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## Moduli spaces of semistable sheaves

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# Moduli spaces of semistable sheaves

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# Abstract

In this thesis we construct moduli spaces of semistable sheaves over a complex smooth projective variety  $X$ , endowed with a fixed polarization  $\mathcal{O}_X(1)$ . Our approach is based on ideas of Le Potier and Jun Li, who independently constructed moduli spaces of slope-semistable torsion-free sheaves over (projective) surfaces. Their spaces are closely related, via the Kobayashi–Hitchin correspondence, to the so-called Donaldson–Uhlenbeck compactification in gauge theory. Here, however, we are mainly interested in the algebraical aspects of their work. In a restrictive sense, this thesis generalizes their construction to higher dimensional pure sheaves, whose support scheme might be singular.

First we introduce a notion of stability for pure coherent sheaves of dimension  $d$  on  $X$ , which lies between slope- and Gieseker-stability. This is defined with respect to the Hilbert polynomial of the sheaf, truncated down to a certain degree. We call it  $\ell$ -(semi)stability, where  $\ell$  marks the level of truncation. In particular, this recovers the classical notion of slope-stability for  $\ell = 1$  and of Gieseker-stability for  $\ell = d$ .

Our construction uses as main ingredient a restriction theorem for (semi)stability, saying that the restriction of an  $\ell$ -semistable (or  $\ell$ -stable) sheaf to a general divisor  $D \in |\mathcal{O}_X(a)|$  of sufficiently large degree in  $X$  is again  $\ell$ -semistable (respectively  $\ell$ -stable). In this regard, in Chapter 2 we prove several restriction theorems for pure sheaves (see Theorems A, B and C). The methods employed in the proofs permit us to give statements in arbitrary characteristic. Furthermore, our results generalize the restriction theorems of Mehta and Ramanathan for slope-(semi)stability, and they apply in particular to Gieseker-semistable sheaves.

Before we give the construction, we take a short detour to generalize the classical Iitaka fibration to the equivariant setting. Given this, we construct projective moduli spaces of  $\ell$ -semistable sheaves in higher dimensions as certain equivariant Iitaka fibrations (see Theorem D). Our construction is new in the literature when  $1 < \ell < d$  or when  $\ell = 1$  and  $d < \dim(X)$ . In particular, in the case of slope-semistable torsion-free sheaves, we recover a result of Huybrechts–Lehn over surfaces and of Greb–Toma in higher dimensions.

Finally, we describe in detail the geometric points of these moduli spaces (see Theorem E). As an application, we show that in the torsion-free case they provide different compactifications of the open locus of slope-stable vector bundles. We can think of these spaces as intermediate compactifications between the Gieseker and the Donaldson–Uhlenbeck compactification.





# Résumé

Dans cette thèse nous construisons des espaces de modules de faisceaux semi-stables sur une variété projective complexe lisse  $X$ , dotée d'une polarisation fixée  $\mathcal{O}_X(1)$ . Notre approche suit les idées de Le Potier et Jun Li, qui ont construit indépendamment des espaces de modules de faisceaux sans torsion, semi-stables par rapport à la pente sur des surfaces (projectives). Leurs espaces sont en relation, par la correspondance Kobayashi–Hitchin, avec la compactification de Donaldson–Uhlenbeck en théorie de jauge. Ici, cependant, nous sommes principalement intéressés par les aspects algébriques de leur travail. En particulier, cette thèse généralise leur construction au cas des faisceaux purs de dimension supérieure, dont le schéma de support peut être singulier.

Nous introduisons d'abord une notion de stabilité pour les faisceaux cohérents purs de dimension  $d$  sur  $X$ , qui se situe entre la stabilité par rapport à la pente et la stabilité de Gieseker. Cette notion est définie par rapport au polynôme de Hilbert du faisceau, tronqué jusqu'à un certain degré. Nous l'appelons  $\ell$ -(semi)stabilité, où  $\ell$  marque le niveau de troncature. En particulier, on retrouve la notion classique de stabilité par rapport à la pente pour  $\ell = 1$  et de stabilité de Gieseker pour  $\ell = d$ .

Notre construction utilise comme ingrédient principal un théorème de restriction pour la (semi-)stabilité, disant que la restriction d'un faisceau  $\ell$ -semistable (ou  $\ell$ -stable) à un diviseur général  $D \in |\mathcal{O}_X(a)|$  de degré suffisamment grand dans  $X$  est à nouveau  $\ell$ -semistable (respectivement  $\ell$ -stable). À cet égard, dans le Chapitre 2, nous prouvons plusieurs théorèmes de restriction pour les faisceaux purs (voir les Théorèmes A, B et C). Les méthodes utilisées dans la preuve nous permettent de donner des énoncés en caractéristique quelconque. De plus, nos résultats généralisent les théorèmes de restriction de Mehta et Ramanathan pour la (semi-)stabilité par rapport à la pente, et ils s'appliquent en particulier aux faisceaux semistables de Gieseker.

Avant de donner la construction, nous faisons un bref détour pour généraliser la fibration d'Itaka classique au cadre équivariant. Nous construisons alors des espaces de modules projectifs de faisceaux  $\ell$ -semistables en dimensions supérieures, comme certaines fibrations d'Itaka équivariantes (voir le Théorème D). Notre construction est nouvelle dans la littérature lorsque  $1 < \ell < d$  ou lorsque  $\ell = 1$  et  $d < \dim(X)$ . En particulier, dans le cas des faisceaux sans torsion, nous récupérons un résultat de Huybrechts–Lehn sur les surfaces et de Greb–Toma en dimensions supérieures.

Enfin, nous décrivons en détail les points géométriques de ces espaces de modules (voir le Théorème E). Comme application, nous montrons que dans le cas sans torsion, ils fournissent des compactifications différentes sur le lieu ouvert des fibrés vectoriels stables par rapport à la pente. Nous pouvons considérer ces espaces comme des compactifications intermédiaires entre la compactification de Gieseker et la compactification de Donaldson–Uhlenbeck.



# Présentation générale

Dans cette section, nous présentons les principaux résultats de cette thèse en français. Soit  $X$  une variété projective lisse (connexe) de dimension  $n \geq 2$  sur  $\mathbb{C}$ . L'objectif de cette thèse est de construire des espaces de modules de faisceaux sur  $X$ . Afin d'obtenir des espaces de modules de type fini, il est nécessaire d'imposer certaines conditions de stabilité sur les faisceaux, telles que la stabilité de Gieseker ou la stabilité par rapport à la pente. La notion de stabilité par rapport à la pente a été introduite pour la première fois par Mumford [66] dans le cadre de l'étude des espaces de modules de faisceaux sur les courbes et a ensuite été généralisée à des dimensions supérieures par Takemoto [81]. D'autre part, la stabilité de Gieseker est une condition plus forte, qui a été introduite par Maruyama [55]. Elle a ensuite été exploitée par Gieseker [21] pour construire des espaces de modules projectifs de faisceaux sans torsion sur des surfaces lisses. Ses méthodes ont été généralisées par Maruyama en dimensions supérieures (voir [56, 57]). En outre, Simpson [78] a par la suite effectué la construction également dans le cas des faisceaux purs de codimension non nulle. Il est important de mentionner que toutes ces constructions utilisent la technique bien connue de la Théorie des Invariants développée par Mumford.

Dans cette thèse, nous introduisons d'abord une notion de stabilité pour les faisceaux cohérents purs de dimension  $d \leq n$  sur  $X$ , qui se situe entre la stabilité par rapport à la pente et la stabilité de Gieseker. Pour la définition, nous fixons un fibré en droites très ample  $\mathcal{O}_X(1)$  sur  $X$ . Cette notion est définie en utilisant le polynôme de Hilbert du faisceau, calculé par rapport à la polarisation  $\mathcal{O}_X(1)$  et tronqué jusqu'à un certain degré (voir Définition 1.2). Nous l'appelons  $\ell$ -(semi)stabilité, où  $\ell$  désigne le niveau de troncature. En particulier, on retrouve la notion classique de stabilité par rapport à la pente pour  $\ell = 1$  et de stabilité de Gieseker pour  $\ell = d$ .

La plupart des propriétés de la (semi)stabilité de Gieseker sont également valables pour la  $\ell$ -(semi)stabilité (voir [36, Sect. 1]), avec des preuves presque identiques. Pour voir certaines de ces propriétés, nous considérons les champs de modules correspondant aux faisceaux  $\ell$ -semistables. Nous choisissons une classe numérique de Grothendieck  $c \in K(X)_{\text{num}}$  de dimension  $d$ , qui fixera le type des faisceaux cohérents. Soit  $\mathfrak{M}_{\mathcal{O}_X(1)}^{[\ell]}(c)$  le champ de modules de faisceaux  $\ell$ -semistables de classe  $c$  sur  $X$ . On remarque la dépendance de la notation par rapport à la polarisation  $\mathcal{O}_X(1)$  choisie. Pour des raisons de facilité, nous omettrons  $c$  et  $\mathcal{O}_X(1)$  dans la notation.

La  $\ell$ -semistabilité étant une propriété ouverte dans les familles plates (voir Lemme 1.1), il est facile de voir que  $\mathfrak{M}^{[\ell]}$  est un sous-champ ouvert et fermé de  $\underline{\text{Coh}}_X$ , le champ de tous les faisceaux cohérents sur  $X$ . Il en résulte que  $\mathfrak{M}^{[\ell]}$  est en fait un champ algébrique localement de type fini, puisque  $\underline{\text{Coh}}_X$  l'est par [80, Tag 09DS]. De plus, par le caractère borné des faisceaux  $\ell$ -semistables (voir Théorème 1.1),  $\mathfrak{M}^{[\ell]}$  est également quasi-compact. Comme nous le montrerons plus tard, ce champ est en fait un champ algébrique quotient.

Afin de construire des espaces de modules propres, une autre propriété importante de  $\mathfrak{M}^{[\ell]}$  est donnée par le résultat de Langton (voir [47] et aussi Théorème 1.2),

qui montre que ce champ est universellement fermé. Cependant, ce champ n'est pas séparé en général, puisque les groupes d'automorphismes des faisceaux  $\ell$ -semistables sont infinis. De plus, il est facile de voir que le champ  $\mathfrak{M}^{[\ell]}$  peut avoir des points fermés avec des stabilisateurs non-réductifs lorsque  $\ell < d$ . Cela ne se produit jamais lorsque  $\ell = d$ , et c'est l'une des différences frappantes entre la  $\ell$ -semistabilité et la semistabilité de Gieseker.

Dans cette thèse, nous voulons construire des espaces de modules de faisceaux  $\ell$ -semistables dans la catégorie des schémas, et non pas simplement comme des champs algébriques. Dans le cas de la semistabilité de Gieseker, on sait que l'on peut construire un *bon* espace de modules  $M_{\mathcal{O}_X(1)}^{Gss}(c)$ , au sens d'Alper [1]. Ceci existe a priori dans la catégorie des espaces algébriques, cependant par les travaux de Gieseker, Maruyama et Simpson, nous savons que  $M^{Gss}$  est en fait un schéma projectif complexe. Ce n'est plus le cas lorsque  $\ell < d$ , puisque l'une des conditions nécessaires à l'existence de bons espaces de modules est que les points fermés possèdent des stabilisateurs réductifs.

Afin de construire des espaces de modules de faisceaux, notre approche suit les idées de Le Potier [49] et Jun Li [53], qui ont construit indépendamment des espaces de modules de faisceaux sans torsion, semi-stables par rapport à la pente sur des surfaces (projectives). Leurs espaces sont en relation, par la correspondance de Kobayashi–Hitchin, avec la compactification de Donaldson–Uhlenbeck en théorie de jauge. Nous devons mentionner que dans le cas sans torsion, Greb et Toma [28] ont généralisé la construction des espaces de modules à des dimensions supérieures, mais ils ont dû restreindre l'étude à la catégorie des variétés faiblement normales. En outre, leurs espaces de modules ont également trouvé des applications dans le contexte de la théorie de la jauge (voir [27]). Ici, cependant, nous sommes principalement intéressés par les aspects algébriques de leur travail. En particulier, cette thèse généralise leur construction au cas des faisceaux purs de dimension supérieure, dont le schéma de support peut être singulier.

## Théorèmes de restriction

L'un des ingrédients clés de la construction est le théorème de restriction de Mehta-Ramanathan, selon lequel la propriété de semistabilité (respectivement de stabilité) par rapport à la pente d'un faisceau est préservée par restriction à une hypersurface générale de degré suffisamment grand. Ces théorèmes jouent un rôle important dans la théorie des faisceaux cohérents, car dans de nombreux cas ils fournissent un instrument utile pour réduire l'étude des faisceaux à des dimensions inférieures. Par exemple, ils ont été utilisés pour montrer l'inégalité de Bogomolov en dimensions supérieures (voir [36, Thm. 7.3.1]).

À ce stade, il existe plusieurs preuves différentes des théorèmes de restriction dans la littérature. Les premiers résultats généraux ont été donnés par Mehta et Ramanathan [61, 62]. Cependant, leurs théorèmes de restriction ne sont pas effectifs, car ils ne fournissent aucune borne sur le degré de l'hypersurface. En caractéristique nulle, Flenner a prouvé plus tard dans [16] un théorème de restriction effectif mais seulement pour la semistabilité. Par une approche différente, Langer a donné dans [42] des théorèmes de restriction effectifs plus forts qui fonctionnent également en caractéristique positive. Nous tenons à souligner que tous les résultats précédents ne s'appliquent qu'aux faisceaux sans torsion sur les variétés lisses.

Dans cette thèse, nous généralisons les théorèmes de restriction au cas des faisceaux purs  $\ell$ -(semi)stables, dont le schéma de support peut être de dimension inférieure ou singulier. Par conséquent, cela résout une conjecture de Langer (voir [44, Conj. 3.13]) par l’affirmative. Plus précisément, nous allons prouver le théorème suivant :

**Théorème** (voir Théorèmes A et B). *Soit  $E$  un faisceau  $\ell$ -semistable (respectivement  $\ell$ -stable) de dimension  $d$  sur  $X$  avec  $\ell < d$ . La restriction  $E|_D$  à un diviseur général  $D \in |\mathcal{O}_X(a)|$  reste  $\ell$ -semistable (respectivement  $\ell$ -stable) pour  $a \gg 0$ .*

Comme conséquence immédiate, nous obtenons également un théorème de restriction pour la semistabilité de Gieseker. Par contre, le résultat est faux pour la stabilité de Gieseker. En effet, le résultat est déjà faux pour un faisceau stable de Gieseker mais non stable par rapport à la pente sur une surface. Voir [7, Exam. 3.1] et [24, Exam. 1.9] pour des exemples explicites de ce genre.

Pour la preuve, nous suivons les méthodes de Mehta et Ramanathan [61, 62], qui fonctionnent en caractéristique quelconque. Malheureusement, cette approche ne donne aucune borne explicite sur le degré à partir duquel la restriction reste  $\ell$ -(semi)stable. A posteriori, par le caractère borné, il existe un degré assez grand qui fonctionne pour tous les faisceaux  $\ell$ -(semi)stables de classe  $c$ . Dans le cas des faisceaux (semi)stables par rapport à la pente, nous pouvons en dire plus à ce sujet. En suivant une approche différente, basée sur les idées de Langer [42], nous prouvons ici un théorème de restriction effectif pour les faisceaux purs (voir Théorème C).

En raison de l’importance géométrique de la stabilité par rapport à la pente, ce théorème de restriction est d’un intérêt indépendant et pourrait avoir des applications dans l’étude des espaces de modules des faisceaux purs supportés en codimension positive. Notre résultat généralise également un théorème de Höring et Peterzell (voir [35, Lem. 2.11]), qu’ils appliquent à l’étude de la décomposition de Beauville-Bogomolov pour les modèles minimaux. Leur résultat et un théorème de Graf [24] sont les seuls autres théorèmes de restriction que nous connaissons sur les espaces singuliers.

## Fibration d’Itaka équivariante

Après avoir prouvé nos théorèmes de restriction, nous digressons pour généraliser la fibration classique d’Itaka au cas équivariant. Ce sera un autre ingrédient utile dans la construction de l’espace de modules. Comme cette partie peut présenter un intérêt indépendant, nous écrivons la construction dans un cadre assez général. Nous fixons un groupe algébrique connexe  $G$  sur  $\mathbb{C}$ .

Notre construction de la fibration d’Itaka équivariante partage certaines similarités avec celle décrite par Lazarsfeld [48, Ch. 2] dans le cas non-équivariant. Puisque nous devons travailler sous certaines hypothèses de compacité, nous introduisons d’abord la notion de  $G$ -propreté (voir Définition 3.1) pour les morphismes de schémas invariants par rapport à  $G$ , qui généralise la notion classique de propreté lorsque  $G$  agit trivialement. Ceci étant, nous définissons ce que nous appelons *la normalisation relative de  $G$*  d’un morphisme  $G$ -propre, une version équivariante de la factorisation bien connue de Stein. Nous allons montrer son existence (voir Proposition 3.1) pour les morphismes  $G$ -propres. Ensuite, nous utilisons la  $G$ -normalisation relative pour construire *la  $G$ -fibration d’Itaka* correspondant à un schéma  $G$ -propre doté d’un fibré en droites  $G$ -équivariant et semi-ample (voir Théorème 3.2).

## Construction des espaces de modules

On peut utiliser les théorèmes de restriction pour construire des espaces de modules de faisceaux semistables. Dans le cas des faisceaux semistables par rapport à la pente, où l'on a les théorèmes de Mehta–Ramanathan, ceci a déjà été fait par Le Potier [49], J. Li [53] et Huybrechts–Lehn [36] sur des surfaces. Dans son article, Le Potier a noté l'obstacle suivant dans sa tentative de généraliser la construction à des dimensions supérieures :

“Faute d'une variante sophistiquée du théorème de restriction de Mehta et Ramanathan, nous n'avons cependant pas pu étendre le théorème 2 en toute dimension.”

Néanmoins, Greb et Toma [28] ont réalisé la construction en dimensions supérieures en utilisant uniquement les théorèmes de restriction de Mehta–Ramanathan. Cependant, dans leur approche, ils ont dû travailler avec certaines hypothèses de normalité faible sur les schémas. Nous verrons dans cette thèse comment la normalisation faible est utilisée dans la construction. Nous renvoyons le lecteur à l'Appendix A pour plus d'informations sur la normalisation faible d'une variété.

Il est important de noter que les travaux mentionnés ci-dessus ne traitent que le cas des faisceaux sans torsion avec un déterminant fixe. Dans cette thèse, nous utiliserons nos théorèmes de restriction pour généraliser la construction au cas des faisceaux purs  $\ell$ -semistables pour  $\ell < d$ . En particulier, nous ne fixons pas le déterminant des faisceaux. Nos résultats sont nouveaux dans la littérature lorsque  $\ell > 1$  ou  $d < \dim(X)$ . Nous allons montrer le théorème suivant :

**Théorème** (voir Théorème D). *Pour  $\ell < d$ , il existe un espace de modules projectif  $M_{\mathcal{O}_X(1)}^{[\ell]}(c)$  de faisceaux  $\ell$ -semistable de classe  $c$  sur  $X$ .*

Plusieurs remarques s'imposent. En premier lieu, nous donnons la construction dans la catégorie des schémas quand  $\ell = d - 1$ , et dans la catégorie des variétés faiblement normales sinon (comme fait par Greb et Toma). On peut alors construire une chaîne de morphismes :

$$(M^{Gss})^{wn} \rightarrow (M^{[d-1]})^{wn} \rightarrow M^{[d-2]} \rightarrow \dots \rightarrow M^{[1]}.$$

Ici,  $M^{Gss}$  et  $M^{[d-1]}$  ne sont pas nécessairement faiblement normaux.

Pour le moment, nous ne savons pas si l'hypothèse de normalité faible est vraiment nécessaire. Dans certains cas, l'espace de modules  $M^{[\ell]}$  est isomorphe à la factorisation de Stein d'un certain morphisme de Quot à Chow (voir Section 4.4). Puisque le schéma de Chow est habituellement construit sur la catégorie des variétés faiblement normales (voir Kollár [40]), cela donne une raison de croire que  $M^{[\ell]}$  a également une structure faiblement normale.

Notez que notre espace  $M^{[\ell]}$  n'est ni un espace de modules grossier ni un bon espace de modules. Cependant, nous donnons une propriété d'unicité pour  $M^{[\ell]}$  (voir Théorème D). En fait, nous construisons cet espace comme une fibration d'Iitaka équivariante appropriée.

## Points géométriques de l'espace de modules

Après la construction, nous étudions en détail les points géométriques des espaces de modules  $M^{[\ell]}$ . Lorsque  $\ell = d - 1$ , nous donnons une description complète des

points géométriques de  $M^{[d-1]}$ , et seulement des résultats partiels dans les autres cas (voir Théorème E). Par conséquent, nous généralisons plusieurs résultats connus dans la littérature au cas des faisceaux  $\ell$ -semistables purs (voir [49, 53, 36, 28]). Voir également [27, Sect. 2.8] pour une discussion détaillée à ce sujet dans le cas sans torsion.

Pour finir, nous donnons quelques applications de notre construction. Tout d'abord, en utilisant le Théorème E, nous montrons que les espaces modulaires  $M^{[\ell]}$  fournissent différentes compactifications du lieu ouvert des faisceaux réflexifs stables par rapport à la pente. On peut considérer ces espaces comme des compactifications intermédiaires entre la compactification de Gieseker et la compactification de Donaldson–Uhlenbeck. Ensuite, nous donnons une application de nos résultats dans le contexte des faisceaux de Higgs. En particulier, nous construisons une compactification projective de l'espace de modules de faisceaux de Higgs stables (par rapport à la pente). Nous notons que cet espace est différent de celui construit par Simpson [79]. Il reste à étudier si cet espace peut être relié aux espaces de modules des connexions Hermite-Einstein en théorie de jauge.





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# Introduction

## Background and motivation

The theory of vector bundles has been extensively developed over the last century. Since the beginning, a strong motivation behind the study has come from physics. Via the pioneering work of Hermann Weyl, among others, physicists have slowly come to realize that many field theories can be mathematically formulated in the language of vector bundles and connections. This line of thought has turned out to be considerably fruitful, providing a unified theoretical framework to describe physical phenomena using gauge theories. In this regard, we have to mention the groundbreaking work of Yang and Mills [85], with their development of non-abelian gauge theories in the study of particle physics.

The Yang–Mills theory has also attracted considerable interest from mathematicians, due to its broad application in various research fields, including hyperkähler geometry, enumerative geometry, moduli theory of vector bundles, etc. At this point, a rather fascinating picture has emerged that unveils deep connections between mathematical physics, differential geometry and algebraic geometry. A remarkable example is the so-called Kobayashi–Hitchin correspondence, which links two apparently distant fields of study: gauge theory and algebraic geometry. In mathematical terms, this says that holomorphic vector bundles over compact complex manifolds admit irreducible Hermite–Einstein metrics (also known as Yang–Mills connections) if and only if they are slope-stable. This important theorem was first established by Narasimhan–Seshadri [68] over compact Riemann surfaces, and it was later generalized to higher dimensions by Donaldson [10] and Uhlenbeck–Yau [83]. In the meantime, it has found many applications in the literature, especially via Donaldson’s theory of invariants (see for example [12, 9, 19, 5]).

Let us fix, for the moment, a compact Kähler manifold  $(X, \omega)$  of dimension  $n$ , where  $\omega$  is the Kähler form. Also, fix  $(E, h)$  a  $\mathcal{C}^\infty$  complex vector bundle on  $X$ , endowed with an Hermitian metric  $h$ . Then, by definition, an  $h$ -unitary connection  $A$  on  $(E, h)$  is called *Hermite–Einstein* if  $A$  is integrable and satisfies

$$\Lambda F_A = -\frac{2\pi i}{(n-1)! \operatorname{vol}(X)} \mu(E) \operatorname{Id}_E.$$

Here  $\Lambda$  is the adjoint of the Lefschetz operator given by wedging with  $\omega$ , and  $\mu(E)$  is called the *slope* of  $E$ , which is defined by

$$\mu(E) = \frac{\operatorname{deg}(E)}{\operatorname{rk}(E)}, \quad \operatorname{deg}(E) := \int_X c_1(E) \wedge \omega^{n-1}.$$

Moreover,  $A$  is called *irreducible* if it has no decomposition  $A = A_1 \oplus A_2$  coming from a splitting  $E = E_1 \oplus E_2$  of the Hermitian smooth vector bundle  $(E, h)$ .

Note that the integrability condition on  $A$  endows  $E$  with a holomorphic structure  $\mathcal{E} = (E, \bar{\partial}_A)$  by the Newlander–Nirenberg theorem. Let  $\mathcal{A}^{1,1}(E, h)$  denote the

set of integrable  $h$ -unitary connections on  $E$ . Then we define the moduli space of irreducible Hermite–Einstein connections on  $(E, h)$  as the quotient set

$$M_{HE}^*(E, h) = \mathcal{A}_{HE}^*(E, h)/\mathcal{U}(E, h),$$

where  $\mathcal{A}^*(E, h) \subset \mathcal{A}^{1,1}(E, h)$  is the subset of irreducible Hermite–Einstein connections, and  $\mathcal{U}(E, h)$  is the group of unitary gauge transformations acting by conjugation. Furthermore, one can show that this moduli space has a (possibly non-reduced) real analytic structure (see [64, Prop. 4.2] and [39, Ch. 7]).

Next we look at slope-stable holomorphic bundle structures on  $E$ . Let us denote by  $\mathcal{A}_{hol}(E)$  the set of holomorphic structures  $\mathcal{E}$  over the underlying smooth vector bundle  $E$ . By definition, a holomorphic vector bundle  $\mathcal{E} \in \mathcal{A}_{hol}(E)$  is called *slope-stable* if  $\mu(\mathcal{F}) < \mu(\mathcal{E})$  for any subbundle  $\mathcal{F} \subset \mathcal{E}$  of rank  $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$ . Then one can form the moduli space of slope-stable vector bundles

$$M^s(E) = \mathcal{A}_{hol}^s(E)/\mathcal{G}$$

where  $\mathcal{A}_{hol}^s(E) \subset \mathcal{A}_{hol}(E)$  is the subset of slope-stable holomorphic structures on  $E$ , and  $\mathcal{G}$  is the group of complex gauge transformations acting by conjugation on  $\mathcal{A}_{hol}^s(E)$ . In this case one can show that  $M^s(E)$  is a complex analytic space. Then, via the Kobayashi–Hitchin correspondence, there is a bijection

$$M_{HE}^*(E, h) \rightarrow M^s(E),$$

which moreover is a real analytic isomorphism (see [54, Ch. 4] for details).

In general the moduli spaces defined above are not compact. In the gauge-theoretic case, one can compactify the moduli space  $M_{HE}^*(E, h)$  of Hermite–Einstein connections via the so-called Donaldson–Uhlenbeck compactification. This was exploited by Donaldson in his study of intersection forms of real 4-manifolds (see [11]). For the construction, he used in an essential manner certain convergence theorems for smooth connections due to Uhlenbeck (see [18, Ch. 8]). There is an analogue construction of the Donaldson–Uhlenbeck compactification in higher dimensions, following Tian [82] (see also [27]).

On the other hand, there is a different compactification of the moduli space  $M^s(E)$  of slope-stable vector bundles by adding Gieseker-semistable torsion-free sheaves at the boundary. In the projective case, this gives a projective compactification, known as the Gieseker–Maruyama–Simpson moduli space. It is natural to wonder if there is an algebro-geometric analogue of the Donaldson–Uhlenbeck compactification. This turns out to be true over smooth projective varieties. In the surface case, Le Potier [49] and J. Li [53] constructed a complex projective moduli space of slope-semistable sheaves. They showed that this moduli space provides a compactification of the open locus of slope-stable vector bundles, which is different from Gieseker’s compactification. Furthermore, Li proved that there is a homeomorphism between this algebraic moduli space and the Donaldson–Uhlenbeck compactification, which restricts to a real analytic isomorphism over the stable locus (see also Morgan [65]). Consequently, one can transfer the complex algebraic structure from the moduli space to the Donaldson–Uhlenbeck compactification.

Following the strategy of Le Potier and J. Li, Huybrechts and Lehn constructed in [36, Ch. 8] a projective scheme  $M^{\mu ss}$  parametrizing slope-semistable sheaves with fixed determinant over a smooth surface. Moreover, their moduli space comes together with a natural transformation  $\mathcal{M}^{[1]} \rightarrow \text{Hom}(-, M^{\mu ss})$  of functors (see (2)

for the definition of  $\mathcal{M}^{[1]}$ ). However, this map is not universal, i.e.  $M^{\mu ss}$  does not corepresent the moduli functor. In the same vein, Greb and Toma [28] later generalized the construction over higher dimensional projective varieties endowed with a multipolarization. Not surprisingly, also their moduli space is related to the Donaldson–Uhlenbeck compactification (see [27]), a reminder of the close ties with gauge theory. It is important to note that the work mentioned above only treats the case of torsion-free sheaves with fixed determinant.

One of the main goals of this thesis is to generalize the construction of Le Potier and Li to the case of pure sheaves supported on a fixed projective variety. Note that such sheaves might have singular support. Besides the strictly mathematical considerations of having such a generalization, another reason for this endeavour comes from gauge theory. This is related to the so-called nonabelian Hodge correspondence, which is a generalized version of the Kobayashi–Hitchin correspondence that we have described above. This time it gives a connection between slope-stable Higgs bundles and certain Hermite–Einstein connections. Recall that a Higgs bundle over  $X$  is given by a pair  $(E, \theta)$  with  $E$  a holomorphic vector bundle and  $\theta : E \rightarrow E \otimes \Omega_X^1$  a homomorphic map such that  $\theta \wedge \theta = 0$ . When  $X$  is projective, the important thing to notice is that a Higgs bundle  $(E, \theta)$  can be thought as a pure coherent sheaf on the cotangent space  $T^*X$  of  $X$  (see Simpson [79, Lem. 6.8]). This fact was exploited by Simpson to construct moduli spaces of Giesker-semistable Higgs sheaves. A similar approach can be taken to construct moduli spaces of slope-semistable Higgs sheaves. As an application, our construction can be used to provide a compactification of the moduli space of Higgs bundles (see Section 4.4), different from Simpson’s compactification.

## Formulating the moduli problem

Before we get to our main results, we want to describe the moduli problem for vector bundles from a purely algebraic perspective. Broadly speaking, the moduli theory studies how vector bundles over a fixed base space  $X$  deform in (continuous, algebraic, differential, etc.) families. Ideally one would like to construct a moduli space of vector bundles on  $X$ , endowed with a geometric structure that will characterize the deformation behaviour of vector bundles. The precise formulation of this moduli problem is usually done in the language of category theory, using the notion of (co)representable functors. In this language, certain properties of the moduli space (such as smoothness, reducedness, separatedness, etc.) can be described in terms of properties of families of vector bundles and vice-versa.

From now on, we place ourselves in the algebraic geometrical setting of scheme theory, and we work over an algebraically closed field  $k$  of characteristic 0. Let  $X$  be a smooth projective variety of dimension  $n$  over  $k$ . The goal is to construct a moduli space of vector bundles on  $X$ , as an algebraic scheme over  $k$ . There are at least two ways of proceeding, using the techniques of geometric invariant theory (GIT) or stack theory. For introducing our moduli problem, we find well suited the second approach. In what follows, we assume the reader is familiar with the basics of stack theory (see [22] for a fast introduction).

When  $X$  is two- or higher-dimensional, families of vector bundles may degenerate at the limit in torsion-free sheaves with “singularities”. This is one reason we prefer to study the deformation theory for torsion-free (coherent) sheaves on  $X$ . As

we are primarily interested in the variation of the algebraic structure, we first fix the numerical type of the coherent sheaves. In this regard, we choose a Grothendieck numerical class  $c \in K(X)_{\text{num}}$  of dimension  $n$ . In order to obtain algebraic families of sheaves keeping the numerical invariants unchanged, the right condition to impose on such families is flatness. Therefore, from now on, we restrict the study to  $S$ -flat families of torsion-free sheaves of class  $c$  on  $X$ , where  $S$  is a  $k$ -scheme of finite type. In other words, these are coherent modules over  $S \times X$ , flat over  $S$  and with torsion-free fibers of class  $c$  on  $X$ . This data is well encoded in the moduli stack  $\mathfrak{M}_X(c)$  of all torsion-free sheaves of class  $c$  on  $X$ .

Via the Hirzebruch–Riemann–Roch formula, one sees that the Euler characteristic of a sheaf on  $X$  is completely determined by its numerical equivalence class. Then, as the Euler characteristic is locally constant in flat families,  $\mathfrak{M}_X(c)$  is an open and closed substack of  $\text{Coh}_X$ , which is the stack of all coherent sheaves on  $X$ . In particular, it follows that  $\mathfrak{M}_X(c)$  is an algebraic (Artin) stack locally of finite type over  $k$  with affine diagonal, since  $\text{Coh}_X$  is so by [80, Tag 09DS].

As the property of being locally free is open in flat families, there is an open substack  $\mathfrak{M}^{lf} \subset \mathfrak{M}_X(c)$  parametrizing only vector bundles. In general this substack of vector bundles is not universally closed. For this reason, we rather work inside the bigger moduli stack of all torsion-free sheaves. It follows by the work of Langton [47] that  $\mathfrak{M}_X(c)$  is in fact universally closed, i.e. it satisfies the existence part of the valuative criterion of properness.

Unfortunately, this stack is not quasi-compact in general. To fix this problem, one has to restrict to a bounded family of sheaves. This can be done by imposing certain stability conditions on sheaves, such as slope- or Gieseker-stability. There is an abundant literature studying these two conditions and their applications in the context of moduli theory (see [36, 63, 45] and references therein). The notion of slope-stability was first introduced by Mumford in [66] while studying moduli of sheaves over curves and was later generalized to higher dimensions by Takemoto [81]. On the other hand, Gieseker-stability is a stronger condition than slope-stability, which was introduced by Maruyama [55]. Then it was exploited by Gieseker [21] to construct projective moduli spaces of torsion-free sheaves over smooth surfaces. His methods were further generalized by Maruyama in higher dimensions (see [56, 57]). In addition, Simpson [78] was later able to work out the construction also in the case of pure sheaves of positive codimension. It is worth mentioning that all these constructions are based on the well-known GIT technique developed by Mumford [67].

Returning to the boundedness problem, it took the efforts of many mathematicians to show that all slope-semistable (respectively Gieseker-semistable) sheaves of fixed numerical class  $c$  on  $X$  form in fact a bounded family (see [3, 81, 58, 78, 42]). In this thesis, we consider a notion of stability for purely  $d$ -dimensional coherent sheaves, with  $d \leq \dim(X)$ , which still satisfies the boundedness property. We call it  $\ell$ -(semi)stability (see Definition 1.2), with  $\ell$  between 1 and  $d$ . It is defined with respect to the Hilbert polynomial of the sheaf, truncated down to degree  $d - \ell$ . Note that this depends on the chosen polarization  $\mathcal{O}_X(1)$  on  $X$  that one uses to compute the Hilbert polynomial.

One recovers the classical notion of slope-(semi)stability for  $\ell = 1$  and of Gieseker-(semi)stability for  $\ell = d$ . This notion of stability is lesser known in the literature for  $1 < \ell < d$ , but it apparently has some applications in the study of

Bridgeland stability conditions on 3-folds (see [75]). In general, there is the following diagram of implications:

$$\begin{array}{ccccccc}
\text{slope-stability} & \implies & \text{2-stability} & \implies & \dots & \implies & \text{Gieseker-stability} \\
\Downarrow & & \Downarrow & & & & \Downarrow \\
\text{slope-semistability} & \iff & \text{2-semistability} & \iff & \dots & \iff & \text{Gieseker-semistability}
\end{array} \tag{1}$$

By using known techniques, one can show that most of the properties of Gieseker-semistability hold also for  $\ell$ -semistability. In particular, we will see in Section 1.2 that the property of being  $\ell$ -semistable is open in flat families of sheaves. In the language of stacks, this means there is an open substack  $\mathfrak{M}_{\mathcal{O}_X(1)}^{[\ell]} \subset \mathfrak{M}_X(c)$  parametrizing all  $\ell$ -semistable sheaves of class  $c$  on  $X$ . We get the following open immersions by (1)

$$\mathfrak{M}_{\mathcal{O}_X(1)}^{G_{ss}} := \mathfrak{M}_{\mathcal{O}_X(1)}^{[n]} \subset \mathfrak{M}_{\mathcal{O}_X(1)}^{[n-1]} \subset \dots \subset \mathfrak{M}_{\mathcal{O}_X(1)}^{[1]} \subset \mathfrak{M}_X(c).$$

Next we list some geometric properties of these moduli stacks (see Table 1 at the end of the Introduction for a summary). As a consequence of the boundedness property,  $\mathfrak{M}_{\mathcal{O}_X(1)}^{[\ell]}$  is in fact an algebraic stack of finite type over  $k$  (with affine diagonal). Furthermore, by Langton's valuative criterion of properness for semistable sheaves [47] (see also Theorem 1.2), this moduli stack is also universally closed.

Unfortunately, the moduli stack  $\mathfrak{M}_{\mathcal{O}_X(1)}^{[\ell]}$  is not proper, since it is not separated. Indeed, as it has affine diagonal, the separatedness property would imply that the moduli stack has finite stabilizers. But this is never the case, since the automorphism groups of semistable sheaves are infinite. There is another reason for the non-separatedness of the moduli stack. This was pointed out by Seshadri [77] in the case of slope-semistable vector bundles over curves, but the same phenomena occurs in our situation. Namely, the problem is that an  $\ell$ -semistable sheaf  $E$  and the graded module  $\text{gr}_{JH}(E)$  corresponding to a Jordan-Hölder filtration of  $E$  can arise as the limit of the same family. Note that  $\text{gr}_{JH}(E)$  is not always unique (see Remark 4.4). This only holds in the case of Gieseker-semistability, that is when  $\ell = \dim(X)$ . In this case, two Gieseker-semistable sheaves are called *S-equivalent* if the corresponding graded modules of their Jordan-Hölder filtrations are isomorphic. As noted, the non-separatedness problem still persists even if one restricts to the open substack  $\mathfrak{M}_{\mathcal{O}_X(1)}^{G_{ss}}$  of Gieseker-stable sheaves, which admit no non-trivial Jordan-Hölder filtrations.

Regarding the moduli problem, the above discussion indicates that the moduli stack of semistable sheaves does not admit in general a *coarse* moduli space, because of the *S*-equivalence relation which identifies non-isomorphic sheaves. Nevertheless, one may still investigate if there exists a *good* moduli space, in the sense of Alper [1], which will parametrize *S*-equivalence classes of semistable sheaves. In zero characteristic, this is equivalent to the fact that the moduli stack is both *S*-complete and  $\Theta$ -reductive (see [2, Thm. A]), which turns out to be true for Gieseker-semistability. Indeed, according to [2, Exam. 7.28], there exists a proper good moduli space of Gieseker-semistable sheaves, endowed with the structure of an algebraic space. In fact, by the work of Gieseker–Maruyama–Simpson, this moduli space is a (projective) scheme. On the other hand, when  $\ell < \dim(X)$ , it is not difficult to see that the moduli stack is neither *S*-complete nor  $\Theta$ -reductive, which obstructs the existence of

a good moduli space in this case. See Langer [46] for a treatment of the slope-stable case.

Now let us reformulate the moduli problem in terms of (co)representable functors. Note that the moduli stack of semistable sheaves is never represented by a scheme, since sheaves have non-trivial automorphisms (multiplication by scalars). Forgetting the groupoid structure, we consider the contravariant moduli functor to sets

$$\mathcal{M}^{[\ell]} := \mathcal{M}_{\mathcal{O}_X(1)}^{[\ell]}(c) : (Sch/k) \rightarrow (Sets) \quad (2)$$

that associates to any  $k$ -scheme  $S$  of finite type the set of all equivalence classes of  $S$ -flat families of  $\ell$ -semistable sheaves of class  $c$  on  $X$ . Here we call two  $S$ -families  $\mathcal{F}$  and  $\mathcal{F}'$  equivalent if and only if  $\mathcal{F} \cong \mathcal{F}' \otimes p_S^*(L)$  for some line bundle  $L \in \text{Pic}(S)$ .

In this language, the representability of the functor  $\mathcal{M}^{[\ell]}$  is equivalent to the existence of a *fine* moduli space. As before, as semistable sheaves have many automorphisms, we should not expect that  $\mathcal{M}^{[\ell]}$  is representable. Instead, one might hope to get representability by restricting the moduli problem to the open subfunctor  $\mathcal{M}^{[\ell],s} \subset \mathcal{M}^{[\ell]}$  corresponding to families of  $\ell$ -stable sheaves. Such stable sheaves are in particular simple, i.e. their automorphism group is  $k^*$  (see Section 1.2). However, even in this case, the representability property often fails. See [36, Sect. 4.6] for a detailed discussion on this topic.

Let us turn to the question of corepresentability for the moduli functor. This holds if one works with the notion of Gieseker-semistability. Indeed, one has the following:

**Theorem** (Mumford, Gieseker, Maruyama, Simpson). *There exists a projective moduli space  $M^{Gss}$  of Gieseker-semistable sheaves of class  $c$  on  $X$  that (universally) corepresents the moduli functor  $\mathcal{M}^{Gss}$ . Geometric points of  $M^{Gss}$  are in bijection with  $S$ -equivalence classes of semistable sheaves. Moreover, there is an open subset  $M^{Gs}$  parametrizing Gieseker-stable sheaves.*

From the point of view of category theory, working with Gieseker-stability gives the right moduli space of sheaves with good functorial properties. Meanwhile, the moduli problem is still open for  $\ell < \dim(X)$ . In this thesis, we will shed some light on this problem and construct a moduli space of  $\ell$ -semistable sheaves in this case.

## Main results

Let us now present the main results of this thesis. In what follows,  $(X, \mathcal{O}_X(1))$  is a fixed polarized connected smooth projective variety of dimension  $n \geq 2$  over an algebraically closed field  $k$ . Choose a Grothendieck numerical class  $c \in K(X)_{\text{num}}$  of dimension  $d \leq n$ , that will fix the numerical type of the coherent sheaves. In this thesis we will construct a moduli space of  $\ell$ -semistable sheaves of class  $c$  on  $X$  when  $\ell < d$  and  $k$  is the field of complex numbers  $\mathbb{C}$ . For this we will follow the methods of Le Potier [49] and J. Li [53]. Some of the results presented below form the content of two preprints [69] and [70] of the author.



# 1 Restriction Theorems

One of the key ingredients in the construction are the Mehta–Ramanathan restriction theorems, saying that the slope-semistability (respectively slope-stability) property of a sheaf is preserved by restriction to a general hypersurface of sufficiently large degree. These theorems play an important role in the theory of coherent sheaves, as in many cases they provide a useful instrument to reduce the study of sheaves to lower dimensions. For example, they were used to show Bogomolov’s inequality in higher dimensions (see [36, Thm. 7.3.1]).

At this point, there are several different proofs of the restriction theorems in the literature. The first general results were given by Mehta and Ramanathan [61, 62]. However, their restriction theorems are not effective, as they provide no bound on the degree of the hypersurface. In zero characteristic, Flenner later proved in [16] an effective restriction theorem but only for slope-semistability. By a different approach, Langer gave in [42] stronger effective restriction theorems that work also in positive characteristic. We wish to emphasize that all these previous results apply only for torsion-free sheaves over smooth varieties.

In this thesis we generalize the restriction theorems to the case of  $\ell$ -(semi)stable pure sheaves, whose support scheme might be lower dimensional or singular. Consequently, this settles a conjecture of Langer (see [44, Conj. 3.13]) in the affirmative. More precisely, we will prove the following two theorems:

**Theorem A.** *Let  $E$  be an  $\ell$ -semistable sheaf of dimension  $d$  on  $X$  with  $\ell < d$ . The restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  remains  $\ell$ -semistable for  $a \gg 0$ .*

**Theorem B.** *Let  $E$  be an  $\ell$ -stable sheaf of dimension  $d$  on  $X$  with  $\ell < d$ . The restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  remains  $\ell$ -stable for  $a \gg 0$ .*

Applying repeatedly Theorem A shows that the  $\ell$ -(semi)stability property of a sheaf is preserved by restriction to general complete intersections of large enough degree. In particular, if we restrict to a complete intersection  $X^{(d-\ell)} = D_1 \cap \dots \cap D_{d-\ell}$  of  $(d-\ell)$  divisors in  $|\mathcal{O}_X(a)|$ , then we recover the notion of Gieseker-(semi)stability over  $X^{(d-\ell)}$ . As an immediate consequence, we obtain a restriction theorem for Gieseker-semistable sheaves (see Corollary 2.1):

**Corollary.** *If  $E$  is a Gieseker-semistable sheaf on  $X$ , then the restriction  $E|_D$  is still Gieseker-semistable for a general divisor  $D \in |\mathcal{O}_X(a)|$  of sufficiently large degree.*

On the other hand, the range for  $\ell$  is sharp in Theorem B. Indeed, the result is already false for a Gieseker-stable but not slope-stable sheaf over a surface. See [7, Exam. 3.1] and [24, Exam. 1.9] for explicit examples of this kind.

For the proof we will follow the methods of Mehta and Ramanathan [61, 62], which work in arbitrary characteristic. Unfortunately, this approach gives no explicit bounds on the degree from which the restriction remains  $\ell$ -(semi)stable. A posteriori, by the boundedness property, there is a large enough degree that works for all  $\ell$ -(semi)stable sheaves of class  $c$ . In the slope-(semi)stable case we can say something more in this regard. Following a different approach, based on ideas of Langer [42], we will prove the following effective restriction theorem:

**Theorem C.** *Let  $E$  be a slope-semistable (resp. slope-stable) sheaf of class  $c$  and multiplicity  $r$ . If*

$$a > \frac{r-1}{r}\delta(c) + \frac{1}{r(r-1)} + r^5(r-1) + \frac{(r-1)\beta_r}{r},$$

*then the restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  is again slope-semistable (resp. slope-stable). See Section 2.3 for the definition of  $\delta(c)$  and  $\beta_r$ .*

Due to the geometric importance of slope-stability, this restriction theorem is of independent interest and might have applications in the study of moduli spaces of pure sheaves supported in positive codimension. Our result also generalizes a theorem of Horing and Peternell (see [35, Lem. 2.11]), which they apply to the study of the Beauville-Bogomolov decomposition for minimal models. Their result and a theorem of Graf [24] are the only other restriction theorems we are aware of over singular spaces.

## 2 Equivariant Iitaka fibration

After proving our restriction theorems, we take a digression to generalize the classical Iitaka fibration to the equivariant case. This will be another useful ingredient in the construction of the moduli space. Since this part can be of independent interest, we write down the construction in a rather general framework. We work over the field of complex numbers  $\mathbb{C}$  and fix a connected algebraic group  $G$ .

Our construction of the equivariant Iitaka fibration shares some similarities with the one described by Lazarsfeld [48, Ch. 2] in the non-equivariant case. Since we need to work under some compactness assumptions, we first introduce the notion of  $G$ -properness (see Definition 3.1) for  $G$ -invariant morphisms of schemes, which generalizes the classical notion of properness when  $G$  acts trivially. Given this, we define what we call *the relative  $G$ -normalization* of a  $G$ -proper morphism, an equivariant version of the well-known Stein factorization. We will show its existence (see Proposition 3.1) for  $G$ -proper morphisms. Then we use the relative  $G$ -normalization to construct *the Iitaka  $G$ -fibration* corresponding to a  $G$ -proper scheme endowed with a  $G$ -equivariant semiample line bundle (see Theorem 3.2).

## 3 Construction of moduli spaces

In what follows we work exclusively over  $\mathbb{C}$ . One can use the restriction theorems to construct moduli spaces of semistable sheaves. In the slope-semistable case, where one has the Mehta–Ramanathan theorems, this was already done by Le Potier [49], J. Li [53] and Huybrechts–Lehn [36] over surfaces. In his article, Le Potier noted the following obstacle in his attempt to generalize the construction to higher dimensions:

“Faute d’une variante sophistiquee du theoreme de restriction de Mehta et Ramanathan, nous n’avons cependant pas pu etendre le theoreme 2 en toute dimension.”

Nevertheless, Greb and Toma [28] were able to work out the construction in higher dimensions by only using the Mehta–Ramanathan restriction theorems. However, in their approach they had to work under some weak normality assumptions

on schemes. We will soon see how the weak normalization comes into play in the construction. We refer the reader to Appendix A for more information on the weak normalization of a variety.

In this thesis we will use our restriction theorems to generalize the work mentioned above to the case of  $\ell$ -semistable pure sheaves for  $\ell < d$ . Our results are new in the literature when  $\ell > 1$  or  $d < \dim(X)$ . We briefly present here the main lines of the construction, see Chapter 4 for the missing details. By the boundedness property, for sufficiently large  $m$ , all  $\ell$ -semistable sheaves of class  $c$  on  $X$  fit inside an open subscheme  $R^{[\ell]}$  of the Quot scheme  $\text{Quot}(V \otimes \mathcal{O}_X(-m), c)$ , where  $V$  is a complex vector space of dimension  $\chi(c(m))$ . Note that any semistable sheaf is uniquely determined in  $R^{[\ell]}$  only up to a base change of  $V$ . We would like to form the quotient  $R^{[\ell]}/\text{SL}(V)$ . For applying the GIT methods, one needs to find an ample  $\text{SL}(V)$ -linearized line bundle on  $R^{[\ell]}$ . As we shall see in Section 4.1, there is a natural method to construct (determinantal) line bundles over  $R^{[\ell]}$ , which reminds of the so-called Fourier–Mukai transform. In this way, we will endow  $R^{[\ell]}$  with a certain  $\text{SL}(V)$ -linearized line bundle  $\mathcal{L}$ . We want to mention in passing that this line bundle is ample when  $\ell = d$ , and it is the one used by Simpson [78] in his GIT construction of the moduli space of Gieseker-semistable sheaves.

Unfortunately, the line bundle  $\mathcal{L}$  is not ample when  $\ell < d$ , so we cannot apply the Hilbert-Mumford criterion in this case, in order to identify the GIT semistable points. Instead, we will show that  $\mathcal{L}$  is  $\text{SL}(V)$ -semiample, i.e. a power of  $\mathcal{L}$  is globally generated by  $\text{SL}(V)$ -invariant sections, after replacing  $R^{[\ell]}$  by its weak normalization if necessary. Here is the step where our restriction theorems come in need, as they play a crucial role in proving the semiampleness of  $\mathcal{L}$ . We give a short outline of the argument below.

The basic idea is to restrict the universal family of quotients over  $R^{[\ell]}$  to a smooth complete intersection  $X^{(d-\ell)} \subset X$  of  $(d - \ell)$  divisors in  $|\mathcal{O}_X(a)|$ . Via this process one may lose the flatness of the family when  $\ell < d - 1$ . Nevertheless, one can show that the flatness property is still preserved outside a closed subset of codimension  $\geq 2$  in  $R^{ss}$ . Moreover, for a fixed quotient  $[E] \in R^{[\ell]}$ , Restriction Theorem A ensures that  $E|_{X^{(d-\ell)}}$  is still  $\ell$ -semistable if we choose  $X^{(d-\ell)}$  sufficiently general of large enough degree. Note that  $\ell$ -(semi)stability coincides with Gieseker-(semi)stability over  $X^{(d-\ell)}$ . Hence  $E|_{X^{(d-\ell)}}$  lies inside the Simpson moduli space  $M_{X^{(d-\ell)}}^{Gss}$  of Gieseker-semistable sheaves on  $X^{(d-\ell)}$ . This will be enough to construct an  $\text{SL}(V)$ -invariant rational map  $\psi_E : R^{[\ell]} \dashrightarrow M_{X^{(d-\ell)}}^{Gss}$ , well-defined around  $[E]$ , such that some power  $\mathcal{L}^\nu$  is the pull-back via  $\psi_E$  of an ample line bundle on  $M_{X^{(d-\ell)}}^{Gss}$ . This way we will obtain a local  $\text{SL}(V)$ -invariant section of  $\mathcal{L}^\nu$  non-vanishing at  $[E]$ , which moreover will be defined outside a closed subset of codimension  $\geq 2$  in  $R^{[\ell]}$  (see Section 4.1 for details). Then, by using the weak normality assumption as in [28], we will extend this section over the whole space  $R^{[\ell]}$ . Repeating this construction around each point of  $R^{[\ell]}$  will show that  $\mathcal{L}$  is  $\text{SL}(V)$ -semiample.

To end the construction, one then considers the Proj-scheme of the  $\mathbb{C}$ -algebra

$$\bigoplus_{k \geq 0} H^0(R^{[\ell]}, \mathcal{L}^{k\nu})^{\text{SL}(V)}$$

for some power  $\nu > 0$  and declares it the moduli space of  $\ell$ -semistable sheaves. However, as  $R^{[\ell]}$  is not proper, some care has to be taken since this algebra is not a priori of finite type over  $\mathbb{C}$ . In contrast to the approach in [36], we follow a different

path for proving the finite-typeness of this algebra, since we are under the impression that there is a missing step in [36, Prop. 8.2.6] (see Remark 3.2). For this, we will use the equivariant Iitaka fibration that we have constructed in Chapter 3.

As a consequence of Langton's valuative criterion of properness [47], one gets almost immediately that  $R^{[\ell]}$  is an  $\mathrm{SL}(V)$ -proper scheme over  $\mathbb{C}$ , in the sense of Definition 3.1. Then we define our moduli space as the  $\mathrm{SL}(V)$ -equivariant Iitaka fibration corresponding to the pair  $(R^{[\ell]}, \mathcal{L})$ . As a culmination of our work, we obtain the following main theorem proving the existence and uniqueness of a moduli space  $M^{[\ell]}$  parametrizing  $\ell$ -semistable sheaves of class  $c$  on  $X$ . In the statement,  $(\mathcal{M}^{[\ell]})^{wn}$  denotes the restriction of  $\mathcal{M}^{[\ell]}$  to the full subcategory  $(Sch^{wn}/\mathbb{C})$  of weakly normal varieties over  $\mathbb{C}$ .

**Theorem D.** *There exists a unique triple  $(M^{[\ell]}, \mathcal{A}, e)$  formed of a weakly normal projective variety  $M^{[\ell]}$  endowed with an ample line bundle  $\mathcal{A}$  and a natural number  $e > 0$  such that there is a natural transformation  $\Psi : (\mathcal{M}^{[\ell]})^{wn} \rightarrow \mathrm{Hom}_{(Sch^{wn}/\mathbb{C})}(-, M^{[\ell]})$ , that associates to any weakly normal variety  $S$  and any  $S$ -flat family  $\mathcal{E}$  of  $\ell$ -semistable sheaves of class  $c$  on  $X$  a classifying morphism  $\Psi_{\mathcal{E}} : S \rightarrow M^{[\ell]}$ , satisfying the following properties:*

- (1) *For any  $S$ -flat family  $\mathcal{E}$  of  $\ell$ -semistable sheaves of class  $c$  on  $X$ , the classifying morphism  $\Psi_{\mathcal{E}}$  satisfies*

$$\Psi_{\mathcal{E}}^*(\mathcal{A}) \cong \mathcal{L}_{\mathcal{E}}^e,$$

*where  $\mathcal{L}_{\mathcal{E}}$  is the determinant line bundle on  $S$  defined in Section 4.1.*

- (2) *For any other triple  $(M', \mathcal{A}', e')$ , with  $M'$  a projective scheme over  $\mathbb{C}$ ,  $\mathcal{A}'$  an ample line bundle on  $M'$  and  $e'$  a natural number satisfying property (1), we have  $e|e'$  and there exists a unique morphism  $\phi : M^{[\ell]} \rightarrow M'$  such that  $\phi^*\mathcal{A}' \cong \mathcal{A}^{(e'/e)}$ .*

*Moreover, if  $\ell = d - 1$ , then the same result as above holds true over the category  $(Sch/\mathbb{C})$  of schemes of finite type over  $\mathbb{C}$ . In particular, the moduli space  $M^{[d-1]}$  is endowed with the geometric structure of a (possibly non-reduced) projective scheme.*

The result above is particularly interesting when  $\ell = d - 1$ , as there is no need to impose the weak normality assumptions in this case. Then one can construct a chain of morphisms:

$$(M^{Gss})^{wn} \rightarrow (M^{[d-1]})^{wn} \rightarrow M^{[d-2]} \rightarrow \dots \rightarrow M^{[1]}.$$

Here  $M^{Gss}$  and  $M^{[d-1]}$  are not necessarily weakly normal. On the other hand, our construction is done under weak normality assumptions when  $\ell < d - 1$ .

## 4 Geometric points of the moduli space

Once the construction is done, we study in detail the geometric points of the moduli spaces  $M^{[\ell]}$ . Consequently, we generalize several known results in the literature to the case of  $\ell$ -semistable (pure) sheaves (see [49, 53, 36, 28]). See also [27, Sect. 2.8] for a detailed discussion on this in the torsion-free case.

The main result here is the following theorem describing the geometric points of these moduli spaces.

**Theorem E.** *Let  $E_1$  and  $E_2$  be two  $\ell$ -polystable sheaves of class  $c$  on  $X$ . For  $i = 1, 2$ , denote by  $E_i^{[\ell]}$  the  $\ell$ -closure of  $E_i$ , defined by (1.1). Also, denote by  $C_{\max}(E_i)$  the Chow-cycle containing the maximal dimensional irreducible components of  $E_i^{[\ell]}/E_i$  counted with their corresponding multiplicities. If either*

- (1)  $E_1^{[\ell]} \not\cong E_2^{[\ell]}$ , or
- (2)  $E_1^{[\ell]} \cong E_2^{[\ell]}$  with  $C_{\max}(E_1) \neq C_{\max}(E_2)$ ,

*then  $E_1$  and  $E_2$  give distinct points in  $M^{[\ell]}$ . Otherwise, if  $E_1^{[\ell]} \cong E_2^{[\ell]} =: F$ ,  $C_{\max}(E_1) = C_{\max}(E_2) =: \xi$  and, furthermore,  $[E_1]$  and  $[E_2]$  lie in the same connected component of the fiber over  $\xi$  of the Quot-to-Chow morphism from the weak normalization of  $\text{Quot}(F, P_F - P_{E_1})$  to the (weakly normal) Chow variety  $\text{Chow}_{d-\ell-1}(X)$  of cycles of dimension  $(d - \ell - 1)$  in  $X$ , then  $E_1$  and  $E_2$  define the same point in  $M^{[\ell]}$ .*

This result is in the same spirit of the one given by Greb–Toma [28] in higher dimensions. The Chow variety referred to in the statement is that constructed by Kollár in [40, Ch. I], and it is naturally endowed with a weakly normal structure. We should mention here that the relation between the geometry of the moduli spaces  $M^{[\ell]}$  and appropriate Quot-to-Chow morphisms is not at all arbitrary. As we shall see in Section 4.4, in some cases our moduli spaces arise as the Stein factorization of certain Quot-to-Chow morphisms.

Furthermore, when  $\ell = d - 1$ , we are able to give a complete description of the geometric points of  $M^{[d-1]}$ . In this case one can show that the corresponding Quot-to-Chow morphism has connected fibers (see Theorem 4.3). This generalizes to higher dimensions the result given by J. Li [53, Thm. 4] over surfaces (see also [36, Thm. 8.2.11]).

In the end, we give some applications of our construction. First, using Theorem E, we show that the moduli spaces  $M^{[\ell]}$  provide different compactifications of the open locus of slope-stable reflexive sheaves. One can think of these spaces as intermediate compactifications between the Gieseker and the Donaldson–Uhlenbeck compactification. Then we give an application of our results in the context of Higgs bundles. Namely, we construct a projective compactification of the moduli space slope-stable Higgs bundles. We note that this space is different than the one constructed by Simpson [79]. It remains to be investigated whether this space can be related to moduli spaces of Hermite–Einstein connections in gauge theory.

## Relations to existing work

As mentioned earlier, our construction of moduli spaces generalizes previous work of Huybrechts–Lehn [36] and Greb–Toma [28] to the case of  $\ell$ -semistable pure sheaves. Even though our approach follows the same strategy as before, it differs in several important aspects that we wish to point out below.

1. Even in the torsion-free case, our approach is slightly different than that in [36] and [28]. There the authors give the construction in the slope-stable case, that is when  $\ell = 1$ . They follow the same strategy as described above, except that they use the Gieseker–Maruyama construction of  $M_{X^{(d-1)}}^{Gss}$ , which endows  $M_{X^{(d-1)}}^{Gss}$  with a relative ample line bundle over  $\text{Pic}(X^{(d-1)})$ . For this reason, they prefer to fix the determinant of sheaves. Our approach overcomes this restriction, we do not fix the determinant since we use an (absolute) ample line bundle over  $M_{X^{(d-1)}}^{Gss}$  in the

corresponding case, as given by Simpson’s construction [78]. In particular, one can immediately derive from our methods a relative moduli space over  $\text{Pic}(X)$ .

Several results in this direction were also obtained by Piwatz, who constructed a relative moduli space over  $\text{Pic}(X)$  in an unpublished manuscript [71]. Her approach is different from ours and follows the relative construction of Le Potier [49]. Furthermore, she was able to show the semiample-ness of  $\mathcal{L}$  by the use of theta sections, generalizing so the methods of Hein [34] to higher dimensions.

2. As already noted, our construction fills in some missing details in the proof of Huybrechts–Lehn (see [36, Prop. 8.2.6] and Remark 3.2). This is noteworthy since there are other sources in the literature that follow quite closely their exposition in [36, Ch. 8] regarding the algebro-geometric construction of the Donaldson–Uhlenbeck compactification (see [6, 74]).

## Further questions and remarks

Here we collect some open problems that this thesis has not yet solved.

1. A classical problem is to construct (slope-)stable vector bundles over smooth projective varieties. So far there are several known construction techniques in the surface case, involving Serre’s correspondence or elementary transformations. As one would expect, the study becomes more complicated in higher dimensions. To show that our results are not completely trivial in this case, we show in Example 1.1 the existence of Gieseker-stable but not slope-stable sheaves over smooth projective varieties with non-zero irregularity.

2. Regarding the restriction theorems, it can be of interest to find some explicit bounds on the degree of the divisor from which the restriction remains  $\ell$ -(semi)stable. Following the methods of Langer [42], we are able to give such bounds only in the slope-(semi)stable case. This problem is also related to the semiample-ness of  $\mathcal{L}$ . Specifically, we do not know a priori which power of  $\mathcal{L}$  becomes base-point-free. In the slope-stable case, this question was investigated by Le Potier in [50] and Popa in [72], where they give effective base-point-freeness statements by the use of theta sections. We think that their approach can be generalized to the case of  $\ell$ -stability with the help of effective restriction theorems.

Another drawback in our Restriction Theorems A and B is that we do not say explicitly what we mean by a “general divisor” in the statement. Again, we are able to give a more precise result only in the slope-(semi)stable case (see Theorem 2.3), otherwise the problem remains open.

3. There are still many open questions concerning the moduli spaces  $M^{[\ell]}$ . We wish to point out some caveats in our construction. First, we do not know if the weak normality assumption is really necessary. In some cases, the moduli space  $M^{[\ell]}$  is isomorphic to the Stein factorization of a certain Quot-to-Chow morphism (see Section 4.4). Since the Chow scheme is usually constructed over the category of weakly normal varieties (see Kollár [40]), this gives reason to believe that  $M^{[\ell]}$  also inherits a weakly normal structure.

Second, maybe even more disturbing, the moduli space  $M^{[\ell]}$  does not corepresent the moduli functor  $\mathcal{M}^{[\ell]}$  whenever  $\ell < d$ . Thus, from the point of view of category theory, the moduli problem remains open.

Finally, one may wish to give a better description of the geometric points of  $M^{[\ell]}$ . As pointed out, one needs to investigate whether certain Quot-to-Chow morphisms have connected fibers, which is known to be a rather difficult problem in general.

4. Note that our construction treats only the case when  $X$  is endowed with a single polarization  $\mathcal{O}_X(1)$ . In the case of slope-semistability it is not very difficult to generalize these methods to work for a multipolarization. This may prove useful if one wants to study the variation of stability with the change of polarization. In dimension two, there is a rich literature on this topic, especially in the case of Gieseker-stability (see [73, 23, 13, 59]). On the other hand, the study becomes more involved in higher dimensions where there are still many open problems worth considering (see [28, 76, 26]). We also recommend [25] for a detailed survey on the subject.

5. As we have mentioned before, the moduli spaces of slope-semistable sheaves constructed following the ideas of Le Potier and J. Li are closely related to the Donaldson–Uhlenbeck compactification of Hermite–Einstein connections, which is partly due to the Kobayashi–Hitchin correspondence. This restricts in particular to a correspondence between Gieseker-stable vector bundles and *almost* Hermite–Einstein metrics (see Leung [52]). Note that the moduli space  $M^{[d-1]}$  gives a compactification of the open locus of Gieseker-stable vector bundles. Via the Kobayashi–Hitchin correspondence, we expect that there is a gauge-theoretic construction of  $M^{[d-1]}$  that will parametrize almost Hermite–Einstein metrics. For this reason, we think that the moduli spaces constructed in this thesis might have some applications in the context of gauge theory.

## Structure of the thesis

In Chapter 1, we introduce the notion of  $\ell$ -stability and recall some of its basic properties. Here we also define the  $\ell$ -closure of a pure sheaf, that will be used later for describing the geometric points of the moduli space.

In Chapter 2, we prove our Restriction Theorems A, B and C. They are one of the main ingredients for constructing moduli spaces.

In Chapter 3, we take a short detour to construct an equivariant analogue of the Iitaka fibration. First we introduce the notion of  $G$ -properness for  $G$ -invariant morphisms and describe some of its basic properties. Given this, we generalize the Stein factorization to the equivariant case in Section 3.3. Afterwards, in Section 3.4, we construct the equivariant Iitaka fibration. Among other things, we also prove here a devissage lemma for equivariant sheaves on  $G$ -proper schemes.

Chapter 4 represents the core part of this thesis. Here is the place where we construct moduli spaces of sheaves. In Section 4.1, we introduce a class of determinant line bundles which correspond functorially to any  $S$ -flat family of slope-semistable sheaves, and then show their semiamplicity, as stated by Theorem 4.1. As in the paper of Greb–Toma [28], in order to prove the semiamplicity result in higher dimensions we need to work under weak normality assumptions on schemes. In Section 4.2, we give the construction of our moduli spaces and prove Theorem D. Then we study the geometric points of these moduli spaces. In Section 4.4, we give several applications of our results. First, we show that our construction provides different compactifications of the open locus of slope-stable reflexive sheaves. Second, we give

a projective compactification of the moduli space of stable Higgs bundles, different than Simpson's compactification. We end the chapter by presenting a class of examples where our moduli spaces arise as the Stein factorization of certain Hilb-to-Chow morphisms.

The thesis ends with two appendices, the first one about the weak normalization of a variety, and the second about the Quot-to-Chow morphism.

## Convention

Throughout this thesis, a separated reduced scheme of finite type over an algebraically closed field will be called an (algebraic) variety. Under this definition a variety might be reducible.

Property	$\mathfrak{M}_{\mathcal{O}_X(1)}^{G_{ss}}(c)$	$\mathfrak{M}_{\mathcal{O}_X(1)}^{[\ell]}(c)$ with $\ell < \dim(c)$	$\mathfrak{M}_X(c)$	$\underline{\text{Coh}}_X$
locally of finite type	yes	yes	yes	yes
quasicompact	yes	yes	no	no
universally closed	yes	yes	yes	yes
separated	no	no	no	no
$S$ -complete	yes	no	no	yes
$\Theta$ -reductive	yes	no	no	yes
admits a good moduli space	yes	no	no	no

Table 1: Properties of some moduli stacks of sheaves



# Chapter 1

## Preliminaries

In this preliminary chapter we set up notation and terminology. First we introduce the notion of  $\ell$ -(semi)stability and then describe some of its properties. Our main reference here is [36, Sect. 1].

Let  $X$  be a connected smooth projective variety over a field  $k$  and  $\mathcal{O}_X(1)$  a very ample line bundle on  $X$ . For the moment,  $k$  is a field of arbitrary characteristic, which is not necessarily algebraically closed. Let  $E$  be a coherent sheaf of dimension  $d$  on  $X$ . The Hilbert polynomial of  $E$  is given by the Euler characteristic  $P(E, m) := \chi(X, E \otimes \mathcal{O}_X(m))$  and has the form

$$P(E, m) = \sum_{j=0}^d \alpha_{d-j}(E) \frac{m^{d-j}}{(d-j)!},$$

with rational coefficients  $\alpha_{d-j}(E)$ . The leading coefficient  $\alpha_d(E)$  is called the *multiplicity* of  $E$ .

**Definition 1.1.** A coherent sheaf  $E$  on  $X$  is *pure of dimension  $d$*  if any non-zero subsheaf  $F \subset E$  has dimension  $d$ . A pure sheaf of dimension  $\dim(X)$  is also called *torsion-free*.

In general, if  $E$  is not pure, then there exists a unique torsion filtration (see [36, Def. 1.1.4])

$$0 \subset T_0(E) \subset \dots \subset T_d(E) = E,$$

where  $T_i(E)$  is the maximal subsheaf of  $E$  of dimension  $\leq i$ . Note that  $E$  is pure if and only if  $T(E) = 0$ . If the support of  $E$  is equidimensional, we call  $T(E) := T_{d-1}(E)$ , with  $d = \dim(E)$ , the *torsion* of  $E$ .

### 1.1 Stability conditions

Let us first recall the notion of stability in the torsion-free case. If  $E$  is torsion-free of dimension  $d = \dim(X)$ , we define its *rank* by  $\text{rk}(E) := \alpha_d(E)/\alpha_d(\mathcal{O}_X)$ , where  $\alpha_d(\mathcal{O}_X)$  measures the *degree* of  $X$  with respect to  $\mathcal{O}_X(1)$ . The  $\mu$ -*slope* of  $E$  is defined as usual by

$$\mu(E) = \frac{\deg(E)}{\text{rk}(E)}, \quad \deg(E) := c_1(E)c_1(\mathcal{O}_X(1))^{d-1}$$

Recall that  $E$  is called *slope-semistable* (resp. *slope-stable*) if  $\mu(F) \leq \mu(E)$  (resp.  $<$ ) for all subsheaves  $F \subset E$  of rank  $0 < \text{rk}(F) < \text{rk}(E)$ .

A simple computation using the Hirzebruch-Riemann-Roch formula yields

$$\deg(E) = \alpha_{d-1}(E) - \operatorname{rk}(E)\alpha_{d-1}(\mathcal{O}_X).$$

Hence one may also use the ratio  $\hat{\mu}(E) := \alpha_{d-1}(E)/\alpha_d(E)$ , called the  $\hat{\mu}$ -slope of  $E$ , to check the slope-semistability of  $E$ . Indeed, this follows immediately since

$$\mu(E) = \alpha_d(\mathcal{O}_X)\hat{\mu}(E) - \alpha_{d-1}(\mathcal{O}_X).$$

This indicates a straightforward way to define slope-semistability for pure sheaves. We can generalize this notion even further as follows:

**Definition 1.2** (Stability conditions). Let  $E$  be a coherent sheaf of dimension  $d$  on  $X$  and choose  $1 \leq \ell \leq d$ . Then  $E$  is called  $\ell$ -semistable (resp.  $\ell$ -stable) if it is pure and

$$\left( \frac{\alpha_{d-1}(F)}{\alpha_d(F)}, \dots, \frac{\alpha_{d-\ell}(F)}{\alpha_d(F)} \right) \leq \left( \frac{\alpha_{d-1}(E)}{\alpha_d(E)}, \dots, \frac{\alpha_{d-\ell}(E)}{\alpha_d(E)} \right) \quad (\text{resp. } <)$$

for all subsheaves  $F \subset E$  such that  $0 < \alpha_d(F) < \alpha_d(E)$ , where  $\leq$  is the lexicographic order.

**Remark 1.1.** 1. With this definition in place, we recover the notion of slope-(semi)stability for  $\ell = 1$  and of Gieseker-(semi)stability for  $\ell = d$ . More generally, one may define  $\ell$ -stability in the quotient category  $\operatorname{Coh}_d(X)/\operatorname{Coh}_{d-\ell-1}(X)$  (see [36, Sect. 1.6]), where  $\operatorname{Coh}_d(X)$  is the category of coherent sheaves of dimension  $\leq d$ . In that case, an  $\ell$ -semistable sheaf is allowed to have torsion in dimension  $< d - \ell$ . However, throughout this thesis, we prefer to work within the category  $\operatorname{Coh}(X)$  of coherent sheaves, where we assume that  $\ell$ -semistable sheaves are pure.

2. Whenever  $\ell < d$ , note that  $\ell$ -semistability is a *weak* stability condition, in the sense of Joyke [37, Def. 4.1].

To simplify the notation, it is convenient to formulate the  $\ell$ -semistability conditions in terms of truncated Hilbert polynomials. If  $E$  is a coherent sheaf of dimension  $d$ , then we denote by

$$P_\ell(E, m) = \sum_{j=0}^{\ell} \alpha_{d-j}(E) \frac{m^{d-j}}{(d-j)!}$$

the  $\ell$ -truncation of the Hilbert polynomial of  $E$  and by  $p_\ell(E) := P_\ell(E)/\alpha_d(E)$  its reduced form. Then, clearly,  $E$  is  $\ell$ -semistable (resp.  $\ell$ -stable) if and only if it is pure and

$$p_\ell(F) \leq p_\ell(E) \quad (\text{resp. } <)$$

for all subsheaves  $F \subset E$  such that  $0 < \alpha_d(F) < \alpha_d(E)$ . Here we compare the polynomials with respect to the natural lexicographic order of their coefficients.

**Example 1.1.** Here we give an example of a Gieseker-stable but not slope-stable sheaf on  $X$ , when  $X$  has non-zero irregularity, i.e.  $H^1(X, \mathcal{O}_X) \neq 0$ . Let  $x \in X$  be a closed point and

$$0 \rightarrow \mathcal{I}_x \rightarrow \mathcal{O}_X \rightarrow k(x) \rightarrow 0$$

the corresponding short exact sequence. From its induced long exact sequence in cohomology, we obtain a surjection

$$H^1(X, \mathcal{I}_x) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0.$$

By assumption  $X$  has non-zero irregularity, hence  $H^1(X, \mathcal{I}_x) \cong \text{Ext}^1(\mathcal{O}_X, \mathcal{I}_x)$  is non-trivial. This means there exists a non-split extension

$$0 \rightarrow \mathcal{I}_x \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0,$$

with  $E$  a rank 2 torsion-free sheaf on  $X$ . Clearly  $E$  is not slope-stable, since  $E$  and  $\mathcal{O}_X$  have the same slope. On the other hand, it is not difficult to see that  $E$  is Gieseker-stable. Indeed, first notice that

$$P(E)/2 = P(\mathcal{I}_x) + 1/2 = P(\mathcal{O}_X) - 1/2.$$

Now let  $F \subset E$  be a rank 1 subsheaf and consider the composition  $F \rightarrow E \rightarrow \mathcal{O}_X$ , which is either zero or injective. In the first case,  $F$  is a subsheaf of  $\mathcal{I}_x$ , so

$$P(F) \leq P(\mathcal{I}_x) < P(E)/2.$$

Otherwise, there is a short exact sequence

$$0 \rightarrow F \rightarrow \mathcal{O}_X \rightarrow Q \rightarrow 0,$$

with  $Q$  supported in codimension  $\geq 2$ . Hence

$$P(F) = P(\mathcal{O}_X) - P(Q) < P(E)/2,$$

from which we conclude that  $E$  is Gieseker-stable.

## 1.2 Properties of stability

Most of the basic properties of Gieseker-(semi)stability, such as the ones treated in [36, Sect. 1], hold also for  $\ell$ -(semi)stability with almost identical proofs. Here we recall only a few of them that will be needed later.

### Boundedness, openness and properness

The following properties regarding families of  $\ell$ -semistable sheaves are particularly important for studying moduli spaces of such sheaves.

**Theorem 1.1.** *Assume that  $k$  is algebraically closed. The family of  $\ell$ -semistable sheaves of fixed Hilbert polynomial on  $X$  is bounded.*

*Proof.* This follows from the boundedness result for slope-semistability (see [42, Thm. 4.4]), since  $\ell$ -semistable sheaves are in particular slope-semistable.  $\square$

**Lemma 1.1.** *The property of being  $\ell$ -semistable, resp. simple, geometrically  $\ell$ -stable, is open in flat families of sheaves.*

*Proof.* The proof is adapted from [36, Prop. 2.3.1]. The property of being simple, i.e.  $\dim_{k(s)}(\text{End}(\mathcal{F}_s)) = 1$ , is open by the semicontinuity properties of the relative Ext sheaves [4, Satz 3]. We treat below the  $\ell$ -semistable case, the proof in the remaining case is similar. Let  $S$  be a  $k$ -scheme of finite type and  $\mathcal{F}$  an  $S$ -flat family of sheaves on  $X$  with Hilbert polynomial  $P$ . Since the property of being pure is open in flat families, we may assume the fibers  $\mathcal{F}_s$  are pure of dimension  $d := \deg(P)$ . We first show that all destabilizing quotients  $\mathcal{F}_s \rightarrow F'$  fit inside a proper Quot scheme. Consider the following set

$$H = \{P(F') \in \mathbb{Q}[T] : \mathcal{F}_s \rightarrow F' \text{ is a pure quotient of dimension } d \text{ for a geometric point } s \in S \text{ such that } p_\ell(F') < p_\ell\}.$$

where  $p_\ell$  denotes the reduced  $\ell$ -truncation of  $P$ . By a boundedness result due to Grothendieck (see [36, Lem. 1.7.9]) we know that  $H$  is finite. Consider the relative Quot scheme  $\varphi : Q := \text{Quot}(\mathcal{F}, H) \rightarrow S$  of quotients  $[\mathcal{F}_s \rightarrow F']$  with  $P(F') \in H$ . Since  $Q$  is proper, its scheme-theoretic image  $\varphi(Q)$  is a closed subset of  $S$ . Then a fiber  $\mathcal{F}_s$  is  $\ell$ -semistable if and only if  $s$  is not contained in  $\varphi(Q)$ .  $\square$

**Theorem 1.2.** *Assume that  $k$  is algebraically closed. Let  $A$  be a discrete valuation ring with residue field  $k$  and quotient field  $K$ . Let  $E$  be an  $A$ -flat family of  $d$ -dimensional sheaves on  $X$  such that  $E_K = E \otimes_A K$  is  $\ell$ -semistable on  $X_K$ . Then there exists a subsheaf  $F \subset E$  such that  $F_K = E_K$  and  $F_k$  is  $\ell$ -semistable on  $X$ .*

*Proof.* The proof in the slope-semistable case is due to Langton [47]. In fact his argument can be adapted to work also for  $\ell$ -semistability (see [36, Thm. 2.B.1] for a proof).  $\square$

## Simple sheaves

If  $k$  is algebraically closed, it is easy to check that Gieseker-stable sheaves are also simple, in the sense of the following definition:

**Definition 1.3.** Let  $E$  be a coherent sheaf of dimension  $d$  on  $X$  and choose  $1 \leq \ell \leq d$ . Then  $E$  is called  $\ell$ -simple if  $\text{End}(E|_{X \setminus Z}) \cong k$  for every closed subset  $Z \subset X$  such that  $\text{codim}(\text{Supp}(E) \cap Z, \text{Supp}(E)) \geq \ell + 1$ . If  $E$  is  $d$ -simple, i.e.  $\text{End}(E) \cong k$ , we just call it *simple*.

One can extend this property to  $\ell$ -stable sheaves as follows.

**Lemma 1.2.** *Assume that  $k$  is algebraically closed. If  $E$  is an  $\ell$ -stable sheaf on  $X$ , then  $E$  is also  $\ell$ -simple.*

*Proof.* Let  $Y := \text{Supp}(E)$  and choose  $Z \subset Y$  a closed subset such that  $\text{codim}(Z, Y) \geq \ell + 1$ . Let  $\varphi : E|_{Y \setminus Z} \rightarrow E|_{Y \setminus Z}$  be a non-trivial map. Set  $F := i_*(E|_{Y \setminus Z})$  and  $\bar{\varphi} := i_*(\varphi)$ , where  $i : Y \setminus Z \rightarrow Y$  denotes the natural inclusion. As  $E$  and  $F$  are isomorphic outside of  $Z$ , we have that  $F$  is  $\ell$ -stable since  $E$  is so. Let  $G$  be the image of  $\bar{\varphi}$ . Note that  $G$  cannot be a proper subsheaf of  $F$ , otherwise by stability  $p_\ell(F) < p_\ell(G) < p_\ell(F)$ . Hence  $G = F$ , which implies that  $\bar{\varphi}$  is an isomorphism, and so  $\varphi$  is also an isomorphism. We obtain that  $\text{End}(E|_{Y \setminus Z})$  is a finite dimensional division algebra over  $k$ . As  $k$  is algebraically closed, we conclude that  $\text{End}(E|_{Y \setminus Z}) \cong k$ .  $\square$

Note that  $\ell$ -simple sheaves are in particular simple, but the converse is not always true. We describe below a class of sheaves for which the converse holds as well. The following construction is treated in greater generality in [31, Ch. IV, 5.9], we slightly change the notation to suit our purposes.

Let  $E$  be a pure sheaf on  $X$  of schematic support  $Y$ . For  $\ell > 0$ , let  $Y^{[\ell]} \subset Y$  be the subset of points  $y \in Y$  such that  $\dim(\mathcal{O}_{Y,y}) \geq \ell + 1$ . Clearly  $Y^{[\ell]}$  is stable under specialization. Then, as described in [31, Ch. IV, 5.10], one can define the  $Y^{[\ell]}$ -closure (or simply  $\ell$ -closure) of  $E$  as the direct limit

$$E^{[\ell]} := \varinjlim_Z (i_Z)_*(E|_{Y \setminus Z}), \quad (1.1)$$

where  $Z \subset Y^{[\ell]}$  runs through all codimension  $\geq \ell + 1$  closed subsets of  $Y$  and  $i_Z : Y \setminus Z \rightarrow Y$  denotes the natural inclusion. Note that, when  $\ell = 1$ , this definition recovers the classical notion of reflexiveness for sheaves, see Kollár [41, Ch. 9]. This is also equivalent to the notion defined by Huybrechts–Lehn [36, Def. 1.1.9].

Let us now list some properties of  $E^{[\ell]}$ . It follows immediately from the criterion in [31, Ch. IV, 5.11.1] that the limit  $E^{[\ell]}$  is still coherent. Also, there exists a natural injection  $E \rightarrow E^{[\ell]}$  that is moreover an isomorphism in codimension  $\ell$ . If this map is a global isomorphism, then we say that  $E$  is  $\ell$ -closed. The following describes some other nice properties of the  $\ell$ -closure.

**Lemma 1.3.** *Let  $E$  be a pure sheaf on  $X$  of schematic support  $Y$ . The following are equivalent:*

- (1)  $E$  is  $\ell$ -closed,
- (2)  $\text{depth}_y(E) \geq 2$  for every  $y \in Y$  such that  $\dim(\mathcal{O}_{Y,y}) \geq \ell + 1$ .

Furthermore, if  $E$  is  $\ell$ -closed then

$$\text{Hom}(E, E) \rightarrow \text{Hom}(E|_{Y \setminus Z}, E|_{Y \setminus Z})$$

is an isomorphism for every closed subset  $Z \subset Y$  of codimension  $\geq \ell + 1$ . In particular, an  $\ell$ -closed sheaf is simple if and only if it is  $\ell$ -simple.

*Proof.* The first part was proved in [31, Ch. IV, 5.10.5]. In particular, this implies that  $E$  is also pure. The second part is a standard result in local cohomology. Let  $Z \subset Y$  be a closed subset of codimension  $\geq \ell + 1$ . We have  $\text{depth}_Z(E) \geq 2$  and the same holds true for  $\mathcal{H}om(E, E)$  by [80, Tag 0AV5]. By [29, Thm. 3.8], this is equivalent to  $\mathcal{E}xt_Z^i(E, E) = 0$  for  $i = 0, 1$ . This further implies

$$\text{Hom}_Z(E, E) = \text{Ext}_Z^1(E, E) = 0,$$

by using the spectral sequence

$$E_2^{pq} = H^p(Y, \mathcal{E}xt_Z^q(E, E)) \Rightarrow \text{Ext}_Z^{p+q}(E, E)$$

given in [30, VI, Thm. 1.6]. From the long exact sequence (see [30, VI, Cor. 1.9])

$$\text{Hom}_Z(E, E) \rightarrow \text{Hom}(E, E) \rightarrow \text{Hom}(E|_{Y \setminus Z}, E|_{Y \setminus Z}) \rightarrow \text{Ext}_Z^1(E, E)$$

it follows that

$$\text{Hom}(E, E) \rightarrow \text{Hom}(E|_{Y \setminus Z}, E|_{Y \setminus Z})$$

is an isomorphism. □

**Lemma 1.4.** *Let  $S$  be an integral scheme of finite type over  $k$  and  $\xi \in S$  its generic point. Let  $f : Y \rightarrow S$  be a smooth projective morphism and  $\mathcal{E}$  an  $S$ -flat family of pure sheaves such that  $\mathcal{E}_s$  is  $\ell$ -simple for every closed point  $s \in S$ . Then  $\mathcal{E}_\xi$  is also  $\ell$ -simple.*

*Proof.* Let  $\mathcal{F}_\xi := \mathcal{E}_\xi^{[\ell]}$  and let  $\iota : \mathcal{E}_\xi \rightarrow \mathcal{F}_\xi$  be the natural injection. By shrinking  $S$  if necessary, we can extend  $\iota$  to an inclusion  $\mathcal{E} \subset \mathcal{F}$  such that  $\mathcal{F}$  is an  $S$ -flat family of pure sheaves and moreover,  $\mathcal{E}_s$  and  $\mathcal{F}_s$  are isomorphic in codimension  $\ell$  for every  $s \in S$ . This second condition implies that  $\mathcal{F}_s$  is  $\ell$ -simple for every closed point  $s \in S$ . Also,  $\mathcal{E}_\xi$  is  $\ell$ -simple if and only if  $\mathcal{F}_\xi$  is  $\ell$ -simple, hence it is enough to prove the statement for the family  $\mathcal{F}$ . By Lemma 1.1, the generic fiber  $\mathcal{F}_\xi$  is also simple, and using Lemma 1.3 we conclude that  $\mathcal{F}_\xi$  is  $\ell$ -simple.  $\square$

**Lemma 1.5.** *Assume that  $E$  is  $\ell$ -closed of support  $Y$ . Then its restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  is still  $\ell$ -closed.*

*Proof.* Let  $y \in Y$  be a point of codimension  $\geq \ell + 1$  in  $Y$ . As  $E$  is  $\ell$ -closed, by Lemma 1.3 we have  $\text{depth}_y(E) \geq 2$ . In other words,  $E_y$  is a reflexive sheaf on  $\mathcal{O}_{Y,y}$ . Then the restriction  $E|_D$  remains reflexive at  $y$  if  $D \in |\mathcal{O}_X(a)|$  is sufficiently general (see [36, Cor. 1.1.14]), so  $\text{depth}_y(E|_D) \geq 2$  and  $\text{codim}(y, D \cap Y) \geq \ell + 1$ . In conclusion, again by Lemma 1.3, we deduce that  $E|_D$  is  $\ell$ -closed.  $\square$

**Lemma 1.6.** *Assume that  $E$  is  $\ell$ -simple and pure. Then its restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  remains  $\ell$ -simple for  $a \gg 0$ .*

*Proof.* Let  $F := E^{[\ell]}$  be the  $\ell$ -closure of  $E$ . As  $E$  and  $F$  are isomorphic in codimension  $\ell$ , it is enough to prove the statement for  $F$ . By construction,  $\text{depth}_x(F) \geq 2$  for any closed point  $x \in X \cap \text{Supp}(F)$ . As  $X$  is regular of dimension  $n := \dim(X)$ , we have by Auslander-Buchsbaum formula

$$\text{hd}_x(F) = \dim(X) - \text{depth}_x(F) \leq n - 2,$$

which further gives  $\mathcal{E}xt_X^{n-1}(F, F) = 0$ .

For  $D \in |\mathcal{O}_X(a)|$ , consider the short exact sequence

$$0 \rightarrow F(-a) \rightarrow F \rightarrow F|_D \rightarrow 0$$

over  $X$ . Applying the functor  $\text{Hom}_X(F, -)$  we obtain a long exact sequence

$$\text{End}(F) \rightarrow \text{End}(F|_D) \rightarrow \text{Ext}^1(F, F(-a)) \rightarrow \dots \quad (1.2)$$

By using Serre duality and the local-to-global Ext spectral sequence, one obtains

$$\text{Ext}_X^1(F, F(-a)) \cong H^0(X, \mathcal{E}xt_X^{n-1}(F, F) \otimes \omega_X(a)) = 0$$

for  $a \gg 0$ , where  $\omega_X$  denotes the dualizing line bundle on  $X$ . As  $F$  is  $\ell$ -simple, it is in particular simple. It follows by (1.2) that  $F|_D$  is also simple for  $a \gg 0$ . For  $D \in |\mathcal{O}_X(a)|$  general we may further assume that  $F|_D$  is still  $\ell$ -closed. Therefore, by Lemma 1.3, we deduce that  $F|_D$  is  $\ell$ -simple.  $\square$

## The Harder-Narasimhan filtration

We next recall the existence of the Harder-Narasimhan filtration. The result is standard, for this reason we omit its proof.

**Lemma 1.7.** *If  $E$  is a pure sheaf on  $X$ , then  $E$  has a unique  $\ell$ -Harder-Narasimhan filtration*

$$0 = E_0 \subset E_1 \subset \dots \subset E_m = E$$

such that its factors  $E_i/E_{i-1}$  are  $\ell$ -semistable and satisfy

$$p_\ell(E_1) > p_\ell(E_2/E_1) > \dots > p_\ell(E/E_{m-1}).$$

We call  $E/E_{m-1}$  the minimal  $\ell$ -destabilizing quotient of  $E$ . Note that if  $E$  is already  $\ell$ -semistable, then  $E$  coincides with its minimal  $\ell$ -destabilizing quotient. We also denote  $p_{\max}(E) := p_\ell(E_1)$  and  $p_{\min}(E) := p_\ell(E/E_{m-1})$ .

**Lemma 1.8.** *Let  $E$  be a pure sheaf of dimension  $d$  on  $X$  and  $E \rightarrow F$  its minimal  $\ell$ -destabilizing quotient. Let  $E \rightarrow G$  be another pure quotient of dimension  $d$  such that  $\alpha_d(F) = \alpha_d(G)$  and  $p_\ell(F) = p_\ell(G)$ . Then  $G$  and  $F$  are isomorphic.*

*Proof.* If  $G$  is not  $\ell$ -semistable, then its minimal  $\ell$ -destabilizing quotient, say  $H$ , satisfies

$$p_\ell(F) = p_\ell(G) > p_\ell(H).$$

This contradicts the minimality of  $p_\ell(F)$ . Thus we may assume that  $G$  is  $\ell$ -semistable. Let  $K := \text{Ker}(E \rightarrow F)$  and consider the composition

$$K \rightarrow E \rightarrow G.$$

As  $p_{\max}(K) > p_\ell(F) = p_\ell(G)$ , it follows that  $\text{Hom}(K, G) = 0$  (see [36, Lem. 1.1.3]). Thus the above map induces a quotient  $F \rightarrow G$  of pure sheaves. By assumption,  $F$  and  $G$  have the same  $\ell$ -truncated Hilbert polynomial, from which we conclude that  $F$  and  $G$  are isomorphic.  $\square$

We also have a relative version of the Harder-Narasimhan filtration, which will later play a key role in the proof of Restriction Theorem A.

**Lemma 1.9.** *Let  $S$  be an integral scheme of finite type over  $k$  and  $\mathcal{F}$  an  $S$ -flat family of  $\ell$ -semistable sheaves of dimension  $d$  on  $X$ . Then there exists a dense open subset  $U \subset S$  and a filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_m = \mathcal{F}$$

such that

- (1) its factors  $\mathcal{F}_i/\mathcal{F}_{i-1}$  are flat over  $U$ ,
- (2) for each closed point  $s \in U$ , the restriction  $0 = \mathcal{F}_{0,s} \subset \mathcal{F}_{1,s} \subset \dots \subset \mathcal{F}_{m,s} = \mathcal{F}_s$  is the  $\ell$ -Harder-Narasimhan filtration of  $\mathcal{F}_s$ .

*Proof.* The proof is very similar to that in the slope-semistable case [36, Thm. 2.3.2], however we provide a proof for the reader's convenience. It is enough to find a dense open subset  $U \subset S$  and a quotient  $\mathcal{F} \rightarrow \mathcal{E}$  such that  $\mathcal{F}_s \rightarrow \mathcal{E}_s$  is the minimal  $\ell$ -destabilizing quotient of  $\mathcal{F}_s$  for  $s \in U$ . Then we can apply the same argument for the kernel of  $\mathcal{F} \rightarrow \mathcal{E}$  and repeat this process until we obtain the relative Harder-Narasimhan filtration.

Let  $P$  be the Hilbert polynomial of the fibers  $\mathcal{F}_s$ ,  $s \in S$ . Consider the following set

$$H = \{P(F') \in \mathbb{Q}[T] : \mathcal{F}_s \rightarrow F' \text{ is a pure quotient of dimension } d \text{ for a geometric point } s \in S \text{ such that } p_\ell(F') \leq p_\ell\},$$

which is finite by a result of Grothendieck (see [36, Lem. 1.7.9]). For each  $P' \in H$ , we can form the relative Quot scheme  $\text{Quot}(\mathcal{F}, P') \rightarrow S$ . Its image, denoted by  $S(P')$ , is a closed subscheme of  $S$ , and we have

$$S = \bigcup_{P' \in H} S(P').$$

We define a total order on  $H'$  as follows:  $P_1 \preceq P_2$  if and only if  $(p_1)_\ell \leq (p_2)_\ell$  and  $P_1 \geq P_2$  if  $(p_1)_\ell = (p_2)_\ell$ . As  $S$  is integral, there is a subset  $H' \subset H$  such that any  $P' \in H'$  satisfies  $S(P') = S$ . Let  $P_-$  be the minimal polynomial in  $H'$  with respect to the above ordering. Then

$$U = S \setminus \bigcup_{P'' \in H, P'' \prec P_-} S(P'')$$

is a non-empty open subset. Consider the universal quotient  $\mathcal{E}$  of  $\pi : \text{Quot}(\mathcal{F}, P_-) \rightarrow S$ , which parametrizes quotients of  $\mathcal{F}_s$  with Hilbert polynomial  $P_-$ . By construction, if  $s \in U$ , then  $\mathcal{E}_s$  is minimally  $\ell$ -destabilizing. Next we show that  $\pi : \pi^{-1}(U) \rightarrow U$  is an isomorphism, which will complete the proof. Since the minimal  $\ell$ -destabilizing quotient is unique and stable under specialization (see [36, Thm. 1.3.7]), this map is bijective and  $k(s) \cong k(t)$  for every point  $t \in \pi^{-1}(U)$  with  $\pi(t) = s$ . Furthermore, the Zariski tangent space of the fiber of  $\pi^{-1}(s)$  at  $t$  is isomorphic to  $\text{Hom}_{X_s}(F'_s, F''_s)$ , where  $0 \rightarrow F'_s \rightarrow F_s \rightarrow F''_s$  is the corresponding short exact sequence. But this vanishes, since  $F''_s$  is the minimal destabilizing quotient. Hence  $\pi$  is also unramified over  $U$ , thus an isomorphism. □

## Jordan-Hölder filtrations

If  $E$  is an  $\ell$ -semistable sheaf on  $X$ , then one can construct a so-called Jordan-Hölder filtration of  $E$

$$0 = E_0 \subset E_1 \subset \dots \subset E_m = E$$

such that its factors  $E_i/E_{i-1}$  are  $\ell$ -stable with  $p_\ell(E_i/E_{i-1}) = p_\ell(E)$ . This filtration is not unique in general, but nevertheless its graded sheaf  $\text{gr}_{JH}(E) = \bigoplus_i E_i/E_{i-1}$  is so in codimension  $\ell$ . In other words, the  $\ell$ -closure  $\text{gr}_{JH}(E)^{[\ell]}$  is uniquely determined by  $E$  (see Remark 4.4 for a proof).



As in the Gieseker-stable case,  $E$  has a unique extended socle, that will be defined precisely below. The following is adapted from [36, Sect. 1.5] and treats the case of  $\ell$ -stability. A similar discussion for slope-stability can be found in [20, Sect. 6].

**Definition 1.4.** We define the  $\ell$ -socle  $\text{Soc}(E)$  of an  $\ell$ -semistable sheaf  $E$  as the saturation of the sum of all  $\ell$ -stable subsheaves  $F \subset E$  with  $p_\ell(F) = p_\ell(E)$ . The *extended  $\ell$ -socle* of  $E$  is the maximal subsheaf  $F \subset E$  with  $p_\ell(F) = p_\ell(E)$  such that every graded factor of  $\text{gr}_{J_H}(F)$  is isomorphic with a graded factor of  $\text{gr}_{J_H}(\text{Soc}(E))$  in codimension  $\ell$ .

**Lemma 1.10.** *Let  $E$  be an  $\ell$ -stable sheaf on  $X$  and  $k \subset K$  the algebraic closure. Then  $E_K = E \otimes_k K$  equals its extended  $\ell$ -socle.*

*Proof.* The proof is identical to that of [20, Lem. 6.2], which was given in the slope-stable case.  $\square$

**Lemma 1.11.** *Let  $E$  be a coherent sheaf on  $X$ . If  $E$  is  $\ell$ -simple,  $\ell$ -semistable and equals its extended  $\ell$ -socle, then  $E$  is  $\ell$ -stable.*

*Proof.* Suppose that  $E$  is not  $\ell$ -stable, so there exists an  $\ell$ -stable quotient  $E \rightarrow F$  with  $p_\ell(F) = p_\ell(E)$ . As  $E$  equals its extended  $\ell$ -socle, there is an open subset  $U \subset \text{Supp}(E)$  with  $\text{codim}(\text{Supp}(E) \setminus U, \text{Supp}(E)) \geq \ell + 1$  such that  $F|_U \subset E|_U$ . This gives a non-trivial composition  $E|_U \rightarrow F|_U \rightarrow E|_U$ . But  $E$  is  $\ell$ -simple, so  $F|_U \cong E|_U$ , which yields a contradiction.  $\square$

**Lemma 1.12.** *If  $E$  is  $\ell$ -simple, then  $E$  is  $\ell$ -stable if and only if  $E$  is geometrically  $\ell$ -stable.*

*Proof.* This follows from Lemma 1.10 and Lemma 1.11.  $\square$



## Chapter 2

# Restriction Theorems

In this chapter, let  $X$  be a connected smooth projective variety over an algebraically closed field  $k$ , and fix  $\mathcal{O}_X(1)$  a very ample line bundle on  $X$ . We will prove the following restriction theorems:

**Theorem A.** *Let  $E$  be an  $\ell$ -semistable sheaf of dimension  $d$  on  $X$  with  $\ell < d$ . The restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  remains  $\ell$ -semistable for  $a \gg 0$ .*

**Theorem B.** *Let  $E$  be an  $\ell$ -stable sheaf of dimension  $d$  on  $X$  with  $\ell < d$ . The restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  remains  $\ell$ -stable for  $a \gg 0$ .*

**Theorem C.** *Let  $c \in K(X)_{num}$  be a class of dimension  $d$  and multiplicity  $r$ , and let  $E$  be a slope-semistable (resp. slope-stable) sheaf of class  $c$  on  $X$ . If*

$$a > \frac{r-1}{r}\delta(c) + \frac{1}{r(r-1)} + r^5(r-1) + \frac{(r-1)\beta_r}{r},$$

*then the restriction  $E|_D$  to a general divisor  $D \in |\mathcal{O}_X(a)|$  is again slope-semistable (resp. slope-stable). See Section 2.3 for the definition of  $\delta(c)$  and  $\beta_r$ .*

**Notation.** Before we proceed with the proofs, let us fix some notation that will be used throughout this chapter. We fix  $E$  a pure sheaf of dimension  $d$  on  $X$ . Denote by  $Y$  the schematic support of  $E$ , and let  $\mathcal{O}_Y(1)$  be the pullback of  $\mathcal{O}_X(1)$  to  $Y$ .

**Lemma 2.1.** *There exists a linear subspace  $W \subset H^0(Y, \mathcal{O}_Y(1))$  of dimension  $d+1$  that gives a surjective finite morphism  $\pi : Y \rightarrow \mathbb{P}(W)$ .*

*Proof.* By using repeatedly the prime avoidance lemma, one can find  $d+1$  hyperplanes  $H_0, \dots, H_d \subset X$  in general position such that

$$Y \cap H_0 \cap \dots \cap H_d = \emptyset.$$

Note that each  $H_i$  is the zero-locus of some section  $s_i \in H^0(X, \mathcal{O}_X(1))$ . Let  $W$  be the vector subspace of  $H^0(Y, \mathcal{O}_Y(1))$  generated by the  $s_i|_Y$ . Then  $W$  has dimension  $d+1$  and induces a finite surjective morphism  $\pi : Y \rightarrow \mathbb{P}(W)$ .  $\square$

Consider the map  $\pi : Y \rightarrow \mathbb{P} := \mathbb{P}(W)$  constructed in the lemma above. Then  $\pi^*(\mathcal{O}_{\mathbb{P}}(1)) = \mathcal{O}_Y(1)$ , where  $\mathcal{O}_{\mathbb{P}}(1)$  denotes as usual the canonical line bundle on  $\mathbb{P}$  corresponding to a hyperplane  $H \in \mathbb{P}(W^\vee)$ . For  $a > 0$ , denote by  $\mathbb{H}_a := \mathbb{P}(S^a W^\vee)$  the linear system of hypersurfaces of degree  $a$  in  $\mathbb{P}$ . Note that  $\mathbb{H}_a$  also parametrizes effective divisors on  $Y$ . To avoid any confusion, whenever we write  $[D] \in \mathbb{H}_a$ , or simply  $D \in \mathbb{H}_a$ , we refer to a hypersurface of degree  $a$  on  $\mathbb{P}$ . Then its scheme-theoretic inverse image  $\pi^{-1}(D)$  gives an effective divisor on  $Y$ . It is important to notice that any such divisor  $\pi^{-1}(D)$  comes as the restriction of a divisor  $\tilde{D}$  on  $X$ . This will be relevant later in the proof.

**Proof Strategy.** Our approach to Theorems A and B is based on the methods of Mehta and Ramanathan [61, 62]. We briefly present here the general strategy of proof. We will argue by contradiction, assuming that the restriction  $E|_{\pi^{-1}(D)}$  is not  $\ell$ -semistable (or  $\ell$ -stable) for general  $D \in \mathbb{H}_a$  and infinitely many  $a$ . We divided the proof in three main steps:

Step I – Variation of the destabilizing quotient: For each  $a > 0$  we will construct a flat family  $E|_{\pi^{-1}(D)} \rightarrow \mathcal{F}_{Y,a}|_{\pi^{-1}(D)}$  of  $\ell$ -destabilizing quotients over a dense open subset  $U_a \subset \mathbb{H}_a$ . Using the direct image of  $\pi$ , we will push these quotients to  $D \subset \mathbb{P}$ . As  $\pi_*$  is finite, the obtained family, denoted by  $\mathcal{F}_a$ , will still have destabilizing quotients. However, note that  $\pi_*(E)$  is not necessarily  $\ell$ -(semi)stable.

Step II – A degeneration argument: Due to our construction, the determinant  $\det(\mathcal{F}_a|_D)$  will be the restriction to  $D$  of a line bundle  $L_a \in \text{Pic}(\mathbb{P})$ . Next we will show that the line bundles  $L_a$  are isomorphic for infinitely many  $a$ . Here is one place where our proof becomes simpler than that in [61]. Since we are on the projective space  $\mathbb{P}$  now, it is enough to study the variation of  $\deg(L_a)$  with  $a$ . Using an ingenious degeneration argument due to Mehta and Ramanathan, we will show that there are infinitely many line bundles  $L_a$  of the same degree, so they are all isomorphic to a certain line bundle  $L \in \text{Pic}(\mathbb{P})$ .

Step III – Extending the destabilizing quotient: By taking the exterior power of  $\pi_*(E)|_D \rightarrow \mathcal{F}_a|_D$ , we will obtain a quotient  $\sigma_D : \Lambda^r \pi_*(E)|_D \rightarrow L|_D$  for  $D \in U_a$  and sufficiently large  $a$ . Here we use in a crucial manner Step II, more precisely the fact that  $\det(\mathcal{F}_a)|_D$  comes as the restriction of  $L$  to  $D$  whenever  $a$  is large enough. Then, by applying Serre Duality, one can extend  $\sigma_D$  to a quotient  $\Lambda^r \pi_*(E) \rightarrow L$  on  $\mathbb{P}$ . This will be used to construct a quotient  $\pi_*(E) \rightarrow F$  destabilizing  $\pi_*(E)$ . The last step will be to show that  $F$  is in fact coming from  $Y$  via the pushforward  $\pi_*$ . This will produce a destabilizing quotient of  $E$ , which will give a contradiction.

Regarding Theorem C, we follow a different approach, which is due to Langer [42]. The proof is more straightforward and uses in an essential manner Bogomolov's inequality. Again, in order to avoid the difficulties associated with working on singular schemes, we use  $\pi$  to transfer the study to the smooth setting.

## 2.1 Proof of Theorem A

We first treat the semistable case. We will prove the following statement, then Restriction Theorem A will come as an easy consequence.

**Theorem 2.1.** *Assume that  $E$  is  $\ell$ -semistable with  $\ell < d$ . Then the restriction  $E|_{\pi^{-1}(D)}$  to a general divisor  $D \in \mathbb{H}_a$  remains  $\ell$ -semistable for  $a \gg 0$ .*

We will argue by contradiction. In what follows, we assume that there is an infinite subset  $N \subset \mathbb{N}$  such that the restriction  $E|_{\pi^{-1}(D)}$  is not  $\ell$ -semistable for any  $D \in \mathbb{H}_a$  and  $a \in N$ . We will make use of this assumption only in Step III of the proof. The basic idea is to extend a proper destabilizing quotient  $E|_{\pi^{-1}(D)} \rightarrow F_D$  over  $Y$ . Once this is achieved, then the following lemma will give a contradiction.

**Lemma 2.2.** *Let  $E \rightarrow F$  be a pure quotient. Then  $p_\ell(E|_{\pi^{-1}(D)}) \geq p_\ell(F|_{\pi^{-1}(D)})$  (resp.  $>$ ) if and only if  $p_\ell(E) \geq p_\ell(F)$  (resp.  $>$ ).*

*Proof.* Consider the short exact sequence

$$0 \rightarrow E(-a) \rightarrow E \rightarrow E|_{\pi^{-1}(D)} \rightarrow 0$$

for some divisor  $D \in \mathbb{H}_a$ . By the additivity of the Hilbert polynomial in short exact sequences we obtain

$$\begin{aligned} \alpha_{d-j}(E|_{\pi^{-1}(D)}) &= \alpha_{d-j}(E) - \alpha_{d-j}(E(-a)) \\ &= \sum_{l=1}^j \frac{(-1)^{l-1} a^l}{l!} \alpha_{d-j+l}(E) \end{aligned} \quad (2.1)$$

for  $1 \leq j \leq d$ . Now the inequalities follow in an iterative manner from (2.1).  $\square$

We divided the proof of Theorem 2.1 in three main steps.

## Step I – Variation of the destabilizing quotient

We want to vary the minimal  $\ell$ -destabilizing quotient of  $E|_{\pi^{-1}(D)}$  in a continuous manner with  $D \in \mathbb{H}_a$ . For this we consider the incidence variety  $Z_a = \{(D, x) \in \mathbb{H}_a \times \mathbb{P} \mid x \in D\}$  with its natural projections:

$$\begin{array}{ccc} Z_a & \xrightarrow{q} & \mathbb{P} \\ \downarrow p & & \\ \mathbb{H}_a & & \end{array}$$

**Remark 2.1.** One can describe  $Z_a$  more explicitly as follows: If  $\mathcal{K}$  denotes the kernel of the evaluation map

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(a)) \otimes \mathcal{O}_{\mathbb{P}} \rightarrow \mathcal{O}_{\mathbb{P}}(a),$$

then  $q : Z_a \rightarrow \mathbb{P}$  is in fact isomorphic to the projective bundle  $\mathbb{P}(\mathcal{K}^\vee) \rightarrow \mathbb{P}$ . Also there exists an isomorphism  $\text{Pic}(Z_a) \cong \text{Pic}(\mathbb{P}) \times \text{Pic}(\mathbb{H}_a)$  such that  $\mathcal{O}_{Z_a}(1) \cong p^*(\mathcal{O}_{\mathbb{H}_a}(1))$  (see the proof in [61, Prop. 2.1]).

Let  $Z_{Y,a}$  denote the pullback of  $q : Z_a \rightarrow \mathbb{P}$  along  $\pi$ , which gives the following cartesian diagram

$$\begin{array}{ccc} Z_{Y,a} & \xrightarrow{\eta} & Z_a \\ \downarrow q_Y & & \downarrow q \\ Y & \xrightarrow{\pi} & \mathbb{P} \end{array} \quad (2.2)$$

There is also a natural projection  $p_Y : Z_{Y,a} \rightarrow \mathbb{H}_a$ . For a geometric point  $[D] \in \mathbb{H}_a$ , we often identify the fiber  $p^{-1}([D]) \subset Z_a$  with  $D \subset \mathbb{P}$  via  $q$ . Also, we canonically identify  $p_Y^{-1}([D]) \subset Z_{Y,a}$  with  $\pi^{-1}(D) \subset Y$  via  $q_Y$ .

By Lemma 1.9 there exists a relative  $\ell$ -Harder-Narasimhan filtration of  $q_Y^*E$ , which one can use to prove the following:

**Lemma 2.3.** *For  $a > 0$  there exists a dense open subset  $U_a \subset \mathbb{H}_a$  and a quotient  $q_Y^*E \rightarrow \mathcal{F}_{Y,a}$  over  $Z_{Y,a}$  such that*

- (1) *each divisor  $D \in U_a$  is smooth and integral in  $\mathbb{P}$ ,*

- (2)  $\mathcal{F}_{Y,a}$  is flat over  $U_a$  or, equivalently, the Hilbert polynomial of  $\mathcal{F}_{Y,a}|_{\pi^{-1}(D)}$  does not vary with  $[D] \in U_a$ ,
- (3) for  $D \in U_a$ ,  $E|_{\pi^{-1}(D)}$  is pure, and  $E|_{\pi^{-1}(D)} \rightarrow \mathcal{F}_{Y,a}|_{\pi^{-1}(D)}$  is the minimal  $\ell$ -destabilizing quotient of  $E|_{\pi^{-1}(D)}$ .

*Proof.* Clearly, for general  $D \in \mathbb{H}_a$ , we may assume that  $E|_{\pi^{-1}(D)}$  is still pure (see [36, Cor. 1.1.14]). Furthermore, (2) and (3) are direct consequences of Lemma 1.9. Note that if  $E|_{\pi^{-1}(D)}$  is already  $\ell$ -semistable for some  $D \in \mathbb{H}_a$ , then we can take  $\mathcal{F}_{Y,a} = q_Y^*(E)$ . Finally, (1) holds by Bertini's Theorem.  $\square$

**Remark 2.2.** If  $E|_{\pi^{-1}(D)}$  is already  $\ell$ -semistable, then  $\mathcal{F}_{Y,a} = q_Y^*(E)$ . However, by our assumption,  $q_Y^*E \rightarrow \mathcal{F}_{Y,a}$  is a proper quotient for infinitely many  $a$ .

Consider the following diagram

$$\begin{array}{ccccc} \pi^{-1}(D) & \xleftarrow{\sim q_Y} & p_Y^{-1}([D]) & \xleftarrow{\iota_Y} & Z_{Y,a} \\ \downarrow \pi & & \downarrow \eta_{[D]} & & \downarrow \eta \\ D & \xleftarrow{\sim q} & p^{-1}([D]) & \xleftarrow{\iota} & Z_a \end{array}$$

for some  $[D] \in \mathbb{H}_a$ . Note that  $\eta$  is finite, since  $\pi$  is so. Thus, as the right-hand square above is cartesian, we have by [80, Lem. 02KG] that

$$\eta_*(\mathcal{F}_{Y,a})|_D \cong i^* \eta_*(\mathcal{F}_{Y,a}) \cong \eta_{[D]*}(\iota_Y^* \mathcal{F}_{Y,a}) \cong \pi_*(\mathcal{F}_{Y,a}|_{\pi^{-1}(D)}).$$

Denote  $\mathcal{F}_a := \eta_*(\mathcal{F}_{Y,a})$ . As  $\mathcal{F}_{Y,a}$  is flat over  $U_a$  and  $\pi_*$  is exact, we deduce that  $\mathcal{F}_{Y,a}|_{\pi^{-1}(D)}$  and  $\mathcal{F}_a|_D$  have the same Hilbert polynomial, which moreover does not vary with  $D \in U_a$ . In this regard, we set  $P(a) := P(\mathcal{F}_a|_D) = P(\mathcal{F}_{Y,a}|_{\pi^{-1}(D)})$ ,  $\alpha_{d-j}(a) := \alpha_{d-j}(\mathcal{F}_a|_D) = \alpha_{d-j}(\mathcal{F}_{Y,a}|_{\pi^{-1}(D)})$  for  $1 \leq j \leq d$ ,  $\text{rk}(a) := \text{rk}(\mathcal{F}_a|_D)$ ,  $\text{deg}(a) := \text{deg}(\mathcal{F}_a|_D)$  and so forth.

## Step II – A degeneration argument

As  $Z_a$  is smooth, we can define the determinant of  $\mathcal{F}_a$ , which we denote by  $\mathcal{L}_a$ . As we saw in Remark 2.1,  $\text{Pic}(Z_a) \cong \text{Pic}(\mathbb{P}) \times \text{Pic}(\mathbb{H}_a)$ , thus  $\mathcal{L}_a \cong p^*M_a \otimes q^*L_a$  for some  $L_a \in \text{Pic}(\mathbb{P})$  and  $M_a \in \text{Pic}(\mathbb{H}_a)$ . In particular  $\det(\mathcal{F}_a|_D) \cong L_a|_D$  for every  $D \in \mathbb{H}_a$ .

In this step we show that the line bundles  $L_a$  are isomorphic for infinitely many  $a$ . As we work over the projective space  $\mathbb{P}$ , all line bundles are completely determined by their degree. So we need to study the variation of  $\text{deg}(L_a)$  with  $a$ . For this we use a degeneration argument due to Mehta and Ramanathan [61].

Let  $D = D_1 + D_2$  be a simple normal crossing divisor of degree  $a = a_1 + a_2$  in  $\mathbb{P}$  with  $D_1 \in U_{a_1}$  and  $D_2 \in U_{a_2}$ . Then one can find a smooth (locally closed) curve  $C \subset \mathbb{H}_a$  passing through  $[D]$  such that  $C \setminus [D] \subset U_a$  and  $Z_C := Z_a \times_{\mathbb{H}_a} C$  is smooth in codimension 2 (see [36, Lem. 7.2.3]). Let  $Z_{Y,C} := Z_{Y,a} \times_{\mathbb{H}_a} C$ . Restricting  $q_Y^*E \rightarrow \mathcal{F}_{Y,a}$  to  $Z_{Y,C \setminus [D]}$  gives a flat family of quotients, which induces a canonical morphism of schemes

$$C \setminus [D] \rightarrow \text{Quot}_{p_Y^{-1}(U_a)/U_a}(q_Y^*E, P(a)).$$

As  $C$  is a smooth curve and the Quot scheme is projective, we can extend the above morphism over  $[D]$  and get a  $C$ -flat quotient  $q_Y^*E|_{Z_{Y,C}} \rightarrow \mathcal{F}_{Y,C}$ .

The natural inclusion  $Z_C \subset Z_a$  induces a cartesian diagram

$$\begin{array}{ccc} Z_{Y,C} & \longrightarrow & Z_{Y,a} \\ \downarrow \eta_C & & \downarrow \eta \\ Z_C & \longrightarrow & Z_a \end{array} \quad (2.3)$$

with  $\eta_C$  finite. Then  $\eta_{C*}$  is exact and thus  $\mathcal{F}_C := \eta_{C*}(\mathcal{F}_{Y,C})$  remains flat over  $C$ . Moreover, using [80, Lem. 02KG], one can check that  $\mathcal{F}_C|_D \cong \pi_*(\mathcal{F}_{Y,C}|_{\pi^{-1}(D)})$ .

We need some preparatory notation. We let  $i = 1, 2$  in the following. Set  $F_Y := (\mathcal{F}_{Y,C}|_{\pi^{-1}(D)})/T(\mathcal{F}_{Y,C}|_{\pi^{-1}(D)})$  and  $F_{Y,i} := F_Y|_{\pi^{-1}(D_i)}$ . Here  $T(-)$  denotes the torsion of the corresponding sheaf (see discussion below Definition 1.1). Also denote  $F := (\mathcal{F}_C|_D)/T(\mathcal{F}_C|_D) = \pi_*(F_Y)$ ,  $T := T(\mathcal{F}_C|_D)$ ,  $F_i := F|_{D_i} = \pi_*(F_{Y,i})$  and  $T_i := T(F_i)$ . For each  $a > 0$ , let

$$\beta(a) := \alpha_{d-2}(a) + \frac{a}{2}\alpha_{d-1}(a) = \alpha_{d-2}(a) + \frac{a^2}{2} \text{rk}(a).$$

**Remark 2.3.** We will soon see why  $\beta(a)$  was defined in this way. For the moment, let us notice its relation with the degree of  $L_a$ . For any  $D \in U_a$ , we have:

$$\begin{aligned} \deg(L_a) &= \frac{\deg(L_a|_D)}{a} = \frac{\deg(\mathcal{F}_a|_D)}{a} \\ &= \frac{\alpha_{d-2}(a) - \text{rk}(a)\alpha_{d-2}(\mathcal{O}_D)}{a} && \text{(by definition)} \\ &= \frac{\alpha_{d-2}(a) - \text{rk}(a)(a\alpha_{d-1}(\mathcal{O}_{\mathbb{P}}) - a^2/2)}{a} \\ &= \frac{\beta(a)}{a} - \text{rk}(a)\alpha_{d-1}(\mathcal{O}_{\mathbb{P}}). \end{aligned}$$

**Lemma 2.4.** For  $a_1, a_2 \gg 0$  we have

$$\frac{\beta(a)}{\text{rk}(a)} \geq \frac{\beta(a_1)}{\text{rk}(a_1)} + \frac{\beta(a_2)}{\text{rk}(a_2)}. \quad (2.4)$$

*Proof.* Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_D \rightarrow \mathcal{O}_{D_1} \oplus \mathcal{O}_{D_2} \rightarrow \mathcal{O}_{D_1 \cap D_2} \rightarrow 0$$

over  $\mathbb{P}$  corresponding to the normal crossing divisor  $D = D_1 + D_2$ . After tensoring it with the torsion-free sheaf  $F$  the sequence

$$0 \rightarrow F \rightarrow F_1 \oplus F_2 \rightarrow F|_{D_1 \cap D_2} \rightarrow 0$$

remains exact. We get by additivity

$$\alpha_{d-2}(F) = \alpha_{d-2}(F_1) + \alpha_{d-2}(F_2) - \alpha_{d-2}(F|_{D_1 \cap D_2}).$$

From the short exact sequence

$$0 \rightarrow T_2 \rightarrow F_2 \rightarrow F_2/T_2 \rightarrow 0, \quad (2.5)$$

we also have

$$\alpha_{d-2}(F_2) = \alpha_{d-2}(T_2) + \alpha_{d-2}(F_2/T_2).$$

As  $F_2/T_2$  has no torsion over  $D_1 \cap D_2$ , the restriction of (2.5) to  $D_1 \cap D_2$  gives the short exact sequence

$$0 \rightarrow T_2|_{D_1 \cap D_2} \rightarrow F|_{D_1 \cap D_2} \rightarrow (F_2/T_2)|_{D_1 \cap D_2} \rightarrow 0.$$

Thus

$$\alpha_{d-2}(F|_{D_1 \cap D_2}) = \alpha_{d-2}(T_2|_{D_1 \cap D_2}) + \alpha_{d-2}((F_2/T_2)|_{D_1 \cap D_2}).$$

Putting together the above identities and using  $\alpha_{d-2}(T_2) \geq \alpha_{d-2}(T_2|_{D_1 \cap D_2})$ , we get

$$\begin{aligned} \alpha_{d-2}(F) &\geq \alpha_{d-2}(F_1) + \alpha_{d-2}(F_2/T_2) - \alpha_{d-2}((F_2/T_2)|_{D_1 \cap D_2}) \\ &\geq \alpha_{d-2}(F_1/T_1) + \alpha_{d-2}(F_2/T_2) - \alpha_{d-2}((F_2/T_2)|_{D_1 \cap D_2}). \end{aligned} \quad (2.6)$$

On the other hand, from the short exact sequence

$$0 \rightarrow (F_2/T_2)(-D_1) \rightarrow F_2/T_2 \rightarrow (F_2/T_2)|_{D_1 \cap D_2} \rightarrow 0,$$

it follows by formula (2.1) that

$$\alpha_{d-2}((F_2/T_2)|_{D_1 \cap D_2}) = a_1 \alpha_{d-1}(F_2/T_2).$$

From this we can rewrite (2.6) as

$$\alpha_{d-2}(F) \geq \alpha_{d-2}(F_1/T_1) + \alpha_{d-2}(F_2/T_2) - a_1 \alpha_{d-1}(F_2/T_2).$$

By a similar computation, we also have

$$\alpha_{d-2}(F) \geq \alpha_{d-2}(F_1/T_1) + \alpha_{d-2}(F_2/T_2) - a_2 \alpha_{d-1}(F_1/T_1).$$

Adding the last two inequalities and dividing by 2 we obtain

$$\begin{aligned} \alpha_{d-2}(F) &\geq \alpha_{d-2}(F_1/T_1) + \alpha_{d-2}(F_2/T_2) - \frac{a_1}{2} \alpha_{d-1}(F_2/T_2) - \frac{a_2}{2} \alpha_{d-1}(F_1/T_1) \\ &\geq \alpha_{d-2}(F_1/T_1) + \alpha_{d-2}(F_2/T_2) - \frac{a_1 a_2}{2} \text{rk}(F_2/T_2) - \frac{a_1 a_2}{2} \text{rk}(F_1/T_1). \end{aligned} \quad (2.7)$$

By the flatness of  $\mathcal{F}_C$  over  $C$ , we have  $\alpha_{d-2}(\mathcal{F}_C|_D) = \alpha_{d-2}(a)$ . Moreover, as  $\text{Supp}(T)$  has codimension  $\geq 1$  in  $D$ , we get

$$\alpha_{d-2}(a) = \alpha_{d-2}(\mathcal{F}_C|_D) \geq \alpha_{d-2}(F).$$

We also have  $\text{rk}(F_i/T_i) = \text{rk}(F) = \text{rk}(a)$  (see [20, Lem. 2.12]). From these last considerations, dividing (2.7) by  $\text{rk}(a)$  we obtain

$$\frac{\alpha_{d-2}(a)}{\text{rk}(a)} \geq \frac{\alpha_{d-2}(F_1/T_1)}{\text{rk}(F_1/T_1)} + \frac{\alpha_{d-2}(F_2/T_2)}{\text{rk}(F_2/T_2)} - a_1 a_2. \quad (2.8)$$

Note that

$$\frac{\alpha_{d-2}(F_i/T_i)}{\alpha_{d-1}(F_i/T_i)} = \frac{\alpha_{d-2}(F_{Y_i}/T(F_{Y_i}))}{\alpha_{d-1}(F_{Y_i}/T(F_{Y_i}))} \geq \frac{\alpha_{d-2}(a_i)}{\alpha_{d-1}(a_i)},$$

where the last inequality holds since the right-hand side measures the slope of the minimal  $\ell$ -destabilizing quotient of  $E|_{\pi^{-1}(D_i)}$ . Then (2.8) becomes

$$\frac{\alpha_{d-2}(a)}{\text{rk}(a)} \geq \frac{\alpha_{d-2}(a_1)}{\text{rk}(a_1)} + \frac{\alpha_{d-2}(a_2)}{\text{rk}(a_2)} - a_1 a_2.$$

Adding  $a^2/2$  to both sides, we finally get (2.4).  $\square$



**Lemma 2.5.**  $\frac{\beta(a)}{a \operatorname{rk}(a)}$  is constant for  $a \gg 0$ .

*Proof.* First, let us see that  $\{\frac{\beta(a)}{a \operatorname{rk}(a)} : a > 0\}$  is a discrete subset of  $\mathbb{Q}$ . Indeed, as noticed in Remark 2.3, we have

$$\frac{\beta(a)}{a \operatorname{rk}(a)} = \frac{\deg(L_a)}{\operatorname{rk}(a)} + \alpha_{d-1}(\mathcal{O}_{\mathbb{P}}) \in \frac{\mathbb{Z}}{\operatorname{rk}(\pi_*(E))!}.$$

Second, let us show that  $\{\frac{\beta(a)}{a \operatorname{rk}(a)} : a > 0\}$  is bounded from above. Clearly

$$\frac{\beta(a)}{a \operatorname{rk}(a)} = \frac{\alpha_{d-2}(a)}{\alpha_{d-1}(a)} + \frac{a}{2} \leq \frac{\alpha_{d-2}(\pi_*(E)|_D)}{\alpha_{d-1}(\pi_*(E)|_D)} + \frac{a}{2}$$

Also, by (2.1) we have

$$\begin{aligned} \alpha_{d-2}(\pi_*(E)|_D) &= a\alpha_{d-1}(\pi_*E) - \frac{a^2}{2}\alpha_d(\pi_*E), \\ \alpha_{d-1}(\pi_*(E)|_D) &= a\alpha_d(\pi_*E). \end{aligned}$$

Putting together the last two observations, we get

$$\frac{\beta(a)}{a \operatorname{rk}(a)} \leq \frac{\alpha_{d-1}(\pi_*E)}{\alpha_d(\pi_*E)}.$$

Finally, we are ready to prove that  $a \mapsto \frac{\beta(a)}{a \operatorname{rk}(a)}$  is constant for  $a \gg 0$ . By Lemma 2.4, there is an integer  $a_0$  such that

$$\frac{\beta(a)}{\operatorname{rk}(a)} \geq \frac{\beta(a_1)}{\operatorname{rk}(a_1)} + \frac{\beta(a_2)}{\operatorname{rk}(a_2)}$$

for  $a_1, a_2 \geq a_0$ . Let

$$b := \max \left\{ \frac{\beta(a)}{a \operatorname{rk}(a)} : a \geq a_0 \right\}.$$

and

$$b' := \max \left\{ \frac{\beta(a)}{a \operatorname{rk}(a)} : a \geq a_0 \text{ and } (a, b) = 1 \right\}.$$

Suppose that  $a = cb + c'b'$  for some integers  $c, c' > 0$ . As  $b, b' \geq a_0$ , by Lemma 2.4 we have

$$\begin{aligned} \frac{\beta(a)}{a \operatorname{rk}(a)} &\geq \frac{c}{a} \frac{\beta(b)}{\operatorname{rk}(b)} + \frac{c'}{a} \frac{\beta(b')}{\operatorname{rk}(b')} \\ &\geq \frac{cb}{a} \frac{\beta(b)}{b \operatorname{rk}(b)} + \frac{c'b'}{a} \frac{\beta(b')}{b' \operatorname{rk}(b')} \\ &\geq \frac{\beta(b')}{b' \operatorname{rk}(b')}. \end{aligned}$$

But  $a$  and  $b'$  are coprime, hence we have equality above, which further gives

$$\frac{\beta(a)}{a \operatorname{rk}(a)} = \frac{\beta(b)}{b \operatorname{rk}(b)}.$$

As  $b$  and  $b'$  are coprime, for  $a \gg 0$  there are always integers  $c, c' \geq c_0$  such that  $a = cb + c'b'$ , which concludes the proof.  $\square$

**Lemma 2.6.** *There exists a line bundle  $L \in \text{Pic}(\mathbb{P})$  such that  $L \cong L_a$  for infinitely many  $a$ .*

*Proof.* As we saw in Remark 2.3,

$$\deg(L_a) = \text{rk}(a) \left( \frac{\beta(a)}{a \text{rk}(a)} - \alpha_{d-1}(\mathcal{O}_{\mathbb{P}}) \right).$$

We may clearly assume that  $\text{rk}(a)$  stays constant for infinitely many  $a$ . Now the result follows by Lemma 2.5.  $\square$

### Step III – Extending the destabilizing quotient

By our assumption, there is a (proper) destabilizing quotient  $\pi_*(E)|_D \rightarrow \mathcal{F}_a|_D$  for  $D \in U_a$  and  $a \in N$ . In this step we extend this quotient over  $\mathbb{P}$  for  $a \in N$  large enough. By Lemma 2.6, we may assume that  $L := L_a$  and  $r := \text{rk}(a)$  for every  $a \in N$ . The proof of the following result is inspired by [36, p. 201].

**Proposition 2.1.** *For  $D \in U_a$  of large enough degree  $a \in N$ , there exists a pure quotient  $E \rightarrow F$  such that  $\pi_*(E) \rightarrow \pi_*(F)$  restricts to  $\pi_*(E)|_D \rightarrow \mathcal{F}_a|_D$ .*

*Proof.* Choose  $D \in U_a$  with  $a \in N$ . As  $D$  is smooth,  $\pi_*(E)|_D$  and  $\mathcal{F}_a|_D$  are both locally free sheaves over an open subset  $D' \subset D$  such that  $\text{codim}(D \setminus D', D) \geq 2$ . Taking the  $r$ -th exterior power of  $\pi_*(E)|_{D'} \rightarrow \mathcal{F}_a|_{D'}$  we obtain a morphism  $\sigma_{D'} : \Lambda^r(\pi_*(E)|_{D'}) \rightarrow L|_{D'}$  which is surjective over  $D'$ . Here we use the assumption that  $L = L_a$  for  $a \in N$ , which gives  $\det(\mathcal{F}_a|_{D'}) \cong L_a|_{D'} \cong L|_{D'}$ . Then  $\sigma_{D'}$  corresponds to a map

$$D' \rightarrow \text{Grass}(\pi_*(E), r) \subset \mathbb{P}(\Lambda^r \pi_*(E))$$

that we wish to extend over an open subset  $U \subset \mathbb{P}$  such that  $\text{codim}(\mathbb{P} \setminus U, \mathbb{P}) \geq 2$ .

As  $D'$  has codimension  $\geq 2$  inside the smooth divisor  $D$ , we can extend  $\sigma_{D'}$  to a morphism  $\sigma_D : \Lambda^r(\pi_*(E)|_D) \rightarrow L|_D$ . Now consider the short exact sequence

$$0 \rightarrow L(-a) \rightarrow L \rightarrow L|_D. \quad (2.9)$$

Applying the functor  $\text{Hom}(\Lambda^r \pi_*(E), -)$  yields the exact sequence

$$\text{Hom}(\Lambda^r \pi_*(E), L) \rightarrow \text{Hom}(\Lambda^r \pi_*(E), L|_D) \rightarrow \text{Ext}^1(\Lambda^r \pi_*(E), L(-a)).$$

By Serre duality

$$\text{Ext}^1(\Lambda^r \pi_*(E), L(-a)) \cong H^{d-1}(\mathbb{P}, \Lambda^r \pi_*(E) \otimes L^\vee(a-d-1)) = 0$$

for  $a \gg 0$ . Thus one can extend  $\sigma_D$  to a morphism  $\sigma : \Lambda^r \pi_*(E) \rightarrow L$ . Note that the support of the cokernel of  $\sigma$  intersects  $D$  in a subset of codimension 2, as  $\sigma_D$  is surjective over  $D'$ . Hence  $\sigma$  is surjective over an open subset  $U \subset \mathbb{P}$  such that  $\text{codim}(\mathbb{P} \setminus U, \mathbb{P}) \geq 2$ . As  $\mathbb{P}$  is smooth, we may further assume that  $\Lambda^r \pi_*(E)$  is also locally free over  $U$ . This induces a map  $j : U \rightarrow \mathbb{P}(\Lambda^r \pi_*(E))$ , and one can show that it factorizes through  $\text{Grass}(\pi_*(E), r)$ . Indeed, let  $\mathcal{I} \subset \text{Sym}^\bullet(\Lambda^r \pi_*(E))$  be the ideal subsheaf corresponding to the embedding  $\text{Grass}(\pi_*(E), r) \subset \mathbb{P}(\Lambda^r \pi_*(E))$ . As  $\mathcal{I}$  is finitely generated and graded, we may assume that  $\mathcal{I}$  is generated by finitely

many  $\mathcal{I}_\nu = \mathcal{I} \cap \text{Sym}^\nu(\Lambda^r \pi_*(E))$ ,  $\nu \leq \nu_0$ . Then  $j$  factors through  $\text{Grass}(\pi_*(E), r)$  if and only if the composition

$$\theta_\nu : \mathcal{I}_\nu \rightarrow \text{Sym}^\nu(\Lambda^r \pi_*(E)) \rightarrow L^\nu$$

is zero for  $\nu \leq \nu_0$ . Consider the following exact sequence

$$0 \rightarrow \text{Hom}(\mathcal{I}_\nu, L^\nu) \rightarrow \text{Hom}(\mathcal{I}_\nu, L^\nu|_D) \rightarrow \text{Ext}^1(\mathcal{I}_\nu, L^\nu(-a)) = 0,$$

obtained by applying the functor  $\text{Hom}(\mathcal{I}_\nu, -)$  to (2.9). As before, the last group is trivial for  $a \gg 0$  by using Serre Duality. Hence  $\theta_\nu$  is zero for  $\nu \leq \nu_0$ , since  $\theta_\nu|_D$  is so by construction.

We get a quotient  $\pi_*(E)|_U \rightarrow G'$  that extends  $\pi_*(E)|_{D'} \rightarrow \mathcal{F}_a|_{D'}$  for  $a \in N$  large enough. Let  $K'$  be its kernel and take  $K := \iota^{-1}(i_* K')$ , where  $i : U \rightarrow \mathbb{P}$  is the inclusion and  $\iota : \pi_*(E) \rightarrow \pi_*(E)^{\vee\vee}$  is the canonical injection. Then we have a short exact sequence

$$0 \rightarrow K \rightarrow \pi_*(E) \rightarrow G \rightarrow 0$$

on  $\mathbb{P}$  such that  $G|_U = G'$ . Restricting this to  $D$ , we get a short exact sequence

$$0 \rightarrow K|_D \rightarrow \pi_*(E)|_D \rightarrow G|_D \rightarrow 0.$$

On the other hand, if  $K_a$  denotes the kernel of  $\pi_*(E)|_D \rightarrow \mathcal{F}_a|_D$ , then we have another short exact sequence

$$0 \rightarrow K_a \rightarrow \pi_*(E)|_D \rightarrow \mathcal{F}_a|_D \rightarrow 0.$$

Note that, by construction,  $K|_{D'}$  is isomorphic to  $K_a|_{D'}$ . Since both  $K|_D$  and  $K_a$  are saturated in  $\pi_*(E)|_D$  and have the same double dual, it follows by the lemma below that they are in fact isomorphic over  $D$ . In particular,  $K|_D$  is a  $\pi_*(\mathcal{O}_{\pi^{-1}(D)})$ -submodule of  $\pi_*(E)|_D$ .

**Lemma 2.7.** *Let  $M$  be a torsion-free sheaf over an integral, normal scheme and  $N \subset M$  a subsheaf. Let  $\iota : M \rightarrow M^{\vee\vee}$  be the canonical injection. Then the saturation of  $N$  in  $M$  is equal to  $\iota^{-1}(N^{\vee\vee})$ .*

*Proof.* Denote by  $S$  the saturation of  $N$  in  $M$  and set  $N' := \iota^{-1}(N^{\vee\vee})$ . Then we have an injection  $M/N' \rightarrow M^{\vee\vee}/N^{\vee\vee}$  such that  $M^{\vee\vee}/N^{\vee\vee}$  is torsion-free, and so  $M/N'$  is also torsion-free. Hence  $S \subset N'$ . We get a diagram as follows

$$\begin{array}{ccccccc} 0 & \longrightarrow & S & \longrightarrow & N' & \longrightarrow & N'/S \longrightarrow 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \\ 0 & \longrightarrow & S^{\vee\vee} & \longrightarrow & N'^{\vee\vee} & \longrightarrow & 0 \longrightarrow 0 \end{array}$$

where  $\alpha$  and  $\beta$  are the canonical maps. By applying Snake's Lemma, we obtain an injection  $N'/S \rightarrow \text{Coker}(\alpha)$ . But  $N'/S \subset E/S$  is torsion-free and  $\text{Coker}(\alpha)$  is supported in positive codimension, thus  $N'/S = 0$ .  $\square$

Let  $M$  be the image of the following composition

$$K \otimes \pi_*(\mathcal{O}_Y) \rightarrow \pi_*(E) \otimes \pi_*(\mathcal{O}_Y) \rightarrow \pi_*(E). \quad (2.10)$$

Then  $M$  is a  $\pi_*(\mathcal{O}_Y)$ -module, which we can write as  $M = \pi_*(R)$  for some subsheaf  $R \subset E$ . Tensoring (2.10) with  $\mathcal{O}_D$  we obtain a morphism

$$K|_D \otimes \pi_*(\mathcal{O}_{\pi^{-1}(D)}) \rightarrow \pi_*(E)|_D,$$

which is a surjection into  $\pi_*(R)|_D$ . As we have seen above,  $K|_D$  is already a  $\pi_*(\mathcal{O}_{\pi^{-1}(D)})$ -submodule of  $\pi_*(E)|_D$ , thus  $K|_D \rightarrow \pi_*(R)|_D$  is surjective. But  $K|_D \subset \pi_*(E)|_D$ , from which we get  $K|_D = \pi_*(R)|_D$ . Consequently, if we take  $F := E/R$ , then  $\pi_*(F)|_D$  is isomorphic to  $\mathcal{F}_a|_D$ .  $\square$

*End of proof of Theorem 2.1.* We can now complete the proof of Theorem 2.1. For  $D \in U_a$  of large enough degree  $a \in N$ , we have shown the existence of a pure quotient  $E \rightarrow F$  such that  $\pi_*(E) \rightarrow \pi_*(F)$  restricts to  $\pi_*(E)|_D \rightarrow \mathcal{F}_a|_D$ . As  $a \in N$ ,  $\mathcal{F}_a|_D$  destabilizes  $\pi_*(E)|_D$  by assumption, thus by Lemma 2.2 we obtain

$$p_\ell(E) = p_\ell(\pi_*E) > p_\ell(\pi_*F) = p_\ell(F).$$

As  $E$  is  $\ell$ -semistable, this gives a contradiction.  $\square$

*Proof of Theorem A.* By Theorem 2.1, the restriction  $E|_{\pi^{-1}(D')}$  remains  $\ell$ -semistable for some divisor  $D' \in \mathbb{H}_a$  with  $a \gg 0$ . By the way we have defined  $\mathbb{H}_a$ ,  $\pi^{-1}(D') = D_0 \cap Y$  for some divisor  $D_0 \in \Pi_a := |\mathcal{O}_X(a)|$  avoiding the associated points of  $E$ .

Consider the incidence variety  $Z_{X,a} = \{(D, x) \in \Pi_a \times X \mid x \in D\}$  with its natural projection  $q_X : Z_{X,a} \rightarrow X$ . If  $U \subset \Pi_a$  is the open subset of divisors avoiding the associated points of  $E$ , then for each  $D \in U$  we have a short exact sequence

$$0 \rightarrow E(-D) \rightarrow E \rightarrow E|_D \rightarrow 0.$$

Hence the Hilbert polynomial of  $E|_D$  remains constant when  $D$  varies through  $U$ , which means that  $q_X^*E$  is flat over  $U$ . But  $D_0 \in U$  and since the  $\ell$ -semistability property is open in flat families (see Lemma 1.1), we conclude that  $E|_D$  is  $\ell$ -semistable for a general divisor  $D \in \Pi_a$ .  $\square$

**Corollary 2.1.** *If  $E$  is a Gieseker-semistable sheaf of dimension  $d$  on  $X$ , then the restriction  $E|_D$  is still Gieseker-semistable for a general divisor  $D \in |\mathcal{O}_X(a)|$  of sufficiently large degree.*

*Proof.* We have the following

$$\begin{aligned} E \text{ Gieseker-semistable} &\Leftrightarrow E \text{ } d\text{-semistable} \Rightarrow E \text{ } (d-1)\text{-semistable} \Rightarrow \\ &\xrightarrow{\text{Thm. A}} E|_D \text{ } (d-1)\text{-semistable} \Leftrightarrow E|_D \text{ Gieseker-semistable,} \end{aligned}$$

where  $D \in |\mathcal{O}_X(a)|$  is a general divisor of sufficiently large degree.  $\square$

## 2.2 Proof of Theorem B

Next we want to show Theorem B. As in the previous case, it is enough to prove the following:

**Theorem 2.2.** *Assume that  $E$  is  $\ell$ -stable with  $\ell < d$ . Then the restriction  $E|_{\pi^{-1}(D)}$  to a general divisor  $D \in \mathbb{H}_a$  remains  $\ell$ -stable for  $a \gg 0$ .*

Recall that any  $\ell$ -stable sheaf is in particular  $\ell$ -simple (see Lemma 1.2). Therefore a necessary condition for the theorem to hold is that the restriction  $E|_{\pi^{-1}(D)}$  to a general divisor of sufficiently large degree remains  $\ell$ -simple. The following confirms that this is indeed the case.

**Lemma 2.8.** *Under the same notation as above, the restriction  $E|_{\pi^{-1}(D)}$  to a general divisor  $D \in \mathbb{H}_a$  remains  $\ell$ -simple for  $a \gg 0$ .*

*Proof.* The proof is similar to that of Lemma 1.6. □

We proceed with the proof of Theorem 2.2. As before, we will argue by contradiction. In what follows, we assume that there is an infinite subset  $N \subset \mathbb{N}$  such that  $E|_{\pi^{-1}(D)}$  is  $\ell$ -unstable,  $\ell$ -semistable (cf. Theorem 2.1) and  $\ell$ -simple (cf. Lemma 2.8) for general  $D \in \mathbb{H}_a$  and  $a \in N$ . We follow the same strategy of proof as in the semistable case. First, we construct a flat family of destabilizing quotients.

**Lemma 2.9.** *For each  $a \in N$  there exists a dense open subset  $W_a \subset \mathbb{H}_a$  and a quotient  $q_Y^*E \rightarrow \mathcal{G}_{Y,a}$  such that*

- (1) *each divisor  $D \in W_a$  is smooth and integral in  $\mathbb{P}$ ,*
- (2)  *$\mathcal{G}_{Y,a}$  is flat over  $W_a$ ,*
- (3) *for  $D \in W_a$ ,  $E|_{\pi^{-1}(D)} \rightarrow \mathcal{G}_{Y,a}|_{\pi^{-1}(D)}$  is  $\ell$ -destabilizing, i.e.  $p_\ell(E|_{\pi^{-1}(D)}) = p_\ell(\mathcal{G}_{Y,a}|_{\pi^{-1}(D)})$ .*

*Proof.* Since  $k$  is algebraically closed,  $E|_{\pi^{-1}(D)}$  is geometrically  $\ell$ -unstable for general  $D \in \mathbb{H}_a$ . Recall that the property of being geometrically  $\ell$ -stable is open in flat families, cf. Lemma 1.1. Thus, if  $D_\xi \in \mathbb{H}_a$  denotes the generic point of  $\mathbb{H}_a$ , then the generic fiber  $E|_{\pi^{-1}(D_\xi)} := (q_Y^*E)_{[D_\xi]}$  must be geometrically  $\ell$ -unstable over  $(Z_{Y,a})_{[D_\xi]}$ . On the other side, from Lemma 1.4 we deduce that  $E|_{\pi^{-1}(D_\xi)}$  must be  $\ell$ -simple. Hence, according to Lemma 1.12,  $E|_{\pi^{-1}(D_\xi)}$  is  $\ell$ -unstable.

Since the  $\ell$ -semistability property is open, we also get that  $E|_{\pi^{-1}(D_\xi)}$  is  $\ell$ -semistable. Hence there is a proper destabilizing quotient  $E|_{\pi^{-1}(D_\xi)} \rightarrow \mathcal{G}_\xi$  such that  $p_\ell(E|_{\pi^{-1}(D_\xi)}) = p_\ell(\mathcal{G}_\xi)$ . Extend  $\mathcal{G}_\xi$  to a quotient  $q_Y^*E \rightarrow \mathcal{G}_{Y,a}$  over  $Z_{Y,a}$ . By generic flatness, there is an open subset  $W_a \subset \mathbb{H}_a$  such that  $\mathcal{G}_{Y,a}$  is flat over  $W_a$  and  $E|_{\pi^{-1}(D)} \rightarrow \mathcal{G}_{Y,a}|_{\pi^{-1}(D)}$  is still  $\ell$ -destabilizing for all  $D \in W_a$ . Moreover, by Bertini's Theorem and shrinking  $W_a$  if necessary, we may assume that each divisor  $D \in W_a$  is smooth and integral in  $\mathbb{P}$ . □

We have constructed above a flat family  $q_Y^*E \rightarrow \mathcal{G}_{Y,a}$  of (proper) destabilizing quotients for every  $a \in N$ . We let  $\mathcal{G}_a := \eta_*(\mathcal{G}_{Y,a})$ , where  $\eta$  is the map given by (2.2). Then, as in the semistable case, there are line bundles  $L_a \in \text{Pic}(\mathbb{P})$  such that  $\det(\mathcal{G}_a|_D) = L_a|_D$  for  $D \in W_a$ .

**Lemma 2.10.** *There exists a line bundle  $L \in \text{Pic}(\mathbb{P})$  such that  $L \cong L_a$  for infinitely many  $a \in N$ .*

*Proof.* It is enough to compute the degree of  $L_a$ . For  $a \in N$ , we have

$$\begin{aligned}
\deg(L_a) &= \frac{\deg(L_a|_D)}{a} = \frac{\deg(\mathcal{G}_a|_D)}{a} \\
&= \frac{\alpha_{d-2}(\mathcal{G}_a|_D) - \text{rk}(\mathcal{G}_a|_D)\alpha_{d-2}(\mathcal{O}_D)}{a} \quad (\text{by definition}) \\
&= \text{rk}(\mathcal{G}_a|_D) \left( \frac{\alpha_{d-2}(\mathcal{G}_a|_D)}{\alpha_{d-1}(\mathcal{G}_a|_D)} - \frac{a\alpha_{d-1}(\mathcal{O}_{\mathbb{P}}) - a^2/2}{a} \right) \\
&= \text{rk}(\mathcal{G}_a|_D) \left( \frac{\alpha_{d-2}(\pi_*(E)|_D)}{\alpha_{d-1}(\pi_*(E)|_D)} - \alpha_{d-1}(\mathcal{O}_{\mathbb{P}}) + \frac{a}{2} \right),
\end{aligned}$$

where the last equality holds since  $p_\ell(\mathcal{G}_a|_D) = p_\ell(\pi_*(E)|_D)$  by construction. By (2.1) we obtain

$$\frac{\alpha_{d-2}(\pi_*(E)|_D)}{\alpha_{d-1}(\pi_*(E)|_D)} = \frac{a\alpha_{d-1}(\pi_*E) - \frac{a^2}{2}\alpha_d(\pi_*E)}{a\alpha_d(\pi_*E)} = \frac{\alpha_{d-1}(\pi_*E)}{\alpha_d(\pi_*E)} - \frac{a}{2}.$$

Hence

$$\deg(L_a) = \text{rk}(\mathcal{G}_a|_D) \left( \frac{\alpha_{d-1}(\pi_*E)}{\alpha_d(\pi_*E)} - \alpha_{d-1}(\mathcal{O}_{\mathbb{P}}) \right).$$

But clearly we may assume that  $\text{rk}(\mathcal{G}_a|_D)$  is constant for infinitely many  $a \in N$ , which concludes the proof.  $\square$

Now one can finish the proof as in the semistable case (see Step III). For  $D \in W_a$  of sufficiently large degree  $a \in N$ , one shows the existence of a pure quotient  $E \rightarrow G$  such that  $\pi_*(E) \rightarrow \pi_*(G)$  restricts to  $\pi_*(E)|_D \rightarrow \mathcal{G}_a|_D$ . By Lemma 2.2 this gives a contradiction and so completes the proof of Theorem 2.2. From this, similarly to the previous case, one can deduce Theorem B.

## 2.3 Proof of Theorem C

In this section we will prove Restriction Theorem C. We continue in the same notation as before. It is enough to show the following:

**Theorem 2.3.** *Let  $c \in K(X)_{num}$  be a class of dimension  $d$  and multiplicity  $r$ . Assume that  $E$  is slope-(semi)stable of class  $c$ . Let  $D \in \mathbb{H}_a$  be a normal divisor on  $\mathbb{P}$  such that  $E|_{\pi^{-1}(D)}$  is pure. If*

$$a > \frac{r-1}{r}\delta(c) + \frac{1}{r(r-1)} + r^5(r-1) + \frac{(r-1)\beta_r}{r},$$

*then the restriction  $E|_{\pi^{-1}(D)}$  is again slope-(semi)stable. See below for notation.*

**Notation.** Let us first clarify the notation in the statement. We denote

$$\beta_r := \begin{cases} 0 & \text{if } \text{char}(k) = 0, \\ \frac{r^2(r-1)^2}{(p-1)^2} & \text{if } \text{char}(k) = p. \end{cases}$$

The definition of  $\beta_r$  is the one given by Langer in [42], specialized to the case of the projective space  $\mathbb{P}$  endowed with its natural polarization  $H$ . The coefficient  $\delta(c)$  is defined by

$$\delta(c) := \Delta(\pi_*E)H^{d-2},$$

where  $\Delta(\pi_*E)$  is the *discriminant* of  $\pi_*E$  in  $\mathbb{P}$ . By definition,

$$\Delta(\pi_*E) := 2rc_2(\pi_*E) - (r-1)c_1(\pi_*E)^2.$$

**Remark 2.4.** Note that the definition of  $\delta(c)$  does not depend on  $E$ , but only on its numerical equivalence class  $c$ . Indeed, as  $\pi_*$  is exact, we have

$$P(\pi_*E, m) = P(E, m) = \chi(c(m)).$$

Then, as we work over the projective space  $\mathbb{P}$ , we can obtain the Chern classes of  $\pi_*(E)$  from its Hilbert polynomial (see [8]).

Before we give the proof of Theorem 2.3 some preparation is needed. Since  $\pi_*$  is exact,  $E$  and  $\pi_*E$  have the same Hilbert polynomial. So  $\alpha_d(E) = \text{rk}(\pi_*E)$  and

$$\hat{\mu}(E) = \mu(\pi_*E) + \hat{\mu}(\mathcal{O}_{\mathbb{P}}) = \mu(\pi_*E) + (d+1)/2,$$

where the slope  $\mu(\pi_*E)$  is computed with respect to  $H$  (see Chapter 1 for the definition of  $\mu$  and  $\hat{\mu}$ ). If the restriction  $E|_{\pi^{-1}(D)}$  remains pure, one gets that

$$\hat{\mu}(E|_{\pi^{-1}(D)}) = \hat{\mu}(\pi_*(E)|_D) = \mu(\pi_*(E)|_D)/a + \hat{\mu}(\mathcal{O}_D). \quad (2.11)$$

Note that the semistability property is not necessarily preserved by pushforward, but we have the following inequalities due to Langer [43, Lem. 6.2.2].

**Lemma 2.11.** *Under the above notation, we have*

- (1)  $\mu_{\max}(\pi_*E) - \mu(\pi_*E) \leq \hat{\mu}_{\max}(E) - \hat{\mu}(E) + \text{rk}(\pi_*E)^2,$
- (2)  $\mu(\pi_*E) - \mu_{\min}(\pi_*E) \leq \hat{\mu}(E) - \hat{\mu}_{\min}(E) + \text{rk}(\pi_*E)^2.$

We will need the following two results for the proof of Theorem 2.3.

**Theorem 2.4.** *If  $F$  is a torsion-free sheaf of rank  $r$  on  $\mathbb{P}$ , then*

$$\Delta(F)H^{d-2} + r^2(\mu_{\max}(F) - \mu(F))(\mu(F) - \mu_{\min}(F)) + \beta_r \geq 0.$$

*Proof.* This is due to Langer, see [42, Thm. 5.1]. □

**Lemma 2.12.** *Let  $G$  be a pure sheaf of schematic support  $Y$  and multiplicity  $r$ . Then*

$$\Delta(\pi_*G)H^{d-2} + r^2(\hat{\mu}_{\max}(G) - \hat{\mu}(G))(\hat{\mu}(G) - \hat{\mu}_{\min}(G)) + \beta_r \geq -r^6.$$

*Proof.* We first introduce some notation. Consider the Harder-Narasimhan filtration of  $G$ :

$$0 = G_0 \subset G_1 \subset \dots \subset G_l = G.$$

We set  $F_i := \pi_*(G_i/G_{i-1})$ ,  $r_i := \text{rk}(F_i)$ ,  $\mu_i := \mu(F_i)$ ,  $\mu := \mu(\pi_*G)$ .

Now a direct computation using the Hodge Index Theorem gives

$$\begin{aligned} \frac{\Delta(\pi_*G)H^{d-2}}{r} &= \sum \frac{\Delta(F_i)H^{d-2}}{r_i} \\ &\quad - \frac{1}{r} \sum_{i<j} r_i r_j \left( \frac{c_1(F_i)}{r_i} - \frac{c_1(F_j)}{r_j} \right)^2 H^{d-2} \\ &\geq \sum \frac{\Delta(F_i)H^{d-2}}{r_i} - \frac{1}{r} \sum_{i<j} r_i r_j (\mu_i - \mu_j)^2. \end{aligned}$$

By using [42, Lem. 1.4] and

$$\frac{\beta_r}{r} \geq \sum_i \frac{\beta_{r_i}}{r_i},$$

we get

$$\Delta(\pi_*G)H^{d-2} + r^2(\mu_1 - \mu)(\mu - \mu_1) + \beta_r \geq \sum \frac{r}{r_i} (\Delta(F_i)H^{d-2} + \beta_{r_i}),$$

or equivalently

$$\Delta(\pi_*G)H^{d-2} + r^2(\hat{\mu}_{\max}(G) - \hat{\mu}(G))(\hat{\mu}(G) - \hat{\mu}_{\min}(G)) + \beta_r \geq \sum \frac{r}{r_i} (\Delta(F_i)H^{d-2} + \beta_{r_i}).$$

Now the result follows from the lemma bellow.  $\square$

**Lemma 2.13.** *Under the above assumptions, we have*

$$\sum \frac{r}{r_i} (\Delta(F_i)H^{d-2} + \beta_{r_i}) \geq -r^6.$$

*Proof.* Applying Lemma 2.4 to each  $F_i$ , we obtain

$$\Delta(F_i)H^{d-2} + r_i^2(\mu_{\max}(F_i) - \mu(F_i))(\mu(F_i) - \mu_{\min}(F_i)) + \beta_{r_i} \geq 0.$$

Also, by Lemma 2.11 we have

$$\mu_{\max}(F_i) - \mu(F_i) \leq r_i^2, \quad \mu(F_i) - \mu_{\min}(F_i) \leq r_i^2.$$

Putting together the above inequalities yields

$$\sum \frac{r}{r_i} (\Delta(F_i)H^{d-2} + \beta_{r_i}) \geq -r \sum r_i^5 \geq -r \left( \sum r_i \right)^5 \geq -r^6.$$

$\square$

We are now ready to prove Theorem 2.3. We give a proof by contradiction following the idea of Langer [42, Thm. 5.2]. For the reader's convenience, we adopt his notation.

*Proof of Theorem 2.3.* Suppose, on contrary, that  $E|_{\pi^{-1}(D)}$  is not stable. Let  $S$  be a saturated destabilizing subsheaf of  $E|_{\pi^{-1}(D)}$ , and denote  $T := (E|_{\pi^{-1}(D)}/S)$ . Let



$G$  be the kernel of the composition  $E \rightarrow E|_{\pi^{-1}(D)} \rightarrow T$ . Then there are two short exact sequences

$$0 \rightarrow G \rightarrow E \rightarrow T \rightarrow 0$$

and

$$0 \rightarrow E(-\pi^{-1}(D)) \rightarrow G \rightarrow S \rightarrow 0.$$

Set  $\rho := \text{rk}(\pi_*S)$ . A direct computation gives

$$\begin{aligned} \Delta(\pi_*G)H^{d-2} &= \Delta(\pi_*E)H^{d-2} - \rho(r - \rho)a^2 \\ &\quad + 2(rc_1(\pi_*T) - (r - \rho)Dc_1(\pi_*E))H^{d-2}. \end{aligned}$$

Since  $T$  is a destabilizing quotient of  $E|_{\pi^{-1}(D)}$ , we have  $\hat{\mu}(T) \leq \hat{\mu}(E|_{\pi^{-1}(D)})$ , or equivalently  $\mu(\pi_*T) \leq \mu(\pi_*(E)|_D)$  by (2.11). Then  $2(rc_1(\pi_*T) - (r - \rho)Dc_1(\pi_*E))H^{d-2} \leq 0$ , and so

$$\Delta(\pi_*G)H^{d-2} \leq \Delta(\pi_*E)H^{d-2} - \rho(r - \rho)a^2.$$

By using Lemma 2.12, we deduce

$$\Delta(\pi_*E)H^{d-2} - \rho(r - \rho)a^2 + r^2(\hat{\mu}_{\max}(G) - \hat{\mu}(G))(\hat{\mu}(G) - \hat{\mu}_{\min}(G)) + \beta_r \geq -r^6. \quad (2.12)$$

Since  $E$  and  $E(-\pi^{-1}(D))$  are stable, we obtain

$$\hat{\mu}_{\max}(G) - \hat{\mu}(G) = \hat{\mu}_{\max}(G) - \hat{\mu}(E) + \frac{r - \rho}{r}a \leq \frac{r - \rho}{r}a - \frac{1}{r(r - 1)} \quad (2.13)$$

and

$$\hat{\mu}(G) - \hat{\mu}_{\min}(G) = \hat{\mu}(E) - \hat{\mu}_{\min}(G) + \frac{\rho}{r}a \leq \frac{\rho}{r}a - \frac{1}{r(r - 1)}. \quad (2.14)$$

Putting together inequalities (2.12)–(2.14) yields

$$\Delta(\pi_*E)H^{d-2} - \rho(r - \rho)a^2 + r^2 \left( \frac{r - \rho}{r}a - \frac{1}{r(r - 1)} \right) \left( \frac{\rho}{r}a - \frac{1}{r(r - 1)} \right) + \beta_r \geq -r^6.$$

Thus

$$\Delta(\pi_*E)H^{d-2} - \frac{r}{r - 1}a + \frac{1}{(r - 1)^2} + \beta_r \geq -r^6,$$

which gives a contradiction.  $\square$

Essentially the same proof works also for strict slope-semistability, as noticed by Langer [42, Cor. 5.4].

**Corollary 2.2.** *Assume that  $E$  is slope-semistable. Let  $0 = E_0 \subset E_1 \subset \dots \subset E_l = E$  be a Jordan-Hölder filtration of  $E$ , and let  $D \in \mathbb{H}_a$  be a normal divisor such that all restrictions  $(E_i/E_{i-1})|_{\pi^{-1}(D)}$  are still pure. If*

$$a > \frac{r - 1}{r} \Delta(\pi_*E)H^{d-2} + \frac{1}{r(r - 1)} + r^5(r - 1) + \frac{(r - 1)\beta_r}{r},$$

*then the restriction  $E|_{\pi^{-1}(D)}$  is again slope-semistable.*

*Proof.* Set  $F_i := E_i/E_{i-1}$  and  $r_i := \alpha_d(F_i)$ . As shown in Lemma 2.12, we have

$$\frac{\Delta(\pi_* E)H^{d-2}}{r} \geq \sum_i \frac{\Delta(\pi_* F_i)H^{d-2}}{r_i}.$$

Then the result follows by applying Theorem 2.3 to each  $F_i$ . □

Finally, the proof of Theorem C goes as in the  $\ell$ -(semi)stable case, using Theorem 2.3 and Corollary 2.2.

## Chapter 3

# Equivariant Iitaka fibration

Throughout this chapter, we fix a connected algebraic group  $G$  over  $\mathbb{C}$ . First we introduce the notion of  $G$ -properness and use it to define what we call the relative  $G$ -normalization. We then construct the Iitaka  $G$ -fibration of a  $G$ -proper scheme endowed with a  $G$ -semiample line bundle. In a sense, we generalize the results described by Lazarsfeld in [48, Ch. 2] to the  $G$ -equivariant setting.

In what follows, we work over the category  $(Sch/\mathbb{C})$  of schemes of finite type over  $\mathbb{C}$ , and so every morphism will be of finite type.

### 3.1 Equivariant properness

**Definition 3.1.** Let  $R$  be a  $G$ -scheme of finite type and  $f : R \rightarrow S$  a  $G$ -invariant morphism. We say that  $f$  is  *$G$ -universally closed* if for every commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{g} & R \\ \downarrow j & \nearrow i & \downarrow f \\ \mathrm{Spec}(A) & \xrightarrow{h} & S \end{array}$$

where  $A$  is a discrete valuation ring over  $\mathbb{C}$  of quotient field  $K$ , there exists a morphism  $i : \mathrm{Spec}(A) \rightarrow R$  such that  $i \circ j$  and  $g$  differ by a group element in  $G(K)$ , and  $h = f \circ i$ . If in addition  $f$  is also separated, we call  $f$  a  *$G$ -proper* morphism.

**Remark 3.1.** When  $G$  acts trivially we recover the classical notion of properness on schemes. As in the classical case, one can easily check the following properties of  $G$ -properness:

1.  $G$ -proper morphisms are stable under base changes  $S' \rightarrow S$ .
  2. If  $f : R \rightarrow Z$  and  $g : Z \rightarrow S$  are two morphisms such that  $g \circ f$  is  $G$ -proper,  $f$  is  $G$ -invariant and  $g$  is separated, then  $f$  is  $G$ -proper.
  3. If  $f : R \rightarrow S$  is  $G$ -proper and  $g : S \rightarrow T$  is proper, then  $g \circ f$  is also  $G$ -proper.
- Similar properties hold also for  $G$ -universally closed morphisms.

**Lemma 3.1.** *Given a commutative diagram*

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ & \searrow p & \swarrow q \\ & & T \end{array}$$

*in the category of finite type  $\mathbb{C}$ -schemes such that  $p$  is  $G$ -universally closed,  $f$  is  $G$ -invariant and  $q$  is separated, then the image of  $f$  is proper over  $T$ .*

*Proof.* By Nagata's Compactification Theorem, we may assume that  $S$  is a proper scheme over  $T$  and reduce the problem to showing that  $\text{Im}(f)$  is a closed subscheme of  $S$ . Since  $f$  is of finite type, it is enough to show that  $\text{Im}(f)$  is stable under specialization, cf. [33, Ch. II, Lem. 4.5]. So let  $x_1 \in R$  be a point and consider a specialization  $s_0 \in S$  of its image  $s_1 = f(x_1)$ . Denote by  $\mathcal{O}$  the local ring of  $s_0$  on  $\{s_1\}$  with its reduced induced structure. Then the quotient field of  $\mathcal{O}$  is  $k(s_1)$ , and we have a finitely generated field extension  $k(s_1) \subset k(x_1)$ . Set  $K := k(x_1)$  and consider a discrete valuation ring  $A$  of  $K$  dominating  $\mathcal{O}$ , which exists cf. [33, Ch. II, Ex. 4.11]. This induces a commutative diagram

$$\begin{array}{ccc} \text{Spec}(K) & \xrightarrow{g} & R \\ \downarrow j & & \downarrow f \\ \text{Spec}(A) & \xrightarrow{h} & S \end{array}$$

such that  $h : \text{Spec}(A) \rightarrow S$  sends the generic point of  $A$  to  $s_1$  and its closed point to  $s_0$ . By Remark 3.1,  $f$  is a  $G$ -universally closed morphism. Thus there exists a morphism  $i : \text{Spec}(A) \rightarrow R$  such that  $f \circ i = h$ , which shows that  $s_0$  is in the image of  $f$ .  $\square$

## 3.2 Dévissage for equivariant coherent sheaves

In this section, let  $R$  be a  $G$ -universally closed scheme over  $\text{Spec}(A)$ , where  $A$  is a finitely generated  $\mathbb{C}$ -algebra. We will prove a devissage result for  $G$ -equivariant coherent sheaves on  $R$ .

We work within the category  $\text{Coh}^G(R)$  of  $G$ -equivariant coherent sheaves on  $R$ . We say an object  $E$  of  $\text{Coh}^G(R)$  has property  $\mathcal{P}$  if the module  $\Gamma(R, E)^G$  of  $G$ -invariant global sections is of finite type over  $A$ . We want to prove that any  $G$ -equivariant sheaf has  $\mathcal{P}$ , for which we need the following two results.

**Lemma 3.2.** *For any short exact sequence of  $G$ -equivariant sheaves on  $R$*

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0,$$

*if  $E'$  and  $E''$  (resp.  $E$  and  $E''$ ) have  $\mathcal{P}$ , then the third has  $\mathcal{P}$ .*

*Proof.* The proof is completely analogue to the one in the non-equivariant case, we omit it.  $\square$

**Lemma 3.3.** *If  $Y$  is an integral  $G$ -stable closed subscheme of  $R$ , then  $\Gamma(Y, \mathcal{O}_Y)^G$  is of finite type over  $A$ .*

*Proof.* Let  $s \in \Gamma(Y, \mathcal{O}_Y)^G$  be a  $G$ -invariant global section, which induces a  $G$ -invariant morphism  $s : Y \rightarrow \text{Spec}(A[T])$ . As  $R$  is  $G$ -universally closed over  $A$ , then so is  $Y$ . By Lemma 3.1 the image of  $s$ , which we denote by  $Z$ , is proper over  $A$ . Then  $Z$  is also affine, so has the form  $\text{Spec}(A[T]/I)$  for some ideal  $I$ . Since  $Z \rightarrow \text{Spec}(A)$  is proper and affine, it is in particular integral. In other words there exists a monic polynomial  $P \in I$  such that  $P(s) = 0$ . Then the ring extension  $A \rightarrow \Gamma(Y, \mathcal{O}_Y)^G$  is integral as well.

Let  $V \cong \text{Spec}(B)$  be an open affine subset of  $Y$ , so  $A \rightarrow B$  is a finite type extension with  $B$  integral. By [80, Tag 03GH] the integral closure of  $A$  in  $B$  is finite over  $A$ . As  $\Gamma(Y, \mathcal{O}_Y)^G$  is an  $A$ -submodule of the integral closure, it is of finite type over  $A$ . This implies that  $A \rightarrow \Gamma(Y, \mathcal{O}_Y)^G$  is also finite.  $\square$

**Theorem 3.1.** *Any  $G$ -equivariant sheaf on  $R$  satisfies property  $\mathcal{P}$ .*

*Proof.* We argue by noetherian induction as follows. Let  $Y$  be a  $G$ -stable closed subscheme of  $R$  and assume that any  $G$ -equivariant sheaf whose support is a proper subscheme  $Z \subset Y$  has  $\mathcal{P}$ . We want to show that any  $G$ -equivariant sheaf supported on  $Y$  has  $\mathcal{P}$ . Let  $E$  be a  $G$ -equivariant sheaf on  $Y$ . First, we reduce to the case when  $Y$  is reduced. In case  $Y$  is not reduced, let  $\mathcal{I}$  be the sheaf ideal corresponding to the reduced induced subscheme structure on  $Y$ . Note that we can naturally endow  $\mathcal{I}$  with the structure of a  $G$ -equivariant sheaf. Also there is an integer  $n > 0$  such that  $\mathcal{I}^n = 0$ . We have a filtration of  $G$ -equivariant sheaves

$$0 = \mathcal{I}^n E \subset \mathcal{I}^{n-1} E \subset \dots \subset \mathcal{I} E \subset E$$

such that the successive quotients  $\mathcal{I}^j E / \mathcal{I}^{j+1} E$  are annihilated by  $\mathcal{I}$ . Hence these quotients are supported on  $Y_{\text{red}}$ , and so they satisfy  $\mathcal{P}$  by assumption. By splitting the filtration above in short exact sequences and using Lemma 3.2, we get that  $E$  has  $\mathcal{P}$ .

Now we assume that  $Y$  is reduced. In case  $Y$  is not irreducible, let  $Y_1, \dots, Y_r$  be its irreducible components. Note that the  $Y_i$  are  $G$ -stable since  $G$  is connected. Let  $\mathcal{I}_1$  be the ideal sheaf of  $Y_1$  with its natural  $G$ -equivariant structure, and consider the exact sequence

$$0 \rightarrow \mathcal{I}_1 E \rightarrow E \rightarrow E/\mathcal{I}_1 E \rightarrow 0.$$

Since  $\mathcal{I}_1 E$  is supported on  $Y_2 \cup \dots \cup Y_r$  and  $E/\mathcal{I}_1 E$  is supported on  $Y_1$ , we may assume by noetherian induction that both sheaves satisfy  $\mathcal{P}$ . By Lemma 3.2, we deduce that  $E$  has also  $\mathcal{P}$ .

Finally, we assume that  $Y$  is integral and show that  $E$  has  $\mathcal{P}$ . We argue by induction on the rank of  $E$ . If  $\text{rank}(E) = 0$ , then  $E$  is supported on a proper subscheme of  $Y$  and so  $E$  has  $\mathcal{P}$ . Now suppose that  $\text{rank}(E) > 0$ . If we denote by  $F$  the maximal torsion subsheaf of  $E$ , then we obtain a short exact sequence in  $\text{Coh}^G(R)$  given by

$$0 \rightarrow F \rightarrow E \rightarrow E/F \rightarrow 0.$$

Note that  $F$  has  $\mathcal{P}$  as it is supported on a proper subscheme. Using again Lemma 3.2 for the above sequence, we reduce to the case  $E$  is pure. We may suppose that  $E$  has at least a  $G$ -invariant global section, say  $s$ . This induces a short exact sequence

$$0 \rightarrow \mathcal{O}_Y \xrightarrow{\cdot s} E \rightarrow Q \rightarrow 0$$

in  $\text{Coh}^G(R)$  such that  $\text{rank}(Q) < \text{rank}(E)$ . Hence  $Q$  has  $\mathcal{P}$  by induction on rank. As  $\mathcal{O}_Y$  has also  $\mathcal{P}$  by Lemma 3.3, we conclude that  $E$  has  $\mathcal{P}$ .  $\square$

### 3.3 Relative equivariant normalization

Next we introduce the notion of relative  $G$ -normalization, which is an equivariant version of the so-called relative normalization (see [80, Tag 035H] and [80, Tag 035I]).

**Definition 3.2.** Let  $R$  be a  $G$ -scheme of finite type over  $\mathbb{C}$  and  $f : R \rightarrow S$  a  $G$ -invariant morphism. The  $G$ -normalization of  $S$  in  $R$  is a  $\mathbb{C}$ -scheme  $S'$  (of finite type) together with a finite morphism  $\pi : S' \rightarrow S$  satisfying the following:

1. there is a factorization  $f : R \xrightarrow{f'} S' \xrightarrow{\pi} S$  with  $f'$   $G$ -invariant,
2. for any other factorization  $f : R \xrightarrow{g} Z \xrightarrow{\nu} S$  with  $g$  a  $G$ -invariant morphism and  $\nu$  finite, there exists a unique morphism  $h : S' \rightarrow Z$  making the diagram

$$\begin{array}{ccc} R & \xrightarrow{g} & Z \\ \downarrow f' & \nearrow h & \downarrow \nu \\ S' & \xrightarrow{\pi} & S \end{array} \quad (3.1)$$

commute.

The following is an equivariant version of the Stein factorization (see [80, Tag 03H0]).

**Proposition 3.1.** *Let  $f : R \rightarrow S$  be a  $G$ -proper surjective morphism in the category of finite type  $\mathbb{C}$ -schemes. Then the  $G$ -normalization of  $S$  in  $R$  exists and gives a factorization*

$$\begin{array}{ccc} R & \xrightarrow{f'} & S' \\ & \searrow f & \downarrow \pi \\ & & S \end{array}$$

with the following properties:

- (1)  $(f'_*\mathcal{O}_R)^G = \mathcal{O}_{S'}$  and  $S' = \underline{\mathrm{Spec}}_S((f_*\mathcal{O}_R)^G)$ ,
- (2) the morphism  $\pi : S' \rightarrow S$  is finite,
- (3)  $f'$  is a surjective  $G$ -invariant morphism.

*Proof.* As  $f$  is of finite type and separated,  $(f_*\mathcal{O}_R)^G$  is quasi-coherent, and so we can define  $S' = \underline{\mathrm{Spec}}_S((f_*\mathcal{O}_R)^G)$ . Then there is clearly a factorization of  $f$  as in the statement. Let us prove that  $(f_*\mathcal{O}_R)^G$  is finite over  $\mathcal{O}_S$ . We argue locally over an open affine subscheme  $U = \mathrm{Spec}(A) \subset S$ . Let  $h \in \Gamma(f^{-1}(U), \mathcal{O}_R)^G$  be a  $G$ -invariant section, which corresponds to a  $G$ -invariant morphism  $h : f^{-1}(U) \rightarrow \mathrm{Spec}(A[T])$  making the diagram

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{h} & \mathrm{Spec}(A[T]) \\ & \searrow f & \downarrow \\ & & U \end{array}$$

commute. By the properties of  $G$ -properness (see Remark 3.1), one gets that  $f|_{f^{-1}(U)}$  is also  $G$ -proper. Therefore, if  $Z$  denotes the image of  $h$ , then  $Z$  is a proper scheme over  $U$  by Lemma 3.1. Also, as a closed subscheme of  $\mathrm{Spec}(A[T])$ ,  $Z = \mathrm{Spec}(A[T]/I)$  for some ideal  $I \subset A[T]$ . Hence  $Z \rightarrow U$  is affine and proper, and so  $Z$  is integral over

$U$ . This means there is a monic polynomial  $P \in I$  such that  $P(h) = 0$ , which proves that  $A \rightarrow \Gamma(f^{-1}(U), \mathcal{O}_R)^G$  is integral. It remains to see that this morphism is also of finite type, but this follows immediately from Theorem 3.1 since  $f^{-1}(U) \rightarrow U$  is  $G$ -universally closed.

We show that  $f'$  is surjective. By Lemma 3.1 the image of  $f'$  is a closed subscheme of  $S'$ , and thus it is determined by a coherent sheaf of ideals  $\mathcal{I} \subset \mathcal{O}_{S'}$  such that  $f'^*\mathcal{I} \rightarrow \mathcal{O}_R$  is zero. Then the composition  $\mathcal{I} \rightarrow \mathcal{O}_{S'} \rightarrow (f'_*\mathcal{O}_R)^G$  is also zero by adjointness. But  $(f'_*\mathcal{O}_R)^G = \mathcal{O}_{S'}$  by (1), which implies that the closed subscheme determined by  $\mathcal{I}$  is exactly  $S'$ .

It remains to show that  $S'$  is in fact the  $G$ -normalization of  $S$  in  $R$ . We use the universal property of the  $G$ -normalization. Suppose there is another factorization  $f : R \xrightarrow{g} Z \xrightarrow{\nu} S$  with  $g$  a  $G$ -invariant morphism and  $\nu$  finite. This gives an inclusion

$$\mathcal{O}_S \rightarrow \nu_*\mathcal{O}_Z \rightarrow f_*(\mathcal{O}_R)^G.$$

Taking the morphism of relative spectra over  $S$  coming from  $\nu_*\mathcal{O}_Z \rightarrow f_*(\mathcal{O}_R)^G$ , we obtain a map  $S' \rightarrow Z$  making diagram (3.1) commute.  $\square$

### 3.4 Equivariant Iitaka fibration

Let  $R$  be a  $G$ -proper scheme over  $\mathbb{C}$  and  $\mathcal{L}$  a  $G$ -equivariant  $G$ -semiample line bundle on  $R$ , i.e. there exists  $\nu > 0$  such that  $\mathcal{L}^\nu$  is globally generated by  $G$ -invariant sections. Then the set

$$M(R, \mathcal{L}) = \{m \in \mathbb{N} \mid \mathcal{L}^{\otimes m} \text{ is generated by } G\text{-invariant sections}\}$$

is nonempty. As  $R$  is  $G$ -proper, for each  $\nu \in M(R, \mathcal{L})$  the vector space  $\Gamma(R, \mathcal{L}^\nu)^G$  is finitely generated by Theorem 3.1 and induces a  $G$ -invariant morphism

$$\varphi_\nu : R \rightarrow \mathbb{P}(\Gamma(R, \mathcal{L}^\nu)^G)$$

such that  $\varphi_\nu^*\mathcal{O}(1) \cong \mathcal{L}^\nu$ . According to Lemma 3.1, the image of  $\varphi_\nu$ , which we denote by  $M_\nu$ , is a projective scheme over  $\mathbb{C}$ . By a slight abuse of notation, we will denote by  $\varphi_\nu$  also the morphism from  $R$  to  $M_\nu$ . Then  $M_\nu$  is endowed with an ample line bundle  $\mathcal{A}_\nu$  such that  $\varphi_\nu^*\mathcal{A}_\nu \cong \mathcal{L}^\nu$ .

For  $k > 0$ , the  $k$ -th symmetric power  $\text{Sym}^k \Gamma(R, \mathcal{L}^\nu)^G$  is a linear subsystem of  $\Gamma(R, \mathcal{L}^{k\nu})^G$  that globally generates  $\mathcal{L}^{k\nu}$ . This gives a commutative diagram

$$\begin{array}{ccc} & R & \\ \varphi_{k\nu} \swarrow & & \searrow \varphi_\nu \\ M_{k\nu} & \xrightarrow{\pi_k} & M_\nu \end{array}$$

with  $\pi_k$  finite such that  $\pi_k^*\mathcal{A}_\nu^k = \mathcal{A}_{k\nu}$ .

**Remark 3.2.** In [36, Prop. 8.2.6], the authors consider the projective limit of a system  $\{M_\nu\}$  as above dominated by a  $G$ -scheme  $R^{\mu ss}$ , with  $G$  a certain algebraic group. Even though this limit exists, in general there is no guarantee that the system eventually stabilizes or, equivalently, that the limit is a priori of finite type. We shall provide an argument for this fact in the proof of Theorem 3.2, as the scheme  $R^{\mu ss}$  they consider is actually  $G$ -proper (see Lemma 4.7).

It should become apparent by now that our approach shares some similarities with the one described by Lazarsfeld for the classical Iitaka fibration [48, Ch. 2], except that  $R$  is neither projective nor normal. We obtain the following theorem, which is a generalized version of [48, Thm. 2.1.27].

**Theorem 3.2.** *Under the same assumptions as above, there exists a unique projective scheme  $M$  over  $\mathbb{C}$  and a  $G$ -invariant surjective morphism  $\varphi : R \rightarrow M$  such that for  $\nu \gg 0$  in  $M(R, \mathcal{L})$  we have  $M_\nu = M$  and  $\varphi_\nu = \varphi$ . Moreover,  $M$  is endowed with an ample line bundle  $\mathcal{A}$  such that  $\varphi^* \mathcal{A} \cong \mathcal{L}^e$ , where  $e$  is the exponent of  $M(R, \mathcal{L})$ , i.e. the largest natural number such that each member of  $M(R, \mathcal{L})$  is a multiple of  $e$ .*

*Proof.* Let  $\nu \in M(R, \mathcal{L})$ . We apply the result of Proposition 3.1 to the morphism  $\varphi_\nu : R \rightarrow M_\nu$ . Denote by  $M$  the  $G$ -normalization of  $M_\nu$  in  $R$ . This gives a natural factorization  $R \xrightarrow{\varphi} M \xrightarrow{\pi_\nu} M_\nu$  of  $\varphi$ , with the following properties:

- (i)  $M = \underline{\text{Spec}}_{M_\nu}((\varphi_{\nu*} \mathcal{O}_R)^G)$  and  $(\varphi_* \mathcal{O}_R)^G = \mathcal{O}_M$ ,
- (ii) the morphism  $\pi_\nu : M \rightarrow M_\nu$  is finite,
- (iii) the morphism  $\varphi$  is surjective and  $M$  is projective over  $\mathbb{C}$ .

By construction, the morphism  $\varphi_\nu : R \rightarrow M_\nu$  satisfies  $\varphi_\nu^* \mathcal{A}_\nu \cong \mathcal{L}^\nu$ . Since  $\pi_\nu$  is finite,  $\mathcal{A}_\nu$  pulls back to an ample line bundle on  $M$ , denoted by  $\mathcal{B}$ . Thus, for  $k > 0$ , we have

$$\varphi^* \mathcal{B}^k \cong \varphi^*(\pi_\nu^* \mathcal{A}_\nu^k) \cong \varphi_\nu^* \mathcal{A}_\nu^k \cong \mathcal{L}^{k\nu}.$$

Then

$$H^0(R, \mathcal{L}^{k\nu})^G \cong H^0(M, \varphi_*(\varphi^* \mathcal{B}^k)^G) \cong H^0(M, \mathcal{B}^k), \quad (3.2)$$

where the last isomorphism follows from (i) and by using the equivariant projection formula (see [28, Lem. 4.9]). Indeed,

$$\varphi_*(\varphi^* \mathcal{B}^k \otimes \mathcal{O}_R)^G \cong \mathcal{B}^k \otimes (\varphi_* \mathcal{O}_R)^G \cong \mathcal{B}^k.$$

For  $k \gg 0$ , the line bundle  $\mathcal{B}^k$  is very ample on  $M$ , and so (3.2) implies that  $\varphi$  is in fact defined by the linear system  $H^0(R, \mathcal{L}^{k\nu})^G$ . Thus  $M_{k\nu} = M$  and  $\varphi_{k\nu} = \varphi$  for  $k \gg 0$ .

The second part of the statement follows exactly as in the proof of [48, Thm. 2.1.27], and for this reason we omit the details.  $\square$



## Chapter 4

# Moduli spaces of semistable sheaves

In this chapter, we will use the restriction theorems to construct moduli spaces of semistable sheaves over  $\mathbb{C}$ . We let  $X$  be a connected smooth projective variety over  $\mathbb{C}$  with a fixed polarization  $\mathcal{O}_X(1)$ .

### Moduli problem

Let us recall the moduli problem. Choose a numerical Grothendieck class  $c \in K(X)_{\text{num}}$  of dimension  $d > 0$ , which will fix the numerical type of sheaves. For each  $1 \leq \ell \leq d$ , we consider the moduli functor

$$\mathcal{M}^{[\ell]} := \mathcal{M}_{\mathcal{O}_X(1)}^{[\ell]}(c) : (\text{Sch}/\mathbb{C}) \rightarrow (\text{Sets})$$

that associates to any  $\mathbb{C}$ -scheme  $S$  of finite type the set of all equivalence classes of  $S$ -flat families of  $\ell$ -semistable sheaves of class  $c$  on  $X$ . Recall that two  $S$ -families  $\mathcal{F}$  and  $\mathcal{F}'$  are equivalent if and only if  $\mathcal{F} \cong \mathcal{F}' \otimes p_S^*(L)$  for some line bundle  $L \in \text{Pic}(S)$ . In the following we aim to construct a moduli space of  $\ell$ -semistable sheaves of class  $c$  on  $X$  when  $\ell < d$ .

### A parametrizing space

We know by Theorem 1.1 that all  $\ell$ -semistable sheaves of class  $c$  form a bounded family. Therefore, for sufficiently large  $m > 0$ , every such semistable sheaf  $E$  is  $m$ -regular (in the sense of Castelnuovo–Mumford). In particular  $E(m)$  is globally generated with  $\chi(c(m)) = h^0(E(m))$ . Then the evaluation map induces a quotient

$$H^0(E(m)) \otimes \mathcal{O}_X(-m) \rightarrow E \rightarrow 0.$$

Let  $V := \mathbb{C}^{\chi(c(m))}$  and consider the subset

$$R^{[\ell]} \subset \text{Quot}(V \otimes \mathcal{O}_X(-m), c) \tag{4.1}$$

of all quotients  $[q : V \otimes \mathcal{O}_X(-m) \rightarrow E]$  on  $X$  such that

- (i)  $E$  is  $\ell$ -semistable,
- (ii)  $q$  induces an isomorphism  $V \rightarrow H^0(E(m))$ .

In this case,  $R^{[\ell]}$  is an open subscheme consisting of all  $\ell$ -semistable quotients of class  $c$  on  $X$ . Note that any semistable sheaf is uniquely determined in  $R^{[\ell]}$  only up to a base change of  $V$  (see [36, Lem. 4.3.2]). In this respect, the  $\text{SL}(V)$ -action on  $V$  induces a natural  $\text{SL}(V)$ -action on  $R^{[\ell]}$  as well. We would like to apply the GIT methods to form the quotient  $R^{[\ell]}/\text{SL}(V)$ . However, this problem does not admit a good GIT quotient in general. We adopt a different path instead, which involves constructing a semiample  $\text{SL}(V)$ -equivariant line bundle  $\mathcal{L}$  on  $R^{[\ell]}$ .

## 4.1 Equivariant determinant line bundles

In this section, we make a detour to construct a class of determinant line bundles which correspond functorially to any  $S$ -flat family of sheaves and are moreover semiample. Throughout this part,  $G$  is a fixed connected algebraic group over  $\mathbb{C}$ .

For any  $G$ -scheme  $S$  of finite type over  $\mathbb{C}$  and any  $G$ -equivariant  $S$ -flat family  $\mathcal{E}$  of sheaves on  $X$ , we define a map  $\lambda_{\mathcal{E}} : K(X) \rightarrow \text{Pic}^G(S)$  by the composition:

$$K(X) \xrightarrow{p_2^*} K^0(S \times X) \xrightarrow{[\mathcal{E}]} K^0(S \times X) \xrightarrow{p_{1!}} K^0(S) \xrightarrow{\det} \text{Pic}^G(S).$$

**Lemma 4.1.** *The determinant map  $\lambda$  has the following properties:*

- (1) *If  $0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0$  is an exact sequence of  $G$ -equivariant  $S$ -flat families, then  $\lambda_{\mathcal{E}} \cong \lambda_{\mathcal{E}'} \otimes \lambda_{\mathcal{E}''}$ .*
- (2) *For any base change  $f : T \rightarrow S$  of finite type over  $\mathbb{C}$  we have  $f^* \lambda_{\mathcal{E}} \cong \lambda_{f^* \mathcal{E}}$ .*
- (3) *If  $\mathcal{E}$  is an  $S$ -flat family of sheaves of class  $c$  and  $u \in K(X)$  is a class satisfying  $\chi(c \cdot u) = 0$ , then  $\lambda_{\mathcal{E} \otimes p_2^*(L)}(u) \cong \lambda_{\mathcal{E}}(u)$  for any  $L \in \text{Pic}^G(S)$ .*

*Proof.* We refer the reader to [49] or [50] for proofs. □

For  $a > 0$  and  $l' > m' > 0$ , consider the following class in  $K(X)$ :

$$w := w(a, l', m') = \chi(c(m') \cdot h^{d-\ell})[\mathcal{O}_X(l')] - \chi(c(l') \cdot h^{d-\ell})[\mathcal{O}_X(m')], \quad (4.2)$$

where  $h = [\mathcal{O}_D]$  denotes the Grothendieck class of a divisor  $D \in \Pi_a := |\mathcal{O}_X(a)|$ . Here the product is in  $K(X)$ . For convenience, we often drop variables  $a, l', m'$  from notation.

**Remark 4.1.** 1. For any  $D \in \Pi_a$  we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-a) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_D \rightarrow 0,$$

which gives

$$h = 1 - [\mathcal{O}_X(-a)]$$

in  $K(X)$ .

2. If  $n = d = 2$  and  $\ell = 1$ , then one can recover the class defined by Huybrechts–Lehn [36, p. 223] in the torsion-free case (up to multiplication by  $l' - m'$ ). More precisely, the authors considered the class

$$w_1 := -\chi(c \cdot h_1 \cdot h) \cdot 1 + \chi(c \cdot h) \cdot h_1 \in K(X)$$

(we slightly change their notation to avoid confusing their class with  $w(a, l', m')$ ), where  $h_1 = [\mathcal{O}_H]$  is the class of a divisor  $H \in |\mathcal{O}_X(1)|$ . Then a straightforward computation yields  $w = (l' - m')w_1$ .

3. The Grothendieck group  $K(X)$  is endowed with a natural quadratic form given by the Euler characteristic  $\chi$ . If  $A \subset K(X)$  is a subset, we denote by  $A^\perp \subset K(X)$  the orthogonal complement of  $A$  with respect to  $\chi$ . If  $u \in K(X)$  is a class such that  $u \in c^\perp \cap \{1, h, \dots, h^n\}^{\perp\perp}$ , then there is a natural determinant line bundle  $\lambda(u)$  over the moduli space  $M^{Gss}(c)$  of Gieseker-semistable sheaves of class  $c$  on  $X$  (see [36, Thm. 8.1.5]). Clearly  $w \cdot h^{d-\ell} \in c^\perp \cap \{1, h, \dots, h^{\dim(X)}\}^{\perp\perp}$ , so one can use these classes to form determinant line bundles on  $M^{Gss}(c)$ .

The main result of this section is the following theorem:

**Theorem 4.1.** *Let  $S$  be a weakly normal  $G$ -variety over  $\mathbb{C}$  and  $\mathcal{E}$  a  $G$ -equivariant  $S$ -flat family of  $\ell$ -semistable sheaves of class  $c$  on  $X$ . Define the  $G$ -linearized determinant line bundle*

$$\mathcal{L}_{\mathcal{E}} := \lambda_{\mathcal{E}}(w(a, l', m') \cdot h^{d-\ell})$$

*over  $S$ . Then, for  $l' \gg m' \gg 0$  and  $a \gg 0$ , there is an integer  $\nu > 0$  such that the line bundle  $\mathcal{L}_{\mathcal{E}}^{\nu}$  is generated by  $G$ -invariant global sections.*

Before we proceed with the proof, we need some preparatory results.

## Restriction of families of sheaves to divisors

Let  $S$  be a  $G$ -scheme of finite type over  $\mathbb{C}$  and  $\mathcal{E}$  an  $S$ -flat family of sheaves on  $X$ . We will see below that  $\mathcal{E}$  remains  $S$ -flat whenever it is restricted to a divisor  $D \subset X$  avoiding the associated points of all geometric fibers of  $\mathcal{E}$  over  $S$ .

**Lemma 4.2.** *Let  $\mathcal{E}$  be an  $S$ -family of sheaves on  $X$  such that each fiber  $\mathcal{E}_s$  has no zero-dimensional associated points. Then, for  $a \gg 0$ , there exists a general divisor  $D \in \Pi_a$  that avoids the associated points of every fiber  $\mathcal{E}_s$  for  $s \in S$ .*

*Proof.* Consider the family  $\Sigma$  of all irreducible closed subschemes of  $X$  corresponding to associated points of  $\mathcal{E}_s$  with  $s \in S$  closed. By a result of Grothendieck (see [32, Prop. 1.3]), the family  $\Sigma$  is bounded, i.e. the set of Hilbert polynomials  $\{P_Y\}_{Y \in \Sigma}$  is finite and for  $a \gg 0$ , every subscheme in  $\Sigma$  is  $a$ -regular. Without loss of generality, we may assume that all elements of  $\Sigma$  have the same Hilbert polynomial, say  $P$ . In this case,  $\Sigma$  is a subset of the Hilbert scheme  $\text{Hilb}_X(P)$ .

Consider the incidence variety

$$I = \{(Y, D) \in \text{Hilb}_X(P) \times \Pi_a \mid Y \subset D\}$$

with its natural projections  $p_1$  and  $p_2$ . Since each  $Y \in \Sigma$  is  $a$ -regular, its Hilbert polynomial satisfies  $P(a) = h^0(\mathcal{O}_Y(a))$ . In this case  $p_1$  is a projective bundle of fiber dimension  $h^0(\mathcal{O}_X(a)) - P(a)$ . Therefore  $\dim(I) = \dim(\text{Hilb}_X(P)) + h^0(\mathcal{O}_X(a)) - P(a)$ . On the other hand, the general fiber of  $p_2$  has dimension

$$\dim(I) - \dim(\Pi_a) = \dim(\text{Hilb}_X(P)) - P(a) < 0.$$

The last inequality is satisfied for  $a \gg 0$ , since  $P$  has degree  $> 0$  by assumption. In other words, for  $a \gg 0$  there is a general divisor  $D \in \Pi_a$  which avoids all closed subschemes in  $\text{Hilb}_X(P)$ .  $\square$

**Lemma 4.3.** *Let  $\mathcal{E}$  be an  $S$ -flat family of sheaves on  $X$  such that each fiber  $\mathcal{E}_s$  has no zero-dimensional associated points. Then the restriction  $\mathcal{E}|_{S \times D}$  to a general divisor  $D \in \Pi_a$  remains  $S$ -flat for  $a \gg 0$ .*

*Proof.* By Lemma 4.2, there exists a general divisor  $D \in \Pi_a$  which is  $\mathcal{E}_s$ -regular, i.e.  $\mathcal{E}_s(-D) \rightarrow \mathcal{E}_s$  is injective, for every  $s \in S$ . By [60, Thm. 22.5] we obtain that  $\mathcal{E}|_{S \times D}$  is  $S$ -flat.  $\square$

The following generalizes the results of [28, Lem. 3.1] and [28, Cor. 3.2] to the case of pure sheaves. For the reader's convenience, we include their proofs.

**Lemma 4.4.** *Let  $\mathcal{E}$  be a  $G$ -equivariant  $S$ -flat family of sheaves on  $X$ , and let  $S_1, \dots, S_k$  be  $G$ -stable irreducible closed subschemes of  $S$ . For each index  $i$ , suppose there exists a point  $s_i \in S_i$  such that  $\mathcal{E}_{s_i}$  is  $\ell$ -semistable of dimension  $d > 0$ . Then, for  $a \gg 0$ , there exists an open subset  $U \subset \Pi_a$  such that for each  $D \in U$ , there exists a  $G$ -stable closed subscheme  $T \subset S$  satisfying:*

- (1) *the restriction  $\mathcal{E}|_{S \times D}$  is flat over  $S \setminus T$ ,*
- (2)  *$D$  is smooth and  $\mathcal{E}_s$ -regular for all  $s \in S \setminus T$ ,*
- (3) *for each  $i$ ,  $\mathcal{E}_{s_i}|_D$  is  $\ell$ -semistable on  $D$ , and*
- (4)  *$T \cap S_i$  has codimension  $\geq 2$  in  $S_i$ .*

*Proof.* Since the property of being pure is open in flat families, there is a largest  $G$ -stable open subset  $V \subset S$ , containing all  $s_i$ , such that  $\mathcal{E}$  has pure fibers of dimension  $d$  over  $V$ . By Lemma 4.3, for  $a \gg 0$  there is an open subset  $U_0 \subset \Pi_a$  such that  $\mathcal{E}|_{S \times D}$  is flat over  $V$  for every  $D \in U_0$ . Moreover, by shrinking  $U_0$  if necessary, we may also assume by Bertini's Theorem that each  $D \in U_0$  is smooth and by Restriction Theorem A that  $\mathcal{E}_{s_i}|_D$  is still  $\ell$ -semistable.

For each non-empty complement  $S_i \setminus V$ , choose a point  $s'_i \in S_i \setminus V$ . By [36, Lem. 1.1.12] and Bertini's Theorem, there is an open subset  $U_i \subset \Pi_a$  such that each  $D \in U_i$  is smooth and  $\mathcal{E}_{s'_i}$ -regular. Therefore, by using [60, Thm. 22.5], for each  $D \in U_i$  we can find a  $G$ -stable open neighborhood  $N(s'_i, D) \subset S$  of  $s'_i$  over which  $\mathcal{E}|_{S \times D}$  is flat. Define the open subset

$$U := U_0 \cap \bigcap_{S_i \setminus V \neq \emptyset} U_i \subset \Pi_a.$$

Then, for each  $D \in U$ , the set

$$T = S \setminus (V \cup \bigcup_{S_i \setminus V \neq \emptyset} N(s'_i, D))$$

satisfies the desired properties. □

**Corollary 4.1.** *Let  $\mathcal{E}$  be a  $G$ -equivariant  $S$ -flat family on  $X$ , and let  $S_1, \dots, S_k$  be  $G$ -stable irreducible closed subschemes of  $S$ . For each index  $i$ , suppose there exists a point  $s_i \in S_i$  such that  $\mathcal{E}_{s_i}$  is  $\ell$ -semistable of dimension  $d > 0$ . Then, for  $a \gg 0$  and a general smooth complete intersection  $X^{(d-\ell)} := D_1 \cap \dots \cap D_{d-\ell}$  of divisors in  $\Pi_a$ , there exists a  $G$ -stable closed subscheme  $T \subset S$  satisfying:*

- (1) *the restriction  $\mathcal{E}|_{S \times X^{(d-\ell)}}$  is flat over  $S \setminus T$ ,*
- (2) *the sequence  $\underline{D} = D_1, \dots, D_{d-\ell}$  is  $\mathcal{E}_s$ -regular for all  $s \in S \setminus T$ ,*
- (3) *for each  $i$ ,  $\mathcal{E}_{s_i}|_{X^{(d-\ell)}}$  is  $\ell$ -semistable on  $X^{(d-\ell)}$ , and*
- (4)  *$T \cap S_i$  has codimension  $\geq 2$  in  $S_i$ .*

*Proof.* Apply inductively Lemma 4.4. □

**Remark 4.2.** A few remarks are in order:

1. For any point  $s \in S$ , one can assume that  $T \subset S$  avoids the  $G$ -orbit of  $s$ . Indeed, it is enough to choose the closure of the  $G$ -orbit of  $s$  among the  $S_i$ .
2. Since  $G$  is connected, the irreducible components of  $S$  are  $G$ -stable, so they may be chosen among the  $S_i$ .

3. If  $\ell = d-1$  and the family  $\mathcal{E}$  has pure fibers over  $S$ , then one can take  $T = \emptyset$ . This is a consequence of Lemma 4.3.

Suppose that  $\mathcal{E}$  is a  $G$ -equivariant  $S$ -flat family of sheaves of class  $c$  on  $X$ . Choose a smooth complete intersection  $X^{(d-\ell)} := D_1 \cap \dots \cap D_{d-\ell}$  of divisors in  $\Pi_a$  satisfying the following:

**Assumption.** The sequence  $\underline{D} = D_1, \dots, D_{d-\ell}$  is  $\mathcal{E}_s$ -regular for every  $s \in S$ .

Set  $\mathcal{E}_i := \mathcal{E} \otimes p_X^*(\Lambda^i \mathbb{C}^{d-\ell} \otimes \mathcal{O}_X(-ia))$  and consider the Koszul complex

$$K_\bullet(\mathcal{E}, \underline{D}) : 0 \rightarrow \mathcal{E}_{d-\ell} \rightarrow \dots \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_0$$

corresponding to  $\underline{D}$ . Note that all the  $\mathcal{E}_i$  are  $S$ -flat and the restriction of  $K_\bullet(\mathcal{E}, \underline{D})$  to each fiber over  $S$  is exact, since  $\underline{D}$  is  $\mathcal{E}_s$ -regular by assumption. According to [80, Tag 00MI], the complex  $K_\bullet(\mathcal{E}, \underline{D})$  is exact and  $\mathcal{E}|_{S \times X^{(d-\ell)}} = \text{Coker}(\mathcal{E}_1 \rightarrow \mathcal{E}_0)$  is  $S$ -flat.

**Lemma 4.5.** *Under the same assumptions as above, there is a  $G$ -equivariant isomorphism*

$$\lambda_{\mathcal{E}}(w \cdot h^{d-\ell}) \cong \lambda_{\mathcal{E}|_{S \times X^{(d-\ell)}}}(w|_{X^{(d-\ell)}})$$

of determinant line bundles over  $S$ .

*Proof.* By applying the map  $\lambda_{(-)}(w)$  to the Koszul resolution  $K_\bullet(\mathcal{E}, \underline{D})$  and using Lemma 4.1, we get the following chain of  $G$ -equivariant isomorphisms:

$$\begin{aligned} \lambda_{\mathcal{E}|_{S \times X^{(d-\ell)}}}(w|_{X^{(d-\ell)}}) &\cong \bigotimes_{i=0}^{d-\ell} \lambda_{\mathcal{E}_i}(w)^{(-1)^i} \\ &\cong \bigotimes_{i=0}^{d-\ell} \lambda_{\mathcal{E}}(w \cdot [\Lambda^i \mathbb{C}^{d-\ell} \otimes \mathcal{O}_X(-ia)])^{(-1)^i} \\ &\cong \lambda_{\mathcal{E}} \left( \sum_{i=0}^{d-\ell} (-1)^i w \cdot [\Lambda^i \mathbb{C}^{d-\ell} \otimes \mathcal{O}_X(-ia)] \right) \\ &\cong \lambda_{\mathcal{E}}(w \cdot h^{d-\ell}). \end{aligned}$$

□

## Grothendieck embedding

As before, choose a complete intersection  $X^{(d-\ell)} := D_1 \cap \dots \cap D_{d-\ell}$  of divisors in  $\Pi_a$ . Set  $c' := c|_{X^{(d-\ell)}}$  and  $V' := \mathbb{C}^{\chi(c'(m'))}$ . Consider the Quot scheme  $\text{Quot}_{X^{(d-\ell)}} := \text{Quot}_{X^{(d-\ell)}}(V' \otimes \mathcal{O}_{X^{(d-\ell)}}(-m'), c')$  endowed with its natural action of  $\text{SL}(V')$ . For  $l' \gg m'$ , by Grothendieck's construction of the Quot scheme [32], there is an  $\text{SL}(V')$ -equivariant closed embedding

$$\varphi_{l'} : \text{Quot}_{X^{(d-\ell)}} \rightarrow \text{Grass}(V' \otimes H^0(\mathcal{O}_{X^{(d-\ell)}}(l' - m')), \chi(c'(l')))$$

that at the level of geometric points associates to a quotient  $[q : V' \otimes \mathcal{O}_{X^{(d-\ell)}}(-m') \rightarrow F]$  its corresponding quotient linear space  $[V' \otimes H^0(\mathcal{O}_{X^{(d-\ell)}}(l' - m')) \rightarrow H^0(F(l'))]$ . If

$\mathcal{F}'$  denotes the universal quotient over  $\text{Quot}_{X^{(d-\ell)}}$ , then the Grothendieck embedding is in fact given by the following very ample determinant line bundle

$$\mathcal{L}_{\mathcal{F}'} := \lambda_{\mathcal{F}'}([\mathcal{O}_{X^{(d-\ell)}}(l')]),$$

cf. [36, Prop. 2.2.5]. Let  $Q \subset \text{Quot}_{X^{(d-\ell)}}$  be the open subscheme consisting of quotients  $[q : V' \otimes \mathcal{O}_{X^{(d-\ell)}}(-m') \rightarrow F]$  such that

- (i)  $H^i(F(m')) = 0$  for all  $i > 0$ ,
- (ii) the induced map  $V' \rightarrow H^0(F(m'))$  is an isomorphism.

Clearly  $Q$  is invariant under the  $\text{SL}(V')$ -action. Furthermore, this action induces a natural  $\text{SL}(V')$ -linearization of  $\mathcal{L}_{\mathcal{F}'}$  over  $Q$  (see [36, p. 101]).

The following is a generalization of [50, Prop. 1.3] to our setting.

**Lemma 4.6.** *We have*

$$\chi(c|_{X^{(d-\ell)}}(m'))\mathcal{L}_{\mathcal{F}'} = \lambda_{\mathcal{F}'}(w|_{X^{(d-\ell)}})$$

in  $\text{Pic}^{\text{SL}(V')}(Q)$  with additive notation.

*Proof.* By construction,  $R^i p_{1*}(\mathcal{F}'(m')) = 0$  for all  $i > 0$  and  $p_{1*}(\mathcal{F}'(m')) \cong V' \otimes \mathcal{O}_Q$  over  $Q$ . Therefore,

$$\lambda_{\mathcal{F}'}([\mathcal{O}_{X^{(d-\ell)}}(m')]) \cong \det(p_{1*}(\mathcal{F}'(m'))) \cong \det(V') \otimes \mathcal{O}_Q$$

is trivial in  $\text{Pic}^{\text{SL}(V')}(Q)$ . We obtain

$$\begin{aligned} \lambda_{\mathcal{F}'}(w|_{X^{(d-\ell)}}) &= \\ &= \chi(c|_{X^{(d-\ell)}}(m'))\lambda_{\mathcal{F}'}([\mathcal{O}_{X^{(d-\ell)}}(l')]) - \chi(c|_{X^{(d-\ell)}}(l'))\lambda_{\mathcal{F}'}([\mathcal{O}_{X^{(d-\ell)}}(m')]) \\ &= \chi(c|_{X^{(d-\ell)}}(m'))\mathcal{L}_{\mathcal{F}'} \end{aligned}$$

in  $\text{Pic}^{\text{SL}(V')}(Q)$  (with additive notation). □

## Proof of the semiampleness theorem

We are now ready to prove Theorem 4.1. Let  $S$  be a weakly normal  $G$ -variety over  $\mathbb{C}$  and  $\mathcal{E}$  a  $G$ -equivariant  $S$ -flat family of  $\ell$ -semistable sheaves of class  $c$  on  $X$ . Consider the  $G$ -equivariant line bundle

$$\mathcal{L}_{\mathcal{E}} := \lambda_{\mathcal{E}}(w(a, l', m') \cdot h^{d-\ell})$$

over  $S$  (see (4.2) for notation). We want to show that  $\mathcal{L}_{\mathcal{E}}$  is  $G$ -semiample for  $a \gg 0$  and  $l' \gg m' \gg 0$ , i.e. there is an integer  $\nu > 0$  such that  $\mathcal{L}_{\mathcal{E}}^\nu$  is generated by  $G$ -invariant global sections. Let  $s_0 \in S$  be a closed point,  $S_1$  its  $G$ -orbit in  $S$  and  $S_2, \dots, S_k$  the irreducible components of  $S$ . By Corollary 4.1, for  $a \gg 0$ , we can choose a general smooth complete intersection  $X^{(d-\ell)} = D_1 \cap \dots \cap D_{d-\ell}$  of divisors in  $\Pi_a$  and a  $G$ -stable closed subvariety  $T \subset S$  avoiding  $s_0$  such that

- (i)  $\mathcal{E}|_{S \times X^{(d-\ell)}}$  is flat over  $S \setminus T$ ,
- (ii) the sequence  $\underline{D} = D_1, \dots, D_{d-\ell}$  is  $\mathcal{E}_s$ -regular for all  $s \in S \setminus T$ ,
- (iii)  $\mathcal{E}_{s_0}|_{X^{(d-\ell)}}$  is Gieseker-semistable and each irreducible component of  $S \setminus T$  contains a dense open subset over which  $\mathcal{E}|_{S \times X^{(d-\ell)}}$  has Gieseker-semistable fibers,

(iv) each  $T \cap S_i$  has codimension  $\geq 2$  in  $S_i$ .

Moreover, by Lemma A.1, for each  $\nu > 0$  we may choose  $T$  satisfying also the following property:

(v) any  $G$ -invariant section in  $H^0(S \setminus T, \mathcal{L}_\mathcal{E}^\nu)^G$  extends to a  $G$ -invariant section in  $H^0(S, \mathcal{L}_\mathcal{E}^\nu)^G$ .

As we saw before, condition (ii) above ensures that the Koszul complex  $K_\bullet(\mathcal{E}, \underline{D})$  is exact over  $S \setminus T$  and so there exists a  $G$ -equivariant isomorphism

$$\mathcal{L}_\mathcal{E} \cong \lambda_{\mathcal{E}|_{S \times_X X^{(d-\ell)}}}(w|_{X^{(d-\ell)}})$$

over  $S \setminus T$  (see Lemma 4.5).

Set  $\mathcal{E}' := \mathcal{E}|_{(S \setminus T) \times X^{(d-\ell)}}$ . For  $m'$  sufficiently large, we may also assume that each fiber of  $\mathcal{E}'$  over  $S \setminus T$  is  $m'$ -regular (cf. [48, Exam. 1.8.7]). In particular  $R^i p_{S*}(\mathcal{E}'(m')) = 0$  for  $i > 0$  and  $p_{S*}(\mathcal{E}'(m'))$  is a locally free  $G$ -equivariant  $\mathcal{O}_{S \setminus T}$ -sheaf of rank  $\chi(c'(m'))$ , where  $c' := c|_{X^{(d-\ell)}} \in K(X^{(d-\ell)})$ . Denote by  $\mathcal{S}$  the projective frame bundle corresponding to  $p_{S*}(\mathcal{E}'(m'))$  and let  $\pi : \mathcal{S} \rightarrow S \setminus T$  be the canonical projection. Then there exists a natural quotient

$$\mathcal{O}_\mathcal{S} \otimes \mathcal{O}_{X^{(d-\ell)}}(-m')^{\chi(c'(m'))} \rightarrow \pi^* \mathcal{E}' \otimes \mathcal{O}_\pi(1) \quad (4.3)$$

Set  $V' := \mathbb{C}^{\chi(c'(m'))}$  and  $\text{Quot}_{X^{(d-\ell)}} := \text{Quot}(V' \otimes \mathcal{O}_{X^{(d-\ell)}}(-m'), c')$ . Then the quotient map (4.3) induces a natural morphism  $\Phi : \mathcal{S} \rightarrow \text{Quot}_{X^{(d-\ell)}}$ . We obtain a diagram as follows

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\Phi} & \text{Quot}_{X^{(d-\ell)}} \\ \downarrow \pi & & \downarrow \varphi' \\ S \setminus T & \dashrightarrow & M_{X^{(d-\ell)}}^{Gss} \end{array} \quad (4.4)$$

where  $M_{X^{(d-\ell)}}^{Gss}$  denotes the Simpson moduli space of Gieseker-semistable sheaves of class  $c'$  on  $X^{(d-\ell)}$ .

**Remark 4.3.** Several remarks about the above construction are in order:

1. Let us recall Simpson's construction [78] of the moduli space  $M_{X^{(d-\ell)}}^{Gss}$ . Let  $\mathcal{F}'$  denote the universal family of quotients over  $\text{Quot}_{X^{(d-\ell)}}$ . As observed in the previous section,  $\mathcal{L}_{\mathcal{F}'} := \lambda_{\mathcal{F}'}([\mathcal{O}_{X^{(d-\ell)}}(l')])$  is a very ample line bundle over  $\text{Quot}_{X^{(d-\ell)}}$  for  $l' \gg 0$ . Also,  $\text{Quot}_{X^{(d-\ell)}}$  is endowed with a natural  $\text{SL}(V')$ -action, which further induces an  $\text{SL}(V')$ -linearization of  $\mathcal{L}_{\mathcal{F}'}$ .

Now consider the  $\text{SL}(V')$ -stable open subscheme  $R_{X^{(d-\ell)}}^{Gss} \subset \text{Quot}_{X^{(d-\ell)}}$  consisting of all quotients  $[q : V' \otimes \mathcal{O}_{X^{(d-\ell)}}(-m') \rightarrow F]$  such that

- (i)  $F$  is Gieseker-semistable,
- (ii) the induced map  $V' \rightarrow H^0(F(m'))$  is an isomorphism.

In fact, for  $m' \gg 0$ , all Gieseker-semistable sheaves of class  $c'$  on  $X^{(d-\ell)}$  appear as quotients in  $R_{X^{(d-\ell)}}^{Gss}$ . In this case, Simpson showed that the GIT quotient  $R_{X^{(d-\ell)}}^{Gss}/\text{SL}(V')$  exists and defines the moduli space of Gieseker-semistable sheaves. More precisely, for  $l' \gg m' \gg 0$ ,  $R_{X^{(d-\ell)}}^{Gss}$  is the locus of GIT-semistable points in the closure  $\overline{R_{X^{(d-\ell)}}^{Gss}} \subset \text{Quot}_{X^{(d-\ell)}}$  with respect to  $\mathcal{L}_{\mathcal{F}'}$ . Furthermore, the rational map  $\varphi'$  is well-defined over  $R_{X^{(d-\ell)}}^{Gss}$ , and  $R_{X^{(d-\ell)}}^{Gss} \rightarrow M_{X^{(d-\ell)}}^{Gss}$  is a good GIT quotient, in the sense of Mumford [67].

2. It is important to note that the image of  $\Phi$  is actually contained in the open subscheme  $Q \subset \text{Quot}_{X^{(d-\ell)}}$  defined in the previous section. That is,  $Q$  is the open subscheme consisting of quotients  $[q : V' \otimes \mathcal{O}_{X^{(d-\ell)}}(-m') \rightarrow F]$  such that

- (i)  $H^i(F(m')) = 0$  for all  $i > 0$ ,
- (ii) the induced map  $V' \rightarrow H^0(F(m'))$  is an isomorphism.

On the other hand, by the choice of  $T$ , the image of  $\Phi$  is also contained in the closure of  $R_{X^{(d-\ell)}}^{Gss}$ .

3. The given  $G$ -action on  $S$  lifts to a natural  $G$ -action on  $\mathcal{S}$ . There is also a natural  $\mathrm{SL}(V')$ -action on  $\mathcal{S}$  induced by that on  $p_{S*}(\mathcal{E}'(m'))$ . Furthermore, one can check that these actions are compatible, and  $\Phi$  is in fact  $G \times \mathrm{SL}(V')$ -equivariant, where  $\mathrm{Quot}_{X^{(d-\ell)}}$  is endowed with its trivial  $G$ -action.

We want to produce a  $G$ -invariant global section of (some power of)  $\mathcal{L}_{\mathcal{E}}$  non-vanishing at  $s_0$ . Recall that we chose the complete intersection  $X^{(d-\ell)}$  such that the restriction  $\mathcal{E}_{s_0}|_{X^{(d-\ell)}}$  is Gieseker-semistable. Hence, by Simpson's construction, there exists an  $\mathrm{SL}(V')$ -invariant section  $\sigma \in H^0(\overline{R_{X^{(d-\ell)}}^{Gss}}, \mathcal{L}_{\mathcal{F}'}^{\nu'})^{\mathrm{SL}(V')}$  non-vanishing at  $[\mathcal{E}_{s_0}|_{X^{(d-\ell)}}]$  for some  $\nu' > 0$ . By Lemma 4.6, there exists an  $\mathrm{SL}(V')$ -equivariant isomorphism

$$\mathcal{L}_{\mathcal{F}'}^{\nu'} \cong \lambda_{\mathcal{F}'}(w|_{X^{(d-\ell)}})^{\nu'}$$

over  $Q$  for some  $\nu > 0$ . Hence the restriction of  $\sigma$  to  $\Phi(\mathcal{S}) \subset Q$  induces an  $\mathrm{SL}(V')$ -invariant section  $\sigma|_{\Phi(\mathcal{S})}$  in  $H^0(\Phi(\mathcal{S}), \lambda_{\mathcal{F}'}(w|_{X^{(d-\ell)}})^{\nu})^{\mathrm{SL}(V')}$ . Consider the following chain of isomorphisms:

$$\begin{aligned} \Phi^*(\lambda_{\mathcal{F}'}(w|_{X^{(d-\ell)}})) &\cong \lambda_{\pi^*\mathcal{E}' \otimes \mathcal{O}_S(1)}(w|_{X^{(d-\ell)}}) && \text{by Lem. 4.1 (2)} \\ &\cong \lambda_{\pi^*\mathcal{E}'}(w|_{X^{(d-\ell)}}) && \text{by Lem. 4.1 (3)} \\ &\cong \pi^*\lambda_{\mathcal{E}'}(w|_{X^{(d-\ell)}}) && \text{by Lem. 4.1 (2)} \\ &\cong \pi^*\mathcal{L}_{\mathcal{E}}|_{S \setminus T} && \text{by Lem. 4.5.} \end{aligned}$$

Clearly the above isomorphisms are compatible with the corresponding group actions of  $G$  and  $\mathrm{SL}(V')$ . Then  $\Phi^*(\sigma|_{\Phi(\mathcal{S})})$  yields a global  $G \times \mathrm{SL}(V')$ -invariant section over  $\mathcal{S}$ , which further descends to a  $G$ -invariant section in  $H^0(S \setminus T, \mathcal{L}_{\mathcal{E}}^{\nu})^G$ , since  $\pi$  is a categorical quotient. As  $S$  is weakly normal, we can extend this section over  $T$  to obtain a global  $G$ -invariant section  $\tilde{\sigma} \in H^0(S, \mathcal{L}_{\mathcal{E}}^{\nu})^G$  non-vanishing at  $s_0$ .

As  $s_0$  was arbitrarily chosen and  $S$  is noetherian, we can find a large enough power  $\nu$  such that  $\mathcal{L}_{\mathcal{E}}^{\nu}$  is globally generated by  $G$ -invariant sections for  $a \gg 0$  and  $l' \gg m' \gg 0$ . This completes the proof of Theorem 4.1.

If  $\ell = d - 1$ , there is no need to impose the weak normality condition on  $S$ . Indeed, as noticed in Remark 4.2, we may choose  $T = \emptyset$  in this case.

**Theorem 4.2.** *Let  $S$  be a  $G$ -scheme of finite type over  $\mathbb{C}$  and  $\mathcal{E}$  a  $G$ -equivariant  $S$ -flat family of  $(d - 1)$ -semistable sheaves of class  $c$  on  $X$ . Then, for  $l' \gg m' \gg 0$  and  $a \gg 0$ , there is an integer  $\nu > 0$  such that the line bundle  $\mathcal{L}_{\mathcal{E}}^{\nu}$  is generated by  $G$ -invariant global sections over  $S$ .*

## 4.2 Construction of the moduli space

We proceed with the construction of the moduli space. Recall that we have defined a complex quasi-projective scheme  $R^{[\ell]}$  parametrizing all  $\ell$ -semistable sheaves of class  $c$  on  $X$ . Even though  $R^{[\ell]}$  is not necessarily proper, we can prove that it is at least  $\mathrm{SL}(V)$ -proper, in the sense of Definition 3.1.



**Lemma 4.7.**  $R^{[\ell]}$  is an  $\mathrm{SL}(V)$ -proper scheme over  $\mathbb{C}$ .

*Proof.* Let  $A$  be a discrete valuation ring over  $\mathbb{C}$  of quotient field  $K$ . Consider a commutative diagram as follows

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{g} & R^{[\ell]} \\ \downarrow j & & \downarrow \\ \mathrm{Spec}(A) & \longrightarrow & \mathrm{Spec}(\mathbb{C}) \end{array}$$

The map  $\mathrm{Spec}(K) \rightarrow R^{[\ell]}$  corresponds to a quotient  $[q_K : V \otimes K \rightarrow E_K] \in R^{[\ell]}$ , with  $E_K$   $\ell$ -semistable. By Theorem 1.2, we can extend  $E_K$  to an  $A$ -flat family  $E$  of  $\ell$ -semistable sheaves on  $X$ . This induces a quotient  $[q_A : V \otimes A \rightarrow E] \in R^{[\ell]}$ , or equivalently an  $A$ -point  $i : \mathrm{Spec}(A) \rightarrow R^{[\ell]}$ . As  $E \otimes_A K \cong E_K$ ,  $g$  and  $i \circ j$  differ by a group element in  $\mathrm{SL}(V)(K)$ , which completes the proof.  $\square$

Let  $\mathcal{F}$  be the universal family of quotients over  $R^{[\ell]}$  and consider the determinant line bundle

$$\mathcal{L} := \lambda_{\mathcal{F}}(w(a, l', m') \cdot h^{d-\ell})$$

over  $R^{[\ell]}$ , where  $w(a, l', m')$  is the Grothendieck class defined in (4.2). For  $l' \gg m' \gg 0$  and  $a \gg 0$ , we may assume that  $\mathcal{L}$  is  $\mathrm{SL}(V)$ -semiample, after replacing  $R^{[\ell]}$  by its weak normalization if necessary. Indeed, if  $\ell = d - 1$ , then this holds by Theorem 4.2. Otherwise, for  $\ell < d - 1$ , in order to ensure that Theorem 4.1 applies for  $(R^{[\ell]}, \mathcal{L})$ , we replace  $R^{[\ell]}$  by its weak normalization and work exclusively over the category  $(\mathrm{Sch}^{wn}/\mathbb{C})$  of weakly normal varieties. Note that the weak normalization of  $R^{[\ell]}$  is still  $\mathrm{SL}(V)$ -proper.

Putting together the above considerations,  $R^{[\ell]}$  is an  $\mathrm{SL}(V)$ -proper scheme over  $\mathbb{C}$  endowed with an  $\mathrm{SL}(V)$ -semiample line bundle  $\mathcal{L}$ . Then one can form the Iitaka  $\mathrm{SL}(V)$ -fibration of  $(R^{[\ell]}, \mathcal{L})$  (see Section 3.4), which will define our moduli space  $M^{[\ell]}$  of  $\ell$ -semistable sheaves of class  $c$  on  $X$ .

**Theorem D.** *There exists a unique triple  $(M^{[\ell]}, \mathcal{A}, e)$  formed of a weakly normal projective variety  $M^{[\ell]}$  endowed with an ample line bundle  $\mathcal{A}$  and a natural number  $e > 0$  such that there is a natural transformation  $\Psi : \mathcal{M}_{wn}^{[\ell]} \rightarrow \mathrm{Hom}_{(\mathrm{Sch}^{wn}/\mathbb{C})}(-, M^{[\ell]})$ , that associates to any weakly normal variety  $S$  and any  $S$ -flat family  $\mathcal{E}$  of  $\ell$ -semistable sheaves of class  $c$  on  $X$  a classifying morphism  $\Psi_{\mathcal{E}} : S \rightarrow M^{[\ell]}$ , satisfying the following properties:*

- (1) *For any  $S$ -flat family  $\mathcal{E}$  of  $\ell$ -semistable sheaves of class  $c$  on  $X$ , the classifying morphism  $\Psi_{\mathcal{E}}$  satisfies*

$$\Psi_{\mathcal{E}}^*(\mathcal{A}) \cong \mathcal{L}_{\mathcal{E}}^e,$$

*where  $\mathcal{L}_{\mathcal{E}}$  is the determinant line bundle on  $S$  defined in Section 4.1.*

- (2) *For any other triple  $(M', \mathcal{A}', e')$ , with  $M'$  a projective scheme over  $\mathbb{C}$ ,  $\mathcal{A}'$  an ample line bundle on  $M'$  and  $e'$  a natural number satisfying property (1), we have  $e|e'$  and there exists a unique morphism  $\phi : M^{[\ell]} \rightarrow M'$  such that  $\phi^*\mathcal{A}' \cong \mathcal{A}^{(e'/e)}$ .*

Moreover, if  $\ell = d - 1$ , then the same result as above holds true over the category  $(\text{Sch}/\mathbb{C})$  of schemes of finite type over  $\mathbb{C}$ . In particular, the moduli space  $M^{[d-1]}$  is endowed with the geometric structure of a (possibly non-reduced) projective scheme.

*Proof.* As  $R^{[\ell]}$  is  $\text{SL}(V)$ -proper and  $\mathcal{L}$  is  $\text{SL}(V)$ -semiample, we can consider the Iitaka  $\text{SL}(V)$ -fibration of  $(R^{[\ell]}, \mathcal{L})$  to obtain a triple  $(M^{[\ell]}, \mathcal{A}, e)$ , as given by Theorem 3.2, together with an  $\text{SL}(V)$ -invariant morphism  $\varphi : R^{[\ell]} \rightarrow M^{[\ell]}$ . In fact, as  $R^{[\ell]}$  is weakly normal, then from the universal property of the weak normalization (see Theorem A.1) it follows that  $M^{[\ell]}$  is also weakly normal.

Now let  $S$  be a weakly normal variety and  $\mathcal{E}$  an  $S$ -flat family of  $\ell$ -semistable sheaves of class  $c$  on  $X$ . By construction, any fiber  $\mathcal{E}_s$  is  $m$ -regular, and so  $p_{1*}(\mathcal{E}(m))$  is a locally free  $\text{SL}(V)$ -equivariant  $\mathcal{O}_S$ -sheaf of rank  $\chi(c(m))$ . Denote by  $\mathcal{S}$  the projective frame bundle associated to  $p_{1*}(\mathcal{E}(m))$  and let  $\pi : \mathcal{S} \rightarrow S$  be the canonical projection. As in Section 4.1, we obtain a diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\Phi_{\mathcal{E}}} & R^{[\ell]} \\ \downarrow \pi & & \downarrow \varphi \\ S & & M^{[\ell]} \end{array}$$

with  $\Phi_{\mathcal{E}}$  an  $\text{SL}(V)$ -equivariant morphism. As  $\pi$  is a categorical quotient, the  $\text{SL}(V)$ -invariant map  $\varphi \circ \Phi_{\mathcal{E}}$  descends to an  $\text{SL}(V)$ -invariant morphism  $\Psi_{\mathcal{E}} : S \rightarrow M^{[\ell]}$ . Furthermore, as the determinant map  $\lambda$  is functorial, we obtain

$$\pi^* \Psi_{\mathcal{E}}^*(\mathcal{A}) \cong \Phi_{\mathcal{E}}^*(\mathcal{L}^e) \cong \lambda_{\pi^* \mathcal{E}}(w \cdot h^{d-\ell})^e \cong \pi^* \mathcal{L}_{\mathcal{E}}^e.$$

But  $\pi^* : \text{Pic}(S) \rightarrow \text{Pic}(\mathcal{S})$  is injective (cf. [49, Lem. 2.14]), and so

$$\Psi_{\mathcal{E}}^*(\mathcal{A}) \cong \mathcal{L}_{\mathcal{E}}^e,$$

which proves the first part of the theorem.

For the second item, consider a triple  $(M', \mathcal{A}', e')$  as given in (2). Since  $M'$  satisfies property (1), there exists a morphism  $\varphi' : R^{[\ell]} \rightarrow M'$  corresponding to the universal family of quotients over  $R^{[\ell]}$  such that  $\varphi'^*(\mathcal{A}') \cong \mathcal{L}^{e'}$ . For  $k > 0$ , this induces a morphism

$$H^0(M', \mathcal{A}'^k) \rightarrow H^0(R^{[\ell]}, \mathcal{L}^{ke'})^{\text{SL}(V)}.$$

For  $k \gg 0$  the line bundle  $\mathcal{A}'^k$  is very ample, therefore  $\varphi'$  factors through the morphism  $\varphi_{ke'} : R^{[\ell]} \rightarrow \mathbb{P}H^0(R^{[\ell]}, \mathcal{L}^{ke'})^{\text{SL}(V)}$ . Denote by  $M_{ke'}$  the image of  $\varphi_{ke'}$ . As in the proof of Theorem 3.2, we obtain  $M^{[\ell]} = M_{ke'}$  for  $k \gg 0$ . This gives a unique morphism  $\phi : M^{[\ell]} \rightarrow M'$  as claimed in (2).

If  $\ell = d - 1$ , then the weak normality condition is not needed to prove the  $\text{SL}(V)$ -semiamplicity of  $\mathcal{L}$  over  $R^{[d-1]}$ . Similarly, the moduli space  $M^{[d-1]}$  is defined as the  $\text{SL}(V)$ -Iitaka fibration of  $(R^{[d-1]}, \mathcal{L})$ .  $\square$

### 4.3 Geometric points of the moduli space

In this section, we describe the geometric points of the moduli space  $M^{[\ell]}$ . As before, we replace  $R^{[\ell]}$  by its weak normalization whenever  $\ell < d - 1$ .

If  $E$  is an  $\ell$ -semistable sheaf on  $X$ , then there exists a Jordan-Hölder filtration

$$0 = E_0 \subset E_1 \subset \dots \subset E_\ell = E$$

such that its factors  $E_i/E_{i-1}$  are  $\ell$ -stable with  $p_\ell(E_i/E_{i-1}) = p_\ell(E)$ . We denote by  $\text{gr}_{JH}(E) := \bigoplus_i E_i/E_{i-1}$  its corresponding graded module. We say that  $E$  is  $\ell$ -*polystable* if  $E$  is isomorphic to a direct sum of  $\ell$ -stable sheaves.

**Lemma 4.8.** *Let  $E$  be an  $\ell$ -semistable sheaf on  $X$  and  $\text{gr}_{JH}(E)$  the graded module corresponding to a Jordan-Hölder filtration of  $E$ . Then  $E$  and  $\text{gr}_{JH}(E)$  give the same point in  $M^{[\ell]}$ .*

*Proof.* One can easily construct an  $\mathbb{A}^1$ -flat family  $\mathcal{E}$  of  $\ell$ -semistable sheaves on  $X$  such that  $\mathcal{E}_0 \cong \text{gr}(E)$  and  $\mathcal{E}_t \cong E$  for all  $t \neq 0$  (see [36, Lem. 4.1.2]). Then the classifying morphism  $\Psi_{\mathcal{E}} : \mathbb{A}^1 \rightarrow M^{[\ell]}$  corresponding to  $\mathcal{E}$  is constant, which shows that  $E$  and  $\text{gr}_{JH}(E)$  define the same point in  $M^{[\ell]}$ .  $\square$

Thus, it will suffice to analyze the separation of polystable sheaves. If  $E$  is  $\ell$ -polystable, let  $E^{[\ell]}$  be the  $\ell$ -closure of  $E$ , as defined by (1.1). We recall below some properties of  $E^{[\ell]}$  (see Lemma 1.3):

- (i)  $E^{[\ell]}$  is  $\ell$ -closed and  $\ell$ -polystable,
- (ii) there is a short exact sequence

$$0 \rightarrow E \rightarrow E^{[\ell]} \rightarrow T_E \rightarrow 0$$

such that  $T$  is supported in codimension  $\geq \ell + 1$  in  $\text{Supp}(E)$ . We denote by  $C_{\max}(E)$  the  $(d - \ell - 1)$ -dimensional cycle corresponding to  $T_E$ , i.e.

$$C_{\max}(E) := \sum_j (\text{length}_{Z_j}(T_E)) \langle Z_j \rangle,$$

where the  $Z_j$  are the irreducible components of maximal dimension of the support of  $T_E$ .

Next we aim to prove the following result:

**Theorem E.** *Let  $E_1$  and  $E_2$  be two  $\ell$ -polystable sheaves of class  $c$  on  $X$ . If either*

- (1)  $E_1^{[\ell]} \not\cong E_2^{[\ell]}$ , or
- (2)  $E_1^{[\ell]} \cong E_2^{[\ell]}$  with  $C_{\max}(E_1) \neq C_{\max}(E_2)$ ,

*then  $E_1$  and  $E_2$  give distinct points in  $M^{[\ell]}$ . Otherwise, if  $E_1^{[\ell]} \cong E_2^{[\ell]} =: F$ ,  $C_{\max}(E_1) = C_{\max}(E_2) =: \xi$  and, furthermore,  $[E_1]$  and  $[E_2]$  lie in the same connected component of the fiber over  $\xi$  of the Quot-to-Chow morphism from the weak normalization of  $\text{Quot}(F, P_F - P_{E_1})$  to the (weakly normal) Chow variety  $\text{Chow}_{d-\ell-1}(X)$  of cycles of dimension  $(d - \ell - 1)$  in  $X$ , then  $E_1$  and  $E_2$  define the same point in  $M^{[\ell]}$ .*

**Remark 4.4.** Let  $E$  be an  $\ell$ -semistable sheaf on  $X$  and  $\text{gr}_{JH}(E)$  the graded module corresponding to a Jordan-Hölder filtration of  $E$ . Note that  $\text{gr}_{JH}(E)$  is uniquely determined only in codimension  $\ell$ . However, one can use the result above to show that the pair  $(\text{gr}_{JH}(E)^{[\ell]}, C_{\max}(\text{gr}_{JH}(E)))$  is in fact determined by  $E$  and the polarization  $\mathcal{O}_X(1)$ . Indeed, suppose we have two graded modules  $\text{gr}_1(E)$  and  $\text{gr}_2(E)$  corresponding to different Jordan-Hölder filtrations of  $E$ . Then, by Lemma 4.8 and Theorem E, we must have  $\text{gr}_1(E)^{[\ell]} \cong \text{gr}_2(E)^{[\ell]}$  and  $C_{\max}(\text{gr}_1(E)) = C_{\max}(\text{gr}_2(E))$ .

We divided the proof of Theorem E in a sequence of lemmas.

**Lemma 4.9.** *Let  $E_1$  and  $E_2$  be two  $\ell$ -polystable sheaves of class  $c$  on  $X$  such that  $E_1^{[\ell]} \not\cong E_2^{[\ell]}$ . Then  $E_1$  and  $E_2$  give distinct points in  $M^{[\ell]}$ .*

*Proof.* Firstly, we assume that  $E_1$  and  $E_2$  have distinct support. Then, for a general complete intersection  $X^{(d-\ell)} \subset X$  of  $(d-\ell)$  divisors in  $\Pi_a$ ,  $E_1|_{X^{(d-\ell)}}$  and  $E_2|_{X^{(d-\ell)}}$  still have distinct support. Furthermore, by Theorem B, for  $a \gg 0$  we may assume that  $E_1|_{X^{(d-\ell)}}$  and  $E_2|_{X^{(d-\ell)}}$  are non-isomorphic  $\ell$ -polystable sheaves, hence they give different points in the Simpson moduli space  $M_{X^{(d-\ell)}}^{Gss}$  of Gieseker semistable sheaves on  $X^{(d-\ell)}$ . Then there exists an  $\mathrm{SL}(V')$ -invariant section  $\sigma \in H^0(\overline{R_{X^{(d-\ell)}}^{Gss}}, \mathcal{L}'_{\mathcal{F}'})^{\mathrm{SL}(V')}$  that separates  $E_1|_{X^{(d-\ell)}}$  and  $E_2|_{X^{(d-\ell)}}$  (see Remark 4.3 for notation). Following the proof of Theorem 4.1, we find a section  $\tilde{\sigma} \in H^0(R^{[\ell]}, \mathcal{L}^\nu)^{\mathrm{SL}(V)}$ , for  $\nu > 0$ , that separates  $[E_1]$  and  $[E_2]$  in  $R^{[\ell]}$ . This implies that  $E_1$  and  $E_2$  give distinct points in  $M^{[\ell]}$ .

Now suppose that  $E_1$  and  $E_2$  have the same support. Let  $D \in \Pi_a$  be a divisor avoiding the associated points of  $E_1$  and  $E_2$ , and consider the short exact sequence

$$0 \rightarrow E_2^{[\ell]}(-D) \rightarrow E_2^{[\ell]} \rightarrow E_2^{[\ell]}|_D \rightarrow 0.$$

Applying the functor  $\mathrm{Hom}_X(E_1^{[\ell]}, -)$  yields a long exact sequence

$$0 \rightarrow \mathrm{Hom}(E_1^{[\ell]}, E_2^{[\ell]}) \rightarrow \mathrm{Hom}(E_1^{[\ell]}|_D, E_2^{[\ell]}|_D) \rightarrow \mathrm{Ext}^1(E_1^{[\ell]}, E_2^{[\ell]}(-D)) \rightarrow \dots,$$

where the first morphism is injective since the above sheaves are pure. By using Serre duality and the local-to-global Ext spectral sequence, one obtains

$$\mathrm{Ext}^1(E_1^{[\ell]}, E_2^{[\ell]}(-D)) \cong H^0(X, \mathcal{E}xt^{\dim(X)-1}(E_1^{[\ell]}, E_2^{[\ell]}) \otimes \omega_X(D)) \quad (4.5)$$

for  $a \gg 0$ , where  $\omega_X$  is the dualizing line bundle on  $X$ . As  $E_2^{[\ell]}$  is  $\ell$ -closed, it satisfies  $\mathcal{E}xt^{\dim(X)-1}(E_1^{[\ell]}, E_2^{[\ell]}) = 0$  (see the proof of Lemma 2.8). Hence, it follows by (4.5) that

$$\mathrm{Hom}(E_1^{[\ell]}, E_2^{[\ell]}) \rightarrow \mathrm{Hom}(E_1^{[\ell]}|_D, E_2^{[\ell]}|_D)$$

is an isomorphism.

One can choose  $D \in \Pi_a$  general enough such that  $E_1^{[\ell]}|_D$  and  $E_2^{[\ell]}|_D$  remain  $\ell$ -closed (see Lemma 1.5). Then, repeating the argument above, we obtain an isomorphism

$$\mathrm{Hom}(E_1^{[\ell]}, E_2^{[\ell]}) \rightarrow \mathrm{Hom}(E_1^{[\ell]}|_{X^{(d-\ell)}}, E_2^{[\ell]}|_{X^{(d-\ell)}})$$

for a general smooth complete intersection  $X^{(d-\ell)}$ . This implies  $E_1^{[\ell]}|_{X^{(d-\ell)}} \not\cong E_2^{[\ell]}|_{X^{(d-\ell)}}$ , since by assumption  $E_1^{[\ell]} \not\cong E_2^{[\ell]}$ . For  $a \gg 0$ , one can choose  $X^{(d-\ell)}$  such that

- (i) the restricted sheaves  $E_1|_{X^{(d-\ell)}}$  and  $E_2|_{X^{(d-\ell)}}$  remain  $\ell$ -polystable,
- (ii)  $E_1|_{X^{(d-\ell)}} \cong E_1^{[\ell]}|_{X^{(d-\ell)}}$  and  $E_2|_{X^{(d-\ell)}} \cong E_2^{[\ell]}|_{X^{(d-\ell)}}$ .

Indeed, the first condition is a consequence of Theorem B and the second is satisfied by a sufficiently general complete intersection. Therefore, for such  $X^{(d-\ell)}$ ,  $E_1|_{X^{(d-\ell)}}$  and  $E_2|_{X^{(d-\ell)}}$  are not  $S$ -equivalent. Now the result follows as in the first part of the proof.  $\square$

**Lemma 4.10.** *Let  $E_1$  and  $E_2$  be two  $\ell$ -polystable sheaves of class  $c$  on  $X$  such that  $E_1^{[\ell]} \cong E_2^{[\ell]} =: F$  with  $C_{\max}(E_1) \neq C_{\max}(E_2)$ . Then  $E_1$  and  $E_2$  give distinct points in  $M$ .*

*Proof.* Choose divisors  $D_1, \dots, D_{d-\ell} \in \Pi_a$  such that  $X^{(d-\ell)} = D_1 \cap \dots \cap D_{d-\ell}$  and  $X^{(d-\ell+1)} = D_1 \cap \dots \cap D_{d-\ell+1}$  are smooth complete intersections in  $X$ . If these divisors are chosen general enough, then

- (i)  $E_1|_{X^{(d-\ell+1)}}$  and  $E_2|_{X^{(d-\ell+1)}}$  are  $\ell$ -polystable and have the same  $\ell$ -closure  $F|_{X^{(d-\ell+1)}}$ ,
- (ii)  $C_{\max}(E_1|_{X^{(d-\ell+1)}}) \neq C_{\max}(E_2|_{X^{(d-\ell+1)}})$ .

The first condition is satisfied by Theorem B and Lemma 1.5, and the second clearly holds for a sufficiently general complete intersection. Moreover, by Corollary 4.1 and Lemma 4.5, we have the following  $\mathrm{SL}(V)$ -isomorphisms

$$\mathcal{L} \cong \lambda_{\mathcal{F}|_{R^{[\ell]} \times X^{(d-\ell+1)}}}(w|_{X^{(d-\ell+1)}} \cdot h|_{X^{(d-\ell+1)}}) \cong \lambda_{\mathcal{F}|_{R^{[\ell]} \times X^{(d-\ell)}}}(w|_{X^{(d-\ell)}})$$

over an  $\mathrm{SL}(V)$ -stable open subset  $V \subset R^{[\ell]}$  containing  $E_1|_{X^{(d-\ell+1)}}$  and  $E_2|_{X^{(d-\ell+1)}}$ , whose complement has codimension  $\geq 2$  in  $R^{[\ell]}$ . Thus, it is enough to find sections of (some power of)  $\lambda_{\mathcal{F}|_{R^{[\ell]} \times X^{(d-\ell+1)}}}(w|_{X^{(d-\ell+1)}} \cdot h|_{X^{(d-\ell+1)}})$  over  $V$  that separate  $E_1|_{X^{(d-\ell+1)}}$  and  $E_2|_{X^{(d-\ell+1)}}$ , since then we can extend them over the whole weakly normal variety  $R^{[\ell]}$ , as in the proof of Theorem 4.1. Therefore, we can restrict ourselves to the case  $\ell = d - 1$ , which we treat in detail below.

In this case, the proof follows closely the one given by Jun Li in [53, Lem. 3.6]. We may assume that there is a point  $y_0 \in \mathrm{Supp}(F)$  such that  $\mathrm{length}_{y_0}(C_{\max}(E_1)) > \mathrm{length}_{y_0}(C_{\max}(E_2)) \geq 0$ . Choose  $D_0 \in \Pi_a$  a smooth divisor passing through  $y_0$  such that  $E_1|_{D_0}$  and  $E_2|_{D_0}$  are  $\ell$ -closed over  $D_0 \setminus \{y_0\}$  and  $F|_{D_0}$  is  $\ell$ -polystable. Now take another smooth divisor  $D_1 \in \Pi_a$  such that  $D_0 \cap D_1$  is a smooth complete intersection avoiding  $y_0$ . By blowing up  $X$  along  $D_0 \cap D_1$  we get a smooth projective scheme  $\tilde{X}$  with a natural map  $q : \tilde{X} \rightarrow X$ . Furthermore, the linear system  $D_t$  generated by  $D_0$  and  $D_1$  extends to a base-point-free pencil  $\tilde{D}_t$  on  $\tilde{X}$ , which induces a morphism  $p : \tilde{X} \rightarrow \mathbb{P}^1$ . Note that  $\tilde{D}_t$  is canonically isomorphic with  $D_t$  via  $q$  for all  $t \in \mathbb{P}^1$ . For this reason, in what follows we freely identify  $q^*E_1|_{\tilde{D}_t}$ ,  $q^*E_2|_{\tilde{D}_t}$  and  $q^*F|_{\tilde{D}_t}$  with  $E_1|_{D_t}$ ,  $E_2|_{D_t}$  and  $F|_{D_t}$  respectively.

By the choice of the linear system, locally around  $0 \in \mathbb{P}^1$ , we can find an open neighborhood  $U \subset \mathbb{P}^1$  such that any fiber  $\tilde{D}_t$  is smooth for  $t \in U$ , and both  $E_1|_{D_t}$  and  $E_2|_{D_t}$  are  $\ell$ -polystable and  $\ell$ -closed for  $0 \neq t \in U$ . We denote by  $p_U : \tilde{X}_U \rightarrow U$  the pullback of  $p$  along the inclusion  $U \subset \mathbb{P}^1$ .

For all  $t \in U$ , the  $F|_{D_t}$  have the same Hilbert polynomial, say  $\tilde{P}$ . According to [36, Thm. 4.3.7], there exists a relative moduli space  $M_{\tilde{X}_U/U}(\tilde{P}) \rightarrow U$  endowed with an ample line bundle  $\mathcal{O}_{M_{\tilde{X}_U/U}}(1)$  such that each fiber  $M_{D_t}$  is the moduli space of Gieseker-semistable sheaves on  $D_t$  with Hilbert polynomial  $\tilde{P}$ . Since  $F|_{D_0}$  is  $\ell$ -semistable, there is a section  $v_0 \in H^0(M_{D_0}, \mathcal{O}_{M_{D_0}}(k))$ , for  $k \gg 0$ , non-vanishing at  $F|_{D_0}$ . By using the base change theorem and shrinking  $U$  if necessary, we can extend  $v_0$  to a section  $v \in H^0(M_{\tilde{X}_U/U}, \mathcal{O}_{M_{\tilde{X}_U/U}}(k))$ .

For each  $t \in U$ , there exists a rational morphism  $\psi_t : R^{[\ell]} \dashrightarrow M_{D_t}$  corresponding to the universal family  $\mathcal{F}$  on  $R^{[\ell]} \times X$ . Moreover, we can pullback the restricted section  $v|_{M_{D_t}}$  via  $\psi_t$  to obtain a canonical section  $\tilde{v}_t \in H^0(R^{[\ell]}, \mathcal{L}^\nu)^{\mathrm{SL}(V)}$  for some  $\nu > 0$ . Since this construction is canonical, we can view the  $\tilde{v}_t$  as restrictions of a section  $\tilde{v} \in H^0(R^{[\ell]} \times U, p_1^* \mathcal{L}^\nu)^{\mathrm{SL}(V)}$ , where  $p_1 : R^{[\ell]} \times U \rightarrow R^{[\ell]}$  is the first projection.

Recall that  $\mathcal{L} = \lambda_{\mathcal{F}}(u)$  on  $R^{[\ell]}$  for an appropriate Grothendieck class  $u \in K(X)$ . Denote by  $\tilde{u}$  the pullback of the class  $u$  to  $K(\tilde{X})$  and take  $\tilde{\mathcal{F}} := (1_{R^{[\ell]} \times \mathbb{P}^1} \times q)^* \mathcal{F}$ . Consider the determinant line bundle

$$\mathcal{L}_{R^{[\ell]} \times \mathbb{P}^1} := \det \left( (1_{R^{[\ell]} \times \mathbb{P}^1} \times p)_! (\tilde{\mathcal{F}} \otimes p_2^* \tilde{u}) \right)$$

on  $R^{[\ell]} \times \mathbb{P}^1$ . For  $i = 1, 2$ , the fiber of  $\mathcal{L}_{R^{[\ell]} \times \mathbb{P}^1}$  over  $[E_i] \in R^{[\ell]}$  gives a line bundle

$$\mathcal{L}_{i, \mathbb{P}^1} := \det(p_!(q^* E_i \otimes p_2^* \tilde{u}))$$

on  $\mathbb{P}^1$ . By using the base change theorem, we get

$$\mathcal{L}_{R^{[\ell]} \times \mathbb{P}^1} \cong p_1^* A \otimes p_2^* \mathcal{O}_{\mathbb{P}^1}(\alpha),$$

where  $A$  is a line bundle on  $R^{[\ell]}$  and  $\mathcal{O}_{\mathbb{P}^1}(\alpha)$  a line bundle on  $\mathbb{P}^1$ . Thus, we have canonical isomorphisms

$$\mathcal{L}_{i, \mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(\alpha).$$

On the other hand, consider the line bundle  $p_1^* \mathcal{L}$  on  $R^{[\ell]} \times U$ . As above, the fiber of  $p_1^* \mathcal{L}$  over  $[E_i] \in R^{[\ell]}$  gives a trivial line bundle  $L_i$  on  $U$ . Thus, if we fix a trivialization of  $\mathcal{O}_{\mathbb{P}^1}(\alpha)$  over  $U$ , then we have the following canonical isomorphisms

$$\gamma_i : \mathcal{L}_{i, \mathbb{P}^1}|_U \rightarrow L_i.$$

For  $i = 1, 2$ , consider the following short exact sequence over  $\tilde{X}_U$ :

$$0 \rightarrow q^* E_i \rightarrow q^* F \rightarrow T_i \rightarrow 0,$$

where  $q : \tilde{X}_U \rightarrow X$  is the natural map, and  $T_i$  is a torsion sheaf on  $\tilde{X}_U$ . This gives us isomorphisms

$$\phi_i : \det(p_!(q^* F \otimes p_2^* \tilde{u})) \otimes \det(p_!(T_i \otimes p_2^* \tilde{u}))^{-1} \rightarrow \mathcal{L}_{i, \mathbb{P}^1}. \quad (4.6)$$

over  $U$ . Since  $E_1|_{D_t} \cong E_2|_{D_t} \cong F|_{D_t}$  for  $0 \neq t \in U$ ,  $T_i$  is supported on  $D_0$  and  $\det(p_!(T_i \otimes p_2^* \tilde{u}))^{-1}$  is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(l_i)|_U$ , where  $l_i = \text{length}(T_i|_{D_0}) = \text{length}_{y_0}(C_{\max}(E_i))$ . Note that  $l_1 > l_2$  by assumption.

Denote by  $\tilde{v}^i \in H^0(U, L_i^{\vee})$  the restriction of  $\tilde{v}$  to  $[E_i] \times U$ . Since  $\gamma_1$  and  $\gamma_2$  are canonical, we get that  $\phi_1^*(\gamma_1^*(\tilde{v}^1))$  and  $\phi_2^*(\gamma_2^*(\tilde{v}^2))$  agree over  $U \setminus \{0\}$  as sections of the line bundle  $\det(p_!(q^* F \otimes p_2^* \tilde{u}))$ . Note that the extension of  $\phi_1^*(\gamma_1^*(\tilde{v}^1))$  over  $U$  cannot be trivial, since we have assumed that  $v_0(F|_{D_0}) \neq 0$ . It follows by (4.6) that  $\tilde{v}^1$  and  $\tilde{v}^2$  have different vanishing order at  $0 \in U$ . Thus, for some  $t_1, t_2 \in U$ , we get

$$\frac{\tilde{v}^1(t_1)}{\tilde{v}^2(t_1)} \neq \frac{\tilde{v}^1(t_2)}{\tilde{v}^2(t_2)},$$

from which it follows that

$$\frac{\tilde{v}_{t_1}([E_1])}{\tilde{v}_{t_2}([E_1])} \neq \frac{\tilde{v}_{t_1}([E_2])}{\tilde{v}_{t_2}([E_2])}.$$

Taking  $\lambda = \tilde{v}_{t_1}([E_1])/\tilde{v}_{t_2}([E_1])$ , then  $\sigma = \tilde{v}_{t_1} - \lambda \tilde{v}_{t_2}$  is an  $\text{SL}(V)$ -invariant section of  $\mathcal{L}^{\vee}$  such that  $\sigma([E_1]) = 0$  and  $\sigma([E_2]) \neq 0$ . Therefore  $E_1$  and  $E_2$  give different points in  $M^{[\ell]}$ .  $\square$

**Lemma 4.11.** *Let  $E_1$  and  $E_2$  be two  $\ell$ -polystable sheaves of class  $c$  on  $X$  such that  $E_1^{[\ell]} \cong E_2^{[\ell]} =: F$ ,  $C_{\max}(E_1) = C_{\max}(E_2) =: \xi$ , and assume that  $[E_1]$  and  $[E_2]$  lie in the same connected component of the fiber over  $\xi$  of the Quot-to-Chow morphism from the weak normalization of  $\text{Quot}(F, P_F - P_{E_1})$  to the (weakly normal) Chow variety  $\text{Chow}_{d-\ell-1}(X)$  of cycles of dimension  $(d - \ell - 1)$  in  $X$ . Then  $E_1$  and  $E_2$  define the same point in  $M^{[\ell]}$ .*

*Proof.* The proof goes as in [28, Lem. 5.7]. For simplicity, we treat here only the case when  $\ell = d - 1$ . Then  $E_1$  and  $E_2$  correspond to two closed points  $y_1$  and  $y_2$  respectively in the Quot scheme  $\text{Quot}(F, l)$ . Denote by  $\rho : \text{Quot}(F, l) \rightarrow S^l(X)$  the Quot-to-Chow morphism that associates to any zero-dimensional quotient of  $F$  of length  $l$  its corresponding zero-cycle. By assumption  $y_1$  and  $y_2$  lie inside a connected component  $C$  of the fiber  $\rho^{-1}(\xi)$ . Let  $\mathcal{U}$  be the universal family of quotients over  $\text{Quot}(F, l)$ , which fits in a short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}_{\text{Quot}} \otimes F \rightarrow \mathcal{U} \rightarrow 0$$

such that  $\mathcal{K}$  is flat over  $\text{Quot}(F, l)$ . As  $F$  is  $\ell$ -polystable, it follows that  $\mathcal{K}|_C$  is a  $C$ -flat family of  $\ell$ -semistable sheaves of class  $c$  on  $X$ . Choose a general divisor  $D \in \Pi_a$  avoiding the support of  $\xi$  such that  $F|_D$  is Gieseker-polystable, which exists for  $a \gg 0$  by Theorem B. Then there exists a short exact sequence

$$0 \rightarrow \mathcal{K}|_C(-D) \rightarrow \mathcal{K}|_C \rightarrow \mathcal{K}|_{C \times D} \rightarrow 0 \quad (4.7)$$

such that  $\mathcal{K}|_{C \times D}$  is a  $C$ -flat family of fiber  $F|_D$ , cf. [60, Thm. 22.5]. This induces a constant morphism  $\Psi_{\mathcal{K}} : C \rightarrow M_D^{Gss}$ , where  $M_D^{Gss}$  denotes the moduli space of Gieseker-semistable sheaves of class  $c|_D$  on  $D$ . As  $w(a, l', m')|_D \in (c|_D)^\perp \cap \{1, h|_D, h|_D^2, \dots\}^{\perp\perp}$ , there exists a natural determinant line bundle  $\lambda(w|_D)$  over  $M_D^{Gss}$  (see Remark 4.1.2) such that

$$\Psi_{\mathcal{K}}^*(\lambda(w|_D)) \cong \lambda_{\mathcal{K}|_{C \times D}}(w|_D) \cong \lambda_{\mathcal{K}|_C}(w \cdot h).$$

The last isomorphism follows by Lemma 4.5. But since  $\Psi_{\mathcal{K}}$  is constant, we obtain that  $\lambda_{\mathcal{K}|_C}(w \cdot h)$  is trivial. Therefore, the classifying morphism  $C \rightarrow M^{[\ell]}$  corresponding to the family  $\mathcal{K}|_C$ , as given by Theorem D, is constant. This shows that  $E_1$  and  $E_2$  define the same point in  $M^{[\ell]}$ .  $\square$

The proof of the following result is inspired by [14].

**Theorem 4.3.** *Let  $F$  be a pure sheaf on  $X$  and  $\xi \in S^l(X)$  a zero-cycle of length  $l$ . Then the fiber  $\text{Quot}_\xi(F, l) := \rho^{-1}(\xi)$  over  $\xi$  of the Quot-to-Chow morphism  $\rho : \text{Quot}(F, l) \rightarrow S^l(X)$  is connected.*

*Proof.* The proof is by induction on  $l$ . It is enough to prove the statement when  $\xi$  is supported at a single closed point of  $\text{Supp}(F)$ . So choose a closed point  $x$  lying in a connected component  $Y$  of  $\text{Supp}(F)$ . If  $l = 1$ , then  $\xi$  corresponds to the closed point  $x \in Y$ , and so  $\text{Quot}_\xi(F, 1)$  is isomorphic to the projective space  $\mathbb{P}(F(x))$ , with  $F(x) = F_x \otimes k(x)$ , which is clearly connected.

Now suppose that  $\text{Quot}_\xi(F, l)$  is connected for some  $l > 0$ . Let  $\mathcal{U}$  be the universal family of quotients over  $\text{Quot}_\xi(F, l)$ , and consider the short exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{O}_{\text{Quot}} \otimes F \rightarrow \mathcal{U} \rightarrow 0$$

over  $\text{Quot}_\xi(F, l)$ . Then  $Z := \mathbb{P}(K)$  is a projective bundle over  $\text{Quot}_\xi(F, l) \times Y$ . Note that  $Z$  is connected, since  $\text{Quot}_\xi(F, l) \times Y$  is so.

Next we want to construct a surjective morphism  $\varphi : Z \rightarrow \text{Quot}_\xi(F, l + 1)$ . Let us first describe this map set-theoretically, at the level of geometric points. By construction, a closed point of  $Z$  corresponds to a quotient  $[\mu : K \rightarrow k(x)] \in \mathbb{P}(K)$ , where  $K$  is the kernel of some closed point  $[F \rightarrow Q] \in \text{Quot}_\xi(F, l)$ . If  $K' := \text{Ker}(\mu)$ , then  $F/K'$  is a zero-dimensional quotient of  $F$  of length  $l + 1$  on  $X$ . We send  $[K \rightarrow k(x)] \in Z$  through  $\varphi$  to the quotient  $[F \rightarrow F/K'] \in \text{Quot}_\xi(F, l + 1)$ . In fact, one can do this construction in families, as described in [14, Prop. 5], to define an algebraic morphism  $\varphi : Z \rightarrow \text{Quot}_\xi(F, l + 1)$ .

It remains to show that  $\varphi$  is surjective. Let  $[q : F \rightarrow T] \in \text{Quot}_\xi(F, l + 1)$  be a closed point and denote by  $K := \text{Ker}(q)$  its kernel. Let  $M \subset T_x$  be the subsheaf of all elements in  $T_x$  annihilated by the maximal ideal in  $\mathcal{O}_{Y,x}$ , and choose a point  $[\lambda : k(x) \rightarrow M] \in \mathbb{P}(M^\vee)$ . This induces a diagram as follows

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & k(x) & \xrightarrow{\lambda} & T & \longrightarrow & Q \longrightarrow 0 \\
& & \uparrow \mu & & \uparrow q & & \parallel \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & Q \longrightarrow 0
\end{array}$$

such that  $[F \rightarrow Q]$  is a zero-dimensional quotient of length  $l$  on  $X$ . Then  $[\mu : K \rightarrow k(x)] \in Z$  is mapped to  $[q : F \rightarrow T]$  by  $\varphi$ , which shows that  $\varphi$  is surjective. As  $Z$  is connected, we conclude that  $\text{Quot}_\xi(F, l + 1)$  must also be connected.  $\square$

**Corollary 4.2.** *Let  $E_1$  and  $E_2$  be two  $(d - 1)$ -polystable sheaves of class  $c$  on  $X$ . If either*

- (1)  $E_1^{[d-1]} \not\cong E_2^{[d-1]}$ , or
- (2)  $E_1^{[d-1]} \cong E_2^{[d-1]}$  with  $C_{\max}(E_1) \neq C_{\max}(E_2)$ ,

*then  $E_1$  and  $E_2$  give distinct points in  $M^{[d-1]}$ . Otherwise, if  $E_1^{[d-1]} \cong E_2^{[d-1]}$  and  $C_{\max}(E_1) = C_{\max}(E_2)$ , then  $E_1$  and  $E_2$  define the same point in  $M^{[d]}$ .*

*Proof.* This follows immediately from Theorem E and Theorem 4.3.  $\square$

## 4.4 Applications and examples

In this section, we give some applications and examples of our construction. To avoid any confusion, throughout this part  $R^{[d]}$  will always denote the scheme defined in (4.1), before taking its weak normalization.

### Compactifications of the locus of stable reflexive sheaves

By using the results of Section 4.3, we can show that the moduli spaces  $M^{[d]}$  provide different compactifications of the open locus of slope-stable reflexive sheaves of class  $c$  on  $X$ . To be precise, let  $M^{Gss}$  be the moduli space of Gieseker-semistable sheaves of class  $c$  on  $X$ , and denote by  $M^{\mu s, refl} \subset M^{Gss}$  the open locus of slope-stable reflexive sheaves. In this context, a sheaf  $E$  is reflexive if and only if it is isomorphic



with its 1-closure  $E^{[1]}$ , defined by (1.1). By the universal property of these moduli spaces, one obtains a chain of maps

$$(M^{Gss})^{wn} \rightarrow (M^{[d-1]})^{wn} \rightarrow M^{[d-2]} \rightarrow \dots \rightarrow M^{[1]}.$$

We wish to emphasize that  $M^{Gss}$  and  $M^{[d-1]}$  are not necessarily weakly normal.

**Remark 4.5.** We want to recall here the construction of the morphism  $(M^{Gss})^{wn} \rightarrow M^{[\ell]}$ . Let  $R^{Gss} \subset R^{[\ell]}$  be the open subset of Gieseker-semistable sheaves. Then, by Simpson's construction,  $M^{Gss}$  is the GIT quotient  $R^{Gss} // \mathrm{SL}(V)$ . Since the weak normality property descends to the GIT quotient, we have

$$(M^{Gss})^{wn} = (R^{Gss})^{wn} // \mathrm{SL}(V)$$

(see [28, Sect. 5.4] for details). Recall that  $M^{[\ell]}$  comes together with a natural transformation  $\mathcal{M}_{wn}^{[\ell]} \rightarrow \mathrm{Hom}_{(\mathrm{Sch}^{wn}/\mathbb{C})}(-, M^{[\ell]})$ . Then, by the universal property of the GIT quotient, there is a natural map  $(M^{Gss})^{wn} \rightarrow M^{[\ell]}$ .

Next we show that every morphism  $(M^{Gss})^{wn} \rightarrow M^{[\ell]}$  is an open embedding over the open locus  $(M^{\mu s, refl})^{wn} \subset (M^{Gss})^{wn}$ . We start with the following result:

**Proposition 4.1.** *The natural morphism  $\rho : M^{\mu s, refl} \rightarrow M^{[d-1]}$  is an open embedding. Consequently,  $\rho^{wn} : (M^{\mu s, refl})^{wn} \rightarrow (M^{[d-1]})^{wn}$  is also an open embedding.*

*Proof.* By Theorem E, we know that  $\rho$  is injective. It remains to show that  $\rho$  is also unramified at each point of  $M^{\mu s, refl}$ . So choose a closed point  $[E] \in M^{\mu s, refl}$  corresponding to a slope-stable reflexive sheaf  $E$  on  $X$ . As we saw in Lemma 4.9, there is a general smooth divisor  $D \in |\mathcal{O}_X(a)|$  such that

$$\mathrm{Ext}^1(E, E) \rightarrow \mathrm{Ext}^1(E|_D, E|_D) \quad (4.8)$$

is an isomorphism. Now, suppose that the restriction  $E|_D$  remains slope-stable, which holds for a general divisor of large degree by Theorem B. By restriction, we obtain a rational morphism  $\theta : M^{\mu s, refl} \dashrightarrow M_D^{Gss}$  well-defined around  $[E]$ , where  $M_D^{Gss}$  denotes the moduli space of Gieseker-semistable sheaves on  $D$ . By the bijectivity of (4.8), it follows that  $\theta$  is an open embedding locally around  $[E]$ . Then, using the following commutative diagram

$$\begin{array}{ccc} M^{\mu s, refl} & \xrightarrow{\rho} & M^{[d-1]} \\ & \searrow \theta & \downarrow \\ & & M_D^{Gss} \end{array}$$

one deduces that  $\rho$  is also an open embedding locally around  $[E] \in M^{\mu s, refl}$ .  $\square$

**Corollary 4.3.** *For  $\ell < d - 1$ , the canonical morphism  $f : (M^{\mu s, refl})^{wn} \rightarrow M^{[\ell]}$  is an open embedding.*

*Proof.* Let  $\rho^{wn}$  be the map defined in Proposition 4.1. Set  $Z := \mathrm{Im}(\rho^{wn})$ ,  $T := f(Z)$  and  $g := f|_Z : Z \rightarrow T$ . Due to the previous result, it remains to show that  $g$  is an isomorphism. By Theorem E, this map is bijective. Since  $f$  is proper, it follows that  $g$  is proper as well. Hence  $g$  is finite by bijectivity, and in particular a homeomorphism. Then it follows by Proposition A.1 that  $g$  is in fact an isomorphism.  $\square$

## Compactification of moduli of Higgs bundles

As we have mentioned in the Introduction, one can use our construction to give a compactification of the moduli space of Higgs bundles.

**Definition 4.1.** A *torsion-free Higgs sheaf* on  $X$  is given by a pair  $(E, \theta)$ , where  $E$  is a torsion-free sheaf on  $X$  and  $\theta : E \rightarrow E \otimes \Omega_X^1$  is a regular map such that  $\theta \wedge \theta = 0$ . Furthermore,  $(E, \theta)$  is called *slope-semistable* (resp. *slope-stable*) if

$$\mu(F) \leq \mu(E) \quad (\text{resp. } <)$$

for any Higgs subsheaf  $F \subset E$ , i.e.  $\theta(F) \subset \Omega_X^1 \otimes F$ , with  $0 < \text{rk}(F) < \text{rk}(E)$ .

Let  $T^*X$  be the cotangent space of  $X$ . This is defined as the relative spectrum  $T^*X = \text{Spec}_X(\text{Sym}^\bullet \mathcal{T}_X)$ , where  $\mathcal{T}_X$  is the tangent sheaf on  $X$ . There is a natural projective compactification of  $T^*X$  by

$$Z := \mathbb{P}(\mathcal{T}_X \oplus \mathcal{O}_X) = \underline{\text{Proj}}_X(\text{Sym}^\bullet(\mathcal{T}_X \oplus \mathcal{O}_X)).$$

Denote by  $D = Z \setminus T^*X$  the divisor at infinity, and let  $\pi : Z \rightarrow X$  be the natural projection. Note that  $Z$  is endowed with a natural ample line bundle  $\mathcal{O}_Z(1) := \pi^* \mathcal{O}_X(k) \otimes \mathcal{O}_Z(D)$  for some  $k > 0$ .

**Theorem 4.4.** *There is an equivalence between the category of slope-(semi)stable (resp. Gieseker-(semi)stable) torsion-free Higgs sheaves on  $X$  and the category of slope-(semi)stable (resp. Gieseker-(semi)stable) pure sheaves of dimension  $\dim(X)$  on  $Z$ , whose support does not meet  $D$ . Here the notions of stability are defined with respect to  $\mathcal{O}_X(1)$  and  $\mathcal{O}_Z(1)$  respectively.*

*Sketch of proof.* This was already proved in detail by Simpson in [79, Lem. 6.8 and Cor. 6.9]. We only sketch the idea here. If  $(E, \theta)$  is a Higgs sheaf, then  $\theta : E \rightarrow E \otimes \Omega^1$  induces a natural action of  $\text{Sym}^\bullet \mathcal{T}_X$  on  $E$  since  $\theta \wedge \theta = 0$ . But this is equivalent to a coherent sheaf  $\mathcal{E}$  on  $T^*X$ , i.e. a coherent  $\mathcal{O}_{T^*X}$ -module. Indeed, this follows from the definition of  $T^*X$  as the relative spectrum  $\text{Spec}_X(\text{Sym}^\bullet \mathcal{T}_X)$ , which gives an equivalence between the category of coherent  $\mathcal{O}_{T^*X}$ -module and that of coherent  $(\text{Sym}^\bullet \mathcal{T}_X)$ -algebras. Conversely, if  $\mathcal{E}$  is a coherent sheaf on  $T^*X$ , then  $E = \pi_*(\mathcal{E})$  is a coherent sheaf of  $(\text{Sym}^\bullet \mathcal{T}_X)$ -algebras. Then this induces an action of  $\text{Sym}^\bullet \mathcal{T}_X$  on  $E$ , which is the same as giving a map  $\theta : E \rightarrow E \otimes \Omega^1$  with  $\theta \wedge \theta = 0$ .

For the other properties, see the cited reference.  $\square$

Let us fix the Hilbert polynomial  $P$  of the Higgs sheaves. If  $\mathcal{E}$  is a sheaf on  $Z$ , whose support does not meet  $D$ , then for  $m \gg 0$

$$\begin{aligned} P(\mathcal{E}, m) &= \chi(\mathcal{E} \otimes \mathcal{O}_Z(m)) = h^0(X, \mathcal{E} \otimes \pi^* \mathcal{O}_X(km)) \\ &= h^0(X, \pi_*(\mathcal{E}) \otimes \mathcal{O}_X(km)) \\ &= P(km). \end{aligned}$$

For  $m$  sufficiently large, set  $V := \mathbb{C}^{P(m)}$  and consider the open subscheme  $R^{ss} \subset \text{Quot}(V \otimes \mathcal{O}_Z(-m), P)$  of all quotients  $[q : V \otimes \mathcal{O}_Z(-m) \rightarrow E]$  such that

- (i)  $E$  is a slope-semistable pure sheaf on  $Z$ , and
- (ii)  $q$  induces an isomorphism  $V \rightarrow H^0(E(m))$ .

Applying the construction from Section 4.2, we obtain a moduli space  $M^{\mu ss}$  of slope-semistable sheaves on  $Z$ , together with a natural map  $\varphi : (R^{ss})^{wn} \rightarrow M^{\mu ss}$ . In this case,  $M^{\mu ss}$  is endowed with a weakly normal structure.

Let  $R_{Higgs} \subset R^{ss}$  be the open subset of quotients whose support avoids  $D$ . Note that, via Theorem 4.4,  $R_{Higgs}$  also parametrizes slope-semistable torsion-free Higgs sheaves of Hilbert polynomial  $P$  on  $X$ . Define

$$M_{Higgs}^{\mu ss} := \varphi((R_{Higgs})^{wn}).$$

If  $E$  is a slope-semistable sheaf on  $Z$ , whose support avoids  $D$ , then the same holds true for  $\text{gr}_{JH}(E)^{[e]}$ . From the characterization of the geometric points of  $M^{\mu ss}$  given by Theorem E, we deduce that

$$\varphi((R_{Higgs})^{wn}) \cap \varphi((R^{ss})^{wn} \setminus (R_{Higgs})^{wn}) = \emptyset.$$

Moreover, as  $(R^{ss})^{wn}$  is  $\text{SL}(V)$ -proper, it follows by Lemma 3.1 that

$$M_{Higgs}^{\mu ss} = M^{\mu ss} \setminus \varphi((R^{ss})^{wn} \setminus (R_{Higgs})^{wn})$$

is an open subset of  $M^{\mu ss}$ .

In [79], Simpson constructed a moduli space  $M_{Higgs}$  of Gieseker-semistable Higgs sheaves of Hilbert polynomial  $P$  on  $X$  as an open subset of the moduli space  $M_Z^{Gss}(P)$  of Gieseker-semistable sheaves on  $Z$ . Let  $M_{Higgs}^{s,lf}$  be the open locus of slope-stable Higgs bundles. By Corollary 4.3, there is an open embedding

$$(M_{Higgs}^{s,lf})^{wn} \rightarrow M_{Higgs}^{\mu ss} \subset M^{\mu ss}.$$

In conclusion,  $M^{\mu ss}$  gives a compactification of the moduli space of Higgs bundles.

## Case of no strictly semistable sheaves

In this section, we study the case when there are no strictly slope-semistable sheaves. Here  $R^{Gss} = R^{[1]}$ ,  $(M^{[1]}, \mathcal{A}, e)$  is the (weakly normal)  $\text{SL}(V)$ -Iitaka fibration of the pair  $((R^{[1]})^{wn}, \mathcal{L})$ , and  $M^{Gss}$  is the GIT quotient  $R^{Gss} // \text{SL}(V)$ .

We will see that  $M^{[1]}$  arises as a certain Iitaka fibration of the moduli space  $M^{Gss}$  of Gieseker semistable sheaves. Consider the following diagram:

$$\begin{array}{ccc} & (R^{Gss})^{wn} & \\ \pi \swarrow & & \searrow \varphi \\ (M^{Gss})^{wn} & \xrightarrow{\Phi} & M^{[1]} \end{array}$$

where the morphisms are the natural ones. By construction, the line bundle  $\mathcal{L}$  descends via  $\pi$  to a semiample line bundle  $\mathcal{N}$  on  $(M^{Gss})^{wn}$  (see Remark 4.1.2) such that  $\Phi^*(\mathcal{A}) \cong \mathcal{N}^e$ . Then, as  $(M^{Gss})^{wn} = (R^{[1]})^{wn} // \text{SL}(V)$ , it is not difficult to see that the Iitaka fibration of  $((M^{Gss})^{wn}, \mathcal{N})$  is isomorphic to the  $\text{SL}(V)$ -Iitaka fibration of  $((R^{[1]})^{wn}, \mathcal{L})$ , which is none other than  $M^{[1]}$ .

## Examples: Stein factorization of a Hilb-to-Chow morphism

Here we give a class of examples where the moduli space  $M^{\mu ss} := M^{[1]}$  of slope-semistable sheaves arises as the Stein factorization of a Hilb-to-Chow morphism. This was already sketched in [25, Exam 3.3].

Let us fix the Hilbert polynomial  $P$  of a closed subscheme of codimension 2 in  $X$ . Denote by  $\text{Hilb}(X, P)$  the Hilbert scheme parametrizing closed subschemes  $Z \subset X$  of Hilbert polynomial  $P_Z = P$ . Note that any such subscheme  $[Z] \in \text{Hilb}(X, P)$  corresponds to a quotient

$$0 \rightarrow \mathcal{I}_Z \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Z \rightarrow 0,$$

where  $\mathcal{I}_Z$  is the ideal sheaf of  $Z$ . Since every ideal sheaf  $\mathcal{I}_Z$  is slope-stable, one can show that  $\text{Hilb}(X, P)$  is in fact isomorphic to the Gieseker moduli space of rank one stable sheaves with trivial determinant and Hilbert polynomial  $P_X - P$  on  $X$ .

Set  $d := \dim(X)$  and consider the following diagram

$$\begin{array}{ccc} \text{Hilb}(X, P)^{wn} & \xrightarrow{\Theta} & M^{\mu ss} \\ \downarrow \Phi & & \\ \text{Chow}_{d-2}(X) & & \end{array}$$

where

- (i)  $M^{\mu ss}$  is the moduli space of slope-stable sheaves of Hilbert polynomial  $P$  on  $X$ , as given by Theorem D,  $\Theta$  is the canonical map, and
- (ii)  $\Phi : \text{Hilb}(X, P)^{wn} \rightarrow \text{Chow}_{d-2}(X)$  is the Hilb-to-Chow morphism, see the discussion in Appendix B.

Let  $\mathcal{Q}$  be the universal family of quotients over  $\text{Hilb}(X, P)^{wn}$ . Then there is a short exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{\text{Hilb}^{wn}} \rightarrow \mathcal{Q} \rightarrow 0,$$

which further induces an isomorphism of line bundles

$$\lambda_{\mathcal{I}}(w \cdot h^{d-1}) \cong \lambda_{\mathcal{Q}}(w \cdot h^{d-1})^\vee.$$

Here  $w := w(a, l', m') \in K(X)$  is the Grothendieck class defined in 4.2 with  $\ell = 1$ . According to Appendix B,  $\text{Chow}_{d-2}(X)$  can be endowed with an ample line bundle  $\mathcal{O}_{\text{Chow}}(1)$  such that

$$\Phi^* \mathcal{O}_{\text{Chow}}(a^{d-1}) \cong \lambda_{\mathcal{Q}}(h^{d-1}).$$

Putting together the last observations, we obtain:

**Lemma 4.12.** *There is an isomorphism*

$$\lambda_{\mathcal{I}}(w \cdot h^{d-1}) \cong \Phi^* \mathcal{O}_{\text{Chow}}(a^{d-1})^{\chi(c(l') \cdot h^{d-1}) - \chi(c(m') \cdot h^{d-1})}$$

over  $\text{Hilb}(X, P)^{wn}$ . In particular, for  $l' \gg m' \gg 0$ ,  $\lambda_{\mathcal{I}}(w \cdot h^{d-1})$  is the pull-back via  $\Phi$  of an ample line bundle.

*Proof.* By definition

$$\begin{aligned} w \cdot h^{d-1} &= \chi(c(m') \cdot h^{d-1})[\mathcal{O}_X(l')] \cdot h^{d-1} - \chi(c(l') \cdot h^{d-1})[\mathcal{O}_X(m')] \cdot h^{d-1} \\ &= (\chi(c(m') \cdot h^{d-1}) - \chi(c(l') \cdot h^{d-1}))h^{d-1} + (\text{higher order terms}). \end{aligned}$$

As  $\mathcal{Q}$  is a family of relative fiber dimension  $d-2$ , one sees that  $\lambda_{\mathcal{Q}}(h^k)$  is trivial for  $k > d-1$ . Therefore

$$\begin{aligned} \lambda_{\mathcal{I}}(w \cdot h^{d-1}) &\cong \lambda_{\mathcal{Q}}(w \cdot h^{d-1})^\vee \cong \lambda_{\mathcal{Q}}((\chi(c(l') \cdot h^{d-1}) - \chi(c(m') \cdot h^{d-1}))h^{d-1}) \\ &\cong \Phi^* \mathcal{O}_{\text{Chow}}(a^{d-1})_{(\chi(c(l') \cdot h^{d-1}) - \chi(c(m') \cdot h^{d-1}))}. \end{aligned}$$

□

From the above considerations, it follows that  $(\text{Chow}_{d-2}(X), \mathcal{O}_{\text{Chow}}(1), e')$  satisfies property (2) of Theorem D for some integer  $e' > 0$ . Let  $S$  be the Stein factorization of  $\Phi$ :

$$\Phi : \text{Hilb}^{wn} \rightarrow S \xrightarrow{\pi} \text{Chow}.$$

As  $\pi$  finite, the line bundle  $\mathcal{O}_S(1) := \pi^* \mathcal{O}_{\text{Chow}}(1)$  is still ample. We obtain that  $(S, \pi^* \mathcal{O}_{\text{Chow}}(1), e')$  also satisfies property (2) of Theorem D. Hence, by the universal property of  $M^{\mu_{ss}}$ , there is a natural map  $f : M^{\mu_{ss}} \rightarrow S$ . On the other hand, according to Theorem E,  $\Theta$  contracts the fibers of  $\Phi$ , so there is a unique finite morphism  $g : S \rightarrow M^{\mu_{ss}}$ . Then, by the universal property of  $M^{\mu_{ss}}$  and the universal property of the Stein factorization  $S$ , it follows that  $g \circ f = \text{id}$  and  $f \circ g = \text{id}$  respectively. In conclusion,  $M^{\mu_{ss}}$  is isomorphic to the Stein factorization of  $\Phi$ .



## Appendix A

# Weak normalization

Here we briefly recall some general facts about the weak normalization of a variety, following [51, 84]. We refer the reader to Fischer's book [15, Appendix to Ch. 2] for more information on this topic.

**Definition A.1.** Let  $R$  be a complex variety and  $\pi : \tilde{R} \rightarrow R$  its normalization. Define the sheaf  $\mathcal{O}_R^c$  of  $c$ -regular functions on  $R$  as follows: for every open subset  $U \subset R$  let

$$\mathcal{O}_R^c(U) = \{f : U \rightarrow \mathbb{C} \mid f \circ \pi \in \Gamma(\pi^{-1}(U), \mathcal{O}_{\tilde{R}})\}.$$

Then  $R$  is called *weakly normal* if and only if  $\mathcal{O}_R = \mathcal{O}_R^c$ .

The following theorem shows the existence and uniqueness of the weak normalization of a variety.

**Theorem A.1.** *Let  $R$  be a complex variety. Then there is a unique pair  $(R^{wn}, f)$ , where  $R^{wn}$  is a weakly normal variety together with a finite birational morphism  $f : R^{wn} \rightarrow R$  which is a homeomorphism. Furthermore, if there exists another pair  $(Z, g)$  with the same properties as above, then there is a unique factorization  $g : Z \rightarrow R^{wn} \xrightarrow{f} R$ .*

*Proof.* See [51, Thm. 2.9] and [84, Prop. 3.12] for the proof. □

**Remark A.1.** If  $R$  is a separated scheme of finite type over  $\mathbb{C}$ , we let  $R^{wn}$  denote the weak normalization of  $R_{red}$ .

**Proposition A.1.** *Let  $f : Z \rightarrow R$  be a finite birational morphism between weakly normal varieties which is a homeomorphism. Then  $f$  is an isomorphism.*

*Proof.* This follows immediately from the universal property stated in Theorem A.1. □

The following extension result was proved by Greb–Toma [28, Lemma 2.12].

**Lemma A.1.** *Let  $G$  be a connected algebraic group,  $S$  a weakly normal  $G$ -variety over  $\mathbb{C}$  and  $\mathcal{L}$  a  $G$ -equivariant line bundle over  $S$ . Then there are finitely many irreducible subvarieties  $S_1, \dots, S_r$  of  $S$  such that the following holds: for any  $G$ -stable closed subvariety  $T \subset S$  satisfying*

(i) *the intersection of  $T$  with each irreducible component  $Z$  of  $S$  has codimension  $\geq 2$  in  $Z$ , and*

(ii)  *$T$  contains none of the  $S_i$ ,*

*any  $G$ -invariant section in  $H^0(S \setminus T, \mathcal{L})^G$  extends to a  $G$ -invariant section in  $H^0(S, \mathcal{L})^G$ .*





## Appendix B

# Quot-to-Chow morphism

In this appendix we first recall the construction of the Quot-to-Chow morphism, and then define a (determinantal) ample line bundle on the Chow scheme. The material presented here is already known, and for this reason we omit many details. Our main reference is [17] (see also [38]).

Let  $X$  be a connected smooth projective variety over  $\mathbb{C}$  and  $X \subset \mathbb{P}^N$  the natural closed embedding induced by a fixed polarization  $\mathcal{O}_X(1)$  on  $X$ . Fix  $P \in \mathbb{Q}[X]$  a polynomial of degree  $e \leq \dim(X)$  and  $G$  a coherent sheaf on  $X$ . Consider the Quot scheme

$$\text{Quot} := \text{Quot}_X(G, P)$$

of quotients  $[q : G \rightarrow E]$  on  $X$  of fixed Hilbert polynomial  $P$ .

Denote by  $\mathbb{G} := \text{Grass}(N - e, N + 1)$  the Grassmannian variety over  $\mathbb{C}$  of  $(N - e - 1)$ -dimensional linear subspaces of  $\mathbb{P}^N$ . We have a diagram as follows

$$\begin{array}{ccc} & X \times \mathbb{G} & \\ p_1 \swarrow & & \searrow p_2 \\ X & & \mathbb{G} \end{array}$$

Consider the incidence variety

$$I = \{(x, V) \in X \times \mathbb{G} \mid x \in V\}$$

and let  $q_1$  and  $q_2$  be the induced projections of  $I$  into  $X$  and  $\mathbb{G}$  respectively. If  $\mathcal{G}$  denotes the universal quotient on  $\text{Quot} \times X$ , then we have a surjection

$$\mathcal{O}_{\text{Quot}} \otimes G \rightarrow \mathcal{G} \rightarrow 0.$$

Since  $\mathcal{O}_X(1)$  is very ample, for  $m$  sufficiently large, there is a surjection (see [17, Sect. 4])

$$p_{13*} p_{12}^*(\mathcal{O}_{\text{Quot}} \otimes G)(m) \rightarrow q_{13*} q_{12}^* \mathcal{G}(m) \rightarrow 0 \tag{B.1}$$

over  $\text{Quot} \times \mathbb{G}$ . Above, we denoted by  $p_{12}$  and  $p_{13}$  the natural projections given by the diagram

$$\begin{array}{ccc} & \text{Quot} \times X \times \mathbb{G} & \\ p_{12} \swarrow & & \searrow p_{13} \\ \text{Quot} \times X & & \text{Quot} \times \mathbb{G} \end{array}$$

and similarly for  $q_{12}$  and  $q_{13}$ . Furthermore, for large values of  $m$ ,  $q_{13*}q_{12}^*\mathcal{G}(m)$  is a Quot-flat family of torsion sheaves, and so we can form its determinant line bundle

$$\mathcal{O}(D_{\mathcal{G}}) := \det(q_{13*}q_{12}^*\mathcal{G}(m))$$

by using a finite locally free resolution, cf. [36, Sect. 2.1]. By taking the determinant of (B.1), we obtain a canonical homomorphism

$$\nabla : \mathcal{O}_{\text{Quot} \times \mathbb{G}} \rightarrow \mathcal{O}(D_{\mathcal{G}})$$

corresponding to a relative Cartier divisor on  $\mathbb{G}$ , i.e. a Quot-flat family of effective Cartier divisors on  $\mathbb{G}$ . We refer the reader to [17, Sect. 7] for the precise definition of  $\nabla$ . In his notation  $\det$  corresponds to  $\text{Inv}$ .

If  $\text{CDiv}(\mathbb{G})$  denotes the scheme parametrizing effective Cartier divisors on  $\mathbb{G}$ , then  $\nabla$  induces a morphism of schemes

$$\Psi : \text{Quot} \rightarrow \text{CDiv}(\mathbb{G}).$$

At the level of geometric points, this map sends a quotient  $[q : G \rightarrow E]$  on  $X$  to the corresponding effective divisor  $\mathcal{O}(D_E) := \det(q_{2*}q_1^*E(m))$  on  $\mathbb{G}$ . According to [17, Sect. 10], one can completely recover from  $D_E$  the associated  $e$ -dimensional cycle  $\langle E \rangle$  of  $E$ , i.e.

$$\langle E \rangle := \sum_j (\text{length}_{Z_j}(E)) \langle Z_j \rangle,$$

where the  $Z_j$  are the irreducible components of maximal dimension of the support of  $E$ . For this reason, we call  $\Psi$  the Quot-to-Chow morphism.

## An ample determinant line bundle on $\text{CDiv}(\mathbb{G})$

Let  $\mathcal{O}_I$  be the structure sheaf of the subscheme  $I \subset X \times \mathbb{G}$ . Then  $p_{12}^*\mathcal{G}(m) \otimes p_{23}^*\mathcal{O}_I$  is flat over  $\text{Quot} \times \mathbb{G}$ , and so we can consider its determinant line bundle

$$\mathcal{L} := \det(p_{13!}(p_{12}^*\mathcal{G}(m) \otimes p_{23}^*\mathcal{O}_I))$$

on  $\text{Quot} \times \mathbb{G}$ . If  $\mathcal{D} \subset \text{CDiv}(\mathbb{G}) \times \mathbb{G}$  denotes the universal Cartier divisor, then by construction there is an isomorphism

$$\mathcal{L} \cong (\Psi \times \text{id}_{\mathbb{G}})^*\mathcal{O}(\mathcal{D})$$

over  $\text{Quot} \times \mathbb{G}$ . Let  $V = H_1 \cap \dots \cap H_{e+1} \in \mathbb{G}$  be a linear subspace of dimension  $N - e - 1$  in  $\mathbb{P}^N$ , with  $H_1, \dots, H_{e+1} \subset \mathbb{P}^N$  hyperplanes. Then the pullback of  $\mathcal{L}$  to  $\text{Quot} \times \{V\}$  can be rewritten as (see Section 4.1 for notation)

$$\mathcal{L}_V \cong \lambda_{\mathcal{G}(m)}(h^{e+1}),$$

where  $h = [\mathcal{O}_H]$  denotes the Grothendieck class of a hyperplane  $H \in |\mathcal{O}_X(1)|$ .

On the other hand, the pullback of  $\mathcal{O}(\mathcal{D})$  to  $\text{CDiv}(\mathbb{G}) \times \{V\}$  is by construction an ample line bundle on  $\text{CDiv}(\mathbb{G})$ . Indeed, it is enough to see this for a projective space  $\mathbb{P}^r$  as there is always a closed immersion

$$\text{CDiv}(\mathbb{G}) \times \mathbb{G} \rightarrow \text{CDiv}(\mathbb{P}^r) \times \mathbb{P}^r$$

induced by the Plücker embedding of the Grassmannian. In this case, if  $\mathcal{H}$  is the universal hyperplane on  $\text{CDiv}(\mathbb{P}^r) \times \mathbb{P}^r$ , then  $\mathcal{H}$  corresponds to a section of  $p_1^*(\mathcal{O}_{\text{CDiv}(\mathbb{P}^r)}(1)) \otimes p_2^*(\mathcal{O}_{\mathbb{P}^r}(1))$ , where  $\mathcal{O}_{\mathbb{P}^r}(1)$  and  $\mathcal{O}_{\text{CDiv}(\mathbb{P}^r)}(1)$  are the natural line bundles on  $\mathbb{P}^r$  and its dual  $\text{CDiv}(\mathbb{P}^r)$  respectively. Clearly the pullback of  $\mathcal{H}$  to  $\text{CDiv}(\mathbb{P}^r) \times \{x\}$  is an ample divisor of determinant  $\mathcal{O}_{\text{CDiv}(\mathbb{P}^r)}(1)$ .

From the above discussion, we conclude that  $\lambda_{\mathcal{G}(m)}(h^{e+1})$  is the pullback via  $\Psi$  of an ample line bundle  $\mathcal{O}_{\text{CDiv}(\mathbb{G})}(1)$  on  $\text{CDiv}(\mathbb{G})$ .

## B.1 Hilb-to-Chow morphism

Now, let us restrict to the case  $G = \mathcal{O}_X$  and consider the Hilbert scheme

$$\text{Hilb} := \text{Quot}_X(\mathcal{O}_X, P)$$

of closed subschemes of  $X$  of Hilbert polynomial  $P$ . Then the weak normalization of  $\Psi : \text{Hilb} \rightarrow \text{CDiv}(\mathbb{G})$  factors in fact through the weakly normal Chow scheme  $\text{Chow}_e(X)$  of effective  $e$ -dimensional cycles on  $X$ , cf. [40, Thm. 6.3]. Thus, we have a commutative diagram

$$\begin{array}{ccc} \text{Hilb}^{wn} & \xrightarrow{\Phi} & \text{Chow}_e(X) \\ & \searrow \Psi^{wn} & \downarrow \iota \\ & & \text{CDiv}(\mathbb{G}) \end{array}$$

with  $\iota$  a closed embedding. Here  $\text{Chow}_e(X)$  is the (weakly normal) Chow scheme constructed by Kollár in [40, Ch. I]. Let  $\mathcal{Q}$  denote the universal quotient over  $\text{Hilb}^{wn}$ . As in the previous section, one can endow  $\text{Chow}_{d-2}(X)$  with an ample line bundle  $\mathcal{O}_{\text{Chow}}(1)$  such that

$$\Phi^* \mathcal{O}_{\text{Chow}}(1) \cong \lambda_{\mathcal{Q}(m)}(h^{e+1}) \cong \lambda_{\mathcal{Q}}(h^{e+1}).$$

The last isomorphism follows since  $\mathcal{Q}$  is a flat family of fiber dimension  $e$ , so  $\lambda_{\mathcal{Q}}(h^k)$  is trivial for  $k > e + 1$ .



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