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Steklov and Neumann eigenvalues : inequalities, asymptotic and mixed problems

Marco Michetti

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Université de Lorraine
Institut Élie Cartan Lorraine
Ecole doctorale IAEM Lorraine

Steklov and Neumann eigenvalues: inequalities, asymptotic and mixed problems

Thèse Présentée par:
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Pour obtenir le grade de
Docteur de l'Université de Lorraine
Spécialité: **Mathématiques.**

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Résumé

Cette thèse est consacrée à l'étude des valeurs propres de Neumann, des valeurs propres de Steklov et des relations entre elles. La motivation initiale de cette thèse était de prouver que, dans le plan, le produit entre le périmètre et la première valeur propre de Steklov est toujours inférieur au produit entre l'aire et la première valeur propre de Neumann. Motivés par la recherche de contre-exemples à cette inégalité, nous donnons, dans la première partie de cette thèse, une description complète du comportement asymptotique des valeurs propres de Steklov dans un domaine en haltère constitué de deux ensembles de Lipschitz reliés par un tube mince de largeur qui va à zéro. En utilisant ces résultats dans le cas bidimensionnel, nous trouvons que l'inégalité n'est pas toujours vraie. Nous étudions l'inégalité dans le cadre convexe, en prouvant une forme plus faible de l'inégalité pour tous les domaines convexes et en prouvant l'inégalité pour une classe spéciale de polygones convexes. Nous donnons également le comportement asymptotique des valeurs propres de Neumann et de Steklov sur des domaines convexes qui s'effondrent, en reliant de cette façon ces deux valeurs propres aux valeurs propres de type Sturm-Liouville. Dans la deuxième partie de cette thèse, en utilisant les résultats concernant le comportement asymptotique des valeurs propres de Neumann sur les domaines effondrés et une analyse fine des fonctions propres de Sturm-Liouville, nous étudions le problème de maximisation des valeurs propres de Neumann sous contrainte de diamètre. Dans la dernière partie de la thèse, nous étudions les valeurs propres de Steklov-Dirichlet. Après une première discussion sur les propriétés de régularité des fonctions propres de Steklov-Dirichlet, nous obtenons un résultat de stabilité pour les valeurs propres. Nous étudions le problème d'optimisation sous une contrainte de mesure sur l'ensemble dans lequel nous imposons des conditions de Steklov, nous prouvons l'existence d'un minimiseur et la non-existence d'un maximiseur. Dans le plan, nous prouvons un résultat de continuité pour les valeurs propres sous une certaine contrainte topologique.

Mots-clés: Valeurs propres de Steklov, valeurs propres de Neumann, géométrie spectrale, analyse asymptotique, optimisation de forme, inégalités spectrales, problèmes de valeurs propres mixtes.

Abstract

This thesis is devoted to the study of Neumann eigenvalues, Steklov eigenvalues and relations between them. The initial motivation of this thesis was to prove that, in the plane, the product between the perimeter and the first Steklov eigenvalue is always less than the product between the area and the first Neumann eigenvalue. Motivated by finding counterexamples to this inequality, in the first part of this thesis, we give a complete description of the asymptotic behavior of the Steklov eigenvalues in a dumbbell domain consisting of two Lipschitz sets connected by a thin tube with vanishing width. Using these results in the two dimensional case we find that the inequality is not always true. We study the inequality in the convex setting, proving a weaker form of the inequality for all convex domains and proving the inequality for a special class of convex polygons. We then also give the asymptotic behavior for Neumann and Steklov eigenvalues on collapsing convex domains, linking in this way these two eigenvalues with Sturm-Liouville type eigenvalues. In the second part of this thesis, using the results concerning the asymptotic behavior of Neumann eigenvalues on collapsing domains and a fine analysis of Sturm-Liouville eigenfunctions we study the maximization problem of Neumann eigenvalues under diameter constraint. In the last part of the thesis we study the mixed Steklov-Dirichlet. After a first discussion about the regularity properties of the Steklov-Dirichlet eigenfunctions we obtain a stability result for the eigenvalues. We study the optimization problem under a measure constraint on the set in which we impose Steklov boundary conditions, we prove the existence of a minimizer and the non-existence of a maximizer. In the plane we prove a continuity result for the eigenvalues under some topological constraint.

Keywords: Steklov eigenvalues, Neumann eigenvalues, spectral geometry, asymptotic analysis, shape optimization, spectral inequalities, mixed eigenvalue problems.

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Résumé des principaux résultats

Le principal objectif de cette thèse est d'étudier les relations entre les valeurs propres de Neumann et de Steklov pour le Laplacien.

Soit $\Omega \subset \mathbb{R}^d$ un ouvert borné, connexe et à bord Lipschitzien, le problème des valeurs propres de Neumann dans Ω consiste en la résolution de (où u est non nulle)

$$\begin{cases} -\Delta u = \mu(\Omega)u & \Omega \\ \partial_\nu u = 0 & \partial\Omega. \end{cases}$$

Comme l'injection de Sobolev $H^1(\Omega) \rightarrow L^2(\Omega)$ est compacte ici, le spectre du problème de Neumann est discret et la suite de valeurs propres (comptées avec leur multiplicité) est croissante vers l'infini:

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \dots \rightarrow +\infty.$$

Le problème de Steklov sur Ω consiste en la résolution du problème de valeurs propres

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \sigma(\Omega)u & \partial\Omega. \end{cases}$$

Comme l'opérateur trace $H^1(\Omega) \rightarrow L^2(\partial\Omega)$ est compact, le spectre du problème de Steklov est discret et la suite de valeurs propres (comptées avec leur multiplicité) est croissante vers l'infini:

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \dots \rightarrow +\infty.$$

Une question naturelle est d'étudier les relations entre ces deux familles de valeurs propres. Plus précisément, nous aimerions savoir s'il existe certaines inégalités entre elles et, dans le cas où elles existent, quel type d'informations géométriques peut-on en tirer sur le domaine Ω .

La première chose à noter est que les deux suites de valeurs propres ont une homogénéité différente vis à vis des homothéties du domaine Ω . Plus précisément, soit t une constante positive, on définit $t\Omega = \{tx | x \in \Omega\}$ alors on a les relations suivantes

$$\mu_k(t\Omega) = \frac{\mu_k(\Omega)}{t^2}, \quad \sigma_k(t\Omega) = \frac{\sigma_k(\Omega)}{t}.$$

A cause de cette différence d'homogénéité, si on souhaite comparer ces valeurs propres, il convient d'en introduire une version invariante par homothétie. Cela s'obtient en multipliant la valeur propre par une quantité géométrique ayant la bonne homogénéité. Introduisons ainsi la normalisation naturelle pour les problèmes de Neumann et de Steklov pour des ouverts plans.

Soit $\Omega \subset \mathbb{R}^2$, comme on peut le voir sur l'équation qui définit les valeurs propres de Neumann, on peut heuristiquement penser que les fonctions propres vivent plutôt à

l'intérieur du domaine Ω , et pour cette raison il est naturel d'introduire la normalisation suivante pour des ouverts Ω du plan:

$$|\Omega|\mu_k(\Omega),$$

où $|\Omega|$ est l'aire de l'ensemble. Pour les mêmes raisons, les fonctions propres de Steklov (étant fonctions propres de l'opérateur Dirichlet-to-Neumann défini sur le bord) vivent plutôt sur le bord du domaine, ainsi une normalisation naturelle consiste à regarder

$$P(\Omega)\sigma_k(\Omega),$$

où $P(\Omega)$ est le périmètre de l'ensemble.

Une première question qui a motivé cette thèse est la suivante:

Soit $\Omega \subset \mathbb{R}^2$ un ouvert connexe, borné et à bord Lipschitzien, est-ce que l'inégalité suivante est vraie?

$$P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega) \quad ?$$

Dans cette thèse, nous considérons directement cette question dans le Chapitre 3 et dans la dernière section du Chapitre 2, les techniques développées pour étudier cette question (en particulier le développement des valeurs propres de Neumann pour des domaines s'aplatissant sur un segment) se sont avérées très utiles pour étudier d'autres problèmes spectraux, notamment ceux présentés dans le Chapitre 4.

Motivé par l'étude des différents comportements des valeurs propres de Neumann et Steklov, nous étudions également un problème mixte. Cela avait déjà été considéré dans [31] où les auteurs considéraient un problème mixte de type Neumann-Dirichlet. Dans le Chapitre 5 nous considérons le même type de questions mais pour un problème mixte Steklov-Dirichlet. Donnons à présent une description rapide des principaux résultats de chaque chapitre.

Chapitre 2

Motivé par l'inégalité spectrale $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ dont il était question ci-dessus, nous avons tout d'abord cherché un contre-exemple à cette inégalité dans le cadre général. Une idée naturelle a été de chercher dans la classe des "haltères", c'est-à-dire deux domaines disjoints reliés par un fin tuyau d'épaisseur ϵ tendant vers zéro.

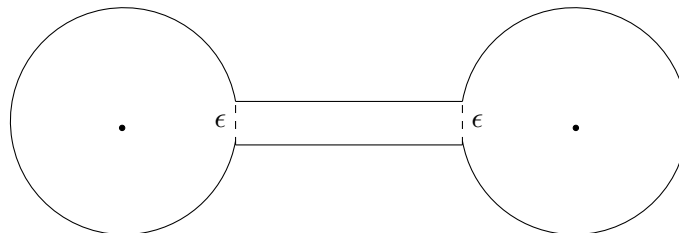


Figure 1: haltère 2-dimensionnelle avec un tuyau droit.

Le comportement asymptotique des valeurs propres de Neumann est bien compris dans ce type de situations, pour une large classe d'haltères différentes, et en toute dimension, nous renvoyons par exemple à [67] pour une telle description.

Dans le Chapitre 2 nous obtenons le développement asymptotique au premier ordre en ϵ pour les valeurs propres de Steklov dans des haltères générales (qui n'était pas connu). Plus précisément, on définit une haltère $\Omega_\epsilon \subset \mathbb{R}^d$ de la façon suivante:

$$\Omega_\epsilon = D_1 \cup T_\epsilon \cup D_2,$$

où D_1 et D_2 sont deux ouverts connexes, bornés, disjoints à bords Lipschitz dans \mathbb{R}^d et T_ϵ peut être écrit comme

$$T_\epsilon = \left\{ x = (x_1, x') \in \mathbb{R}^d \mid -\frac{L}{2} \leq x_1 \leq \frac{L}{2}, |x'| < \epsilon \rho(x_1) \right\},$$

où $L > 0$ et $\rho \in C^0\left(-\frac{L}{2}, \frac{L}{2}\right] \cap C^\infty\left(-\frac{L}{2}, \frac{L}{2}\right)$ est une fonction (strictement) positive.

Dans ce chapitre on donne un développement au premier ordre en ϵ des quantités $\sigma_k(\Omega_\epsilon)$ quand $\epsilon \rightarrow 0$. En particulier, on voit que toutes les valeurs propres tendent vers zéro, mais le taux de convergence et la constante en facteur du terme principal dépendent à la fois de la dimension et du rang de la valeur propre. Ces résultats sont énoncés dans le théorème ci-dessous dans lequel nous notons ω_d le volume de la boule unité d -dimensionnelle et nous introduisons l'application $\Phi_\epsilon : T_1 \rightarrow T_\epsilon$, $\Phi_\epsilon(x_1, x') = (x_1, \epsilon x')$.

Théorème. ($d = 2$) Soit $\Omega_\epsilon \subset \mathbb{R}^2$ le domaine de type haltère défini ci-dessus. Alors

$$\sigma_k^\epsilon \sim \mu_k \epsilon + o(\epsilon) \quad \text{quand } \epsilon \rightarrow 0,$$

où μ_k est la k -ième valeur propre du problème suivant

$$\begin{cases} -\frac{d}{dx} \left(\rho(x) \frac{dV_k}{dx}(x) \right) = \mu_k V_k(x) & x \in \left(-\frac{L}{2}, \frac{L}{2} \right) \\ \rho\left(-\frac{L}{2}\right) \frac{dV_k}{dx}\left(-\frac{L}{2}\right) = -\frac{\mu_k}{2} P(D_1) V_k\left(-\frac{L}{2}\right) \\ \rho\left(\frac{L}{2}\right) \frac{dV_k}{dx}\left(\frac{L}{2}\right) = \frac{\mu_k}{2} P(D_2) V_k\left(\frac{L}{2}\right). \end{cases} \quad (1)$$

Pour toute sous-suite $\{\epsilon_n\}_{n=1}^\infty$ telle que $\epsilon_n \rightarrow 0$, nous avons

$$u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_k \quad \text{dans } H^1(T_1),$$

où \bar{V}_k est la k -ième valeur propre du problème (3) étendue de façon constante dans la variable x_2 .

Théorème. ($d \geq 3$, $k \geq 2$) Soit $\Omega_\epsilon \subset \mathbb{R}^d$ le domaine de type haltère défini ci-dessus et $d \geq 3$. Alors, pour tout $k \geq 2$ on a

$$\sigma_k^\epsilon \sim \alpha_{k-1} \epsilon + o(\epsilon) \quad \text{quand } \epsilon \rightarrow 0,$$

où α_{k-1} est la $(k-1)$ -ième valeur propre (comptée à partir de 0) de

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} \left(\rho^{d-1}(x) \frac{dV_k}{dx}(x) \right) = \alpha_k \omega_{d-2} \rho^{d-2}(x) V_k(x) & x \in \left(-\frac{L}{2}, \frac{L}{2} \right) \\ V_k\left(-\frac{L}{2}\right) = 0 \\ V_k\left(\frac{L}{2}\right) = 0. \end{cases}$$

Pour toute sous-suite $\{\epsilon_n\}_{n=1}^\infty$ telle que $\epsilon_n \rightarrow 0$, on a

$$\epsilon_n^{\frac{d-2}{2}} u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_{k-1} \quad \text{dans } H^1(T_1),$$

où \bar{V}_{k-1} est une fonction propre correspondant à α_{k-1} , étendue de façon constante dans les variables x_i pour $2 \leq i \leq d$.

Théorème. ($d \geq 3$, $k = 1$) Soit $\Omega_\epsilon \subset \mathbb{R}^d$ le domaine de type haltère défini ci-dessus et $d \geq 3$. La première valeur propre Steklov a le comportement asymptotique suivant

$$\sigma_1^\epsilon \sim \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}) \quad \text{quand } \epsilon \rightarrow 0,$$

où σ_1 est l'unique nombre positif tel que l'équation différentielle ci-dessous ait une solution non triviale:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases} \quad (2)$$

Pour toute sous-suite $\{\epsilon_n\}_{n=1}^\infty$ telle que $\epsilon_n \rightarrow 0$, on a

$$u_1^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_1 \quad \text{in } H^1(T_1),$$

où \bar{V}_1 est la solution de l'équation (4) étendue de façon constante dans les variables x_i pour $2 \leq i \leq d$.

En utilisant ce développement asymptotique quand $\Omega_\epsilon \subset \mathbb{R}^2$ est une haltère bi-dimensionnelle construite de telle façon que D_1 et D_2 sont deux disques identiques et $\rho = 1$, pour L assez grand et ϵ assez petit, on obtient bien que $P(\Omega_\epsilon) \sigma_1(\Omega_\epsilon) > |\Omega_\epsilon| \mu_1(\Omega_\epsilon)$.

Chapitre 3

Dans le Chapitre 3 nous étudions de façon plus approfondie l'inégalité $P(\Omega) \sigma_1(\Omega) \leq |\Omega| \mu_1(\Omega)$ où $\Omega \subset \mathbb{R}^2$. Plus précisément, on introduit la fonctionnelle suivante :

$$F(\Omega) = \frac{\mu_1(\Omega) |\Omega|}{\sigma_1(\Omega) P(\Omega)}.$$

Il s'agit alors de chercher des bornes pour cette fonctionnelle et d'étudier sa minimisation/maximisation dans les cas où Ω est seulement supposé Lipschitzien, puis dans le cas où on suppose Ω convexe. Nous montrons que dans le cas Lipschitzien, les problèmes de minimisation/maximisation sont mal posés au sens où

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ ouvert borné Lipschitz}\} = 0,$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ ouvert borné Lipschitz}\} = +\infty.$$

Ensuite, nous étudions le cas convexe. Nous commençons par analyser la situation d'une suite de convexes convergeant vers un segment. Pour cela, nous analysons le comportement asymptotique des valeurs propres de Neumann et de Steklov dans cette situation: en particulier, nous introduisons la classe suivante de domaines "fins":

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\},$$

et nous montrons les résultats suivants

Théorème. Soit $h^+ \in L^\infty(0, 1)$ et $h^- \in L^\infty(0, 1)$ deux fonctions positives. Soit $h = h^+ + h^-$ et soit Ω_ϵ le domaine défini par

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

S'il existe $K > 0$ et $p < 2$ tels que $h(x) \geq K(x(1-x))^p$ p.p. dans $(0, 1)$, alors:

$$\mu_k(\Omega_\epsilon) = \mu_k(h) + o(1) \quad \text{quand } \epsilon \rightarrow 0,$$

où $\mu_k(h)$ est la k -ième valeur propre non triviale du problème de Sturm-Liouville

$$\begin{cases} -\frac{d}{dx}(h(x)\frac{du}{dx}(x)) = \mu(h)h(x)u(x) & x \in (0, 1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0, \end{cases}$$

et nous prouvons le résultat analogue pour les valeurs propres de Steklov

Théorème. Soit $h^+ \in L^\infty(0, 1)$ et $h^- \in L^\infty(0, 1)$ deux fonctions positives. Soit $h = h^+ + h^-$ et soit Ω_ϵ le domaine défini par

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

S'il existent $K > 0$ et $p < 2$ tels que $h(x) \geq K(x(1-x))^p$ p.p. dans $(0, 1)$, alors

$$\sigma_k(\Omega_\epsilon) = \frac{\sigma_k(h)}{2}\epsilon + o(\epsilon) \quad \text{quand } \epsilon \rightarrow 0,$$

où $\sigma_k(h)$ est la k -ième valeur propre non triviale du problème de Sturm-Liouville

$$\begin{cases} -\frac{d}{dx}(h(x)\frac{dv}{dx}(x)) = \sigma(h)v(x) & x \in (0, 1) \\ h(0)\frac{dv}{dx}(0) = h(1)\frac{dv}{dx}(1) = 0. \end{cases}$$

À l'aide de ces résultats, nous pouvons obtenir le comportement asymptotique de la fonctionnelle $F(\Omega_\epsilon)$ et nous obtenons à la limite une nouvelle fonctionnelle $F(h)$ qui ne dépend que de la géométrie de la suite de domaines s'écrasant vers un segment. Nous pouvons alors étudier les problèmes de minimisation et maximisation de cette fonctionnelle $F(h)$.

Dans la seconde partie de ce chapitre, nous trouvons des bornes inférieures et supérieures explicites pour $F(\Omega)$ lorsque Ω est un convexe plan, plus précisément nous prouvons

Théorème. Il existe une constante explicite C_1 telle que, pour tout ouvert convexe borné et non vide $\Omega \subset \mathbb{R}^2$, on ait l'inégalité suivante:

$$\frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) \leq C_1 \leq 9.04.$$

Nous étudions alors le diagramme de Blaschke-Santaló défini par

$$\mathcal{E} = \{(x, y) \text{ avec } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2\}$$

$$\mathcal{E}^C = \{(x, y) \text{ avec } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2, \Omega \text{ convexe.}\},$$

et nous prouvons

Théorème.

$$\bar{\mathcal{E}} = [0, 8\pi] \times [0, \mu_1(\mathbb{D})\pi]$$

où $\mu_1(\mathbb{D}) = j_{11}^2$ est la première valeur propre (non triviale) de Neumann du disque unité.

Dans la dernière partie de ce chapitre, nous introduisons une nouvelle façon de prouver l'inégalité $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ pour les domaines ayant deux axes de symétrie. Cette approche est basée sur la preuve d'une version L^2 d'une conjecture de type hot spot pour la première fonction propre de Neumann d'un domaine convexe. En particulier, l'un des résultats que nous montrons est le suivant

Théorème. *Soit Ω un polygone circonscrit à un cercle et soit u sa première fonction propre de Neumann (non triviale), alors*

$$\frac{1}{P(\Omega)} \int_{\partial\Omega} u^2 ds \geq \frac{1}{|\Omega|} \int_{\Omega} u^2 dx.$$

Une conséquence directe de ce théorème est alors

Théorème. *Soit Ω un polygone circonscrit à un cercle avec deux axes de symétries, alors*

$$P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega).$$

Chapitre 4

NOTE: *Ce chapitre correspond à la première version de l'article [56]. Juste avant que nous soumettions cet article, le Professeur Pedro Freitas a porté à notre connaissance l'article de P. Kröger [74]. que nous ne connaissions pas: nous lui en sommes très reconnaissants! Cet article a obtenu les mêmes résultats (en dimension deux) par une approche assez différente. L'originalité de notre approche est d'utiliser le formalisme développé notamment dans [79] pour démontrer le Théorème . Comme nous caractérisons de cette manière les points critiques de la fonctionnelle $h \mapsto \mu_k(h)$, notre méthode peut également être utilisée dans d'autres contextes.*

Ce chapitre est consacré à l'étude des valeurs propres de Neumann sous contrainte de diamètre. Ce problème avait déjà été étudié dans [8, 29, 74], et dans ce chapitre nous montrons d'une part, qu'en général, la quantité $D(\Omega)^2\mu_k(\Omega)$ est non bornée si $\Omega \subset \mathbb{R}^d$ est un domaine à bord simplement Lipschitz. D'autre part, nous donnons des bornes optimales pour cette quantité lorsque $\Omega \subset \mathbb{R}^2$ est un domaine convexe. Nous donnons également des suites maximisantes qui saturent cette borne supérieure. Le point de départ de cette analyse consiste à comprendre le lien profond entre les valeurs propres de Neumann $\mu_k(\Omega)$ et le problème de Sturm-Liouville suivant

$$\begin{cases} -\frac{d}{dx}\left(h(x)\frac{du}{dx}(x)\right) = \mu(h)h(x)u(x) & x \in (0, 1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0. \end{cases}$$

L'idée principale, heuristiquement, est d'utiliser la propriété (prouvée au Chapitre 3), que pour tout domaine régulier Ω il existe une fonction h telle que $\mu_k(\Omega) < \mu_k(h)$ et réciproquement, pour toute fonction h il existe une suite de domaines convexes Ω_ϵ dont l'adhérence converge au sens de Hausdorff vers un segment, et telle que $\mu_k(\Omega_\epsilon) \rightarrow \mu_k(h)$.

En utilisant cette connection nous montrons d'abord, sans l'hypothèse de convexité, que nous pouvons construire une suite de domaines avec un diamètre fixé et dont la première valeur propre de Neumann μ_1 est arbitrairement grande, en particulier:

Théorème. *Pour tout $d \geq 2$ on a l'égalité suivante*

$$\sup\{D(\Omega)^2\mu_1(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ Ouvert et Lipschitz}\} = +\infty.$$

Nous étudions alors le cas où Ω est un domaine convexe du plan. Cette question nous conduit naturellement à l'étude d'un problème d'optimisation pour les valeurs propres d'un problème de Sturm-Liouville avec contrainte de concavité sur la fonction h , plus précisément, on étudie

$$\sup\{\mu_k(h) \mid h \text{ concave, positive sur } [0, 1]\}.$$

Le résultat principal du Chapitre 4 est le théorème suivant

Théorème. *Pour tout $k \geq 1$, le problème $\max\{\mu_k(h), h \text{ concave, positive sur } [0, 1]\}$ a une solution h_k^* , de plus*

- $\max\{\mu_1(h), h \text{ concave, positive sur } [0, 1]\} = \mu_1(h_1^*) = (2j_{0,1})^2$ et

$$h_1^* = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ 2(1-x) & x \in [\frac{1}{2}, 1] \end{cases}$$

- Soit $k \geq 2$ alors $\max\{\mu_k(h), h \text{ concave, positive sur } [0, 1]\} = \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2$

$$h_k^* = \begin{cases} \frac{x(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [0, \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ 1 & x \in [\frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ \frac{(1-x)(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1] \end{cases}$$

Le résultat clé (et aussi le plus difficile) pour obtenir ce théorème est le suivant, que nous pouvons prouver en utilisant l'écriture des conditions d'optimalité avec contraintes de concavité telles que développées dans [79]

Théorème. *Pour tout $k \geq 1$, le problème $\max\{\mu_k(h), h \in \mathcal{L}\}$ a une solution h_k^* . Celle-ci a un graphe qui est une ligne polygonale composée (au plus) de $k+1$ segments.*

Comme application de ce résultat nous retrouvons les inégalités de Kröger :

Théorème. *Pour tout $\Omega \subset \mathbb{R}^2$ convexe*

$$D(\Omega)^2 \mu_k(\Omega) \leq \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2.$$

L'égalité est obtenue à la limite par une suite de convexes fins Ω_ϵ avec ϵ allant vers zéro, où Ω_ϵ est une suite de domaine définie par

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

avec $h^+ + h^- = h_k^*$.

Chapitre 5

Ce chapitre est consacré à l'étude du problème mixte de valeurs propres Steklov-Dirichlet. Soit $\Omega \subset \mathbb{R}^d$ un ouvert borné, connexe à bord Lipschitzien. Soit $\Gamma_S \subset \partial\Omega$ une partie relativement ouverte de son bord (également supposée Lipschitz) et on définit aussi $\Gamma_D = \partial\Omega \setminus \Gamma_S$. On considère le problème mixte dit de Steklov-Dirichlet suivant:

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \lambda(\Omega, \Gamma_S)u & \Gamma_S \\ u = 0 & \Gamma_D. \end{cases}$$

Ce type de problème de valeurs propres a été déjà étudié, nous renvoyons par exemple à [10, 51]. Le principal but de ce chapitre est d'étudier la dépendance des valeurs propres $\lambda_k(\Omega, \Gamma_S)$ en fonction de $\Gamma_S \subset \partial\Omega$. En particulier, nous analysons les problèmes variationnels suivants, où $0 < m < 1$ est une constante positive

$$\inf\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\},$$

$$\sup\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\},$$

Ce type de problème variationnel a déjà été étudié dans le cas d'un problème mixte Dirichlet-Neumann, voir [31]. Dans la première section nous commençons par analyser les propriétés de régularité pour le problème de Steklov-Dirichlet. Puis nous montrons un résultat de stabilité pour les valeurs propres $\lambda_k(\Omega, \Gamma_S)$. Enfin, nous prouvons des résultats d'existence et de non-existence pour les problèmes variationnels ci-dessus, plus précisément

Théorème. *Soit $\Omega \subset \mathbb{R}^d$ un ouvert Lipschitzien et soit $0 < m < 1$ une constante, alors pour tout k , le problème variationnel suivant*

$$\inf\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\},$$

a une solution.

Théorème. *Soit $\Omega \subset \mathbb{R}^d$ un ouvert Lipschitzien et soit $0 < m < 1$ une constante, alors on a l'égalité suivante*

$$\sup\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\} = +\infty.$$

Dans la seconde partie de ce chapitre, nous montrons un résultat que l'on peut voir comme un analogue du fameux résultat de V. Šverák concernant les valeurs propres de Dirichlet (voir [58, 96]) cette fois pour les valeurs propres de Steklov-Dirichlet, en particulier on montre

Théorème. *Soit $\Omega \subset \mathbb{R}^2$ un ouvert de classe $\mathcal{C}^{1,1}$, Soit $\Gamma_{D,n} \subset \partial\Omega$ une suite de sous-ensembles compacts qui converge pour la distance de Hausdorff vers un compact $\Gamma_D \subset \partial\Omega$. On définit les deux ensembles $\Gamma_{S,n} = \partial\Omega \setminus \Gamma_{D,n}$ et $\Gamma_S = \partial\Omega \setminus \Gamma_D$. Si on suppose que le nombre de composantes connexes de $\Gamma_{D,n}$ est uniformément borné, alors pour tout k*

$$\lambda_k(\Omega, \Gamma_{S,n}) \rightarrow \lambda_k(\Omega, \Gamma_S).$$

En utilisant ce résultat de continuité, on prouve l'existence d'un maximiseur pour $\lambda_k(\Omega, \Gamma_S)$ si on se restreint à la classe des ensembles qui sont le complémentaire de compacts de mesure donnée et avec un nombre fini donné de composantes connexes.

Summary of the main results

The main object of the present thesis is to investigate the relations between the Neumann and the Steklov eigenvalues.

Let $\Omega \subset \mathbb{R}^d$ open, bounded, connected and Lipschitz set, the Neumann eigenvalue problem on Ω consists in solving the eigenvalue problem

$$\begin{cases} -\Delta u = \mu(\Omega)u & \Omega \\ \partial_\nu u = 0 & \partial\Omega. \end{cases}$$

As the Sobolev embedding $H^1(\Omega) \rightarrow L^2(\Omega)$ is compact here, the spectrum of the Neumann problem is discrete and the eigenvalues (counted with their multiplicities) go to infinity

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \dots \rightarrow +\infty.$$

The Steklov problem on Ω consists in solving the eigenvalue problem

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \sigma(\Omega)u & \partial\Omega. \end{cases}$$

As the trace operator $H^1(\Omega) \rightarrow L^2(\partial\Omega)$ is compact, the spectrum of the Steklov problem is discrete and the eigenvalues (counted with their multiplicities) go to infinity

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \dots \rightarrow +\infty.$$

A natural question is to investigate the relations between the Neumann and Steklov eigenvalues. More precisely we want to know if there are some inequalities relating the two spectra and if these inequalities do occur, what is the geometric information that we can have about the domain Ω .

The first thing to notice is that the two eigenvalues have a different homogeneity under the rescaling of the domain Ω . More precisely let t be a positive constant, we define $t\Omega = \{tx|x \in \Omega\}$, then the following relations hold

$$\mu_k(t\Omega) = \frac{\mu_k(\Omega)}{t^2}, \quad \sigma_k(t\Omega) = \frac{\sigma_k(\Omega)}{t}.$$

Thanks to this difference in the homogeneity, in order to compare these two spectral quantities, we need to introduce a scale invariant version of these eigenvalues. That means to multiply the eigenvalue by a geometric quantity related to the set Ω in order to obtain that this product is invariant under scaling of the set. We now introduce the natural normalization of the Neumann eigenvalue and the natural normalization of the Steklov eigenvalue in the plane.

Let $\Omega \subset \mathbb{R}^2$, as we can see from the equation that defines the Neumann eigenvalues, we can heuristically think that the Neumann eigenvalues live inside the domain Ω , for

this reason it is natural to introduce the following normalized version of the Neumann eigenvalues in the plane

$$|\Omega|\mu_k(\Omega),$$

where $|\Omega|$ is the area of the set. For the same reason we can heuristically think that the Steklov eigenvalues live on the boundary of the domain, for this reason a natural normalized version of the Steklov eigenvalues in the plane is the following

$$P(\Omega)\sigma_k(\Omega),$$

where $P(\Omega)$ is the perimeter of the set.

A first question that motivates this thesis is the following:

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, connected and Lipschitz set, is the following inequality true

$$P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega) \quad ?$$

In this Thesis we directly address this question in Chapter 3 and in the last section of Chapter 2, but the techniques we developed in order to study this question (in particular the asymptotic expansion of Neumann eigenvalues on collapsing domains) are very flexible and it turns out to be very useful to study other spectral problems that are presented in Chapter 4.

Motivated by the study of the different behavior between Neumann and Steklov eigenvalue we analyse also what happens to mixed eigenvalue problem, in [31] the authors studied optimization problems for mixed Neumann-Dirichlet eigenvalue problem, in Chapter 5 we address the same questions for the mixed Steklov-Dirichlet eigenvalues. We now give a short description of the main results presented in this thesis.

Chapter 2

Motivated by the conjectured spectral inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ we wanted, at first, to find some counterexample at this inequality in the general case. In order to find such counterexample we started to analyze what happens if we consider a dumbbell domain in the plane, when the width of the channel is ϵ , and we send ϵ to zero.

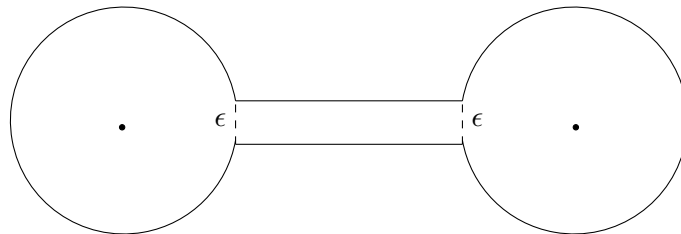


Figure 2: 2–dimensional dumbbell shape domain with straight channel.

The asymptotic behavior of the Neumann eigenvalues is completely known in all dimensions and for a huge class of different dumbbell domains, this behavior was presented e.g. in [67].

In Chapter 2 we find the first order asymptotic expansion in ϵ for the Steklov eigenvalues in general dumbbell domains. More precisely we define the dumbbell domains $\Omega_\epsilon \subset \mathbb{R}^d$

in the following way:

$$\Omega_\epsilon = D_1 \cup T_\epsilon \cup D_2,$$

where D_1 and D_2 are disjoint, bounded, open, connected sets in \mathbb{R}^d with Lipschitz boundary and T_ϵ is expressed as

$$T_\epsilon = \left\{ x = (x_1, x') \in \mathbb{R}^d \mid -\frac{L}{2} \leq x_1 \leq \frac{L}{2}, |x'| < \epsilon \rho(x_1) \right\},$$

where $L > 0$ and $\rho \in C^0([-\frac{L}{2}, \frac{L}{2}]) \cap C^\infty((-\frac{L}{2}, \frac{L}{2}))$ is a positive function.

In this chapter we give a first order expansion of the quantities $\sigma_k(\Omega_\epsilon)$ when $\epsilon \rightarrow 0$. In particular we see that all the eigenvalues go to zero when ϵ goes to zero, but the speed of convergence and the constant in front of the main factor in ϵ strongly depends on the dimension d and on the index of the eigenvalue k . This asymptotic behavior are stated in the following theorems. We denote by ω_d the Lebesgue measure of the d -dimensional unit ball and we define the following map $\Phi_\epsilon : T_1 \rightarrow T_\epsilon$, $\Phi_\epsilon(x_1, x') = (x_1, \epsilon x')$.

Theorem. ($d = 2$) *Let $\Omega_\epsilon \subset \mathbb{R}^2$ be the dumbbell shape domain defined as above. Then*

$$\sigma_k^\epsilon \sim \mu_k \epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where μ_k is the k -th eigenvalue of the following problem

$$\begin{cases} -\frac{d}{dx}(\rho(x) \frac{dV_k}{dx}(x)) = \mu_k V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho(-\frac{L}{2}) \frac{dV_k}{dx}(-\frac{L}{2}) = -\frac{\mu_k}{2} P(D_1) V_k(-\frac{L}{2}) \\ \rho(\frac{L}{2}) \frac{dV_k}{dx}(\frac{L}{2}) = \frac{\mu_k}{2} P(D_2) V_k(\frac{L}{2}). \end{cases} \quad (3)$$

For every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_k \quad \text{in } H^1(T_1),$$

where \bar{V}_k is a k -th eigenfunction of the problem (3) constantly extended in the variable x_2 .

Theorem. ($d \geq 3$, $k \geq 2$) *Let $\Omega_\epsilon \subset \mathbb{R}^d$ be the dumbbell shape domain defined as above and $d \geq 3$. Then for all $k \geq 2$ we have*

$$\sigma_k^\epsilon \sim \alpha_{k-1} \epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where α_{k-1} is the $(k-1)$ -th eigenvalue (counting from zero) of

$$\begin{cases} -\omega_{d-1} \frac{d}{dx}(\rho^{d-1}(x) \frac{dV_k}{dx}(x)) = \alpha_k \omega_{d-2} \rho^{d-2}(x) V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ V_k(-\frac{L}{2}) = 0 \\ V_k(\frac{L}{2}) = 0. \end{cases}$$

For every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$\epsilon_n^{\frac{d-2}{2}} u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_{k-1} \quad \text{in } H^1(T_1),$$

where \bar{V}_{k-1} is an eigenfunction corresponding to α_{k-1} , constantly extended into the variables x_i for $2 \leq i \leq d$.

Theorem. ($d \geq 3, k = 1$) Let $\Omega_\epsilon \subset \mathbb{R}^d$ be the dumbbell shape domain defined as above and $d \geq 3$. The first Steklov eigenvalue has the following asymptotic behavior

$$\sigma_1^\epsilon \sim \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}) \quad \text{as } \epsilon \rightarrow 0,$$

where σ_1 is the unique positive number such that the following differential equation has a non-trivial solution:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases} \quad (4)$$

For every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$u_1^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_1 \quad \text{in } H^1(T_1),$$

where \bar{V}_1 is the solution of the equation (4) constantly extended to the variables x_i for $2 \leq i \leq d$.

Using this asymptotic results, in the specific case when $\Omega_\epsilon \subset \mathbb{R}^2$ is a dumbbell domain constructed in such a way that D_1 and D_2 are two equal balls and $\rho = 1$, if L is large enough and ϵ is small enough we obtain that $P(\Omega_\epsilon) \sigma_1(\Omega_\epsilon) > |\Omega_\epsilon| \mu_1(\Omega_\epsilon)$.

Chapter 3

In Chapter 3 we study in a more deep way the inequality $P(\Omega) \sigma_1(\Omega) \leq |\Omega| \mu_1(\Omega)$ for $\Omega \subset \mathbb{R}^2$. In particular we introduce the following spectral shape functional

$$F(\Omega) = \frac{\mu_1(\Omega) |\Omega|}{\sigma_1(\Omega) P(\Omega)}.$$

We want to study maximization and minimization problems for this functional, in the first place when $\Omega \subset \mathbb{R}^2$ is a Lipschitz set and then we analyze the case when $\Omega \subset \mathbb{R}^2$ is convex. We show that in the class of Lipschitz domains the problem of maximization and minimization of $\Omega \rightarrow F(\Omega)$ is ill posed. Indeed we have

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open set and Lipschitz}\} = 0,$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open set and Lipschitz}\} = +\infty.$$

We then analyze the problem in the class of convex sets. We start by studying what happens to a sequence of collapsing convex domains. The main result in this context is the proof of an asymptotic expansion for the Steklov eigenvalues and Neumann eigenvalues on convex thin domains. In particular we introduce the following class of convex thin domains

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\},$$

and precisely we prove the following theorems

Theorem. Let $h^+ \in L^\infty(0,1)$ and $h^- \in L^\infty(0,1)$ be two non-negative functions. Let $h = h^+ + h^-$ and let Ω_ϵ be the following thin domain

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

If there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0,1)$, then:

$$\mu_k(\Omega_\epsilon) = \mu_k(h) + o(1) \quad \text{as } \epsilon \rightarrow 0,$$

Where $\mu_k(h)$ is the k -th non trivial eigenvalue of

$$\begin{cases} -\frac{d}{dx}(h(x)\frac{du}{dx}(x)) = \mu(h)h(x)u(x) & x \in (0,1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0, \end{cases}$$

and we prove the following result for the Steklov eigenvalues

Theorem. Let $h^+ \in L^\infty(0,1)$ and $h^- \in L^\infty(0,1)$ be two non-negative functions. Let $h = h^+ + h^-$ and let Ω_ϵ be the following thin domain

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

If there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0,1)$, then:

$$\sigma_k(\Omega_\epsilon) = \frac{\sigma_k(h)}{2}\epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where $\sigma_k(h)$ is the k -th non trivial eigenvalue of

$$\begin{cases} -\frac{d}{dx}(h(x)\frac{dv}{dx}(x)) = \sigma(h)v(x) & x \in (0,1) \\ h(0)\frac{dv}{dx}(0) = h(1)\frac{dv}{dx}(1) = 0. \end{cases}$$

Using this results we are able to find the asymptotic behavior of the functional $F(\Omega_\epsilon)$ and we find, at the limit, a new functional $F(h)$ that depends only on the geometry in which the sequence of the collapsing domains converge to a segment. We then perform an analysis for the maximizer and the minimizer for the functional $F(h)$.

In the second part of this chapter we find explicit upper and lower bound for $F(\Omega)$ when Ω is a convex set, more precisely we prove the following theorem

Theorem. There exists an explicit constant C_1 such that, for every convex open set $\Omega \subset \mathbb{R}^2$, the following inequalities hold

$$\frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) \leq C_1 \leq 9.04.$$

We then study the following Blaschke-Santaló diagrams

$$\mathcal{E} = \{(x, y) \text{ where } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2\}$$

$$\mathcal{E}^C = \{(x, y) \text{ where } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2, \Omega \text{ convex.}\},$$

we prove the following theorem

Theorem.

$$\bar{\mathcal{E}} = [0, 8\pi] \times [0, \mu_1(\mathbb{D})\pi]$$

where $\mu_1(\mathbb{D}) = j_{11}'^2$ is the first Neumann eigenvalue of the unit disk.

In the last part of this chapter we introduce a different way to approach the inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ for domains with two axis of symmetries. This approach is based on the proof of an L^2 version of the famous hot spot conjecture for the first Neumann eigenfunction. In particular, one of the results of the work in progress [57] is the following theorem

Theorem. *Let Ω be a circumscribed polygon in the plane, and let u be the first non trivial Neumann eigenfunction of Ω , then*

$$\frac{1}{P(\Omega)} \int_{\partial\Omega} u^2 ds \geq \frac{1}{|\Omega|} \int_{\Omega} u^2 dx.$$

A direct consequence of this theorem is the following

Theorem. *Let Ω be a circumscribed polygon in the plane with two axis of symmetries then*

$$P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega).$$

Chapter 4

NOTE: *this chapter corresponds to the first version of the paper [56]. Just before to submit this paper Pedro Freitas brought to the authors' attention the work by P. Kröger [74]. We warmly thanks Pedro Freitas for pointing out this reference. In our work we partially recover the results present in [74] using another approach. In particular the novelty of our work is to use the abstract formalism developed in [79] to prove the optimal upper bounds for $D(\Omega)^2\mu_k(\Omega)$. Our approach is based on finding critical points and then compare them, for this reason our approach can be also used in order to detect the minimum for the Sturm-Liouville eigenvalues. In the new version of the work [56] we will include these results.*

This chapter is devoted to the study of the Neumann eigenvalues under diameter constraint. This problem was already study in [8, 29, 74], in this chapter we prove that in general the quantity $D(\Omega)^2\mu_k(\Omega)$ is unbounded when $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain, then we give sharp upper bounds for this quantity when $\Omega \subset \mathbb{R}^2$ is a convex domain and we also characterize the sequence of domains that at the limit saturates the upper bound. The starting point of the analysis is to understand the deep link between the Neumann eigenvalues $\mu_k(\Omega)$ and the following Sturm-Liouville eigenvalues

$$\begin{cases} -\frac{d}{dx}\left(h(x)\frac{du}{dx}(x)\right) = \mu(h)h(x)u(x) & x \in (0, 1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0. \end{cases}$$

The main idea, heuristically, is to use the fact, proved in Chapter 3, that for every regular enough domain Ω there exists a function h such that $\mu_k(\Omega) < \mu_k(h)$ and for every h there exists a sequence of convex domains Ω_ϵ that converges in Hausdorff sense to a segment, such that $\mu_k(\Omega_\epsilon) \rightarrow \mu_k(h)$.

Using this connection we first prove that without the convexity assumption, we can construct domains with a fixed diameter and for which the Neumann eigenvalues μ_1 is arbitrary large, in particular we prove the following theorem

Theorem. For every $d \geq 2$ the following equality holds

$$\sup\{D(\Omega)^2 \mu_1(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ Open and Lipschitz}\} = +\infty.$$

We then study the case when Ω is a plane convex domain. This problem leads us to the study of an optimization problem for the Sturm-Liouville eigenvalue under concavity constraint on the function h , more precisely we deal with the following maximization problem

$$\sup\{\mu_k(h) \mid h \text{ concave, non negative on } [0, 1]\}.$$

The main result of Chapter 4 is the following theorem

Theorem. For any $k \geq 1$, the problem $\max\{\mu_k(h), h \text{ concave, non negative on } [0, 1]\}$ has a solution h_k^* , moreover

- $\max\{\mu_1(h), h \text{ concave, non negative on } [0, 1]\} = \mu_1(h_1^*) = (2j_{0,1})^2$ and

$$h_1^* = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ 2(1-x) & x \in [\frac{1}{2}, 1] \end{cases}$$

- let $k \geq 2$ then $\max\{\mu_k(h), h \text{ concave, non negative on } [0, 1]\} = \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2$

$$h_k^* = \begin{cases} \frac{x(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [0, \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ 1 & x \in [\frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ \frac{(1-x)(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1] \end{cases},$$

The key result (and also the difficult step) in order to prove this theorem is the following result, that we proved deeply using the abstract formulation for optimality conditions proved in [79].

Theorem. For any $k \geq 1$, the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ has a solution h_k^* . This one has a graph that is a polygonal line composed of (at most) $k+1$ segments.

As an application of this result we recover the Kröger inequalities.

Theorem. For every $\Omega \subset \mathbb{R}^2$ convex

$$D(\Omega)^2 \mu_k(\Omega) \leq \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2.$$

The equality is achieved by the sequence of thin convex domains Ω_ϵ when ϵ goes to zero, where Ω_ϵ is the following collapsing set

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

with $h^+ + h^- = h_k^*$.

Chapter 5

This chapter is devoted to the study of the mixed Steklov-Dirichlet eigenvalues. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, connected set with Lipschitz boundary. Let $\Gamma_S \subset \partial\Omega$ be a

relative open submanifold with Lipschitz boundary and we define also $\Gamma_D = \partial\Omega \setminus \Gamma_S$. We consider the following mixed Steklov-Dirichlet eigenvalue problem:

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \lambda(\Omega, \Gamma_S)u & \Gamma_S \\ u = 0 & \Gamma_D. \end{cases}$$

This kind of eigenvalues have been already studied see e.g. [10, 51]. and references therein. The main purpose of this chapter is to study the dependence of the eigenvalues $\lambda_k(\Omega, \Gamma_S)$ with respect to $\Gamma_S \subset \partial\Omega$. In particular we analyze the following variational problems, let $0 < m < 1$ be a constant

$$\begin{aligned} & \inf\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\}, \\ & \sup\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\}, \end{aligned}$$

this kind of variational problems were already studied for the mixed Neumann-Dirichlet eigenvalues in [31]. In the first section we analyze the regularity property for the Steklov-Dirichlet eigenfunctions. Then we prove a stability result for the the Steklov Dirichlet eigenvalues $\lambda_k(\Omega, \Gamma_S)$. We then prove existence and non-existence results for the variational problems, more precisely we prove that

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $0 < m < 1$ be a constant, then, for all k , the following variational problem*

$$\inf\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\},$$

has a solution.

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $0 < m < 1$ be a constant, then the following equality holds*

$$\sup\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\} = +\infty.$$

In the second part of this Chapter we prove a result in the same spirit of the famous result by V. Šverák concerning the Dirichlet eigenvalues (see [58, 96]) for the Steklov-Dirichlet eigenvalues, in particular we prove the following theorem

Theorem. *Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^{1,1}$ open domain, let $\Gamma_{D,n} \subset \partial\Omega$ be a sequence of compact subdomains converging for the Hausdorff metric to a compact set $\Gamma_D \subset \partial\Omega$. We define the two sets $\Gamma_{S,n} = \partial\Omega \setminus \Gamma_{D,n}$ and $\Gamma_S = \partial\Omega \setminus \Gamma_D$, assume that the number of connected components of $\Gamma_{D,n}$ is uniformly bounded, then for all k*

$$\lambda_k(\Omega, \Gamma_{S,n}) \rightarrow \lambda_k(\Omega, \Gamma_S).$$

Using this continuity property we prove the existence of a maximizer in the class of sets with given measure and bounded number of connected components.

Chapter 1

Introduction

1.1 The Steklov and the Neumann eigenvalues

1.1.1 The Neumann eigenvalue problem

Let $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, be an open, bounded and Lipschitz set. The Neumann eigenvalue problem on Ω consists in finding parameters $\mu(\Omega)$ for which the following equation has a solution in $H^1(\Omega)$ not identically equal to zero

$$\begin{cases} -\Delta u = \mu(\Omega)u & \Omega \\ \partial_\nu u = 0 & \partial\Omega, \end{cases} \quad (1.1)$$

where ∂_ν is the outward normal derivative along $\partial\Omega$. We consider the Neumann Laplacian operator, as the operator associated to the following symmetric bilinear form $a : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{R}$

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx.$$

The parameters $\mu(\Omega)$ coincide with the spectrum of the Neumann Laplacian operator. Thanks to the compact embedding $H^1(\Omega) \rightarrow L^2(\Omega)$ (that in this case holds because we assumed Ω to be Lipschitz) we can conclude that the Neumann Laplacian operator is a positive, self-adjoint and compact operator. In particular, thanks to the spectral theorem, its spectrum is given by a sequence of eigenvalues, called the Neumann eigenvalues of the domain Ω

$$0 = \mu_0(\Omega) \leq \mu_1(\Omega) \leq \mu_2(\Omega) \leq \dots \rightarrow +\infty,$$

where we count the eigenvalues with their multiplicities. The first eigenvalue $\mu_0(\Omega)$ is always equal to zero and it is associated to the constant eigenfunction, and if Ω is connected then $0 < \mu_1(\Omega)$.

It is also known that the Neumann eigenvalues admit the following variational characterization, also known as Poincaré principle or Courant-Fischer formulae (see [30])

$$\mu_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx}, \quad (1.2)$$

where the infimum is taken over all k -dimensional subspaces of the Sobolev space $H^1(\Omega)$ which are L^2 -orthogonal to constants on Ω . We notice, from the variational characterization given in (1.2), that if we make an homothety of the domain Ω with a ratio $t > 0$

we obtain a new domain $t\Omega = \{tx|x \in \Omega\}$ then the Neumann eigenvalues scale in the following way

$$\mu_k(t\Omega) = \frac{\mu_k(\Omega)}{t^2}. \quad (1.3)$$

A central question in spectral geometry is to study shape optimization problems for Neumann eigenvalues with geometric constraints, and it is interesting to analyze the behavior of the solutions with different geometric constraints. The first result in this direction is the solution of the following variational problem

$$\sup\{|\Omega|^{\frac{2}{d}}\mu_1(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ open and Lipschitz}\}, \quad (1.4)$$

we notice that, in order to have an interesting shape optimization problems we need to consider shape functional that are scale invariant. For this reason we need to consider the normalized Neumann eigenvalue $|\Omega|^{\frac{2}{d}}\mu_1(\Omega)$. We notice that, thanks to the scale invariance of the functional $|\Omega|^{\frac{2}{d}}\mu_1(\Omega)$, solving problem (1.4) is equivalent to solving the maximization problem for $\mu_1(\Omega)$ under a volume constraint on the domain Ω . The solution of problem (1.4) was first proved by Szegő in [97] for simply connected and two dimensional domain and then Weinberger in [105] removed the topological constraint and the dimensional restriction. This result is nowadays known as the Szegő-Weinberger inequality

Theorem 1.1 (Szegő-Weinberger inequality [97, 105]). *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and Lipschitz set then the following inequality holds*

$$|\Omega|^{\frac{2}{d}}\mu_1(\Omega) \leq |B|^{\frac{2}{d}}\mu_1(B)$$

where B is a ball in \mathbb{R}^d .

A natural question is to study the same variational problem for higher eigenvalues. Despite the simplicity of the statement, solving this problem is really hard. Indeed only recently the following maximization problem

$$\sup\{|\Omega|^{\frac{2}{d}}\mu_2(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ open and Lipschitz}\},$$

has been solved, more precisely A. Girouard, N. Nadirashvili and I. Polterovich in [45] solved the problem in two dimensions for simply connected domains and then in [24] D. Bucur and A. Henrot solved the problem in full generality and in all dimensions

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and Lipschitz set then the following inequality holds*

$$|\Omega|^{\frac{2}{d}}\mu_2(\Omega) \leq |\Omega^*|^{\frac{2}{d}}\mu_2(\Omega^*)$$

where Ω^* is the union of two equals disjoint balls.

For $k \geq 3$ also the existence of a maximizer for the following problem

$$\sup\{|\Omega|^{\frac{2}{d}}\mu_k(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ open and Lipschitz}\},$$

is an open question, see for instance [26].

It is now interesting to analyze the solution of the maximization problem with the perimeter constraint. The first big difference is that, at least in the plane, the maximization problem under perimeter constraint does not have a solution, in particular the following equality holds

$$\sup\{P(\Omega)^2\mu_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = +\infty.$$

In order to obtain the existence of a maximizer we need to restrict the class of admissible domains, in the class of plane, convex, and symmetric domain A. Henrot, A. Lemenant and I. Lucardesi in [54] proved the following result

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a convex domain with two axis of symmetry then the following inequality is true:*

$$P(\Omega)^2 \mu_1(\Omega) \leq P(T)^2 \mu_1(T) = 16\pi^2$$

where T is either a square or an equilateral triangle

We remark that the maximization problem under volume constraint and the maximization problem under perimeter constraint are different in nature. In particular the problem with the perimeter constraint has no solution in general and if we restrict in the class of convex and symmetric domains the solution is not given by the ball (as in the case of the volume constraint) and moreover is not unique.

In this Thesis, in Chapter 4, we study the maximization problem for the Neumann eigenvalues under Diameter constraint on Ω , see Section 1.5 in this Introduction

1.1.2 The Steklov eigenvalue problem

Let $\Omega \subset \mathbb{R}^d$, with $d \geq 2$, be an open, bounded and Lipschitz set. The Steklov eigenvalue problem on Ω consists in finding parameters $\sigma(\Omega)$ for which the following equation has a solution in $H^1(\Omega)$ not identically equal to zero

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \sigma(\Omega)u & \partial\Omega, \end{cases}$$

where ∂_ν is the outward normal derivative along $\partial\Omega$. We introduce the Dirichlet-to-Neumann (or Steklov-Poincaré) operator $\mathcal{D} : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega)$, which is defined by

$$\mathcal{D}(f) = \partial_\nu Hf$$

where Hf is the harmonic extension of f in Ω , i. e. Hf satisfies $\Delta Hf = 0$ on Ω and $Hf = f$ on $\partial\Omega$. The Dirichlet-to-Neumann map is a first order self-adjoint elliptic pseudodifferential operator (see [98]). The parameters $\sigma(\Omega)$ coincides with the spectrum of the Dirichlet-to-Neumann map \mathcal{D} . Thanks to the compactness of the trace operator $H^1(\Omega) \rightarrow L^2(\partial\Omega)$ (that in this case holds because we assumed Ω to be Lipschitz) we can conclude that the Dirichlet-to-Neumann map is a positive, self-adjoint and compact operator. In particular its spectrum is given by a sequence of eigenvalues, called the Steklov eigenvalues of the domain Ω

$$0 = \sigma_0(\Omega) \leq \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \dots \rightarrow +\infty,$$

where we count the eigenvalues with their multiplicities. The first eigenvalue $\sigma_0(\Omega)$ is always equal to zero and it is associated to the constant eigenfunction, and if Ω is connected then $0 < \sigma_1(\Omega)$.

Also the Steklov eigenvalues admit a variational characterization thanks to the Courant-Fischer formulae (see [30])

$$\sigma_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_\Omega |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 d\mathcal{H}^{d-1}}, \quad (1.5)$$

where the infimum is taken over all k -dimensional subspaces of the Sobolev space $H^1(\Omega)$ which are L^2 -orthogonal to constants on $\partial\Omega$. Thanks to this formulation is easy to see that the Steklov eigenvalues scales in the following way

$$\sigma_k(t\Omega) = \frac{\sigma_k(\Omega)}{t}. \quad (1.6)$$

As for the Neumann eigenvalues also for the Steklov eigenvalues the problem of shape optimization under different geometric constraint has been deeply studied. The problem of maximization of the first (non trivial) Steklov eigenvalue, thanks to the scaling property, reads as follows

$$\sup\{|\Omega|^{\frac{1}{d}}\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ open and Lipschitz}\}.$$

This problem has been solved by F. Brock in [19]

Theorem 1.4 (F. Brock inequality [19]). *Let $\Omega \subset \mathbb{R}^d$ be an open, bounded and Lipschitz set then the following inequality holds*

$$|\Omega|^{\frac{1}{d}}\sigma_1(\Omega) \leq |B|^{\frac{1}{d}}\sigma_1(B) = \omega_d^{\frac{1}{d}}$$

where B is a ball in \mathbb{R}^d and ω_d is the volume of the unit ball in \mathbb{R}^d .

The problem of the existence of a solution for the problem

$$\sup\{|\Omega|^{\frac{1}{d}}\sigma_k(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ open and Lipschitz}\},$$

has been studied in [15].

Another really interesting problem is to study the maximization problem under perimeter constraint. Heuristically we can think that this kind of constraint is more natural for the Steklov eigenvalues because these are eigenvalues that live on the boundary of the domain. The first result in this direction is the Weinstock inequality

Theorem 1.5 (Weinstock inequality [105]). *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, simply-connected and Lipschitz set then the following inequality holds*

$$P(\Omega)\sigma_1(\Omega) \leq P(\mathbb{D})\sigma_1(\mathbb{D}) = 2\pi$$

where \mathbb{D} is a disk.

In [22] Weinstock inequality in all dimensions for convex domains is proved. Weinstock used conformal maps, that is an argument similar to the argument used by Szegő in [97] in order to prove that for every open, bounded, simply-connected and Lipschitz set then $|\Omega|\mu_1(\Omega) \leq |\mathbb{D}|\mu_1(\mathbb{D})$. As we already discussed in Section 1.1.1 the proof of Szegő can be generalized and the Szegő-Weinberger inequality (Theorem 1.1) is true in all dimensions and also for non simply connected domain, this generalization is not possible for the Weinstock inequality. Indeed the Weinstock inequality is not true for non simply connected domain, in [47], the Steklov eigenvalues for the annulus $A_\epsilon = \mathbb{D} \setminus B(0, \epsilon)$ (where $B(0, \epsilon)$ is a ball centered in zero with radius ϵ) have been computed and it turns out that for ϵ small enough $P(A_\epsilon)\sigma_1(A_\epsilon) > P(\mathbb{D})\sigma_1(\mathbb{D})$. A natural question is to prove the existence or non-existence of a solution of the following maximization problem

$$\sup\{P(\Omega)\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\}.$$

This problem has been solved only recently by A. Girouard, M. Karpukhin and J. Lagacé in [44] where they proved that the maximizer does not exist and the maximizing sequence is obtained by an homogenization procedure (see Section 1.4).

Theorem 1.6 (Girouard-Karpukhin-Lagacé [44]). *The following equality holds*

$$\sup\{P(\Omega)\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = 8\pi.$$

Moreover for every simply-connected and Lipschitz domain $\Omega \subset \mathbb{R}^2$ there exists a sequence of subdomains $\Omega_\epsilon \subset \Omega$ such that $\partial\Omega \subset \partial\Omega_\epsilon$ and

$$P(\Omega_\epsilon)\sigma_1(\Omega_\epsilon) \rightarrow 8\pi$$

In this thesis we study the relations between the normalized Steklov eigenvalue $P(\Omega)\sigma_1(\Omega)$ and the normalized Neumann eigenvalue $|\Omega|\mu_1(\Omega)$, see Section 1.3 of this Introduction and Chapter 3

1.2 Continuity under domain variations

In order to address shape optimization problems related to Neumann and Steklov eigenvalues it is crucial to understand the continuity property of these eigenvalues under domain variation. In this section we present some results related to the continuity of eigenvalues. Let us begin with the introduction of Hausdorff distance between two compact sets, see [58]

Definition 1.1. *Let K_1 and K_2 be two compact subsets of \mathbb{R}^d , we define the Hausdorff distance between this two sets in the following way*

$$d(K_1, K_2) := \max \left\{ \sup_{x \in K_1} \inf_{y \in K_2} |x - y|, \sup_{x \in K_2} \inf_{y \in K_1} |x - y| \right\}.$$

Let D be a bounded, closed domain in \mathbb{R}^d , let $\Omega_1 \subset D$ and $\Omega_2 \subset D$ be two open sets, then we define the Hausdorff distance between this two sets in the following way

$$d_O(\Omega_1, \Omega_2) := d(D \setminus \Omega_1, D \setminus \Omega_2).$$

It turns out that this distance does not depend on the box D . We say that Ω_n converge in the Hausdorff sense or in the Hausdorff distance to Ω if $d(\Omega_n, \Omega) \rightarrow 0$.

In sections 1.1.1 and 1.1.2 we introduce the Steklov and Neumann eigenvalues and, in particular, we show that these eigenvalues are well defined only under the assumption that the trace operator $H^1(\Omega) \rightarrow L^2(\partial\Omega)$ is compact for Steklov eigenvalues and the embedding $H^1(\Omega) \rightarrow L^2(\Omega)$ is compact for Neumann eigenvalues. Heuristically this suggests that, in order to have the continuity of these eigenvalues, we have at least to preserve this compactness, in a uniform way, along the sequence. For this reason we introduce the following class of domains, that have a uniform cone condition

Definition 1.2. *Let y be a point in \mathbb{R}^d , ξ a unit vector, and ϵ a positive real number. Let $C(y, \xi, \epsilon)$ be the cone of vertex y , of direction ξ and dimension ϵ , defined by*

$$C(y, \xi, \epsilon) = \{z \in \mathbb{R}^d, (z - y, \xi) \geq \cos(\epsilon)|z - y|, \text{ and } 0 < |z - y| < \epsilon\}.$$

Let $B(x, r)$ be the ball centered in x with radius r . An open set Ω is said to have ϵ -cone property if, $\forall x \in \partial\Omega$, exists ξ_x unit vector such that $\forall y \in \overline{\Omega} \cap B(x, \epsilon)$, $C(y, \xi_x, \epsilon) \subset \Omega$.

Let D be a fixed compact set in \mathbb{R}^d and let $\Omega_n \subset D$ be a sequence of open domains. In order to have the continuity for Neumann eigenvalues is crucial that the sequence of domains $\Omega_n \subset D$ satisfies the following property: there exists a constant $M > 0$ and a sequence of extension operators $P_n : H^1(\Omega_n) \rightarrow H^1(D)$ such that $\|P_n\| \leq M$. In order to have this property the sequence Ω_n must satisfy a uniform regularity property, in particular D. Chenaïs in [28] proved the following result

Theorem 1.7 (D. Chenaïs [28]). *Let D be a fixed compact set in \mathbb{R}^d and let $\Omega_n \subset D$ be a sequence of open domains that satisfies an uniform ϵ -condition (i. e. ϵ is uniform along the sequence) then there exists a constant $M > 0$ and a sequence of extension operators $P_n : H^1(\Omega_n) \rightarrow H^1(D)$ such that $\|P_n\| \leq M$.*

A direct consequence of this thesis is the following continuity results for Neumann eigenvalues (see [40])

Theorem 1.8. *Let D be a fixed compact set in \mathbb{R}^d and $\Omega_n \subset D$ be a sequence of domains that satisfies an uniform ϵ -condition (i. e. ϵ is uniform along the sequence) and converge in Hausdorff distance to Ω . Then, for all k fixed, $\mu_k(\Omega_n) \rightarrow \mu_k(\Omega)$.*

As one can expect this continuity result is crucial in order to prove the existence of solutions for variational problems involving the Neumann eigenvalues, but we will see that the uniform ϵ -cone condition is a strong assumption. Indeed we will see in Chapter 4 that the maximizing sequence of the following problem

$$\sup\{D(\Omega)^2 \mu_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and convex}\}$$

does not satisfy this property and the Neumann eigenvalues are not continuous under the maximizing sequence. There is a huge family of interesting singular sequences along which the Neumann eigenvalues are not continuous and it's really interesting to study how the geometry of the singularity of the sequence will affect the behavior of the Neumann eigenvalues. A first explicit example of a sequence of domains for which the Neumann eigenvalues are not continuous goes back to R. Courant and D. Hilbert (see [30]). We present some results in this direction in Section 1.4, and in Chapters 3 and 4 we give a full characterization of the behavior of Neumann eigenvalues on domains that converges in Hausdorff distance to a segment.

The results for the continuity of the Steklov eigenvalues are more recent and the geometric conditions that one must impose on the sequence in order to have continuity take deeply into account the behavior of the boundary $\partial\Omega_n$ of the domains.

The first result that we present is a result proved by B. Bogosel in [14] where he proved a result in the same spirit of Theorem (1.8), assuming the uniform ϵ -cone condition

Theorem 1.9. *Let D be a fixed compact set in \mathbb{R}^d and $\Omega_n \subset D$ be a sequence of domains that satisfies an uniform ϵ -condition (i. e. ϵ is uniform along the sequence) and converge in Hausdorff distance to Ω .*

- For all k the following inequality holds

$$\limsup_{n \rightarrow \infty} \sigma_k(\Omega_n) \leq \sigma_k(\Omega).$$

- If $P(\Omega_n) \rightarrow P(\Omega)$ then for all k the following holds

$$\sigma_k(\Omega_n) \rightarrow \sigma_k(\Omega).$$

We give another continuity result for the Steklov eigenvalues, this time do we not assume that the sequence satisfies an uniform ϵ -cone condition but the result strongly relies on a uniform bound of some trace operators, this continuity result was proved in [23].

Let $\mathcal{M}_b(\Omega)$ be the space of Radon measures in Ω , we introduce the space of functions of bounded variation $BV(\Omega)$, i. e. $u \in BV(\Omega)$ if and only if $u \in L^1(\Omega)$ and its distributional gradient is a Radon measure $\nabla u \in \mathcal{M}_b(\Omega)$. If Ω is a bounded open domain with Lipschitz boundary we know that there exists the following linear and continuous trace operator

$$T : BV(\Omega) \rightarrow L^1(\partial\Omega),$$

we denote $C_{trace}(\Omega)$ the operator norm of T . We already discussed the fact that the continuity of the Neumann eigenvalues is strongly related to the uniform bound of the extension operators P_n , we now see that the continuity of the Steklov eigenvalues are strongly related to the uniform bound on the trace operators T_n .

Theorem 1.10. *Let $\Omega_n \subset \mathbb{R}^d$ be a sequence of Lipschitz domain such that*

- *the characteristic functions converge, $\mathbb{1}_{\Omega_n} \rightarrow \mathbb{1}_{\Omega}$ in $L^1(\mathbb{R}^d)$*
- *$P(\Omega_n) \rightarrow P(\Omega)$*
- *$\sup_{n \in \mathbb{N}} C_{trace}(\Omega_n) < +\infty$*

where $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain. Then for all k the following holds

$$\sigma_k(\Omega_n) \rightarrow \sigma_k(\Omega).$$

This continuity property for Steklov eigenvalues are crucial in order to solve maximum problems involving Steklov eigenvalues, but as in the case of Neumann eigenvalues, the condition in order to have the continuity under a maximizing sequence are really restrictive. In particular the Steklov eigenvalues are not continuous on the maximizing sequence, obtained by an homogenization procedure, of the following problem

$$\sup\{P(\Omega)\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\}.$$

As we already notice for the Neumann problem, it is really interesting and useful to understand what is the behavior of the Steklov eigenvalues on a sequence of singular domains. The study of this problem is more recent compared to the study of the behavior of Neumann eigenvalues on singular domains. In Section 1.4 we present some results on this direction, in particular we present the study of the Steklov eigenvalues under homogenization, this results are crucial to study the link between Neumann and Steklov eigenvalues. In this Thesis, in Chapter 2, we prove an asymptotic expansion for the Steklov eigenvalues on dumbbell domain and in Chapter 3 we prove the asymptotic expansion of Steklov eigenvalues on thin domains.

1.3 Behavior on singular domains

We have seen in Section 1.2 that in order to have the continuity of the Steklov and Neumann eigenvalue we need to impose geometric constraint on the sequence. In this section we present results of non continuity of the eigenvalues. We recall the classical example by R. Courant and D. Hilbert (see [30]) where we lose the continuity of the Neumann eigenvalues (see also [53]). Let R_ϵ be the following set

$$R_\epsilon = R \cup l_\epsilon \cup S_\epsilon,$$

where $R = \{(x, y) \mid x \in (-1, 0) \text{ and } y \in (-1, 0)\}$, $l_\epsilon = \{(x, y) \mid x \in (0, 1) \text{ and } y \in (-\epsilon^3, \epsilon^3)\}$ and $S_\epsilon = \{(x, y) \mid x \in (1, 1 + \epsilon) \text{ and } y \in (-\epsilon, \epsilon)\}$.

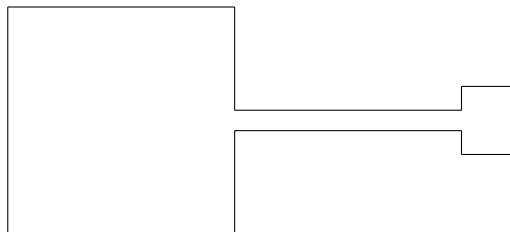


Figure 1.1: Courant-Hilbert example

We have that R_ϵ converge in Hausdorff sense to the square R . We introduce the following test function that is constant on the two squares and linear on the channel

$$\phi_\epsilon = \begin{cases} c_1 := \epsilon^2 + \frac{\epsilon^3}{2} & R \\ (c_2 - c_1)x + c_1 & l_\epsilon \\ c_2 := -1 - \frac{\epsilon^3}{2} & S_\epsilon. \end{cases}$$

We have that $\int_{R_\epsilon} \phi_\epsilon = 0$, we can use this as a test function in the variational formulation (1.2) and we obtain

$$\mu_1(R_\epsilon) \leq \frac{\int_{R_\epsilon} |\nabla \phi_\epsilon|^2}{\int_{R_\epsilon} \phi_\epsilon^2} \leq \frac{(c_2 - c_1)^2 \epsilon^3}{c_1^2 + c_2^2 \epsilon^2}.$$

This shows that $\mu_1(R_\epsilon) \rightarrow 0$ when ϵ goes to zero, but R_ϵ converge in Hausdorff sense to R and $\mu_1(R) = \pi^2$. It is clear that the sequence R_ϵ does not satisfy an uniform ϵ -cone condition, one can also adapt this example in order to see that also the Steklov eigenvalues are not continuous on this kind of singular sequences (see [23]). In this thesis we study the behavior of Neumann and Steklov eigenvalues on different class of singular sequences of domains. In particular we study sequences of dumbbell domains with vanishing channel and thin domains that collapses to a segment.

1.3.1 Dumbbell domains

We present the behavior of Neumann and Steklov eigenvalues on dumbbell domains, we highlight from now that the behaviors are totally different. In the case of Neumann eigenvalues we have a phenomenon of spectral pollution and in the case of Steklov eigenvalues we have a collapsing behavior of the spectrum. We present now the results achieved in [67] about the behavior of Neumann eigenvalues on dumbbell domains.

Let $\Omega_\epsilon \subset \mathbb{R}^d$ be a dumbbell shape domain defined in the following way (see Chapter 3 for more details of the definition)

$$\Omega_\epsilon = D_1 \cup T_\epsilon \cup D_2,$$

where D_1 and D_2 are disjoint, bounded, open, connected sets in \mathbb{R}^d with Lipschitz boundary and T_ϵ is expressed as

$$T_\epsilon = \left\{ x = (x_1, x') \in \mathbb{R}^d \mid -\frac{L}{2} \leq x_1 \leq \frac{L}{2}, |x'| < \epsilon \rho(x_1) \right\},$$

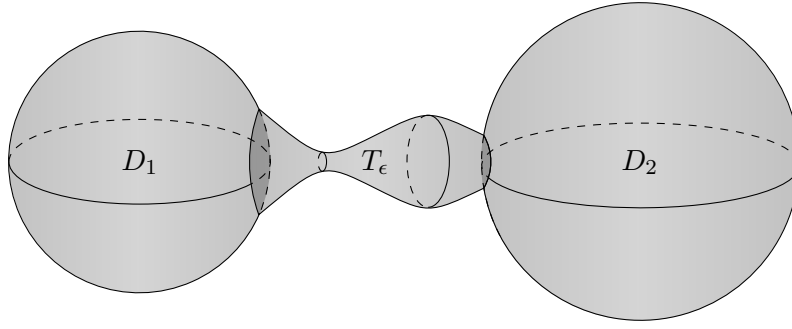
where $L > 0$ and $\rho \in C^0\left(\left[-\frac{L}{2}, \frac{L}{2}\right]\right) \cap C^\infty\left(\left(-\frac{L}{2}, \frac{L}{2}\right)\right)$ is a positive function.

S. Jimbo and Y. Morita in [67] studied the asymptotic behavior of the Neumann eigenvalue $\mu_k(\Omega_\epsilon)$ when ϵ goes to zero, they proved the results for a network of dumbbell domains, here we present the result for a simple dumbbell. What happens is that we have a pollution of the spectrum, it means that the spectrum $\mu_k(\Omega_\epsilon)$ converge to a set of eigenvalues given by the eigenvalues $\mu_k(D_1 \cup D_2)$ joined with a set of eigenvalues related to the thin channel T_ϵ .

We define λ_k to be the eigenvalues of the following Sturm-Liouville eigenvalue problem with Dirichlet boundary conditions

$$\begin{cases} -\frac{d}{dx}(\rho^{d-1}(x) \frac{dS_k}{dx}(x)) = \lambda_k \rho^{d-1}(x) S_k(x) & x \in \left(-\frac{L}{2}, \frac{L}{2}\right) \\ S_k\left(-\frac{L}{2}\right) = S_k\left(\frac{L}{2}\right) = 0. \end{cases}$$

The following theorem characterize the limit of the Neumann spectrum on dumbbell domains

Figure 1.2: Dumbbell shape domain Ω_ϵ .

Theorem 1.11 (S. Jimbo-Y. Morita [67]). *Rearrange $\{\mu_k(D_1 \cup D_2)\}_{k=1}^\infty \cup \{\lambda_k\}_{k=1}^\infty$ in increasing order (counting multiplicity) and denote them by $\{\mu_k\}_{k=1}^\infty$, then we have*

$$\mu_k(\Omega_\epsilon) \rightarrow \mu_k \quad \text{as } \epsilon \rightarrow 0.$$

Moreover, let $\mu_1(\Omega_\epsilon)$ be the first non zero Neumann eigenvalue of the dumbbell domains, we have:

$$\mu_1(\Omega_\epsilon) \sim \alpha_1 \epsilon^{d-1} + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0.$$

Where α_1 is the non zero eigenvalue of the following 2×2 real symmetric matrix A

$$A = \begin{pmatrix} |D_1|^{-1} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{1-d} \right)^{-1} & |D_1|^{-\frac{1}{2}} |D_2|^{-\frac{1}{2}} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{1-d} \right)^{-1} \\ |D_1|^{-\frac{1}{2}} |D_2|^{-\frac{1}{2}} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{1-d} \right)^{-1} & |D_2|^{-1} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{1-d} \right)^{-1} \end{pmatrix}$$

In this thesis we present the asymptotic behavior of the Steklov eigenvalues on dumbbell domains. In Chapter 2 we give full characterization of the behavior of the Steklov eigenvalues on dumbbell domains. Using suitable test function in the variational characterization (1.5) we can prove that (see [46])

$$\forall k \in \mathbb{N}, \quad \sigma_k^\epsilon \rightarrow 0 \quad \text{when } \epsilon \rightarrow 0,$$

that is a totally different behavior with respect to the Neumann spectrum. The Neumann eigenvalues actually converge to the set of eigenvalues $\{\mu_k(D_1 \cup D_2)\}_{k=1}^\infty \cup \{\lambda_k\}_{k=1}^\infty$, the Steklov spectrum collapses to zero. In Chapter 2 we give a first order expansion of the eigenvalues $\sigma_k(\Omega_\epsilon)$ and we study also the behavior of the Steklov eigenfunctions.

This asymptotic expansion strongly depends on the dimension of the ambient space and we have to distinguish the case where $d = 2$ and the case $d \geq 3$. We denote by ω_d the Lebesgue measure of the d -dimensional unit ball, we denote by $P(\Omega)$ the perimeter of the set Ω and we define the following map $\Phi_\epsilon : T_1 \rightarrow T_\epsilon$, $\Phi_\epsilon(x_1, x') = (x_1, \epsilon x')$. The first theorem concerns the case $d = 2$.

Theorem 1.12. ($d = 2$) *Let $\Omega_\epsilon \subset \mathbb{R}^2$ be the dumbbell shape domain defined as above. Then*

$$\sigma_k^\epsilon \sim \mu_k \epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where μ_k is the k -th eigenvalue of the following problem

$$\begin{cases} -\frac{d}{dx} \left(\rho(x) \frac{dV_k}{dx}(x) \right) = \mu_k V_k(x) & x \in \left(-\frac{L}{2}, \frac{L}{2} \right) \\ \rho\left(-\frac{L}{2}\right) \frac{dV_k}{dx}\left(-\frac{L}{2}\right) = -\frac{\mu_k}{2} P(D_1) V_k\left(-\frac{L}{2}\right) \\ \rho\left(\frac{L}{2}\right) \frac{dV_k}{dx}\left(\frac{L}{2}\right) = \frac{\mu_k}{2} P(D_2) V_k\left(\frac{L}{2}\right). \end{cases}$$

Let u_k^ϵ be the k -th Steklov eigenfunction of the dumbbell domain Ω_ϵ , then for every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_k \quad \text{in } H^1(T_1),$$

where \bar{V}_k is a k -th eigenfunction of the problem (2.1) constantly extended in the variable x_2 .

In order to state the results for the case $d \geq 3$ we need to distinguish two cases, indeed the first eigenvalue will go to zero more rapidly than the other ones.

Theorem 1.13. ($d \geq 3, k \geq 2$) Let $\Omega_\epsilon \subset \mathbb{R}^d$ be the dumbbell shape domain defined as above and $d \geq 3$. Then for all $k \geq 2$ we have

$$\sigma_k^\epsilon \sim \alpha_{k-1}\epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where α_{k-1} is the $(k-1)$ -th eigenvalue (counting from zero) of

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_k}{dx}(x)) = \alpha_k \omega_{d-2} \rho^{d-2}(x) V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ V_k(-\frac{L}{2}) = 0 \\ V_k(\frac{L}{2}) = 0. \end{cases}$$

Let u_k^ϵ be the k -th Steklov eigenfunction of the dumbbell domain Ω_ϵ , then for every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$\epsilon_n^{\frac{d-2}{2}} u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_{k-1} \quad \text{in } H^1(T_1),$$

where \bar{V}_{k-1} is an eigenfunction corresponding to α_{k-1} , constantly extended into the variables x_i for $2 \leq i \leq d$.

Therefore, in the case $d \geq 3, k \geq 2$ we end up with a classical Sturm-Liouville eigenvalue problem with Dirichlet conditions. The result that concerns the first eigenvalue is the following

Theorem 1.14. ($d \geq 3, k = 1$) Let $\Omega_\epsilon \subset \mathbb{R}^d$ be the dumbbell shape domain defined as above and $d \geq 3$. The first Steklov eigenvalue has the following asymptotic behavior

$$\sigma_1^\epsilon \sim \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}) \quad \text{as } \epsilon \rightarrow 0,$$

where σ_1 is the unique positive number such that the following differential equation has a non-trivial solution:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases}$$

For every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$u_1^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_1 \quad \text{in } H^1(T_1),$$

where \bar{V}_1 is the solution of the equation (2.3) constantly extended to the variables x_i for $2 \leq i \leq d$.

In Chapter 2 we present this problem in a more precise way and we prove the above theorems.

1.3.2 Collapsing thin domains

Another interesting class of singular sequence of domains are the so called thin domains. Let h^+ and h^- be two positive functions, we define the thin domain Ω_ϵ associated to the function $h = h^+ + h^-$ in the following way

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

The goal is to study the asymptotic behavior of the Steklov and Neumann eigenvalues on these domains when ϵ goes to zero. For every function h the sequence Ω_ϵ does not satisfy an uniform ϵ -cone condition and when ϵ goes to zero the sequence Ω_ϵ converge in Hausdorff distance to a segment.

This class of collapsing domains are crucial in the study of shape optimization problems under convexity constraint in the plane. Indeed if we consider a maximizing/minimizing sequence in this setting, thanks to the Blaschke selection principle (see [71]), only two things can happen

1. The maximizing/minimizing sequence converge in Hausdorff distance to an open convex set
2. The maximizing/minimizing sequence converge in Hausdorff distance to a segment

and a huge class of convex collapsing sequence in the plane can be parametrize by choosing the two functions h^+ and h^- to be two positive and concave functions.

In Chapter 3 we are dealing to the maximization and minimization of the following functional

$$F(\Omega) = \frac{\mu_1(\Omega)|\Omega|}{\sigma_1(\Omega)P(\Omega)},$$

among plane convex domains. For this reason, it is important to understand what is the behavior of the Neumann and Steklov eigenvalues on this collapsing domains. In particular in Chapter 3 we prove the following asymptotic behavior for the Neumann eigenvalues.

Theorem 1.15. *Let $h^+ \in L^\infty(0, 1)$ and $h^- \in L^\infty(0, 1)$ be two non negative functions. Let $h = h^+ + h^-$ and let Ω_ϵ be the following thin domain*

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

If there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0, 1)$, then:

$$\mu_k(\Omega_\epsilon) = \mu_k(h) + o(1) \quad \text{as } \epsilon \rightarrow 0.$$

Where $\mu_k(h)$ is the k -th non trivial eigenvalue of

$$\begin{cases} -\frac{d}{dx}\left(h(x)\frac{du}{dx}(x)\right) = \mu(h)h(x)u(x) & x \in (0, 1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0, \end{cases}$$

and we prove the following result for the Steklov eigenvalues.

Theorem 1.16. *Let $h^+ \in L^\infty(0, 1)$ and $h^- \in L^\infty(0, 1)$ be two non negative functions. Let $h = h^+ + h^-$ and let Ω_ϵ be the following thin domain*

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

If there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0, 1)$, then:

$$\sigma_k(\Omega_\epsilon) = \frac{\sigma_k(h)}{2}\epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where $\sigma_k(h)$ is the k -th non trivial eigenvalue of

$$\begin{cases} -\frac{d}{dx}(h(x)\frac{dv}{dx}(x)) = \sigma(h)v(x) & x \in (0, 1) \\ h(0)\frac{dv}{dx}(0) = h(1)\frac{dv}{dx}(1) = 0. \end{cases}$$

Similar results for the Neumann eigenvalues have already been proved in a different geometric context in [18] and [73]. The novelty of our theorems is the fact that our function h can vanish, this means that the Sturm-Liouville eigenvalue problems that we found at the limit are not classical in the sense that they are not uniformly elliptic. In Chapter 3 we will see that the hypothesis on the function h that we assume are crucial in order to have that the limit is a well defined Sturm-Liouville eigenvalue problem.

1.4 Relations between Neumann and Steklov eigenvalues

1.4.1 From from Neumann to Steklov and from Steklov to Neumann

An interesting question is to understand what are the link between the Steklov eigenvalues and the Neumann eigenvalues. In this section we will study the problem of constructing a sequence of Neumann type eigenvalues (by constructing a sequence of domains or a sequence of densities as we will see) that converge to Steklov eigenvalues and vice versa.

A crucial role in study this problems is played by the weighted Neumann eigenvalues. Heuristically we can think that the classical Neumann eigenvalues as the frequency of oscillation of an homogeneous membrane (see for instance [30]). The idea is now to study the frequencies of an non-homogeneous membrane (that will be the weighted Neumann problem) and, in order to recover the Steklov eigenvalues, we will concentrate all the mass of this non-homogeneous membrane on the boundary. This idea can be made rigorous (see [6, 32]) in particular we can construct a sequence of densities for which the sequence of the corresponding weighted Neumann eigenvalues converges to a Steklov eigenvalue. Let Ω be a sufficiently smooth domain (see [32]) we introduce the following weighted Neumann eigenvalues

$$\begin{cases} -\Delta u = \mu(\Omega, \rho_\epsilon)\rho_\epsilon u & \Omega \\ \partial_\nu u = 0 & \partial\Omega, \end{cases}$$

the function ρ_ϵ is a function defined in the following way

$$\rho_\epsilon = \begin{cases} \epsilon & \Omega \setminus \bar{\delta}_\epsilon \\ \frac{P(\Omega) - \epsilon|\Omega \setminus \bar{\delta}_\epsilon|}{|\bar{\delta}_\epsilon|} & \bar{\delta}_\epsilon. \end{cases}$$

where

$$\bar{\delta}_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \epsilon\}.$$

The functions ρ_ϵ are densities that accumulates on the boundary $\partial\Omega$ when ϵ goes to zero, so it is natural to guess that $\mu_k(\Omega, \rho_\epsilon)$ converge to $\sigma_k(\Omega)$. Indeed using the results proved in [6, 32] one can prove the following result

$$\mu_k(\Omega, \rho_\epsilon) \rightarrow \sigma_k(\Omega).$$

We want to present the link also in the reverse sense, in particular we present the results contained in [43] in which the Neumann problem is realized as a sequence of Steklov type problems. Also in this case the heuristically point of view in which we can see the Neumann eigenvalues as the frequencies of a membrane with an homogeneous density, and the Steklov eigenvalues as the frequencies of a membrane with all the mass concentrated to the boundary is useful in order to understand the construction. Let Ω be a simply connected and Lipschitz domain and we consider the Steklov eigenvalues $\sigma_k(\Omega)$, now in order to construct a sequence of domains Ω_ϵ s.t. the sequence $\sigma_k(\Omega_\epsilon)$ converges to a Neumann type eigenvalue the idea is to accumulate uniformly distributed boundary components inside the domain Ω . This is done by perforating the interior of the domain Ω by small holes uniformly distributed. This process is known as an *homogenization process*. The main idea is to construct a periodic grid inside the domain Ω with hyper cubes with sides of length ϵ and at the center of each hyper cube we make an hole of radius $r_\epsilon < \epsilon$. By a careful choose of the radius r_ϵ of the holes, in [43], the authors proved that the Steklov eigenvalues converge to the eigenvalue of the following problem

$$\begin{cases} -\Delta U = \Sigma_\gamma(\Omega)\omega_d\gamma U & \Omega \\ \partial_\nu U = \Sigma_\gamma U & \partial\Omega, \end{cases} \quad (1.7)$$

where $\gamma \geq 0$ is a parameter linked to the choice of the radius r_ϵ , i. e. $\lim_{\epsilon \rightarrow 0} \epsilon^{-d} r_\epsilon^{d-1} = \gamma$. Thanks to the regularity of the domain the problem (1.7) has the spectrum given by a sequence of eigenvalues

$$0 = \Sigma_{\gamma,0}(\Omega) < \Sigma_{\gamma,1}(\Omega) \leq \Sigma_{\gamma,2}(\Omega) \leq \dots \rightarrow +\infty,$$

at the end we recover the Neumann eigenvalues by sending γ to infinity. We can summaries this results in the following theorem.

Theorem 1.17 (A. Girouard-A. Henrot-J. Lagacé [43]). *Let γ be a positive parameter and $\Omega \subset \mathbb{R}^d$ be a simply connected and Lipschitz domain. Then there exists a sequence of domains Ω_ϵ (obtained by a periodic homogenization) such that*

$$\sigma_k(\Omega_\epsilon) \rightarrow \Sigma_{\gamma,k}(\Omega),$$

moreover

$$\lim_{\gamma \rightarrow +\infty} \Sigma_{\gamma,k}(\Omega) = \mu_k(\Omega).$$

This theorem shows how to reach a Neumann eigenvalue problem starting from a Steklov eigenvalue problem. The procedure is given in two steps: the first step consist in performing a periodic homogenization of the domain obtaining in this way a new eigenvalue problem depending on a parameter γ , then, in the second step, we tend this parameter to infinity in order to find at the limit a Neumann eigenvalue.

We present another important result proved by M. Karpukhin, A. Girouard and J. Lagacé in [44], where, using a different homogenization process they found a new link between Steklov eigenvalues and a generalization of Neumann eigenvalue. This results is important and indeed, as an application, provides a solution of the following problem:

$$\sup\{P(\Omega)\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\}.$$

In order to present the results in full generality we need to introduce the space of functions $L^1(\log L^1)(\Omega)$ that consists of functions f such that

$$\int_{\Omega} |f|(\log(2 + |f|)) dx < \infty,$$

this space can be endowed with a suitable norm (see [12]) in order to obtain a Banach space. Let β be a nonnegative and non trivial function such that $\beta \in L^{\frac{d}{2}}(\Omega)$ if $d \geq 3$, and $\beta \in L^1(\log L^1)(\Omega)$ if $d = 2$, we consider the weighted Neumann problem

$$\begin{cases} -\Delta u = \mu(\Omega, \beta)\beta u & \Omega \\ \partial_\nu u = 0 & \partial\Omega. \end{cases}$$

In [44] the authors proved that for every domain Ω and for every function β there exists a sequence of domains Ω_ϵ such that $\sigma_k(\Omega_\epsilon)$ converges to $\mu_k(\Omega, \beta)$ when ϵ goes to zero. Also in this case, as in the case of Theorem 1.17, the domain Ω_ϵ is constructed via an homogenization method, but this time the process is more involved and in particular is not a periodic homogenization and the construction of the grid of homogenization is deeply linked to the function β . The main theorem proved in [44] is the following

Theorem 1.18 (A. Girouard-M. Karpukhin-J. Lagacé [44]). *For any simply connected and Lipschitz domain $\Omega \subset \mathbb{R}^d$, and any nonnegative and nontrivial function $\beta \in L^{\frac{d}{2}}(\Omega)$ (if $d \geq 3$), and $\beta \in L^1(\log L^1)(\Omega)$ (if $d = 2$) there exists a family of domains Ω_ϵ (obtained by a non periodic homogenization) such that*

$$\begin{aligned} \sigma_k(\Omega_\epsilon) &\rightarrow \mu_k(\Omega, \beta), \\ \mu_k(\Omega_\epsilon) &\rightarrow \mu_k(\Omega), \\ |\Omega_\epsilon| &\rightarrow |\Omega|. \end{aligned}$$

A consequence of this theorem (see [44]) is the fact that

$$\sup\{P(\Omega)\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = 8\pi,$$

and moreover the solution of this supremum problem does not exist and the maximizing sequence can be constructed from any simply connected and Lipschitz domain $\Omega \subset \mathbb{R}^2$ by performing a specific non periodic homogenization procedure (we still refer to [44] for the details of the construction).

1.4.2 Initial motivation of this thesis

The problem that motivate this thesis comes from the study of the existence of a solution of a maximizer for the Steklov eigenvalues under Perimeter constraint. As we already mention this problem was solved by M. Karpukhin, A. Girouard and J. Lagacé in [44] in 2021. A first attempt to prove the non existence of the maximizer come out during the preparation of [43]. Indeed with the periodic homogenization process that we briefly described in the section above we can prove that, given a planar domain Ω , we can construct a sequence of domains Ω_ϵ s. t.

$$P(\Omega_\epsilon)\sigma_1(\Omega_\epsilon) \rightarrow (P(\Omega) + 2\pi\gamma|\Omega|)\Sigma_{\gamma,1}(\Omega),$$

where γ is a positive parameter and $\Sigma_{\gamma,k}(\Omega)$ is the eigenvalue defined in (1.7). Now If we send γ to infinity in the quantity $(P(\Omega) + 2\pi\gamma|\Omega|)\Sigma_{\gamma,k}(\Omega)$, from the results obtained in [43], we conclude that

$$\lim_{\gamma \rightarrow \infty} (P(\Omega) + 2\pi\gamma|\Omega|)\Sigma_{\gamma,k}(\Omega) = |\Omega|\mu_1(\Omega).$$

This thesis started in October 2019, at that time the non existence for the maximizer in the maximization problem $\sup\{P(\Omega)\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\}$ was a conjecture, and a natural way to attack it was to use the results available in [43]. Using the results that we explained above the non existence of a solution for the problem $\sup\{P(\Omega)\sigma_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\}$ directly follow from a positive answer to the following question

Question 1.4.1. *Is it true that for all $\Omega \subset \mathbb{R}^2$ the following inequality holds $P(\Omega)\sigma_1(\Omega) < |\Omega|\mu_1(\Omega)$?*

Indeed suppose that Ω is a maximizer, then we can construct a sequence of domain such that

$$P(\Omega_\epsilon)\sigma_1(\Omega_\epsilon) \rightarrow |\Omega|\mu_1(\Omega) > P(\Omega)\sigma_1(\Omega)$$

that is a contradiction. The question of whether the inequality $P(\Omega)\sigma_1(\Omega) < |\Omega|\mu_1(\Omega)$ is true or not came at first from this problem. But it turns out that the techniques that we develop in order to answer this question were useful also to study different problems. For example searching for counter examples bring us to the work [25], the direct study of this question bring us to the work [55], the study of the behavior of collapsing domain bring us to the solution of a maximization problem for Sturm-Liouville eigenvalue problem in [56] and the idea of this inequality motivate us to start the collaboration [57].

1.4.3 Validity of inequalities for Lipschitz domains

In this thesis we will show that, if we don't put any restriction, then there are no inequalities that link the quantity $P(\Omega)\sigma_1(\Omega)$ and the quantity $|\Omega|\mu_1(\Omega)$ (we will be more precise about this later). But now we start by presenting two classical inequalities, proved in [76], linking the first Steklov eigenvalue and the first Neumann eigenvalue. The first inequality involve only geometric quantities, more precisely let $\Omega \subset \mathbb{R}^2$ be a star shaped domain, let $r(\Omega)$ be its inradius and $D(\Omega)$ be its diameter, then the following inequality holds

$$\sigma_1(\Omega) \geq \frac{\mu_1(\Omega)r(\Omega)}{2(1 + \sqrt{\mu_1(\Omega)}D(\Omega))}.$$

The second inequality between $\mu_1(\Omega)$ and $\sigma_1(\Omega)$ does not involve geometric quantities, but it involves also the first non trivial eigenvalue, denoted by ξ_1 , of the following biharmonic Steklov problem

$$\begin{cases} \Delta^2 u = 0 & \Omega \\ \partial_\nu u = 0 & \partial\Omega \\ \partial_\nu \Delta u = -\xi(\Omega)u & \partial\Omega, \end{cases}$$

where here Δ^2 is the Bilaplace operator. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, then the following inequality holds

$$\mu_1(\Omega)\sigma_1(\Omega) \leq \xi_1(\Omega).$$

This two inequalities are an example of possible relations between $\mu_1(\Omega)$ and $\sigma_1(\Omega)$, but, as one can notice, this inequalities in some sense does not give a direct comparison between the $\mu_1(\Omega)$ and $\sigma_1(\Omega)$.

From the scale properties described in (1.3) and in (1.6) we see that the scale factor between this two eigenvalues is different and, in order to be able to compare them, we need to normalize these eigenvalues. From the definition of the first Neumann eigenvalue, we can heuristically say, that it lives inside the domain Ω so the more natural normalization for this eigenvalue is the following

$$|\Omega|\mu_1(\Omega).$$

From the same reason, we can think that the first Steklov eigenvalue, we can heuristically say, that it lives on the boundary of the domain Ω so the more natural normalization for this eigenvalue is the following

$$P(\Omega)\sigma_1(\Omega).$$

The first result that we can achieve is a negative answer to the question 1.4.1 (see Chapter 2 for the proofs). Indeed we can consider the dumbbell domain $\Omega_\epsilon \subset \mathbb{R}^2$ constructed as follow, D_1 and D_2 are two balls such that $|D_1| = |D_2| = 1$ and $\rho = 1$ constantly.

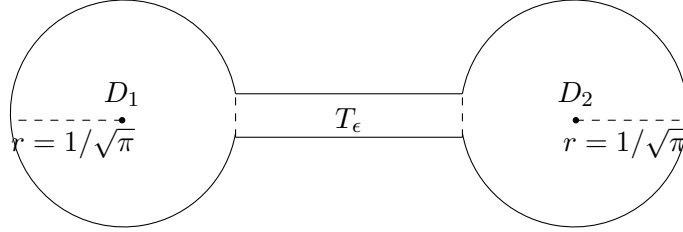


Figure 1.3: Dumbbell shape domain in $d = 2$, with $\rho \equiv 1$.

By choosing, in the variational formulation of $\mu_1(\Omega_\epsilon)$, a test function which is constant on each disk and affine in the tube, we can prove that $\mu_1(\Omega_\epsilon) \leq \frac{4}{L}\epsilon$, so we conclude that

$$\frac{|\Omega_\epsilon|}{P(\Omega_\epsilon)}\mu_1(\Omega_\epsilon) \leq \frac{4}{(2\sqrt{\pi} + L)L}\epsilon + o(\epsilon).$$

Now using the result that we achieved Theorem 1.12 we have a characterization of the constant α_1 in the asymptotic expansion $\sigma_1(\Omega_\epsilon) \sim \alpha_1\epsilon$ and in particular, if the length of the channel L is large enough, we conclude that

$$\frac{4}{(2\sqrt{\pi} + L)L} < \alpha_1.$$

This will imply that there exists a dumbbell domain Ω_ϵ with a long and thin enough channel, such that

$$P(\Omega_\epsilon)\sigma_1(\Omega_\epsilon) > |\Omega_\epsilon|\mu_1(\Omega_\epsilon).$$

At the beginning of this section we stated that there are no inequalities that link the quantity $P(\Omega)\sigma_1(\Omega)$ and the quantity $|\Omega|\mu_1(\Omega)$, more precisely this means that there are no bounds from above and below of the following functional

$$F(\Omega) = \frac{\mu_1(\Omega)|\Omega|}{\sigma_1(\Omega)P(\Omega)},$$

in the class of Lipschitz domains. In particular in Chapter 3 we prove that

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open set and Lipschitz}\} = 0,$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open set and Lipschitz}\} = +\infty.$$

In order to construct the maximizing sequence and the minimizing sequence for the problems above a crucial role is played by the results about the total instability of the Weinstock inequality proved by D. Bucur and M. Nahon in [27]

Theorem 1.19 (Bucur-Nahon [27]). *Let $\Omega, \omega \subset \mathbb{R}^2$ be two smooth, conformal open sets. Then there exists a sequence of smooth open sets $(\Omega_\epsilon)_{\epsilon>0}$ with uniformly bounded perimeter and satisfying a uniform ε -cone condition such that*

$$\lim_{\epsilon \rightarrow 0} d_H(\partial\Omega_\epsilon, \partial\Omega) = 0, \quad \lim_{\epsilon \rightarrow 0} P(\Omega_\epsilon)\sigma_k(\Omega_\epsilon) = P(\omega)\sigma_k(\omega), \quad \lim_{\epsilon \rightarrow 0} |\Omega_\epsilon|\mu_k(\Omega_\epsilon) = |\Omega|\mu_k(\Omega).$$

Moreover in Chapter 3 we will also study the following Blaschke-Santaló diagram

$$\mathcal{E} = \{(x, y) \text{ where } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2\},$$

In particular we prove the following theorem

Theorem 1.20. *The following equality holds*

$$\bar{\mathcal{E}} = [0, 8\pi] \times [0, \mu_1(\mathbb{D})\pi]$$

where $\mu_1(\mathbb{D}) = j'_{11}{}^2$ is the first Neumann eigenvalue of the unit disk.

1.4.4 Inequalities for convex domains

A natural question is to find upper and lower bound for the functional $F(\Omega)$ in a restricted class of domains Ω . We state the results that we prove in Chapter 3 concerning the class of convex domains. We are interested by the following problems

$$\begin{aligned} & \inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open and convex set}\}, \\ & \sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open and convex set}\}. \end{aligned}$$

Thanks to the Blaschke selection principle (see [71]) we know that a maximizing/minimizing sequence can have only two behaviors: or it converges in Hausdorff sense to an open convex set or it converges in Hausdorff sense to a segment. In order to study this second situation we use the asymptotic behavior that we stated in Theorems 1.15 and 1.16 about Neumann and Steklov eigenvalues on thin collapsing domain. This gives, almost directly, the asymptotic behavior of the functional $F(\Omega)$ on thin domains

Theorem 1.21. *Let h^+ and h^- be two positive, bounded and concave functions, let $h = h^+ + h^-$ and let Ω_ϵ be the following thin domain*

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

Then the following asymptotic behavior holds:

$$F(\Omega_\epsilon) \xrightarrow{\epsilon \rightarrow 0} F(h) := \frac{\mu_1(h) \int_0^1 h(x) dx}{\sigma_1(h)}.$$

Where $\mu_1(h)$ is the first non zero eigenvalue of

$$\begin{cases} -\frac{d}{dx} \left(h(x) \frac{du_k}{dx}(x) \right) = \mu_k(h) h(x) u_k(x) & x \in (0, 1) \\ h(0) \frac{du_k}{dx}(0) = h(1) \frac{du_k}{dx}(1) = 0, \end{cases}$$

and $\sigma_1(h)$ is the first non zero eigenvalue of

$$\begin{cases} -\frac{d}{dx} \left(h(x) \frac{dv_k}{dx}(x) \right) = \sigma_k(h) v_k(x) & x \in (0, 1) \\ h(0) \frac{dv_k}{dx}(0) = h(1) \frac{dv_k}{dx}(1) = 0. \end{cases}$$

Thanks to the theorem above we have that the study of the limit behavior of the functional $F(\Omega)$ on thin domains reduces to the study of the spectral functional $F(h)$ that depends only on 1-dimensional Sturm-Liouville eigenvalues. We studied this functional and we obtained the following results.

Theorem 1.22. *The minimization problem (resp. the maximization problem)*

$$\inf\{F(h) : h \in \mathcal{L}\}, \quad (\text{resp. } \sup\{F(h) : h \in \mathcal{L}\})$$

has a solution, moreover the constant function $h \equiv 1$ is a local minimizer.

We come back now to the question of finding upper and lower bounds for the functional $F(\Omega)$ in the class of convex set, in particular these bounds will provide inequalities between $P(\Omega)\sigma_1(\Omega)$ and $|\Omega|\mu_1(\Omega)$. We first prove the following upper and lower bound for the limit functional $F(h)$

Theorem 1.23. *For every non negative and concave function $h \in L^\infty(0,1)$ the following inequalities hold*

$$\frac{\pi^2}{12} \leq F(h) \leq 4$$

Then we will prove the main theorem of Chapter 3 that gives upper and lower bounds for the functional $F(\Omega)$

Theorem 1.24. *There exists an explicit constant C_1 such that, for every convex open set $\Omega \subset \mathbb{R}^2$, the following inequalities hold*

$$\frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) \leq C_1 \leq 9.04.$$

The explicit constant C_1 will be described in Section 3.4. As a direct consequence of this theorem is the existence of direct inequalities between the normalized Neumann eigenvalue and the normalized Steklov eigenvalue, indeed we proved that, for every $\Omega \subset \mathbb{R}^2$ open convex domain, the following inequalities hold

$$\frac{\pi^2}{6\sqrt[3]{18}} P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega), \quad P(\Omega)\sigma_1(\Omega) \leq C_1|\Omega|\mu_1(\Omega) \leq 9.04|\Omega|\mu_1(\Omega).$$

In the end of Chapter 3, supported by numerical evidences, we state some conjecture. The main of this conjecture concerns the sharp inequalities between $P(\Omega)\sigma_1(\Omega)$ and $|\Omega|\mu_1(\Omega)$

Conjecture 1.1. *For every bounded, convex and open set $\Omega \subset \mathbb{R}^2$ the following sharp bounds hold*

$$1 \leq F(\Omega) \leq 2.$$

In the last part of Chapter 3 we use a different approach in order to prove the inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$, we study this inequality using, heuristically, a localization property for the Neumann eigenfunctions. In particular, by an easy test function argument, the inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ for convex plane domains with two axis of symmetry follows directly from this conjecture

Conjecture 1.2 (L^2 hot spot conjecture). *Let $\Omega \subset \mathbb{R}^2$ be a convex domain and let u be the first non trivial Neumann eigenfunction of the domain Ω , then*

$$\frac{1}{P(\Omega)} \int_{\partial\Omega} u^2 ds \geq \frac{1}{|\Omega|} \int_{\Omega} u^2 dx.$$

In Chapter 3 we prove the following theorem

Theorem 1.25. *Let $\Omega \subset \mathbb{R}^2$ circumscribed polygon and let u be the first non trivial Neumann eigenfunction of the polygon Ω , then*

$$\frac{1}{P(\Omega)} \int_{\partial\Omega} u^2 ds \geq \frac{1}{|\Omega|} \int_{\Omega} u^2 dx.$$

As a corollary we obtain that the inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ is true for a specific class of convex plane domains.

Corollary 1.26. *Let Ω be a circumscribed polygon in the plane with two axis of symmetries then*

$$P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega).$$

1.5 Sturm-Liouville eigenvalues and Neumann eigenvalues under diameter constraint

Upper bounds for the Neumann eigenvalues in terms of the diameter was proved by S. Y. Cheng in [29] in the contest of complete and smooth Riemannian manifolds. Then R. Banuelos and K. Burdzy in [8] solve the following problem

$$\sup\{D(\Omega)^2\mu_1(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ open and convex}\}.$$

They proved that this problem has no solution and the maximizing sequence is given by collapsing rhombi

Theorem 1.27 (S. Y. Cheng [29], R. Banuelos-K. Burdzy [8]). *For any plane convex domain Ω*

$$D(\Omega)^2\mu_1(\Omega) \leq 4j_{0,1}^2$$

where $j_{0,1} \simeq 2.405$ is the first zero of the Bessel function J_0 . Moreover, this bound is sharp, asymptotically attained by collapsing isosceles triangles (or rhombi).

Then P. Kröger give the following upper bounds for all the Neumann eigenvalues.

Theorem 1.28 (P. Kröger [74]). *For any plane convex domain Ω*

$$D(\Omega)^2\mu_k(\Omega) \leq (2j_{0,1} + (k-1)\pi)^2$$

where $j_{0,1} \simeq 2.405$ is the first zero of the Bessel function J_0 .

In Chapter 4 we study the maximization problem of Neumann eigenvalues under Diameter constraint by analyzing a maximization problem for Sturm-Luoiville eigenvalues.

1.5.1 Connection between $\mu_1(\Omega)$ and $\mu_1(h)$, unboundedness of $D(\Omega)^2\mu_1(\Omega)$

Let $h \in L^\infty(0,1)$ be a non negative function, in the first part of Chapter 4 we prove a connection between the first Neumann eigenvalue problem and the following quantity

$$\mu_1(h) := \inf \left\{ \frac{\int_0^1 (u')^2 h dx}{\int_0^1 u^2 h dx} \mid u \in H^1(0,1) \text{ and } \int_0^1 u h dx = 0 \right\}.$$

In particular we prove the following proposition

Proposition 1.29. *We have equality between the two quantities*

$$\begin{aligned} S_1 &:= \sup\{\mu_1(\Omega), \Omega \subset \mathbb{R}^d, \text{ bounded, Lipschitz}, D(\Omega) = 1\} \\ S_2 &:= \sup\{\mu_1(h), h \in L^\infty(0, 1), h \geq 0, h \neq 0\}. \end{aligned}$$

We can explain the idea behind the identification between the two maximizing problems in the following way:

1. the first thing to notice is the following: given a domain $\Omega \subset \mathbb{R}^d$ we can describe this domain by means of a one dimensional function h defined in the following way

$$h(x_1) = \mathcal{H}^{d-1}(\{x' \in \mathbb{R}^{d-1} \mid (x_1, x') \in \Omega\}).$$

Then using in the variational characterization of $\mu_1(\Omega)$ a test function that depends only on the first variable x_1 , one can prove that

$$\mu_1(\Omega) \leq \mu_1(h).$$

2. If we take a positive non trivial function $h \in L^\infty(0, 1)$, then we can construct the following axysymmetric domain

$$\Omega_h := \{(x_1, x') \in \mathbb{R}^d \mid |x'| < h(x_1), \forall x_1 \in (0, 1)\}.$$

Of course, from the previous point, $\mu_1(\Omega_h) \leq \mu_1(h^{d-1})$ but if we consider the following sequence of domains $\mu_1(\Omega_{\epsilon h})$ when ϵ goes to zero we have the following inequality

$$\mu_1(\Omega_{\epsilon h}) \geq \mu_1(h^{d-1}) + f(\epsilon, h) \tag{1.8}$$

where $f(\epsilon, h) \rightarrow 0$ when $\epsilon \rightarrow 0$.

We described in a non rigorous way the identification given in Proposition 1.29, a rigorous proof of this identifications follows from Lemmas 4.6, 4.7 and 4.8.

Thanks to this result we can explicitly construct a sequence of domains Ω_ϵ such that

$$\liminf_{\epsilon \rightarrow 0} D(\Omega_\epsilon)^2 \mu_1(\Omega_\epsilon) = +\infty.$$

We construct this sequence as a sequence of axysymmetric domains with a profile described by the following function

$$g_a(x) = \begin{cases} e^{2w_a(x-a)} & 0 \leq x < a \\ 1 & a \leq x \leq 1-a \\ e^{2w_a(1-x-a)} & 1-a < x \leq 1, \end{cases}$$

where $0 < a < \frac{1}{2}$, and w_a is the first positive root of the following equation

$$\tan\left(x\left(\frac{1}{2} - a\right)\right) = 1 + \frac{1}{ax}.$$

We prove the following proposition

Proposition 1.30. *Let g_a be the function defined above. Then $\mu_1(g_a) = w_a$ and*

$$\lim_{a \rightarrow \frac{1}{2}} \mu_1(g_a) = +\infty.$$

Now the constructions of the domain of the sequence Ω_ϵ follow easily, indeed from the proposition above and from the inequality (1.8) we conclude that

$$\liminf_{\epsilon \rightarrow 0} D(\Omega_{\epsilon(g_a)}^{\frac{1}{d-1}})^2 \mu_1(\Omega_{\epsilon(g_a)}^{\frac{1}{d-1}}) = +\infty.$$

1.5.2 Analysis of a Sturm-Liouville eigenvalue problem

We described in the last section that we have a deep connection between $\mu_1(\Omega)$ and $\mu_1(h)$, we would like to identify $\mu_1(h)$ as a first eigenvalue of a Sturm-Liouville eigenvalue problem and then extend our study, identifying $\mu_k(\Omega)$ with the k-th Sturm-Liouville eigenvalue $\mu_k(h)$.

Unfortunately, in general, if we only assume that h is a bounded and non negative function, the quantity $\mu_1(h)$ cannot be identify as a first (non trivial) eigenvalue of a Sturm-Liouville eigenvalue problem. The problem is that, if we do not assume some condition on how fast h vanishes the Green kernel of the Sturm-Liouville problem is not well defined (see [55]) and then the existence of Sturm-Liouville eigenfunctions is not ensured. But if we assume that there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0, 1)$, then $\mu_1(h)$ is actually the first eigenvalue of the following eigenvalue problem (see [55])

$$\begin{cases} -\frac{d}{dx}\left(h(x)\frac{du}{dx}(x)\right) = \mu(h)h(x)u(x) & x \in (0, 1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0. \end{cases} \quad (1.9)$$

In particular the identification, that we described in the previous section can be extended to all the eigenvalues in the case, for instance, of plane convex domains. Heuristically we have that, given a convex plane domain Ω there exists a concave and non negative function h such that $\mu_k(\Omega) \leq \mu_k(h)$ (that will be the function that describes the boundary of Ω , see Chapter 4 for more details), and for every concave and non negative function h there exists a sequence of collapsing domains Ω_ϵ such that $\mu_k(\Omega_\epsilon) \rightarrow \mu_k(h)$ when ϵ goes to zero. This means that we have the following equality

$$\begin{aligned} \sup\{\mu_k(\Omega), \Omega \text{ plane convex domain}, D(\Omega) = 1\} &= \\ &= \sup\{\mu_k(h), h \text{ concave, non negative on } [0, 1]\}. \end{aligned}$$

Motivated by this equality in Chapter 4 we study the following maximization problem

$$\sup\{\mu_k(h) \mid h \text{ concave, non negative on } [0, 1]\}, \quad (1.10)$$

where $\mu_k(h)$ is the k-th eigenvalue of the eigenvalue problem (1.9). The existence of a maximizer is a straightforward application of results we achieved in [55].

The main result of Chapter 4 is the following theorem

Theorem 1.31. *For any $k \geq 1$, the problem $\max\{\mu_k(h), h \text{ concave, non negative on } [0, 1]\}$ has a solution h_k^* , moreover*

- $\max\{\mu_1(h), h \text{ concave, non negative on } [0, 1]\} = \mu_1(h_1^*) = (2j_{0,1})^2$ and

$$h_1^* = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ 2(1-x) & x \in [\frac{1}{2}, 1] \end{cases}$$

- let $k \geq 2$ then $\max\{\mu_k(h), h \text{ concave, non negative on } [0, 1]\} = \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2$

$$h_k^* = \begin{cases} \frac{x(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [0, \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ 1 & x \in [\frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ \frac{(1-x)(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1] \end{cases}$$

The key result (and also the difficult step) in order to prove this theorem is the following result, that we proved deeply using the abstract formulation for optimality conditions proved in [79]

Theorem 1.32. *For any $k \geq 1$, the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ has a solution h_k^* . This one has a graph that is a polygonal line composed of (at most) $k + 1$ segments.*

Our approach to prove this theorem strongly relies on the work by J. Lamboley and A. Novruzi [79]. In this work they were interested in solving shape optimization problems of the following kind

$$\min\{J(\Omega) \mid \Omega \text{ convex}, \Omega \in \mathcal{S}_{ad}\} \quad (1.11)$$

where \mathcal{S}_{ad} is a class of 2-dimensional admissible shape and $J : \mathcal{S}_{ad} \rightarrow \mathbb{R}$ is a shape functional. They derive some general optimality conditions and they prove that, for a large class of shape functional, the solution of the optimization problem is a polygon.

The two optimization problems (1.10) and (1.11), at first sight, seems to be quite different, but they have an important properties in common: in problem (1.11) we are searching for polygonal solution and we want to prove that the solution of (1.10) is given by a concave function with a graph given by a polygonal line. In order to understand better this similarity we need to introduce the concept of support function associated to a convex set $\Omega \subset \mathbb{R}^2$. We will introduce these functions and we describe the properties, we refer to the book [93] for all the details. Let $\Omega \subset \mathbb{R}^2$ be a convex domain, the support function g_Ω associated to Ω is defined in the following way

$$g_\Omega(x) = \sup_{y \in \Omega} (x, y).$$

The support function is a Lipschitz and positively homogeneous of degree one function on \mathbb{R}^2 . Thanks to this homogeneity property, without loss of generality, we can restrict the function on \mathbb{S}^1 obtaining a 2π -periodic function and the support function also have the property that $g''_\Omega + g_\Omega \geq 0$ in the sense that $g''_\Omega + g_\Omega$ is a positive Radon measure.

The converse is also true in the sense that given 2π -periodic function $g \in W^{1,\infty}(\mathbb{S}^1)$ such that $g'' + g \geq 0$ then there exists a convex domain Ω such that the function g is the support function of the set Ω . This means that (modulo fixing an origin in \mathbb{R}^2) the following equality holds

$$\min\{J(\Omega) \mid \Omega \text{ convex}, \Omega \in \mathcal{S}_{ad}\} = \min\{j(u) \mid u \in W^{1,\infty}(\mathbb{S}^1) \text{ s. t. } u'' + u \geq 0\},$$

where j is a suitable integral functional. Moreover we have also the following property, let $Suppt(g''_\Omega + g_\Omega)$ be the support of the positive measure $g''_\Omega + g_\Omega$ then

$$\Omega \subset \mathbb{R}^2 \text{ is a convex polygon with } n \text{ vertex} \iff Suppt(g''_\Omega + g_\Omega) \text{ contains only } n \text{ points.}$$

This means that prove that a solutions for the problem (1.11) is a polygon is equivalent to prove that if u^* is the solution of the following problem

$$\min\{j(u) \mid u \in W^{1,\infty}(\mathbb{S}^1) \text{ s. t. } u'' + u \geq 0\},$$

then the set $Suppt(u^{*''} + u)$ is a discrete set. The analogy with our optimization problem is now clear, indeed prove Theorem 1.32 is equivalent to prove that if h^* is a solution of the following maximization problem

$$\sup\{\mu_k(h) \mid h'' \leq 0 \text{ non negative on } [0, 1]\},$$

then the set $\text{Suppt}(h^{**})$ contains at most k points.

Having this analogy in mind, in Chapter 4, we reformulate the optimality condition present in [79] to our setting and in a first place we prove that $\text{Suppt}(h^{**})$ must be a discrete set. Then with a careful analysis on the nodal intervals of the Sturm-Liouville eigenfunctions we are able to prove that the set $\text{Suppt}(h^{**})$ must have at most k elements. Without entering into the details we want to give a general idea on how the interplay between the optimality condition we get by adapting the one given in [79], and the analysis of the nodal intervals of Sturm-Liouville eigenfunctions brings us to the proof of Theorem 1.32, the detailed proofs are presented in Chapter 4.

The first step in order to obtain the optimality condition is to compute the first derivative of the Sturm-Liouville eigenvalues with respect the function h , this means to compute the following quantity (we can compute the derivative in this way because the eigenvalue are simple)

$$\dot{\mu}_\phi := \left. \frac{d}{dt} \mu_k(h + t\phi) \right|_{t=0},$$

for this reason we prove the following lemma

Lemma 1.33. *The derivative of μ_k in the direction ϕ is given by*

$$\dot{\mu}_\phi = \int_0^1 (u'^2 - \mu_k u^2) \phi dx$$

where u is the eigenfunction associated to $\mu_k(h)$ normalized by $\int_0^1 hu^2 = 1$.

With this formula we can explicit the first order optimality condition by using the formalism of [79], (Proposition 2.3.2)

Proposition 1.34. *Let h^* be a maximizer of $\mu_k(h)$ in the class \mathcal{H} . We denote by S the support of (the negative measure) h^{**} . Then there exist*

- a function $\xi \in H^1(0, 1)$, such that $\xi \geq 0$, $\xi = 0$ on S ,
- two non-negative Radon measures ν_0, ν_1 such that $\text{suppt}(\nu_0) \subset \{x | h^*(x) = 0\}$ and $\text{suppt}(\nu_1) \subset \{x | h^*(x) = 1\}$

such that, for any $\phi \in H^1(0, 1)$

$$\langle \dot{\mu}_\phi, \phi \rangle = - \langle \xi'', \phi \rangle + \int_0^1 \phi d\nu_0 - \int_0^1 \phi d\nu_1. \quad (1.12)$$

We define the following function

$$f := u'^2 - \mu_k u^2.$$

The support of the measure h^{**} (that we denote with S) is a closed set, so its complement S^c is a union of intervals: $S^c = \bigcup_{i \in I} (\alpha_i, \beta_i)$. We will distinguish the internal intervals (those for which $\alpha_i > 0$ and $\beta_i < 1$) and the boundary intervals: we prove that the number of this intervals is finite, thus we will have only two boundary intervals that we will denote $(0, \beta_b)$ and $(\alpha_b, 1)$. Using the optimality condition given in Proposition 1.34 we prove the following lemma

Lemma 1.35. *The function f vanishes at least two times on each internal interval (α_i, β_i) and at least one each interval $(0, \beta_b)$ and $(\alpha_b, 1)$.*

We then need to count the number of zeros of the function f inside the nodal interval of the Sturm-Liouville eigenfunction associated to $\mu_k(h^*)$. We distinguish here the "internal" nodal intervals, those where u vanishes at the two extremities and the two "boundary" nodal intervals (where u vanishes only at one extremity). The key proposition is the following:

Proposition 1.36. *The function f vanishes exactly two times on an internal nodal interval of u and exactly one time on a boundary nodal interval.*

Since u is the $k + 1$ -th eigenfunction of a Sturm-Liouville problem, its number of nodal domains is exactly $k + 1$ ($k - 1$ internal plus two boundary nodal intervals). Following Proposition 1.36, it implies that the function f has exactly $2k$ zeros.

Combining Lemma 1.35 and Proposition 1.36 we obtain an upper bound of the number of intervals that compose S^c , we conclude that S^c must be the union of at most $k + 1$ intervals.

From the optimality conditions we have information about the behavior of the Sturm-Liouville eigenfunctions on the vertex of the optimal function h_k^* . Thanks to this information we can improve the result above obtaining that h_1^* has the graph that is a polygonal line composed of (at most) 2 segments and for all $k \geq 2$ h_k^* has the graph that is a polygonal line composed of (at most) 3 segments. We then obtain the explicit formulas for the maximizers h_k^* from an analysis of the zeros of Bessel functions, proving in this way Theorem 1.31. As an application we give an alternative proof of the Kröger inequalities

Theorem 1.37 (P. Kröger [74]). *For any plane convex domain Ω*

$$D(\Omega)^2 \mu_k(\Omega) \leq (2j_{0,1} + (k - 1)\pi)^2$$

where $j_{0,1} \simeq 2.405$ is the first zero of the Bessel function J_0 . Moreover, this bound is sharp, asymptotically attained by collapsing trapezoids (or hexagons).

1.6 Mixed eigenvalue problems

As we already mentioned, the eigenvalues, such as the Neumann eigenvalues and the Steklov eigenvalues and also the more classical Dirichlet eigenvalues, have a very deep physical interpretation. They can be interpreted, for instance, as a mode of vibration, and the boundary condition can be interpreted as physical conditions on the system (see [30]). The most elementary example of this is the vibrating membrane. The Dirichlet eigenvalues correspond to the modes of a membrane with constant density, that is fixed on the boundary. The Neumann eigenvalues correspond to the modes of a membrane with constant density without any force applied on the boundary. A natural question is to understand what is the mode of a membrane with a given part of the boundary that is fastened and the other part that is free to move. The simplest mathematical model that described the mode of this physical system is given by a mixed Dirichlet-Neumann eigenvalue problem. Given a Lipschitz domain $\Omega \subset \mathbb{R}^d$ and a subset of his boundary $\Gamma_D \subset \partial\Omega$ the mixed Dirichlet-Neumann eigenvalue $\xi(\Omega, \Gamma_D)$ is given by the following problem:

$$\begin{cases} -\Delta u = \xi(\Omega, \Gamma_D)u & \Omega \\ u = 0 & \Gamma_D, \\ \partial_\nu u = 0 & \partial\Omega \setminus \Gamma_D. \end{cases}$$

This kind of eigenvalue has been studied by S. Cox in [31], more precisely in this work the author studied the dependence of this eigenvalues with respect to the boundary part Γ_D in which Dirichlet boundary conditions is imposed.

Another important mixed eigenvalue problem is the mixed eigenvalue problem with a part of the boundary in which we impose Steklov boundary condition. We start by exploiting the mixed Steklov-Neumann eigenvalue problem. More precisely let $\Omega \subset \mathbb{R}^d$ be a bounded, open, connected set with Lipschitz boundary and let $\Gamma_S \subset \partial\Omega$ be a relative open submanifold with Lipschitz boundary we define the mixed Steklov-Neumann eigenvalues $\mu_k(\Omega, \Gamma_S)$ in the following way

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \mu(\Omega, \Gamma_S)u & \Gamma_S \\ \partial_\nu u = 0 & \partial\Omega \setminus \Gamma_S. \end{cases} \quad (1.13)$$

The eigenvalue problem (1.13) has important applications in hydrodynamics, it is deeply connected to the sloshing of a liquid, for this reason this problem is also known as sloshing problem. The first to study this problem was Euler (see [36]) in 1761 and since then a huge amount of research has been done on this topic. Without claiming to be complete we give an introduction to the physical meaning of the problem (1.13), we refer to [38] for an historical overview of the problem and we refer to the books [60, 72, 77] for an extensive treatment of the sloshing problem. In the mathematical model of the sloshing problem Ω can be seen as a container of a liquid. The boundary of this container is divided in two parts: a free surfaces Γ_S where the liquid is not touching the container (in the physical application it will be horizontal, for example the part of the surface of a liquid in a mug that is not touching the mug) and a contact surface $\partial\Omega \setminus \Gamma_S$ where the liquid actually touch the container. The liquid motion is assumed to be irrotational and the surface tension is neglected on Γ_S . In the framework of linear water wave theory one can identify the modes and the frequencies of the liquid respectively with the eigenfunctions and the eigenvalues of the sloshing problem (1.13).

In this thesis we focus on the mixed Steklov-Dirichlet eigenvalues. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, connected set with Lipschitz boundary and let $\Gamma_S \subset \partial\Omega$ be a relative open submanifold with Lipschitz boundary we define the mixed Steklov-Dirichlet eigenvalues $\lambda_k(\Omega, \Gamma_S)$ in the following way

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \lambda(\Omega, \Gamma_S)u & \Gamma_S \\ u = 0 & \partial\Omega \setminus \Gamma_S. \end{cases} \quad (1.14)$$

Also this mixed eigenvalue problem has a deep physical meaning, it describes the stationary heat distribution in Ω under the condition that the heat flux through Γ_S is proportional to the temperature and the part $\partial\Omega \setminus \Gamma_S$ is kept under zero temperature. Problem (1.14) has also a deep connection with Brownian motion, more precisely it is linked to Cauchy process, that are traces of the $(d + 1)$ -dimensional Brownian motion, this connection is described in details in [9].

In recent years mixed eigenvalues problem have attracted considerable attention. In [51] bounds for the Riesz mean has been obtained, in [10] the authors obtained inequalities between Steklov-Dirichlet eigenvalues and Steklov-Neumann eigenvalues and in [80] the authors proved a two terms asymptotic for the Weyl's law. Optimization problems for these eigenvalues are also interesting. For instance in [41, 39, 101] optimization problems for Steklov-Dirichlet eigenvalues have been studied, in particular the case where we have

a convex domain Ω with a spherical hole, with Steklov conditions imposed in $\partial\Omega$ and Dirichlet conditions imposed on the boundary of the hole.

Chapter 5 is devoted to the study of the Steklov-Dirichlet eigenvalues. The first result that we prove is the following stability result of $\lambda_k(\Omega, \Gamma_S)$ with respect to Γ_S

Theorem 1.38. *Let Ω be a uniform $\mathcal{C}^{1,1}$ open set of \mathbb{R}^d , let $\Gamma_S \subset \partial\Omega$ and $\Gamma'_S \subset \partial\Omega$ two $\mathcal{C}^{1,1}$ relative open submanifolds such that $\mathcal{H}^{d-1}(\Gamma_S \cap \Gamma'_S) > 0$. We define the two sets $\Gamma_D = \partial\Omega \setminus \Gamma_S$ and $\Gamma'_D = \partial\Omega \setminus \Gamma'_S$ then*

- For $d \geq 3$ there exists a constant C_1 such that:

$$|\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)| \leq C_1 (\mathcal{H}^{d-1}(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2d}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}).$$

- For $d = 2$ there exists a constant C_2 such that:

$$|\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)| \leq C_2 (\mathcal{H}^1(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}).$$

Where $d(\Gamma_D, \Gamma'_D)$ is the Hausdorff distance between the two sets.

In this introduction we are not going to give more information about the constants C_1 and C_2 , for more information see Theorem 5.7. We are also not going to give the precise definition of a uniform $\mathcal{C}^{1,1}$ open set (see Definition 5.1) but we want to explain why an hypothesis of regularity of the domain Ω is needed.

One of the main difficulties that we have to face when we study mixed Steklov-Dirichlet eigenvalues is the lack of regularity of the corresponding eigenfunctions. Indeed it is well known (see for instance the monographs [34, 48]) that a singularity can appear when the boundary condition change. We present a classical example (that we also recall in the introduction of Chapter 5) let $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and we consider the following problem

$$\begin{cases} \Delta u = 0 & \mathbb{R}_+^2 \\ \partial_\nu u = 0 & \{(x, y) \in \mathbb{R}^2 \mid x < 0, y = 0\} \\ u = 0 & \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y = 0\}. \end{cases}$$

In polar coordinates a solution for the above problem is given by the following function:

$$u(r, \theta) = r^{\frac{1}{2}} \sin\left(\frac{\theta}{2}\right) \quad r \geq 0, 0 \leq \theta \leq 2\pi.$$

We have that $u \in \mathcal{C}^{\frac{1}{2}}(\overline{\mathbb{R}_+^2})$, but not more regular than that. We measure the regularity of Steklov-Dirichlet eigenfunctions in the class of Besov spaces

Definition 1.3 (Besov spaces, see [2, 13]). *Let $\Omega \subset \mathbb{R}^d$ and let $f \in L^p(\Omega)$ with $p \geq 1$ we define the m -th difference operator in the following way*

$$\Delta_h^m f(x) = \sum_{k=0}^m \binom{m}{k} (-1)^k f(x + kh),$$

we define the m -th order modulus of continuity of f in L^p in the following way

$$w_p^m(f, t) = \sup_{|h| < t} \|\Delta_h^m f\|_{L^p(\Omega)}.$$

Let n be a non negative integer, $p, q \in [1, \infty)$ and we define $s = n + \alpha$ with $0 < \alpha \leq 1$, we define the following quasi-seminorm

$$|f|_{B_{p,q}^s(\Omega)} = \begin{cases} \left(\int_0^\infty [t^{-\alpha} w_p^m(f, t)]^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 1 \leq q < \infty \\ \sup_{t \in (0, \infty)} t^{-\alpha} w_p^m(f, t) & \text{if } q = \infty \end{cases}.$$

The Besov space $B_{p,q}^s(\Omega)$ is defined as the following space

$$B_{p,q}^s(\Omega) = \{f \in W^{n,p}(\Omega) \mid |f|_{B_{p,q}^s(\Omega)} < \infty\},$$

endowed with the following quasi-seminorm

$$\|f\|_{B_{p,q}^s(\Omega)} = \|f\|_{W^{n,p}(\Omega)} + |f|_{B_{p,q}^s(\Omega)}.$$

We see in Chapter 5 that the Steklov-Dirichlet eigenfunctions are in the Besov space $B_{2,\infty}^{\frac{3}{2}}(\Omega)$, but in order to have this regularity we need to have a regular enough domain Ω . More precisely we prove the following theorem

Theorem 1.39. *Let Ω be a uniform $\mathcal{C}^{1,1}$ open set of \mathbb{R}^d , let $\Gamma_S \subset \partial\Omega$ be a $\mathcal{C}^{1,1}$ relative open submanifold (as in Definition 5.1) and let u_k be an eigenfunction corresponding to the eigenvalue $\lambda_k(\Omega, \Gamma_S)$ then $u_k \in B_{2,\infty}^{\frac{3}{2}}(\Omega)$. Moreover if $d = 2$ then $u_k \in \mathcal{C}^{\frac{1}{2}}(\overline{\Omega})$.*

In order to prove Theorem 1.38 we need this regularity for the eigenfunctions and moreover we need a stability result for solution of partial differential equations with mixed boundary conditions (Theorem 4 in [92]), in both of this results the regularity of the domain is needed.

In Chapter 5 we prove also the two following existence theorem

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $0 < m < 1$ be a constant, then, for all k , the following variational problem*

$$\inf\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\},$$

has a solution.

Actually we prove the existence in a relaxed sense and, in general, thanks to the non regularity of the eigenfunctions it is not clear if the relaxed minimizer Γ_S can be actually a relative open submanifold of $\partial\Omega$.

We also study the maximization problem, and in this case we prove the non existence of a maximizer, in particular we prove the following theorem

Theorem 1.40. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $0 < m < 1$ be a constant, then the following equality holds*

$$\sup\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\} = +\infty.$$

From the proof of this theorem it is clear that a possible maximizing sequence is given by a sequence $\Gamma_{S,n}$ that have an unbounded number of connected components. We give also an explicit example in dimension two and, motivated by this example, we study the existence of a maximizer in two dimension with a constraint on the maximal number of connected components. Actually we prove more than that, we prove that under this topological constraint we have continuity of the eigenvalues

Theorem 1.41. *Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^{1,1}$ open domain, let $\Gamma_{D,n} \subset \partial\Omega$ be a sequence of compact subdomains converging for the Hausdorff metric to a compact set $\Gamma_D \subset \partial\Omega$. We define the two sets $\Gamma_{S,n} = \partial\Omega \setminus \Gamma_{D,n}$ and $\Gamma_S = \partial\Omega \setminus \Gamma_D$, assume that the number of connected components of $\Gamma_{D,n}$ is uniformly bounded, then for all k*

$$\lambda_k(\Omega, \Gamma_{S,n}) \rightarrow \lambda_k(\Omega, \Gamma_S).$$

This result can be interpreted as the analog for the Steklov-Dirichlet eigenvalues of the well known result by Šverák concerning the classical Dirichlet eigenvalues (see [58, 96]). Indeed the Šverák 's theorem also gives a continuity result for the Dirichlet eigenvalue in the plane, under a constraint on the number of connected components, more precisely the theorem state the following

Theorem 1.42 (V. Šverák [58, 96]). *Let $\Omega_n \subset \mathbb{R}^2$ be a sequence of bounded open sets converging for the Hausdorff metric to an open set Ω . Assume that the number of connected component of the complement of sequence Ω_n is uniformly bounded, then, for all k , the Dirichlet eigenvalues converge*

$$\lambda_k(\Omega_n) \rightarrow \lambda_k(\Omega).$$

Chapter 2

Steklov problem on dumbbell domains

In this chapter we present the results we achieved in [25].

2.1 Introduction

The purpose of this Chapter is to analyse the asymptotic behavior of the eigenvalues and eigenfunctions of the Steklov problem in a dumbbell domain.

Let $\Omega_\epsilon \subset \mathbb{R}^d$ be a dumbbell shape domain given by (see Figure 2.1)

$$\Omega_\epsilon = D_1 \cup T_\epsilon \cup D_2,$$

where D_1 and D_2 are disjoint, bounded, open, connected sets in \mathbb{R}^d with Lipschitz boundary and T_ϵ is expressed as

$$T_\epsilon = \{x = (x_1, x') \in \mathbb{R}^d \mid -\frac{L}{2} \leq x_1 \leq \frac{L}{2}, |x'| < \epsilon \rho(x_1)\},$$

where $L > 0$ and $\rho \in C^0([-\frac{L}{2}, \frac{L}{2}]) \cap C^\infty((-\frac{L}{2}, \frac{L}{2}))$ is a positive function.

The connection between the channel and the two regions D_1 and D_2 occurs as follows: we assume that there exist an orthogonal system of coordinate $x = (x_1, x_2, \dots, x_d) =$

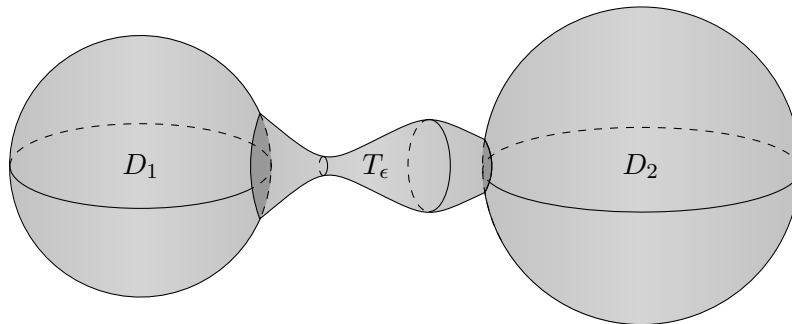


Figure 2.1: Dumbbell shape domain Ω_ϵ .

$(x_1, x') \in \mathbb{R}^d$ and two constants $L, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \overline{D_1} \cap \{x = (x_1, x') \in \mathbb{R}^d | x_1 \geq -\frac{L}{2}, |x'| \leq \delta\} &= \{x = (-\frac{L}{2}, x') \in \mathbb{R}^d | |x'| \leq \delta\} \\ \overline{D_2} \cap \{x = (x_1, x') \in \mathbb{R}^d | x_1 \leq \frac{L}{2}, |x'| \leq \delta\} &= \{x = (\frac{L}{2}, x') \in \mathbb{R}^d | |x'| \leq \delta\}. \end{aligned}$$

The eigenvalues of the Steklov problem in Ω_ϵ are denoted by

$$0 = \sigma_0^\epsilon < \sigma_1^\epsilon \leq \sigma_2^\epsilon \leq \dots \nearrow \infty \quad \forall \epsilon > 0,$$

multiplicity being counted, and the corresponding eigenfunctions by u_k^ϵ , which are normalized in $L^2(\partial\Omega_\epsilon)$, $\|u_k^\epsilon\|_{L^2(\partial\Omega_\epsilon)} = 1$.

The main purpose of this chapter is to study what is the behavior of $(\sigma_k^\epsilon, u_k^\epsilon)$ when ϵ goes to 0. The first thing to notice is that, if $\epsilon \rightarrow 0$, the channel T_ϵ collapse to a line and the norm of the trace operator blows up. One can easily observe that

$$\forall k \in \mathbb{N}, \quad \sigma_k^\epsilon \rightarrow 0 \quad \text{when } \epsilon \rightarrow 0,$$

our objective being to give precise estimates of the asymptotic behavior of σ_k^ϵ when ϵ goes to 0. We shall prove that σ_k^ϵ behaves, roughly speaking, as $\mu_k \epsilon^\gamma$, where μ_k is the k -th eigenvalue of some one dimensional problem and $\gamma \in \{1, d-1\}$.

As an interesting feature, we notice that the behavior strongly depends on the dimension of the ambient space. Indeed we have to distinguish between the cases $d = 2$ and $d \geq 3$, as we shall see below. This fact is due to the presence of the boundary energy in the Rayleigh quotient of the Steklov problem and to the fact that in dimension three, or higher, the surface area measure of the boundary of the tube is vanishing with ϵ .

Below, we denote by $P(D)$ the surface area measure of the boundary of D and ω_d is the Lebesgue measure of the d -dimensional unit ball. Let $\Phi_\epsilon : T_1 \rightarrow T_\epsilon$, $\Phi_\epsilon(x_1, x') = (x_1, \epsilon x')$.

Here are our main results. The first theorem concerns the case $d = 2$.

Theorem 2.1. ($d = 2$) *Let $\Omega_\epsilon \subset \mathbb{R}^2$ be the dumbbell shape domain defined as above. Then*

$$\sigma_k^\epsilon \sim \mu_k \epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where μ_k is the k -th eigenvalue of the following problem

$$\begin{cases} -\frac{d}{dx}(\rho(x) \frac{dV_k}{dx}(x)) = \mu_k V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho(-\frac{L}{2}) \frac{dV_k}{dx}(-\frac{L}{2}) = -\frac{\mu_k}{2} P(D_1) V_k(-\frac{L}{2}) \\ \rho(\frac{L}{2}) \frac{dV_k}{dx}(\frac{L}{2}) = \frac{\mu_k}{2} P(D_2) V_k(\frac{L}{2}). \end{cases} \quad (2.1)$$

For every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \overline{V}_k \quad \text{in } H^1(T_1),$$

where \overline{V}_k is a k -th eigenfunction of the problem (2.1) constantly extended in the variable x_2 .

This kind of eigenvalue problem (in any dimension) where the eigenvalue μ_k appears both inside the domain and in the boundary condition is sometimes called a *dynamical eigenvalue problem*. It appears at different places in the literature. We refer for example to [102] where a complete study of this eigenvalue problem has been done. See also [43] where a similar problem appears in the homogenization of the Steklov problem.

The next two theorems concern the case $d \geq 3$. We shall distinguish between the behavior of the first non-zero eigenvalue, and the others.

Theorem 2.2. ($d \geq 3, k \geq 2$) Let $\Omega_\epsilon \subset \mathbb{R}^d$ be the dumbbell shape domain defined as above and $d \geq 3$. Then for all $k \geq 2$ we have

$$\sigma_k^\epsilon \sim \alpha_{k-1}\epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where α_{k-1} is the $(k-1)$ -th eigenvalue (counting from zero) of

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_k}{dx}(x)) = \alpha_k \omega_{d-2} \rho^{d-2}(x) V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ V_k(-\frac{L}{2}) = 0 \\ V_k(\frac{L}{2}) = 0. \end{cases} \quad (2.2)$$

For every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$\epsilon_n^{\frac{d-2}{2}} u_k^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_{k-1} \quad \text{in } H^1(T_1),$$

where \bar{V}_{k-1} is an eigenfunction corresponding to α_{k-1} , constantly extended into the variables x_i for $2 \leq i \leq d$.

Therefore, in the case $d \geq 3, k \geq 2$ we end up with a classical Dirichlet eigenvalue problem.

Theorem 2.3. ($d \geq 3, k = 1$) Let $\Omega_\epsilon \subset \mathbb{R}^d$ be the dumbbell shape domain defined as above and $d \geq 3$. The first Steklov eigenvalue has the following asymptotic behavior

$$\sigma_1^\epsilon \sim \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}) \quad \text{as } \epsilon \rightarrow 0,$$

where σ_1 is the unique positive number such that the following differential equation has a non-trivial solution:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases} \quad (2.3)$$

For every subsequence $\{\epsilon_n\}_{n=1}^\infty$ such that $\epsilon_n \rightarrow 0$, we have

$$u_1^{\epsilon_n} \circ \Phi_{\epsilon_n} \rightharpoonup \bar{V}_1 \quad \text{in } H^1(T_1),$$

where \bar{V}_1 is the solution of the equation (2.3) constantly extended to the variables x_i for $2 \leq i \leq d$.

Let us now comment on the existing literature. A similar problem for the eigenvalues of the Neumann Laplacian has been deeply studied, in particular in a series of papers by S. Jimbo. A first characterization of the eigenvalues in the Neumann case was given in [11]. In [64] there is a complete description of the behavior of the Neumann eigenfunctions and in [67] there is a complete description of the Neumann eigenvalues when the channel collapse to a segment. Other references for the Neumann problem in dumbbell shape domains are [4, 62, 63, 65, 66]. These results turn out to be very useful in the study of the solutions of reaction diffusion systems in singular domains (see for instance [5, 37, 49, 84]). Perturbations of the geometric domain for the Steklov problem have been considered in [46]. For an asymptotic behavior of the Steklov problem on a singular perturbation somehow close to our analysis, we refer to the result of Nazarov [86] where he studies

a two dimensional domain obtained by the junction of two rectangles (see also [87] for a perturbation by a small whole). At last, let us mention that in the case of Dirichlet boundary conditions, singular perturbations of this type are less interesting, since the spectrum is stable to this geometric perturbation. Indeed, it can be proved that the dumbbell γ -converges to the union of the two sets $D_1 \cup D_2$ which means that its Dirichlet eigenvalues converge to the union of the spectrum of D_1 and D_2 . We refer to the books [20] and [58] for more details.

2.2 The case $d = 2$. Proof of Theorem 2.1.

In this section we will prove Theorem 2.1. We define $\partial T_\epsilon^e \subset \partial \Omega_\epsilon$ in the following way

$$\partial T_\epsilon^e = \left\{ x = (x_1, x') \in \mathbb{R}^2 \mid -\frac{L}{2} \leq x_1 \leq \frac{L}{2}, x' = \epsilon |\rho(x_1)| \right\}$$

In two dimensions, the set ∂T_ϵ^e is not connected and we decompose it

$$\partial T_\epsilon^e = \Gamma_\epsilon^- \cup \Gamma_\epsilon^+,$$

where

$$\begin{aligned} \Gamma_\epsilon^+ &= \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \mid -\frac{L}{2} \leq x_1 \leq \frac{L}{2}, x_2 = \epsilon \rho(x_1) \right\} \\ \Gamma_\epsilon^- &= \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \mid -\frac{L}{2} \leq x_1 \leq \frac{L}{2}, x_2 = -\epsilon \rho(x_1) \right\}. \end{aligned}$$

2.2.1 Upper bound for Steklov eigenvalues

First of all we prove that there exists a constant $C > 0$ such that the following upper bound holds for ϵ small enough:

$$\sigma_k^\epsilon \leq C\epsilon. \quad (2.4)$$

Precisely, we prove the following.

Lemma 2.4. *Let μ_k the k -th eigenvalue of (2.1) then we have*

$$\sigma_k^\epsilon \leq \mu_k \epsilon + o(\epsilon). \quad (2.5)$$

Proof. In order to obtain this upper bound, we use the variational formulation

$$\sigma_k^\epsilon = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_{\Omega_\epsilon} |\nabla u|^2 dx}{\int_{\partial \Omega_\epsilon} u^2 ds},$$

where the infimum is taken over all k -dimensional subspace of the Sobolev space $H^1(\Omega_\epsilon)$ which are orthogonal to constants on $\partial \Omega_\epsilon$. We choose a particular subspace E_k in order to obtain the upper bound.

We consider the eigenvalue problem (2.1) and take a basis of eigenfunctions $\{\phi_i\}_{i \in \mathbb{N}}$ normalized in the following way

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_i \phi_j dx_1 + \frac{1}{2} P(D_2) \phi_i \left(\frac{L}{2} \right) \phi_j \left(\frac{L}{2} \right) + \frac{1}{2} P(D_1) \phi_i \left(-\frac{L}{2} \right) \phi_j \left(-\frac{L}{2} \right) = \delta_{ij}. \quad (2.6)$$

Then

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho \phi'_i \phi'_j dx_1 = 0 \text{ if } i \neq j \quad (2.7)$$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho (\phi'_i)^2 dx_1 = \mu_i. \quad (2.8)$$

From the variational formulation of the eigenvalue problem we know that $\forall v \in H^1(-\frac{L}{2}, \frac{L}{2})$

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \rho \phi'_i v' dx_1 = \frac{\mu_i}{2} P(D_2) \phi_i\left(\frac{L}{2}\right) v\left(\frac{L}{2}\right) + \frac{\mu_i}{2} P(D_1) \phi_i\left(-\frac{L}{2}\right) v\left(-\frac{L}{2}\right) + \mu_i \int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_i v dx_1,$$

we choose $v = 1$ we obtain:

$$\frac{1}{2} P(D_2) \phi_i\left(\frac{L}{2}\right) + \frac{1}{2} P(D_1) \phi_i\left(-\frac{L}{2}\right) + \int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_i dx_1 = 0. \quad (2.9)$$

We now introduce our test functions that are the basis of our test subspace E_k . We define

$$\Phi_i = \begin{cases} \phi_i\left(-\frac{L}{2}\right) & \text{if } (x_1, x_2) \in D_1 \\ \phi_i(x_1) & \text{if } (x_1, x_2) \in T_\epsilon \\ \phi_i\left(\frac{L}{2}\right) & \text{if } (x_1, x_2) \in D_2, \end{cases}$$

and we introduce its mean value

$$m_i^\epsilon = \frac{1}{|\partial\Omega_\epsilon|} \int_{\partial\Omega_\epsilon} \Phi_i ds.$$

The mean goes to zero if $\epsilon \rightarrow 0$, indeed

$$\int_{\partial\Omega_\epsilon} \Phi_i ds = P(D_2) \phi_i\left(\frac{L}{2}\right) + P(D_1) \phi_i\left(-\frac{L}{2}\right) + 2 \int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_i \sqrt{1 + \epsilon^2 \rho'^2} dx_1,$$

from equation (2.9), dominated convergence and the fact that $|\partial\Omega_\epsilon| \rightarrow P(D_1) + P(D_2) + 2L > 0$ we obtain

$$m_i^\epsilon \rightarrow 0 \quad \forall i \in \mathbb{N}. \quad (2.10)$$

We introduce now our basis elements

$$\Psi_i = \Phi_i - m_i^\epsilon,$$

and our subspace will be $E_k = \text{Span} \langle \Psi_1, \dots, \Psi_k \rangle$. Now we compute all the quantities we need for the Rayleigh quotient. We start by the numerator, if $i \neq j$:

$$\int_{\Omega_\epsilon} \nabla \Psi_i \cdot \nabla \Psi_j ds = \int_{T_\epsilon} \nabla \Phi_i \cdot \nabla \Phi_j ds = 2\epsilon \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho \phi'_i \phi'_j dx_1 = 0,$$

where the last equality is given by (2.7), and

$$\int_{\Omega_\epsilon} |\nabla \Psi_i|^2 ds = \int_{T_\epsilon} |\nabla \Phi_i|^2 ds = 2\epsilon \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho (\phi'_i)^2 dx_1 = 2\epsilon \mu_i,$$

where the last equality is given by (2.8). Now we compute the terms in the denominator,

$$\begin{aligned} f_{i,j}(\epsilon) &:= \int_{\partial\Omega_\epsilon} \Psi_i \Psi_j ds = \int_{\partial\Omega_\epsilon} (\Phi_i - m_i^\epsilon)(\Phi_j - m_j^\epsilon) ds = \int_{\partial\Omega_\epsilon} \Phi_i \Phi_j ds - m_i^\epsilon m_j^\epsilon P(\Omega_\epsilon) \\ &= \frac{1}{2} P(D_2) \phi_i\left(\frac{L}{2}\right) \phi_j\left(\frac{L}{2}\right) + \frac{1}{2} P(D_1) \phi_i\left(-\frac{L}{2}\right) \phi_j\left(-\frac{L}{2}\right) \\ &\quad + \int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_i \phi_j \sqrt{1 + \epsilon^2 \rho'^2} dx_1 - m_i^\epsilon m_j^\epsilon P(\Omega_\epsilon). \end{aligned}$$

From (2.10), (2.6) and the dominated convergence we obtain

$$\lim_{\epsilon \rightarrow 0} f_{i,j}(\epsilon) = 0 \quad i \neq j. \quad (2.11)$$

Similarly,

$$\begin{aligned} f_{i,i}(\epsilon) &:= \int_{\partial\Omega_\epsilon} \Psi_i^2 ds = \int_{\partial\Omega_\epsilon} (\Phi_i - m_i^\epsilon)^2 ds = \int_{\partial\Omega_\epsilon} \Phi_i^2 ds - (m_i^\epsilon)^2 P(\Omega_\epsilon) \\ &= 2 \int_{-\frac{L}{2}}^{\frac{L}{2}} \phi_i^2 \sqrt{1 + \epsilon^2 \rho'^2} dx_1 + P(D_2) \phi_i\left(\frac{L}{2}\right)^2 + P(D_1) \phi_i\left(-\frac{L}{2}\right)^2 - (m_i^\epsilon)^2 P(\Omega_\epsilon), \end{aligned}$$

now from (2.10), (2.6) and dominated convergence we obtain,

$$\lim_{\epsilon \rightarrow 0} f_{i,i}(\epsilon) = 2. \quad (2.12)$$

Now if we use the test subspace E_k in the variational characterization we obtain

$$\sigma_k^\epsilon \leq \sup_{(x_1, \dots, x_k) \in \mathbb{R}^k} \frac{2\epsilon \sum_{i=1}^k x_i^2 \mu_i}{\sum_{i=1}^k x_i^2 f_{i,i}(\epsilon) + \sum_{i < j} 2x_i x_j f_{i,j}(\epsilon)},$$

if ϵ is small enough from (2.11) and (2.12) we obtain

$$\sigma_k^\epsilon \leq \sup_{(x_1, \dots, x_k) \in \mathbb{R}^k} \frac{\epsilon \sum_{i=1}^k x_i^2 \mu_i}{\sum_{i=1}^k x_i^2} + o(\epsilon) = \mu_k \epsilon + o(\epsilon). \quad (2.13)$$

□

2.2.2 Convergence of eigenfunctions

We start by showing the convergence on the two regions D_i where $i = 1, 2$.

Lemma 2.5. *Let $k \geq 1$ we have (up to a sub-sequence that we still denote by u_k^ϵ)*

$$\begin{aligned} u_k^\epsilon &\rightharpoonup c_{i,k} \quad \text{in } H^1(D_i), \\ u_k^\epsilon &\rightarrow c_{i,k} \quad \text{locally uniformly in } D_i. \end{aligned}$$

where $c_{i,k} \in \mathbb{R}$ are constants

Proof. First of all we know that $\sigma_k^\epsilon \rightarrow 0$ as ϵ goes to 0, and from $\|u_k^\epsilon\|_{L^2(\partial\Omega_\epsilon)} = 1$ we conclude that

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega_\epsilon} |\nabla u_k^\epsilon|^2 dx = 0,$$

so it means that $\|\nabla u_k^\epsilon\|_{L^2(D_1)} \leq \|\nabla u_k^\epsilon\|_{L^2(\Omega_\epsilon)} \leq C$. Now we want to bound $\|u_k^\epsilon\|_{L^2(D_1)}$ uniformly on ϵ . Using Poincaré-Friedrichs inequality we obtain

$$\int_{D_1} (u_k^\epsilon)^2 dx \leq \int_{\Omega_\epsilon} (u_k^\epsilon)^2 dx \leq C_{\Omega_\epsilon} \left[\int_{\Omega_\epsilon} |\nabla u_k^\epsilon|^2 dx + \int_{\partial\Omega_\epsilon} (u_k^\epsilon)^2 ds \right],$$

we know that $\|u_k^\epsilon\|_{L^2(\partial\Omega_\epsilon)} = 1$ and $\|\nabla u_k^\epsilon\|_{L^2(\Omega_\epsilon)} \leq C$, we have only to check that $C_{\Omega_\epsilon} \leq C < \infty$ if ϵ is small enough.

We have the following variational characterization for the constant C_{Ω_ϵ}

$$\frac{1}{C_{\Omega_\epsilon}} = \inf_{v \in H^1(\Omega_\epsilon)} \frac{\int_{\Omega_\epsilon} |\nabla v|^2 dx + \int_{\partial\Omega_\epsilon} v^2 ds}{\int_{\Omega_\epsilon} v^2 dx} = \lambda_1(\Omega_\epsilon, 1),$$

where $\lambda_1(\Omega_\epsilon, 1)$ is the first Robin eigenvalue with boundary parameter 1 (see [33]). We denote by B_{R_ϵ} the ball with the same measure of Ω_ϵ , now, using the Bossel-Daners inequality and the rescaling property of the Robin eigenvalue (see [33]), we obtain

$$\frac{1}{C_{\Omega_\epsilon}} = \lambda_1(\Omega_\epsilon, 1) \geq \lambda_1(B_{R_\epsilon}, 1) = \frac{1}{R_\epsilon^2} \lambda_1(B_1, R_\epsilon).$$

Now, for ϵ small enough, we have $|D_1| + |D_2| \leq |\Omega_\epsilon| \leq |D_1| + |D_2| + 1$ so, by monotonicity of the Robin eigenvalue on balls we finally obtain

$$\frac{1}{C_{\Omega_\epsilon}} = \lambda_1(\Omega_\epsilon, 1) \geq \lambda_1(B_{R_\epsilon}, 1) = \frac{\pi}{|D_1| + |D_2| + 1} \lambda_1\left(B_1, \sqrt{\frac{|D_1| + |D_2|}{\pi}}\right) > 0.$$

Finally we conclude that $C_{\Omega_\epsilon} \leq C < \infty$ for ϵ small enough. We conclude that

$$\|u_k^\epsilon\|_{H^1(D_1)} \leq C < \infty,$$

so exist a sequence, that we still denote by u_k^ϵ , and $u_k^0 \in H^1(D_1)$ such that

$$u_k^\epsilon \rightharpoonup u_k^0 \quad \text{in } H^1(D_1).$$

We also know that $\|\nabla u_k^\epsilon\|_{L^2(D_1)} \rightarrow 0$, so we conclude that there exists a constant $c_{1,k} \in \mathbb{R}$ such that

$$u_k^\epsilon \rightharpoonup c_{1,k} \quad \text{in } H^1(D_1).$$

We can improve this convergence since u_k^ϵ are harmonic. Fix a compact set $K \subset D_1$ and take $\delta > 0$ such that $B_\delta(x) \subset D_1$ for all $x \in K$. By the average properties of harmonic functions and the Cauchy-Schwartz inequality we have:

$$|u_k^\epsilon(x)| = \frac{1}{|B_\delta(x)|} \int_{B_\delta(x)} |u_k^\epsilon(y)| dy \leq |B_\delta(x)|^{-\frac{1}{2}} \|u_k^\epsilon\|_{L^2(D_1)} \leq C.$$

Up to a subsequence, we get that u_k^ϵ uniformly converges on K to a constant. We conclude that for $i = 1, 2$

$$\begin{aligned} u_k^\epsilon &\rightharpoonup c_{i,k} \quad \text{in } H^1(D_i), \\ u_k^\epsilon &\rightarrow c_{i,k} \quad \text{locally uniformly in } D_i. \end{aligned}$$

□

Now we study the behavior of the eigenfunctions in the tube T_ϵ . We define the following functions

$$v_k^\epsilon(x_1, x_2) = u_k^\epsilon(x_1, \epsilon x_2) \quad \forall (x_1, x_2) \in T_1$$

Lemma 2.6. *Let $k \geq 1$. There exists $\bar{V}_k \in H^1(T_1)$ such that*

$$v_k^\epsilon \rightharpoonup \bar{V}_k \quad \text{in } H^1(T_1),$$

(up to a sub-sequence, still denoted by v_k^ϵ), where \bar{V}_k depends only on the variable x_1 .

Proof. We start with the bound of $\|\nabla v_k^\epsilon\|_{L^2(T_1)}$

$$\int_{T_1} |\nabla v_k^\epsilon|^2 dx \leq \int_{T_1} \left(\frac{\partial v_k^\epsilon}{\partial x_1} \right)^2 + \frac{1}{\epsilon^2} \left(\frac{\partial v_k^\epsilon}{\partial x_2} \right)^2 dx = \frac{1}{\epsilon} \int_{T_\epsilon} |\nabla u_k^\epsilon|^2 dy \leq C$$

where we did the change of coordinates $y_1 = x_1$, $y_2 = \epsilon x_2$ and the last inequality is true because of (2.4). We want now to bound $\|v_k^\epsilon\|_{L^2(T_1)}$. By the Poincaré-Friedrichs inequality we get

$$\int_{T_1} (v_k^\epsilon)^2 dx \leq C_{T_1} \left[\int_{T_1} |\nabla v_k^\epsilon|^2 dx + \int_{\Gamma_1^+ \cup \Gamma_1^-} (v_k^\epsilon)^2 ds \right].$$

Now $\|\nabla v_k^\epsilon\|_{L^2(T_1)}$ is bounded, so it remains to bound the second term in the r.h.s. of the inequality. Since $\sqrt{1 + \rho'^2}$ is a bounded function, for ϵ small enough we obtain

$$\begin{aligned} \int_{\Gamma_1^+} (v_k^\epsilon)^2 ds &= \int_{-\frac{L}{2}}^{\frac{L}{2}} (v_k^\epsilon(x_1, \rho(x_1)))^2 \sqrt{1 + \rho'^2} dx_1 \\ &\leq C \int_{-\frac{L}{2}}^{\frac{L}{2}} (u_k^\epsilon(x_1, \epsilon \rho(x_1)))^2 \sqrt{1 + \epsilon^2 \rho'^2} dx_1 \\ &\leq C \int_{\Gamma_\epsilon^+} (u_k^\epsilon)^2 ds \leq C \end{aligned}$$

where the last inequality is true because $\|u_k^\epsilon\|_{L^2(\partial\Omega_\epsilon)} = 1$. The same computation is true for the integral over Γ_1^- .

We conclude that there exists $\bar{V}_k \in H^1(T_1)$ such that (up to a sub-sequence that we still denote by v_k^ϵ)

$$v_k^\epsilon \rightharpoonup \bar{V}_k \quad \text{in } H^1(T_1).$$

We finish the proof by showing that \bar{V}_k does not depend on x_2 . Indeed

$$\int_{T_1} \left(\frac{\partial v_k^\epsilon}{\partial x_2} \right)^2 dx = \epsilon \int_{T_\epsilon} \left(\frac{\partial u_k^\epsilon}{\partial x_2} \right)^2 dx \leq C \epsilon^2 \rightarrow 0.$$

□

2.2.3 Limit eigenvalue problem

From (2.5) we know that there exists $0 \leq \beta \leq \mu_k$ such that

$$\frac{\sigma_k^\epsilon}{\epsilon} \rightarrow \beta.$$

First, we prove that there exists $j \in \mathbb{N}$ such that $0 \leq j \leq k$ and $\beta = \mu_j$ and, in a second step, we prove that $j = k$.

Step 1. We begin with the following.

Lemma 2.7. *There exists $j \in \mathbb{N}$ such that $0 \leq j \leq k$ and*

$$\frac{\sigma_k^\epsilon}{\epsilon} \rightarrow \mu_j,$$

where μ_j is the j -th eigenvalue of the problem (2.1).

Proof. We define β in such a way that

$$\frac{\sigma_k^\epsilon}{\epsilon} \rightarrow \beta,$$

from (2.5) we know that $0 \leq \beta \leq \mu_k$. We use the variational formulation of the Steklov problem with the following test function $\phi \in C_c^\infty(-\frac{L}{2}, \frac{L}{2})$ (we constantly extend ϕ in the last variable x_2), we obtain:

$$\int_{T_\epsilon} \frac{\partial u_k^\epsilon}{\partial x_1} \frac{\partial \phi}{\partial x_1} dx = \sigma_k^\epsilon \int_{\Gamma_\epsilon^+ \cup \Gamma_\epsilon^-} u_k^\epsilon \phi ds.$$

Now we make the change of variable $y_1 = x_1$ and $y_2 = \epsilon x_2$ and we write the integral in the right hand side by the integral in the graph of $\epsilon \rho$

$$\begin{aligned} \int_{T_1} \frac{\partial v_k^\epsilon}{\partial y_1} \frac{\partial \phi}{\partial y_1} dy &= \frac{\sigma_k^\epsilon}{\epsilon} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} u_k^\epsilon(x_1, \epsilon \rho(x_1)) \phi \sqrt{1 + \epsilon^2 \rho'^2} dx_1 + \int_{-\frac{L}{2}}^{\frac{L}{2}} u_k^\epsilon(x_1, -\epsilon \rho(x_1)) \phi \sqrt{1 + \epsilon^2 \rho'^2} dx_1 \right) \\ &= \frac{\sigma_k^\epsilon}{\epsilon} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} v_k^\epsilon(x_1, \rho(x_1)) \phi \sqrt{1 + \epsilon^2 \rho'^2} dx_1 + \int_{-\frac{L}{2}}^{\frac{L}{2}} v_k^\epsilon(x_1, -\rho(x_1)) \phi \sqrt{1 + \epsilon^2 \rho'^2} dx_1 \right) \end{aligned}$$

We know that $v_k^\epsilon \rightharpoonup \bar{V}_k$ in $H^1(T_1)$ and \bar{V}_k depends only on the variable x_1 , we introduce the function V_k that is the restriction of \bar{V}_k to the variable x_1 . We let ϵ goes to 0 and we obtain

$$\int_{T_1} \frac{\partial V_k}{\partial y_1} \frac{\partial \phi}{\partial y_1} dy = 2\beta \int_{-\frac{L}{2}}^{\frac{L}{2}} V_k \phi dx_1.$$

Integrating by parts in the left hand side, we finally obtain

$$-2 \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{d}{dx_1} \left(\rho \frac{d}{dx_1} V_k \right) \phi dx_1 = 2\beta \int_{-\frac{L}{2}}^{\frac{L}{2}} V_k \phi dx_1.$$

This relation is true for every test function $\phi \in C_c^\infty(-\frac{L}{2}, \frac{L}{2})$ so we have that V_k and β must have to satisfy the following differential equation

$$-\frac{d}{dx} \left(\rho(x) \frac{dV_k}{dx}(x) \right) = \beta V_k(x) \quad x \in \left(-\frac{L}{2}, \frac{L}{2} \right). \quad (2.14)$$

We find now the boundary conditions associated to this equation. Fix a real number $\xi > 0$ and define the extended function $\bar{\rho}$ in the following way

$$\bar{\rho} = \begin{cases} \rho(-\frac{L}{2}) & \text{if } -\frac{L}{2} - \xi \leq x_1 \leq -\frac{L}{2} \\ \rho(x_1) & \text{if } -\frac{L}{2} \leq x_1 \leq \frac{L}{2} \\ \rho(\frac{L}{2}) & \text{if } \frac{L}{2} \leq x_1 \leq \frac{L}{2} + \xi. \end{cases}$$

We define the extended tube E_ϵ

$$E_\epsilon = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \mid -\frac{L}{2} - \xi \leq x_1 \leq \frac{L}{2} + \xi, |x_2| < \epsilon \bar{\rho}(x_1) \right\}, \quad (2.15)$$

and we choose ξ in such a way that $E_\epsilon \subset \Omega_\epsilon$. Now, repeating all the arguments in Lemma 2.6, we obtain that

$$v_k^\epsilon \rightharpoonup \bar{V}_k \quad \text{in } H^1(E_1)$$

and \bar{V}_k depends only on x_1 . We also know from Lemma 2.5 that u_k^ϵ locally uniformly converge to $c_{1,k}$ in D_1 . From this fact we have that

$$v_k^\epsilon(-\frac{L}{2} - \delta, 0) \rightarrow c_{1,k} = V_k(-\frac{L}{2} - \delta) \quad \forall \xi \geq \delta > 0,$$

where V_k is the restriction of \bar{V}_k to the variable x_1 . We know that $\bar{V}_k \in H^1(E_1)$, from embedding theorem V_k is continuous so we finally obtain

$$V_k(-\frac{L}{2}) = c_{1,k} = \lim_{\delta \rightarrow 0} V_k(-\frac{L}{2} - \delta). \quad (2.16)$$

Similarly,

$$V_k(\frac{L}{2}) = c_{2,k}.$$

We use the variational formulation of the Steklov eigenvalue with a test function ψ defined on all Ω_ϵ and that depends only on x_1 ,

$$\int_{\Omega_\epsilon} \frac{\partial u_k^\epsilon}{\partial x_1} \frac{\partial \psi}{\partial x_1} dx = \sigma_k^\epsilon \int_{\partial \Omega_\epsilon} u_k^\epsilon \psi ds.$$

We repeat all the computations above and letting ϵ goes to 0, we obtain

$$2 \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho \frac{dV_k}{dx_1} \frac{d\psi}{dx_1} dx_1 = \beta \left(V_k(-\frac{L}{2}) \int_{\partial D_1} \psi ds + V_k(\frac{L}{2}) \int_{\partial D_2} \psi ds + 2 \int_{-\frac{L}{2}}^{\frac{L}{2}} V_k \psi dx_1 \right).$$

Integrating by parts the left hand side and recalling the equation (2.14) we finally get

$$\rho(\frac{L}{2}) \frac{dV_k}{dx}(\frac{L}{2}) \psi(\frac{L}{2}) - \rho(-\frac{L}{2}) \frac{dV_k}{dx}(-\frac{L}{2}) \psi(-\frac{L}{2}) = \beta \left(V_k(-\frac{L}{2}) \int_{\partial D_1} \psi ds + V_k(\frac{L}{2}) \int_{\partial D_2} \psi ds \right).$$

Choosing a test function such that $\psi = 1$ in D_1 and $\psi = 0$ in D_2 , we get the first boundary condition

$$\rho(-\frac{L}{2}) \frac{dV_k}{dx}(-\frac{L}{2}) = -\frac{\beta}{2} P(D_1) V_k(-\frac{L}{2})$$

and, similarly, choosing a test function such that $\psi = 0$ in D_1 and $\psi = 1$ in D_2 we get the second boundary condition

$$\rho(\frac{L}{2}) \frac{dV_k}{dx}(\frac{L}{2}) = \frac{\beta}{2} P(D_2) V_k(\frac{L}{2}).$$

We finally obtain the following eigenvalue problem for β

$$\begin{cases} -\frac{d}{dx}(\rho(x) \frac{dV_k}{dx}(x)) = \beta V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho(-\frac{L}{2}) \frac{dV_k}{dx}(-\frac{L}{2}) = -\frac{\beta}{2} P(D_1) V_k(-\frac{L}{2}) \\ \rho(\frac{L}{2}) \frac{dV_k}{dx}(\frac{L}{2}) = \frac{\beta}{2} P(D_2) V_k(\frac{L}{2}). \end{cases} \quad (2.17)$$

To be able to conclude, it remains to prove that V_k is not the zero function. If V_k would be zero, from the normalization $\int_{\partial \Omega_\epsilon} u_k^\epsilon{}^2 = 1$ and the convergence on the extended tube, we would have

$$1 = P(D_1) c_{1,k}^2 + P(D_2) c_{2,k}^2 + 2 \int_{L/2}^{L/2} V_k^2.$$

Therefore, $c_{1,k}$ or $c_{2,k}$ would not be zero yielding a contradiction since the function V_k being in $H^1(-\frac{L}{2} - \delta, \frac{L}{2} + \delta)$ is continuous and thus cannot be constant (different from zero) on $(-\frac{L}{2} - \delta, -\frac{L}{2})$ and zero after $-\frac{L}{2}$. Therefore we have proved that there exist $j \in \mathbb{N}$ such that $0 \leq j \leq k$ and $\beta = \mu_j$, where μ_j is the j -th eigenvalue of the problem (2.1). \square

Step 2. We have just proved that $\sigma_k^\epsilon \sim \mu_j \epsilon$, with $0 \leq j \leq k$. In this step we justify that $j = k$. We denote by \tilde{V}_k the function constructed by taking the k -th eigenfunction of problem (2.1) and extending it constantly in the x_2 variable and equally constant to $V_k(-\frac{L}{2})$ in D_1 and to $V_k(\frac{L}{2})$ in D_2 .

We proceed by induction on the eigenvalue rank k . The case $k = 0$ is obvious. Now suppose that for all $0 \leq j \leq k - 1$ we have that $v_j^\epsilon \rightharpoonup \bar{V}_j$ in $H^1(T_1)$ and $\sigma_j^\epsilon \sim \mu_j \epsilon$.

Now we will prove that $\mu_k \epsilon + o(\epsilon) \leq \sigma_k^\epsilon$. By contradiction we suppose that there exists $j \in \mathbb{N}$ such that $0 \leq j \leq k - 1$, $v_j^\epsilon \rightharpoonup \bar{V}_j$ in $H^1(T_1)$ and $\sigma_k^\epsilon \sim \mu_j \epsilon$. From the orthogonality of the Steklov eigenfunctions we have the following equality

$$0 = \lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_\epsilon} u_k^\epsilon \tilde{V}_j ds + \int_{\partial\Omega_\epsilon} u_k^\epsilon (u_j^\epsilon - \tilde{V}_j) ds.$$

For the first term, from (2.6), we have that $\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_\epsilon} u_k^\epsilon \tilde{V}_j = 2$. For the second term, we recall the inductive hypothesis $v_j^\epsilon \rightharpoonup \bar{V}_j$, using the same argument in the proof of Lemma 2.7 we conclude also the equality (2.16) and, using Cauchy-Schwartz inequality, we obtain

$$|\lim_{\epsilon \rightarrow 0} \int_{\partial\Omega_\epsilon} u_k^\epsilon (u_j^\epsilon - \bar{V}_j) ds| \leq \lim_{\epsilon \rightarrow 0} \|u_k^\epsilon\|_{L^2(\partial\Omega_\epsilon)} \|u_j^\epsilon - \bar{V}_j\|_{L^2(\partial\Omega_\epsilon)} = 0.$$

This is a contradiction, we conclude that $\mu_k \epsilon + o(\epsilon) \leq \sigma_k^\epsilon$. Now recalling that $\sigma_k^\epsilon \leq \mu_k \epsilon + o(\epsilon)$ (see inequality (2.5)) we can conclude that

$$\sigma_k^\epsilon \sim \mu_k \epsilon + o(\epsilon).$$

We also conclude that

$$u_k^\epsilon(x_1, \epsilon x_2) \rightharpoonup \bar{V}_k(x_1, x_2) \quad \text{in } H^1(T_1),$$

where \bar{V}_k is the k -th eigenfunction of the problem (2.1) constantly extended to x_2 . We end the proof by proving that the convergence is true not only up to a subsequence but is true for all the sequence. We have seen that the only possible accumulation point is \bar{V}_k eigenfunction of Problem (2.17). Now it is a classical result for Sturm-Liouville type problem that any eigenfunction is simple: use the ODE to prove that the Wronskian is constant and the boundary conditions to prove that it is zero, yielding the result.

From the uniqueness of the accumulation point we conclude that the convergence holds for the whole sequence. This concludes the proof of Theorem 2.1

2.3 The case $d \geq 3$ and $k \geq 2$. Proof of Theorem 2.2.

In this section we will prove the second part of Theorem 2.2. We will use the following notation, take $x \in \mathbb{R}^d$ then we write $x = (x_1, x')$ where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{d-1}$

2.3.1 Upper bound for the Steklov eigenvalue

In this section we prove an upper bound for all the Steklov eigenvalues. In the following lemma we give an estimate from above of the speed of convergence to zero of the Steklov eigenvalues.

Lemma 2.8. *Let $d \geq 3$ and let $\Omega_\epsilon \subset \mathbb{R}^d$ be a dumbbell shape domain then*

- for the first Steklov eigenvalue

$$\sigma_1^\epsilon \leq \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}) \quad (2.18)$$

where $\sigma_1 > 0$ is the unique positive number such that the following differential equation has a non-trivial solution:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases}$$

- For all the other Steklov eigenvalues ($k \geq 2$)

$$\sigma_k^\epsilon \leq \lambda_k^\epsilon \epsilon + o(\epsilon) \quad (2.19)$$

where λ_k^ϵ is defined by the following 1-dimensional eigenvalue problem:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_k}{dx}(x)) = \lambda_k^\epsilon \omega_{n-2} \rho^{d-2}(x) V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_k}{dx}(-\frac{L}{2}) = -\frac{\lambda_k^\epsilon}{\epsilon^{d-2}} P(D_1) V_k(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_k}{dx}(\frac{L}{2}) = \frac{\lambda_k^\epsilon}{\epsilon^{d-2}} P(D_2) V_k(\frac{L}{2}). \end{cases} \quad (2.20)$$

Proof. We introduce the following functional space

$$H_{co}^1(\Omega_\epsilon) = \{u \in H^1(\Omega_\epsilon) \mid u \equiv c_i \text{ in } D_i \int_{\partial\Omega_\epsilon} u = 0, \text{ and } u \text{ depends only on } x_1 \text{ in } T_\epsilon\} \quad (2.21)$$

and denote $\sigma_k^{co}(\Omega_\epsilon)$ the (pseudo) k -th Steklov eigenvalue computed by replacing the Sobolev space $H^1(\Omega_\epsilon)$ with $H_{co}^1(\Omega_\epsilon)$ in the variational formulation using the Rayleigh quotient. Since $H_{co}^1(\Omega_\epsilon)$ is a subspace of $H^1(\Omega_\epsilon)$, we obtain:

$$\begin{aligned} \sigma_1^\epsilon &\leq \sigma_1^{co}(\Omega_\epsilon) = \inf_{0 \neq u \in H_{co}^1(\Omega_\epsilon)} \frac{\int_{\Omega_\epsilon} |\nabla u|^2 dx}{\int_{\partial\Omega_\epsilon} u^2 d\mathcal{H}^{d-1}} \\ &\leq \inf_{0 \neq u \in H_{co}^1(\Omega_\epsilon)} \frac{\epsilon^{d-1} \omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1}(u')^2 dx_1}{P(D_1) u^2(-\frac{L}{2}) + P(D_2) u^2(\frac{L}{2})} + o(\epsilon^{d-1}) \\ &\leq \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}). \end{aligned}$$

The last inequality is true because the quantity

$$\inf_{0 \neq u \in H_{co}^1(\Omega_\epsilon)} \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1}(u')^2 dx_1}{P(D_1) u^2(-\frac{L}{2}) + P(D_2) u^2(\frac{L}{2})}$$

is equal to σ_1 that is the unique positive number such that the following differential equation has a non-trivial solution:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases}$$

We now prove the second part of the lemma. We start by noticing that from the geometric properties of ∂T_ϵ^e we can compute the surface measure and we obtain that

$$d\mathcal{H}^{d-1} \llcorner \partial T_\epsilon^e = \epsilon^{d-2} \rho^{d-2} \sqrt{1 + \epsilon^2 \rho'^2} dx_1 d\varphi_1 \dots d\varphi_{d-2} \quad (2.22)$$

Let S_k be the family of all the k -dimensional subspaces of the functional space $H_{co}^1(\Omega_\epsilon)$ with $k \geq 2$, as above we have the following inequalities

$$\begin{aligned} \sigma_k^\epsilon &\leq \sigma_k^{co}(\Omega_\epsilon) = \inf_{E \in S_{k+1}} \sup_{u \in E} \frac{\int_{\Omega_\epsilon} |\nabla u|^2 dx}{\int_{\partial \Omega_\epsilon} u^2 d\mathcal{H}^{d-1}} \\ &\leq \inf_{E \in S_{k+1}} \sup_{u \in E} \frac{\epsilon^{d-1} \omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} (u')^2 dx_1}{P(D_1) u^2(-\frac{L}{2}) + P(D_2) u^2(\frac{L}{2}) + \epsilon^{d-2} \omega_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} u^2 \rho^{d-2} \sqrt{1 + \epsilon^2 \rho'^2} dx_1} + o(\epsilon) \\ &\leq \epsilon \inf_{E \in S_{k+1}} \sup_{u \in E} \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} (u')^2 dx_1}{\frac{P(D_1)}{\epsilon^{d-2}} u^2(-\frac{L}{2}) + \frac{P(D_2)}{\epsilon^{d-2}} u^2(\frac{L}{2}) + \omega_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} u^2 \rho^{d-2} dx_1} + o(\epsilon) \\ &\leq \lambda_k^\epsilon \epsilon + o(\epsilon). \end{aligned}$$

Where the last inequality is true because the quantity

$$\frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} (u')^2 dx_1}{\frac{P(D_1)}{\epsilon^{d-2}} u^2(-\frac{L}{2}) + \frac{P(D_2)}{\epsilon^{d-2}} u^2(\frac{L}{2}) + \omega_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} u^2 \rho^{d-2} dx_1}$$

is the Rayleigh quotient of the following eigenvalue problem that depends on ϵ

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1} (x) \frac{dV_k}{dx} (x)) = \lambda_k^\epsilon \omega_{d-2} \rho^{d-2} (x) V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1} (-\frac{L}{2}) \frac{dV_k}{dx} (-\frac{L}{2}) = -\frac{\lambda_k^\epsilon}{\epsilon^{d-2}} P(D_1) V_k(-\frac{L}{2}) \\ \rho^{d-1} (\frac{L}{2}) \frac{dV_k}{dx} (\frac{L}{2}) = \frac{\lambda_k^\epsilon}{\epsilon^{d-2}} P(D_2) V_k(\frac{L}{2}). \end{cases} \quad (2.23)$$

□

2.3.2 Convergence of eigenfunctions

We begin with the convergence on the two regions D_i where $i = 1, 2$. The proof of the following lemma is the same of the proof of Lemma 2.5, so we do not repeat it.

Lemma 2.9. *Let $k \geq 1$ we have (up to a sub-sequence that we still denote by u_k^ϵ)*

$$\begin{aligned} u_k^\epsilon &\rightharpoonup c_{i,k} \quad \text{in } H^1(D_i), \\ u_k^\epsilon &\rightarrow c_{i,k} \quad \text{locally uniformly in } D_i. \end{aligned}$$

where $c_{i,k}$ are constants

We study the behavior of the eigenfunctions in the tube T_ϵ . For every $k \geq 2$ we define the following rescaled functions

$$v_k^\epsilon(x_1, x') = \epsilon^{\frac{d-2}{2}} u_k^\epsilon(x_1, \epsilon x') \quad \forall (x_1, x') \in T_1$$

Lemma 2.10. *Let $d \geq 3$ and $k \geq 2$. There exists $\bar{V}_k \in H^1(T_1)$ which depends only on the variable x_1 such that*

$$v_k^\epsilon \rightharpoonup \bar{V}_k \quad \text{in } H^1(T_1),$$

up to a sub-sequence (that we still denote by v_k^ϵ).

Proof. Below, by C we denote a constant which may change from line to line. We start with the bound of $\|\nabla v_k^\epsilon\|_{L^2(T_1)}$

$$\int_{T_1} |\nabla v_k^\epsilon|^2 dx \leq \int_{T_1} \left(\frac{\partial v_k^\epsilon}{\partial x_1} \right)^2 + \frac{1}{\epsilon^2} |\nabla_{x'} v_k^\epsilon|^2 dx = \frac{1}{\epsilon} \int_{T_\epsilon} |\nabla u_k^\epsilon|^2 dy \leq C$$

where we performed the change of coordinates $y_1 = x_1$, $y' = \epsilon x'$. The last inequality is true because of (2.19). We want now to bound $\|v_k^\epsilon\|_{L^2(T_1)}$. By Fubini Tonelli we have:

$$\int_{T_1} (v_k^\epsilon)^2 dx = \frac{1}{\epsilon} \int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{B_{\epsilon\rho(x_1)}^{d-1}(x_1)} (u_k^\epsilon(x_1, x'))^2 d\mathcal{H}^{d-1} dx_1, \quad (2.24)$$

where $B_{\epsilon\rho(x_1)}^{d-1}(x_1)$ is the $n - 1$ dimensional ball centered in x_1 with radius $\epsilon\rho(x_1)$. Using the characterization of Robin eigenvalues, we obtain for all $x_1 \in (-\frac{L}{2}, \frac{L}{2})$:

$$\int_{B_{\epsilon\rho(x_1)}^{d-1}(x_1)} (u_k^\epsilon)^2 d\mathcal{H}^{d-1} \leq \frac{\int_{B_{\epsilon\rho(x_1)}^{d-1}(x_1)} |\nabla u_k^\epsilon|^2 d\mathcal{H}^{d-1} + \int_{\partial B_{\epsilon\rho(x_1)}^{d-1}(x_1)} (u_k^\epsilon)^2 d\mathcal{H}^{d-2}}{\lambda_1(B_{\epsilon\rho(x_1)}^{d-1}(x_1), 1)} \quad (2.25)$$

where $\lambda_1(B_{\epsilon\rho(x_1)}^{d-1}(x_1), 1)$ is the first Robin eigenvalue with parameter 1 of the ball $B_{\epsilon\rho(x_1)}^{d-1}(x_1)$.

Now we recall that (see [42])

$$\lambda_1(B_\epsilon, 1) \sim \frac{d}{\epsilon}$$

where B_ϵ is the d -dimensional ball of radius ϵ . In particular we have

$$\lambda_1(B_{\epsilon\rho(x_1)}^{d-1}(x_1), 1) \sim \frac{d-1}{\epsilon\rho(x_1)}.$$

This asymptotic formulae together with (2.25) and (2.24) give

$$\int_{T_1} (v_k^\epsilon)^2 dx \leq C \left(\int_{T_\epsilon} (u_k^\epsilon)^2 dx + \int_{\partial T_\epsilon} (u_k^\epsilon)^2 ds \right) \leq C,$$

where the last inequality is true because $\|\nabla u_k^\epsilon\|_{L^2(D_1)} \leq C\epsilon$ and $\|u_k^\epsilon\|_{L^2(\partial\Omega_\epsilon)} = 1$.

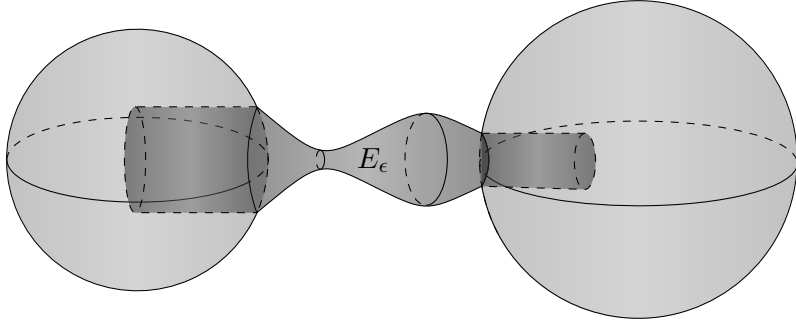
We conclude that there exists $\bar{V}_k \in H^1(T_1)$ such that (up to a sub-sequence that we still denote by v_k^ϵ)

$$v_k^\epsilon \rightharpoonup \bar{V}_k \quad \text{in } H^1(T_1).$$

We finish the proof by showing that \bar{V}_k does not depend on x_i for all $i \geq 2$, indeed

$$\int_{T_1} \left(\frac{\partial v_k^\epsilon}{\partial x_i} \right)^2 dx = \epsilon \int_{T_\epsilon} \left(\frac{\partial u_k^\epsilon}{\partial x_i} \right)^2 dx \leq C\epsilon^2 \rightarrow 0.$$

□

Figure 2.2: Extended tube E_ϵ .

Let now V_k be the restriction of \bar{V}_k to the variable x_1 , the main goal now is to prove that V_k is not the zero function and $V_k(-\frac{L}{2}) = V_k(\frac{L}{2}) = 0$. In order to reach this result we start by some consideration about the constants $c_{i,k}$ in Lemma 2.9.

We know that $\int_{\partial\Omega_\epsilon} (u_k^\epsilon)^2 d\mathcal{H}^{d-1} = 1$ for all ϵ and it is easy to see that, by change of variable, we have that $\int_{\partial T_\epsilon} (u_k^\epsilon)^2 d\mathcal{H}^{d-1} = \int_{\partial T_1} (v_k^\epsilon)^2 d\mathcal{H}^{d-1}$. Now by Lemmas 2.9 and 2.10 we obtain that for all $k \geq 2$,

$$\int_{\partial\Omega_\epsilon} (u_k^\epsilon)^2 d\mathcal{H}^{d-1} \rightarrow c_{1,k}^2 P(D_1) + c_{2,k}^2 P(D_2) + w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} V_k^2(x_1) \rho^{d-2}(x_1) dx_1,$$

so if we prove that $c_{1,k} = c_{2,k} = 0$, we conclude that V_k is not identically zero. We now prove that $c_{1,k} = c_{2,k} = 0$ for $k \geq 2$.

We note that for all $k \geq 1$, by Cauchy-Schwarz inequality

$$\left| \int_{\partial T_\epsilon} u_k^\epsilon d\mathcal{H}^{d-1} \right| \leq P(T_\epsilon)^{\frac{1}{2}} \|u_k^\epsilon\|_{L^2(\partial T_\epsilon)} \rightarrow 0,$$

and we also know that $\int_{\partial\Omega_\epsilon} u_k^\epsilon d\mathcal{H}^{d-1} = 0$, from Lemmas 2.9 and 2.10 we obtain that for all $k \geq 1$

$$c_{1,k} P(D_1) + c_{2,k} P(D_2) = 0. \quad (2.26)$$

Like in the proof of Lemma 2.7, we introduce the extended tube E_ϵ . We fix a real number $\xi > 0$ and we define the extended function $\bar{\rho}$ in the following way

$$\bar{\rho} = \begin{cases} \rho(-\frac{L}{2}) & \text{if } -\frac{L}{2} - \xi \leq x_1 \leq -\frac{L}{2} \\ \rho(x_1) & \text{if } -\frac{L}{2} \leq x_1 \leq \frac{L}{2} \\ \rho(\frac{L}{2}) & \text{if } \frac{L}{2} \leq x_1 \leq \frac{L}{2} + \xi. \end{cases}$$

We define the extended tube E_ϵ

$$E_\epsilon = \{x = (x_1, x') \in \mathbb{R}^d \mid -\frac{L}{2} - \xi \leq x_1 \leq \frac{L}{2} + \xi, |x'| < \epsilon \bar{\rho}(x_1)\}, \quad (2.27)$$

and we choose ξ in such a way that $E_\epsilon \subset \Omega_\epsilon$.

Repeating all the arguments in Lemma 2.10, we obtain that for all $k \geq 2$

$$v_k^\epsilon \rightharpoonup \bar{V}_k \quad \text{in } H^1(E_1)$$

and \bar{V}_k depends only on x_1 . We also know from Lemma 2.9 that, for all $k \geq 1$, u_k^ϵ locally uniformly converge to $c_{1,k}$ in D_1 and to $c_{2,k}$ in D_2 .

The following lemma contains a key result on the asymptotic behavior of the first eigenfunctions.

Lemma 2.11. *Let $d \geq 3$ and let $\Omega_\epsilon \subset \mathbb{R}^d$ be a dumbbell shape domain then the following holds*

$$\lim_{\epsilon \rightarrow 0} \int_{\partial T_\epsilon} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} = 0. \quad (2.28)$$

In particular $c_{1,1} \neq 0, c_{2,1} \neq 0$.

Proof. By contradiction we assume that there exists $\alpha > 0$ such that for ϵ small enough

$$\int_{\partial T_\epsilon} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} \geq \alpha > 0. \quad (2.29)$$

For every $x_1 \in (-\frac{L}{2}, \frac{L}{2})$ we consider the ball $B_{\epsilon\rho(x_1)}^{d-1}(x_1)$, that is x_1 -section of the tube T_ϵ^e . We use the Trace Theorem in any section of the tube T_1^e and we rescale to the sections of the tube T_ϵ^e , we obtain, for all $x_1 \in (-\frac{L}{2}, \frac{L}{2})$:

$$\int_{\partial B_{\epsilon\rho(x_1)}^{d-1}(x_1)} (u_1^\epsilon)^2 d\mathcal{H}^{d-2} \leq \frac{1}{\epsilon} \int_{B_{\epsilon\rho(x_1)}^{d-1}(x_1)} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} + \epsilon \int_{B_{\epsilon\rho(x_1)}^{d-1}(x_1)} |\nabla_{x'} u_1^\epsilon|^2 d\mathcal{H}^{d-1}.$$

Integrating this inequality in x_1 we obtain

$$\epsilon \int_{\partial T_\epsilon^e} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} \leq \int_{T_\epsilon^e} (u_1^\epsilon)^2 dx + \epsilon^2 \int_{T_\epsilon^e} |\nabla_{x'} u_1^\epsilon|^2 dx,$$

from this inequality, the fact that $\sigma_1^\epsilon \leq C\epsilon^{d-1}$ and inequality (2.29) we finally have

$$\epsilon \frac{\alpha}{2} \leq \int_{T_\epsilon^e} (u_1^\epsilon)^2 dx. \quad (2.30)$$

From the fact that $\int_{\partial\Omega_\epsilon} u_1^\epsilon d\mathcal{H}^{d-1} = 0$ for all ϵ we have two cases: either $c_{1,1} = c_{2,1} = 0$ or $c_{1,1}$ and $c_{2,1}$ have opposite sign, say $c_{1,1} > 0 > c_{2,1}$.

If $c_{1,1} > 0 > c_{2,1}$, then at x' fixed the function $x_1 \rightarrow u_1^\epsilon(x_1, x')$ change sign. In particular there exist $C > 0$ such that:

$$\int_{-\frac{L}{2}-\xi}^{\frac{L}{2}+\xi} \left(\frac{\partial u_k^\epsilon}{\partial x_1}\right)^2(x_1, x') dx_1 \geq C \int_{-\frac{L}{2}}^{\frac{L}{2}} (u_k^\epsilon)^2(x_1, x') dx_1,$$

by integrating in x' we finally obtain

$$\int_{E_\epsilon} |\nabla u_k^\epsilon|^2 dx \geq C\epsilon \frac{\alpha}{2}.$$

which is a contradiction to the fact that $\sigma_1^\epsilon \leq C\epsilon^{d-1}$.

If $c_{1,1} = c_{2,1} = 0$, we define $g_1^\epsilon(x_1, x') = \epsilon^{\frac{d-2}{2}} u_1^\epsilon(x_1, \epsilon x')$. Using the same argument in Lemma 2.10 we conclude that there exist $\bar{G}_1 \in H^1(E_1)$ such that

$$g_1^\epsilon \rightharpoonup \bar{G}_1 \quad \text{in } H^1(E_1).$$

From this convergence, the fact that $\int_{\partial\Omega_\epsilon} (u_k^\epsilon)^2 d\mathcal{H}^{d-1} = 1$ for all ϵ and the assumption $c_{1,1} = c_{2,1} = 0$ we get

$$\int_{\partial T_1^e} \bar{G}_1^2 d\mathcal{H}^{d-1} = 1 \quad \text{and} \quad \int_{\partial D_1 \cup \partial D_2} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} \rightarrow 0.$$

From this consideration we conclude that

$$C\epsilon^{d-1} \geq \sigma_1^\epsilon \geq \int_{E_1} |\nabla \bar{G}_1|^2 dx,$$

hence $\int_{E_1} |\nabla \bar{G}_1|^2 dx = 0$. In particular \bar{G}_1 is almost everywhere constant $C \neq 0$. Consider a point in the extremity on the extended tube E_1 , where we know that u_1^ϵ uniformly converge to a constant. More precisely we find $\bar{x} = (\bar{x}_1, \bar{x}') \in E_1$ such that the following holds

$$g_1^\epsilon(\bar{x}_1, \bar{x}') \rightarrow C > 0 \quad \text{and} \quad g_1^\epsilon(\bar{x}_1, \bar{x}') = \epsilon^{\frac{d-2}{2}} u_1^\epsilon(\bar{x}_1, \epsilon \bar{x}') \rightarrow 0.$$

This is a contradiction. □

We now prove that $c_{1,k} = c_{2,k} = 0$ for all $k \geq 2$, concluding that V_k is not identically zero for all $k \geq 2$.

Lemma 2.12. *Let $d \geq 3$, $k \geq 2$ and let $\Omega_\epsilon \subset \mathbb{R}^d$ be a dumbbell shape domain. The following holds*

$$\begin{aligned} u_k^\epsilon &\rightarrow 0 \quad \text{in } H^1(D_i), \\ u_k^\epsilon &\rightarrow 0 \quad \text{locally uniformly in } D_i. \end{aligned}$$

Proof. We start by fixing $k \geq 2$. From the orthogonality condition of the Steklov eigenfunctions we know that

$$\int_{\partial\Omega_\epsilon} u_1^\epsilon u_k^\epsilon d\mathcal{H}^{d-1} = \int_{\partial D_1} u_1^\epsilon u_k^\epsilon d\mathcal{H}^{d-1} + \int_{\partial T_\epsilon} u_1^\epsilon u_k^\epsilon d\mathcal{H}^{d-1} + \int_{\partial D_2} u_1^\epsilon u_k^\epsilon d\mathcal{H}^{d-1} = 0.$$

From this equality and Lemmas 2.9, 2.10 and 2.11 we obtain the equality

$$c_{1,k}c_{1,1}P(D_1) + c_{2,k}c_{2,1}P(D_2) = 0$$

which, together with

$$\begin{cases} c_{1,k}P(D_1) + c_{2,k}P(D_2) = 0 \\ c_{1,1}P(D_1) + c_{2,1}P(D_2) = 0, \end{cases}$$

lead to a system of equations.

We know that $c_{1,1} \neq 0$ and $c_{2,1} \neq 0$ and (without loss of generality) $c_{1,1} > 0 > c_{2,1}$. Suppose by contradiction that $c_{2,k} \neq 0$, from the system above we have the following

$$\frac{c_{1,1}}{c_{2,1}} = \frac{c_{1,k}}{c_{2,k}} = \frac{c_{1,1}c_{1,k}}{c_{2,1}c_{2,k}},$$

it means that $c_{1,1} = c_{2,1}$ and $c_{1,k} = c_{2,k}$ that is a contradiction. We obtain the same conclusion if we suppose that $c_{1,k} \neq 0$. □

Let V_k be the restriction to x_1 of the limit eigenfunction \bar{V}_k in Lemma 2.10, in the next lemma we prove that V_k is not constant and we find the boundary conditions of V_k .

Lemma 2.13. *Let $d \geq 3$, $k \geq 2$ and let V_k be the restriction to x_1 of the limit eigenfunction \bar{V}_k in Lemma 2.10. Then*

- V_k is continuous

- $V_k(-\frac{L}{2}) = V_k(\frac{L}{2}) = 0$
- V_k is not constant

Proof. The first point is immediate because we know that, if we consider the extended tube E_1 , $V_k \in H^1((-\frac{L}{2} - \xi, \frac{L}{2} + \xi))$, by classical embedding theorem we have $V_k \in C((-\frac{L}{2} - \xi, \frac{L}{2} + \xi))$. We prove the second point, we know, from Lemma 2.12 that u_k^ϵ locally uniformly converge to 0 in D_1 . From this fact we have that

$$v_k^\epsilon(-\frac{L}{2} - \delta, 0) = \epsilon^{\frac{d-2}{2}} u_k^\epsilon(-\frac{L}{2} - \delta, 0) \rightarrow 0 = V_k(-\frac{L}{2} - \delta) \quad \forall \xi \geq \delta > 0.$$

From the continuity of V_k we conclude that

$$V_k(-\frac{L}{2}) = \lim_{\delta \rightarrow 0} V_k(-\frac{L}{2} - \delta) = 0.$$

The same is true for $V_k(\frac{L}{2})$. The third point is a direct consequence of this, indeed if V_k is constant then V_k must be equal to zero and this is a contradiction with Lemma 2.12 \square

2.3.3 Proof of Theorem 2.2

In this section we prove Theorem 2.2. We will first prove a bound from below for the asymptotic of σ_k^ϵ . Then we will prove that this bound from below is also a bound from above finding in that way the right asymptotic of σ_k^ϵ .

Lemma 2.14. *Let $d \geq 3$ and $k \geq 2$ then, for ϵ small enough*

$$\sigma_k^\epsilon \geq \alpha_{k-1}\epsilon + o(\epsilon) \tag{2.31}$$

where α_k is defined by the following 1-dimensional Dirichlet eigenvalue problem:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_k}{dx}(x)) = \alpha_k w_{d-2} \rho^{d-2}(x) V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ V_k(-\frac{L}{2}) = 0 \\ V_k(\frac{L}{2}) = 0. \end{cases}$$

Proof. We start by showing that there exists a constant $C_k > 0$ such that $\sigma_k^\epsilon \geq C_k \epsilon$. Indeed we have

$$\begin{aligned} \sigma_k^\epsilon &= \frac{\int_{\Omega_\epsilon} |\nabla u_k^\epsilon|^2 dx}{\int_{\partial\Omega_\epsilon} (u_k^\epsilon)^2 d\mathcal{H}^{d-1}} \geq \frac{\epsilon^{d-2} \int_{T_\epsilon} |\nabla u_k^\epsilon|^2 dx}{\epsilon^{d-2} \int_{\partial D_1 \cup \partial D_2} (u_k^\epsilon)^2 d\mathcal{H}^{d-1} + \epsilon^{d-2} \int_{\partial T_\epsilon} (u_k^\epsilon)^2 d\mathcal{H}^{d-1}} \\ &\geq \epsilon \frac{\int_{T_\epsilon} |\nabla v_k^\epsilon|^2 dx}{\int_{\partial D_1 \cup \partial D_2} (u_k^\epsilon)^2 d\mathcal{H}^{d-1} + \int_{\partial T_\epsilon} (v_k^\epsilon)^2 d\mathcal{H}^{d-1}}. \end{aligned}$$

Now from Lemmas 2.10 and 2.12, recalling that V_k is the restriction to x_1 of the limit eigenfunction \bar{V}_k in Lemma 2.10 and recalling also the geometry of the tube, we finally obtain

$$\sigma_k^\epsilon = \frac{\int_{\Omega_\epsilon} |\nabla u_k^\epsilon|^2 dx}{\int_{\partial\Omega_\epsilon} (u_k^\epsilon)^2 d\mathcal{H}^{d-1}} \geq \epsilon \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} (V_k')^2 \rho^{d-1} dx_1}{w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} V_k^2 \rho^{d-2} dx_1}. \tag{2.32}$$

From this inequality and Lemma 2.13 follow that there exists $C_k > 0$ such that $\sigma_k^\epsilon \geq C_k \epsilon$. From the variational characterization of the Steklov eigenvalues we have

$$\sigma_k^\epsilon = \max_{u \in \langle u_1^\epsilon, u_2^\epsilon, \dots, u_k^\epsilon \rangle} \frac{\int_{\Omega_\epsilon} |\nabla u|^2 dx}{\int_{\partial\Omega_\epsilon} u^2 d\mathcal{H}^{d-1}},$$

where $\langle u_1^\epsilon, u_2^\epsilon, \dots, u_k^\epsilon \rangle$ is the subspace of $H^1(\Omega_\epsilon)$ generated by $u_1^\epsilon, u_2^\epsilon, \dots, u_{k-1}^\epsilon$ and u_k^ϵ . We know that that the maximum is achieved when $u = u_k^\epsilon$, so we have

$$\begin{aligned} \sigma_k^\epsilon &= \max_{u \in \langle u_2^\epsilon, \dots, u_k^\epsilon \rangle} \frac{\int_{\Omega_\epsilon} |\nabla u|^2 dx}{\int_{\partial\Omega_\epsilon} u^2 d\mathcal{H}^{d-1}} \\ &\geq \max_{\{\beta_j\}_{j=2}^k} \frac{\sum_{j=2}^k \beta_j^2 \int_{T_\epsilon^\epsilon} |\nabla u_j^\epsilon|^2 dx}{\sum_{j=2}^k \beta_j^2 \int_{\partial\Omega_\epsilon} (u_j^\epsilon)^2 d\mathcal{H}^{d-1}}. \end{aligned}$$

From the inequality above and inequality (2.32) we obtain

$$\begin{aligned} \sigma_k^\epsilon &\geq \epsilon \max_{\{\beta_j\}_{j=2}^k} \frac{\sum_{j=2}^k \beta_j^2 \omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} (V_j')^2 \rho^{d-1} dx_1}{\sum_{j=2}^k \beta_j^2 \omega_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} V_j^2 \rho^{d-2} dx_1} \\ &\geq \epsilon \max_{V \in \langle V_2, \dots, V_k \rangle} \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} (V')^2 \rho^{d-1} dx_1}{\omega_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} V^2 \rho^{d-2} dx_1}. \end{aligned} \quad (2.33)$$

From Lemma 2.13 we know that $V_j \in H_0^1((-\frac{L}{2}, \frac{L}{2}))$ for all $2 \leq j \leq k$ and from the orthogonality of the Steklov eigenfunctions we also know that $\dim \langle V_2, \dots, V_k \rangle = k - 1$. It is easy to check that the ratio (2.33) is the Rayleigh quotient of the following eigenvalue problem

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_k}{dx}(x)) = \alpha_k \omega_{d-2} \rho^{d-2}(x) V_k(x) & x \in (-\frac{L}{2}, \frac{L}{2}) \\ V_k(-\frac{L}{2}) = 0 \\ V_k(\frac{L}{2}) = 0. \end{cases}$$

From (2.33) we finally conclude that

$$\sigma_k^\epsilon \geq \alpha_{k-1} \epsilon + o(\epsilon)$$

□

We prove that the bound from below given in (2.31) is in fact also a bound from above.

Lemma 2.15. *Let α_k be the k -th eigenvalue of the problem (2.2) and let λ_k^ϵ be k -th eigenvalue of the problem (2.20), then for ϵ small enough we have*

$$\lambda_k^\epsilon \leq \alpha_{k-1} + o(\epsilon). \quad (2.34)$$

Proof. We choose V_1, \dots, V_{k-1} eigenfunctions of the problem (2.2), in particular $V_i(-\frac{L}{2}) = V_i(\frac{L}{2}) = 0$ for all $i = 1, \dots, k - 1$. Now we define a function $\psi \in C^\infty(-\frac{L}{2}, \frac{L}{2})$ such that

$\psi(-\frac{L}{2}) = 1$, $\psi(\frac{L}{2}) = 0$ and $\dim\langle \psi, V_1, \dots, V_{k-1} \rangle = k$. Using the variational characterization of the eigenvalue λ_k^ϵ we obtain

$$\begin{aligned} \lambda_k^\epsilon &\leq \max_{u \in \langle \psi, V_1, \dots, V_{k-1} \rangle} \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} (u')^2 dx_1}{\frac{P(D_1)}{\epsilon^{d-2}} u^2(-\frac{L}{2}) + \frac{P(D_2)}{\epsilon^{d-2}} u^2(\frac{L}{2}) + w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} u^2 \rho^{d-2} dx_1} \\ &\leq \max_{\beta_0(\epsilon) \cup \{\beta_j\}_{j=1}^{k-1}} \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} (\beta_0(\epsilon)\psi' + \sum_{i=1}^{k-1} \beta_i V_i')^2 dx_1}{\frac{P(D_1)}{\epsilon^{d-2}} \beta_0(\epsilon)^2 + w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-2} (\beta_0(\epsilon)\psi + \sum_{i=1}^{k-1} \beta_i V_i)^2 dx_1}. \end{aligned} \quad (2.35)$$

In the inequality above, in order to study the cases in full generality, we must impose that the first real coefficient β_0 depends on ϵ , because the boundary conditions of the eigenvalue problem (2.20) depends on ϵ . We define the following quantity

$$A_\beta(\epsilon) = \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} (\beta_0(\epsilon)\psi' + \sum_{i=1}^{k-1} \beta_i V_i')^2 dx_1}{\frac{P(D_1)}{\epsilon^{d-2}} \beta_0(\epsilon)^2 + w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-2} (\beta_0(\epsilon)\psi + \sum_{i=1}^{k-1} \beta_i V_i)^2 dx_1}.$$

It is easy to check that in order to get the maximum (2.35) we must have that $\lim_{\epsilon \rightarrow 0} \frac{\beta_0(\epsilon)^2}{\epsilon^{d-2}} = C < \infty$. Otherwise if $\lim_{\epsilon \rightarrow 0} \frac{\beta_0(\epsilon)}{\epsilon^{d-2}} = \infty$ we will have that $\lim_{\epsilon \rightarrow 0} A_\beta(\epsilon) = 0$, so we don't reach the maximum in this case.

We conclude that $\beta_0(\epsilon) \sim \beta_0 \epsilon^{\frac{d-2}{2}} + o(\epsilon^{\frac{d-2}{2}})$, (if $\beta_0(\epsilon) < \beta_0 \epsilon^{\frac{d-2}{2}} + o(\epsilon^{\frac{d-2}{2}})$ we have the same results) from (2.35) we obtain that

$$\begin{aligned} \lambda_k^\epsilon &\leq \max_{\{\beta_j\}_{j=0}^{k-1}} \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} ((\beta_0 \epsilon^{\frac{d-2}{2}} + o(\epsilon^{\frac{d-2}{2}}))\psi' + \sum_{i=1}^{k-1} \beta_i V_i')^2 dx_1}{P(D_1)\beta_0^2 + w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-2} ((\beta_0 \epsilon^{\frac{d-2}{2}} + o(\epsilon^{\frac{d-2}{2}}))\psi + \sum_{i=1}^{k-1} \beta_i V_i)^2 dx_1} \\ &\leq \max_{\{\beta_j\}_{j=1}^{k-1}} \frac{\sum_{j=1}^{k-1} \beta_j^2 \omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} (V_j')^2 \rho^{d-1} dx_1}{\sum_{j=1}^{k-1} \beta_j^2 w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} V_j^2 \rho^{d-2} dx_1} + o(\epsilon). \end{aligned}$$

Recalling that V_1, \dots, V_{k-1} are eigenfunctions of the problem (2.2), by the variational characterization of the eigenvalue of the problem (2.2) we finally conclude that

$$\alpha_{k-1} = \max_{\{\beta_j\}_{j=1}^{k-1}} \frac{\sum_{j=1}^{k-1} \beta_j^2 \omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} (V_j')^2 \rho^{d-1} dx_1}{\sum_{j=1}^{k-1} \beta_j^2 w_{d-2} \int_{-\frac{L}{2}}^{\frac{L}{2}} V_j^2 \rho^{d-2} dx_1}$$

and this concludes the proof. \square

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. From the second part of Lemma 2.8, Lemma 2.14 and Lemma 2.15, we finally conclude that for all $k \geq 2$ we have

$$\sigma_k^\epsilon \sim \alpha_{k-1} \epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0.$$

We prove the convergence of the eigenfunctions, showing that V_k must be the $(k-1)$ -th eigenfunction of problem (2.3). From Lemma 2.13 we know that V_k satisfies the Dirichlet

boundary conditions, it remains to prove the fact that V_k satisfies the eigenvalue equation. We use the variational formulation of the Steklov problem using the following test function $\phi \in C_c^\infty(-\frac{L}{2}, \frac{L}{2})$ (we constantly extend ϕ in the last variables x'), we obtain:

$$\int_{T_\epsilon} \frac{\partial u_k^\epsilon}{\partial x_1} \frac{\partial \phi}{\partial x_1} dx = \sigma_k^\epsilon \int_{\partial T_\epsilon} u_k^\epsilon \phi d\mathcal{H}^{d-1}.$$

We make the following change of variable $y_1 = x_1$ and $y' = \epsilon x'$ in the right hand side of the variational formulation, using the formula (2.22) for the surface measure in the right hand side we obtain

$$\begin{aligned} \int_{T_1} \frac{\partial v_k^\epsilon}{\partial y_1} \frac{\partial \phi}{\partial y_1} dy &= \frac{\sigma_k^\epsilon \epsilon^{\frac{d-2}{2}}}{\epsilon} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{\partial B_{\epsilon \rho(x_1)}^{d-1}(x_1)} u_k^\epsilon(x_1, \epsilon \tilde{\rho}(x_1)) \phi(x_1) \rho^{d-2} \sqrt{1 + \epsilon^2 \rho'^2} dx_1 d\varphi_1 \dots d\varphi_{d-2} \right) \\ &= \frac{\sigma_k^\epsilon}{\epsilon} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{\partial B_{\rho(x_1)}^{d-1}(x_1)} v_k^\epsilon(x_1, \tilde{\rho}(x_1)) \phi(x_1) \rho^{d-2} \sqrt{1 + \epsilon^2 \rho'^2} dx_1 d\varphi_1 \dots d\varphi_{d-2} \right), \end{aligned}$$

where $\tilde{\rho}(x_1)$ are the spherical coordinates that describes $\partial B_{\rho(x_1)}^{d-1}(x_1)$. Now we let ϵ goes to 0, recalling that $v_k^\epsilon \rightharpoonup \bar{V}_k$ in $H^1(T_1)$ and $\sigma_k^\epsilon \sim \alpha_{k-1} \epsilon + o(\epsilon)$ we obtain

$$\int_{T_1} \frac{\partial \bar{V}_k}{\partial y_1} \frac{\partial \phi}{\partial y_1} dy = \alpha_{k-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} w_{d-2} \rho^{d-2} V_k \phi dx_1.$$

Now integrating by part the left hand side, we finally obtain

$$-\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{d}{dx_1} \left(\rho^{d-1} \frac{d}{dx_1} V_k \right) \phi dx_1 = \alpha_{k-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} w_{d-2} \rho^{d-2} V_k \phi dx_1.$$

This relation is true for every test function $\phi \in C_c^\infty(-\frac{L}{2}, \frac{L}{2})$ so we have

$$-\omega_{d-1} \frac{d}{dx} \left(\rho^{d-1}(x) \frac{dV_k}{dx}(x) \right) = \alpha_{k-1} w_{d-2} \rho^{d-2}(x) V_k(x) \quad x \in \left(-\frac{L}{2}, \frac{L}{2} \right),$$

and $V_k(-\frac{L}{2}) = V_k(\frac{L}{2}) = 0$. Using the same argument as at the end of proof of Theorem 2.1 we conclude that the result is true for all the sequence $\{\epsilon_n\}_{n=1}^\infty$. This concludes the proof. \square

2.4 The case $d \geq 3$ and $k = 1$. Proof of Theorem 2.3.

In this section we prove Theorem 2.3.

2.4.1 Convergence of the eigenfunctions

From Lemma 2.9, we know that

$$\begin{aligned} u_1^\epsilon &\rightharpoonup c_{i,1} \quad \text{in } H^1(D_i), \\ u_1^\epsilon &\rightarrow c_{i,2} \quad \text{locally uniformly in } D_i, \end{aligned}$$

We also know that $c_{1,1} > 0 > c_{2,1}$, this information letting us to improve Lemma 2.11.

Lemma 2.16. *Let $d \geq 3$ and let $\Omega_\epsilon \subset \mathbb{R}^d$ be a dumbbell shape domain. There exists a constant $C > 0$ such that*

$$\limsup_{\epsilon \rightarrow 0} \frac{\int_{\partial T_\epsilon} (u_1^\epsilon)^2 d\mathcal{H}^{d-1}}{\epsilon^{d-2}} \leq C. \quad (2.36)$$

Proof. By contradiction we suppose

$$N\epsilon^{d-2} \leq \int_{\partial T_\epsilon} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} \quad \forall N \in \mathbb{N}. \quad (2.37)$$

Using the same argument in the proof of Lemma 2.11 we conclude that

$$\epsilon \int_{\partial T_\epsilon} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} \leq \int_{T_\epsilon} (u_1^\epsilon)^2 dx + \epsilon^2 \int_{T_\epsilon} |\nabla_{x'} u_1^\epsilon|^2 dx.$$

from this inequality, the fact that $\sigma_1^\epsilon \leq C\epsilon^{d-1}$ and inequality (2.37) we finally have

$$\epsilon^{d-1} \frac{N}{2} \leq \int_{T_\epsilon} (u_1^\epsilon)^2 dx \quad \forall N \in \mathbb{N}.$$

We know that $c_{1,1} > 0 > c_{2,1}$, repeating all the arguments in the first part of the proof of Lemma 2.11, we obtain that

$$CN\epsilon^{d-1} \leq \int_{E_\epsilon} |\nabla u_1^\epsilon|^2 dx \quad \forall N \in \mathbb{N},$$

where E_ϵ is the extended tube defined in (2.27). This is a contradiction with the fact that $\sigma_1^\epsilon \leq C\epsilon^{d-1}$. \square

We introduce the following function

$$v_1^\epsilon(x_1, x') = u_1^\epsilon(x_1, \epsilon x') \quad \forall (x_1, x') \in T_1.$$

Lemma 2.17. *Let $d \geq 3$ then there exists $\bar{V}_1 \in H^1(T_1)$ such that (up to a sub-sequence that we still denote by v_1^ϵ)*

$$v_1^\epsilon \rightharpoonup \bar{V}_1 \quad \text{in } H^1(T_1).$$

and \bar{V}_1 depends only on the variable x_1 .

Proof. We start with the bound of $\|\nabla v_1^\epsilon\|_{L^2(T_1)}$

$$\int_{T_1} |\nabla v_1^\epsilon|^2 dx \leq \int_{T_1} \left(\frac{\partial v_1^\epsilon}{\partial x_1} \right)^2 + \frac{1}{\epsilon^2} |\nabla_{x'} v_1^\epsilon|^2 dx = \frac{1}{\epsilon^{d-1}} \int_{T_\epsilon} |\nabla u_1^\epsilon|^2 dy \leq C$$

where we did the change of coordinates $y_1 = x_1$, $y' = \epsilon x'$ and the last inequality is true because of (2.18). We want now to bound $\|v_1^\epsilon\|_{L^2(T_1)}$. Following the computations in the proof of Lemma 2.11 we obtain

$$\begin{aligned} \int_{T_1} (v_1^\epsilon)^2 dx &= \frac{1}{\epsilon^{d-1}} \int_{T_\epsilon} (u_1^\epsilon)^2 dx \leq \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\int_{B_{\epsilon\rho(x_1)}^{d-1}(x_1)} |\nabla u_1^\epsilon|^2 d\mathcal{H}^{d-1} + \int_{\partial B_{\epsilon\rho(x_1)}^{d-1}(x_1)} (u_1^\epsilon)^2 d\mathcal{H}^{d-2}}{\epsilon^{d-1} \lambda_1(B_{\epsilon\rho(x_1)}^{d-1}(x_1), 1)} \\ &\leq \frac{C}{\epsilon^{d-2}} \left(\int_{T_\epsilon} |\nabla u_1^\epsilon|^2 dx + \int_{\partial T_\epsilon} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} \right) \\ &\leq C, \end{aligned}$$

where the last inequality is true because of (2.36) and (2.18). We conclude that there exist $\bar{V}_1 \in H^1(T_1)$ such that (up to a sub-sequence that we still denote by v_1^ϵ)

$$v_1^\epsilon \rightharpoonup \bar{V}_1 \quad \text{in } H^1(T_1).$$

We finish the proof by showing that \bar{V}_1 does not depend on x_i for all $i \geq 2$, indeed

$$\int_{T_1} \left(\frac{\partial v_1^\epsilon}{\partial x_i} \right)^2 dx = \frac{1}{\epsilon^{d-3}} \int_{T_\epsilon} \left(\frac{\partial u_1^\epsilon}{\partial x_i} \right)^2 dx \leq C\epsilon^2 \rightarrow 0.$$

\square

We denote by V_1 the restriction to the x_1 variable of the function \bar{V}_1 and we introduce the extended tube E_ϵ (see (2.27)). In the next Lemma we find the boundary conditions of V_k .

Lemma 2.18. *Let $d \geq 3$ and let V_1 be the restriction to x_1 of the limit eigenfunction \bar{V}_1 in Lemma 2.17 then V_1 is continuous and*

$$V_1\left(-\frac{L}{2}\right) = c_{1,1} \quad \text{and} \quad V_1\left(\frac{L}{2}\right) = c_{2,1}.$$

Proof. The first point is immediate because we know that, if we consider the extended tube E_1 , $V_1 \in H^1\left(\left(-\frac{L}{2} - \xi, \frac{L}{2} + \xi\right)\right)$, by classical embedding theorem we have $V_1 \in C\left(\left(-\frac{L}{2} - \xi, \frac{L}{2} + \xi\right)\right)$. We prove the second point, we know, from Lemma 2.16 that u_1^ϵ locally uniformly converge to $c_{1,1}$ in D_1 . From this fact we have that

$$v_1^\epsilon\left(-\frac{L}{2} - \delta, 0\right) = u_1^\epsilon\left(-\frac{L}{2} - \delta, 0\right) \rightarrow c_{1,1} = V_1\left(-\frac{L}{2} - \delta\right) \quad \forall \xi \geq \delta > 0.$$

From the continuity of V_1 we conclude that

$$V_1\left(-\frac{L}{2}\right) = \lim_{\delta \rightarrow 0} V_1\left(-\frac{L}{2} - \delta\right) = c_{1,1}.$$

Using the same techniques we obtain also $V_1\left(\frac{L}{2}\right) = c_{2,1}$. □

2.4.2 Proof of Theorem 2.3

In this section we prove Theorem 2.3. We will show that the bound from above given in the first part of Lemma 2.8 is actually the right asymptotics. In particular the following result holds.

Lemma 2.19. *Let $d \geq 3$ then, for ϵ small enough*

$$\sigma_1^\epsilon \geq \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}) \tag{2.38}$$

where σ_1 is the positive number such that the following differential equation has a solution:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} \left(\rho^{d-1}(x) \frac{dV_1}{dx}(x) \right) = 0 & x \in \left(-\frac{L}{2}, \frac{L}{2} \right) \\ \rho^{d-1}\left(-\frac{L}{2}\right) \frac{dV_1}{dx}\left(-\frac{L}{2}\right) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1\left(-\frac{L}{2}\right) \\ \rho^{d-1}\left(\frac{L}{2}\right) \frac{dV_1}{dx}\left(\frac{L}{2}\right) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1\left(\frac{L}{2}\right). \end{cases}$$

Proof. From the variational characterization of the first Steklov eigenfunction we have:

$$\sigma_1^\epsilon = \frac{\int_{\Omega_\epsilon} |\nabla u_1^\epsilon|^2 dx}{\int_{\partial\Omega_\epsilon} (u_1^\epsilon)^2 d\mathcal{H}^{d-1}} \geq \epsilon^{d-1} \frac{\int_{T_1^\epsilon} |\nabla v_1^\epsilon|^2 dx}{\int_{\partial D_1 \cup \partial D_2} (u_1^\epsilon)^2 d\mathcal{H}^{d-1} + o(\epsilon)}.$$

From this inequality and the convergence results, in particular Lemmas 2.9, 2.17 and 2.18 we obtain

$$\sigma_1^\epsilon \geq \epsilon^{d-1} \frac{\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} (V_1')^2 \rho^{d-1} dx}{P(D_1) V_1^2\left(-\frac{L}{2}\right) + P(D_2) V_1^2\left(\frac{L}{2}\right)} + o(\epsilon^{d-1}) \geq \epsilon^{d-1} \sigma_1 + o(\epsilon^{d-1}),$$

where $\sigma_1 > 0$ is the positive number such that the following differential equation has a solution:

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases}$$

□

Now we are ready to prove Theorem 2.3

Proof of Theorem 2.3. From the first part of Lemma 2.8 and Lemma 2.19, we get that

$$\sigma_1^\epsilon \sim \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1}) \quad \text{as } \epsilon \rightarrow 0,$$

We use the variational formulation of the Steklov problem using the following test function $\phi \in C_c^\infty(-\frac{L}{2}, \frac{L}{2})$ (we constantly extend ϕ in the last variables x'), we obtain:

$$\int_{T_\epsilon} \frac{\partial u_1^\epsilon}{\partial x_1} \frac{\partial \phi}{\partial x_1} dx = \sigma_k^\epsilon \int_{\partial T_\epsilon} u_1^\epsilon \phi d\mathcal{H}^{d-1}.$$

We perform the following change of variable $y_1 = x_1$ and $y' = \epsilon x'$ in the right hand side of the variational formulation, using the formula (2.22) for the surface measure in the right hand side we obtain

$$\begin{aligned} \int_{T_1} \frac{\partial v_1^\epsilon}{\partial y_1} \frac{\partial \phi}{\partial y_1} dy &= \frac{\sigma_k^\epsilon}{\epsilon} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{\partial B_{\epsilon \rho(x_1)}^{d-1}(x_1)} u_k^\epsilon(x_1, \tilde{\rho}(x_1)) \phi(x_1) \rho^{d-2} \sqrt{1 + \epsilon^2 \rho'^2} dx_1 d\varphi_1 \dots d\varphi_{d-2} \right) \\ &= \frac{\sigma_k^\epsilon}{\epsilon} \left(\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_{\partial B_{\rho(x_1)}^{d-1}(x_1)} v_k^\epsilon(x_1, \tilde{\rho}(x_1)) \phi(x_1) \rho^{d-2} \sqrt{1 + \epsilon^2 \rho'^2} dx_1 d\varphi_1 \dots d\varphi_{d-2} \right), \end{aligned}$$

where $\tilde{\rho}(x_1)$ are the spherical coordinates that describes $\partial B_{\rho(x_1)}^{d-1}(x_1)$.

We let ϵ goes to 0, recalling that $v_1^\epsilon \rightarrow \bar{V}_1$ in $H^1(T_1)$ and the fact that $\sigma_1^\epsilon \sim \sigma_1 \epsilon^{d-1} + o(\epsilon^{d-1})$ we obtain

$$\int_{T_1} \frac{\partial \bar{V}_1}{\partial y_1} \frac{\partial \phi}{\partial y_1} dy = 0.$$

Integrating by parts in the left hand side, we finally obtain

$$-\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{d}{dy} (\rho^{d-1} \frac{d}{dy} V_1) \phi dy = 0.$$

This relation is true for every test function $\phi \in C_c^\infty(-\frac{L}{2}, \frac{L}{2})$ so we have that V_1 and α_1 must have to satisfy the following differential equation

$$-\frac{d}{dx} (\rho(x)^{d-1} \frac{dV_1}{dx}(x)) = 0 \quad x \in (-\frac{L}{2}, \frac{L}{2}). \quad (2.39)$$

In order to find the boundary conditions for this equation we use the variational formulation with a test function ψ defined on all Ω_ϵ and which depends only on x_1 ,

$$\int_{\Omega_\epsilon} \frac{\partial u_1^\epsilon}{\partial x_1} \frac{\partial \psi}{\partial x_1} dx = \sigma_1^\epsilon \int_{\partial \Omega_\epsilon} u_1^\epsilon \psi d\mathcal{H}^{d-1}.$$

We recall that u_1^ϵ uniformly converge to $c_{1,1}$ in D_1 and to $c_{2,1}$ in D_2 , so we extend the functions v_1^ϵ to be equal to u_1^ϵ in D_1 and the same for D_2 . From Lemma 2.18 we have that $v_1^\epsilon \rightarrow c_{1,1} = V_1(-\frac{L}{2})$ in D_1 and $v_1^\epsilon \rightarrow c_{2,1} = V_1(\frac{L}{2})$ in D_2 . We repeat all the computations that we did above and we obtain

$$\omega_{d-1} \int_{-\frac{L}{2}}^{\frac{L}{2}} \rho^{d-1} \frac{d\bar{V}_1}{dx_1} \frac{d\psi}{dx_1} dx_1 = \sigma_1 (V_1(-\frac{L}{2}) \int_{\partial D_1} \psi d\mathcal{H}^{d-1} + V_1(\frac{L}{2}) \int_{\partial D_2} \psi d\mathcal{H}^{d-1}).$$

Integrating by parts in the left hand side and recalling (2.39), we finally obtain

$$\begin{aligned} & \rho^{d-1} \left(\frac{L}{2}\right) \frac{dV_1}{dx} \left(\frac{L}{2}\right) \psi \left(\frac{L}{2}\right) - \rho^{d-1} \left(-\frac{L}{2}\right) \frac{dV_1}{dx} \left(-\frac{L}{2}\right) \psi \left(-\frac{L}{2}\right) = \\ & = \sigma_1 (V_1(-\frac{L}{2}) \int_{\partial D_1} \psi d\mathcal{H}^{d-1} + V_1(\frac{L}{2}) \int_{\partial D_2} \psi d\mathcal{H}^{d-1}). \end{aligned}$$

We choose the test function such that $\psi = 1$ in D_1 and $\psi = 0$ in D_2 and we deduce the first boundary condition

$$\rho^{d-1} \left(-\frac{L}{2}\right) \frac{dV_1}{dx} \left(-\frac{L}{2}\right) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1 \left(-\frac{L}{2}\right).$$

Similarly if we choose the test function such that $\psi = 0$ in D_1 and $\psi = 1$ in D_2 we get the second boundary condition

$$\rho^{d-1} \left(\frac{L}{2}\right) \frac{dV_1}{dx} \left(\frac{L}{2}\right) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1 \left(\frac{L}{2}\right).$$

We finally obtain the following differential equation

$$\begin{cases} -\omega_{d-1} \frac{d}{dx} (\rho^{d-1}(x) \frac{dV_1}{dx}(x)) = 0 & x \in (-\frac{L}{2}, \frac{L}{2}) \\ \rho^{d-1}(-\frac{L}{2}) \frac{dV_1}{dx}(-\frac{L}{2}) = -\frac{\sigma_1}{\omega_{d-1}} P(D_1) V_1(-\frac{L}{2}) \\ \rho^{d-1}(\frac{L}{2}) \frac{dV_1}{dx}(\frac{L}{2}) = \frac{\sigma_1}{\omega_{d-1}} P(D_2) V_1(\frac{L}{2}). \end{cases}$$

Using the same argument as at the end of proof of Theorem 2.1 we conclude that the result is true for all the sequence $\{\epsilon_n\}_{n=1}^\infty$. This concludes the proof. \square

2.5 Application: counter-example to a Spectral Inequality

We consider the Neumann eigenvalue problem

$$\begin{cases} -\Delta v_k = \mu_k v_k & \Omega \\ \partial_\nu v_k = 0 & \partial\Omega. \end{cases}$$

During the writing of the paper [43] came the following question: is it true that the inequality

$$\mu_1 |\Omega| \geq \sigma_1 P(\Omega),$$

holds for any plane domains? For several domains like balls, annulus, rectangles, convex sets with a ratio between the inradius and circumradius large enough, this inequality turns out to be true.

Nevertheless, the results of [27, 44] implicitly show that the inequality can not be true in general, its failure coming either from highly oscillating boundaries or from the presence of a large number of small holes.

Our aim in this section is to provide another counter-example which is simply connected and do not have an oscillating boundary, for which the reverse inequality holds:

$$\mu_1|\Omega| < \sigma_1 P(\Omega).$$

Consider the domain $\Omega_\epsilon \subset \mathbb{R}^2$ constructed as follow, D_1 and D_2 are two balls such that $|D_1| = |D_2| = 1$ and $\rho = 1$ constantly.

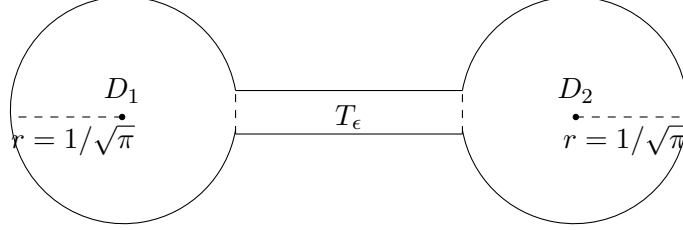


Figure 2.3: Dumbbell shape domain in $d = 2$, with $\rho \equiv 1$.

One can compute the eigenvalues of problem (2.1), which becomes

$$\begin{cases} -\frac{d^2 V_k}{dx^2}(x) = \alpha_k V_k(x) & x \in \left(-\frac{L}{2}, \frac{L}{2}\right) \\ \frac{dV_k}{dx}\left(-\frac{L}{2}\right) = -\frac{\alpha_k}{2} P(D_1) V_k\left(-\frac{L}{2}\right) \\ \frac{dV_k}{dx}\left(\frac{L}{2}\right) = \frac{\alpha_k}{2} P(D_2) V_k\left(\frac{L}{2}\right). \end{cases}$$

The general solution has the following form $V_k = A \cos(w_k x) + B \sin(w_k x)$ where $w_k^2 = \alpha_k$ and the boundary conditions give us the following equations for the unknowns A and B ,

$$\begin{cases} A(\sin(w_k \frac{L}{2}) + w_k \frac{P(D_1)}{2} \cos(w_k \frac{L}{2})) + B(\cos(w_k \frac{L}{2}) - w_k \frac{P(D_1)}{2} \sin(w_k \frac{L}{2})) = 0 \\ A(-\sin(w_k \frac{L}{2}) - w_k \frac{P(D_2)}{2} \cos(w_k \frac{L}{2})) + B(\cos(w_k \frac{L}{2}) - w_k \frac{P(D_2)}{2} \sin(w_k \frac{L}{2})) = 0. \end{cases}$$

In order to have non trivial solutions the determinant of this 2×2 system must be equal to 0, so we obtain that w_k must be satisfies the following transcendental equation

$$\cot(w_k L) = \frac{w_k^2 P(D_1) P(D_2) - 4}{2w_k (P(D_1) + P(D_2))}. \quad (2.40)$$

We know that $\sigma_1^\epsilon \sim \alpha_1 \epsilon$, where $\alpha_1 = w_1^2$ where w_1 is the first value for which the equation (2.40) holds.

By choosing, in the variational formulation, a test function which is constant on each disk and affine in the tube, we can prove that $\mu_1^\epsilon \leq \frac{4}{L} \epsilon$, so we conclude that

$$\frac{|\Omega_\epsilon|}{P(\Omega_\epsilon)} \mu_1^\epsilon \leq \frac{4}{(2\sqrt{\pi} + L)L} \epsilon + o(\epsilon).$$

If we prove that there exists $L > 0$ such that:

$$\frac{4}{(2\sqrt{\pi} + L)L} < \alpha_1$$

we conclude that there exists $\bar{\epsilon}$ such that

$$\mu_1^{\bar{\epsilon}} |\Omega_{\bar{\epsilon}}| < \sigma_1^{\bar{\epsilon}} P(\Omega_{\bar{\epsilon}}).$$

We introduce the following function

$$f(w) = \cot(wL) - \frac{w^2\pi - 1}{2\sqrt{\pi}w},$$

an easy computation shows that

$$0 < f(x) \quad \forall \quad 0 < x < \frac{3\pi}{4L} \Leftrightarrow \quad L > \frac{3}{4}(\sqrt{2} + 1)\pi^{\frac{3}{2}}.$$

So we conclude that for all L such that $L > \frac{3}{4}(\sqrt{2} + 1)\pi^{\frac{3}{2}}$ we have

$$\alpha_1 \geq \frac{9\pi^2}{16L^2} > \frac{4}{(2\sqrt{\pi} + L)L}.$$

providing the desired counter-example.

Chapter 3

Comparison between Steklov and Neumann eigenvalues

The main part of the results that we present in this chapter already appeared in [55] and the last section of this chapter will appear in the forthcoming work [57].

3.1 Introduction

Recently several papers study the link between these two families of eigenvalues, let us mention for example [43], [44], [52], [78]. A natural question is to compare the first (non-trivial) eigenvalues suitably normalized, that is to say to compare $|\Omega|\mu_1(\Omega)$ and $P(\Omega)\sigma_1(\Omega)$ where $\Omega \subset \mathbb{R}^2$ is an open Lipschitz set in the plane, $|\Omega|$ is its Lebesgue measure, $P(\Omega)$ is its perimeter. More precisely, in this paper we study the following spectral shape functional:

$$F(\Omega) = \frac{\mu_1(\Omega)|\Omega|}{\sigma_1(\Omega)P(\Omega)}. \quad (3.1)$$

In this Chapter we want to find bounds for $F(\Omega)$ (if possible optimal) in the two following cases: the set $\Omega \subset \mathbb{R}^2$ is just bounded and Lipschitz or the set $\Omega \subset \mathbb{R}^2$ is bounded and convex.

We now present the main results and the structure of this Chapter. In Section 3.2 we show that, if we do not put any restriction on the class of sets, the problem of maximization and minimization of $F(\Omega)$ is ill posed, indeed we have

$$\begin{aligned} \inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open set and Lipschitz}\} &= 0, \\ \sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ bounded open set and Lipschitz}\} &= +\infty. \end{aligned}$$

Thus we will study the problem of minimizing or maximizing $F(\Omega)$ in the class of convex plane domains. It is well known that minimizing (or maximizing) sequences of plane convex domains

- either converge (in the Hausdorff sense) to an open convex set and we will see that, in this case, this set will be the minimizer or maximizer,
- or shrink to a segment which leads us to consider such particular sequences of convex domains.

Therefore, in Section 3.3 we will study the behavior of the functional $F(\Omega_\epsilon)$ where Ω_ϵ is a special class of domains, called thin domains (see (3.4)). The main theorem of this section gives the precise asymptotic behavior of the functional $F(\Omega_\epsilon)$

Theorem 3.1. *Let $\Omega_\epsilon \subset \mathbb{R}^2$ be a sequence of thin domains that converges to a segment in the Hausdorff sense. Then there exists a non negative and concave function $h \in L^\infty(0, 1)$ such that the following asymptotic behavior holds:*

$$F(\Omega_\epsilon) \xrightarrow{\epsilon \rightarrow 0} F(h) := \frac{\mu_1(h) \int_0^1 h(x) dx}{\sigma_1(h)}.$$

Where $\mu_1(h)$ is the first non zero eigenvalue of

$$\begin{cases} -\frac{d}{dx}(h(x) \frac{du_k}{dx}(x)) = \mu_k(h) h(x) u_k(x) & x \in (0, 1) \\ h(0) \frac{du_k}{dx}(0) = h(1) \frac{du_k}{dx}(1) = 0, \end{cases}$$

and $\sigma_1(h)$ is the first non zero eigenvalue of

$$\begin{cases} -\frac{d}{dx}(h(x) \frac{dv_k}{dx}(x)) = \sigma_k(h) v_k(x) & x \in (0, 1) \\ h(0) \frac{dv_k}{dx}(0) = h(1) \frac{dv_k}{dx}(1) = 0. \end{cases}$$

In order to obtain this result in Lemma 3.8 and in Lemma 3.10 we prove general asymptotic behaviors for Neumann and Steklov eigenvalues on collapsing domains. Similar results for the Neumann eigenvalues, but in a different geometrical context, were proved in [18] and [73]. We want to highlight the fact that the limit eigenvalues problems in Lemma 3.8 and in Lemma 3.10 are non-standard: since the function h can vanish at the boundary, they are non-uniformly elliptic. We are not aware of similar asymptotic behavior in the literature.

In the rest of Section 3.3 we are interested in studying in which way a sequence of thin domains Ω_ϵ must collapse in order to obtain the lowest possible value of the limit $F(\Omega_\epsilon)$. From Theorem 3.1 this problem is equivalent to study the minimization problem for the one-dimensional spectral functional $F(h)$ in the class of $L^\infty(0, 1)$, concave and non negative functions. In particular in Theorem 3.12 we will show that there exists a minimizer and also that the function $h \equiv 1$ is a local minimizer.

Section 3.4 is devoted to the study of upper and lower bounds for the functionals $F(h)$ and $F(\Omega)$. We start by showing the following bounds for the functional $F(h)$

Theorem 3.2. *For every non negative and concave function $h \in L^\infty(0, 1)$ the following inequalities hold*

$$\frac{\pi^2}{12} \leq F(h) \leq 4$$

Then we will prove the following bounds for the functional $F(\Omega)$

Theorem 3.3. *There exists an explicit constant C_1 such that, for every convex open set $\Omega \subset \mathbb{R}^2$, the following inequalities hold*

$$\frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) \leq C_1 \leq 9.04$$

The explicit constant C_1 will be described in Section 3.4.

In the last Section we are interested in plotting the Blaschke–Santaló diagrams

$$\mathcal{E} = \{(x, y) \text{ where } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2\}$$

$$\mathcal{E}^C = \{(x, y) \text{ where } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2, \Omega \text{ convex.}\}$$

This kind of diagrams for spectral quantities has been recently studied by different authors, let us mention for example [3], [21], [100], [40], [81]. In this section, we show that the diagram \mathcal{E} is, in some sense, trivial while the diagram \mathcal{E}^C is more complicated delimited by two unknown curves. We present some numerical experiments and give some conjectures for this diagram.

3.2 Existence or non-existence of extremal domains

We show that, in general, the problem of minimization and maximization of the functional $F(\Omega)$ is ill posed, in the sense that one can construct sequences of domains for which $F(\Omega_\epsilon)$ converge to 0 and sequences of domains for which $F(\Omega_\epsilon)$ converge to $+\infty$.

Proposition 3.4. *The following equalities hold*

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = 0,$$

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = +\infty.$$

In order to prove that the infimum is 0 we construct a sequence of domains Ω_ϵ for which $\sigma_1(\Omega_\epsilon)P(\Omega_\epsilon) \rightarrow c > 0$ and $\mu_1(\Omega_\epsilon)|\Omega_\epsilon| \rightarrow 0$. We use similar ideas in order to construct another sequence Ω_ϵ for which $\sigma_1(\Omega_\epsilon)P(\Omega_\epsilon) \rightarrow 0$ and $\mu_1(\Omega_\epsilon)|\Omega_\epsilon| \rightarrow c > 0$, proving in this way that the supremum is $+\infty$.

We construct the desired sequences Ω_ϵ by perturbing a given set Ω by adding oscillations on the boundary (see [27] for the details of the construction). Given two compact sets $\Omega_1, \Omega_2 \in \mathbb{R}^2$ we denote by $d_H(\Omega_1, \Omega_2)$ the Hausdorff distance between the two sets (see [58]), the key result is the following

Theorem 3.5 (Bucur-Nahon [27]). *Let $\Omega, \omega \subset \mathbb{R}^2$ be two smooth, conformal open sets. Then there exists a sequence of smooth open sets $(\Omega_\epsilon)_{\epsilon>0}$ with uniformly bounded perimeter and satisfying a uniform ϵ -cone condition (see [58]) such that*

$$\lim_{\epsilon \rightarrow 0} d_H(\partial\Omega_\epsilon, \partial\Omega) = 0, \quad \lim_{\epsilon \rightarrow 0} P(\Omega_\epsilon)\sigma_k(\Omega_\epsilon) = P(\omega)\sigma_k(\omega), \quad \lim_{\epsilon \rightarrow 0} |\Omega_\epsilon|\mu_k(\Omega_\epsilon) = |\Omega|\mu_k(\Omega). \quad (3.2)$$

Proof of Proposition 3.4. Let $\delta > 0$, let Ω be a simply connected domain for which $\mu_1(\Omega)|\Omega| \leq \delta$ (for example a dumbbell shape domain with the channel very thin see [67]). Let ω be a disc, we know that $\sigma_1(\omega)P(\omega) = 2\pi$. Using Theorem 3.5 we can perturb the domain Ω in such a way that

$$\lim_{\epsilon \rightarrow 0} P(\Omega_\epsilon)\sigma_1(\Omega_\epsilon) = 2\pi, \quad \lim_{\epsilon \rightarrow 0} |\Omega_\epsilon|\mu_1(\Omega_\epsilon) \leq 2\delta$$

Thus we can conclude that, for ϵ small enough

$$F(\Omega_\epsilon) \leq \frac{2\delta}{2\pi - 1}$$

since δ was arbitrary small we conclude that:

$$\inf\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = 0.$$

For the other case, we choose Ω as the unit disc, then $\mu_1(\Omega)|\Omega| = \pi j_{11}'^2$ (j_{11}' is the first zero of the derivative of the Bessel function J_1). Let ω be a set for which $\sigma_1(\omega)P(\omega) \leq \delta$ (for

example a dumbbell shape domain with the channel very thin see [25]), using arguments similar at the ones above we conclude that

$$\frac{\pi j_{11}^2 - 1}{2\delta} \leq F(\Omega_\epsilon),$$

since δ was arbitrary small we conclude that:

$$\sup\{F(\Omega) : \Omega \subset \mathbb{R}^2 \text{ open and Lipschitz}\} = +\infty.$$

□

We mention that there exists another way to construct a sequence of domains such that $F(\Omega_\epsilon) \rightarrow 0$, this method is based on an homogenization technique, the key result is the following (see Theorem 1.14 in [44]):

Theorem 3.6 (Girouard-Karpukhin-Lagacé [44]). *There exists a sequence of domains $\Omega_\epsilon \subset \mathbb{R}^2$ such that for every $k \in \mathbb{N}$ the following holds*

$$\begin{aligned} \sigma_k(\Omega_\epsilon)P(\Omega_\epsilon) &\rightarrow 8\pi k \\ \mu_k(\Omega_\epsilon)|\Omega_\epsilon| &\rightarrow 0 \end{aligned}$$

From now on we will restrict ourselves to the class of convex domains. As recalled in the Introduction, a minimizing (or a maximizing) sequence of plane convex domains Ω_ϵ has the following behavior:

- i either the minimizing (maximizing) sequence Ω_ϵ converges to a segment (for the Hausdorff metric).
- ii or the minimizing (maximizing) sequence Ω_ϵ converges to a convex open set Ω

In the second case (ii), we deduce that there exists a minimizer (maximizer) for the functional $F(\Omega)$ in the class of convex domains. Indeed, the four quantities area, perimeter, μ_1 and σ_1 are continuous for Hausdorff convergence of plane convex domains (see [58] for the first three and [14] or [23] for Steklov eigenvalues).

3.3 Convex case: Thin Domains

We start by defining the following space of functions

$$\mathcal{L} := \{h \in L^\infty(0, 1) : h \text{ non negative, concave and } \int_0^1 h = 1\}. \quad (3.3)$$

Given two functions $h^- \in \mathcal{L}$ and $h^+ \in \mathcal{L}$, we define the class of thin domains Ω_ϵ in the following way (see Remark 2):

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}. \quad (3.4)$$

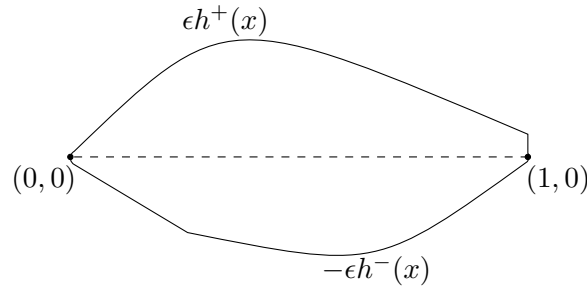


Figure 3.1: Description of the thin domain Ω_ϵ

We notice that the functional $F(\Omega)$ is scale invariant so without loss of generality we can consider domains that have diameter $D(\Omega_\epsilon) \rightarrow 1$ when $\epsilon \rightarrow 0$.

In the next lemma we give a compactness result for the space of functions \mathcal{L}

Lemma 3.7. *Let $h_n \in \mathcal{L}$ be a sequence of functions, then there exists a function $h \in \mathcal{L}$ such that, up to a subsequence that we still denote by h_n , we have*

$$\begin{aligned} h_n &\rightarrow h \quad \text{in } L^2(0,1) \\ h_n &\rightarrow h \quad \text{uniformly on every compact subset of } (0,1). \end{aligned}$$

Proof. From the concavity of the functions h_n and from the fact that $\|h_n\|_{L^1(0,1)} = 1$, we conclude that $\|h_n\|_{L^\infty(0,1)} \leq 2$. Let us assume first that the functions h_n are smooth, say C^1 inside $(0,1)$. We fix a parameter $0 < \delta < 1$ and we consider the interval $I_\delta = [\delta, 1 - \delta]$. The functions h_n being uniformly bounded in I_δ , from the concavity and the uniform bound we conclude

$$-\frac{2}{\delta} \leq -\frac{h_n(x)}{\delta} \leq h'_n(x) \leq \frac{h_n(x)}{\delta} \leq \frac{2}{\delta} \quad \forall x \in I_\delta.$$

We can now apply Ascoli-Arzelà Theorem and we conclude that there exists a function $h \in C([0,1])$ such that, for every $0 < \delta < 1$, up to a subsequence that we still denote by h_n

$$h_n \rightarrow h \quad \text{uniformly in } I_\delta.$$

From the convergence above and from the fact that h_n is concave for every n we infer that h is also concave in I_δ . So for every interval of the type I_δ we found the limit function h .

Now we need to analyze what happens on the two extremities of the interval $[0,1]$. We consider the bounded sequence $h_n(0)$, up to a subsequence, this sequence has a limit, we extend the function h that we found above to be equal at that limit in $x = 0$, so $h(0) = \lim_{n \rightarrow \infty} h_n(0)$. We use the same argument for the point $x = 1$. Now it is straightforward to check (by passing to the limit in the concavity inequality for h_n) that h is a concave function on the interval $[0,1]$ and that

$$h_n \rightarrow h \quad \text{in } L^2(0,1).$$

We finally argue by density to extend the previous result to a general sequence h_n . \square

3.3.1 Asymptotic behavior of eigenvalues

In this section we present some general results concerning the asymptotic behavior of σ_k and μ_k in a wide class of collapsing domains. We then apply this results in the particular case of thin domains in order to obtain the asymptotics given in Theorem 3.1.

We start with the analysis of the Steklov eigenvalues:

Lemma 3.8. *Let $h^+ \in L^\infty(0,1)$ and $h^- \in L^\infty(0,1)$ be two non negative functions, we define the following collapsing domains:*

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

Let $h = h^+ + h^-$, if there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0,1)$, then

$$\sigma_k(\Omega_\epsilon) = \frac{\sigma_k(h)}{2} \epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

where $\sigma_k(h)$ is the k -th non trivial eigenvalue of

$$\begin{cases} -\frac{d}{dx}\left(h(x)\frac{dv}{dx}(x)\right) = \sigma(h)v(x) & x \in (0,1) \\ h(0)\frac{dv}{dx}(0) = h(1)\frac{dv}{dx}(1) = 0. \end{cases} \quad (3.5)$$

Remark 1. In the previous Lemma the problem (3.5) is understood in the weak sense. The function h is allowed to vanish at the extremities of the interval, therefore the operator $-\frac{d}{dx}\left(h(x)\frac{dv}{dx}\right)$ is not uniformly elliptic and the existence of eigenvalues and eigenfunctions does not follow in a classical way. For this reason in the first part of the proof we will prove the existence of the eigenvalues, under the assumption that we made on the function h .

Proof of Lemma 3.8. Let $f \in L^2(0,1)$, the inverse of the operator $-\frac{d}{dx}\left(h(x)\frac{dv}{dx}\right)$ with the boundary conditions $h(0)v'(0) = h(1)v'(1) = 0$ is given by the following integral representation (see [99]):

$$v(x) = \int_0^1 g(x,y)f(y)dy \text{ with } g(x,y) = \int_0^{\min(x,y)} \frac{t}{h(t)}dt + \int_{\max(x,y)}^1 \frac{1-t}{h(t)}dt. \quad (3.6)$$

From the assumption on the function h it follows that $g(x,y) \in L^2([0,1] \times [0,1])$. We conclude that the integral operator defined in (3.6) is an Hilbert-Schmidt integral operator and so problem (3.5) posses a sequence of eigenvalues and eigenfunctions. In particular the eigenvalue $\sigma_k(h)$ admits the following variational characterization:

$$\sigma_k(h) = \inf_{E_k} \sup_{0 \neq v \in E_k} \frac{\int_0^1 (v')^2 h dx_1}{\int_0^1 v^2 dx_1}, \quad (3.7)$$

where the infimum is taken over all k -dimensional subspaces of $H^1(0,1)$ which are L^2 -orthogonal to constants.

Let f_k be the eigenfunction of the problem (3.5) associated to the eigenvalue $\sigma_k(h)$, we define the function $F_k(x_1, x_2) = f_k(x_1)$ for every $(x_1, x_2) \in \Omega_\epsilon$. We define the mean value of the function F_k on $\partial\Omega_\epsilon$:

$$MF_{k,\epsilon} := \frac{1}{P(\Omega_\epsilon)} \int_{\partial\Omega_\epsilon} F_k ds = \frac{1}{P(\Omega_\epsilon)} \int_0^1 f_k(\sqrt{1 + (\epsilon h^+) ^2} + \sqrt{1 + (\epsilon h^-)^2}) dx_1.$$

From (3.5) it is straightforward to check that $\int_0^1 f_k = 0$, so we have the following limit

$$\lim_{\epsilon \rightarrow 0} MF_{k,\epsilon} = 0. \quad (3.8)$$

We introduce the following subspace $E_k = \text{Span}[F_1 - MF_{1,\epsilon}, \dots, F_k - MF_{k,\epsilon}]$, we can use this as a test subspace in the variational characterization (1.5), we obtain

$$\sigma_k(\Omega_\epsilon) \leq \max_{v \in E_k} \frac{\int_{\Omega_\epsilon} |\nabla v|^2 dx}{\int_{\partial\Omega_\epsilon} v^2 ds} = \max_{\beta \in \mathbb{R}^k} \frac{\epsilon \sum_{i=1}^k \beta_i^2 \int_0^1 (f'_i)^2 h dx_1}{\int_0^1 \left(\sum_{i=1}^k \beta_i (f_i - MF_{i,\epsilon}) \right)^2 \left((1 + (\epsilon h^+) ^2)^{\frac{1}{2}} + (1 + (\epsilon h^-)^2)^{\frac{1}{2}} \right) dx_1}.$$

From (3.8) and the above inequality we can conclude that for ϵ small enough

$$\sigma_k(\Omega_\epsilon) \leq \frac{\epsilon}{2} \max_{\beta \in \mathbb{R}^k} \frac{\sum_{i=1}^k \beta_i^2 \int_0^1 (f'_i)^2 h dx_1}{\sum_{i=1}^k \beta_i^2 \int_0^1 f_i^2 dx_1} + o(\epsilon) = \frac{\sigma_k(h)}{2} \epsilon + o(\epsilon), \quad (3.9)$$

where the last equality is true because f_k is the eigenfunction corresponding to $\sigma_k(h)$

On the other hand, let us denote by Ω_1 the convex domain corresponding to $\epsilon = 1$. Let $v_{k,\epsilon}$ be a Steklov eigenfunction associated to $\sigma_k(\Omega_\epsilon)$, normalized in such a way that $\|v_{k,\epsilon}\|_{L^2(\partial\Omega_\epsilon)} = 1$. We define the following function

$$\bar{v}_{k,\epsilon}(x_1, x_2) = v_{k,\epsilon}(x_1, \epsilon x_2) \quad \forall (x_1, x_2) \in \Omega_1.$$

We start with the bound of $\|\nabla \bar{v}_{k,\epsilon}\|_{L^2(\Omega_1)}$,

$$\int_{\Omega_1} |\nabla \bar{v}_{k,\epsilon}|^2 dx \leq \int_{\Omega_1} \left(\frac{\partial \bar{v}_{k,\epsilon}}{\partial x_1} \right)^2 + \frac{1}{\epsilon^2} \left(\frac{\partial \bar{v}_{k,\epsilon}}{\partial x_2} \right)^2 dx = \frac{1}{\epsilon} \int_{\Omega_\epsilon} |\nabla v_{k,\epsilon}|^2 dy = \frac{\sigma_k(\Omega_\epsilon)}{\epsilon} \leq C$$

where we did the change of coordinates $y_1 = x_1$, $y_2 = \epsilon x_2$ and the last inequality is true because of (3.9). We want now to bound $\|\bar{v}_{k,\epsilon}\|_{L^2(\Omega_1)}$. By the Poincaré-Friedrichs inequality or the variational characterization of Robin eigenvalues (we denote by $\lambda_1^R(\Omega, \beta)$ the first Robin eigenvalue of the domain Ω with the boundary parameter β), we get

$$\int_{\Omega_\epsilon} v_{k,\epsilon}^2 dx \leq \frac{1}{\lambda_1^R(\Omega_\epsilon, 1)} \left[\int_{\Omega_\epsilon} |\nabla v_{k,\epsilon}|^2 dx + \int_{\partial\Omega_\epsilon} v_{k,\epsilon}^2 ds \right]. \quad (3.10)$$

Using Bossel's inequality, see [17], we infer $\lambda_1^R(\Omega_\epsilon, 1) \geq h(\Omega_\epsilon) - 1$ where $h(\Omega_\epsilon)$ is the Cheeger constant of Ω_ϵ . Now by monotonicity of the Cheeger constant with respect to inclusion, we have $h(\Omega_\epsilon) \geq h(R_\epsilon)$ where R_ϵ is a rectangle of length 1 and width 4ϵ . Now the Cheeger constant of such a rectangle can be computed explicitly, see [70] and it turns out that, for any ϵ , $h(R_\epsilon) \geq 2/\epsilon$. Therefore, using (3.10) and the normalization $\int_{\partial\Omega_\epsilon} v_{k,\epsilon}^2 ds = 1$ we finally get

$$\int_{\Omega_\epsilon} v_{k,\epsilon}^2 dx \leq \epsilon(C\epsilon + 1) \leq 2\epsilon.$$

Now, coming back to $\bar{v}_{k,\epsilon}$, we have

$$\int_{\Omega_1} \bar{v}_{k,\epsilon}^2 dx = \frac{1}{\epsilon} \int_{\Omega_\epsilon} v_{k,\epsilon}^2 dx \leq 2$$

therefore we conclude that there exists $\bar{V}_k \in H^1(\Omega_1)$ such that (up to a sub-sequence that we still denote by $\bar{v}_{k,\epsilon}$)

$$\bar{v}_{k,\epsilon} \rightharpoonup \bar{V}_k \quad \text{in } H^1(\Omega_1), \quad \text{and strongly in } L^2. \quad (3.11)$$

We also know that \bar{V}_k does not depend on x_2 , indeed

$$\int_{\Omega_1} \left(\frac{\partial \bar{v}_{k,\epsilon}}{\partial x_2} \right)^2 dx = \epsilon \int_{\Omega_\epsilon} \left(\frac{\partial v_{k,\epsilon}}{\partial x_2} \right)^2 dx \leq C\epsilon^2 \rightarrow 0.$$

We define the function V_k as the restriction of \bar{V}_k to the variable x_1 . We want to prove that $\int_0^1 V_k dx_1 = 0$ and V_k is not a constant function. By definition of $\bar{v}_{k,\epsilon}$ and $v_{k,\epsilon}$ the following equality holds

$$0 = \int_{\partial\Omega_\epsilon} v_{k,\epsilon} ds = \int_0^1 \bar{v}_{k,\epsilon}(x_1, h^+(x_1)) \sqrt{1 + (\epsilon h^+)' ^2} dx_1 + \int_0^1 \bar{v}_{k,\epsilon}(x_1, h^-(x_1)) \sqrt{1 + (\epsilon h^-)' ^2} dx_1.$$

Now, $\bar{v}_{k,\epsilon}$ converges strongly in L^2 to \bar{V}_k while $\sqrt{1 + (\epsilon h^+)' ^2}$ converges weakly in L^2 to 1, thus passing to the limit yields

$$\int_0^1 V_k dx_1 = 0. \quad (3.12)$$

Now from the fact that $\|v_{k,\epsilon}\|_{L^2(\partial\Omega_\epsilon)} = 1$, using similar arguments we conclude that:

$$\int_0^1 V_k^2 dx_1 = 2,$$

from this equality and (3.12) we conclude that V_k cannot be a constant function.

Using the convergence given in (3.11), the variational characterization and the relations that we have just obtained, we conclude that for ϵ small enough we have the following lower bound

$$\sigma_k(\Omega_\epsilon) = \max_{\beta \in \mathbb{R}^k} \frac{\sum_{i=1}^k \beta_i^2 \int_{\Omega_\epsilon} |\nabla v_{i,\epsilon}|^2 dx}{\sum_{i=1}^k \beta_i^2 \int_{\partial\Omega_\epsilon} v_{i,\epsilon}^2 ds} \geq \frac{\epsilon}{2} \max_{\beta \in \mathbb{R}^k} \frac{\sum_{i=1}^k \beta_i^2 \int_0^1 V_i'^2 h dx_1}{\sum_{i=1}^k \beta_i^2 \int_0^1 V_i^2 dx_1} + o(\epsilon) \geq \frac{\sigma_k(h)}{2} \epsilon + o(\epsilon). \quad (3.13)$$

The last inequality is true because of the variational characterization (3.7) for $\sigma_k(h)$. From (3.9) and (3.13) we finally conclude that

$$\sigma_k(\Omega_\epsilon) = \frac{\sigma_k(h)}{2} \epsilon + o(\epsilon) \quad \text{as } \epsilon \rightarrow 0,$$

□

We now specify the result above in the case of thin domains and we give also some continuity results for $\sigma_k(h)$.

Lemma 3.9. *Let Ω_ϵ be a sequence of thin domains then Lemma 3.8 holds. Moreover let $h_n \in \mathcal{L}$ and $h \in \mathcal{L}$ be such that $h_n \rightarrow h$ in $L^2(0,1)$, then, we have*

$$\sigma_k(h_n) \rightarrow \sigma_k(h).$$

Proof. From the concavity and positivity of $h \in \mathcal{L}$ it follows that there exists a constant $K > 0$ such that

$$h(x) \geq Kx(1-x) \quad \text{for a. e. } 0 \leq x \leq 1. \quad (3.14)$$

In particular the hypothesis of Lemma 3.8 are satisfied.

Let $h_n \in \mathcal{L}$ and $h \in \mathcal{L}$ be such that $h_n \rightarrow h$ in $L^2(0,1)$, we define

$$v_n(x) = \int_0^1 g_n(x,y) f(y) dy \quad \text{with } g_n(x,y) = \int_0^{\min(x,y)} \frac{t}{h_n(t)} dt + \int_{\max(x,y)}^1 \frac{1-t}{h_n(t)} dt.$$

The aim is to prove that $v_n \rightarrow v$ in $L^2(0,1)$, this, by classical results (see [53]), will imply the convergence of the spectrum. We know that up to a subsequence $h_n \rightarrow h$ a. e. in $[0,1]$, now using the lower bound (3.14) we obtain an upper bound $g_n(x,y) \leq C$, for every $n \in \mathbb{N}$ and for every $(x,y) \in [0,1] \times [0,1]$. We can apply the dominated convergence on the sequence $g_n(x,y)$ and we conclude that $g_n(x,y) \rightarrow g(x,y)$ for every $(x,y) \in [0,1] \times [0,1]$. Similarly we can conclude also that $v_n(x) \rightarrow v(x)$ for every $x \in [0,1]$. Combining this convergence with the uniform bound on $g_n(x,y)$ we can use the dominated convergence to conclude that

$$\int_0^1 (v_n(x) - v(x))^2 dx \rightarrow 0.$$

□

We now study the asymptotic behavior for the Neumann eigenvalues:

Lemma 3.10. *Let $h^+ \in L^\infty(0,1)$ and $h^- \in L^\infty(0,1)$ be two non negative functions, we define the following collapsing domains:*

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}.$$

Let $h = h^+ + h^-$, if there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0,1)$, then:

$$\mu_k(\Omega_\epsilon) = \mu_k(h) + o(1) \quad \text{as } \epsilon \rightarrow 0,$$

Where $\mu_k(h)$ is the k -th non trivial eigenvalue of

$$\begin{cases} -\frac{d}{dx}\left(h(x)\frac{du}{dx}(x)\right) = \mu(h)h(x)u(x) & x \in (0,1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0, \end{cases} \quad (3.15)$$

Proof. Let $f \in L^2(0,1)$, the inverse of the operator $-\frac{1}{h(x)}\frac{d}{dx}\left(h(x)\frac{du}{dx}\right)$ with the boundary conditions $h(0)u'(0) = h(1)u'(1) = 0$ is given by the integral representation:

$$u(x) = \int_0^1 g(x,y)h(y)f(y)dy \quad \text{with } g(x,y) = \int_0^{\min(x,y)} \frac{t}{h(t)}dt + \int_{\max(x,y)}^1 \frac{1-t}{h(t)}dt.$$

We can adapt the proof of Lemma 3.8 at this integral operator and we conclude that the problem (3.15) posses a sequence of eigenvalues and eigenfunctions. In particular the eigenvalue $\mu_k(h)$ admit the following variational characterization:

$$\mu_k(h) = \inf_{E_k} \sup_{0 \neq v \in E_k} \frac{\int_0^1 (v')^2 h dx_1}{\int_0^1 v^2 h dx_1}, \quad (3.16)$$

where the infimum is taken over all k -dimensional subspaces of $H^1(0,1)$ which are orthogonal in L^2 to the function h .

Let g_k be the eigenfunction associated to the eigenvalue $\mu_k(h)$, we define the function $G_k(x_1, x_2) = g_k(x_1)$ for every $(x_1, x_2) \in \Omega_\epsilon$. We define the mean value of the function G_k

$$MG_{k,\epsilon} := \frac{1}{|\Omega_\epsilon|} \int_{\Omega_\epsilon} G_k dx = \frac{1}{|\Omega_1|} \int_0^1 g_k h dx_1.$$

From (3.15) it is straightforward to check that $\int_0^1 g_k h dx_1 = 0$, so we have

$$MG_{k,\epsilon} = 0. \quad (3.17)$$

We introduce the following subspace $E_k = \text{Span}[G_1, \dots, G_k]$, we can use this as a test subspace in the variational characterization (1.2), we obtain

$$\mu_k(\Omega_\epsilon) \leq \max_{\beta \in \mathbb{R}^k} \frac{\sum_{i=1}^k \beta_i^2 \int_{\Omega_\epsilon} |\nabla G_i|^2 dx}{\sum_{i=1}^k \beta_i^2 \int_{\Omega_\epsilon} G_i^2 dx} = \max_{\beta \in \mathbb{R}^k} \frac{\sum_{i=1}^k \beta_i^2 \int_0^1 (u_i')^2 h dx_1}{\sum_{i=1}^k \beta_i^2 \int_0^1 u_i^2 h dx_1} = \mu_k(h). \quad (3.18)$$

where the last equality is true by the variational characterization (3.16) for the eigenvalue $\mu_k(h)$.

Let $u_{k,\epsilon}$ be a Neumann eigenfunction associated to $\mu_k(\Omega_\epsilon)$, normalized in such a way that $\|u_{k,\epsilon}\|_{L^2(\Omega_\epsilon)} = 1$, we define the following function

$$\bar{u}_{k,\epsilon}(x_1, x_2) = \epsilon^{\frac{1}{2}} u_{k,\epsilon}(x_1, \epsilon x_2) \quad \forall (x_1, x_2) \in \Omega_1.$$

We start with the bound of $\|\nabla \bar{u}_{k,\epsilon}\|_{L^2(\Omega_1)}$,

$$\int_{\Omega_1} |\nabla \bar{u}_{k,\epsilon}|^2 dx \leq \epsilon \int_{\Omega_1} \left(\frac{\partial u_{k,\epsilon}}{\partial x_1} \right)^2 + \frac{1}{\epsilon^2} \left(\frac{\partial u_{k,\epsilon}}{\partial x_2} \right)^2 dx \leq \int_{\Omega_\epsilon} |\nabla u_{k,\epsilon}|^2 dy \leq \mu_k(h)$$

here we performed the change of coordinates $y_1 = x_1$, $y_2 = \epsilon x_2$. Using the same change of variable we obtain $\|\bar{u}_\epsilon\|_{L^2(\Omega_1)} = 1$.

We conclude that there exists $\bar{U}_k \in H^1(\Omega_1)$ such that (up to a sub-sequence that we still denote by $\bar{u}_{k,\epsilon}$)

$$\bar{u}_{k,\epsilon} \rightharpoonup \bar{U}_k \quad \text{in } H^1(\Omega_1) \quad \text{and strongly in } L^2. \quad (3.19)$$

We also know that \bar{U}_k does not depend on x_2 , indeed

$$\int_{\Omega_1} \left(\frac{\partial \bar{U}_k}{\partial x_2} \right)^2 dx \leq \liminf \int_{\Omega_1} \left(\frac{\partial \bar{u}_{k,\epsilon}}{\partial x_2} \right)^2 dx = \liminf \epsilon^2 \int_{\Omega_\epsilon} \left(\frac{\partial u_{k,\epsilon}}{\partial x_2} \right)^2 dx = 0.$$

We define the function U_k that is the restriction of \bar{U}_k to the variable x_1 . We want to prove that $\int_0^1 U_k h dx_1 = 0$ and U_k is not a constant function. By definition of $\bar{u}_{k,\epsilon}$ and $u_{k,\epsilon}$ the following equality holds

$$\forall \epsilon \quad \int_{\Omega_1} \bar{u}_{k,\epsilon} dx = \frac{1}{\epsilon^{\frac{1}{2}}} \int_{\Omega_\epsilon} u_{k,\epsilon} = 0.$$

From the convergence results (3.19) we know that, up to a subsequence, $\bar{u}_{k,\epsilon}$ converge a. e. to \bar{U}_k so passing to the limit as ϵ goes to zero in the above equality we conclude that

$$\int_0^1 U_k h dx_1 = 0. \quad (3.20)$$

Now from the fact that $\|\bar{u}_{k,\epsilon}\|_{L^2(\partial\Omega_1)} = 1$, using similar arguments we conclude that:

$$\int_0^1 U_k^2 h dx_1 = 1,$$

from this equality (3.20) and the fact that $\int_0^1 h = 1$ we conclude that U cannot be a constant function.

Using the convergence given in (3.19) and the relations that we have just obtained, we conclude that for ϵ small enough we have the following lower bound

$$\mu_k(\Omega_\epsilon) = \max_{\beta \in \mathbb{R}^k} \frac{\sum_{i=1}^k \beta_i^2 \int_{\Omega_\epsilon} |\nabla u_{i,\epsilon}|^2 dx}{\sum_{i=1}^k \beta_i^2 \int_{\Omega_\epsilon} u_{i,\epsilon}^2 dx} \geq \max_{\beta \in \mathbb{R}^k} \frac{\sum_{i=1}^k \beta_i^2 \int_0^1 (U_i')^2 h dx_1}{\sum_{i=1}^k \beta_i^2 \int_0^1 U_i^2 h dx_1} + o(1) \geq \mu_k(h) + o(1). \quad (3.21)$$

The last inequality is true because because of the variational characterization (3.16) for $\mu_k(h)$. From (3.18) and (3.21) we finally conclude that

$$\mu_k(\Omega_\epsilon) = \mu_k(h) + o(1) \quad \text{as } \epsilon \rightarrow 0,$$

□

As we did for the Steklov eigenvalues, we now specify the result above in the case of thin domains and we give also some continuity results for $\mu_k(h)$.

Lemma 3.11. *Let Ω_ϵ be a sequence of thin domains then Lemma 3.10 holds. Moreover let $h_n \in \mathcal{L}$ and $h \in \mathcal{L}$ be such that $h_n \rightarrow h$ in $L^2(0,1)$, then we have*

$$\mu_k(h_n) \rightarrow \mu_k(h)$$

Proof. Let $f \in L^2(0,1)$, the inverse of the operator $-\frac{1}{h(x)}\frac{d}{dx}\left(h(x)\frac{du}{dx}\right)$ with the boundary conditions $h(0)u'(0) = h(1)u'(1) = 0$ is given by the integral representation:

$$u(x) = \int_0^1 g(x,y)h(y)f(y)dy \text{ with } g(x,y) = \int_0^{\min(x,y)} \frac{t}{h(t)}dt + \int_{\max(x,y)}^1 \frac{1-t}{h(t)}dt.$$

The proof is a straightforward adaptation of the proof of Lemma 3.9 at this integral operator. \square

Remark 2. *We can consider the most general class of collapsing thin domains given by the following parametrization:*

$$\Omega_\epsilon = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -g^-(\epsilon)h^-(x) \leq y \leq g^+(\epsilon)h^+(x)\}.$$

Where $h^+ \in L^\infty(0,1)$ and $h^- \in L^\infty(0,1)$ are two non negative functions that satisfy the conditions in Lemma 3.8 and Lemma 3.10 and $g^-(\epsilon), g^+(\epsilon)$ are positive functions that go to zero when ϵ goes to zero. We define the following limit

$$\lim_{\epsilon \rightarrow 0} \frac{g^-(\epsilon)}{g^+(\epsilon)} = K < +\infty,$$

(if the limit above is $+\infty$ we consider the inverse and in what follows we replace $g^+(\epsilon)$ with $g^-(\epsilon)$). In this case the asymptotics of the eigenvalues $\sigma_k(\Omega_\epsilon)$ and $\mu_k(\Omega_\epsilon)$ become:

$$\begin{aligned} \sigma_k(\Omega_\epsilon) &\sim \frac{\sigma_k(h^+ + Kh^-)}{2}g^+(\epsilon) + o(g^+(\epsilon)) \quad \text{as } \epsilon \rightarrow 0 \\ \mu_k(\Omega_\epsilon) &\sim \mu_k(h^+ + Kh^-) + o(1) \quad \text{as } \epsilon \rightarrow 0. \end{aligned}$$

The proof of this asymptotics use the same arguments of the proofs of Lemma 3.8 and Lemma 3.10. We prefer to give the statements and the proofs for $g^+(\epsilon) = g^-(\epsilon) = \epsilon$ in order to simplify the exposition and also because this kind of generality is not needed to study the asymptotic behavior of $F(\Omega_\epsilon)$.

3.3.2 Study of the asymptotic behavior of $F(\Omega_\epsilon)$

The proof of Theorem 3.1 immediately follows from the above results.

Proof of Theorem 3.1. Without loss of generality we can rescale the sequence Ω_ϵ in such a way that $D(\Omega_\epsilon) = 1$. we consider the sequence $F(\Omega_\epsilon)$, from Lemma 3.8 and Lemma 3.10 we obtain the desired result by sending ϵ to zero. \square

Let $h \in \mathcal{L}$, by Theorem 3.1, the functional

$$F(h) = \frac{\mu_1(h) \int_0^1 h(x)dx}{\sigma_1(h)}$$

describes the behavior of the functional $F(\Omega_\epsilon)$, when Ω_ϵ is a sequence of thin domains that converges to a segment in the Hausdorff sense. We want to study the problem of finding in which way a sequence of thin domains Ω_ϵ must collapse in order to obtain the lowest possible value of the limit $F(\Omega_\epsilon)$. For this reason we prove the following theorem:

Theorem 3.12. *The minimization problem (resp. the maximization problem)*

$$\inf\{F(h) : h \in \mathcal{L}\}, \quad (\text{resp. } \sup\{F(h) : h \in \mathcal{L}\}) \quad (3.22)$$

has a solution, moreover the constant function $h \equiv 1$ is a local minimizer.

Proof. The existence of the minimizer or the maximizer follows directly from the compactness result given in Lemma 3.7, the continuity results given in Lemma 3.9 and Lemma 3.11.

The proof of the fact that $h \equiv 1$ is a local minimizer is divided in two steps where we use first and second derivative respectively. In the first step, using the first derivative, we prove that $h \equiv 1$ satisfies a first order optimality condition and in the second step, using second derivative, we prove that it also satisfies the second order optimality condition. First of all, we recall that the eigenvalues $\mu_{0,\phi}$ and $\sigma_{0,\phi}$, being the eigenvalues of a Sturm-Liouville problem, are simple eigenvalues, see e.g. [35, chapter 5]. In particular they are twice differentiable. Before we start the proof we fix the notation, we consider $t > 0$ a positive number, and we define the following derivatives:

- for every $\phi \in \mathcal{L}$ we define $\mu_{t,\phi} := \mu_1(1+t\phi)$ and we denote by $u_{t,\phi}$ the corresponding eigenfunction. We use the following notation for the derivatives of the eigenvalues:

$$\dot{\mu}_\phi := \frac{d}{dt}\mu_1(1+t\phi)\Big|_{t=0} \quad \ddot{\mu}_\phi := \frac{d^2}{dt^2}\mu_1(1+t\phi)\Big|_{t=0},$$

and the following notation for the derivative of the eigenfunctions:

$$\dot{u}_\phi := \frac{d}{dt}u_{t,\phi}\Big|_{t=0} \quad \ddot{u}_\phi := \frac{d^2}{dt^2}u_{t,\phi}\Big|_{t=0}.$$

- for every $\phi \in \mathcal{L}$ we define $\sigma_{t,\phi} := \sigma_1(1+t\phi)$ and we denote by $v_{t,\phi}$ the corresponding eigenfunction. We use the following notation for the derivatives of the eigenvalues:

$$\dot{\sigma}_\phi := \frac{d}{dt}\sigma_1(1+t\phi)\Big|_{t=0} \quad \ddot{\sigma}_\phi := \frac{d^2}{dt^2}\sigma_1(1+t\phi)\Big|_{t=0},$$

and the following notation for the derivative of the eigenfunctions:

$$\dot{v}_\phi := \frac{d}{dt}v_{t,\phi}\Big|_{t=0} \quad \ddot{v}_\phi := \frac{d^2}{dt^2}v_{t,\phi}\Big|_{t=0}.$$

We notice that

$$\mu_{0,\phi} = \sigma_{0,\phi} = \pi^2 \quad \text{and} \quad u_{0,\phi}(x) = v_{0,\phi}(x) = \sqrt{2} \cos(\pi x) \quad (3.23)$$

Step 1. We start by proving the following inequality

$$\frac{d}{dt}F(1+t\phi)\Big|_{t=0} \geq 0 \quad \forall \phi \in \mathcal{L}.$$

The derivative of $F(h)$ has the following expression

$$\frac{d}{dt}F(1+t\phi)\Big|_{t=0} = \frac{\dot{\mu}_\phi}{\pi^2} + \int_0^1 \phi dx - \frac{\dot{\sigma}_\phi}{\pi^2}. \quad (3.24)$$

Since this kind of perturbation is classical, see e.g. [58, section 5.7] we just perform a formal computation here, the complete justification would involve an implicit function

theorem together with Fredholm alternative. We start by computing $\dot{\sigma}_\phi$, from (3.5) we know that

$$\frac{d}{dt} \left[-\frac{d}{dx} \left((1+t\phi) \frac{dv_{t,\phi}}{dx} \right) \right] \Big|_{t=0} = \frac{d}{dt} [\sigma_{t,\phi} v_{t,\phi}] \Big|_{t=0},$$

so we obtain the following differential equation satisfied by \dot{v}_ϕ

$$-(\phi' v'_{0,\phi} + \phi v''_{0,\phi} + \dot{v}''_\phi) = \dot{\sigma}_\phi v_{0,\phi} + \sigma_{0,\phi} \dot{v}_\phi. \quad (3.25)$$

Multiplying both side of the above equation by $v_{0,\phi}$ and integrating, recalling (3.23), we obtain

$$\dot{\sigma}_\phi = 2\pi^2 \int_0^1 \phi \sin^2(\pi x) dx. \quad (3.26)$$

We now compute $\dot{\mu}_\phi$, from (3.15) we know that

$$\frac{d}{dt} \left[-\frac{d}{dx} \left((1+t\phi) \frac{du_{t,\phi}}{dx} \right) \right] \Big|_{t=0} = \frac{d}{dt} [\mu_{t,\phi} (1+t\phi) u_{t,\phi}] \Big|_{t=0},$$

so we obtain the following differential equation satisfied by \dot{u}_ϕ

$$-(\phi' u'_{0,\phi} + \dot{u}''_\phi) = \dot{\mu}_\phi u_{0,\phi} + \mu_{0,\phi} \dot{u}_\phi. \quad (3.27)$$

Multiplying both side of the above equation by $u_{0,\phi}$ and integrating, recalling (3.23), we obtain

$$\dot{\mu}_\phi = 2\pi^2 \int_0^1 \phi (\sin^2(\pi x) - \cos^2(\pi x)) dx. \quad (3.28)$$

Using the explicit formulas given by (5.1) and (4.14) in (3.24) we finally obtain

$$\frac{d}{dt} F(1+t\phi) \Big|_{t=0} = - \int_0^1 \phi \cos(2\pi x) dx \quad \forall \phi \in \mathcal{L}.$$

Now it is well known (see [103]) that the first cosine Fourier coefficient of a concave function is non positive. Moreover it is easy to check that if $\phi \in \mathcal{L}$ then $\int_0^1 \phi \cos(2\pi x) dx = 0$ if and only if ϕ is a linear function. So we have two cases

i The function $\phi \in \mathcal{L}$ is not a linear function. In this case

$$\frac{d}{dt} F(1+t\phi) \Big|_{t=0} > 0$$

and we conclude that $h \equiv 1$ is a local minimizer for this kind of perturbation.

ii The function ϕ is of the form $\phi(x) = B + Ax$, in this case

$$\frac{d}{dt} F(1+t(B+Ax)) \Big|_{t=0} = 0.$$

In order to conclude the proof we need to study the second variation of the functional $F(h)$ for perturbation of the form $\phi(x) = B + Ax$.

Step 2. Given two real numbers $(A, B) \in \mathbb{R}^2 \setminus (0, 0)$, we want to prove that

$$\frac{d^2}{dt^2} F(1+t(B+Ax)) \Big|_{t=0} > 0. \quad (3.29)$$

We start by noticing that for every $k \in \mathbb{R}$ different from zero we have that $F(kh) = F(h)$, so in order to prove inequality (3.29) it is enough to prove that

$$\frac{d^2}{dt^2} F(1 + tAx) \Big|_{t=0} > 0. \quad (3.30)$$

This second derivative has the following expression

$$\frac{d^2}{dt^2} F(1 + tAx) \Big|_{t=0} = \frac{\ddot{\mu}_{Ax}}{\pi^2} + \frac{\dot{\mu}_{Ax}A}{\pi^2} - \frac{\dot{\sigma}_{Ax}A}{\pi^2} - \frac{2\dot{\sigma}_{Ax}\dot{\mu}_{Ax}}{\pi^4} - \frac{\ddot{\sigma}_{Ax}}{\pi^2} + \frac{2\dot{\sigma}_{Ax}^2}{\pi^4}. \quad (3.31)$$

From (5.1) and (4.14) it is easy to check that:

$$\dot{\mu}_{Ax} = 0 \quad \text{and} \quad \dot{\sigma}_{Ax} = \frac{A\pi^2}{2}. \quad (3.32)$$

We start by computing $\ddot{\sigma}_{Ax}$, from (3.5) we know that

$$\frac{d^2}{dt^2} \left[-\frac{d}{dx} \left((1 + tAx) \frac{dv_{t,Ax}}{dx} \right) \right] \Big|_{t=0} = \frac{d^2}{dt^2} [\sigma_{t,Ax} v_{t,Ax}] \Big|_{t=0}.$$

After a similar computation as the one we did in order to compute $\dot{\sigma}_\phi$ we obtain

$$\ddot{\sigma}_{Ax} = 2 \int_0^1 Ax \dot{v}'_{Ax} v'_{0,Ax} - \dot{\sigma}_{Ax} \dot{v}_{Ax} v_{0,Ax} dx. \quad (3.33)$$

Now we have to find the function \dot{v}_{Ax} and then compute the integral above. From (3.25), (3.23) and (3.32) we can conclude that \dot{v}_{Ax} satisfies the following differential equation

$$-\dot{v}''_{Ax}(x) - \pi^2 \dot{v}_{Ax}(x) = \left(\frac{A\pi^2}{\sqrt{2}} - Ax\sqrt{2}\pi^2 \right) \cos(\pi x) - A\sqrt{2}\pi \sin(\pi x).$$

We are free to choose a normalization for the eigenfunctions of the problem (3.5), so we can assume that, for every t , we have $\int_0^1 v_{t,Ax}^2 dx = 1$. From this we conclude that:

$$2 \int_0^1 \dot{v}_{Ax} v_{0,Ax} dx = \frac{d}{dt} \left[\int_0^1 v_{t,Ax}^2 dx = 1 \right] \Big|_{t=0} = 0.$$

From the boundary conditions of the problem (3.5) we obtain the following boundary conditions for \dot{v}_{Ax}

$$\begin{aligned} \dot{v}'_{Ax}(0) &= \frac{d}{dt} [v'_{t,Ax}(0)] \Big|_{t=0} = 0 \\ \dot{v}'_{Ax}(1) &= \frac{d}{dt} [(1 + tA)v'_{t,Ax}(1)] \Big|_{t=0} = 0. \end{aligned}$$

We finally obtain that \dot{v}_{Ax} must satisfy

$$\begin{cases} -\dot{v}''_{Ax}(x) - \pi^2 \dot{v}_{Ax}(x) = \left(\frac{A\pi^2}{\sqrt{2}} - Ax\sqrt{2}\pi^2 \right) \cos(\pi x) - A\sqrt{2}\pi \sin(\pi x) & x \in (0, 1) \\ \dot{v}'_{Ax}(0) = \dot{v}'_{Ax}(1) = 0 \\ \int_0^1 \dot{v}_{Ax} v_{0,Ax} dx = 0. \end{cases}$$

This problem admits a unique solution given by the following function:

$$\dot{v}_{Ax}(x) = \left(\frac{A}{4\sqrt{2}} - \frac{A}{2\sqrt{2}}x\right) \cos(\pi x) + \left(\frac{A}{2\sqrt{2}\pi} + \frac{A\pi}{2\sqrt{2}}(x^2 - x)\right) \sin(\pi x). \quad (3.34)$$

Putting the expressions given by (3.23) and (3.34) in the formula (3.33) we finally obtain

$$\ddot{\sigma}_{Ax} = \frac{A^2}{8}(3 - \pi^2). \quad (3.35)$$

We now compute $\ddot{\mu}_{Ax}$, from (3.15) we know that

$$\frac{d^2}{dt^2} \left[-\frac{d}{dx} \left((1 + tAx) \frac{du_{t,Ax}}{dx} \right) \right] \Big|_{t=0} = \frac{d}{dt} [\mu_{t,Ax}(1 + tAx)u_{t,Ax}] \Big|_{t=0},$$

After a similar computation as the one we did in order to compute $\dot{\mu}_\phi$ we obtain

$$\ddot{\mu}_{Ax} = 2 \int_0^1 Ax(\dot{u}'_{Ax}u'_{0,Ax} - \pi^2\dot{u}_{Ax}u_{0,Ax})dx. \quad (3.36)$$

Now we have to find the function \dot{u}_{Ax} and then compute the integral above. From (4.14), (3.23) and (3.32) we can conclude that \dot{u}_{Ax} must satisfy the following differential equation

$$-\dot{u}''_{Ax}(x) - \pi^2\dot{u}_{Ax}(x) = -A\sqrt{2}\pi \sin(\pi x).$$

We are free to choose a normalization for the eigenfunction of the problem (3.15), so we can assume that for every t we have $\int_0^1 (1 + tAx)u_{t,Ax}^2 dx = 1$, by differentiating with respect to t this relation and computing the derivative at zero we conclude that

$$\int_0^1 \dot{u}_{Ax}u_{0,Ax}dx = A \int_0^1 x \cos(\pi x)dx.$$

Using the same argument as above for the boundary conditions for \dot{u}_{Ax} we can conclude that \dot{u}_{Ax} must satisfy

$$\begin{cases} -\dot{u}''_{Ax}(x) - \pi^2\dot{u}_{Ax}(x) = -A\sqrt{2}\pi \sin(\pi x) & x \in (0, 1) \\ \dot{u}'_{Ax}(0) = \dot{u}'_{Ax}(1) = 0 \\ \int_0^1 \dot{u}_{Ax}u_{0,Ax}dx = A \int_0^1 x \cos(\pi x)dx. \end{cases}$$

This problem admits a unique solution given by the following function:

$$\dot{u}_{Ax}(x) = \frac{A}{\sqrt{2}} \left(\frac{1}{\pi} \sin(\pi x) - x \cos(\pi x) \right). \quad (3.37)$$

Putting the expressions given by (3.23) and (3.37) in the formula (3.36) we finally obtain

$$\ddot{\mu}_{Ax} = \frac{3}{2}A^2. \quad (3.38)$$

Finally putting (3.35), (3.38) and (3.32) inside (3.31) we obtain

$$\frac{d^2}{dt^2} F(1 + tAx) \Big|_{t=0} = \frac{A^2(9 + \pi^2)}{8\pi^2} > 0 \quad (3.39)$$

This concludes the proof. \square

3.4 Convex case: upper and lower bounds for $F(h)$ and $F(\Omega)$

In this section we prove Theorem 3.2 and Theorem 3.3. For every $0 < x_0 < 1$ we define the following triangular shape function

$$T_{x_0} = \begin{cases} \frac{x}{x_0} & x \in [0, x_0] \\ \frac{1-x}{1-x_0} & x \in [x_0, 1]. \end{cases}$$

Before proving Theorem 3.2 let us state the following Lemma, that will be crucial in the proof of the upper bound for $F(h)$

Lemma 3.13. *For every $0 < x_0 < 1$ the following equality holds*

$$\frac{\mu_1(T_{x_0})}{\sigma_1(T_{x_0})} = 4.$$

Proof. We want to compute the eigenvalue $\sigma_1(T_{x_0})$, we introduce the parameter σ and we want to find a function $v \in C^1(0, 1)$ such that

$$\begin{cases} xv''(x) + v'(x) + x_0\sigma v(x) = 0 & x \in [0, x_0] \\ (1-x)v''(x) - v'(x) + (1-x_0)\sigma v(x) = 0 & x \in [x_0, 1]. \end{cases} \quad (3.40)$$

The idea will be to solve the equation first on the interval $[0, x_0]$ then on the interval $[x_0, 1]$ and then find the condition on the parameter σ in order to have a good matching in the point x_0 . Let J_0, Y_0 be the Bessel functions of the first and second kind respectively with parameter 0, we start by noticing that all the solutions of the second order ODE (3.40) (1st line) are given in the interval $[0, x_0]$ by

$$v_l = C_1 J_0(2\sqrt{\sigma x_0 x}) + \hat{C}_1 Y_0(2\sqrt{\sigma x_0 x}).$$

Now, since $uY_0'(u) \rightarrow 2/\pi$ when $u \rightarrow 0$ we see that, in order the boundary condition $T_{x_0}(x)v_l'(x) \rightarrow 0$ be satisfied, we must choose $\hat{C}_1 = 0$. Using the change of variable $y = 1 - x$ is straightforward to check that, the solution of (3.40) (2nd line) is given in the interval $[x_0, 1]$ by

$$v_r = C_2 J_0(2\sqrt{\sigma(1-x_0)(1-x)}).$$

Now, we impose the following matching condition $v_l(x_0) = v_r(x_0)$ and $v_l'(x_0) = v_r'(x_0)$, this condition is equivalent to say that there exists a parameter σ for which the following system has a solution

$$\begin{cases} C_1 J_0(2\sqrt{\sigma}x_0) = C_2 J_0(2\sqrt{\sigma}(1-x_0)) \\ C_1 J_0'(2\sqrt{\sigma}x_0) = -C_2 J_0'(2\sqrt{\sigma}(1-x_0)). \end{cases}$$

The system above has a solution if and only if the parameter σ is a root of the following transcendental equation

$$J_0(2\sqrt{\sigma}x_0)J_0'(2\sqrt{\sigma}(1-x_0)) + J_0(2\sqrt{\sigma}(1-x_0))J_0'(2\sqrt{\sigma}x_0) = 0, \quad (3.41)$$

so $\sigma_1(T_{x_0})$ will be the smallest non zero root of the above equation.

Now we want to compute the eigenvalue $\mu_1(T_{x_0})$, we introduce the parameter μ and we want to find a function $u \in C^1(0, 1)$ such that

$$\begin{cases} xu''(x) + u'(x) + \mu xu(x) = 0 & x \in [0, x_0] \\ (1-x)u''(x) - u'(x) + \mu(1-x)u(x) = 0 & x \in [x_0, 1]. \end{cases} \quad (3.42)$$

We will find the conditions on μ by using the same arguments as before. For every constant C_1 the function

$$u_l = C_1 J_0(\sqrt{\mu}x)$$

is a solution for (3.42) in the interval $[0, x_0]$ (we can rule out the function Y_0 by the same argument). Using the change of variable $y = 1 - x$ is straightforward to check that, for every constant C_2 , the function

$$u_r = C_2 J_0(\sqrt{\mu}(1-x))$$

is a solution for (3.42) in the interval $[x_0, 1]$. We impose the following matching condition $u_l(x_0) = u_r(x_0)$ and $u'_l(x_0) = u'_r(x_0)$, this condition is equivalent to say that there exists a parameter μ for which the following system has a solution

$$\begin{cases} C_1 J_0(\sqrt{\mu}x_0) = C_2 J_0(\sqrt{\mu}(1-x_0)) \\ C_1 J'_0(\sqrt{\mu}x_0) = -C_2 J'_0(\sqrt{\mu}(1-x_0)). \end{cases}$$

The system above has a solution if and only if the parameter μ is a root of the following transcendental equation

$$J_0(\sqrt{\mu}x_0)J'_0(\sqrt{\mu}(1-x_0)) + J_0(\sqrt{\mu}(1-x_0))J'_0(\sqrt{\mu}x_0) = 0, \quad (3.43)$$

so $\mu_1(T_{x_0})$ will be the smallest non zero root of the above equation.

Now comparing the transcendental equations (3.41) and (3.43) we can conclude that

$$\frac{\mu_1(T_{x_0})}{\sigma_1(T_{x_0})} = 4.$$

□

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. We start by the lower bound:

Lower bound. Let $h^* = 6x(1-x)$, it is known (see for instance [99]) that, for every $h \in \mathcal{L}$, the following inequality holds

$$\sigma_1(h) \leq \sigma_1(h^*) = 12. \quad (3.44)$$

Now we want to prove that, for every $h \in \mathcal{L}$, the following inequality holds

$$\mu_1(h) \geq \pi^2. \quad (3.45)$$

Suppose by contradiction that there exists $\bar{h} \in \mathcal{L}$ such that

$$\mu_1(\bar{h}) < \pi^2,$$

by Lemma 3.10 we conclude that, for ϵ small enough, there exists a thin domain Ω_ϵ such that:

$$\mu_1(\Omega_\epsilon) < \pi^2.$$

We reach a contradiction because we know from Payne inequality (see [89]) that for every convex domain Ω with diameter 1

$$\mu_1(\Omega) \geq \pi^2.$$

From (3.44) and (3.45) we conclude that, for every $h \in \mathcal{L}$, the following lower bound holds

$$\frac{\pi^2}{12} \leq F(h).$$

Upper bound. We start by proving that, for every $h \in \mathcal{L}$, the following inequality holds

$$\mu_1(h) \leq \mu_1(T_{\frac{1}{2}}). \quad (3.46)$$

Suppose by contradiction that there exists $\bar{h} \in \mathcal{L}$ such that

$$\mu_1(\bar{h}) > \mu_1(T_{\frac{1}{2}}). \quad (3.47)$$

We introduce the following family of thin domains, first Ω_ϵ defined thanks to this function \bar{h} and then R_ϵ defined as follows:

$$R_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon \frac{1}{2} T_{\frac{1}{2}} \leq y \leq \epsilon \frac{1}{2} T_{\frac{1}{2}}\},$$

this class of domains R_ϵ can be seen as flattering rhombi. By Lemma 3.10 and (3.47) we conclude that, for ϵ small enough, we have:

$$\mu_1(\Omega_\epsilon) > \mu_1(R_\epsilon),$$

we reach a contradiction because we know from [8], [29] that for every thin domain Ω_ϵ and for every ϵ small enough

$$\mu_1(\Omega_\epsilon) \leq \lim_{\epsilon \rightarrow 0} \mu_1(R_\epsilon) = 4j_{01}^2.$$

Now we prove that, for every $h \in \mathcal{L}$, the following lower bound for $\sigma_1(h)$ holds

$$\sigma_1(h) \geq h\left(\frac{1}{2}\right)\sigma_1(T_{\frac{1}{2}}). \quad (3.48)$$

Let v be an eigenfunction associated to $\sigma_1(h)$, using the variational characterization for $\sigma_1(h)$ and using the fact that h is concave and positive we conclude that

$$\sigma_1(h) = \frac{\int_0^1 (v')^2 h dx}{\int_0^1 v^2 dx} \geq h\left(\frac{1}{2}\right) \frac{\int_0^1 (v')^2 T_{\frac{1}{2}} dx}{\int_0^1 v^2 dx} \geq h\left(\frac{1}{2}\right)\sigma_1(T_{\frac{1}{2}}),$$

where in the last inequality we used the variational characterization for $\sigma_1(T_{\frac{1}{2}})$. From (3.46) and (3.48) we conclude that:

$$F(h) \leq \frac{\mu_1(T_{\frac{1}{2}})}{\sigma_1(T_{\frac{1}{2}})} \frac{\int_0^1 h dx}{h\left(\frac{1}{2}\right)} \leq 4, \quad (3.49)$$

where the last inequality comes from the fact that $h \in \mathcal{L}$ and Lemma 3.13. □

We turn to the proof of Theorem 3.3. Let $\tau \in [0, 1]$ be a parameter, in order to prove the upper bound in Theorem 3.3, we need to introduce the following family of polynomials of degree four:

$$P_\tau(y) = \frac{1}{4}\tau y^4 - 2y^3 + 5\tau y^2 - 4\tau^2 y + \tau^3.$$

In the next Lemma we prove that the polynomials P_τ have always positive roots and we give some explicit estimates on its roots, this estimates will be useful in the proof of the upper bound for $F(\Omega)$.

Lemma 3.14. *Let $0 < \tau < 1$, then the polynomial P_τ has four positive roots. Let $\{y_1(\tau), y_2(\tau), y_3(\tau), y_4(\tau)\}$ be its roots ordered in increasing order, then the following holds:*

i if $0 < \tau \leq \frac{\sqrt{3}}{2}$, then $y_1(\tau) \in (0, \frac{2}{3}\tau)$, $y_2(\tau) \in (\frac{2}{3}\tau, \tau + \frac{1}{2}\tau^2)$, $y_3(\tau) \in (\tau + \frac{1}{2}\tau^2, 2 + \sqrt{2})$ and $y_4(\tau) \in (2 + \sqrt{2}, +\infty)$

ii if $\frac{\sqrt{3}}{2} \leq \tau \leq 0.9$, then $y_1(\tau) \in (0, \frac{1}{2})$, $y_2(\tau) \in (\frac{1}{2}, \tau + \frac{1}{2}\tau^2)$, $y_3(\tau) \in (\tau + \frac{1}{2}\tau^2, 2 + \sqrt{2})$ and $y_4(\tau) \in (2 + \sqrt{2}, +\infty)$,

iii if $0.9 \leq \tau < 1$, then $y_1(\tau) \in (0, 2 - \sqrt{2})$, $y_2(\tau) \in (2 - \sqrt{2}, \tau + \frac{1}{2}\tau^2)$, $y_3(\tau) \in (\tau + \frac{1}{2}\tau^2, 2 + \sqrt{2})$ and $y_4(\tau) \in (2 + \sqrt{2}, +\infty)$.

Moreover $P_\tau(y) \geq 0$ in $[0, y_1(\tau)] \cup [y_2(\tau), y_3(\tau)] \cup [y_4(\tau), +\infty)$.

Proof. We start by noticing that for every $0 < \tau < 1$ we have that $P_\tau(0) > 0$ and $\lim_{y \rightarrow +\infty} P_\tau(y) = +\infty$, the idea of the proof will be to find three consecutive points $0 < a < b < c < +\infty$ for which $P_\tau(a) < 0$, $P_\tau(b) > 0$ and $P_\tau(c) < 0$. Before passing to the three different cases, we give some inequalities that are true for every $0 < \tau < 1$. It is straightforward to check that the following inequalities hold:

$$P_\tau(\tau + \frac{1}{2}\tau^2) = \frac{1}{4}\tau^6(1 + \frac{3}{2}\tau + \frac{1}{2}\tau^2 + \frac{1}{4}\tau^3) > 0 \quad \forall 0 < \tau < 1, \quad (3.50)$$

$$P_\tau(2 + \sqrt{2}) = (\tau - 1)(\tau^2 - (7 + 4\sqrt{2})\tau + 40 + 28\sqrt{2}) < 0 \quad \forall 0 < \tau < 1. \quad (3.51)$$

We now prove separately the three cases.

i If $0 < \tau \leq \frac{\sqrt{3}}{2}$, then the following inequality holds

$$P_\tau(\frac{2}{3}\tau) = \frac{4}{9}\tau^3(\frac{\tau^2}{9} - \frac{1}{12}) < 0,$$

the result follows from this inequality combined with (3.50) and (3.51).

ii if $\frac{\sqrt{3}}{2} \leq \tau \leq 0.9$, then the following inequalities hold

$$P_\tau(\frac{1}{2}) = \tau^3 - 2\tau^2 + \frac{81}{64}\tau - \frac{1}{4} < 0,$$

$$\frac{1}{2} < \tau + \frac{1}{2}\tau^2,$$

the result follows from the inequalities above combined with (3.50) and (3.51).

iii If $0.9 \leq \tau < 1$, then the following inequalities hold

$$P_\tau(2 - \sqrt{2}) = (\tau - 1)(\tau^2 - (7 - 4\sqrt{2})\tau + 40 - 28\sqrt{2}) < 0,$$

$$2 - \sqrt{2} < \tau + \frac{1}{2}\tau^2.$$

the result follows from the inequalities above combined with (3.50) and (3.51). \square

We now state Theorem 3.3 in a more precise way, in order to give more information about the explicit constant C_1 .

Theorem 3.15. *Let K be the following constant*

$$K = \max_{\tau \in [0,1]} \frac{2\pi\tau}{y_2(\tau)[2\sqrt{1-\tau^2} + 2\tau \arcsin(\tau)]}.$$

Then for every bounded convex open set $\Omega \subset \mathbb{R}^2$, the following inequalities hold

$$\frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) \leq 2(1 + K) \leq 9.04.$$

Proof. We start by proving the lower bound

Lower bound. Let $\delta \in [2, \pi]$, we define the following class of bounded convex domains

$$\mathcal{C}_\delta := \{\Omega \subset \mathbb{R}^2 : \Omega \text{ is convex and } P(\Omega) \leq \delta D(\Omega)\}. \quad (3.52)$$

We recall that the functional $F(\Omega)$ is invariant under translation and rotation, so without loss of generality, we can assume that the origin is the center of mass of the boundary of Ω and the x_1 axis is parallel to (one of) the diameter(s). We know the following inequalities for $\mu_1(\Omega)$ and $\sigma_1(\Omega)$

$$\mu_1(\Omega) \geq \frac{\pi^2}{D(\Omega)^2}, \quad \sigma_1(\Omega) \leq \frac{|\Omega|}{\int_{\partial\Omega} x_1^2 ds} \leq \frac{6|\Omega|}{D(\Omega)^3}.$$

The inequality for μ_1 is Payne inequality (see [89]) and the inequality for $\sigma_1(\Omega)$ is obtained by using the function $u(x_1, x_2) = x_1$ as a test function in (1.2) and then using the fact that $\int_{\partial\Omega} x_1^2 ds \geq \int_{-\frac{D}{2}}^{\frac{D}{2}} x_1^2 dx_1$. Let $\Omega \in \mathcal{C}_\delta$, using the inequalities above we obtain

$$F(\Omega) \geq \frac{\pi^2}{6\delta}. \quad (3.53)$$

Now we consider the class of domains \mathcal{C}_δ^c , i. e. convex domains such that $P(\Omega) > \delta D(\Omega)$. We start by recalling the following result (see [91] for a geometric proof or [50] for a proof based on Fourier series)

$$\min \left\{ \frac{\int_{\partial\Omega} (x_1^2 + x_2^2) ds}{P(\Omega)^3} : \Omega \subset \mathbb{R}^2 \text{ convex} \right\} = \frac{1}{54}, \quad (3.54)$$

and the minimum is achieved by the equilateral triangle. Assuming that the origin is at the center of mass of the boundary, and using in the variational characterization (1.5) the coordinates functions x_1 and x_2 we obtain after summing

$$\sigma_1(\Omega) \leq \frac{2|\Omega|}{\int_{\partial\Omega} (x_1^2 + x_2^2) ds}. \quad (3.55)$$

Now from Payne inequality, $(\mu_1(\Omega) \geq \pi^2/D^2)$, (3.54) and (3.55) we conclude that for every $\Omega \in \mathcal{C}_\delta^c$ the following holds

$$F(\Omega) \geq \frac{\delta^2 \pi^2}{108}. \quad (3.56)$$

We notice that the lower bounds in (3.53) and (3.56) coincide when $\delta = \sqrt[3]{18}$, so we finally obtain:

$$F(\Omega) \geq \frac{\pi^2}{6\sqrt[3]{18}}.$$

Upper bound. Given a bounded convex set $\Omega \subset \mathbb{R}^2$, we denote by $r(\Omega)$ its inradius and by $w(\Omega)$ its minimal width. We know the following estimate from below for $\sigma_1(\Omega)$ (see [76])

$$\sigma_1(\Omega) \geq \frac{\mu_1(\Omega)r(\Omega)}{2(1 + \sqrt{\mu_1(\Omega)D(\Omega)}),$$

we also know the following upper bound for $\mu_1(\Omega)$, see [54]:

$$\mu_1(\Omega) \leq \pi^2 \frac{w(\Omega)^2}{|\Omega|^2},$$

we also use the following geometric inequality (see [16])

$$\frac{|\Omega|}{r(\Omega)P(\Omega)} \leq 1.$$

Using the three inequalities above we conclude that

$$F(\Omega) \leq 2 \left(1 + \frac{\pi w(\Omega)D(\Omega)}{r(\Omega)P(\Omega)} \right). \quad (3.57)$$

We introduce the parameter $\tau = \frac{w(\Omega)}{D(\Omega)}$, we know the following geometric inequality (see [75], [94])

$$\frac{D(\Omega)}{P(\Omega)} \leq \frac{1}{2\sqrt{1-\tau^2} + 2\tau \arcsin(\tau)} =: g(\tau).$$

Now, in order to obtain an upper bound for the functional $F(\Omega)$, we need an upper bound for the quantity $\frac{w(\Omega)}{r(\Omega)}$ where the quantity $\tau = \frac{w(\Omega)}{D(\Omega)}$ is fixed.

The complete system of inequalities for the triplet $(w(\Omega), D(\Omega), r(\Omega))$ is known, in [59] we can find the Blaschke–Santaló diagram where $x(\Omega) = \tau = \frac{w(\Omega)}{D(\Omega)}$ and $y(\Omega) = \frac{2r(\Omega)}{D(\Omega)}$. Let us fix the quantity τ , in order to obtain an upper bound for $\frac{w(\Omega)}{r(\Omega)}$ it is enough to obtain a lower bound for $y(\Omega)$. From [59] we know that the following inequality holds

$$P_\tau(y(\Omega)) = \frac{1}{4}\tau y(\Omega)^4 - 2y(\Omega)^3 + 5\tau y(\Omega)^2 - 4\tau^2 y(\Omega) + \tau^3 \geq 0.$$

In particular from Lemma 3.14 we know that $y(\Omega) \in [0, y_1(\tau)] \cup [y_2(\tau), y_3(\tau)] \cup [y_4(\tau), +\infty)$, we now prove that $y(\Omega) \geq y_2(\tau)$. Suppose by contradiction that $y(\Omega) \in [0, y_1(\tau)]$, from the Blaschke–Santaló diagram $(w(\Omega), D(\Omega), r(\Omega))$ we see that $y(\Omega) \geq \frac{2}{3}\tau$, but now from Lemma 3.14 we know that $y_1(\tau) < \frac{2}{3}\tau$ and this is a contradiction. Note that we can prove in the same way that $y(\Omega) < y_4(\tau)$.

We conclude that $y(\Omega) \geq y_2(\tau)$, so we finally obtain the following upper bound

$$\frac{\pi w(\Omega)D(\Omega)}{r(\Omega)P(\Omega)} \leq \frac{2\pi g(\tau)\tau}{y_2(\tau)} =: f(\tau).$$

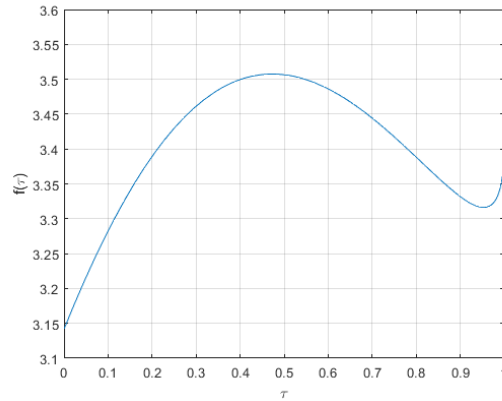
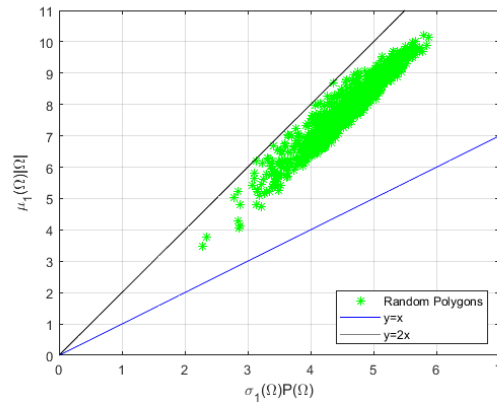
Figure 3.2: Plot of the function $f(\tau)$ 

Figure 3.3: Blaschke–Santaló diagram with random convex polygons

We introduce the following constant

$$K = \max_{\tau \in [0,1]} f(\tau)$$

numerically one can check that $K \leq 3.52$ (see Figure 3.2), from (3.57) we finally conclude that

$$\frac{\pi^2}{6\sqrt[3]{18}} \leq F(\Omega) \leq 2(1 + K) \leq 9.04.$$

□

3.5 Blaschke–Santaló diagrams and open problems

A Blaschke-Santaló diagram is a convenient way to represent in the plane the possible values taken by two quantities (geometric or spectral). As mentioned in the Introduction,

such a diagram has been recently established for quantities like $(\lambda_1(\Omega), \lambda_2(\Omega))$ (the Dirichlet eigenvalues) in [3], [21], $(\mu_1(\Omega), \mu_2(\Omega))$ (the Neumann eigenvalues) in [3], $(\lambda_1(\Omega), \mu_1(\Omega))$ in [40] or $(\lambda_1(\Omega), T(\Omega))$ (where $T(\Omega)$ is the torsion) in [100], [81].

Here we are interested in plotting the set of points (x, y) with

$$\begin{aligned}\mathcal{E} &= \{(x, y) \text{ where } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2\} \\ \mathcal{E}^C &= \{(x, y) \text{ where } x = \sigma_1(\Omega)P(\Omega), y = \mu_1(\Omega)|\Omega|, \Omega \subset \mathbb{R}^2, \Omega \text{ convex.}\}.\end{aligned}$$

3.5.1 The Blaschke-Santaló diagram \mathcal{E}

We start with the diagram \mathcal{E} (no constraint on the sets Ω).

Theorem 3.16. *The following equality holds*

$$\overline{\mathcal{E}} = [0, 8\pi] \times [0, \mu_1(\mathbb{D})\pi]$$

where $\mu_1(\mathbb{D}) = j'_{11}{}^2$ is the first Neumann eigenvalue of the unit disk.

Proof. We recall the following classical result by Szegő (for the simply connected case) and Weinberger [97] and [105].

$$\max\{\mu_1(\Omega)|\Omega| \mid \Omega \subset \mathbb{R}^2 \text{ bounded, open and Lipschitz}\} = \mu_1(\mathbb{D})\pi,$$

from [44] we also know that

$$\sup\{\sigma_1(\Omega)P(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ bounded, open and Lipschitz}\} = 8\pi.$$

From the inequalities above it is clear that $\mathcal{E} \subset [0, 8\pi] \times [0, \mu_1(\mathbb{D})\pi]$, now we want to prove that $[0, 8\pi] \times [0, \mu_1(\mathbb{D})\pi] \subseteq \overline{\mathcal{E}}$.

We start by proving that for every $y \in [0, \mu_1(\mathbb{D})\pi]$ there exists a simply connected domain Ω_y for which $\mu_1(\Omega_y)|\Omega_y| = y$. For that purpose, let us consider a dumbbell domain D_ϵ , we know that we can choose the width of the channel in order to have $\mu_1(D_\epsilon)|D_\epsilon| = \epsilon$ where ϵ is a small quantity, (see [67]). Now we can gradually enlarge the channel (preserving the ϵ -cone condition) until we reach a stadium, then we can modify this stadium continuously until we reach the ball. In all that process, the eigenvalue μ_1 and the area vary continuously. So we constructed a continuous path for the value $\mu_1(\Omega_y)|\Omega_y|$ starting from ϵ and arriving to $\mu_1(\mathbb{D})\pi$, we conclude because ϵ was arbitrary small. Using the same argument (and [25]) we can prove that for every $x \in [0, 2\pi]$ there exists a simply connected domain Ω_x for which $\sigma_1(\Omega_x)P(\Omega_x) = x$ (2π is the value of $P(\mathbb{D})\sigma_1(\mathbb{D})$).

Let $(x, y) \in [0, 8\pi] \times [0, \mu_1(\mathbb{D})\pi]$ we want to prove that there exists a sequence of domains Ω_ϵ such that $\sigma_1(\Omega_\epsilon)P(\Omega_\epsilon) \rightarrow x$ and $\mu_1(\Omega_\epsilon)|\Omega_\epsilon| \rightarrow y$. From the discussion above we know that there exists a simply connected domain Ω_y for which $\mu_1(\Omega_y)|\Omega_y| = y$, now we divide the proof in two cases:

Case 1. Suppose $x > \sigma_1(\Omega_y)P(\Omega_y)$, let β be a non negative and non trivial function, we introduce the following weighted Neumann eigenvalue

$$\mu_1(\Omega, \beta) = \min \left\{ \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 \beta dx} : u \in H^1(\Omega), \int_{\Omega} u \beta dx = 0 \right\}.$$

From Theorem 1.11 in [44] we know that for every domain Ω and every non negative and non trivial function $\beta \in L^1(\log L)^1$ (this space is a Orlicz space see [44] for the details) there exists a sequence of subdomains $\Omega_\epsilon \subseteq \Omega$ such that

$$\begin{aligned}\sigma_1(\Omega_\epsilon)P(\Omega_\epsilon) &\rightarrow \mu_1(\Omega, \beta) \int_{\Omega} \beta dx, \\ \mu_1(\Omega_\epsilon)|\Omega_\epsilon| &\rightarrow \mu_1(\Omega)|\Omega|.\end{aligned}$$

Let us fix a parameter δ , from [44] we know that there exists a function β_1 such that $\mu_1(\Omega, \beta_1) \int_{\Omega} \beta_1 dx \leq 8\pi - \delta$, we also know (see [78]) that there exists a function β_2 such that $|\mu_1(\Omega, \beta_2) \int_{\Omega} \beta_2 dx - \sigma_1(\Omega)P(\Omega)| \leq \delta$. Let $0 \leq t \leq 1$, we consider the following family of functions $\beta_t = t\beta_1 + (1-t)\beta_2$ and we introduce the measures $d\mu_t = \beta_t dx$. It is straightforward to check that the family of measures $d\mu_t$ satisfies the conditions **M1**, **M2** and **M3** in page 26 of [44], in particular for every $z \in [\sigma_1(\Omega)P(\Omega) + \delta, 8\pi - \delta]$ there exists $t \in [0, 1]$ such that $\mu_1(\Omega, \beta_t) \int_{\Omega} \beta_t dx = z$.

We know that $x \in [\sigma_1(\Omega_y)P(\Omega_y) + \delta, 8\pi - \delta]$, let t_0 be such that $\mu_1(\Omega_y, \beta_{t_0}) \int_{\Omega_y} \beta_{t_0} dx = x$, from the previous results we conclude that there exists a sequence of domains $\Omega_\epsilon \subseteq \Omega_y$ such that

$$\begin{aligned} \sigma_1(\Omega_\epsilon)P(\Omega_\epsilon) &\rightarrow \mu_1(\Omega_y, \beta_{t_0}) \int_{\Omega} \beta_{t_0} dx = x \\ \mu_1(\Omega_\epsilon)|\Omega_\epsilon| &\rightarrow \mu_1(\Omega_y)|\Omega_y| = y. \end{aligned}$$

The result follows because δ was arbitrary.

Case 2. Suppose $x \leq \sigma_1(\Omega_y)P(\Omega_y)$, from the fact that Ω_y is simply connected we know from [106] that $x \leq 2\pi$. By a previous step we know that there exists a simply connected domain ω such that $\sigma_1(\omega)P(\omega) = x$, now from Theorem 3.5 (see [27] for details) we know that there exists a sequence of smooth open sets Ω_ϵ such that

$$\begin{aligned} \sigma_1(\Omega_\epsilon)P(\Omega_\epsilon) &\rightarrow \sigma_1(\omega)P(\omega) = x \\ \mu_1(\Omega_\epsilon)|\Omega_\epsilon| &\rightarrow \mu_1(\Omega_y)|\Omega_y| = y. \end{aligned}$$

This concludes the proof. □

We can give the following more precise conjecture:

Conjecture 3.1. *Prove that $\mathcal{E} = (0, 8\pi) \times (0, \pi\mu_1(\mathbb{D})) \cup \{(0, 0)\} \cup \{(2\pi, \pi\mu_1(\mathbb{D}))\}$.*

The point $\{(0, 0)\}$ is attained by any disconnected domain. Moreover the segments $\{0\} \times (0, \pi\mu_1(\mathbb{D}))$ and $(0, 8\pi) \times \{0\}$ cannot be in the set \mathcal{E} because if μ_1 or σ_1 are zero, it means that the domain is disconnected, thus $(\sigma_1, \mu_1) = (0, 0)$. The segment $(0, 8\pi) \times \{\pi\mu_1(\mathbb{D})\}$ only contains the point corresponding to the disk because the disk is the only domain providing equality in the Szegő-Weinberger inequality. Finally, the segment $\{8\pi\} \times (0, \pi\mu_1(\mathbb{D}))$ is not included in the diagram because the inequality $P(\Omega)\sigma_1(\Omega) < 8\pi$ is strict, see [44]. Thus the conjecture means that except these "boundary lines", every point (x, y) such that $0 < x < 8\pi$ and $0 < y < \pi\mu_1(\mathbb{D})$ should correspond to a set Ω in the sense that $x = P(\Omega)\sigma_1(\Omega)$ and $y = |\Omega|\mu_1(\Omega)$.

3.5.2 The Blaschke-Santaló diagram \mathcal{E}^c

Now we turn to the convex case. To have some idea about the shape of this diagram, we produced random convex polygons in the plane and plot the corresponding quantities $x = \sigma_1(\Omega)P(\Omega)$, $y = \mu_1(\Omega)|\Omega|$.

Figure 3.3 shows the values of these quantities for 1000 random convex polygons. Each of this polygon is constructed by choosing 15 random points in the plane and then we compute the convex hull of this points. From Figure 3.3 it is natural to conjecture that $1 \leq F(\Omega) \leq 2$.

Now we show some experiments that will give us informations about the behavior of the extremal sets in the class of convex domains. In the Figure 3.4 we plotted the quantities $\sigma_1(\Omega)P(\Omega)$ and $\mu_1(\Omega)|\Omega|$ for random triangles in the plane.

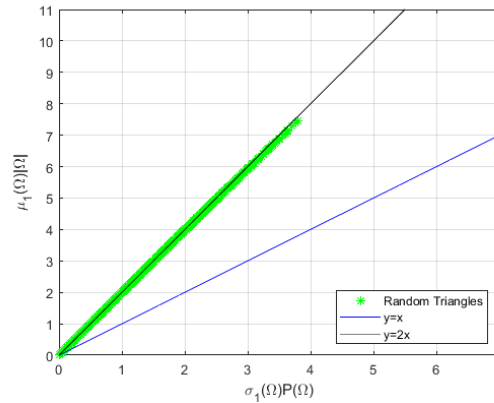


Figure 3.4: Blaschke–Santaló diagram with random triangles

From Figure 3.4 we see that for every triangle $T \subset \mathbb{R}^2$ we have that $F(T)$ is slightly less than (and very close to) 2. Actually a more precise numerical computation shows that it is not true that $F(T) = 2$ for every triangles. For example, let T_1 be an equilateral triangle of length 1, we know that $\mu_1(T_1) = \frac{16\pi^2}{9}$ and let T_2 be a right triangle with both cathetus equal to 1, we know that $\mu_1(T_2) = \pi^2$. A precise numerical computation of the first Steklov eigenvalue for T_1 and T_2 (using $P2$ finite element methods) gives us the following values $\sigma_1(T_1) \approx 1.2908$ and $\sigma_1(T_2) \approx 0.7310$. Using these values inside the functional $F(\Omega)$ we finally obtain

$$\begin{aligned} F(T_1) &\approx 1.962 < 2, \\ F(T_2) &\approx 1.977 < 2. \end{aligned}$$

The value 2 can be reached asymptotically, let us consider the following sequence of collapsing triangles

$$\Omega_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq \epsilon T_{\frac{1}{2}}\},$$

from Theorem 3.1 and Lemma 3.13 we conclude that

$$F(\Omega_\epsilon) \rightarrow F(T_{\frac{1}{2}}) = 2.$$

We remark that, from Theorem 3.1 and Lemma 3.13, $F(\Omega_\epsilon) \rightarrow 2$ for every sequence Ω_ϵ of collapsing thin domains for which $h = h^+ + h^- = T_{x_0}$, where $0 < x_0 < 1$.

It remains to characterize the behavior of the minimizing sequence. We introduce the following family of collapsing rectangles:

$$C_\epsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq \epsilon\}.$$

We plot the values of $\sigma_1(C_\epsilon)P(C_\epsilon)$ and $\mu_1(C_\epsilon)|C_\epsilon|$ when ϵ is approaching zero.

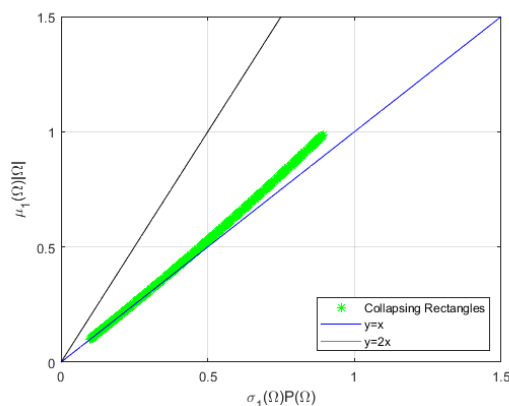


Figure 3.5: Blaschke–Santaló diagram with collapsing rectangles

We know from Theorem 3.1 that $F(C_\epsilon) \rightarrow 1$ but from Figures 3.5 and 3.3 it seems that $F(\Omega) > 1$ for every $\Omega \subset \mathbb{R}^2$ convex and the only way to approach the value 1 is given by a sequence of collapsing rectangles.

Supported by these numerical evidences we state the following conjectures:

Conjecture 3.2. *For every bounded, convex and open set $\Omega \subset \mathbb{R}^2$ the following bounds hold*

$$1 \leq F(\Omega) \leq 2.$$

Conjecture 3.3. *The following minimization problem has no solution*

$$\inf\{F(\Omega) \mid \Omega \subset \mathbb{R}^2 \text{ bounded, convex and open}\}.$$

In particular every minimizing sequence Ω_ϵ must be of the form of collapsing rectangles.

We now consider only convex quadrilaterals in \mathbb{R}^2 , in the following numerical experiment we will have in red random convex quadrilaterals and in green collapsing rectangles, starting from a square S of unit area (corresponding to the farthest green point from the origin) and asymptotically approach the segment.

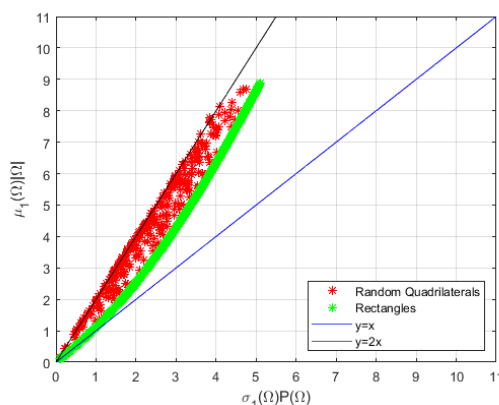


Figure 3.6: Blaschke–Santaló diagram with random convex quadrilaterals and collapsing rectangles

From Figure 3.6 it is natural to state the following conjecture.

Conjecture 3.4. *For every $0 < C \leq 4\sigma_1(\mathbb{S})$ the solution of the minimization problem*

$$\inf\{\mu_1(\Omega)|\Omega| \mid \Omega \subset \mathbb{R}^2 \text{ convex quadrilateral s.t. } \sigma_1(C_\epsilon)P(C_\epsilon) = C\},$$

is given by a rectangle.

3.6 Inequality between Neumann and Steklov eigenvalues: approach via L^2 hot spot conjecture

In this section we prove the inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ when Ω is a circumscribed polygon in the plane with two axis of symmetries.

This result is a direct consequence of the results that we are obtaining in the work in progress [57] in collaboration with A. Henrot and E. Parini.

We now present an approach to prove the inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$ for convex domains with two axis of symmetries. We introduce what we call the L^2 version of the hot spot conjecture. A famous conjecture concerning the Neumann eigenfunction is the Hot spot conjecture. This conjecture has a long story, it was first stated by J. Rauch in 1974 in the context of the asymptotic behavior of the points with higher temperature of a heated body Ω when the time goes to infinity. J. Rauch conjectured that the points with higher temperature of an heated body approach asymptotically the boundary when we let the body cool down. This conjecture can be rigorous stated in the following way:

Conjecture 3.5 (Hot spot conjecture). *Let $\Omega \subset \mathbb{R}^2$ be a convex domain and let u be the first non trivial Neumann eigenfunction of the domain Ω , then*

$$\inf_{x \in \partial\Omega} u(x) \leq u(y) \leq \sup_{x \in \partial\Omega} u(x) \quad \forall y \in \Omega.$$

This conjectured in full generality is still open, we refer to [8, 61, 88, 68] for solutions of this conjecture in some specific cases.

In this section we want to study a similar conjecture, that relates the mean value of the Neumann eigenfunction on the boundary with it's mean value inside, we call this conjecture L^2 hot spot conjecture.

Conjecture 3.6 (L^2 hot spot conjecture). *Let $\Omega \subset \mathbb{R}^2$ be a convex domain and let u be the first non trivial Neumann eigenfunction of the domain Ω , then*

$$\frac{1}{P(\Omega)} \int_{\partial\Omega} u^2 ds \geq \frac{1}{|\Omega|} \int_{\Omega} u^2 dx.$$

We see that this conjecture is really important in order to prove the inequality $P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega)$. Indeed suppose we have a plane convex domain Ω with two axis of symmetries, thanks to this symmetry we know that there exists a Neumann eigenfunction u such that $\int_{\partial\Omega} u ds = 0$, so we can use the Neumann eigenfunction as a test function for the Steklov eigenvalue and assuming the L^2 hot spot conjecture is true, we have that

$$P(\Omega)\sigma_1(\Omega) \leq P(\Omega) \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\partial\Omega} u^2 ds} \leq |\Omega| \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx} = |\Omega|\mu_1(\Omega).$$

We prove the following theorem.

Theorem 3.17. *Let $\Omega \subset \mathbb{R}^2$ circumscribed polygon and let u be the first non trivial Neumann eigenfunction of the polygon Ω , then*

$$\frac{1}{P(\Omega)} \int_{\partial\Omega} u^2 ds \geq \frac{1}{|\Omega|} \int_{\Omega} u^2 dx.$$

The key result in order to prove this theorem is the following lemma regarding an explicit formula for a Neumann torsion function in the class of circumscribed polygon.

Lemma 3.18. *Let $\Omega \subset \mathbb{R}^2$ circumscribed polygon and let a be the inradius of the polygon, then the function*

$$w(x, y) = \frac{1}{2a}(x^2 + y^2),$$

is a weak solution of the following problem

$$\begin{cases} \Delta w = \frac{P(\Omega)}{|\Omega|} & \Omega \\ \partial_\nu w = 1 & \partial\Omega, \end{cases}$$

that means

$$\int_{\Omega} \nabla w \cdot \nabla \phi dx = \int_{\partial\Omega} \phi ds - \frac{P(\Omega)}{|\Omega|} \int_{\Omega} \phi dx, \quad \forall \phi \in H^1(\Omega).$$

Proof. Let B_a be the ball of radius a centered in the origin and let Ω be a circumscribed polygon with incircle B_a , and we also define the following function

$$w(x, y) = \frac{1}{2a}(x^2 + y^2),$$

it is easy to check that $\frac{P(\Omega)}{|\Omega|} = \frac{P(B_a)}{|B_a|} = \frac{2}{a}$ and also that

$$\Delta w(x, y) = \frac{2}{a} = \frac{P(\Omega)}{|\Omega|}.$$

We check the boundary conditions, we can parametrize a generic side l_i of the polygon in the following way

$$l_i = \{\alpha_i x + \beta_i y = a\sqrt{\alpha_i^2 + \beta_i^2} \mid x \in [a_i, b_i], y \in [c_i, d_i]\}$$

where $\alpha_i, \beta_i, a_i, b_i, c_i$ and d_i are real numbers. It is now straightforward to check that

$$\partial_\nu w(x, y) = \begin{pmatrix} x \\ y \\ a \end{pmatrix} \cdot \begin{pmatrix} \frac{\alpha_i}{\sqrt{\alpha_i^2 + \beta_i^2}} \\ \frac{\beta_i}{\sqrt{\alpha_i^2 + \beta_i^2}} \\ 1 \end{pmatrix} = 1 \quad \forall (x, y) \in l_i.$$

This can be repeated for all the sides of the polygon. Let $\phi \in H^1(\Omega)$, we can integrate by parts the quantity $\int_{\Omega} \nabla w \cdot \nabla \phi dx$ and we finally obtain

$$\int_{\Omega} \nabla w \cdot \nabla \phi dx = \int_{\partial\Omega} \phi ds - \frac{P(\Omega)}{|\Omega|} \int_{\Omega} \phi dx.$$

□

We can use now this Lemma in order to prove Theorem 3.17.

Proof of Theorem 3.17. Let $\Omega \subset \mathbb{R}^2$ circumscribed polygon and let u be the eigenfunction associated to $\mu_1(\Omega)$, let $F \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^2)$ be a smooth vector field, by the Rellich formula (see Theorem 3.1 in [52])

$$\begin{aligned} 0 &= \int_{\Omega} (\Delta u + \mu_1(\Omega)u)(F \cdot \nabla u) dx = \int_{\partial\Omega} \partial_{\nu} u (F \cdot \nabla u) ds - \frac{1}{2} \int_{\partial\Omega} |\nabla u|^2 (F \cdot \nu) ds + \\ &+ \frac{\mu_1(\Omega)}{2} \int_{\partial\Omega} u^2 (F \cdot \nu) ds + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \operatorname{div} F dx - \frac{\mu_1(\Omega)}{2} \int_{\Omega} u^2 \operatorname{div} F dx - \int_{\Omega} JF(\nabla u, \nabla u) dx \\ &= -\frac{1}{2} \int_{\partial\Omega} |\nabla_{\Gamma} u|^2 (F \cdot \nu) ds + \frac{\mu_1(\Omega)}{2} \int_{\partial\Omega} u^2 (F \cdot \nu) ds + \frac{1}{2} \int_{\Omega} |\nabla u|^2 \operatorname{div} F dx + \\ &- \frac{\mu_1(\Omega)}{2} \int_{\Omega} u^2 \operatorname{div} F dx - \int_{\Omega} JF(\nabla u, \nabla u) dx, \end{aligned}$$

where $\nabla_{\Gamma} u$ is the tangential gradient on the boundary $\partial\Omega$ and JF is the Jacobian of the vector field F . We choose the following vector field $F = \nabla w$, where w is the weak solution of the following Neumann problem

$$\begin{cases} \Delta w = \frac{P(\Omega)}{|\Omega|} & \Omega \\ \partial_{\nu} w = 1 & \partial\Omega, \end{cases}$$

in the qualities above and we obtain

$$\frac{\mu_1(\Omega)}{2} \int_{\partial\Omega} u^2 ds = \int_{\Omega} \langle D^2 w \cdot \nabla u, \nabla u \rangle dx + \frac{1}{2} \int_{\partial\Omega} |\nabla_{\Gamma} u|^2 ds.$$

From the equality above, in order to prove following inequality

$$\frac{1}{P(\Omega)} \int_{\partial\Omega} u^2 ds \geq \frac{1}{|\Omega|} \int_{\Omega} u^2 dx$$

is enough to prove that

$$\int_{\Omega} \langle D^2 w \cdot \nabla u, \nabla u \rangle dx \geq \frac{\mu_1(\Omega)}{2} \frac{P(\Omega)}{|\Omega|} \int_{\Omega} u^2 dx. \quad (3.58)$$

Let a be the inradius of the circumscribed polygon Ω we know that $\frac{P(\Omega)}{|\Omega|} = \frac{2}{a}$, then from Lemma 3.18 we have that

$$D^2 w = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{a} \end{pmatrix},$$

from (3.58) we finally conclude that

$$\int_{\Omega} \langle D^2 w \cdot \nabla u, \nabla u \rangle dx \geq \frac{1}{a} \int_{\Omega} |\nabla u|^2 dx \geq \frac{1}{2} \frac{P(\Omega)}{|\Omega|} \int_{\Omega} |\nabla u|^2 dx \geq \frac{\mu_1(\Omega)}{2} \frac{P(\Omega)}{|\Omega|} \int_{\Omega} u^2 dx,$$

this concludes the proof. □

A direct consequence of Theorem 3.17, using the argument we used in the introduction we obtain that

Corollary 3.19. *Let Ω be a circumscribed polygon in the plane with two axis of symmetries then*

$$P(\Omega)\sigma_1(\Omega) \leq |\Omega|\mu_1(\Omega).$$

Chapter 4

Optimization of Neumann and Sturm-Liouville eigenvalues

NOTE: this chapter correspond to the first version of the paper [56]. Just before to submit this paper Pedro Freitas point to the authors attention the work by P. Kröger [74]. We warmly thanks Pedro Freitas for pointing out this reference. In our work we partially recover the results present in [74] using another approach. In particular the novelty of our work is to use the abstract formalism developed in [79] to prove Theorem 4.12. Our approach is based on finding critical points and then compare them, for this reason our approach can be also used in order to detect the minimum for the Sturm-Liouville eigenvalues. In the new version of the work [56] we will include these results.

4.1 Introduction to the Chapter

A central question in spectral geometry is to find upper bounds for the Neumann eigenvalues involving geometric quantities related to the domain. The best result that one can achieve in this direction is to prove an explicit and sharp upper bound for the eigenvalues and to give a characterization of the domains for which the equality is achieved.

One of the first result in this direction is the Szegő-Weinberger inequality:

$$|\Omega|^{\frac{2}{d}}\mu_1(\Omega) \leq |B|^{\frac{2}{d}}\mu_1(B),$$

where B is a ball. This inequality was first proved by G. Szegő in [97] for planar, simply connected and Lipschitz sets and then H. F. Weinberger in [105] removed the dimensional and topological constraints. An explicit and sharp upper bound involving the volume of the domain is also known for the second Neumann eigenvalue, this was proved in 2 dimensions for simply connected domains by A. Girouard, N. Nadirashvili and I. Polterovich in [45] and in all dimensions without any restriction on the topology by D. Bucur and the first author in [24]. The inequality reads as follows:

$$|\Omega|^{\frac{2}{d}}\mu_2(\Omega) \leq |\Omega^*|^{\frac{2}{d}}\mu_2(\Omega^*),$$

where Ω^* is the union of two disjoint equal balls. The existence of a maximizer for the quantity $|\Omega|^{\frac{2}{d}}\mu_k(\Omega)$ with $k \geq 3$ is not known in general.

Another related question is to find inequalities for the Neumann eigenvalues involving the perimeter of the set. Recently A. Henrot, A. Lemenant and I. Lucardesi in [54] proved that for every Ω plane convex domain with two axis of symmetry the following inequality is true:

$$P(\Omega)^2\mu_1(\Omega) \leq 16\pi^2,$$

with equality for a square and an equilateral triangle. It is conjectured that this inequality holds true for any plane convex domains. Note that, without convexity constraint, we have $\sup P^2(\Omega)\mu_1(\Omega) = +\infty$.

An important result concerns the upper bound for planar convex domains in terms of the diameter $D(\Omega)$ of the set. Results in this direction is given by S. Y. Cheng in [29], where he gives general upper bounds involving the diameter for smooth and complete Riemannian manifolds. The given bound is sharp for μ_1 but not for the other eigenvalues. Later, this sharp and explicit upper bound for the first eigenvalue has been generalized by R. Banuelos and K. Burdzy in [8] to (slightly) more general domains than convex ones. To sum up, the theorem, for plane convex domains, is the following:

Theorem 4.1 (S.Y. Cheng [29], R. Banuelos-K. Burdzy [8]). *For any plane convex domain Ω*

$$D(\Omega)^2\mu_1(\Omega) \leq 4j_{0,1}^2$$

where $j_{0,1} \simeq 2.405$ is the first zero of the Bessel function J_0 . Moreover, this bound is sharp, asymptotically attained by collapsing isosceles triangles (or rhombi).

Then P. Kröger give the following upper bounds for all the Neumann eigenvalues

Theorem 4.2 (P. Kröger [74]). *For any plane convex domain Ω*

$$D(\Omega)^2\mu_k(\Omega) \leq (2j_{0,1} + (k-1)\pi)^2$$

where $j_{0,1} \simeq 2.405$ is the first zero of the Bessel function J_0 .

In this chapter, motivated by the study of explicit upper bounds for the Neumann eigenvalues involving the diameter $D(\Omega)$, we prove connections between the Neumann eigenvalues and the following one dimensional Sturm-Liouville eigenvalues

$$\begin{cases} -\frac{d}{dx}\left(h(x)\frac{du}{dx}(x)\right) = \mu(h)h(x)u(x) & x \in (0, 1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0. \end{cases}$$

Note that the function h is allowed to vanish at the extremities of the interval, making this eigenvalue problem not completely classical.

The main idea, heuristically, is to use the fact, proved in [55], that for every convex domain Ω there exists a concave function h such that $\mu_k(\Omega) < \mu_k(h)$ and for every concave function h there exists a sequence of convex domains Ω_ϵ that converges to a segment, such that $\mu_k(\Omega_\epsilon) \rightarrow \mu_k(h)$.

In this way we are led to study the maximization problem $\sup\{\mu_k(h), h \in \mathcal{S}\}$ where \mathcal{S} is a class of admissible functions. In Section 4.2 we prove that, if we do not put a concavity constraint on h , we have that $\sup\{\mu_k(h)\} = +\infty$ and, using this result, we prove that

Theorem 4.3. *For every $d \geq 2$ the following equality holds*

$$\sup\{D(\Omega)^2\mu_1(\Omega) \mid \Omega \subset \mathbb{R}^d \text{ open and Lipschitz}\} = +\infty.$$

The main result of this chapter is presented in Section 4.3, in this section we study the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ where \mathcal{L} is a class of concave functions. The main result of this Chapter is the following:

Theorem 4.4. *For any $k \geq 1$, the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ has a solution h_k^* , moreover*

- $\max\{\mu_1(h), h \in \mathcal{L}\} = \mu_1(h_1^*) = (2j_{0,1})^2$ and

$$h_1^* = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ 2(1-x) & x \in [\frac{1}{2}, 1] \end{cases}$$

- let $k \geq 2$ then $\max\{\mu_k(h), h \in \mathcal{L}\} = \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2$

$$h_k^* = \begin{cases} \frac{x(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [0, \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ 1 & x \in [\frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ \frac{(1-x)(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1]. \end{cases}$$

The key result (and also the difficult step) in order to prove this theorem is the following result, that we proved deeply using the abstract formulation for optimality conditions proved in [79], (see also Section 1.5).

Theorem 4.5. *For any $k \geq 1$, the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ has a solution h_k^* . This one has a graph that is a polygonal line composed of (at most) $k+1$ segments.*

In Section 4.3.1, From the optimality conditions we have information about the behavior of the Sturm-Liouville eigenfunctions on the vertex of the optimal function h_k^* . Thanks to this information we can improve the result above obtaining that h_1^* has the graph that is a polygonal line composed of (at most) 2 segments and for all $k \geq 2$ h_k^* has the graph that is a polygonal line composed of (at most) 3 segments. We then obtain the explicit formulas for the maximizers h_k^* from an analysis of the zeros of Bessel functions.

4.2 The maximization problem for $D(\Omega)^2\mu_k(\Omega)$ is ill posed

The aim of this section is to prove Theorem 4.3. In order to prove this theorem we need to analyze the asymptotic behavior of Neumann eigenvalues on collapsing domain. We start by defining the following quantity, that will be crucial in order to understand the asymptotic behavior of the Neumann eigenvalues

Definition 4.1. *Let $h \in L^\infty(0, 1)$ be a non negative function, then we define the following quantity*

$$\mu_1(h) = \inf \left\{ \frac{\int_0^1 (u')^2 h dx}{\int_0^1 u^2 h dx} \mid u \in H^1(0, 1) \text{ and } \int_0^1 u h dx = 0 \right\}.$$

Remark 3. *If we also assume that there exist $K > 0$ and $p < 2$ such that $h(x) \geq K(x(1-x))^p$ a. e. in $(0, 1)$, then $\mu_1(h)$ defined in Definition 4.1 is the first eigenvalue of the following eigenvalue problem (see [55])*

$$\begin{cases} -\frac{d}{dx} \left(h(x) \frac{du}{dx} \right) = \mu(h) h(x) u(x) & x \in (0, 1) \\ h(0) \frac{du}{dx}(0) = h(1) \frac{du}{dx}(1) = 0, \end{cases}$$

This is the case, in particular, for a concave (non negative) function h since such a function satisfies $h(x) \geq K(x(1-x))$ for a positive K .

We prove the following bound for the supremum of the eigenvalue $\mu_1(h)$ in a particular class of functions h .

Lemma 4.6. *For every $K > 0$ and $A > 0$, $K \leq A$, there exists a constant $C_1(K, A)$ such that the following holds*

$$\sup\{\mu_1(h) \mid \|h\|_{L^\infty(0,1)} \leq A \text{ and } \|h\|_{L^1(0,1)} \geq K\} \leq C_1(K, A).$$

Proof. We estimate from above the quantity $\mu_1(h)$ by using the following test function

$$\phi(x) = x - \frac{\int_0^1 th(t)dt}{\int_0^1 h(t)dt}.$$

From Definition 4.1 we obtain that, for every function h such that $0 \leq h(x) \leq A$ and $\int_0^1 h(x)dx \geq K$, the following inequality holds:

$$\mu_1(h) \leq \frac{\int_0^1 h(x)dx}{\int_0^1 \left(x - \frac{\int_0^1 th(t)dt}{\int_0^1 h(t)dt}\right)^2 h(x)dx}.$$

In particular we have that

$$\begin{aligned} & \sup\{\mu_1(h) \mid \|h\|_{L^\infty(0,1)} \leq A \text{ and } \|h\|_{L^1(0,1)} \geq K\} \leq \\ & \leq \sup\left\{ \frac{\int_0^1 h(t)dt}{\int_0^1 \left(x - \frac{\int_0^1 th(t)dt}{\int_0^1 h(t)dt}\right)^2 h(x)dx} \mid \|h\|_{L^\infty(0,1)} \leq A \text{ and } \|h\|_{L^1(0,1)} \geq K \right\}. \end{aligned} \quad (4.1)$$

Now from the constraint $\|h\|_{L^\infty(0,1)} \leq A$ on the admissible functions, we can extract a maximizing sequence (maximizing for the right hand side of (4.1)) with the following properties

$$h_n \xrightarrow{*} \bar{h} \text{ in } L^\infty(0,1) \quad (4.2)$$

and, moreover, from the constraint $\|h\|_{L^1(0,1)} \geq K$ we can conclude that $\|\bar{h}\|_{L^1(0,1)} \geq K$. It is straightforward to check that the functional

$$h \rightarrow \frac{\int_0^1 h(t)dt}{\int_0^1 \left(x - \frac{\int_0^1 th(t)dt}{\int_0^1 h(t)dt}\right)^2 h(x)dx}$$

is continuous under the convergence (4.2). In particular the maximization problem on the right hand side of (4.1) has a solution, this concludes the proof. \square

In the next two lemmas we explain the relations between the first Neumann eigenvalue and the quantity $\mu_1(h)$. In the first lemma we prove that, for every domain $\Omega \subset \mathbb{R}^d$, we can find a function h such that $\mu_1(\Omega) \leq \mu_1(h)$. In the second lemma we will prove that, for every function h , we can construct a sequence of domains Ω_ϵ such that $\mu_1(\Omega_\epsilon)$ approach $\mu_1(h)$.

Lemma 4.7. *Let Ω be an open and Lipschitz domain such that $D(\Omega) = 1$ and we assume also that the diameter of the set lies on the x_1 axis. Let h be the function defined in the following way:*

$$h(x_1) = \mathcal{H}^{d-1}(\{x' \in \mathbb{R}^{d-1} \mid (x_1, x') \in \Omega\}),$$

then

$$\mu_1(\Omega) \leq \mu_1(h).$$

Proof. From the variational characterization, using a test function that depends only on x_1 , we recover the variational characterization for the eigenvalue $\mu_1(h)$, indeed we have that

$$\mu_1(\Omega) \leq \inf \left\{ \frac{\int_0^1 (u')^2 h dx}{\int_0^1 u^2 h dx} \mid u \in H^1(0,1) \text{ and } \int_0^1 u h dx = 0 \right\} = \mu_1(h). \quad (4.3)$$

□

Now we are interested in a reverse inequality. For that purpose, we construct a sequences of domains that can approach the eigenvalue $\mu_1(h)$ for any h . More precisely given an $h \in L^\infty(0,1)$ we construct a sequence of collapsing domains $\Omega_{\epsilon h}$ such that when ϵ goes to zero, $\mu_1(\Omega_{\epsilon h})$ approaches $\mu_1(h^{d-1})$. Let $h \in L^\infty(0,1)$ be a non negative function (not identically zero), then we define the following axisymmetric domain:

$$\Omega_h := \{(x_1, x') \in \mathbb{R}^d \mid |x'| < h(x_1), \forall x_1 \in (0,1)\}.$$

Lemma 4.8. *Let $h \in L^\infty(0,1)$ be a non negative function (not identically zero), then*

$$\mu_1(\Omega_{\epsilon h}) \geq \mu_1(h^{d-1}) + f(\epsilon, h)$$

where $f(\epsilon, h) \rightarrow 0$ when $\epsilon \rightarrow 0$.

Proof. Let ω_{d-1} be the volume of the $d-1$ dimensional unit ball, then according to Lemma 4.7 we have that:

$$\mu_1(\Omega_{\epsilon h}) \leq \mu_1(\omega_{d-1} \epsilon^{d-1} h^{d-1}) = \mu_1(h^{d-1}),$$

the last equality is true because from Definition 4.1 the eigenvalue $\mu_1(h)$ is invariant under multiplication of h with a positive constant. In particular from Lemma 4.6 we conclude that there exists a constant $C_2(h)$, that depends on h but does not depend on ϵ , such that

$$\mu_1(\Omega_{\epsilon h}) \leq C_2(h).$$

Let u_ϵ be the eigenfunction associated to the eigenvalue $\mu_1(\Omega_{\epsilon h})$, normalized in such a way that $\|u_\epsilon\|_{L^2(\Omega_{\epsilon h})} = 1$, we introduce the following functions

$$\bar{u}_\epsilon(x_1, x') = \epsilon^{\frac{d-1}{2}} u_\epsilon(x_1, \epsilon x') \quad \forall (x_1, x') \in \Omega_h.$$

We want to prove that the sequence of functions \bar{u}_ϵ is bounded in $H^1(\Omega_h)$. We start by bound the quantity $\|\nabla \bar{u}_\epsilon\|_{L^2(\Omega_h)}$

$$\int_{\Omega_h} |\nabla \bar{u}_\epsilon|^2 dx = \epsilon^{d-1} \int_{\Omega_h} \left(\frac{\partial u_\epsilon}{\partial x_1} \right)^2 + \epsilon^2 |\nabla_{x'} u_\epsilon|^2 dx \leq \int_{\Omega_{\epsilon h}} |\nabla u_\epsilon|^2 dy = \mu_1(\Omega_{\epsilon h}) \leq C_2(h),$$

in the inequalities above we use the change of variables $y_1 = x_1$ and $y' = \epsilon x'$. Using the same change of variable we also conclude that $\|\bar{u}_\epsilon\|_{L^2(\Omega_h)} = \|u_\epsilon\|_{L^2(\Omega_{\epsilon h})} = 1$.

We conclude that there exists a function $\bar{U} \in H^1(\Omega_h)$ such that, up to a subsequence,

$$\begin{aligned} \bar{u}_\epsilon &\rightharpoonup \bar{U} \text{ in } H^1(\Omega_h) \\ \bar{u}_\epsilon &\rightarrow \bar{U} \text{ in } L^2(\Omega_h). \end{aligned} \quad (4.4)$$

We now prove that the limit function \bar{U} does not depend on x_i where $i = 2, \dots, d$, indeed we have that

$$\int_{\Omega_h} \left(\frac{\partial \bar{U}}{\partial x_i} \right)^2 dx \leq \liminf \int_{\Omega_h} \left(\frac{\partial \bar{u}_\epsilon}{\partial x_i} \right)^2 dx = \liminf \epsilon^2 \int_{\Omega_{\epsilon h}} \left(\frac{\partial u_\epsilon}{\partial x_i} \right)^2 dx = 0.$$

We denote by U the restriction of the function \bar{U} to the variable x_1 , from the convergences given in (4.4) we conclude that, for ϵ small enough we have that

$$\mu_1(\Omega_{\epsilon h}) = \frac{\int_{\Omega_{\epsilon h}} |\nabla u_\epsilon|^2 dx}{\int_{\Omega_{\epsilon h}} u_\epsilon^2 dx} \geq \frac{\int_{\Omega_h} |\nabla \bar{u}_\epsilon|^2 dx}{\int_{\Omega_h} \bar{u}_\epsilon^2 dx} \geq \frac{\int_0^1 U'^2 h^{d-1} dx}{\int_0^1 U^2 h^{d-1} dx} + f(\epsilon, h) \quad (4.5)$$

where $f(\epsilon, h)$ goes to 0 when $\epsilon \rightarrow 0$. From the fact that u_ϵ is a Neumann eigenfunction and from the convergences (4.4) it is clear that $\int_0^1 U h^{d-1} = 0$, from Definition 4.1 and from (4.5), using the function U as a test function for $\mu_1(h^{d-1})$ we obtain that

$$\mu_1(\Omega_{\epsilon h}) \geq \mu_1(h^{d-1}) + f(\epsilon, h).$$

□

From the two previous lemmas, we easily deduce

Proposition 4.9. *We have equality between the two quantities*

$$\begin{aligned} S_1 &:= \sup\{\mu_1(\Omega), \Omega \subset \mathbb{R}^d, \text{ bounded, Lipschitz}, D(\Omega) = 1\} \\ S_2 &:= \sup\{\mu_1(h), h \in L^\infty(0, 1), h \geq 0, h \neq 0\}. \end{aligned}$$

Now we are going to prove that the quantity S_2 is infinite. For that purpose, we want to construct a particular sequence of functions: let $0 < a < \frac{1}{2}$ and let denote w_a the first positive root of the following equation

$$\tan\left(x\left(\frac{1}{2} - a\right)\right) = 1 + \frac{1}{ax}. \quad (4.6)$$

It is straightforward to obtain the following estimates for w_a :

$$\frac{\pi}{4\left(\frac{1}{2} - a\right)} < w_a < \frac{\pi}{3\left(\frac{1}{2} - a\right)}. \quad (4.7)$$

Now we define the following function

$$g_a(x) = \begin{cases} e^{2w_a(x-a)} & 0 \leq x < a \\ 1 & a \leq x \leq 1-a \\ e^{2w_a(1-x-a)} & 1-a < x \leq 1. \end{cases}$$

We prove the following proposition

Proposition 4.10. *Let g_a be the function defined above, then $\mu_1(g_a) = w_a$ and therefore*

$$\lim_{a \rightarrow \frac{1}{2}} \mu_1(g_a) = +\infty$$

Proof. The function g_a is strictly positive, so we can identify the quantity $\mu_1(g_a)$ as the first non trivial eigenvalue of the following eigenvalue problem

$$\begin{cases} -\frac{d}{dx}\left(g_a(x)\frac{du}{dx}(x)\right) = \mu(g_a)g_a(x)u(x) & x \in (0, 1) \\ \frac{du}{dx}(0) = \frac{du}{dx}(1) = 0. \end{cases} \quad (4.8)$$

In order to simplify the notation we will define $\mu_a = \mu_1(g_a)$. We divide the interval $(0, \frac{1}{2}]$ in two intervals $(0, a]$ and $[a, \frac{1}{2}]$. We solve the equation in $(0, \frac{1}{2}]$ and, by symmetry of g_a , the associated eigenfunction u must be odd with respect to the point $x = \frac{1}{2}$.

1. In the interval $[a, \frac{1}{2}]$, knowing that $u(\frac{1}{2}) = 0$, we have that the eigenfunction, in this interval, must be of the form

$$u_r = \alpha \sin\left(\sqrt{\mu_a}\left(x - \frac{1}{2}\right)\right),$$

where $\alpha \in \mathbb{R}$.

2. In the region $(0, a]$ we have to solve the differential equation

$$\frac{d^2u}{dx^2} + 2w_a \frac{du}{dx} + \mu_a u = 0. \quad (4.9)$$

We have two possibilities, the first is when $\mu_a \neq w_a^2$ and the second is when $\mu_a = w_a^2$. We now continue the analysis assuming that the second case happens, we will exclude the first possibility later in the proof. So in the case when $\mu_a = w_a^2$, knowing that $u'(0) = 0$ we obtain that

$$u_l(x) = Ae^{-\sqrt{\mu_a}x}(1 + \sqrt{\mu_a}x),$$

where $A \in \mathbb{R}$.

We impose the conditions on α and A in order to have $u_l(a) = u_r(a)$ and $u'_l(a) = u'_r(a)$, so we need to solve the following system

$$\begin{cases} (1 + \sqrt{\mu_a}a)Ae^{-\sqrt{\mu_a}a} - \alpha \sin\left(\sqrt{\mu_a}\left(a - \frac{1}{2}\right)\right) = 0 \\ -a\mu_a e^{-\sqrt{\mu_a}a} - \alpha\sqrt{\mu_a} \cos\left(\sqrt{\mu_a}\left(a - \frac{1}{2}\right)\right) = 0. \end{cases}$$

Necessarily the determinant of this linear system is equal to zero, this gives us a transcendental equation, the first non trivial root of this equation will be the first eigenvalue μ_a . The system above has a solution if and only if

$$\tan\left(\sqrt{\mu_a}\left(a - \frac{1}{2}\right)\right) = 1 + \frac{1}{a\sqrt{\mu_a}}.$$

We recover Equation (4.6) confirming that, in the case, $\mu_a = w_a^2$ we have obtained an eigenfunction of Problem (4.8).

It remains to prove that we cannot find a smaller eigenvalue. Let us assume, for a contradiction, that there exists an eigenvalue $\mu_a < w_a^2$. Let $\rho_{1,2} = -w_a \pm \sqrt{w_a^2 - \mu_a}$ then the solution of the differential equation (4.9) in $(0, a]$, knowing that $u'(0) = 0$, must be of the form

$$u_r(x) = A(\rho_1 e^{\rho_1 x} - \rho_2 e^{\rho_2 x}),$$

where $A \in \mathbb{R}$. Imposing the condition that the eigenfunction must be \mathcal{C}^1 we obtain the following transcendental equation that μ_a must solve

$$\sqrt{\mu_a} \tan\left(\sqrt{\mu_a}\left(\frac{1}{2} - a\right)\right) = w_a + \frac{\sqrt{w_a^2 - \mu_a}}{\tanh(a\sqrt{w_a^2 - \mu_a})}. \quad (4.10)$$

Let us introduce the function

$$\phi(x) = x \tan\left(x\left(\frac{1}{2} - a\right)\right) - w_a - \frac{\sqrt{w_a^2 - x^2}}{\tanh(a\sqrt{w_a^2 - x^2})} < 0,$$

It is immediate to check that this function is strictly increasing in the interval $(0, w_a)$. Moreover, by definition of w_a , we have

$$\lim_{x \rightarrow w_a} \phi(x) = 0$$

thus, $\phi(x) < 0$ for all $x \in (0, w_a)$ showing that there no eigenvalue in the interval $(0, w_a)$. We have proved that $\mu_a = w_a^2$ and the last claim of the proposition comes from the estimate (4.7). \square

We are now ready to prove Theorem 4.3

Proof of Theorem 4.3. We fix a constant $M > 0$, from Lemma 4.10 we know that there exists $a < \frac{1}{2}$ such that

$$\mu_1(g_a) > M,$$

with $\|g_a\|_{L^\infty} = 1$ and $\|g_a\|_{L^1} > 1 - 2a$. From Lemma 4.8 we know that for ϵ small enough we have that

$$\mu_1(\Omega_{\epsilon(g_a)^{\frac{1}{d-1}}}) \geq \mu_1(g_a) + f(g_a, \epsilon) \geq M.$$

This concludes the proof because M is arbitrary and, by construction, for all ϵ the sets that we constructed have diameter equal to 1. \square

4.3 Maximization of $\mu_k(h)$

We consider the following class of functions

$$\mathcal{L} := \{h : [0, 1] \rightarrow [0, 1], h \text{ concave}, \max h = 1\}.$$

We have seen in Section 2 that the Sturm-Liouville eigenvalue problem

$$\begin{cases} -\frac{d}{dx}(h(x)\frac{du}{dx}(x)) = \mu(h)h(x)u(x) & x \in (0, 1) \\ h(0)\frac{du}{dx}(0) = h(1)\frac{du}{dx}(1) = 0, \end{cases} \quad (4.11)$$

is well-posed when h is concave (even if h vanishes at the extremities of the interval $[0, 1]$). We will denote by $\mu_k(h)$ the $k + 1$ -th eigenvalue of this problem ($\mu_0(h) = 0$ associated to constant eigenfunctions). Proposition 4.9 extends in a straightforward way and we can prove (see a more precise statement in Theorem 4.21):

$$\begin{aligned} \sup\{\mu_k(\Omega), \Omega \text{ plane convex domain}, D(\Omega) = 1\} = \\ = \sup\{\mu_k(h), h \text{ concave, non negative on } [0, 1]\}. \end{aligned}$$

Therefore, we are interested, in this section, in the maximization problem $\max\{\mu_k(h), h \in \mathcal{L}\}$. We recall two important properties for the class \mathcal{L} and the eigenvalues $\mu_k(h)$, see Lemma 3.1 and Lemma 3.6 in [55] (see also Chapter 3).

Proposition 4.11. • *For any sequence h_n in \mathcal{L} , there exists a subsequence (still denoted h_n) and a function $h \in \mathcal{L}$ such that h_n converge to h in $L^2(0, 1)$ and uniformly on every compact subset of $(0, 1)$.*

- *If $h_n \rightarrow h$ in $L^2(0, 1)$ then, for any k , $\mu_k(h_n) \rightarrow \mu_k(h)$.*

Let us give the main result of this section.

Theorem 4.12. *For any $k \geq 1$, the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ has a solution h_k^* , moreover*

- $\max\{\mu_1(h), h \in \mathcal{L}\} = \mu_1(h_1^*) = (2j_{0,1})^2$ and

$$h_1^* = \begin{cases} 2x & x \in [0, \frac{1}{2}], \\ 2(1-x) & x \in [\frac{1}{2}, 1] \end{cases}$$

- let $k \geq 2$ then $\max\{\mu_k(h), h \in \mathcal{L}\} = \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2$

$$h_k^* = \begin{cases} \frac{x(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [0, \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ 1 & x \in [\frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}] \\ \frac{(1-x)(2j_{0,1} + (k-1)\pi)}{j_{0,1}} & x \in [1 - \frac{j_{0,1}}{(2j_{0,1} + (k-1)\pi)}, 1]. \end{cases}$$

We prove this theorem in several steps, the main result in order to prove this result is the following theorem:

Theorem 4.13. *For any $k \geq 1$, the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ has a solution. This one has a graph that is a polygonal line composed of (at most) $k+1$ segments.*

The existence of a maximizer follows immediately from the compactness of the class \mathcal{L} and the continuity of the eigenvalues stated in Proposition 4.11. We will denote by h_k^* (or simply h^* if no confusion can occur) a maximizer.

For the qualitative properties of the maximizer, we will mainly use the optimality conditions but we have to take into account the concavity constraint and the bound constraints $0 \leq h(x) \leq 1$. Let us first give the derivative of the eigenvalue. Let us fix the notations, we consider $t > 0$ a positive number, and we define the following derivatives: for every $\phi \in \mathcal{H}$ we define $\mu_{t,\phi} := \mu_k(h + t\phi)$ and we denote by $u_{t,\phi}$ the corresponding eigenfunction. We use the following notation for the derivative of the eigenvalue (since this is a Sturm-Liouville problem, we know that the eigenvalue is simple and then differentiable):

$$\dot{\mu}_\phi := \left. \frac{d}{dt} \mu_k(h + t\phi) \right|_{t=0}$$

and the following notation for the derivative of the eigenfunction :

$$\dot{u}_\phi := \left. \frac{d}{dt} u_{t,\phi} \right|_{t=0}.$$

Lemma 4.14. *The derivative of μ_k in the direction ϕ is given by*

$$\dot{\mu}_\phi = \int_0^1 (u'^2 - \mu_k u^2) \phi dx \quad (4.12)$$

where u is the eigenfunction associated to $\mu_k(h)$ normalized by $\int_0^1 hu^2 = 1$.

Proof. Since this kind of perturbation is classical, see e.g. [58, section 5.7] we just perform a formal computation here, the complete justification would involve an implicit function theorem together with Fredholm alternative. Let us compute $\dot{\mu}_\phi$, from (4.11) we know that

$$\frac{d}{dt} \left[-\frac{d}{dx} \left((h + t\phi) \frac{du_{t,\phi}}{dx} \right) \right] \Big|_{t=0} = \frac{d}{dt} [\mu_{t,\phi}(h + t\phi) u_{t,\phi}] \Big|_{t=0},$$

so we obtain the following differential equation satisfied by \dot{u}_ϕ

$$-\frac{d}{dx} \left(h \frac{d\dot{u}_\phi}{dx} \right) - \frac{d}{dx} \left(\phi \frac{du}{dx} \right) = \dot{\mu}_\phi h u + \mu_k h \dot{u}_\phi + \mu_k \phi u. \quad (4.13)$$

Multiplying both side of the above equation by u and integrating we obtain

$$\int_0^1 h u' \dot{u}'_\phi - u h \dot{u}'_\phi \Big|_{x=0}^{x=1} + \int_0^1 \phi u'^2 - u \phi u'' \Big|_{x=0}^{x=1} = \dot{\mu}_\phi + \mu_k \int_0^1 \phi u^2 + \mu_k \int_0^1 h u \dot{u}_\phi, \quad (4.14)$$

where we have used $\int_0^1 h u^2 = 1$. Now from the variational formulation satisfied by u we see that

$$\int_0^1 h u' \dot{u}'_\phi = \mu_k \int_0^1 h u \dot{u}_\phi.$$

On the other hand, differentiating w.r.t. t the boundary condition $(h+t\phi)(0)u_{t,\phi}(0)$ yields

$$\phi(0)u(0) + h(0)\dot{u}_\phi(0) = 0 \quad \text{and} \quad \phi(1)u(1) + h(1)\dot{u}_\phi(1) = 0,$$

therefore the boundary terms cancel and formula (4.12) follows. \square

Let us now write the first order optimality condition, taking into account the concavity constraint (i.e. $h'' \leq 0$) and the bound constraints $0 \leq h \leq 1$. For that purpose, we follow the formalism of [79], (Proposition 2.3.2) see also the abstract framework in [82]:

Proposition 4.15. *Let h^* be a maximizer of $\mu_k(h)$ in the class $\mathcal{H}_.$. We denote by S the support of (the negative measure) h'' . Then there exist*

- a function $\xi \in H^1(0, 1)$, such that $\xi \geq 0$, $\xi = 0$ on S ,
- two non-negative Radon measures ν_0, ν_1 such that $\text{suppt}(\nu_0) \subset \{x|h^*(x) = 0\}$ and $\text{suppt}(\nu_1) \subset \{x|h^*(x) = 1\}$

such that, for any $\phi \in H^1(0, 1)$

$$\langle \dot{\mu}_\phi, \phi \rangle = - \langle \xi'', \phi \rangle + \int_0^1 \phi d\nu_0 - \int_0^1 \phi d\nu_1. \quad (4.15)$$

Note that, since h^* is non-negative and concave with $\max h^* = 1$ we have the following properties of the sets $\{x/h^*(x) = 0\}$ and $\{x/h^*(x) = 1\}$:

$$\{x/h^*(x) = 0\} \subset \{0\} \cup \{1\} \quad (4.16)$$

$$\{x/h^*(x) = 1\} \text{ is empty or an interval } [a_1, a_2] \text{ (with } 0 < a_1 \leq a_2 < 1). \quad (4.17)$$

Moreover, since the eigenvalue $\mu_k(h)$ is invariant when we multiply h by a positive constant, we can even assume, if needed, that the bound constraint $h^* = 1$ is not saturated.

Let us introduce the function $f := u'^2 - \mu_k u^2$, then, according to (4.12), $\langle \dot{\mu}_\phi, \phi \rangle = \int_0^1 f \phi dx$. From the ODE satisfied by the eigenfunction, we see that $u \in H^2(0, 1) \subset C^1([0, 1])$, therefore f is continuous on $[0, 1]$.

Taking ϕ compactly supported in $(0, a_1) \cup (a_2, 1)$ in (4.15) yields

$$\int_0^1 f \phi dx = - \langle \xi'', \phi \rangle$$

therefore, ξ satisfies, in the sense of distributions

$$-\xi'' = f \quad \text{on } (0, a_1) \cup (a_2, 1). \quad (4.18)$$

We also know that ξ vanishes on S the support of $h^{*''}$. This support being closed, its complement S^c is a union of intervals: $S^c = \bigcup_{i \in I} (\alpha_i, \beta_i)$. We will distinguish the internal intervals (those for which $\alpha_i > 0$ and $\beta_i < 1$) and the boundary intervals: it will be a consequence of the proof below that the number of internal intervals is finite, thus we will have only two boundary intervals that we will denote $(0, \beta_b)$ and $(\alpha_b, 1)$.

Let us start with the internal intervals, on such an interval (α_i, β_i) , ξ is solution of

$$\begin{cases} -\xi'' = f & \text{on } (\alpha_i, \beta_i) \\ \xi(\alpha_i) = \xi(\beta_i) = 0. \end{cases} \quad (4.19)$$

We infer:

Lemma 4.16. *The function f vanishes at least two times on each internal interval (α_i, β_i) .*

Proof. Let us denote $(\alpha, \beta) = (\alpha_i, \beta_i)$. From equation (4.19) we can get an explicit expression of ξ on the interval (α, β) , namely

$$\xi(x) = \frac{x - \alpha}{\beta - \alpha} \int_{\alpha}^{\beta} (\beta - t) f(t) dt - \int_{\alpha}^x (x - t) f(t) dt. \quad (4.20)$$

Now, since f is continuous, the function ξ belongs to $H^2((0, a_1) \cup (a_2, 1))$ and then it is C^1 (and even C^2). But ξ being non-negative on $(0, a_1) \cup (a_2, 1)$, we necessarily have $\xi'(\alpha) = \xi'(\beta) = 0$. With the explicit expression (4.20), we get $\xi'(x) = \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} (\beta - t) f(t) dt - \int_{\alpha}^x f(t) dt$ thus, this provides the two relations

$$\int_{\alpha}^{\beta} (\beta - t) f(t) dt = 0 \quad \int_{\alpha}^{\beta} f(t) dt = 0$$

or

$$\int_{\alpha}^{\beta} f(t) dt = 0 \quad \int_{\alpha}^{\beta} t f(t) dt = 0. \quad (4.21)$$

These two relations imply the thesis. Indeed from the first one f must vanish at least one time, say at $x = \gamma$. Assuming, that f vanishes only at γ would provide $\int_{\alpha}^{\beta} (t - \gamma) f(t) dt \neq 0$ a contradiction. \square

In order to get the result announced in Theorem 4.13, we need to prove that the complement S^c of the support of $h^{*''}$ has only $k + 1$ components (including possibly the interval (a_1, a_2) if it is not empty and two boundary intervals). For that purpose, we study the function f on any nodal domain of u . We distinguish here the "internal" nodal intervals, those where u vanishes at the two extremities and the two "boundary" nodal intervals (where u vanishes only at one extremity). The key proposition is the following:

Proposition 4.17. *The function f vanishes exactly two times on an internal nodal interval of u and exactly one time on a boundary nodal interval.*

Since u is the $k + 1$ -th eigenfunction of a Sturm-Liouville problem, its number of nodal domains is exactly $k + 1$ ($k - 1$ internal plus two boundary nodal intervals). Following Proposition 4.17, it implies that the function f has exactly $2k$ zeros.

Proof of Proposition 4.17. In the proof, the optimal function h^* will be simply denoted h and the eigenvalue is μ . Let us consider first an internal nodal interval $[a, b]$ of the eigenfunction u . Without loss of generality, we can assume that $u > 0$ on (a, b) and we also assume that $(a, b) \subset (0, a_1)$ (i.e. this nodal interval is before the maximum of h^* (where possibly $h^* = 1$). For the other case, $(a, b) \subset [a_2, 1)$, the proof will follow the same lines. Therefore, on this interval h' is decreasing (by concavity) and h is increasing, in particular, $h'/h > 0$. The differential equation satisfied by u can be written

$$u'' = -\frac{h'}{h}u' - \mu u. \quad (4.22)$$

This shows that u is $W_{loc}^{2,\infty}$ in $(0, 1)$. More precisely, u'' can be discontinuous at discontinuity points of h' (except if u' vanishes at these points).

1st step: u is unimodal on (a, b) in the sense that u is increasing on some interval (a, c) and then decreasing on (c, b) .

Let us consider a point c where u achieves its maximum on $[a, b]$. We prove now that u is decreasing after c . Let us assume, for a contradiction, that there is a point $x_0 \geq c$ with $u'(x_0) > 0$. Let us introduce the point

$$x_1 := \max\{x \in [a, x_0], u'(x) \leq 0\}.$$

By continuity, we have $u'(x_1) = 0$ and by definition of x_1 , we have $u' > 0$ on $(x_1, x_0]$. Therefore, using the equation (4.22), we see that $u'' < 0$ on $[x_1, x_0]$ yielding $0 < u'(x_0) \leq u'(x_1) = 0$ a contradiction.

In the same way, we prove by contradiction that u is increasing on $[a, c]$ assuming that there exists $x_0 < c$ such that $u'(x_0) < 0$ and introducing

$$x_1 := \max\{x \in [x_0, c], u'(x) < 0\},$$

we get $u'(x_1) = 0$, $u' \geq 0$ on $[x_1, c]$, therefore $u'' < 0$ on (x_1, c) providing a contradiction since $u'(c) = u'(x_1) = 0$.

Another way to express these properties is the following: when $u > 0$, the only points where u' vanishes correspond to (local) maxima, since, by the equation $u'' < 0$, there.

Now let us remark that $f(a) = u'(a)^2 \geq 0$, $f(c) = -\mu u^2(c) < 0$ and $f(b) = u'(b)^2 \geq 0$, thus f vanishes at least once between a and c and at least once between c and b . Thus we need to prove:

2nd step: The function f cannot vanish more than one time on each interval $[a, c]$ and $[c, b]$.

On the first interval $[a, c]$ it is clear: since $u' > 0$ there, we have by (4.22) $u'' < 0$ on (a, c) . Now f' being given by

$$f' = 2u'(u'' - \mu u) = -2u'\left(\frac{h'}{h}u' + 2\mu u\right), \quad (4.23)$$

we see that $f' < 0$ on (a, c) and therefore f vanishes only once.

Now we look at the interval (c, b) . Since $f(c) < 0$ and $f(b) > 0$ f vanishes an odd number of time between c and b . Assume, for a contradiction, that f vanishes at (least) three times and take three consecutive zeros x_1, x_2, x_3 . Writing $f = (u' + \sqrt{\mu}u)(u' - \sqrt{\mu}u)$ and observing that $u' - \sqrt{\mu}u < 0$ on (c, b) , we infer that the function $g = u' + \sqrt{\mu}u$ vanishes at the same points x_1, x_2, x_3 . Let us also observe that, since h is positive and increasing while h is decreasing by concavity, the function h'/h is decreasing on (a, b) .

Let us now introduce the function $\varphi := u/u'$. By definition, $\varphi < 0$ on the interval (c, b) . The differential equation satisfied by φ is

$$\varphi' = 1 - \frac{uu''}{u'^2} = 1 + \frac{h'}{h} \varphi + \mu\varphi^2. \quad (4.24)$$

Now, $g(x_i) = 0$ implies $\varphi(x_1) = \varphi(x_2) = \varphi(x_3) = -1/\sqrt{\mu}$. Therefore, there exist two points $y_1 \in (x_1, x_2)$ and $y_2 \in (x_2, x_3)$ such that $\varphi'(y_1) = \varphi'(y_2) = 0$. Moreover, we can assume that $|\varphi| < 1/\sqrt{\mu}$ on (x_1, x_2) (and $|\varphi| > 1/\sqrt{\mu}$ on (x_2, x_3)). Coming back to (4.24), we see that $\varphi(y_1)$ and $\varphi(y_2)$ must be solutions of the quadratic

$$1 + \frac{h'}{h} X + \mu X^2 = 0.$$

This implies that

$$\frac{h'}{h}(y_1) \geq \frac{h'}{h}(y_2) \geq 2\sqrt{\mu}.$$

Now, let us introduce $z_1 \in (x_1, x_2)$ a maximum of f on this interval. From $f'(z_1) = 0$, we infer $u''(z_1) = \mu u(z_1)$ and therefore

$$\varphi'(z_1) = 1 - \frac{\mu u^2(z_1)}{u'(z_1)^2} = 1 - \mu\varphi(z_1)^2$$

that yields with (4.24)

$$\frac{h'}{h}(z_1) = -2\mu\varphi(z_1). \quad (4.25)$$

More precisely, we see that

$$f' \leq 0 \Leftrightarrow \frac{h'}{h} \geq -2\mu\varphi$$

now, by (4.25) and the fact that $|\varphi| < 1/\sqrt{\mu}$ on (x_1, x_2) , we see that $f'(y_1) \leq 0$ thus $y_1 \geq z_1$. We use now the fact that h'/h is decreasing to write

$$2\sqrt{\mu} < \frac{h'}{h}(y_1) \leq \frac{h'}{h}(z_1) = -2\mu\varphi(z_1) < 2\mu \frac{1}{\sqrt{\mu}}$$

yielding a contradiction.

It remains to look at a nodal interval of the kind $I = [0, a_1]$. For that purpose, we need the following lemma.

Lemma 4.18. *let h_k^* be a maximizer for the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$ then $h_k^*(0) = h_k^*(1) = 0$.*

Proof. In the proof, the optimal function h_k^* will be simply denoted by h , the eigenvalue is μ and the eigenfunction is u . Suppose by contradiction that $h(0) > 0$, let $\eta \in (0, 1)$ then we define the following function

$$\phi_\eta = \begin{cases} x - \eta & x \in [0, \eta) \\ 0 & x \in [\eta, 1]. \end{cases}$$

for η small enough we have that the function $h + t\phi_\eta$ is in \mathcal{L} for t small enough (we recall that we can assume $h < 1$ because $\mu(h)$ is invariant by scaling) in particular we can compute the derivative of the eigenvalue, and we obtain that

$$\dot{\mu}_{\phi_\eta} = \int_0^1 (u'^2 - \mu_k u^2) \phi_\eta dx = 0. \quad (4.26)$$

From the fact that $h(0) > 0$ and the boundary conditions we know that $u'(0) = 0$. Now assume that $u(0) \neq 0$, so there exists η_1 for which $u'^2 - \mu_k u^2 < 0$ in $[0, \eta_1]$ and this will imply that

$$\dot{\mu}\phi_{\eta_1} = \int_0^1 (u'^2 - \mu_k u^2)\phi_{\eta_1} dx > 0, \quad (4.27)$$

that is a contradiction, so we conclude that $u(0) = 0$. In particular the eigenfunction u must solve the following Cauchy problem

$$\begin{cases} u'' + \frac{h'}{h}u' + \mu u = 0 & x \in [0, \eta_1) \\ u(0) = 0 \\ u'(0) = 0. \end{cases}$$

this will imply that $u \equiv 0$ in an interval of positive measure, that is a contradiction. We can adapt the same argument in order to prove that $h(1) = 0$. \square

We come back to the analysis of $f = u'^2 - \mu u^2$ on the first nodal interval $[0, a_1]$. Since $h(0) = 0$, we can prove that $u'(0) = 0$. Indeed, using the representation of the eigenfunction given in [99], namely

$$u(x) = \mu \int_0^1 g(x, y)h(y)u(y)dy \quad \text{where } g(x, y) = \int_0^{\min(x, y)} \frac{t dt}{h(t)} + \int_{\max(x, y)}^1 \frac{(1-t) dt}{h(t)}$$

we get by differentiation

$$u'(x) = \mu \left(\frac{x}{h(x)} \int_x^1 h(y)u(y)dy - \frac{1-x}{h(x)} \int_0^x h(y)u(y)dy \right).$$

Since h is concave (non identically zero) and $h(0) = 0$, we have $h(x) \geq cx$ for some positive c in a neighborhood of 0, therefore passing to the limit in the expression above, and using $\int_0^1 h(y)u(y)dy = 0$, we get $u'(0) = 0$. We infer $f(0) < 0$, $f(a_1) > 0$ and we can follow exactly the proof of the general case to obtain that f vanishes only once in the interval $[0, a_1]$. The same holds for the last nodal interval. \square

We are now ready to prove Theorem 4.13

Proof of Theorem 4.13. we have seen that the function f vanishes exactly $2k$ times on $(0, 1)$. Moreover, according to Lemma 4.16, f vanishes at least two times on each internal interval contained in the complement of the support of the measure h'' . Therefore, there are at most k such internal intervals denoted (α_i, β_i) , $i = 1, \dots, M-1$, $M \leq k+1$ and possibly two boundary intervals $(0, x_1)$ and $(x_M, 1)$. Let us first prove that the support S is discrete and contains only the extremities of the previous intervals, namely the x_i . Assume, for a contradiction that, for some i , $\beta_i < \alpha_{i+1}$. In that case, the interval $J = [\beta_i, \alpha_{i+1}]$ is contained in the support S and, according to the optimality condition given in Proposition 4.15, we have

$$u'^2 - \mu u^2 = 0 \text{ on } J \implies u' = \pm \sqrt{\mu}u \text{ on } J.$$

This implies immediately that $u'' = \mu u$ on J and coming back to the equation satisfied by the eigenfunction, we infer $h'/h = \pm 2\sqrt{\mu}$ (on a subinterval J' where u does not vanish), or $h(x) = \exp(2\sqrt{\mu}x)$ on J' , but this is impossible since we assumed h to be concave. We have proved that the support of h'' is the discrete set $S = \{0, x_1, \dots, x_M, 1\}$ and then h is affine on each interval (x_i, x_{i+1}) (we set $x_0 = 0$ and $x_{M+1} = 1$).

Now we are going to prove that the function f vanishes at least once on each boundary interval $(0, x_1)$ and $(x_M, 1)$. For that purpose, we still use the optimality condition given in Proposition 4.15 on this interval. We have now to take into account the non-negative Radon measure ν_0 . Since its support is restricted to the two points 0 and 1 (as a consequence of Lemma 4.18), it can be written as a sum of two Dirac measures: $\nu_0 = t_0\delta_0 + t_1\delta_1$ with $t_0 \geq 0$ and $t_1 \geq 0$. Now coming back to the equation (4.15) satisfied by ξ , we get for any $\phi \in H^1(0, 1)$:

$$\int_0^1 f\phi = \int_0^1 \xi'\phi' + t_0\phi(0) + t_1\phi(1).$$

This implies, choosing a ϕ that vanishes on $[x_1, 1]$:

$$\begin{cases} \xi'' = -f & \text{in } (0, x_1) \\ \xi'(0) = t_0 \\ \xi'(x_1) = 0 \end{cases}$$

We already know that $f(0) = -\mu u^2(0) \leq 0$, thus if f does not change sign on $(0, x_1)$, we would have ξ convex, but this is incompatible with the two boundary conditions since $\xi'(0) = t_0 \geq 0$. Therefore, f must vanish on this interval $(0, x_1)$. The same holds true on $(x_M, 1)$. Now if we count the number of zeros of f , seen from the side of these intervals, we have *at least* $1 + 1$ (for each boundary interval $(0, x_1)$ and $(x_M, 1)$) and $2(M - 1)$ for all internal intervals, that makes at least $2M$ zeros. But, we know from Proposition 4.17 that the number of zeros of f is exactly $2k$, therefore $M \leq k$ and the number of segments in the graph of h^* is at most $k + 1$. \square

In order to prove Theorem 4.12 we need to prove two lemmas about the behavior of the Sturm-Liouville eigenfunctions on the vertex of the optimal function h_k^* .

Lemma 4.19. *Let h_k^* be a maximizer for the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$, let u_k be the eigenfunction associated to $\mu_k(h_k^*)$, and let $\{x_0, x_1, \dots, x_m, x_{m+1}\} = \text{suppt}(h_k^{*\prime\prime})$ then:*

1. $u_k(x_i)u'(x_i) = 0$ for all $i = 0, \dots, m + 1$,
2. if $h_k^{*\prime} > 0$ in (x_i, x_{i+1}) then $u'(x_i) = 0$ and if $h_k^{*\prime} < 0$ in (x_i, x_{i+1}) then $u'(x_{i+1}) = 0$

Proof. In the proof, the optimal function h_k^* will be simply denoted by h , the eigenvalue is μ and the eigenfunction by u . From the previous proof, we already know that $u'(x_0) = u'(0) = 0$ and the same at the point $x_{m+1} = 1$. Now from the optimality conditions (4.21), and recalling that h must be a piecewise linear function, we obtain that

$$\int_{x_i}^{x_{i+1}} (u'^2 - \mu_k u^2) h dx = -h(x_{i+1})u(x_{i+1})u'(x_{i+1}) + h(x_i)u(x_i)u'(x_i) = 0.$$

Now we conclude by induction, indeed from the relations above, starting from $x_0 = 0$ and x_1 we obtain that $u(x_2)u'(x_2) = 0$, because $h(x_2) > 0$ (we know that $x_2 < x_{m+1} = 1$). Repeating this argument for all the intervals we conclude that $u(x_i)u'(x_i) = 0$ for all $i < m$, this concludes the first part of the proof.

We now prove the second part of the Lemma. We already know that $u'(0) = 0$ and $u'(1) = 0$. Now we consider the interval $[x_i, x_{i+1}]$ with $0 < x_i < x_{i+1} < 1$ and we assume that $h' > 0$ on this interval, so we assume that we are in the increasing part of the function h . We know that $u(x_i)u'(x_i) = 0$, we prove that $u(x_i) \neq 0$. We integrate by parts the

optimality condition $\int_{x_i}^{x_{i+1}} (u'^2 - \mu_k u^2) dx = 0$ and we use the relation $u'' = -\frac{h'}{h}u' - \mu u$ (we recall that $h > 0$) in order to obtain the following equality

$$\int_{x_i}^{x_{i+1}} u' \frac{u}{h} dx = 0. \quad (4.28)$$

We integrate by parts the relation above and we obtain

$$\frac{u^2(x_{i+1})}{h(x_{i+1})} - \frac{u^2(x_i)}{h(x_i)} - \int_{x_i}^{x_{i+1}} u \frac{u'h - uh'}{h^2} dx = 0,$$

using (4.28) we obtain that

$$\int_{x_i}^{x_{i+1}} u^2 \frac{h'}{h^2} dx = \frac{u^2(x_i)}{h(x_i)} - \frac{u^2(x_{i+1})}{h(x_{i+1})}, \quad (4.29)$$

we know that $h' > 0$ in $[x_i, x_{i+1}]$ so we conclude that $u(x_i)$ must be different from zero. We can adapt the same proof in the decreasing part of h , in this case we will have that $h' < 0$ and we conclude that $u(x_{i+1})$ must be different from zero. \square

Lemma 4.20. *Let h_k^* be a maximizer for the problem $\max\{\mu_k(h), h \in \mathcal{L}\}$, let u_k be the eigenfunction associated to $\mu_k(h_k^*)$, and let $\{x_0, x_1, \dots, x_m, x_{m+1}\} = \text{suppt}(h_k^{*\prime\prime})$ then $u_k(x_i) = 0$ for all $i = 0, \dots, m+1$.*

Proof. In the proof, the optimal function h_k^* will be simply denoted by h , the eigenvalue is μ and the eigenfunction by u . Suppose by contradiction that $u(x_i) \neq 0$, from the point 1 in Lemma 4.19 we know that $u'(x_i) = 0$. This means that there exists an interval $[a_i, b_i]$ s. t. $u'^2 - \mu u(x) < 0$ inside the interval.

We choose a function $\phi \leq 0$ s. t. $\text{Suppt}(\phi) \subset [a_i, b_i]$ and $h + t\phi \in \mathcal{L}$, from Lemma 4.12 we obtain that

$$\dot{\mu}_\phi = \int_0^1 (u'^2 - \mu_k u^2) \phi dx = \int_{a_i}^{b_i} (u'^2 - \mu_k u^2) \phi dx > 0$$

that is a contradiction with the assumption that h is a maximizer. \square

We are now ready to prove Theorem 4.12

Proof of Theorem 4.12. In the proof, the optimal function h_k^* will be simply denoted by h , the eigenvalue is μ and the eigenfunction by u . The existence follow directly from Proposition 4.11. We start by analyzing the case $k = 1$, in this case, from Theorem 4.13, we know that h can be or a constant function or a function that has a graph given by 2 segments. From Lemma 4.18 we now that h cannot be constant, otherwise it must be constantly equal to zero. We have that h must be of the following form

$$h = \begin{cases} \frac{x}{x_a} & x \in [0, x_a], \\ \frac{1-x}{1-x_a} & x \in [x_a, 1]. \end{cases}$$

We introduce the following parameter w that depends on x_a such that $w^2 = \mu$. The explicit expression of the eigenfunction u in the interval $[0, x_a]$ and in the interval $[x_a, 1]$ is known (see [55]). In particular there exist two constants A_1 and B_1 such that the eigenfunction u must have the following form

$$u = \begin{cases} A_1 J_0(w(x_a)x) & x \in [0, x_a] \\ B_1 J_0(w(x_a)(1-x)) & x \in [x_a, 1], \end{cases} \quad (4.30)$$

where J_0 is the Bessel function with parameter 0. From Lemma 4.20 we know that w and x_a must be linked by the following conditions

$$\begin{aligned} wx_a &= j_{0,m_1} & m_1 &\in \mathbb{N} \\ w(1-x_a) &= j_{0,m_2} & m_2 &\in \mathbb{N} \\ w &= j_{0,m_1} + j_{0,m_2}, \end{aligned}$$

where $j_{0,m}$ is the m -th zero of the Bessel function J_0 . We have that $u(x_a) = 0$ and with the expression of u given in (4.30) we see that u vanishes at least $m_1 - 1$ times in $(0, x_a)$ and at least $m_2 - 1$ times in $(x_a, 1)$. The only possibility, in order not to contradict Courant nodal domain Theorem is $m_1 = 1$ and $m_2 = 1$. We obtain the following

$$w = 2j_{0,1},$$

and this gives that $x_a = \frac{1}{2}$, this concludes the case $k = 1$.

We study the case where $k \geq 2$. From Theorem 4.13 we know that h has a graph that is a polygonal line composed of (at most) $k + 1$ segments. From the two lemmas 4.19 and 4.20 we can conclude that h must have the graph that is a polygonal line composed of (at most) 3 segments. Indeed if we have more than 3 segments this will imply that there exists a segment that is not the plateau and it is neither the first segment nor the last. Combining the second point of Lemma 4.19 and Lemma 4.20 we conclude that there exists a point x_i s. t. $u(x_i) = u'(x_i) = 0$ and this is a contradiction with the fact that u is a Sturm-Liouville eigenfunction.

This means that the graph of h is composed by 2 segments or 2 segments plus a central plateau, we start by analyzing this second case. Let $x_a < x_b$ we know that h must be of the following form

$$h = \begin{cases} \frac{x}{x_a} & x \in [0, x_a] \\ 1 & x \in [x_a, x_b] \\ \frac{(1-x)}{1-x_b} & x \in [x_b, 1]. \end{cases}$$

In order to simplify the notation we will define $a = x_a$ and $b = 1 - x_b$.

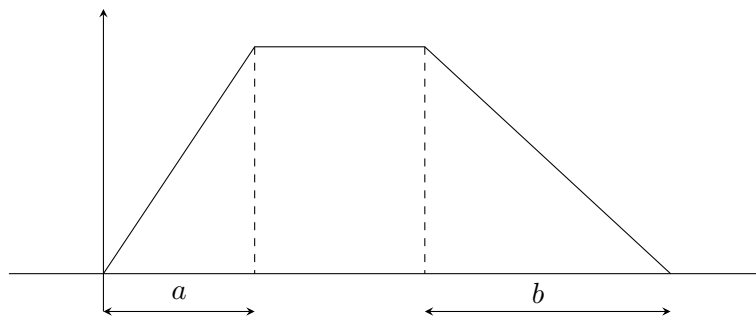


Figure 4.1: Shape of the function h , with respect the quantities a and b .

We introduce the following parameter $w = w(a, b)$ that depends on a and b and such that $w^2 = \mu$. The idea of the proof is to give an implicit formula for the parameter w and then, using the optimality conditions combined with an analysis of the nodal domains of the eigenfunctions u , we will find that only one triplet of values for w , a and b is possible. The explicit expression of the eigenfunction u in each interval $[0, a]$, $[a, b]$ and $[b, 1]$ is known (we recover the Bessel equation on $(0, a)$ and $(b, 1)$ see [55]). Let J_α be the Bessel

function with parameter α , we know that there exist four constants A_1 , B_1 , B_2 and C_1 such that the eigenfunction u has the following form

$$u = \begin{cases} A_1 J_0(wx) & x \in [0, x_a] \\ B_1 \cos(wx) + B_2 \sin(wx) & x \in [x_a, x_b] \\ C_1 J_0(w(1-x)) & x \in [x_b, 1]. \end{cases} \quad (4.31)$$

In order to find an implicit formula for w we proceed in a classical way, we know that the eigenfunction is C^1 , we write the compatibility condition and this gives a homogeneous linear system that the constants A_1 , B_1 , B_2 and C_1 must satisfy. The parameters w are exactly the parameters for which the homogeneous linear system has a solution, so are the parameters for which the determinant of the homogeneous linear system is zero. After a straightforward computation (recalling that $J'_0 = -J_1$) we find the following transcendental equation that w must solve

$$\tan(w(1-a-b)) = \frac{J_1(wa)J_0(wb) + J_1(wb)J_0(wa)}{J_1(wa)J_1(wb) - J_0(wb)J_0(wa)}. \quad (4.32)$$

From Lemma 4.20 we conclude that

$$\begin{aligned} wa &= j_{0,m_1} & m_1 &\in \mathbb{N} \\ wb &= j_{0,m_2} & m_2 &\in \mathbb{N}, \end{aligned}$$

where $j_{0,m}$ is the m -th zero of the Bessel function J_0 . In this case we have m_1 nodal interval in the interval $[0, x_a]$ and m_2 nodal intervals in the interval $[x_b, 1]$. From the transcendental equation (4.32) we obtain an explicit condition that w must satisfy, indeed we have that $\tan(w - j_{0,m_1} - j_{0,m_2}) = 0$, so we obtain

$$w = j_{0,m_1} + j_{0,m_2} + l\pi \quad l \in \mathbb{Z}.$$

If $l < 0$ then $a + b > 1$ and this is not possible. If $l = 0$ we have that $a + b = 1$, but this is a contradiction with the fact that $x_a < x_b$. We finally obtain that

$$w = j_{0,m_1} + j_{0,m_2} + l\pi \quad l \in \mathbb{N} \setminus \{0\}.$$

We analyze the behavior of the eigenfunction in the interval $[x_a, x_b]$, using the fact that $u(x_a) = u(x_b) = 0$ and the formula (4.31) we obtain that

$$u(x) = \frac{B_2}{\cos(j_{0,m_1})} (\sin(wx - j_{0,m_1})).$$

It is clear that $wx_a - j_{0,m_1} = 0$ and $wx_b - j_{0,m_1} = l\pi$, this means that the eigenfunction u has l nodal domains in the interval (x_a, x_b) . We finally have that the eigenfunction u has m_1 nodal interval in the interval $[0, x_a]$, m_2 nodal intervals in the interval $[x_b, 1]$ and l nodal domains in the interval (x_a, x_b) , from Courant Theorem on nodal intervals we have that u has exactly $k + 1$ nodal interval (Sturm-Liouville eigenfunctions are Courant sharp) so we conclude that

$$m_1 + m_2 + l = k + 1.$$

From the relation above we obtain the following expression for the parameter w ,

$$w = j_{0,m_1} + j_{0,m_2} + (k + 1 - m_1 - m_2)\pi.$$

We know that two consecutive zeros of the Bessel function J_0 always have a difference less than π (see [104]), we conclude that the optimal m_1 and m_2 in order to maximize w are $m_1 = m_2 = 1$ and we obtain the following competitor

$$w_1 = 2j_{0,1} + (k-1)\pi.$$

We now study the case when $x_a = x_b$, in this case, using the same argument we used in order to prove the case $k = 1$, we conclude that the parameter w is given by the following formula

$$w = j_{0,m_1} + j_{0,m_2}, \text{ with } m_1 + m_2 = k + 1,$$

From the fact that two consecutive zeros of Bessel functions have a difference less than π (see [104]), we conclude that for all m_1 and m_2 s. t. $m_1 + m_2 = k + 1$ the following inequality holds

$$w = j_{0,m_1} + j_{0,m_2} \leq 2j_{0,1} + (k-1)\pi.$$

this concludes the proof. \square

4.3.1 Sharp upper bounds for $D(\Omega)^2\mu_k(\Omega)$ for plane convex domains

Let $\Omega \subset \mathbb{R}^2$ be a convex domain, we prove sharp upper bounds for the quantity $D(\Omega)^2\mu_k(\Omega)$. Before presenting the results we introduce the class of thin domains, that are the sequences of sets that provide an asymptotic equality in the upper bounds that we give below. Given two functions $h^- \in \mathcal{L}$ and $h^+ \in \mathcal{L}$, let $h = h^+ + h^-$ we define the class of thin domains $T_{h,\epsilon}$ in the following way:

$$T_{h,\epsilon} = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -\epsilon h^-(x) \leq y \leq \epsilon h^+(x)\}. \quad (4.33)$$

As an application of Theorem 4.12 we give an alternative proof of Kröger inequalities ([74])

Theorem 4.21. *For every $\Omega \subset \mathbb{R}^2$ convex*

$$D(\Omega)^2\mu_k(\Omega) \leq \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2. \quad (4.34)$$

The equality is achieved by the sequence of thin convex domains $T_{h_k^,\epsilon}$ when ϵ goes to zero*

Proof. The quantity $D(\Omega)^2\mu_k(\Omega)$ is scale invariant, so we can assume that $D(\Omega) = 1$. Let $h^- \in \mathcal{L}$ and $h^+ \in \mathcal{L}$ such that we can parametrize the convex set Ω in the following way

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, -h^-(x) \leq y \leq h^+(x)\}.$$

We define $h = h^+ + h^-$, it is straightforward to check (see Lemma 4.7 and (4.3)) that

$$\mu_k(\Omega) \leq \mu_k(h).$$

Now from Theorem 4.12 we know that there exists a concave function h_k^* such that:

$$\max\{\mu_k(h), h \in \mathcal{L}\} = \mu_k(h_k^*) = (2j_{0,1} + (k-1)\pi)^2.$$

We end the proof by noticing that (see Lemma 4.7 and Lemma 4.8) $\mu_k(T_{h_k^*,\epsilon}) \rightarrow \mu_k(h_k^*)$ when $\epsilon \rightarrow 0$. \square

Chapter 5

Mixed Steklov-Dirichlet Problem

In this chapter we present the results that we achieved in [83].

5.1 Introduction

We start by recalling the definition of the mixed Steklov-Dirichlet eigenvalues, see Section 1.6 for more details. Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, connected set with Lipschitz boundary. Let $\Gamma_S \subset \partial\Omega$ be a relative open submanifold with Lipschitz boundary and we define also $\Gamma_D = \partial\Omega \setminus \Gamma_S$. We consider the following mixed Steklov-Dirichlet eigenvalue problem:

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = \lambda(\Omega, \Gamma_S)u & \Gamma_S \\ u = 0 & \Gamma_D, \end{cases} \quad (5.1)$$

where ν stands for the outer unit normal. It is known (see [2]) that the Steklov-Dirichlet eigenvalue problem (5.1) has a discrete spectrum $\{\lambda_k(\Omega, \Gamma_S)\}_{k=1}^\infty$

$$0 < \lambda_1(\Omega, \Gamma_S) \leq \lambda_2(\Omega, \Gamma_S) \leq \lambda_3(\Omega, \Gamma_S) \leq \dots \rightarrow +\infty,$$

and the eigenvalues admit the following variational characterization:

$$\lambda_k(\Omega, \Gamma_S) = \inf_{V_k \subset H_0^1(\Omega, \Gamma_S)} \sup_{v \in V_k} \frac{\int_\Omega |\nabla v|^2 dx}{\int_{\Gamma_S} v^2 d\mathcal{H}^{d-1}}, \quad (5.2)$$

where the infimum is taken over all k -dimensional subspaces V_k of the space $H_0^1(\Omega, \Gamma_S) = \{v \in H^1(\Omega) \mid v \equiv 0 \text{ on } \Gamma_D\}$, where the equality on the boundary is intended in the sense of trace. We denote the corresponding eigenfunctions by $\{u_k\}_{k=1}^\infty$ and we know that the following relation holds

$$\lambda_k(\Omega, \Gamma_S) = \frac{\int_\Omega |\nabla u_k|^2 dx}{\int_{\Gamma_S} u_k^2 d\mathcal{H}^{d-1}}.$$

The main purpose of this chapter is to study the dependence of the eigenvalues $\lambda_k(\Omega, \Gamma_S)$ with respect to $\Gamma_S \subset \partial\Omega$.

This kind of mixed eigenvalues has been deeply studied. For instance in [51] bounds for the Riesz mean has been obtained, in [10] the authors obtained inequalities between Steklov-Dirichlet eigenvalues and Steklov-Neumann eigenvalues, in [80] the authors proved a two terms asymptotic formula and in [41, 39, 101] optimization of the first Steklov-Dirichlet eigenvalue on doubly connected domains has been studied.

The Steklov-Dirichlet eigenvalues and eigenfunctions are used to model some important physical process (see [80, 10, 19]) e. g. they describe the stationary heat distribution in Ω when the flux through Γ_S is proportional to the temperature itself and the part Γ_D is kept under zero temperature. The boundary value problem (5.1) has also interesting probabilistic interpretation (see [10, 9]).

We now describe the structure of the chapter and the main results. In Section 5.2 we recall some useful results about the regularity of solutions of mixed boundary value problems, in particular we prove that the Steklov-Dirichlet eigenfunctions are Besov functions.

In Section 5.3 we prove a stability result for the Steklov-Dirichlet eigenvalues. More precisely we prove the following theorem (see Theorem 5.7 in order to have more information about the constants C_1 and C_2)

Theorem 5.1. *Let Ω be a uniform $\mathcal{C}^{1,1}$ open set of \mathbb{R}^d , let $\Gamma_S \subset \partial\Omega$ and $\Gamma'_S \subset \partial\Omega$ two $\mathcal{C}^{1,1}$ relative open submanifolds such that $\mathcal{H}^{d-1}(\Gamma_S \cap \Gamma'_S) > 0$. We define the two sets $\Gamma_D = \partial\Omega \setminus \Gamma_S$ and $\Gamma'_D = \partial\Omega \setminus \Gamma'_S$ then*

- For $d \geq 3$ there exists a constant C_1 such that:

$$|\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)| \leq C_1 (\mathcal{H}^{d-1}(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2d}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}).$$

- For $d = 2$ there exists a constant C_2 such that:

$$|\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)| \leq C_2 (\mathcal{H}^1(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}).$$

Where $d(\Gamma_D, \Gamma'_D)$ is the Hausdorff distance between the two sets.

In the first part of Section 5.4 we study maximization and minimization problems for the Steklov-Dirichlet eigenvalues, when we put a measure constraint on Γ_S . Similar problems were already studied in [31] for Dirichlet-Neumann eigenvalues.

In particular we prove the following theorems

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $0 < m < 1$ be a constant, then, for all k , the following variational problem*

$$\inf\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\},$$

has a solution.

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $0 < m < 1$ be a constant, then the following equality holds*

$$\sup\{\lambda_k(\Omega, \Gamma_S) \mid \Gamma_S \subset \partial\Omega, \mathcal{H}^{d-1}(\Gamma_S) = m\mathcal{H}^{d-1}(\partial\Omega)\} = +\infty.$$

Then we focus on the planar case. We show an explicit example of a sequence of domains $\Gamma_{S,n}$ for which $\lim_{n \rightarrow \infty} \lambda_1(\Omega, \Gamma_{S,n}) = \infty$. We see that this phenomenon is linked to the fact that the sequence $\Gamma_{S,n}$ has an unbounded number of connected component.

Motivated by this example, in the second part of Section 5.4, we study the continuity properties of the Steklov-Dirichlet eigenvalues. More precisely we prove the following theorem, that is a result in the same spirit of the famous result by V. Šverák concerning the Dirichlet eigenvalues (see [58, 96]) for the Steklov-Dirichlet eigenvalues

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^{1,1}$ open domain, let $\Gamma_{D,n} \subset \partial\Omega$ be a sequence of compact subdomains converging for the Hausdorff metric to a compact set $\Gamma_D \subset \partial\Omega$. We define the two sets $\Gamma_{S,n} = \partial\Omega \setminus \Gamma_{D,n}$ and $\Gamma_S = \partial\Omega \setminus \Gamma_D$, assume that the number of connected components of $\Gamma_{D,n}$ is uniformly bounded, then for all k*

$$\lambda_k(\Omega, \Gamma_{S,n}) \rightarrow \lambda_k(\Omega, \Gamma_S).$$

Using this continuity property we prove the existence of a maximizer in the class of sets with given measure and bounded number of connected components.

5.2 Regularity of eigenfunctions

Let $\Omega \subset \mathbb{R}^d$ be a bounded, open, connected set. Let $\Gamma_S \subset \partial\Omega$ be a relative open submanifolds and we define also $\Gamma_D = \partial\Omega \setminus \Gamma_S$. In order to investigate the regularity properties of the Steklov-Dirichlet eigenfunctions u_k we are lead to study the following mixed boundary value problem:

$$\begin{cases} \Delta u = 0 & \Omega \\ \partial_\nu u = g & \Gamma_S \\ u = 0 & \Gamma_D, \end{cases} \quad (5.3)$$

in particular we want to study the regularity of the solution depending on the regularity of the function g .

The big issue for this type of boundary value problem comes from the singularities that can appear when the boundary conditions change. The following example shows that the solution of (5.3) is not smooth in general, no matter how the data are regular.

Example 5.2.1. *We define the following set $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ and we consider the following problem*

$$\begin{cases} \Delta u = 0 & \mathbb{R}_+^2 \\ \partial_\nu u = 0 & \{(x, y) \in \mathbb{R}^2 \mid x < 0, y = 0\} \\ u = 0 & \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y = 0\}. \end{cases}$$

In polar coordinates a solution for the above problem is given by the following function:

$$u(r, \theta) = r^{\frac{1}{2}} \sin\left(\frac{\theta}{2}\right) \quad r \geq 0, 0 \leq \theta \leq 2\pi.$$

We have that $u \in \mathcal{C}^{\frac{1}{2}}(\overline{\mathbb{R}_+^2})$.

In general we cannot expect that the solutions of (5.3) are more regular than $\mathcal{C}^{\frac{1}{2}}(\overline{\Omega})$. There is a huge literature concerning the problem of regularity of solutions of mixed boundary value problems, without claiming to be exhaustive we can refer to the following monographs [34, 48], to the articles [7, 92, 69] and the references therein. We will investigate the case when both Ω and Γ_S are smooth, and we will measure the regularity of the solutions in the class of Besov spaces, deeply using the results from [92].

Before presenting the results in our setting we want to comment the fact that, counterintuitively, the case when Ω is smooth is the case where the least regularity is expected. Indeed when Γ_S and Γ_D do not meet tangentially then we have a better situation and we have more regularity for the solutions (see[34]), but the gain of regularity vanishes as the angle between Γ_S and Γ_D approach π .

We introduce the Besov space $B_{2\infty}^{\frac{3}{2}}(\Omega)$ via its characterization as an interpolation space between two Sobolev spaces, let $(\cdot, \cdot)_{\theta, p}$ be the real interpolation functor (see [13]), then we have that:

$$B_{2\infty}^{\frac{3}{2}}(\Omega) = (H^1(\Omega), H^2(\Omega))_{\frac{1}{2}, \infty}.$$

Under some smoothness assumption on Ω and Γ_S we can conclude that if u is a solution of (5.3) then $u \in B_{2\infty}^{\frac{3}{2}}(\Omega)$ (see [92]), and we will use this results in order to show that $u_k \in B_{2\infty}^{\frac{3}{2}}(\Omega)$.

We define what are the precise regularity property of Ω and Γ_S (see [1, 95, 92])

Definition 5.1. *We say that Ω is a uniform $\mathcal{C}^{1,1}$ open set of \mathbb{R}^d and $\Gamma_S \subset \partial\Omega$ is a $\mathcal{C}^{1,1}$ relative open submanifold if there exists an $\epsilon > 0$, an integer l , an $M > 0$ and a possible finite sequence U_1, \dots, U_k, \dots of open sets of \mathbb{R}^d so that*

1. *if $x \in \partial\Omega$ then $\overline{B_\epsilon(x)} \subset U_k$ for some k ;*
2. *no point of \mathbb{R}^d is contained in more than l of the U_k 's;*
3. *there exist $\mathcal{C}^{1,1}$ diffeomorphisms*

$$\begin{cases} \Phi_k : U_k \rightarrow V_k = B_1(x_k) \subset \mathbb{R}^d, & \Psi_k = \Phi_k^{-1}, \\ \|\Phi_k\|_{\mathcal{C}^{1,1}(U_k)} \leq M & \|\Psi_k\|_{\mathcal{C}^{1,1}(U_k)} \leq M \end{cases}$$

with $x_k \in \partial\mathbb{R}_+^d$, and

$$\begin{cases} \Phi_k(U_k \cap \Omega) = V_k \cap \mathbb{R}_+^d, \\ \Phi_k(U_k \cap \partial\Omega) = V_k \cap \partial\mathbb{R}_+^d, \\ \Phi_k(U_k \cap \Gamma_D) = V_k \cap \mathbb{R}_+^{d-1} \times \{0\}, \\ \Phi_k(U_k \cap \partial\Gamma_D) = V_k \cap \mathbb{R}^{d-2} \times \{(0, 0)\}. \end{cases}$$

We can now state the result about the regularity of eigenfunctions

Theorem 5.5. *Let Ω be a uniform $\mathcal{C}^{1,1}$ open set of \mathbb{R}^d , let $\Gamma_S \subset \partial\Omega$ be a $\mathcal{C}^{1,1}$ relative open submanifold (as in Definition 5.1) and let u_k be an eigenfunction corresponding to the eigenvalue $\lambda_k(\Omega, \Gamma_S)$ then $u_k \in B_{2\infty}^{\frac{3}{2}}(\Omega)$. Moreover if $d = 2$ then $u_k \in \mathcal{C}^{\frac{1}{2}}(\overline{\Omega})$.*

Proof. We know that $u_k \in H^1(\Omega)$ so in particular $u_k \in H^{\frac{1}{2}}(\Gamma_S)$ and from Theorem 1 in [92] we conclude that $u_k \in B_{2\infty}^{\frac{3}{2}}(\Omega)$. Moreover if $d = 2$ the Sobolev-Besov embedding Theorem gives us that $B_{2\infty}^{\frac{3}{2}}(\Omega) \subset \mathcal{C}^{\frac{1}{2}}(\overline{\Omega})$ and in particular $u_k \in \mathcal{C}^{\frac{1}{2}}(\overline{\Omega})$ \square

Remark 4. *We notice that in dimension $d = 2$ the Steklov-Dirichlet eigenfunctions u_k reach the maximal Holder regularity allowed by Example 5.2.1.*

5.3 Stability of the Steklov-Dirichlet eigenvalues

In this section we investigate the stability properties of the eigenvalues $\lambda_k(\Omega, \Gamma_S)$. More precisely we state Theorem 5.1 in a more precise way, explaining the precise dependence of the constants C_1 and C_2 from Ω , Γ_S and Γ'_S , and we prove the stability result.

In order to do so we need to introduce the following Lemma about the existence of a family of test functions.

Lemma 5.6. *Let u_k be an eigenfunction associated to the eigenvalue $\lambda_k(\Omega, \Gamma_S)$ and let M be a constant independent from Γ_S and Γ'_S , then there exist a family of functions $g_i \in C^\infty(\Gamma'_S \setminus \Gamma_S)$ with $i = 1, \dots, k$, such that $\|g_i\|_{C^0(\Omega)} \leq M$ and the solutions of the following mixed boundary value*

$$\begin{cases} \Delta \tilde{v}_i = 0 & \Omega \\ \partial_\nu \tilde{v}_i = \lambda_k(\Omega, \Gamma_S) u_k & \Gamma'_S \cap \Gamma_S \\ \partial_\nu \tilde{v}_i = g_i & \Gamma'_S \setminus \Gamma_S \\ \tilde{v}_i = 0 & \partial\Omega \setminus \Gamma'_S \end{cases} \quad (5.4)$$

have the following properties: for all $i \neq j$ we have that

$$\int_{\Omega} \nabla \tilde{v}_i \cdot \nabla \tilde{v}_j dx = \int_{\partial\Omega} \tilde{v}_i \tilde{v}_j d\mathcal{H}^{d-1} = 0. \quad (5.5)$$

The same result holds if we exchange the role of Γ_S and Γ'_S

Proof. We define the functions g_i via an induction procedure. In the proof M will be a constant independent from Γ_S and Γ'_S that can change from line to line. We choose the first function $g_1 \in C^\infty(\Gamma'_S \setminus \Gamma_S)$ such that $\|g_1\|_{C^0(\Gamma'_S \setminus \Gamma_S)} \leq M$ and such that the function

$$\Phi_1(x) = \begin{cases} \lambda_k(\Omega, \Gamma_S) u_k(x) & x \in \Gamma_S \cap \Gamma'_S \\ g_1(x) & x \in \Gamma'_S \setminus \Gamma_S \end{cases}$$

is in $H^{\frac{1}{2}}(\Gamma'_S)$, so we introduce the first test function \tilde{v}_1

$$\begin{cases} \Delta \tilde{v}_1 = 0 & \Omega \\ \partial_\nu \tilde{v}_1 = \lambda_k(\Omega, \Gamma_S) u_k & \Gamma'_S \cap \Gamma_S \\ \partial_\nu \tilde{v}_1 = g_1 & \Gamma'_S \setminus \Gamma_S \\ \tilde{v}_1 = 0 & \partial\Omega \setminus \Gamma'_S. \end{cases}$$

We now show how to construct the function g_2 with the desired properties, and then we generalize this procedure, constructing the function g_{i+1} knowing the function g_1, g_2, \dots, g_i . We introduce two functions φ_1 and φ_2 in $C_c^\infty(\Gamma'_S \setminus \Gamma_S)$ such that $\|\varphi_1\|_{C^0(\Gamma'_S \setminus \Gamma_S)} \leq M$ and $\|\varphi_2\|_{C^0(\Gamma'_S \setminus \Gamma_S)} \leq M$, two real parameters t_1 and t_2 , we will choose more precisely the functions and the parameters later in the proof. We define also v_{φ_i} with $i = 1, 2$ to be the solution of the following mixed boundary value problem

$$\begin{cases} \Delta v_{\varphi_i} = 0 & \Omega \\ \partial_\nu v_{\varphi_i} = \varphi_i & \Gamma'_S \\ v_{\varphi_i} = 0 & \partial\Omega \setminus \Gamma'_S. \end{cases}$$

We denote by \tilde{v}_2 the solution of the following problem:

$$\begin{cases} \Delta \tilde{v}_2 = 0 & \Omega \\ \partial_\nu \tilde{v}_2 = \lambda_k(\Omega, \Gamma_S) u_k & \Gamma'_S \cap \Gamma_S \\ \partial_\nu \tilde{v}_2 = g_1 + t_1 \varphi_1 + t_2 \varphi_2 & \Gamma'_S \setminus \Gamma_S \\ \tilde{v}_2 = 0 & \partial\Omega \setminus \Gamma'_S. \end{cases} \quad (5.6)$$

Now we want to choose the functions φ_1 and φ_2 and the parameters t_1 and t_2 in such a way that the conditions (5.5) are satisfied. More precisely, from the linearity of the equation

(5.6) and from the linearity of the Dirichlet-to-Neumann map, we obtain the following equalities:

$$\begin{aligned} \int_{\Omega} \nabla \tilde{v}_1 \cdot \nabla \tilde{v}_2 dx &= \int_{\Gamma'_S \cap \Gamma_S} \lambda_k(\Omega, \Gamma_S)^2 u_k^2 d\mathcal{H}^{d-1} + \int_{\Gamma'_S \setminus \Gamma_S} g_1^2 d\mathcal{H}^{d-1} + \\ &\quad + t_1 \int_{\Gamma'_S \setminus \Gamma_S} g_1 \varphi_1 d\mathcal{H}^{d-1} + t_2 \int_{\Gamma'_S \setminus \Gamma_S} g_1 \varphi_2 d\mathcal{H}^{d-1}, \\ \int_{\partial\Omega} \tilde{v}_1 \tilde{v}_2 dx &= \int_{\Gamma'_S} \tilde{v}_1^2 d\mathcal{H}^{d-1} + t_1 \int_{\Gamma'_S} \tilde{v}_1 v_{\varphi_1} d\mathcal{H}^{d-1} + t_2 \int_{\Gamma'_S} \tilde{v}_1 v_{\varphi_2} d\mathcal{H}^{d-1}. \end{aligned}$$

Now we introduce the following matrix and the following vector

$$\begin{aligned} A_{2 \times 2} &= \begin{pmatrix} \int_{\Gamma'_S \setminus \Gamma_S} g_1 \varphi_1 d\mathcal{H}^{d-1} & \int_{\Gamma'_S \setminus \Gamma_S} g_1 \varphi_2 d\mathcal{H}^{d-1} \\ \int_{\Gamma'_S} \tilde{v}_1 v_{\varphi_1} d\mathcal{H}^{d-1} & \int_{\Gamma'_S} \tilde{v}_1 v_{\varphi_2} d\mathcal{H}^{d-1} \end{pmatrix} \\ b_2 &= \begin{pmatrix} - \int_{\Gamma'_S \cap \Gamma_S} \lambda_k(\Omega, \Gamma_S)^2 u_k^2 d\mathcal{H}^{d-1} - \int_{\Gamma'_S \setminus \Gamma_S} g_1^2 d\mathcal{H}^{d-1} \\ - \int_{\Gamma'_S} \tilde{v}_1^2 d\mathcal{H}^{d-1} \end{pmatrix} \end{aligned}$$

We choose φ_1 and φ_2 in such a way that $A_{2 \times 2}^{-1}$ exists, $\|A_{2 \times 2}^{-1}\| \leq M$ and $\|b_2\| \leq M$ (we recall that the constant M can change). Now we choose the parameters $t = (t_1, t_2)$ to be the solutions of the following linear system $A_{2 \times 2} t = b_2$. Now we choose g_2 to be the following function:

$$g_2 = g_1 + t_1 \varphi_1 + t_2 \varphi_2.$$

With the choice we made for φ_1 , φ_2 and $t = (t_1, t_2)$ it is clear that $\|g_2\|_{C^0(\Omega)} \leq M$ and the function \tilde{v}_2 satisfies the conditions (5.5).

Now suppose we know the functions g_1, \dots, g_i , we can construct the function g_{i+1} using the same algorithm. This time we need to introduce the functions $\{\varphi_j\}_{j=1}^{2i}$ such that $\varphi_j \in C_c^\infty(\Gamma'_S \setminus \Gamma_S)$ and $\|\varphi_j\|_{C^0(\Gamma'_S \setminus \Gamma_S)} \leq M$ for all $j = 1, \dots, 2i$, and the parameters $\{t_j\}_{j=1}^{2i}$. We introduce the function

$$\begin{cases} \Delta \tilde{v}_{i+1} = 0 & \Omega \\ \partial_\nu \tilde{v}_{i+1} = \lambda_k(\Omega, \Gamma_S) u_k & \Gamma'_S \cap \Gamma_S \\ \partial_\nu \tilde{v}_{i+1} = g_1 + \sum_{j=1}^{2i} t_j \varphi_j & \Gamma'_S \setminus \Gamma_S \\ \tilde{v}_{i+1} = 0 & \partial\Omega \setminus \Gamma'_S. \end{cases}$$

Now, by imposing $2i$ orthogonality conditions with the function $\{\tilde{v}_j\}_{j=1}^i$ we will obtain a similar system of linear equations, in particular we will define in a similar way the matrix $A_{2i \times 2i}$ and the vector b_{2i} . We choose the functions $\{\varphi_j\}_{j=1}^{2i}$ in such a way that $A_{2i \times 2i}^{-1}$ exists, $\|A_{2i \times 2i}^{-1}\| \leq M$ and $\|b_{2i}\| \leq M$ (we recall that the constant M can change). Now we choose the vector $t = (t_1, t_2, \dots, t_{2i})$ to be the solution of the following system $A_{2i \times 2i} t = b_{2i}$ and we define g_{i+1} in the following way

$$g_{i+1} = g_1 + \sum_{j=1}^{2i} t_j \varphi_j.$$

With the choice we made for $\{\varphi_j\}_{j=1}^{2i}$ and $t = (t_1, t_2, \dots, t_{2i})$ it is clear that $\|g_{i+1}\|_{C^0(\Omega)} \leq M$ and the function \tilde{v}_{i+1} satisfies the conditions (5.5). The same construction is possible when we exchange the two sets Γ_S and Γ'_S . \square

We are now ready to state in a more precise way Theorem 5.1 and to prove it. Let $K_1 \subset \mathbb{R}^d$ and $K_2 \subset \mathbb{R}^d$ be two compact sets, we denote by $d(K_1, K_2)$ the Hausdorff distance between the two sets.

Theorem 5.7. *Let Ω be a uniform $\mathcal{C}^{1,1}$ open set of \mathbb{R}^d , let $\Gamma_S \subset \partial\Omega$ and $\Gamma'_S \subset \partial\Omega$ two $\mathcal{C}^{1,1}$ relative open submanifolds (see Definition 5.24) such that $\mathcal{H}^{d-1}(\Gamma_S \cap \Gamma'_S) > 0$. We define the two sets $\Gamma_D = \partial\Omega \setminus \Gamma_S$ and $\Gamma'_D = \partial\Omega \setminus \Gamma'_S$ and let v_k be a linear combination of the functions $\{\tilde{v}_i\}_{i=1}^k$ defined in Lemma 5.6 then*

- For $d \geq 3$ there exists a constant C_1 , that depends on Ω , on $\|v_k\|_{L^2(\partial\Omega)}$, on $\|v_k\|_{H^1(\Omega)}$ and on $\max\{\lambda_k(\Omega, \Gamma_S), \lambda_k(\Omega, \Gamma'_S)\}$, such that:

$$|\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)| \leq C_1(\mathcal{H}^{d-1}(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2d}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}). \quad (5.7)$$

- For $d = 2$ there exists a constant C_1 , that depends on Ω , on $\|v_k\|_{L^2(\partial\Omega)}$, on $\|v_k\|_{H^1(\Omega)}$ and on $\max\{\lambda_k(\Omega, \Gamma_S), \lambda_k(\Omega, \Gamma'_S)\}$, such that:

$$|\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)| \leq C_2(\mathcal{H}^1(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}). \quad (5.8)$$

We will prove the upper bound for the quantity $\lambda_k(\Omega, \Gamma'_S) - \lambda_k(\Omega, \Gamma_S)$, from the proof of this upper bound we will easily obtain the desired estimates also for the quantity $\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)$ by simply exchanging the role of Γ_S and Γ'_S .

Proof. In the proof we will denote by C a constant, which can change from line to line, that depends on Ω , on $\|v_k\|_{L^2(\partial\Omega)}$, on $\|v_k\|_{H^1(\Omega)}$ and on $\max\{\lambda_k(\Omega, \Gamma_S), \lambda_k(\Omega, \Gamma'_S)\}$. We start by estimating the following quantity:

$$\lambda_k(\Omega, \Gamma'_S) - \lambda_k(\Omega, \Gamma_S). \quad (5.9)$$

We use the variational characterization (5.2) and the test functions constructed in Lemma 5.6 and we obtain that

$$\lambda_k(\Omega, \Gamma'_S) \leq \max_{\alpha \in \mathbb{R}^k} \frac{\int_{\Omega} |\nabla(\sum_{i=1}^k \alpha_i \tilde{v}_i)|^2 dx}{\int_{\Gamma'_S} (\sum_{i=1}^k \alpha_i \tilde{v}_i)^2 d\mathcal{H}^{d-1}} = \max_{\alpha \in \mathbb{R}^k} \frac{\sum_{i=1}^k \alpha_i^2 \int_{\Omega} |\nabla \tilde{v}_i|^2 dx}{\sum_{i=1}^k \alpha_i^2 \int_{\Gamma'_S} \tilde{v}_i^2 d\mathcal{H}^{d-1}},$$

where the last equality is true because of the conditions (5.5). Let $\tilde{\alpha} \in \mathbb{R}^k$ be the solution of the maximization problem, from the last equality it is clear that we can assume $\tilde{\alpha}_i \geq 0$ for all $i = 1, \dots, k$, in particular $\sum_{i=1}^k \tilde{\alpha}_i > 0$, because otherwise $\tilde{\alpha}_i = 0$ for all $i = 1, \dots, k$ and this is not possible. The Rayleigh quotient is invariant under multiplication by a scalar, so we can also assume that $\sum_{i=1}^k \tilde{\alpha}_i = 1$ and we define the following function:

$$v_k = \sum_{i=1}^k \tilde{\alpha}_i \tilde{v}_i.$$

From the linearity of the equations (5.4) we know that v_k satisfies the following mixed boundary value problem

$$\begin{cases} \Delta v_k = 0 & \Omega \\ \partial_\nu v_k = \lambda_k(\Omega, \Gamma_S) u_k & \Gamma'_S \cap \Gamma_S \\ \partial_\nu v_k = \sum_{i=1}^k \tilde{\alpha}_i g_i & \Gamma'_S \setminus \Gamma_S \\ v_k = 0 & \partial\Omega \setminus \Gamma'_S. \end{cases} \quad (5.10)$$

From the variational characterization, using the function v_k as a test function for $\lambda_k(\Omega, \Gamma'_S)$ and let u_k be an eigenfunction corresponding to $\lambda_k(\Omega, \Gamma_S)$, we obtain the following inequality:

$$\lambda_k(\Omega, \Gamma'_S) - \lambda_k(\Omega, \Gamma_S) \leq \frac{\int_{\Gamma_S} u_k^2 d\mathcal{H}^{d-1} \int_{\Omega} |\nabla v_k|^2 dx - \int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1} \int_{\Omega} |\nabla u_k|^2 dx}{\int_{\Gamma_S} u_k^2 d\mathcal{H}^{d-1} \int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1}}. \quad (5.11)$$

We start by giving an upper bound for the quantity $\int_{\Gamma_S} u_k^2$. From Cauchy-Schwarz inequality we know that

$$\int_{\Gamma_S} u_k^2 d\mathcal{H}^{d-1} - \int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1} \leq \|u_k + v_k\|_{L^2(\partial\Omega)} \|u_k - v_k\|_{L^2(\partial\Omega)},$$

and we also know that

$$\|v_k\|_{L^2(\partial\Omega)} \leq \|u_k - v_k\|_{L^2(\partial\Omega)} + \|u_k\|_{L^2(\partial\Omega)},$$

so we conclude that

$$\int_{\Gamma_S} u_k^2 d\mathcal{H}^{d-1} - \int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1} \leq (\|u_k - v_k\|_{L^2(\partial\Omega)} + 2\|u_k\|_{L^2(\partial\Omega)}) \|u_k - v_k\|_{L^2(\partial\Omega)}. \quad (5.12)$$

From the regularity assumption on the domain Ω we know that the trace operator is compact and we denote its norm by C_t . Using this fact, combined with (5.12) we obtain the following upper bound for $\int_{\Gamma_S} u_k^2$:

$$\int_{\Gamma_S} u_k^2 d\mathcal{H}^{d-1} \leq (C_t \|u_k - v_k\|_{H^1(\Omega)} + 2\|u_k\|_{L^2(\partial\Omega)}) C_t \|u_k - v_k\|_{H^1(\Omega)} + \int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1},$$

using this estimate in (5.11) we finally obtain

$$\begin{aligned} \lambda_k(\Omega, \Gamma'_S) - \lambda_k(\Omega, \Gamma_S) &\leq \frac{1}{\int_{\Gamma_S} u_k^2 d\mathcal{H}^{d-1} \int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1}} \left[\int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1} \left(\int_{\Omega} |\nabla v_k|^2 dx - \int_{\Omega} |\nabla u_k|^2 dx \right) + \right. \\ &\quad \left. + (C_t^2 \|u_k - v_k\|_{H^1(\Omega)}^2 + 2\|u_k\|_{L^2(\partial\Omega)} C_t \|u_k - v_k\|_{H^1(\Omega)}) \int_{\Omega} |\nabla v_k|^2 dx \right]. \end{aligned} \quad (5.13)$$

We want to bound the right hand side of (5.13). We start by noticing that the function v_k is in $H^1(\Omega)$ and also that there exists a constant $C > 0$ such that

$$C \leq \int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1}, \quad (5.14)$$

indeed if we have that $\int_{\Gamma'_S} v_k^2 d\mathcal{H}^{d-1} = 0$, we will have that the trace of the function v_k is a function constantly equal to zero, and this is a contradiction with the equation (5.10) that v_k must satisfy.

We now estimate the quantity $\int_{\Omega} |\nabla v_k|^2 dx - \int_{\Omega} |\nabla u_k|^2 dx$ in (5.13), we obtain that

$$\begin{aligned} \int_{\Omega} |\nabla v_k|^2 dx - \int_{\Omega} |\nabla u_k|^2 dx &\leq \left| \|v_k\|_{H^1(\Omega)}^2 - \|u_k\|_{H^1(\Omega)}^2 \right| + \left| \|v_k\|_{L^2(\Omega)}^2 - \|u_k\|_{L^2(\Omega)}^2 \right| \\ &\leq \left| \|v_k\|_{H^1(\Omega)} - \|u_k\|_{H^1(\Omega)} \right| \left(\|v_k\|_{H^1(\Omega)} + \|u_k\|_{H^1(\Omega)} \right) + \\ &\quad + \left| \|v_k\|_{L^2(\Omega)} - \|u_k\|_{L^2(\Omega)} \right| \left(\|v_k\|_{L^2(\Omega)} + \|u_k\|_{L^2(\Omega)} \right) \\ &\leq C \|u_k - v_k\|_{H^1(\Omega)} \end{aligned} \quad (5.15)$$

The last remaining term to estimate is the quantity $\|u_k - v_k\|_{H^1(\Omega)}$, in order to estimate this term we use Theorem 4 in [92] and (5.10) to conclude that there exists a constant C that depends only on Ω such that the following inequality holds

$$\begin{aligned} \|u_k - v_k\|_{H^1(\Omega)} &\leq C \left(\sup_{\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}=1} \left| \lambda_k(\Omega, \Gamma_S) \int_{\Gamma_S} (\partial_\nu u_k) w d\mathcal{H}^{d-1} - \int_{\Gamma'_S} (\partial_\nu v_k) w d\mathcal{H}^{d-1} \right| + \right. \\ &\quad \left. + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}} \right) \\ &\leq C \left(\sup_{\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}=1} \int_{\Gamma_S \setminus \Gamma'_S} |u_k w| d\mathcal{H}^{d-1} + \int_{\Gamma'_S \setminus \Gamma_S} |w| \left(\sum_{i=1}^k \tilde{\alpha}_i g_i \right) d\mathcal{H}^{d-1} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}} \right) \\ &\leq C \left(\|u_k\|_{L^2(\Gamma_S \setminus \Gamma'_S)} + \left\| \sum_{i=1}^k \tilde{\alpha}_i g_i \right\|_{L^2(\Gamma'_S \setminus \Gamma_S)} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}} \right). \end{aligned}$$

Let us now consider separately the case $d \geq 3$ and $d = 2$. For $d \geq 3$ it is enough to know that $u_k \in H^{\frac{1}{2}}(\partial\Omega)$, indeed by classical embedding theorem for Sobolev spaces we have that $u_k \in L^{\frac{2d}{d-1}}(\partial\Omega)$ by Lemma 5.6 we also know that $\sum_{i=1}^k \tilde{\alpha}_i g_i \in L^{\frac{2d}{d-1}}(\partial\Omega)$. Using Hölder inequality we conclude that

$$\begin{aligned} \|u_k - v_k\|_{H^1(\Omega)} &\leq C \left(\|u_k\|_{L^{\frac{2d}{d-1}}(\partial\Omega)} \mathcal{H}^{d-1}(\Gamma_S \setminus \Gamma'_S)^{\frac{1}{2d}} + \left\| \sum_{i=1}^k \tilde{\alpha}_i g_i \right\|_{L^{\frac{2d}{d-1}}(\partial\Omega)} \mathcal{H}^{d-1}(\Gamma'_S \setminus \Gamma_S)^{\frac{1}{2d}} + \right. \\ &\quad \left. + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}} \right) \end{aligned} \quad (5.16)$$

From Lemma 5.6 we know that there exists a constant C , that does not depend on Γ_S and Γ'_S , such that $\left\| \sum_{i=1}^k \tilde{\alpha}_i g_i \right\|_{L^{\frac{2d}{d-1}}(\partial\Omega)}^{\frac{1}{2}} \leq C$. From the continuity of the Sobolev embedding and the trace operator we have that there exists a constant C , that depends only on Ω , such that $\|u_k\|_{L^{\frac{2d}{d-1}}(\partial\Omega)} \leq C \|u_k\|_{H^1(\Omega)}$. Now using this inequality together with the Poincaré -Friedrichs inequality, assuming that $\|u_k\|_{L^2(\Omega)} = 1$, we obtain

$$\|u_k\|_{L^{\frac{2d}{d-1}}(\partial\Omega)} \leq C \sqrt{\lambda_k(\Omega, \Gamma_S) \left(\frac{1}{\lambda_{1,1}(\Omega)} + 1 \right) + \frac{1}{\lambda_{1,1}(\Omega)}},$$

where $\lambda_{1,1}(\Omega)$ is the first Robin eigenvalue with parameter 1. From the inequality above and from (5.16) we conclude that there exists a constant C that depends only on Ω and $\lambda_k(\Omega, \Gamma_S)$ such that

$$\|u_k - v_k\|_{H^1(\Omega)} \leq C \left(\mathcal{H}^{d-1}(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2d}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}} \right). \quad (5.17)$$

For $d = 2$, using the regularity results given in Theorem 5.5, we know that $u_k \in \mathcal{C}^{\frac{1}{2}}(\bar{\Omega})$, by Lemma 5.6 we also know that $\sum_{i=1}^k \tilde{\alpha}_i g_i \in \mathcal{C}^{\frac{1}{2}}(\partial\Omega)$, so we obtain that

$$\|u_k - v_k\|_{H^1(\Omega)} \leq C \left(\|u_k\|_{\mathcal{C}^0(\bar{\Omega})}^{\frac{1}{2}} \mathcal{H}^1(\Gamma_S \setminus \Gamma'_S)^{\frac{1}{2}} + \left\| \sum_{i=1}^k \tilde{\alpha}_i g_i \right\|_{\mathcal{C}^0(\partial\Omega)}^{\frac{1}{2}} \mathcal{H}^1(\Gamma'_S \setminus \Gamma_S)^{\frac{1}{2}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}} \right). \quad (5.18)$$

From Lemma 5.6 we know that there exists a constant C , that does not depend on Γ_S and Γ'_S , such that $\left\| \sum_{i=1}^k \tilde{\alpha}_i g_i \right\|_{\mathcal{C}^0(\partial\Omega)} \leq C$. From the continuity of the Sobolev-Besov

embedding (see [1, 13, 92]) we know that there exists a constant C , that depends only on Ω , such that $\|u_k\|_{C^{\frac{1}{2}}(\bar{\Omega})} \leq C\|u_k\|_{B_{2\infty}^{\frac{3}{2}}(\Omega)}$. From this inequality and from Theorem 1 in [92] we conclude that there exists a constant C , that depends only on Ω , such that

$$\|u_k\|_{C^{\frac{1}{2}}(\bar{\Omega})} \leq C\sqrt{\|u_k\|_{H^1(\Omega)}(1 + \|u_k\|_{H^1(\Omega)})}.$$

Now using a similar argument as the one we used in the case $d \geq 3$ we can bound from above the quantity $\|u_k\|_{H^1(\Omega)}$ and, from (5.18), we conclude that there exists a constant C , that depends only on Ω and $\lambda_k(\Omega, \Gamma_S)$, such that

$$\|u_k - v_k\|_{H^1(\Omega)} \leq C(\mathcal{H}^1(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}). \quad (5.19)$$

For $d \geq 3$ using (5.14), (5.15) and (5.17) in (5.13) we conclude that there exists a constant C_1 , that depends on Ω , on $\|v_k\|_{L^2(\partial\Omega)}$, on $\|v_k\|_{H^1(\Omega)}$ and on $\lambda_k(\Omega, \Gamma_S)$, such that:

$$\lambda_k(\Omega, \Gamma'_S) - \lambda_k(\Omega, \Gamma_S) \leq C_1(\mathcal{H}^{d-1}(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2d}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}).$$

For $d = 2$ using using (5.14), (5.15) and (5.19) in (5.13) we conclude that there exists a constant C_2 , that depends on Ω , on $\|v_k\|_{L^2(\partial\Omega)}$, on $\|v_k\|_{H^1(\Omega)}$ and on $\lambda_k(\Omega, \Gamma_S)$, such that:

$$\lambda_k(\Omega, \Gamma'_S) - \lambda_k(\Omega, \Gamma_S) \leq C_2(\mathcal{H}^1(\Gamma_S \Delta \Gamma'_S)^{\frac{1}{2}} + d(\Gamma_D, \Gamma'_D)^{\frac{1}{2}}).$$

Now exchanging the role of Γ_S and Γ'_S we obtain the same kind of estimates for the quantity $\lambda_k(\Omega, \Gamma_S) - \lambda_k(\Omega, \Gamma'_S)$, this concludes the proof \square

5.4 Optimization problems and continuity properties

In the first part of this section we study maximization and minimization problems for the Steklov-Dirichlet eigenvalues with a measure constraint on the set Γ_S . In the second part we study the continuity properties of the Steklov-Dirichlet eigenvalues.

5.4.1 Optimization problems

We start by proving the existence of a minimizer, in particular we prove Theorem 5.2. We now state an elementary lemma about the monotonicity property of the Steklov-Dirichlet eigenvalues.

Lemma 5.8. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain and let $\Gamma_S \subset \partial\Omega$ and $\Gamma'_S \subset \partial\Omega$ two relative open subsets such that $\Gamma_S \subset \Gamma'_S$ then, for all k , the following inequality holds*

$$\lambda_k(\Omega, \Gamma'_S) \leq \lambda_k(\Omega, \Gamma_S)$$

Proof. Let u_i be an eigenfunction associated to $\lambda_i(\Omega, \Gamma_S)$ with $i = 1, \dots, k$. From the assumption $\Gamma_S \subset \Gamma'_S$, it is clear that $V = \text{span}[u_1, u_2, \dots, u_k]$ is a subspace of the Hilbert space $H_0^1(\Omega, \Gamma'_S)$, in particular we can use the test subspace V in the variational characterization of $\lambda_k(\Omega, \Gamma'_S)$ and this concludes the proof. \square

This simple lemma is crucial in order to prove the existence of a minimizer. Indeed, using this property, we can construct a minimizer by using a classical procedure based on the concept of weak γ -convergence (see [20] for more details).

Proof of Theorem 5.2. We consider a minimizing sequence $\Gamma_{S,\epsilon}$, and we consider the normalized eigenfunctions $\|u_{k,\epsilon}\|_{L^2(\partial\Omega)} = 1$. We want to obtain a uniform bound in $H^1(\Omega)$ for the eigenfunctions $u_{k,\epsilon}$. From the minimality assumption for the sequence $\Gamma_{S,\epsilon}$ we can assume that:

$$\int_{\Omega} |\nabla u_{k,\epsilon}|^2 dx = \lambda_k(\Omega, \Gamma_{S,\epsilon}) \leq C < \infty \quad \forall \epsilon > 0.$$

Now the bound for $\int_{\Omega} u_{k,\epsilon}^2 dx$ follows directly from Poincare-Friedrichs inequality

$$\int_{\Omega} u_{k,\epsilon}^2 dx \leq \frac{1}{\lambda_{1,1}(\Omega)} \left[\int_{\Omega} |\nabla u_{k,\epsilon}|^2 dx + \int_{\partial\Omega} u_{k,\epsilon}^2 d\mathcal{H}^{d-1} \right],$$

where $\lambda_{1,1}(\Omega)$ is the first Robin eigenvalue with parameter 1.

We conclude that there exists a function $u_k \in H^1(\Omega)$ such that, up to a subsequence, the following convergences hold:

$$\begin{aligned} u_{k,\epsilon} &\rightharpoonup u_k && \text{in } H^1(\Omega), \\ u_{k,\epsilon} &\rightarrow u_k && \text{in } L^2(\partial\Omega), \\ u_{k,\epsilon} &\rightarrow u_k && \text{a. e. in } \partial\Omega. \end{aligned} \tag{5.20}$$

Let $j \neq k$, we can use the same argument as above for $u_{j,\epsilon}$ and we obtain that there exists a function $u_j \in H^1(\Omega)$ such that the same convergences above held. Now from the orthogonality of the eigenfunctions in $L^2(\partial\Omega)$ we have that

$$0 = \int_{\partial\Omega} u_{j,\epsilon} u_{k,\epsilon} \rightarrow \int_{\partial\Omega} u_j u_k = 0 \tag{5.21}$$

Now we want to prove the orthogonality of the gradient of the eigenfunctions. We can use the concept of compensated compactness (see [85]), indeed we have that $\operatorname{div}(\nabla u_{j,\epsilon}) = \Delta u_{j,\epsilon} = 0$ and $\operatorname{rot}(\nabla u_{k,\epsilon}) = 0$, in particular, from the convergences (5.20), we can conclude that

$$0 = \int_{\Omega} \nabla u_{k,\epsilon} \cdot \nabla u_{j,\epsilon} dx \rightarrow \int_{\Omega} \nabla u_k \cdot \nabla u_j dx = 0. \tag{5.22}$$

From the convergence of the eigenfunctions $u_{i,\epsilon}$, with $i = 1, \dots, k$ we can conclude that, up to a subsequence

$$\sum_{i=1}^k u_{i,\epsilon}^2 \rightarrow \sum_{i=1}^k u_i^2 \quad \text{a. e. in } \partial\Omega. \tag{5.23}$$

We define the following set

$$\Gamma = \left\{ x \in \partial\Omega \mid \sum_{i=1}^k u_i(x)^2 > 0 \right\}. \tag{5.24}$$

We can use the test subspace $V = \operatorname{span}[u_1, u_2, \dots, u_k]$ in the variational characterization (5.2) for the eigenvalue $\lambda_k(\Omega, \Gamma)$. Recalling the orthogonality conditions (5.21), (5.22), the

normalization $\|u_{i,\epsilon}\|_{L^2(\partial\Omega)} = 1$ and the convergences (5.20) we obtain

$$\begin{aligned} \lambda_k(\Omega, \Gamma) &\leq \max_{\substack{\alpha \in \mathbb{R}^k \\ \sum_{i=1}^k \alpha_i^2 = 1}} \frac{\int_{\Omega} |\nabla(\sum_{i=1}^k \alpha_i u_i)|^2 dx}{\int_{\Gamma} (\sum_{i=1}^k \alpha_i u_i)^2 d\mathcal{H}^{d-1}} \\ &\leq \sum_{i=1}^k \bar{\alpha}_i^2 \int_{\Omega} |\nabla u_i|^2 dx \\ &\leq \liminf_{\epsilon \rightarrow 0} \sum_{i=1}^k \bar{\alpha}_i^2 \int_{\Omega} |\nabla u_{i,\epsilon}|^2 dx \\ &\leq \liminf_{\epsilon \rightarrow 0} \sum_{i=1}^k \bar{\alpha}_i^2 \lambda_i(\Omega, \Gamma_{S,\epsilon}) \\ &\leq \liminf_{\epsilon \rightarrow 0} \lambda_k(\Omega, \Gamma_{S,\epsilon}). \end{aligned}$$

From the definition (5.24), the convergence (5.23) and from Fatou Lemma we have:

$$\mathcal{H}^{d-1}(\Gamma) \leq \liminf_{\epsilon \rightarrow 0} \mathcal{H}^{d-1}(\{x \in \partial\Omega \mid \sum_{i=1}^k u_{i,\epsilon}(x)^2 > 0\}) = m\mathcal{H}^{d-1}(\partial\Omega).$$

Now if $\mathcal{H}^{d-1}(\Gamma) = m\mathcal{H}^{d-1}(\partial\Omega)$ the proof is finished. If instead $\mathcal{H}^{d-1}(\Gamma) < m\mathcal{H}^{d-1}(\partial\Omega)$, we define a new set Γ_1 such that $\mathcal{H}^{d-1}(\Gamma_1) = m\mathcal{H}^{d-1}(\partial\Omega)$ and $\Gamma \subset \Gamma_1$, from Lemma 5.8 we know that $\lambda_k(\Omega, \Gamma_1) \leq \lambda_k(\Omega, \Gamma)$. This concludes the proof. \square

Remark 5. *We actually proved the existence of a minimizer in a relaxed framework. Indeed the minimizing set $\Gamma \subset \partial\Omega$ that we defined is not a relative open set, but we can still define the Steklov-Dirichlet eigenvalue $\lambda_k(\Omega, \Gamma)$. Indeed by construction we have that $\partial\Omega \setminus \Gamma = \cup_{i=1}^k \{x \in \partial\Omega \mid u_i(x) = 0\}$, with $u_i \in H^1(\Omega)$ for all $i = 1, \dots, k$, and in particular the space $H_0^1(\Omega, \Gamma)$ is a well defined Hilbert space. We can define the eigenvalues $\lambda_k(\Omega, \Gamma)$ via the variational characterization (5.2).*

We now study the maximization problem, in particular we prove Theorem 5.3. In order to prove that a maximizer does not exist it is enough to construct a sequence of domains $\Gamma_{S,n} \subset \partial\Omega$ such that $\mathcal{H}^{d-1}(\Gamma_{S,n}) = m\mathcal{H}^{d-1}(\partial\Omega)$ for all n and $\lambda_1(\Omega, \Gamma_{S,n}) \rightarrow +\infty$. The following geometric Lemma is crucial for the construction of the maximizing sequence. This Lemma is classical (see [58]), for this reason we will give only a sketch of the proof.

Lemma 5.9. *Let Ω be a Lipschitz domain and let $0 < m < 1$ be a constant, then there exists a sequence of domains $\Gamma_n \subset \partial\Omega$ such that*

$$\begin{aligned} \mathcal{H}^{d-1}(\Gamma_n) &= m\mathcal{H}^{d-1}(\partial\Omega) \quad \forall n, \\ \chi_{\Gamma_n} &\xrightarrow{*} m\chi_{\partial\Omega} \quad \text{in } L^\infty(\partial\Omega). \end{aligned}$$

Sketch of the Proof. In order to give a more clear idea of the construction of the set Γ_n , we assume that there exists $c \in \mathbb{N}$ such that $m = \frac{1}{c}$.

Without loss of generality we can assume that $\mathcal{H}^{d-1}(\partial\Omega) = 1$. Let $(\partial\Omega, g)$ be the manifold given by the boundary of Ω endowed with the metric g induced by the euclidean metric on \mathbb{R}^d . We denote by $B_g(x, r) \subset \partial\Omega$ the ball with respect to the metric g , with center x and radius r . We fix $n \in \mathbb{N}$ and we consider a set of points $\{x_k\}_{k=1}^{c^{n-1}}$ and a set of

radii $\{r_{k,n}\}_{k=1}^{c^{n-1}}$ such that $x_j \notin B_g(x_k, r_{k,n})$ if $x_k \neq x_j$ and $\mathcal{H}^{d-1}(B_g(x_k, r_{k,n})) = \frac{1}{c^{n-1}}$ for all $k = 1, 2, \dots, c^{n-1}$. We define the following set

$$\Gamma_n = \bigcup_{k=1}^{c^{n-1}} B_g(x_k, r_{k,n}),$$

now we have that $\mathcal{H}^{d-1}(\Gamma_n) = m$ for all n . Let $s \in L^1(\partial\Omega)$ be a step function, it is straightforward to check that $\int_{\partial\Omega} \chi_{\Gamma_n} s \rightarrow m \int_{\partial\Omega} s$ and we conclude by density (see [58]). In the more general case where $c \notin \mathbb{N}$ we must define the radii $\{r_{k,n}\}_{k=1}^{c^{n-1}}$ in such a way that:

$$\sum_{k=1}^{\lfloor c^{n-1} \rfloor} \mathcal{H}^{d-1}(B_g(x_k, r_{k,n})) = c.$$

□

We are now ready to prove Theorem 5.3.

Proof of Theorem 5.3. Let Γ_n be the sequence of subdomains defined in Lemma 5.9 and we define $\Gamma_{S,n} = \Gamma_n$. Let $u_{1,n}$ be a first Steklov-Dirichlet eigenfunction associated to $\lambda_1(\Omega, \Gamma_{S,n})$ and we assume that $\|u_{1,n}\|_{L^2(\partial\Omega)} = 1$.

Suppose by contradiction that there exists a constant C such that:

$$\sup_{n \in \mathbb{N}} \lambda_1(\Omega, \Gamma_{S,n}) \leq C.$$

Now we know that $\lambda_1(\Omega, \Gamma_{S,n}) = \|\nabla u_{1,n}\|_{L^2(\Omega)}^2 \leq C$ and, from Poincaré-Friedrichs inequality, we also know the following bound

$$\int_{\Omega} u_{1,n}^2 dx \leq \frac{1}{\lambda_{1,1}(\Omega)} \left[\int_{\Omega} |\nabla u_{1,n}|^2 dx + \int_{\partial\Omega} u_{1,n}^2 d\mathcal{H}^{d-1} \right],$$

where $\lambda_{1,1}(\Omega)$ is the first Robin eigenvalue with parameter 1. We conclude that the functions $u_{1,n}$ are bounded in $H^1(\Omega)$ and, from the fact that Ω is a Lipschitz domain, we can conclude also that there exists $u \in L^2(\partial\Omega)$ such that, up to a subsequence

$$u_{1,n} \rightarrow u \quad \text{in } L^2(\partial\Omega).$$

Now we reach a contradiction, indeed we know that $\|u_{1,n}\|_{L^2(\partial\Omega)} = 1$ for all n , this implies

$$1 = \int_{\partial\Omega} u_{1,n}^2 d\mathcal{H}^{d-1} \rightarrow \int_{\partial\Omega} u^2 d\mathcal{H}^{d-1} = 1,$$

but from Lemma 5.9 we have that

$$1 = \int_{\partial\Omega} u_{1,n}^2 d\mathcal{H}^{d-1} = \int_{\partial\Omega} \chi_{\Gamma_{S,n}} u_{1,n}^2 d\mathcal{H}^{d-1} \rightarrow m \int_{\partial\Omega} u^2 d\mathcal{H}^{d-1} = m < 1,$$

this is a contradiction. □

We want now to construct an explicit example in the plane where we can understand more deeply the divergence of the sequence $\lambda_1(\Omega, \Gamma_{S,n})$. This example actually show that the divergence of the sequence $\lambda_1(\Omega, \Gamma_{S,n})$ is linked to the unboundedness of the number of connected components of the sequence $\Gamma_{S,n}$.

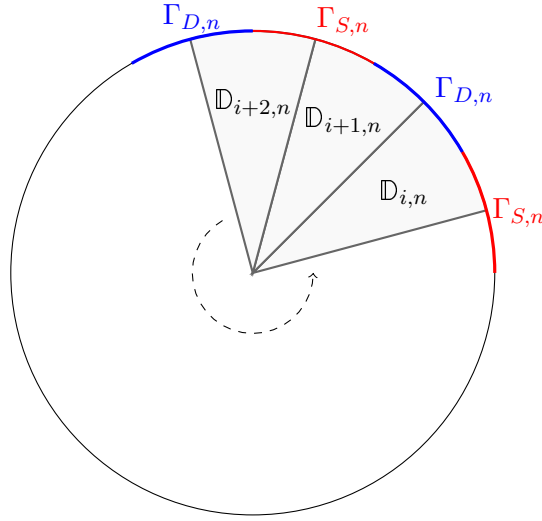


Figure 5.1: Example of the domain constructed in Example 5.4.1

Example 5.4.1. We consider the unit disk \mathbb{D} in the plane and let $n \in \mathbb{N}$ be an even number. We define $\Gamma_{S,n}$ in the following way

$$\Gamma_{S,n} = \bigcup_{k=0}^{\frac{n}{2}-1} \{e^{i\phi} \mid (2k)\frac{2\pi}{n} < \phi < (2k+1)\frac{2\pi}{n}\},$$

and, for all $k = 0, \dots, n-1$, we define

$$\mathbb{D}_{k,n} = \{re^{i\phi} \mid 0 < r < 1, k\frac{2\pi}{n} + \frac{\pi}{n} < \phi < (k+1)\frac{2\pi}{n} + \frac{\pi}{n}\}.$$

Let $u_{1,n}$ be the eigenfunction associated to $\lambda_1(\mathbb{D}, \Gamma_{S,n})$, it is easy to check that there exists an index \bar{j} such that the following holds

$$\lambda_1(\mathbb{D}, \Gamma_{S,n}) = \frac{\int_{\mathbb{D}} |\nabla u_{1,n}|^2 dx}{\int_{\partial\mathbb{D}} u_{1,n}^2 ds} = \frac{\sum_{k=0}^{n-1} \int_{\mathbb{D}_{k,n}} |\nabla u_{1,n}|^2 dx}{\sum_{k=0}^{n-1} \int_{\partial\mathbb{D} \cap \partial\mathbb{D}_{k,n}} u_{1,n}^2 ds} \geq \frac{\int_{\mathbb{D}_{\bar{j},n}} |\nabla u_{1,n}|^2 dx}{\int_{\partial\mathbb{D} \cap \partial\mathbb{D}_{\bar{j},n}} u_{1,n}^2 ds}. \quad (5.25)$$

Now by induction we define a new function \bar{u}_n , in \mathbb{D} , we define \bar{u}_n to be equal to $u_{1,n}$ on $\mathbb{D}_{\bar{j},n}$, on $\mathbb{D}_{\bar{j}+1,n}$ we define \bar{u}_n in the following way

$$\bar{u}_n(r, \phi) = \bar{u}_n(r, (\bar{j}+1)\frac{4\pi}{n} + \frac{2\pi}{n} - \phi) \quad \forall (r, \phi) \in \mathbb{D}_{\bar{j}+1,n},$$

and, for all k , we will define \bar{u}_n on $\mathbb{D}_{k+1,n}$ knowing the function on $\mathbb{D}_{k,n}$ in the following way

$$\bar{u}_n(r, \phi) = \bar{u}_n(r, (k+1)\frac{4\pi}{n} + \frac{2\pi}{n} - \phi) \quad \forall (r, \phi) \in \mathbb{D}_{k+1,n}.$$

From the definition we know that $\bar{u}_n \in H^1(\Omega)$, for all $0 < r_0 \leq 1$ the function $\bar{u}_n(r_0, \phi)$ is periodic in $[0, 2\pi]$ with period $\frac{2\pi}{n}$ and also, from (5.25), we know that

$$\lambda_1(\mathbb{D}, \Gamma_{S,n}) \geq \frac{\int_{\mathbb{D}_{\bar{j},n}} |\nabla u_{1,n}|^2 dx}{\int_{\partial\mathbb{D} \cap \partial\mathbb{D}_{\bar{j},n}} u_{1,n}^2 ds} = \frac{\int_{\mathbb{D}_{\bar{j},n}} |\nabla \bar{u}_n|^2 dx}{\int_{\partial\mathbb{D} \cap \partial\mathbb{D}_{\bar{j},n}} \bar{u}_n^2 ds} = \frac{\int_{\mathbb{D}} |\nabla \bar{u}_n|^2 dx}{\int_{\partial\mathbb{D}} \bar{u}_n^2 ds}. \quad (5.26)$$

The inequality above prove that the function \bar{u}_n is an eigenfunction associated to $\lambda_1(\mathbb{D}, \Gamma_{S,n})$, in particular is an harmonic function on \mathbb{D} .

Finally we know that \bar{u}_n is harmonic in \mathbb{D} and $\bar{u}_n|_{\partial\mathbb{D}}$ is periodic in $[0, 2\pi]$ with period $\frac{2\pi}{n}$, so let c_l be the Fourier coefficients for the function $\bar{u}_n|_{\partial\mathbb{D}}$, the following equalities holds

$$\int_{\partial\mathbb{D}} \bar{u}_n^2 ds = \sum_{l=1}^{\infty} c_l^2,$$

$$\int_{\mathbb{D}} |\nabla \bar{u}_n|^2 dx = \pi n \sum_{l=1}^{\infty} l c_l^2.$$

From the inequalities above and (5.26) we finally conclude that:

$$\lambda_1(\mathbb{D}, \Gamma_{S,n}) = \frac{\int_{\mathbb{D}} |\nabla \bar{u}_n|^2 dx}{\int_{\partial\mathbb{D}} \bar{u}_n^2 ds} \geq \pi n$$

so we finally have that $\lim_{n \rightarrow \infty} \lambda_1(\mathbb{D}, \Gamma_{S,n}) = +\infty$.

Now let Ω be a simply connected domain and let $f : \Omega \rightarrow \mathbb{D}$ be a conformal map. Suppose that Ω is regular enough so that $\min_{z \in \partial\Omega} |f'(z)| > 0$, for instance Ω is a inner domain of a Dini-Jordan curve (see [90]). We define $\hat{\Gamma}_{S,n} := f^{-1}(\Gamma_{S,n})$, let $v_{1,n}$ be the eigenfunction associated to $\lambda_1(\Omega, \hat{\Gamma}_{S,n})$ and let $\hat{v}_n = v_{1,n} \circ f^{-1}$, using the test function \hat{v}_n in the variational characterization of $\lambda_1(\mathbb{D}, \Gamma_{S,n})$ we obtain that

$$\lambda_1(\mathbb{D}, \Gamma_{S,n}) \leq \frac{\int_{\mathbb{D}} |\nabla \hat{v}_n|^2 dx}{\int_{\partial\mathbb{D}} \hat{v}_n^2 ds} = \frac{\int_{\Omega} |\nabla v_{1,n}|^2 dx}{\int_{\partial\Omega} v_{1,n}^2 |f'| ds}.$$

So we have that

$$\min_{z \in \partial\Omega} |f'(z)| \lambda_1(\mathbb{D}, \Gamma_{S,n}) \leq \frac{\int_{\Omega} |\nabla v_{1,n}|^2 dx}{\int_{\partial\Omega} v_{1,n}^2 ds} = \lambda_1(\Omega, \hat{\Gamma}_{S,n})$$

and finally we obtain that $\lim_{n \rightarrow \infty} \lambda_1(\Omega, \hat{\Gamma}_{S,n}) = +\infty$.

5.4.2 Continuity of eigenvalues under topological constraints

Motivated by the Example 5.4.1, and also by the explicit construction that we use to prove Theorem 5.3, it is natural to ask what happened if we put some topological constraint that prevent the phenomenon of the diffusion of $\Gamma_{S,n}$ over all the boundary. In dimension $d = 2$ it is natural to put the constraint on the maximal number of connected components.

With this constraint in dimension $d = 2$ we can compare the L^1 distance of two sets in $\partial\Omega$ with the Hausdorff distance. This comparison, in general without any topological constraint, is not possible (see [58]). Due to this comparison, that is now possible thanks to the topological constraint, we also have a nice control on the constant C_2 in Theorem 5.7 and we can prove a continuity result for the Steklov-Dirichlet eigenvalues.

A similar continuity result for Dirichlet eigenvalues was proved by V. Šverák in [96]. More precisely he proved the following result: let $\lambda_k(\Omega)$ the k -th Dirichlet eigenvalue of the set Ω and we define the following

$$\#(\Omega) = \text{number of connected components of } \Omega$$

Theorem 5.10 (V. Šverák [58, 96]). *Let $\Omega_n \subset \mathbb{R}^2$ be a sequence of bounded open sets converging for the Hausdorff metric to an open set Ω . Assume that there exists a constant C such that $\#(\mathbb{R}^2 \setminus \Omega_n) \leq C$ for all n , then, for all k , the Dirichlet eigenvalues converge*

$$\lambda_k(\Omega_n) \rightarrow \lambda_k(\Omega).$$

We prove now Theorem 5.4 that is a result in the same spirit of Theorem 5.10 in the context of Steklov-Dirichlet eigenvalues.

Proof of Theorem 5.4. Let M be a constant such that $\#(\Gamma_{D,n}) \leq M$ for all n , let $(\partial\Omega, s)$ be the 1-dimensional manifold endowed with the length distance s and let $d_{(\partial\Omega, s)}$ be the Hausdorff distance with respect to the length distance. From the boundedness of the connected components and from the regularity assumption on Ω (the boundary is parametrized by smooth functions with uniform bounded derivatives, see Definition 5.1) we conclude that there exists a constant $C_1(M)$ that depends only on M and a constant $C_2(\Omega)$ that depends only on Ω , such that

$$\mathcal{H}^1(\Gamma_{D,n} \Delta \Gamma_D) \leq C_1(M) d_{(\partial\Omega, s)}(\Gamma_{D,n}, \Gamma_D) \leq C_1(M) C_2(\Omega) d(\Gamma_{D,n}, \Gamma_D)$$

and in particular

$$\begin{aligned} \mathcal{H}^1(\Gamma_{D,n} \Delta \Gamma_D) &\rightarrow 0, \\ \mathcal{H}^1(\Gamma_{S,n} \Delta \Gamma_S) &\rightarrow 0. \end{aligned} \tag{5.27}$$

From Theorem 5.7 we have that there exists a sequence of constants $C_{2,n}$ such that

$$|\lambda_1(\Omega, \Gamma_{S,n}) - \lambda_1(\Omega, \Gamma_S)| \leq C_{2,n} (\mathcal{H}^1(\Gamma_{S,n} \Delta \Gamma_S)^{\frac{1}{2}} + d(\Gamma_{D,n}, \Gamma_D)^{\frac{1}{2}}). \tag{5.28}$$

Now we prove that there exists a constant L such that $C_{2,n} \leq L$ for all n . We call $v_{k,n}$ the function constructed in Theorem 5.7 when we consider $\Gamma'_S = \Gamma_{S,n}$. From the proof of Theorem 5.7 it is clear that $C_{2,n} \leq L$ for all n if and only if there exist a constant $C > 0$ such that, the three following estimates hold

$$\begin{aligned} \int_{\partial\Omega} v_{k,n}^2 d\mathcal{H}^1 &> \frac{1}{C}, \\ \|v_{k,n}\|_{H^1(\Omega)} &\leq C \end{aligned}$$

and

$$\max\{\lambda_k(\Omega, \Gamma_{S,n}), \lambda_k(\Omega, \Gamma_S)\} \leq C.$$

Suppose by contradiction that

$$\liminf_{n \rightarrow \infty} \int_{\partial\Omega} v_{k,n}^2 d\mathcal{H}^1 = 0. \tag{5.29}$$

Let u_k be the Steklov-Dirichlet eigenfunction associated to $\lambda_k(\Omega, \Gamma_S)$. From Theorem 4 in [92], and using similar arguments as in (5.19), we have that:

$$\|u_k - v_{k,n}\|_{H^1(\Omega)} \leq C (\mathcal{H}^1(\Gamma_{S,n} \Delta \Gamma_S)^{\frac{1}{2}} + d(\Gamma_{D,n}, \Gamma_D)^{\frac{1}{2}}),$$

in particular

$$\begin{aligned} v_{k,n} &\rightarrow u_k \quad \text{in } H^1(\Omega), \\ v_{k,n} &\rightarrow u_k \quad \text{in } L^2(\partial\Omega). \end{aligned}$$

From (5.29) we conclude that $\int_{\partial\Omega} u_k^2 d\mathcal{H}^1 = 0$ that is a contradiction, because u_k is a Steklov-Dirichlet eigenfunction. From the convergence above it is also clear that $\|v_{k,n}\|_{H^1(\Omega)}$ is bounded.

Now we prove an upper bound for the quantity $\max\{\lambda_k(\Omega, \Gamma_{S,n}), \lambda_k(\Omega, \Gamma_S)\}$. From (5.27) we know that there exist a set Γ such that $\mathcal{H}^1(\Gamma) > 0$ and a natural number N such that

$$\Gamma \subset \Gamma_{S,n} \cap \Gamma_S \quad \forall n \geq N.$$

Now by Lemma 5.8 we conclude that:

$$\max\{\lambda_k(\Omega, \Gamma_{S,n}), \lambda_k(\Omega, \Gamma_S)\} \leq \lambda_k(\Omega, \Gamma) \quad \forall n \geq N.$$

We finally proved that $C_{2,n} \leq L$ for all n . From the boundedness of $C_{2,n}$, from (5.28), from (5.27) and from the assumption that $\Gamma_{D,n}$ Hausdorff converge to Γ_D we conclude that

$$\limsup_{n \rightarrow \infty} |\lambda_1(\Omega, \Gamma_{S,n}) - \lambda_1(\Omega, \hat{\Gamma}_S)| \leq \limsup_{n \rightarrow \infty} C_{2,n} (\mathcal{H}^1(\Gamma_{S,n} \Delta \hat{\Gamma}_S)^{\frac{1}{2}} + d(\Gamma_{D,n}, \hat{\Gamma}_D)^{\frac{1}{2}}) = 0.$$

This concludes the proof \square

Using this continuity result we can now prove the following existence Theorem:

Theorem 5.11. *Let $\Omega \subset \mathbb{R}^2$ be a $\mathcal{C}^{1,1}$ open domain, let $0 < m_1 < 1$ and $m_2 \in \mathbb{N}$ be two constants, then the following problem*

$$\sup\{\lambda_k(\Omega, \Gamma_S) \mid \mathcal{H}^1(\Gamma_S) = m_1 \mathcal{H}^1(\partial\Omega) \text{ and } \#(\Gamma_S) \leq m_2\}$$

has a solution.

Proof. We define $\mathcal{A} = \{\Gamma_S \subset \partial\Omega \mid \mathcal{H}^1(\Gamma_S) = m_1 \mathcal{H}^1(\partial\Omega) \text{ and } \#(\Gamma_S) \leq m_2\}$ and we consider a maximizing sequence $\Gamma_{S,n}$, we define also the following compact sets $\Gamma_{D,n} = \partial\Omega \setminus \Gamma_{S,n}$. For all n we know that $\Gamma_{S,n} \in \mathcal{A}$, so from the constraint on the measure and the constraint on the number of connected components we have that

$$\|\chi_{\Gamma_{S,n}}\|_{BV(\partial\Omega)} = \mathcal{H}^1(\Gamma_{S,n}) + \mathcal{H}^0(\Gamma_{S,n}) \leq m_1 \mathcal{H}^1(\partial\Omega) + 2m_2 \quad \forall n.$$

From the compactness property of the space $BV(\partial\Omega)$ we have that there exists a set $\hat{\Gamma}_S \subset \partial\Omega$ such that, up to a subsequence

$$\chi_{\Gamma_{S,n}} \rightarrow \chi_{\hat{\Gamma}_S} \quad \text{in } L^1(\partial\Omega).$$

We define the set $\hat{\Gamma}_D = \partial\Omega \setminus \hat{\Gamma}_S$, from the convergence above we conclude that:

$$\mathcal{H}^1(\Gamma_{S,n} \Delta \hat{\Gamma}_S) \rightarrow 0, \tag{5.30}$$

$$\mathcal{H}^1(\Gamma_{D,n} \Delta \hat{\Gamma}_D) \rightarrow 0.$$

Let s be the length distance on $\partial\Omega$, for all $x_1 \in \partial\Omega$ and $x_2 \in \partial\Omega$ we have that $|x_1 - x_2| \leq s(x_1, x_2)$ and in particular

$$d(\Gamma_{D,n}, \hat{\Gamma}_D) \leq d_{(\partial\Omega, s)}(\Gamma_{D,n}, \hat{\Gamma}_D). \tag{5.31}$$

Where we denoted by $d_{(\partial\Omega, s)}(\Gamma_1, \Gamma_2)$ the Hausdorff distance with respect the length distance on $\partial\Omega$. The boundary $\partial\Omega$ is a 1-dimensional manifold, so it is clear that

$$d_{(\partial\Omega, s)}(\Gamma_{D,n}, \hat{\Gamma}_D) \leq \mathcal{H}^1(\Gamma_{D,n} \Delta \hat{\Gamma}_D), \tag{5.32}$$

so from (5.30) we conclude that $\Gamma_{D,n}$ Hausdorff converge to $\hat{\Gamma}_D$. From Theorem 5.4 we conclude that

$$\lambda_k(\Omega, \Gamma_{S,n}) \rightarrow \lambda_k(\Omega, \hat{\Gamma}_S).$$

The fact that $\hat{\Gamma}_S \in \mathcal{A}$ is straightforward, the measure constraint comes directly from (5.30) and the bounds on the number of connected components is preserved by the Hausdorff distance (see [58]). \square

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