3):701–725, 2010.
- [BL04] John C. Baez and Aaron D. Lauda. Higher-dimensional algebra. V. 2-groups. *Theory Appl. Categ.*, 12:423–491, 2004.
- [Bry08] Jean-Luc Brylinski. *Loop spaces, characteristic classes and geometric quantization*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1993 edition.
- [BS07] John C. Baez and Urs Schreiber. Higher gauge theory. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 7–30. Amer. Math. Soc., Providence, RI, 2007.
- [BSS18] Severin Bunk, Christian Sämann, and Richard J. Szabo. The 2-Hilbert space of a prequantum bundle gerbe. *Rev. Math. Phys.*, 30(1):1850001, 101, 2018.
- [Bun] Severin Bunk. Gerbes in Geometry, Field Theory, and Quantisation. *arXiv e-prints arXiv:2102.10406*.

- [BX11] Kai Behrend and Ping Xu. Differentiable stacks and gerbes. *J. Symplectic Geom.*, 9(3):285–341, 2011.
- [CC00] Alberto Candel and Lawrence Conlon. *Foliations I*, volume 23 of *Graduate Texts in Mathematics*. American Mathematical Society, 2000.
- [CFRZ16] Martin Callies, Yaël Frégier, Christopher L. Rogers, and Marco Zambon. Homotopy moment maps. *Adv. Math.*, 303:954–1043, 2016.
- [CI96] Frans Cantrijn, Alberto Ibort, and Manuel de León. Hamiltonian structures on multisymplectic manifolds. *Rend. Sem. Mat. Univ. Politec. Torino* 54.3, 1996.
- [CI99] Frans Cantrijn, Alberto Ibort, and Manuel de León. On the geometry of multisymplectic manifolds. *Journal of the Australian Mathematical Society. Series A. Pure Mathematics and Statistics*, 66(3):303–330, 1999.
- [CnCI91] José F. Cariñena, Michael Crampin, and Alberto Ibort. On the multisymplectic formalism for first order field theories. *Diff. Geom. Appl.*, 1(4):345–374, 1991.
- [Col] Braxton L. Collier. Infinitesimal Symmetries of Dixmier-Douady Gerbes. *arXiv e-prints arXiv:2008.05184*.
- [DD63] Jacques Dixmier and Adrien Douady. Champs continus d’espaces hilbertiens et de c^* -algèbres. *Bull. Soc. Math. France*, 91:227–284, 1963.
- [dds03] Manuel de León, David Martín de Diego, and Aitor Santamaría-Merino. Tulczyjew’s triples and lagrangian submanifolds in classical field theories. In: *Applied Differential Geometry and Mechanics*. Universiteit Gent, Ghent, Academia Press, 2003.
- [FPR03] Michael Forger, Cornelius Paufler, and Hartmann Römer. The Poisson Bracket for Poisson Forms in Multisymplectic Field Theory. *Rev. Math. Phys.*, 15:705–743, 2003.
- [FPR05] Michael Forger, Cornelius Paufler, and Hartmann Römer. Hamiltonian multivector fields and Poisson forms in multisymplectic field theory. *Journal of Mathematical Physics*, 46(11):112903, November 2005.

- [FRS14] Domenico Fiorenza, Christopher L. Rogers, and Urs Schreiber. L_∞ -algebras of local observables from higher prequantum bundles. *Homology Homotopy Appl.*, 16(2):107–142, 2014.
- [FY13] Michael Forger and Sandra Z. Yepes. Lagrangian distributions and connections in multisymplectic and polysymplectic geometry. *Diff. Geom. Appl.*, 31:775–807, 2013.
- [GG73] Marty Golubitsky and Victor Guillemin. *Stable Mappings and Their Singularities*. Graduate Texts in Mathematics. Springer-Verlag New York, 1973.
- [Got91] Mark J. Gotay. A multisymplectic framework for classical field theory and the calculus of variations. I. Covariant Hamiltonian formalism. In *Mechanics, analysis and geometry: 200 years after Lagrange*, North-Holland Delta Ser., pages 203–235. North-Holland, Amsterdam, 1991.
- [GS77] Victor Guillemin and Shlomo Sternberg. *Geometric asymptotics*. American Mathematical Society, 1977.
- [GS15] Grégory Ginot and Mathieu Stiénon. G -gerbes, principal 2-group bundles and characteristic classes. *J. Symplectic Geom.*, 13(4):1001–1047, 2015.
- [H12] Frédéric Hélein. Multisymplectic formalism and the covariant phase space. In *Variational problems in differential geometry*, volume 394 of *London Math. Soc. Lecture Note Ser.*, pages 94–126. Cambridge Univ. Press, Cambridge, 2012.
- [Hep09] Richard Hepworth. Vector fields and flows on differentiable stacks. *Theory Appl. Categ.*, 22:542–587, 2009.
- [HJJS08] Dale Husemöller, Michael Joachim, Branislav Jurčo, and Martin Schottenloher. *Basic bundle theory and K-cohomology invariants*, volume 726 of *Lecture Notes in Physics*. Springer, Berlin, 2008. With contributions by Siegfried Echterhoff, Stefan Fredenhagen and Bernhard Krötz.
- [Kij73] Jerzy Kijowski. A finite-dimensional canonical formalism in the classical field theory. *Comm. Math. Phys.*, 30:99–128, 1973.
- [KM97] Andreas Kriegl and Peter W. Michor. *The convenient setting of global analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1997.

- [KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I.* Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.
- [Koh02] Toshitake Kohno. *Conformal field theory and topology*, volume 210 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2002. Translated from the 1998 Japanese original by the author, Iwanami Series in Modern Mathematics.
- [Kos70] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. 1970.
- [KV] Derek Krepski and Jennifer Vaughan. Multiplicative vector fields on bundle gerbes. *arXiv e-prints arXiv:2003.12874*.
- [Mac05] Kirill C. H. Mackenzie. *General theory of Lie groupoids and Lie algebroids*, volume 213 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2005.
- [Mar88] Geoffrey Martin. A Darboux theorem for multi-symplectic manifolds. *Letters in Mathematical Physics*, 16(2):133–138, 1988.
- [Mic08] Peter W. Michor. *Topics in differential geometry*, volume 93 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [Mos65] Jürgen Moser. On the volume elements on a manifold. *Transactions of the American Mathematical Society*, 120:286–294, 1965.
- [MS00] Michael K. Murray and Daniel Stevenson. Bundle gerbes: stable isomorphism and local theory. *J. London Math. Soc. (2)*, 62(3):925–937, 2000.
- [Mur96] Michael K. Murray. Bundle gerbes. *J. London Math. Soc. (2)*, 54(2):403–416, 1996.
- [Mur10] Michael K. Murray. An introduction to bundle gerbes. In *The many facets of geometry*, pages 237–260. Oxford Univ. Press, Oxford, 2010.

- [MX98] Kirill C. H. Mackenzie and Ping Xu. Classical lifting processes and multiplicative vector fields. *Quart. J. Math. Oxford Ser. (2)*, 49(193):59–85, 1998.
- [Noo13] Behrang Noohi. Integrating morphisms of Lie 2-algebras. *Compos. Math.*, 149(2):264–294, 2013.
- [NW13] Thomas Nikolaus and Konrad Waldorf. Four equivalent versions of nonabelian gerbes. *Pacific J. Math.*, 264(2):355–419, 2013.
- [OW19] Cristian Ortiz and James Waldron. On the Lie 2-algebra of sections of an LA -groupoid. *J. Geom. Phys.*, 145:103474, 34, 2019.
- [Rog11] Christopher L. Rogers. *Higher Symplectic Geometry*. 2011. Thesis (Ph.D.)—University of California, Riverside. arXiv:1106.4068.
- [Rog12] Christopher L. Rogers. L-infinity algebras from multisymplectic geometry. *Letters in Mathematical Physics*, 100(1):29–50, Apr 2012.
- [Rog13] Christopher L. Rogers. 2-plectic geometry, Courant algebroids, and categorified prequantization. *J. Symplectic Geom.*, 11(1):53–91, 2013.
- [Roy99] Dmitry Roytenberg. *Courant algebroids, derived brackets and even symplectic supermanifolds*. 1999. Thesis (Ph.D.)—University of California, Berkeley. arXiv:math/9910078.
- [RR09] Narciso Román-Roy. Multisymplectic Lagrangian and Hamiltonian formalisms of classical field theories. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 5:Paper 100, 25, 2009.
- [RW19] Leonid Ryvkin and Tilmann Wurzbacher. An invitation to multisymplectic geometry. *J. Geom. Phys.*, 142:9–36, 2019.
- [Ryv] Leonid Ryvkin. A multisymplectic manifold not covered by Darboux charts. *arXiv e-prints arXiv:1608.07424*.
- [Ryv16] Leonid Ryvkin. *Observables and Symmetries of n -Plectic Manifolds*. BestMasters. Springer Spektrum, 2016.
- [Sou66] J.-M. Souriau. Quantification géométrique. *Comm. Math. Phys.*, 1:374–398, 1966.

- [Spa81] Edwin H. Spanier. *Algebraic topology*. Springer-Verlag, New York-Berlin, 1981. Corrected reprint.
- [SS13] Christian Sämann and Richard J. Szabo. Groupoids, loop spaces and quantization of 2-plectic manifolds. *Rev. Math. Phys.*, 25(3):1330005, 72, 2013.
- [Ste00] Daniel Stevenson. *The Geometry of Bundle Gerbes*. 2000. Thesis (Ph.D.)—University of Adelaide. arXiv:math/0004117.
- [SZ12] Yunhe Sheng and Chenchang Zhu. Integration of semidirect product Lie 2-algebras. *Int. J. Geom. Methods Mod. Phys.*, 9(5):1250043, 31, 2012.
- [Wal18] Konrad Waldorf. A global perspective to connections on principal 2-bundles. *Forum Math.*, 30(4):809–843, 2018.
- [Wei71] Alan Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Advances in Mathematics*, 6:329–346, 1971.
- [Woo92] Nicholas M. J. Woodhouse. *Geometric quantization*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1992. Oxford Science Publications.
- [Zam12] Marco Zambon. L_∞ -algebras and higher analogues of Dirac structures and Courant algebroids. *J. Symplectic Geom.*, 10(4):563–599, 2012.