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IAEM



UNIVERSITÉ DE LORRAINE

INSTITUT ÉLIE CARTAN DE LORRAINE

UNIVERSAL HIGHER LIE ALGEBRAS OF SINGULAR SPACES
AND THEIR SYMMETRIES

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Thèse

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(mention Mathématiques)

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RÉSUMÉ

Cette thèse se décompose en deux parties principales.

1. Nous montrons qu'il existe une équivalence de catégories entre les algèbres de Lie-Rinehart sur une algèbre commutative \mathcal{O} et classes d'équivalence d'homotopie d'algébroïdes de Lie infinie acycliques graduées négativement. Par conséquent, ce résultat donne un sens à l'algébroïde de Lie infinie universel d'un feuilletage singulier, sans hypothèse supplémentaire, et pour les algébroïdes de Lie singulières d'Androulidakis-Zambon. Ceci étend à un cadre purement algébrique la construction de la Q -variété universelle d'un feuilletage singulier localement réel analytique de [LLS20, Lav17]. Aussi, à tout idéal \mathcal{I} de \mathcal{O} préservé par l'application d'ancre d'une algèbre de Lie-Rinehart \mathcal{A} , on associe une classe d'équivalence d'homotopie d'algébroïdes de Lie infinie négativement graduée sur des complexes calculant $\mathrm{Tor}_{\mathcal{O}}(\mathcal{A}, \mathcal{O}/\mathcal{I})$. Plusieurs exemples explicites sont donnés.
2. La deuxième partie est consacrée à quelques applications des résultats sur les algèbres de Lie-Rinehart.
 - (a) On associe à toute variété affine une algébroïde de Lie infinie universelle de l'algèbre de Lie-Rinehart de ses champs de vecteurs. Nous étudions l'effet de certaines opérations courantes sur des variétés affines telles que les éclatements, germes en un point, etc.
 - (b) Nous donnons une interprétation de l'éclatement d'un feuilletage singulier \mathfrak{F} au sens de Mohsen [Moh21] en termes de l'algébroïde de Lie infinie universelle de \mathfrak{F} .
 - (c) Nous introduisons la notion de champs de vecteurs longitudinaux sur une variété graduée sur un feuilletage, et étudier leur cohomologie. Nous prouvons que les groupes de cohomologie de ce dernier sont nuls.
 - (d) Nous étudions les symétries de feuilletages singuliers à travers des algébroïdes de Lie infinie universels. Plus précisément, nous prouvons qu'une action par symétrie faible d'une algèbre de Lie \mathfrak{g} sur un feuilletage singulier \mathfrak{F} (qui est moralement une action de \mathfrak{g} sur l'espace des feuilles M/\mathfrak{F}) induit un unique morphisme de Lie infini à homotopie près de \mathfrak{g} vers l'algèbre de Lie différentielle graduée (DGLA) des champs de vecteurs sur un algébroïde de Lie infinie universel de \mathfrak{F} . On déduit de ce résultat général plusieurs conséquences. Par exemple, nous donnons un exemple d'action d'algèbre de Lie sur une sous-variété affine qui ne peut s'étendre à l'espace ambiant. Enfin, nous présentons la notion de tour de bisubmersions sur un feuilletage singulier et relève des symétries à celles-ci.

Mots clés en français: Algèbres à homotopie près, Algébroïdes à homotopie près, Singularités, Géométrie différentielle et algébriques, Feuilletages, dg-variétés.

ABSTRACT

This thesis breaks into two main parts. Let me describe them.

- We show that there is an equivalence of categories between Lie-Rinehart algebras over a commutative algebra \mathcal{O} and homotopy equivalence classes of negatively graded acyclic Lie ∞ -algebroids. Therefore, this result makes sense of the universal Lie ∞ -algebroid of every singular foliation, without any additional assumption, and for Andrroulidakis-Zambon singular Lie algebroids. This extends to a purely algebraic setting the construction of the universal Q -manifold of a locally real analytic singular foliation of [LLS20, Lav17]. Also, to any ideal $\mathcal{I} \subset \mathcal{O}$ preserved by the anchor map of a Lie-Rinehart algebra \mathcal{A} , we associate a homotopy equivalence class of negatively graded Lie ∞ -algebroids over complexes computing $\mathrm{Tor}_{\mathcal{O}}(\mathcal{A}, \mathcal{O}/\mathcal{I})$. Several explicit examples are given.
- The second part is dedicated to some applications of the results on Lie-Rinehart algebras.
 1. We associate to any affine variety a universal Lie ∞ -algebroid of the Lie-Rinehart algebra of its vector fields. We study the effect of some common operations on affine varieties such as blow-ups, germs at a point, etc.
 2. We give an interpretation of the blow-up of a singular foliation \mathfrak{F} in the sense of Mohsen [Moh21] in terms of the universal Lie ∞ -algebroid of \mathfrak{F} , in fact an almost Lie algebroid over a geometric resolution of \mathfrak{F} .
 3. We introduce the notion of longitudinal vector fields on a graded manifold over a singular foliation, and study their cohomology. We prove that the cohomology groups of the latter vanish.
 4. We study symmetries of singular foliations through universal Lie ∞ -algebroids. More precisely, we prove that a weak symmetry action of a Lie algebra \mathfrak{g} on a singular foliation \mathfrak{F} (which is morally an action of \mathfrak{g} on the leaf space M/\mathfrak{F}) induces a unique up to homotopy Lie ∞ -morphism from \mathfrak{g} to the Differential Graded Lie Algebra (DGLA) of vector fields on a universal Lie ∞ -algebroid of \mathfrak{F} (such morphism is known under the name " L_{∞} -algebra action" in [MZ12]). We deduce from this general result several geometrical consequences. For instance, We give an example of a Lie algebra action on an affine sub-variety which cannot be extended on the ambient space. Last, we present the notion of bi-submersion towers over a singular foliation and lift symmetries to those.

Keywords: Homotopy algebras, Lie ∞ -algebroids, dg-manifolds, singular foliations, algebraic geometry, singularities.

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Les études récentes sur les ∞ -algèbres de Lie ou les ∞ -groupes de Lie, leurs morphismes et leurs équivalents en -oïdes (c'est-à-dire les ∞ -algèbroïdes de Lie [Cam19, Vor05, Vor10] et les "groupoïdes supérieurs" [SS20]) est généralement justifié par leur utilisation dans divers domaines de la physique théorique et des mathématiques. Lie ∞ -algèbres ou -oïdes apparaissent souvent là où, à première vue, ils ne semblent pas faire partie de l'histoire, mais finissent par être nécessaires pour répondre à des questions naturelles, en particulier des questions où aucun concept de structure supérieure ne semble a priori impliqué. Parmi les exemples d'une telle situation, citons la quantification de déformation des variétés de Poisson [Kon03] et de nombreux développements récents de la théorie des opérateurs BV, par ex. [Cam17], déformations de sous-variétés coisotropes [CF04], problèmes d'intégration d'algèbroïdes de Lie par stacky-groupoïdes [TZ06], sous-variétés complexes et classes d'Atiyah [CSX16, Kap99, LGSX21]. La liste pourrait être plus longue.

Par exemple, il apparaît dans la théorie des feuilletages singuliers. Les feuilletages singuliers apparaissent fréquemment en géométrie différentielle ou algébrique. Ici, suivant [AS09, AZ14, Cer79, Deb01, LLS20] nous définissons un feuilletage singulier sur une variété analytique lisse, complexe, algébrique, réelle, M avec le faisceau de fonctions \mathcal{O} comme sous-faisceau $\mathfrak{F}: \mathcal{U} \rightarrow \mathfrak{F}(\mathcal{U})$ du faisceau de champs de vecteurs \mathfrak{X} , qui est stable par crochet de Lie et est localement fini engendré en tant qu'un \mathcal{O} -module. D'après le théorème d'Hermann [Her62], cela suffit pour induire une partition de la variété M en sous-variétés de dimensions éventuellement différentes, appelées *feuilles* du feuilletage singulier. Les feuilletages singuliers apparaissent par exemple comme des orbites d'actions de groupe de Lie avec des dimensions éventuellement différentes ou comme des feuilles symplectiques d'une structure de Poisson. Lorsque toutes les feuilles ont la même dimension, on retrouve les "feuilletages réguliers" [DHH86, LGLR22].

Dans [LLS20]-[Lav17], il est prouvé que "derrière" certains feuilletages singuliers \mathfrak{F} il existe une classe naturelle d'homotopie ∞ -algèbroïdes de Lie, appelée *∞ -algebroid de Lie universelle de \mathfrak{F}* , et que ce dernier répond à des questions fondamentales naturelles sur l'existence de "good" générateurs et relations pour un feuilletage singulier. La première partie de cette thèse est principalement une algébrisation de [LLS20]-[Lav17], qui permet d'élargir largement les classes d'exemples. Plus précisément, les théorèmes 4.2.1 et 4.2.4 sont similaires aux théorèmes principaux Théorème 2.8. et Théorème 2.9 dans [LLS20] :

1. Le théorème 4.2.1 équipe toute \mathcal{O} -résolution libre d'une algèbre de Lie-Rinehart \mathcal{A} avec une structure ∞ -algébroïde de Lie (Théorème 2.8. dans [LLS20] était que pour les résolutions géométriques du feuilletage singulier analytique localement réel sur un sous-ensemble ouvert avec adhérence compacte). C'est une sorte de théorème de transfert d'homotopie, sauf qu'aucun théorème de transfert d'homotopie existant ne s'applique dans le contexte des \mathcal{O} -modules générique (par exemple, [Cam19] ne traite que des projectifs \mathcal{O} -modules). La difficulté est que nous ne pouvons pas appliquer les formules de transfert explicites qui apparaissent dans le lemme de perturbation homologique, car il n'y a en général pas de section \mathcal{O} -linéaire de \mathcal{A} à ses résolutions projectives.
2. Theorem 4.2.4 dit que toute structure ∞ -algébroïde de Lie qui se termine dans \mathcal{A} est équipée d'un unique morphisme ∞ -algébroïde de Lie à homotopie à n'importe quelle structure comme dans le premier item (le théorème 2.8. dans [LLS20] était une similaire pour Lie ∞ -algébroïdes dont l'ancre prend des valeurs dans un feuilletage singulier donné).

Comme dans [LLS20], un corollaire immédiat du résultat est que deux ∞ -algébroïdes de Lie comme dans l'item 1 sont homotopiques équivalents d'une manière unique à l'homotopie près, définissant ainsi une classe canoniquement associée à l'algèbre de Lie-Rinehart qui mérite au vu de l'item 2 d'être appelée "universelle."

Cependant :

1. Alors que [LLS20] traitait des ∞ -algèbres de Lie sur des résolutions projectives de longueur et de dimension finies, nous travaillons ici avec des ∞ -algèbroïdes de Lie sur n'importe quelle résolution *libre* -même celles de longueur infinie et de dimension infinie à chaque degré.
2. En particulier, comme on est dans un contexte où prendre deux fois le dual ne ramène pas l'espace initial, on ne peut pas travailler avec des Q -variétés (celles-ci étant les "dual" ∞ -algébroïdes de Lie) : il est beaucoup plus compliqué de traiter les morphismes et les homotopies.

Ce faisant, plusieurs limitations de [LLS20] sont surmontées. Alors que [LLS20] ne s'appliquait qu'aux feuilletages singuliers qui étaient analytiques, algébriques ou localement réels sur un sous-ensemble ouvert relativement compact, la présente thèse associe une classe d'homotopie naturelle ∞ -algébroïdes de Lie à toute algèbre de Lie-Rinehart, et en particulier

- a) à tout feuilletage singulier sur une variété lisse, (finiment engendré ou non). Cette construction fonctionne toujours avec des feuilletages singuliers au sens de Stefan-Sussmann par exemple, ou avec n'importe quel module involutif $C^\infty(M)$ dans $\Gamma(A)$.
- b) à toute variété affine, à laquelle on associe son algèbre de Lie-Rinehart des champs de vecteurs), et plus généralement aux dérivations de toute algèbre commutative,
- c) aux algébroïdes de Lie singulière au sens d'Androulidakis et de Zambon [AZ18],
- d) à divers contextes inattendus, par ex. Champs de vecteurs de Poisson d'une variété de Poisson, vue comme une algèbre de Lie-Rinehart sur des fonctions de Casimir, ou symétries d'un feuilletage singulier, vue comme une algèbre de Lie-Rinehart sur des fonctions constantes sur les feuilles.

Ces ∞ -algèbroïdes de Lie sont construits sur des résolutions par des \mathcal{O} -modules libres de l'algèbre de Lie-Rinehart initiale sur \mathcal{O} . Ils sont universels dans un certain sens (voir Section 4.2), et ils sont aussi en particulier uniques à homotopie près.

Une algébrisation similaire des principaux résultats de [LLS20], utilisant la catégorie des semi-modèles, apparue récemment dans Yaël Frégier et Rigel A. Juarez-Ojeda [FJO18]. Il y a de fortes similitudes entre nos résultats et les leurs, mais les morphismes et les homotopies dans [FJO18] ne correspondent pas aux nôtres. Il est cependant fort possible que le théorème 4.2.1 puisse être récupéré à l'aide de leurs résultats. Luca Vitagliano [Vit15] a également construit des structures ∞ -algèbres de Lie à partir de feuilletages réguliers, qui sont bien sûr un cas particulier de l'algèbre de Lie-Rinehart. Ces constructions n'ont pas les mêmes finalités. Pour les feuilletages réguliers, notre structure ∞ -algèbroïde de Lie est triviale dans le sens où c'est un algèbroïde de Lie, alors que ses structures deviennent triviales lorsqu'une bonne sous-variété transverse existe. Lars Kjeseth [Kje01b, Kje01a] a aussi une notion de résolution des algèbres de Lie-Rinehart. Mais sa construction est davantage dans l'esprit de la résolution de Koszul-Tate : la définition 1. dans [Kje01b] définit les résolutions des algèbres de Lie-Rinehart comme des résolutions de leur coalgèbre de Chevalley-Eilenberg, et non de l'algèbre de Lie-Rinehart elle-même en tant que module. Il répond à une autre catégorie de questions, liées à BRST et à la recherche de modèle cohomologique pour la cohomologie de l'algèbre de Lie-Rinehart.

Ces résultats peuvent être utilisés pour comprendre la géométrie des feuilletages singuliers tels que leurs symétries. Plus précisément, Soit (M, \mathfrak{F}) une variété feuilletée. Une *symétrie globale* d'un feuilletage singulier \mathfrak{F} sur M est un difféomorphisme $\phi: M \rightarrow M$ qui préserve \mathfrak{F} , soit $\phi_*(\mathfrak{F}) = \mathfrak{F}$. L'image d'une feuille à travers une symétrie globale est à nouveau une feuille (pas nécessairement la même feuille). Pour G un groupe de Lie, une *action stricte par symétrie* de G sur une variété feuilletée (M, \mathfrak{F}) est une action lisse $G \times M \rightarrow M$ qui agit par symétries globales [GZZ1]. Infinitésimalement, il correspond à un morphisme d'algèbre de Lie $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ entre l'algèbre de Lie $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ de G et l'algèbre de Lie des symétries de \mathfrak{F} .

Une action de symétrie stricte de G sur M passe au quotient sur l'espace feuille M/\mathfrak{F} , même si ce dernier espace n'est pas une variété. La direction opposée est plus sophistiquée, puisqu'une action de symétrie stricte de G sur M/\mathfrak{F} n'induit pas d'action stricte sur M en général. Cependant, il est logique de considérer des applications linéaires $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ qui satisfont $[\varrho(x), \mathfrak{F}] \subset \mathfrak{F}$ pour tout $x \in \mathfrak{g}$, et qui sont des morphismes d'algèbre de Lie à \mathfrak{F} près, à savoir, $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathfrak{F}$ pour tous les $x, y \in \mathfrak{g}$. Ces dernières applications linéaires sont appelées "*actions faibles par symétrie*". Ces actions induisent une "action stricte" sur l'espace des feuilles, c'est-à-dire un morphisme d'algèbre de Lie $\mathfrak{g} \rightarrow \mathfrak{X}(M/\mathfrak{F})$, chaque fois que M/\mathfrak{F} est une variété.

Insistons sur l'observation suivante : Une action infinitésimale d'une algèbre de Lie \mathfrak{g} sur une variété différentielle M est un morphisme d'algèbre de Lie $\mathfrak{g} \rightarrow \mathfrak{X}(M)$. En remplaçant M par une Q -variété (E, Q) , on s'attend à les définir comme des morphismes de ∞ -algèbres de Lie $\mathfrak{g} \rightarrow \mathfrak{X}(E, Q)$, ce dernier espace étant un DGLA. Divers résultats à ce sujet sont donnés dans Mehta-Zambon [MZ12]. En particulier, ces auteurs en donnent plusieurs définitions et interprétations équivalentes.

Au vu de [LLS20, Lav17], derrière chaque feuilletage singulier ou plus généralement toute algèbre de Lie-Rinehart (voir [LGL22b]) il existe une structure ∞ -algèbroïde de Lie qui est unique à homotopie près appelée l' ∞ -algèbroïde de Lie *universelle*. Voici une question naturelle : qu'induit une symétrie

d'un feuilletage singulier \mathfrak{F} sur une ∞ -algèbroïde de Lie universelle de \mathfrak{F} ? Le théorème 8.3.1 de cette thèse donne une réponse à cette question. Il dit que toute action de symétrie faible d'une algèbre de Lie sur un feuilletage singulier \mathfrak{F} peut être relevée en un ∞ -morphisme de Lie à valeur dans le DGLA des champs de vecteurs sur un ∞ -algèbroïde de Lie universel de \mathfrak{F} . Dans cette thèse, un tel ∞ -morphisme de Lie est appelé un *relèvement de l'action par symétrie*. De tels ∞ -morphisms de Lie ont été étudiés par Mehta et Zambon [MZ12] comme "actions de L_∞ -algèbres". Cela va dans le même sens que [GZ21] qui soulignait déjà les structures à 2-groupes de Lie associées à l'action stricte par symétrie de groupes de Lie. De plus, le théorème 8.3.1 dit que ce relevé est une équivalence à homotopie près.

Ce résultat donne plusieurs conséquences géométriques. Voici une question élémentaire : une action d'algèbre de Lie $\mathfrak{g} \rightarrow \mathfrak{X}(W)$ sur une variété affine $W \subset \mathbb{C}^d$ peut-elle être étendue à une action d'algèbre de Lie $\mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{C}^d)$ sur \mathbb{C}^d ? En d'autres termes : il est trivial que tout champ de vecteurs sur W s'étende à \mathbb{C}^d , mais cette extension peut-elle être faite de telle manière qu'elle préserve le crochet de Lie ? Bien qu'aucun " ∞ -oids" n'apparaisse dans la question, qui semble être une pure question de géométrie algébrique, nous affirmons que la réponse passe par les ∞ -algèbroïds de Lie et les feuilletages singuliers. Plus précisément, l'idée est alors de dire que toute \mathfrak{g} -action sur W induit une action faible par symétrie sur le feuilletage singulier $\mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ de tous les champs de vecteurs s'annulant sur W (ici \mathcal{I}_W est l'idéal qui définit W). Par le théorème 8.3.1, nous savons qu'il est possible de relever toute action faible par symétrie de feuilletage singulier en un ∞ -morphisme de Lie sur l'universelle. Mais est-il possible de construire un tel ∞ -morphisme de Lie où le degré polynomial -1 du coefficient de Taylor du second ordre est nul ? Il existe des obstructions cohomologiques. Dans certains cas spécifiques, des classes d'obstruction apparaissent sur certaines cohomologies, bien qu'en général l'obstruction soit plutôt une équation de type Maurer-Cartan qui peut ou non avoir des solutions. Nous montrons que les deux questions sont en fait liées.

Voici une autre question intéressante à laquelle nous aimerions appliquer nos résultats : Peut-on désingulariser une variété affine singulière $W \subseteq \mathbb{C}^d$ en utilisant l' ∞ -algèbroïde de Lie universelle du feuilletage singulier $\mathfrak{F} = \mathfrak{X}(W)$ de champs de vecteurs tangents à W ? Dans la section 7.2.2, nous utilisons la résolution géométrique du feuilletage singulier (c'est-à-dire la résolution sur laquelle est construit le ∞ -algèbroïde de Lie universel) pour retrouver plusieurs notions de résolution des singularités : l'une étant due à Nash [LU81] et une seconde à Mohsen [Moh21]. Mais la signification de ces espaces n'est pas claire. Nous voudrions les mettre en relation avec les crochets supérieurs de l' ∞ -algèbroïde de Lie universelle, par ex. comprendre le rôle du 3-crochet dans cette procédure.

Dans le dernier chapitre de la thèse, nous introduisons la notion de "tours de bi-submersions" sur des feuilletages singuliers que nous notons \mathcal{T}_B . Cette dernière notion, comme son nom l'indique, est une famille de "bi-submersions" qui se construisent les unes sur les autres. Le concept de bi-submersion sur des feuilletages singuliers a été introduit dans [AS09] et il est utilisé dans la K -théorie [AS19] ou la géométrie différentielle [AS11, AZ14, GZ19]. Nous montrons qu'une telle tour de bi-submersion sur un feuilletage singulier \mathfrak{F} existe si \mathfrak{F} admet une résolution géométrique. Sous réserve qu'elle existe, on montre dans le théorème 10.1.26 que toute action infinitésimale d'une algèbre de Lie \mathfrak{g} sur le feuilletage singulier \mathfrak{F} se relève sur la tour de bi-submersions. Ce relevé ressemble au début d'une sorte d' ∞ -morphisme de Lie. Nous nous demandons si nous pouvons continuer la construction du théorème 10.1.26 vers un ∞ -morphisme de Lie.

La thèse est organisée comme suit. Au chapitre 1, nous rappelons quelques bases sur les structures de co-algèbres graduées, sur l'algèbre de symétrie graduée et leurs morphismes. Nous passons également en revue la notion de co-dérivations et leurs propriétés. Au chapitre 2, nous introduisons la notion de Lie ∞ -algebroid sur une algèbre unitaire commutative arbitraire \mathcal{O} sur un corps de caractéristique nulle, et définissons également leurs morphismes en termes de co-dérivations. Nous définissons la notion d'homotopie entre eux. Quelques Lemmes et Propositions techniques sont donnés. Au chapitre 3, nous fixons les notations et passons en revue les définitions, exemples, et donnent les principales propriétés des algèbres de Lie-Rinehart. De plus, nous construisons une structure 2-algèbroïde de Lie pour toute algèbre de Lie-Rinehart qui admet une résolution libre de longueur inférieure ou égale à 2. Au chapitre 4, nous énonçons et prouvons les principaux résultats de la première partie de la thèse, c'est-à-dire l'équivalence de catégories entre les algèbres de Lie-Rinehart et les classes d'homotopie d' ∞ -algèbroïdes de Lie acycliques libres, ce qui justifie le nom ∞ -algèbroïde de Lie universelle d'une algèbre de Lie-Rinehart. Aussi, nous décrivons les ∞ -algèbroïdes de Lie universelle de plusieurs algèbres de Lie-Rinehart. La complexité atteinte par les crochets supérieure dans ces exemples devraient nous convaincre qu'il ne s'agit pas d'une structure triviale, même pour des algèbres de Lie-Rinehart relativement simples. Le chapitre 5 est consacré aux applications des résultats des chapitres précédents aux variétés affines. Nous rappelons quelques définitions et théorèmes de base sur les variétés affines W . Nous présentons trois constructions principales de Lie-Rinehart associées à W et relient ensemble leurs ∞ -algèbroïdes de Lie universelle. De plus, quelques exemples d' ∞ -algèbroïdes de Lie universels sont donnés, tels que les éclatements, des champs de vecteurs qui s'annulent sur une intersection complète, etc. Au chapitre 6, nous rappelons la notion de Q -variétés et appliquons les résultats sur les algèbres de Lie-Rinehart pour retrouver les ∞ -algèbroïdes de Lie universelles d'un feuilletage singulier [LLS20]. Ce chapitre se termine par un résultat sur la cohomologie des champs de vecteurs longitudinaux. Au chapitre 7, on rappelle la définition de l'isotropie d'Androulidakis et de Skandalis Algèbre de Lie d'un feuilletage singulier en un point et rappelons de [LLS20] sa relation avec les ∞ -algèbroïdes de Lie universelle d'un feuilletage singulier. Nous terminons par une procédure d'éclatement d'un feuilletage singulier inspiré de O. Mohsen. Dans le chapitre 8, nous étudions les symétries des feuilletages singuliers. Nous présentons quelques définitions et faits sur les actions faibles par symétrie des algèbres de Lie sur les feuilletages singuliers et donnons quelques exemples. Nous énonçons les résultats principaux de la deuxième partie de la thèse et présentons leurs preuves. Dans le chapitre 9, nous définissons une classe d'obstruction pour étendre une action d'algèbre de Lie sur une variété affine à l'espace ambiant et donnons également quelques exemples. Dans le chapitre 10, nous nous intéressons aux symétries des bi-submersions. Ensuite, nous introduisons la notion de tours de bi-submersions sur un feuilletage singulier et soulignons quelques observations liées à leurs symétries.

Enfin, en annexe A, nous rappelons la définition de l'algèbre tensorielle et fixons les notations. En annexe B, nous rappelons quelques faits généraux sur l'algèbre homologique et donnons quelques constructions géométriques.

Part I

**Lie-Rinehart algebras \equiv Acyclic Lie
 ∞ -algebroids**

This chapter sets the ground for the whole thesis, especially chapters 2, 3 and 4. It fixes terminologies, conventions and notations. For more details, we also refer the reader to Appendix A and B.

Throughout this thesis, \mathbb{K} is a field of characteristic zero, and \mathcal{O} is an associative commutative unital \mathbb{K} -algebra unless otherwise mentioned. Also, an \mathcal{O} -module \mathcal{E} is seen as \mathbb{K} -vector space in the natural way, $\lambda \cdot e := (\lambda \cdot 1_{\mathcal{O}}) \cdot e$, where $1_{\mathcal{O}} \equiv 1$ is the unit of \mathcal{O} . In the sequel, we will drop the notation " \cdot ".

Geometrically, \mathcal{O} can be understood as the algebra of smooth functions on a manifold M , or on an open subset $\mathcal{U} \subset M$ of a complex manifold, or the coordinate ring of an affine variety W .

1.1 Graded symmetric algebras

For $\mathcal{E} = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i$ be a \mathbb{Z} -graded module, we denote by $|x| \in \mathbb{Z}$ the degree of a homogeneous element $x \in \mathcal{E}$.

- We denote by $\odot^{\bullet} \mathcal{E}$ and call *(reduced) graded symmetric algebra of \mathcal{E} over \mathcal{O}* the quotient

$$T_{\mathcal{O}}^{\bullet} \mathcal{E} / \langle x \otimes_{\mathcal{O}} y - (-1)^{|x||y|} y \otimes_{\mathcal{O}} x \rangle$$

of the tensor algebra (see Appendix A) over \mathcal{O} of \mathcal{E} , namely

$$T_{\mathcal{O}}^{\bullet} \mathcal{E} := \bigoplus_{k=1}^{\infty} \underbrace{\mathcal{E} \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \mathcal{E}}_{k \text{ times}}$$

by the ideal generated by $x \otimes_{\mathcal{O}} y - (-1)^{|x||y|} y \otimes_{\mathcal{O}} x$, with x, y arbitrary homogeneous elements of \mathcal{E} . This quotient is a graded commutative algebra. We denote its product by \odot .

- Similarly, we denote by $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ and call *(reduced) graded symmetric algebra of \mathcal{E} over the field \mathbb{K}* the quotient of the tensor algebra (over \mathbb{K}) of \mathcal{E} , i.e.,

$$T_{\mathbb{K}}^{\bullet} \mathcal{E} := \bigoplus_{k=1}^{\infty} \underbrace{\mathcal{E} \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} \mathcal{E}}_{k \text{ times}}$$

by the ideal generated by $x \otimes_{\mathbb{K}} y - (-1)^{|x||y|} y \otimes_{\mathbb{K}} x$, with x, y arbitrary homogeneous elements of \mathcal{E} . We denote by \cdot the product in $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$.

The algebras $\odot^{\bullet} \mathcal{E}$ and $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ come equipped with two different "grading". Let us make them explicit.

1. We define the *degree* of $x = x_1 \odot \cdots \odot x_n \in \odot^n \mathcal{E}$ or $x = x_1 \cdots \cdots x_n \in S_{\mathbb{K}}^n(\mathcal{E})$ by

$$|x_1 \cdots \cdots x_n| = |x_1 \odot \cdots \odot x_n| = |x_1| + \cdots + |x_n|$$

for any homogeneous $x_1, \dots, x_n \in \mathcal{E}$. With respect to this degree, $\odot^{\bullet} \mathcal{E}$ and $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ are graded commutative algebras.

2. The second grading is called "polynomial-degree". The *polynomial-degree* of $x_1 \odot \cdots \odot x_n \in \odot^n \mathcal{E}$ or $x_1 \cdots \cdots x_n \in S_{\mathbb{K}}^n(\mathcal{E})$ is defined to be n . We have the following polynomial-degree decomposition $\odot^{\bullet} \mathcal{E} = \bigoplus_{k \geq 1} \odot^k \mathcal{E}$ and $S_{\mathbb{K}}^{\bullet}(\mathcal{E}) = \bigoplus_{k \geq 1} S_{\mathbb{K}}^k(\mathcal{E})$, where $\odot^k \mathcal{E}$ and $S_{\mathbb{K}}^k(\mathcal{E})$ stand for the \mathcal{O} -module of elements of polynomial-degree k and the \mathbb{K} -vector space of elements of polynomial-degree k , respectively.

Convention 1.1.1. For \mathcal{E} a graded \mathcal{O} -module, the set of elements of polynomial-degree k and degree d in $\odot^{\bullet} \mathcal{E}$ (resp. $S_{\mathbb{K}}^k(\mathcal{E})$) shall be denoted by $\odot^k \mathcal{E}_{|d}$ (resp. $S_{\mathbb{K}}^k(\mathcal{E})_{|d}$). For example,

$$\odot^k \mathcal{E}_{|d} = \bigoplus_{i_1 + \cdots + i_k = d} \mathcal{E}_{i_1} \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \mathcal{E}_{i_k}.$$

- For any homogeneous elements $x_1, \dots, x_k \in \mathcal{E}$ and $\sigma \in \mathfrak{S}_k$ a permutation of $\{1, \dots, k\}$, the *Koszul sign* $\epsilon(\sigma; x_1, \dots, x_k)$ is defined by:

$$x_{\sigma(1)} \odot \cdots \odot x_{\sigma(k)} = \epsilon(\sigma; x_1, \dots, x_k) x_1 \odot \cdots \odot x_k.$$

We often write $\epsilon(\sigma)$ for $\epsilon(\sigma; x_1, \dots, x_k)$.

- For $i, j \in \mathbb{N}$, a (i, j) -*shuffle* is a permutation $\sigma \in \mathfrak{S}_{i+j}$ such that $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(i+j)$, and the set of all (i, j) -shuffles is denoted by $\mathfrak{S}(i, j)$. More generally, a (i_1, \dots, i_k) -*shuffle* is a permutation $\sigma \in \mathfrak{S}_{i_1 + \dots + i_k}$ such that

$$\begin{aligned} \sigma(1) &< \cdots < \sigma(i_1) \\ \sigma(i_1 + 1) &< \cdots < \sigma(i_1 + i_2), \\ &\vdots \\ \sigma(i_1 + i_2 + \cdots + i_{j-1} + 1) &< \cdots < \sigma(i_1 + i_2 + \cdots + i_j), \\ &\vdots \\ \sigma(i_1 + i_2 + \cdots + i_{k-1} + 1) &< \cdots < \sigma(i_1 + i_2 + \cdots + i_k) \end{aligned}$$

The set of all (i_1, \dots, i_k) -shuffles is denoted by $\mathfrak{S}(i_1, \dots, i_k)$.

1.2 Graded symmetric co-algebras and their morphisms

Graded co-algebra structures are the dual version of graded algebra structures, i.e., they are obtained by reversing all arrows and permutes the order of composition. Let us recall the definition of a co-algebra structure on an \mathcal{O} -module. We refer the reader to, e.g [JLL, Kas12, Man] or Chapter 8 of [Man05b] for more details on this topic.

Definition 1.2.1. A *coassociative graded cocommutative co-algebra* structure on an graded \mathcal{O} -module $\mathcal{C} = \bigoplus_{i \in \mathbb{Z}} \mathcal{C}_i$ is

- a linear map of degree zero called (*graded*) *coproduct* $\Delta: \mathcal{C} \longrightarrow \mathcal{C} \otimes \mathcal{C}$ with

$$\Delta(\mathcal{C}_k) \subset \bigoplus_{i+j=k} \mathcal{C}_i \otimes \mathcal{C}_j$$

is such that $\Delta(\mathcal{C}_k)$ has non-zero¹ intersection with only finitely many spaces $\mathcal{C}_i \otimes \mathcal{C}_j$

- $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta: \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$, called *graded coassociativity*.
- $\tau \circ \Delta = \Delta$, called *graded cocommutativity*,

where $\tau: \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$ is defined by $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$.

Equivalently, the following diagrams commute

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \otimes \mathcal{C} \\ \Delta \downarrow & & \downarrow \Delta \otimes \text{id} \\ \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\text{id} \otimes \Delta} & \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \end{array} \quad \begin{array}{ccc} \mathcal{C} & & \\ \Delta \downarrow & \searrow \Delta & \\ \mathcal{C} \otimes \mathcal{C} & \xrightarrow{\tau} & \mathcal{C} \otimes \mathcal{C}. \end{array}$$

The pair (\mathcal{C}, Δ) is called *graded \mathcal{O} -co-algebra*. We say that (\mathcal{C}, Δ) is *counital* if there is a morphism of graded vector spaces $u: \mathcal{C} \rightarrow \mathbb{K}$ such that $(u \otimes \text{id}) \circ \Delta = (\text{id} \otimes u) \circ \Delta = \text{id}$ i.e. the diagram below commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \otimes \mathcal{C} \\ \Delta \downarrow & \searrow \text{id} & \downarrow \text{id} \otimes u \\ \mathcal{C} \otimes \mathcal{C} & \xrightarrow{u \otimes \text{id}} & \mathbb{K} \otimes \mathcal{C} \simeq \mathcal{C} \simeq \mathcal{C} \otimes \mathbb{K}. \end{array}$$

Proposition 1.2.2. Let (\mathcal{C}, Δ) and (\mathcal{C}', Δ') be two co-algebras over \mathcal{O} . Then, the tensor product $(\mathcal{C} \otimes \mathcal{C}', (\text{id}_{\mathcal{C}} \otimes \tau \otimes \text{id}_{\mathcal{C}'})) \circ \Delta \otimes \Delta'$ is a co-algebra over \mathcal{O} .

Example 1.2.3. The polynomial ring $\mathbb{K}[t]$ in one indeterminate t is a co-algebra with the coproduct Δ given on monomials $t^n, n \geq 0$ by:

$$\Delta(t^n) = \sum_{k=0}^n \binom{n}{k} t^k \otimes t^{n-k}$$

and extends by linearity.

The graded symmetric algebra is also an example of co-algebra.

¹Note that this condition is automatically satisfied when the module is concentrated in positive degree.

Lemma 1.2.4. *Let \mathcal{E} be a \mathcal{O} -module. Both $\odot^\bullet \mathcal{E}$ and $S_{\mathbb{K}}^\bullet(\mathcal{E})$ admit natural co-algebra structures with respect to the deconcatenation Δ defined by:*

$$\Delta(x_1 \odot \cdots \odot x_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in \mathfrak{S}(i, n-i)} \epsilon(\sigma) x_{\sigma(1)} \odot \cdots \odot x_{\sigma(i)} \otimes x_{\sigma(i+1)} \odot \cdots \odot x_{\sigma(n)} \quad (1.1)$$

for every $x_1, \dots, x_n \in \mathcal{E}$.

Example 1.2.5. For small n . The formula of Equation (1.1) on homogeneous elements $x, y, z \in \mathcal{E}$ means that

- $\Delta(x) = 0$
- $\Delta(x \odot y) = x \otimes y + (-1)^{|x||y|} y \otimes x$
- $\Delta(x \odot y \odot z) = x \odot y \otimes z + (-1)^{|x||z|} x \odot z \otimes y + (-1)^{|x|(|z|+|y|)} y \odot z \otimes x + x \otimes y \odot z + (-1)^{|x||y|} y \otimes x \odot z + (-1)^{|z|(|x|+|y|)} z \otimes x \odot y$

Remark 1.2.6. According to Definition 1.2.1, the coproduct Δ is graded with respect to the positive grading given by the "polynomial-degree" of monomials in the symmetric algebras $\odot^\bullet \mathcal{E}$ and $S_{\mathbb{K}}^\bullet(\mathcal{E})$.

Example 1.2.7. Consider $M = \mathbb{R}^d$ with global coordinates (x_1, \dots, x_d) . The algebra of differential forms $(\Omega(M), \wedge)$ over M has a co-algebra structure $\Delta_\Omega: \Omega(M) \rightarrow \Omega(M) \otimes \Omega(M)$ given as in Lemma 1.2.4. Namely,

$$\Delta(dx_1 \wedge \cdots \wedge dx_n) = \sum_{i=1}^{n-1} \sum_{\sigma \in \mathfrak{S}(i, n-i)} \epsilon(\sigma) dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(i)} \otimes dx_{\sigma(i+1)} \wedge \cdots \wedge dx_{\sigma(n)}.$$

This matches (1.1) if we consider $\mathcal{E} = \Omega^1(M)$ is concentrated in degree +1.

1.2.1 Co-morphisms and co-derivations

We recall the definitions of co-morphisms, co-derivations of graded co-algebra structures.

Definition 1.2.8. Let (\mathcal{C}', Δ') and (\mathcal{C}, Δ) be two graded co-algebras.

1. A *morphism of co-algebras* or *co-morphism* from (\mathcal{C}', Δ') to (\mathcal{C}, Δ) is a linear map $\Phi: \mathcal{C}' \rightarrow \mathcal{C}$ of degree 0 such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\Phi} & \mathcal{C} \\ \Delta' \downarrow & & \downarrow \Delta \\ \mathcal{C}' \otimes \mathcal{C}' & \xrightarrow{\Phi \otimes \Phi} & \mathcal{C} \otimes \mathcal{C} \end{array} \quad (1.2)$$

In formula: $\Delta \circ \Phi = (\Phi \otimes \Phi) \circ \Delta'$.

2. Let $\Phi: \mathcal{C}' \rightarrow \mathcal{C}$ be a co-morphism. A Φ -*co-derivation of degree l* is a linear map $\mathcal{H}: \mathcal{C}' \rightarrow \mathcal{C}$ of degree l such that the following diagram commutes,

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\mathcal{H}} & \mathcal{C} \\ \Delta' \downarrow & & \downarrow \Delta \\ \mathcal{C}' \otimes \mathcal{C}' & \xrightarrow{\mathcal{H} \otimes \Phi + \Phi \otimes \mathcal{H}} & \mathcal{C} \otimes \mathcal{C} \end{array} \quad (1.3)$$

that is, the so-called *co-Leibniz identity* is satisfied:

$$\Delta \circ \mathcal{H} = (\mathcal{H} \otimes \Phi + \Phi \otimes \mathcal{H}) \circ \Delta'. \quad (1.4)$$

When $\mathcal{C}' = \mathcal{C}$ and $\Phi = \text{id}$ we speak of a *co-derivation* (of degree l).

Remark 1.2.9. It is easily checked that the composition of two co-morphisms $\mathcal{C}'' \xrightarrow{\Psi} \mathcal{C}' \xrightarrow{\Phi} \mathcal{C}$ is again a co-morphism.

Proposition 1.2.10. *We denote by $\text{CoDer}(\mathcal{C})$ the linear space of co-derivations of (\mathcal{C}, Δ) . The set $\text{CoDer}(\mathcal{C})$ is a graded Lie sub-algebra of $\text{Hom}(\mathcal{C}, \mathcal{C})$ when equipped with the graded commutator.*

Proof. For two co-derivations $\mathcal{H}_1, \mathcal{H}_2 \in \text{CoDer}(\mathcal{C})$ one has,

$$\begin{aligned} \Delta \circ \mathcal{H}_1 \circ \mathcal{H}_2 &= (\mathcal{H}_1 \otimes \text{id} + \text{id} \otimes \mathcal{H}_1) \circ (\mathcal{H}_2 \otimes \text{id} + \text{id} \otimes \mathcal{H}_2) \circ \Delta \\ &= (\mathcal{H}_1 \circ \mathcal{H}_2 \otimes \text{id} + \mathcal{H}_1 \otimes \mathcal{H}_2 + (-1)^{|\mathcal{H}_1||\mathcal{H}_2|} \mathcal{H}_2 \otimes \mathcal{H}_1 + \text{id} \otimes \mathcal{H}_1 \circ \mathcal{H}_2) \circ \Delta. \end{aligned}$$

Also, we obtain a similar equation for $(-1)^{|\mathcal{H}_1||\mathcal{H}_2|} \Delta \circ \mathcal{H}_2 \circ \mathcal{H}_1$ by changing the roles of \mathcal{H}_1 and \mathcal{H}_2 . Subtracting both terms, one get the co-Leibniz identity for $[\mathcal{H}_1, \mathcal{H}_2] = \mathcal{H}_1 \circ \mathcal{H}_2 - (-1)^{|\mathcal{H}_1||\mathcal{H}_2|} \mathcal{H}_2 \circ \mathcal{H}_1$. \square

Let us describe Definition 1.2.8 in the context of the thesis. Let \mathcal{E}' and \mathcal{E} be graded \mathcal{O} -modules.

Definition 1.2.11. A linear map $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ is said to be of *polynomial-degree/degree* $r \in \mathbb{Z}$, if it sends polynomials of $S_{\mathbb{K}}^k(\mathcal{E})$ to those of $S_{\mathbb{K}}^{k-r}(\mathcal{E})$ /elements of $S_{\mathbb{K}}^{\bullet}(\mathcal{E})|_d$ to $S_{\mathbb{K}}^{\bullet}(\mathcal{E})|_{d+r}$. In formulas, for very $k \geq 0$, (resp. $d \in \mathbb{Z}$) one has

$$\Phi \left(S_{\mathbb{K}}^k(\mathcal{E}) \right) \subseteq S_{\mathbb{K}}^{k-r}(\mathcal{E}), \quad \text{resp.} \quad \Phi \left(S_{\mathbb{K}}^{\bullet}(\mathcal{E})|_d \right) \subseteq S_{\mathbb{K}}^{\bullet}(\mathcal{E})|_{d+r}.$$

Remark 1.2.12. Any linear map $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ of degree l can be decomposed with respect to the polynomial-degree as formal sum:

$$\Phi = \sum_{r \in \mathbb{Z}} \Phi^{(r)} \quad (1.5)$$

where, for all $k \in \mathbb{N}_0$, $\Phi^{(r)}: S_{\mathbb{K}}^k(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{k-r}(\mathcal{E})$ is a linear map of polynomial-degree r and of degree l .

- It should be understood that $\Phi^{(r)}(v) = 0$ for all $v \in S_{\mathbb{K}}^k(\mathcal{E}')$ with $k \leq r$. Also, the decomposition (1.5) makes sense in particular when the polynomial-degrees of Φ are lower bounded that is, there is an integer $N \in \mathbb{Z}$, such that $\Phi^{(r)} = 0$ for all $r < N$. In that case, for every $v \in S_{\mathbb{K}}^k(\mathcal{E})$,

$$\sum_{r \in \mathbb{Z}} \Phi^{(r)}(v) = \sum_{N \leq r < k} \Phi^{(r)}(v),$$

is a finite sum. Hence, Equation (1.5) becomes

$$\Phi = \sum_{r \geq N} \Phi^{(r)},$$

for some $N \in \mathbb{Z}$, since the l.h.s of (1.5) is finite. Notice that a linear map $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ is of polynomial-degree N if and only if $\Phi^{(N)}$ is the unique non-zero term, namely $\Phi^{(r)} = 0$, for $r \neq N$.

- Also, for the composition of two linear maps $S_{\mathbb{K}}^{\bullet}(\mathcal{E}'') \xrightarrow{\Psi} S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \xrightarrow{\Phi} S_{\mathbb{K}}^{\bullet}(\mathcal{E})$, we have

$$(\Phi \circ \Psi)^{(r)} = \sum_{i+j=r} \Phi^{(i)} \circ \Psi^{(j)}. \quad (1.6)$$

for every $r \in \mathbb{Z}$. Here, although the sum is infinite, it becomes finite when applied to a given element in $S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$.

Co-morphisms on symmetric algebras

Let us have a discussion on co-morphisms from the symmetric graded co-algebras $(S_{\mathbb{K}}^{\bullet}(\mathcal{E}'), \Delta')$ and $(S_{\mathbb{K}}^{\bullet}(\mathcal{E}), \Delta)$, where Δ' and Δ are the respective coproducts like in Lemma 1.2.4. Given any linear map $F: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow \mathcal{E}$. Denoting by $F_r: S_{\mathbb{K}}^{r+1}(\mathcal{E}') \rightarrow \mathcal{E}$ for $r \in \mathbb{N}_0$, the restriction of F to $S_{\mathbb{K}}^{r+1}(\mathcal{E}')$. The linear map F can be extended to a unique co-morphism $\bar{F}: (S_{\mathbb{K}}^{\bullet}(\mathcal{E}'), \Delta') \rightarrow (S_{\mathbb{K}}^{\bullet}(\mathcal{E}), \Delta)$ by taking for $k \in \mathbb{N}$ the component on $S_{\mathbb{K}}^k(\mathcal{E})$ to be, for any homogeneous $x_1, \dots, x_n \in \mathcal{E}'$

$$\sum_{i_1 + \dots + i_k = n} \sum_{\sigma \in \mathfrak{S}(i_1, \dots, i_k)} \epsilon(\sigma) \frac{1}{k!} \prod_{j=1}^k F_{i_j}(x_{\sigma(i_1 + \dots + i_{j-1} + 1)}, \dots, x_{\sigma(i_1 + \dots + i_j)}). \quad (1.7)$$

where $\mathfrak{S}(i_1, \dots, i_k)$ is the set of (i_1, \dots, i_k) -shuffles, with $i_1, \dots, i_k \in \mathbb{N}$.

Example 1.2.13. Let us compute $\bar{F}: (S_{\mathbb{K}}^k(\mathcal{E}'), \Delta') \rightarrow (S_{\mathbb{K}}^{\bullet}(\mathcal{E}), \Delta)$ for $k = 1, 2, 3$. For $x, y, z \in \mathcal{E}'$,

- $\bar{F}(x) = F_0(x)$.
- $\bar{F}(x \cdot y) = F_1(x \cdot y) + F_0(x) \cdot F_0(y)$.
- $\bar{F}(x \cdot y \cdot z) = F_2(x \cdot y \cdot z) + F_0(x) \cdot F_1(y \cdot z) + (-1)^{|x||y|} F_0(y) \cdot F_1(x \cdot z) + (-1)^{(|x|+|y|)|z|} F_0(z) \cdot F_1(x \cdot y) + F_0(x) \cdot F_0(y) \cdot F_0(z)$.

We can easily check that $\Delta \circ \bar{F} = (\bar{F} \otimes \bar{F}) \circ \Delta'$, e.g

- $\Delta \circ \bar{F}(x) = \Delta \circ F_0(x) = 0 = (\bar{F} \otimes \bar{F}) \circ \Delta'(x)$
- On one side,

$$\begin{aligned} \Delta \circ \bar{F}(x \cdot y) &= \Delta \circ F_1(x \cdot y) + \Delta(F_0(x) \cdot F_0(y)) \\ &= F_0(x) \otimes F_0(y) + (-1)^{|x||y|} F_0(y) \otimes F_0(x). \end{aligned}$$

On the other side,

$$\begin{aligned} (\bar{F} \otimes \bar{F}) \circ \Delta'(x, y) &= (\bar{F} \otimes \bar{F})(x \otimes y + (-1)^{|x||y|} y \otimes x) \\ &= (\bar{F} \otimes \bar{F})(x \otimes y) + (-1)^{|x||y|} (\bar{F} \otimes \bar{F})(y \otimes x) \\ &= F_0(x) \otimes F_0(y) + (-1)^{|x||y|} F_0(y) \cdot F_0(x). \end{aligned}$$

The following proposition claims that every co-morphism from $S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$ to $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ is of the form described in (1.7). For a proof, see [JLL, Kas12, Man05b].

Proposition 1.2.14. *Let $\mathcal{E}', \mathcal{E}$ be two graded \mathcal{O} -modules. For any morphism of graded vector space $F: S_{\mathbb{K}}(\mathcal{E}') \rightarrow \mathcal{E}$ there exists an (unique) morphism of graded co-algebras $\bar{F}: (S_{\mathbb{K}}(\mathcal{E}'), \Delta') \rightarrow (S_{\mathbb{K}}(\mathcal{E}), \Delta)$ that satisfies $\text{pr} \circ \bar{F} = F$.*

Here pr is the canonical projection onto the term of polynomial-degree 1, i.e., $\text{pr}: S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \rightarrow S_{\mathbb{K}}^1(\mathcal{E}) \simeq \mathcal{E}$.

Remark 1.2.15. The following facts are crucial, and will be used in the sequel.

1. By Proposition 1.2.14, a co-morphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ is entirely determined by the collection indexed by $r \in \mathbb{N}_0$ of maps called its r -th *Taylor coefficients*:

$$\Phi_r: S_{\mathbb{K}}^{r+1}(\mathcal{E}') \xrightarrow{\Phi} S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \xrightarrow{\text{pr}} \mathcal{E}, \quad (1.8)$$

Notice that the component $\Phi^{(r)}$ of polynomial-degree $r \geq 0$ of Φ coincides with r -th Taylor coefficient Φ_r on $S_{\mathbb{K}}^{r+1}(\mathcal{E}')$. In fact we have, $\Phi_r = (\text{pr} \circ \Phi^{(r)})|_{S_{\mathbb{K}}^{r+1}(\mathcal{E}'})$.

2. By item 1, a co-morphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ is completely determined by its components of polynomial-degree $r \geq 0$, hence it admits a polynomial decomposition of the form:

$$\Phi = \sum_{r \geq 0} \Phi^{(r)}. \quad (1.9)$$

3. Similar results as in Proposition 1.2.14 hold for Φ -co-derivations or for co-derivations, e.g. see Corollary VIII.34 in [Man05b] or [JLL]. Given a co-morphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ a Φ -co-derivation \mathcal{H} of degree l is entirely determined by the *Taylor coefficients* defined as in (1.8)

$$\mathcal{H}_r: S_{\mathbb{K}}^{r+1}(\mathcal{E}') \rightarrow \mathcal{E}, \quad r \in \mathbb{N}_0.$$

For $k \in \mathbb{N}_0$ and $x_1, \dots, x_n \in \mathcal{E}'$, the component on $S_{\mathbb{K}}^k(\mathcal{E})$ of $\mathcal{H}(x_1 \cdots x_n)$ takes the form

$$\sum_{i_1 + \dots + i_k = n} \sum_{\sigma \in \mathfrak{S}(i_1, \dots, i_k)} \epsilon(\sigma) \frac{1}{k!} \mathcal{H}_{i_1}(x_{\sigma(1)}, \dots, x_{\sigma(i_1)}) \cdot \prod_{j=1}^{k-1} \Phi_{i_j-1}(x_{\sigma(i_1+\dots+i_{j-1}+1)}, \dots, x_{\sigma(i_1+\dots+i_j)}). \quad (1.10)$$

Also, \mathcal{H} has a decomposition into polynomial-degree of the form :

$$\mathcal{H} = \sum_{r \geq 0} \mathcal{H}^{(r)}. \quad (1.11)$$

Example 1.2.16. Every linear map $H: S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \rightarrow \mathcal{E}$ of degree l admits an unique extension to a co-derivation $\bar{H}: S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ of degree l . The latter is given explicitly by the formula

$$\bar{H}(x_1 \cdots x_n) = \sum_{i=1}^n \sum_{\sigma \in \mathfrak{S}(i, n-i)} \epsilon(\sigma) H(x_{\sigma(1)} \cdots x_{\sigma(i)}) \cdot x_{\sigma(i+1)} \cdots x_{\sigma(n)}. \quad (1.12)$$

The following lemma-definition is helpful in the sequel.

Lemma 1.2.17 (Definition). *We say that a co-algebra morphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ is \mathcal{O} -multilinear when the equivalent conditions below are satisfied:*

- (i) *For every $n \geq 0$, the n -th Taylor coefficient $\Phi^{(n)}: S_{\mathbb{K}}^{n+1}(\mathcal{E}') \rightarrow \mathcal{E}$ of Φ is \mathcal{O} -multilinear*
- (ii) *There exists an induced co-algebra morphism $\Phi_{\mathcal{O}}: \odot^{\bullet} \mathcal{E}' \rightarrow \odot^{\bullet} \mathcal{E}$ making the following diagram commutative :*

$$\begin{array}{ccc} S_{\mathbb{K}}^{\bullet}(\mathcal{E}') & \xrightarrow{\Phi} & S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \odot^{\bullet} \mathcal{E}' & \xrightarrow{\Phi_{\mathcal{O}}} & \odot^{\bullet} \mathcal{E} \end{array}$$

When it is the case we still write Φ for $\Phi_{\mathcal{O}}$.

Proof. The equivalence is straightforward. □

Remark 1.2.18. Similarly, the formula (1.2.16) shows that the Taylor coefficients of \mathcal{H} are \mathcal{O} -multilinear if and only if \mathcal{H} induces a $\Phi_{\mathcal{O}}$ -co-derivation $\mathcal{H}_{\mathcal{O}}: \odot^{\bullet} \mathcal{E}' \rightarrow \odot^{\bullet} \mathcal{E}$.

1.2.2 Richardson-Nijenhuis bracket and co-derivations

We need to use the Richardson-Nijenhuis bracket to express our results and explain some proofs in the coming chapters. This bracket was defined mainly to understand Lie algebra structures on a vector space and their deformations [NR66] as well as Poisson structures and their generalizations. Let us recall the definition in our context.

Let \mathcal{E} be an \mathcal{O} -module. For $k \in \mathbb{N}_0$, homomorphisms of degree j from $S_{\mathbb{K}}^{k+1}(\mathcal{E})$ to \mathcal{E} shall be, by definition, the space

$$\mathrm{Hom}_{\mathbb{K}}^j \left(S_{\mathbb{K}}^{k+1}(\mathcal{E}), \mathcal{E} \right) := \bigoplus_{m \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{K}} \left(S_{\mathbb{K}}^{k+1}(\mathcal{E})|_{m-j}, \mathcal{E}_m \right).$$

We refer the reader for example to [KMS93, KS04] for the following definition.

Definition 1.2.19. The *Richardson-Nijenhuis bracket* is a \mathbb{K} -bilinear map

$$[\cdot, \cdot]_{\mathrm{RN}} : \mathrm{Hom}_{\mathbb{K}}^i \left(S_{\mathbb{K}}^{p+1}(\mathcal{E}), \mathcal{E} \right) \otimes \mathrm{Hom}_{\mathbb{K}}^j \left(S_{\mathbb{K}}^{q+1}(\mathcal{E}), \mathcal{E} \right) \rightarrow \mathrm{Hom}_{\mathbb{K}}^{i+j} \left(S_{\mathbb{K}}^{p+q+1}(\mathcal{E}), \mathcal{E} \right)$$

which is defined on homogeneous elements $P \in \mathrm{Hom}_{\mathbb{K}}^{\bullet} \left(S_{\mathbb{K}}^{p+1}(\mathcal{E}), \mathcal{E} \right)$ and $R \in \mathrm{Hom}_{\mathbb{K}}^{\bullet} \left(S_{\mathbb{K}}^{q+1}(\mathcal{E}), \mathcal{E} \right)$ by

$$[P, R]_{\mathrm{RN}} = P \circ R - (-1)^{|P||R|} R \circ P, \quad (1.13)$$

with the understanding that $P \circ R$ is defined on $x_0 \cdots x_{p+q} \in S_{\mathbb{K}}^{p+q+1}(\mathcal{E})$ by

$$(P \circ R)(x_0 \cdots x_{p+q}) := \sum_{\sigma \in \mathfrak{S}(q+1, p)} \epsilon(\sigma) P(R(x_{\sigma(0)} \cdots x_{\sigma(q)}) \cdot x_{\sigma(q+1)} \cdots x_{\sigma(p+q)}). \quad (1.14)$$

The bracket is extended by bilinearity.

Remark 1.2.20. The bracket $[\cdot, \cdot]_{\mathrm{RN}}$ should not be confused with the graded commutator $[\cdot, \cdot]$ of the graded vector space $\mathrm{Hom}_{\mathbb{K}}(S_{\mathbb{K}}^{\bullet}(\mathcal{E}), S_{\mathbb{K}}^{\bullet}(\mathcal{E}))$. In fact, the graded commutator of two elements $P, R \in \mathrm{Hom}_{\mathbb{K}}^{\bullet}(S_{\mathbb{K}}^{\bullet}(\mathcal{E}), S_{\mathbb{K}}^{\bullet}(\mathcal{E}))$ is $[P, R] = P \circ R - (-1)^{|P||R|} R \circ P$, but here $P \circ R$ is the usual composition of homomorphism.

It is easy to check that

Lemma 1.2.21. *The graded vector space $\mathrm{Hom}_{\mathbb{K}}^{\bullet}(S_{\mathbb{K}}^{\bullet}(\mathcal{E}), \mathcal{E})$ together with the Richardson-Nijenhuis bracket $[\cdot, \cdot]_{\mathrm{RN}}$ is a graded Lie algebra structure.*

For a given $i \in \mathbb{Z}$, and a given $P = \sum_{k \geq 0} P_k$ with $P_k \in \mathrm{Hom}_{\mathbb{K}}^i \left(S_{\mathbb{K}}^{k+1}(\mathcal{E}), \mathcal{E} \right)$, we denote by $\bar{P} \in \mathrm{CoDer}(S_{\mathbb{K}}^{\bullet}(\mathcal{E}))$ the unique co-derivation of degree i with Taylor coefficients $(P_k)_{k \in \mathbb{N}_0}$. The relation between the Richardson-Nijenhuis bracket and co-derivations is described in the following lemma:

Lemma 1.2.22. *The map $P \mapsto \bar{P}$ is a graded Lie algebra morphism that is, for every P, R of degrees l, r as above, we have*

$$\overline{[P, R]_{\mathrm{RN}}} = [\bar{P}, \bar{R}] = \bar{P} \circ \bar{R} - (-1)^{lr} \bar{R} \circ \bar{P}.$$

Proof. Linearity follows from uniqueness of the extension \bar{P} described in Proposition 1.2.14. Indeed, for P, R as above, one should have $\mathrm{pr} \circ (\bar{P} + \bar{Q}) = P + R = \mathrm{pr} \circ (\overline{P + R})$. By unicity, we must have $\overline{P + R} = \bar{P} + \bar{R}$. Likewise, one has for any $\lambda \in \mathbb{K}$, $\overline{\lambda P} = \lambda \bar{P}$.

To show that it is a graded Lie algebra morphism, it suffices to check that $\text{pr} \circ [\bar{P}, \bar{R}] = [P, R]_{\text{RN}}$. For $x = x_0 \cdots x_r \in S_{\mathbb{K}}^{r+1}(\mathcal{E})$

$$(\text{pr} \circ [\bar{P}, \bar{R}])_r(x) = P \circ \bar{R}(x) - (-1)^{|P||R|} R \circ \bar{P}(x) \quad (1.15)$$

By using Formula (1.12), we obtain for example

$$\begin{aligned} P \circ \bar{R}(x) &= \sum_{j=0}^k \sum_{\sigma \in \mathfrak{S}(j+1, k-j)} \epsilon(\sigma) P(R_j(x_{\sigma(0)} \cdots x_{\sigma(j)})) \cdot x_{\sigma(j+1)} \cdots x_{\sigma(k)} \\ &= \sum_{\substack{i+j=k \\ i, j \geq 0}} \sum_{\sigma \in \mathfrak{S}(j+1, i)} \epsilon(\sigma) P_i(R_j(x_{\sigma(0)} \cdots x_{\sigma(j)})) \cdot x_{\sigma(j+1)} \cdots x_{\sigma(r)} \\ &= \sum_{\substack{i+j=r \\ i, j \geq 0}} (P_i \circ R_j)(x) \end{aligned}$$

with the understanding that $P_i(x_0 \cdots x_m) = 0$ for $m \neq i$. As a consequence, the right-hand side of Equation (1.15) becomes

$$\begin{aligned} \sum_{\substack{i+j=r \\ i, j \geq 0}} \left((P_i \circ R_j)(x) - (-1)^{|P||R|} (R_j \circ P_i)(x) \right) &= \sum_{\substack{i+j=r \\ i, j \geq 0}} [P_i, R_j]_{\text{RN}}(x) \\ &= ([P, R]_{\text{RN}})_r(x). \end{aligned}$$

□

Conclusion:

In this chapter, we recalled the co-algebra language, in particular comorphisms, Richardson-Nijenhuis bracket. An important point is not to confuse the tensor products $\otimes_{\mathbb{K}}$ and $\otimes_{\mathcal{O}}$, hence the graded symmetric algebra $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ and $S_{\mathcal{O}}^{\bullet}(\mathcal{E})$, which is denoted by $S^{\bullet}(\mathcal{E})$. This introduction is completed in Appendix A.

2.1 Definitions

In this section, we present the notion Lie ∞ -algebroid over the algebra \mathcal{O} . We also give geometrical examples. We refer the reader to Appendix B for some generalities on graded modules.

By definition, Lie ∞ -algebras on a graded vector space V are co-derivations of degree -1 squaring to 0 of the graded symmetric algebra $S(V)$ (e.g see [LS93, Ryv16]). Lie ∞ -algebroids generalize Lie ∞ -algebras and Lie algebroids, but the situation is more sophisticated, because the 2-ary bracket is not \mathcal{O} -linear. Let us make some preparations before introducing the definition.

Definition 2.1.1. A *derivation* of \mathcal{O} is a \mathbb{K} -linear map $X: \mathcal{O} \rightarrow \mathcal{O}$ that satisfies the so-called Leibniz identity

$$X(fg) = X(f)g + fX(g)$$

for all $f, g \in \mathcal{O}$. The set of all derivations of \mathcal{O} inherits a Lie \mathbb{K} -algebra structure from the Lie algebra $\text{End}_{\mathbb{K}}(\mathcal{O})$, whose Lie bracket is the commutator, that is,

$$[X, Y] = X \circ Y - Y \circ X,$$

for all X, Y derivations of \mathcal{O} .

We denote by $\text{Der}(\mathcal{O})$ the Lie algebra of derivations of \mathcal{O} . Also, for $X \in \text{Der}(\mathcal{O})$, $X[f]$ stands for the derivation X applied to $f \in \mathcal{O}$.

Example 2.1.2. Geometrically, derivations of \mathcal{O} can be understood as follows:

1. when $\mathcal{O} = C^\infty(M)$ is the algebra of functions of a smooth manifold M , as vector fields on a smooth manifold M ;

2. when \mathcal{O} is the algebra of holomorphic functions on an open subset \mathcal{U} of \mathbb{C}^d , as vector fields on \mathcal{U} ;
3. when $\mathcal{O} = \mathcal{O}_W$ is the coordinate ring of an affine variety W , as vector fields on the affine variety W .

2.1.1 dg -almost Lie algebroids over \mathcal{O}

Definition 2.1.3. [Lav17, LLS20] A (negatively graded) almost differential graded Lie algebroid $(\mathcal{E}_\bullet, \ell_1, \ell_2, \rho)$ over \mathcal{O} is a complex

$$(\mathcal{E}, \ell_1, \rho) : \cdots \xrightarrow{\ell_1} \mathcal{E}_{-3} \xrightarrow{\ell_1} \mathcal{E}_{-2} \xrightarrow{\ell_1} \mathcal{E}_{-1} \xrightarrow{\rho} \text{Der}(\mathcal{O}).$$

of projective \mathcal{O} -modules equipped with a graded symmetric degree +1 \mathbb{K} -bilinear *bracket*

$$\ell_2 = [\cdot, \cdot] : \odot^2 \mathcal{E} \rightarrow \mathcal{E}$$

such that:

1. ℓ_2 satisfies the *Leibniz identity* with respect to $\rho: \mathcal{E}_{-1} \rightarrow \text{Der}(\mathcal{O})$, i.e.

$$\ell_2(x, fy) = f\ell_2(x, y) + \rho(x)[f]y \quad (2.1)$$

for all $x \in \mathcal{E}_{-1}, y \in \mathcal{E}$ and $f \in \mathcal{O}$.

2. ℓ_1 is degree +1-derivation of ℓ_2 , i.e. for all $x \in \mathcal{E}_{-i}, y \in \mathcal{E}$:

$$\ell_1(\ell_2(x, y)) + \ell_2(\ell_1(x), y) + (-1)^i \ell_2(x, \ell_1(y)) = 0,$$

3. ρ is a morphism, i.e. for all $x, y \in \mathcal{E}_{-1}$

$$\rho(\ell_2(x, y)) = [\rho(x), \rho(y)].$$

The \mathcal{O} -linear map ρ is called the *anchor map*, and ℓ_1 the *differential*.

Remark 2.1.4. For any almost graded Lie-algebroid $(\mathcal{E}_\bullet, \ell_1, \ell_2, \rho)$ define the *Jacobiator*

$$Jac(x, y, z) = \ell_2(\ell_2(x, y), z) + (-1)^{|y||z|} \ell_2(\ell_2(x, z), y) + (-1)^{|x|(|y|+|z|)} \ell_2(\ell_2(y, z), x), \quad x, y, z \in \mathcal{E} \quad (2.2)$$

For any triple of elements $x, y, z \in \mathcal{E}_{-1}$ of degree -1 . Equivalently:

$$Jac(x, y, z) = \ell_2(\ell_2(x, y), z) + \mathcal{O}(x, y, z),$$

where $\mathcal{O}(x, y, z)$ means that we sum on the circular permutations of x, y, z with Koszul signs. It is graded symmetric of degree -2 . By axiom 3, the Jacobiator takes values in the kernel of the anchor map, since

$$\begin{aligned} \rho(Jac(x, y, z)) &= \rho(\ell_2(\ell_2(x, y), z)) + \mathcal{O}(x, y, z) \\ &= [\rho(\ell_2(x, y)), \rho(z)] + \mathcal{O}(x, y, z) \\ &= [[\rho(x), \rho(y)], \rho(z)] + \mathcal{O}(x, y, z) = 0, \quad \text{since } [\cdot, \cdot] \text{ satisfies Jacobi identity.} \end{aligned}$$

It is easy to see that the bracket ℓ_2 goes to the quotient and endows, $\frac{\mathcal{E}_{-1}}{\ker \rho}$ and $\frac{\mathcal{E}_{-1}}{\ell_1(\mathcal{E}_{-2})}$ with a Lie \mathbb{K} -algebra bracket $\bar{\ell}_2$.

Remark 2.1.5. The map $(x, y, z) \mapsto Jac(x, y, z)$ is \mathcal{O} -trilinear: indeed, it is clear if $x, y, z \in \mathcal{E}_{\leq -2}$. By graded symmetry, we need to verify this point when the at least one of the variables is of degree -1 . Let $f \in \mathcal{O}$,

Case 1: $x \in \mathcal{E}_{-1}$ and $y, z \in \mathcal{E}_{\leq -2}$

$$\begin{aligned} Jac(x, y, fz) &= \ell_2(\ell_2(x, y), fz) + (-1)^{|z||y|} \ell_2(\ell_2(x, fz), y) + (-1)^{|z|+|y|} \ell_2(\ell_2(y, fz), x) \\ &= f\ell_2(\ell_2(x, y), z) + (-1)^{|z||y|} (\ell_2(f\ell_2(x, z), y) + \ell_2(\rho(x)[f]z, y)) + (-1)^{|z|+|y|} f\ell_2(\ell_2(y, z), x) + \\ &\quad (-1)^{|z|+|y|} (-1)^{|z|+|y|+1} \rho(x)[f] \ell_2(y, z) \\ &= fJac(x, y, z) + \cancel{(-1)^{|z||y|} \rho(x)[f] \ell_2(z, y)} - \cancel{\rho(x)[f] \ell_2(y, z)} \end{aligned}$$

Case 2: $x, y \in \mathcal{E}_{-1}$ and $z \in \mathcal{E}_{\leq -2}$

$$\begin{aligned} Jac(x, y, fz) &= \ell_2(\ell_2(x, y), fz) + (-1)^{|z|} \ell_2(\ell_2(x, fz), y) + (-1)^{|z|+1} \ell_2(\ell_2(y, fz), x) \\ &= f\ell_2(\ell_2(x, y), z) + \rho(\ell_2(x, y))[f]z + (-1)^{|z|} \ell_2(f\ell_2(x, z) + \rho(x)[f]z, y) + \\ &\quad (-1)^{|z|+1} \ell_2(f\ell_2(y, z) + \rho(y)[f]z, x) \\ &= fJac(x, y, z) + \rho(\ell_2(x, y))[f]z + (-1)^{|z|} ((-1)^{|z|} \cancel{\rho(y)[f] \ell_2(x, z)} + \cancel{\rho(x)[f] \ell_2(z, y)}) \\ &\quad + (-1)^{|z|} \rho(y)[\rho(x)[f]]z + (-1)^{|z|+1} ((-1)^{|z|} \cancel{\rho(x)[f] \ell_2(y, z)} + \cancel{\rho(y)[f] \ell_2(z, x)}) \\ &\quad + (-1)^{|z|} \rho(x)[\rho(y)[f]]z \\ &= fJac(x, y, z) + \cancel{(\rho(\ell_2(x, y)) - [\rho(x), \rho(y)])} [f]z \\ &= fJac(x, y, z). \end{aligned}$$

Case 3: for $x, y, z \in \mathcal{E}_{-1}$, the proof is identical to case 2.

Example 2.1.6. A *differential graded Lie algebra (DGLA)* (see e.g [Man05a]) is a graded vector space $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ over \mathbb{K} together with a bilinear map called *graded Lie bracket* $[\cdot, \cdot] : \mathfrak{g}_i \times \mathfrak{g}_j \rightarrow \mathfrak{g}_{i+j}$ and a *differential map* $d : \mathfrak{g}_i \rightarrow \mathfrak{g}_{i-1}$ such that for all homogeneous elements $v_1, v_2, v_3 \in \mathfrak{g}$ has the properties

- **graded commutativity:** $[v_1, v_2] = -(-1)^{|v_1||v_2|} [v_2, v_1]$,
- **Jacobi's identity:** $(-1)^{|v_1||v_3|} [v_1, [v_2, v_3]] + (-1)^{|v_2||v_1|} [v_2, [v_3, v_1]] + (-1)^{|v_3||v_2|} [v_3, [v_1, v_2]] = 0$,
- **Leibniz's identity:** $d[v_1, v_2] = [dv_1, v_2] + (-1)^{|v_1|} [v_1, dv_2]$.

DGLAs are examples of almost differential graded Lie algebroid (ADGLA) with $\mathcal{O} = \mathbb{K}$ (and $\mathfrak{g}_j = 0$ for $j \geq 0$). We take $\mathcal{E}_{-i} := \mathfrak{g}_i[1]$ for $i \geq 1$, $\ell_1(x) := -d(x)$, and $\ell_2(x, y) := (-1)^{|x|} [x, y]$. Of course, here $\rho = 0$. Note that ADGLAs are more general than DGLAs, since it does not impose Jacobi identity.

Example 2.1.7. [Hue03] An *almost-Lie algebroid over a manifold M* is a triple $(A \rightarrow M, [\cdot, \cdot]_A, \rho_A)$ made of a vector bundle $A \rightarrow M$, a skew-symmetric bracket $[\cdot, \cdot]_A : \Gamma(A) \times \Gamma(A) \rightarrow \Gamma(A)$, fulfilling the Leibniz identity, i.e. for all $a, b \in \Gamma(A)$, $f \in C^\infty(M)$

$$[a, fb]_A = f[a, b]_A + \rho_A(a)[f]b,$$

and a vector bundle morphism $\rho: A \rightarrow TM$, that satisfies,

$$\rho([a, b]_A) = [\rho_A(a), \rho_A(b)].$$

We say that ρ_A is the *anchor* of $(A \rightarrow M, [\cdot, \cdot]_A, \rho_A)$. When $[\cdot, \cdot]_A$ satisfies Jacobi's identity, we speak of a *Lie algebroid over M* [Mac05].

Almost Lie algebroids over a manifold M give examples of almost differential graded Lie algebroid over $\mathcal{O} = C^\infty(M)$ with $\mathcal{E}_{-1} = \Gamma(A)$, $\mathcal{E}_{-i} = 0$ for $i \neq 1$, $\ell_2 = [\cdot, \cdot]_A$, and the anchor is ρ_A .

2.1.2 Lie ∞ -algebroids over \mathcal{O}

Lie ∞ -algebroids over manifolds were introduced (explicitly or implicitly) by various authors, e.g. [SSS12], [Vor10], and [Še01]. We refer to Giuseppe Bonavolontà and Norbert Poncin for a complete overview of the matter [BP13]. It extends the notion of almost differential graded Lie algebroids.

Definition 2.1.8. [LS93] A *negatively graded Lie ∞ -algebroid over \mathcal{O}* is a collection $\mathcal{E} = (\mathcal{E}_{-i})_{i \geq 1}$ of projective \mathcal{O} -modules, equipped with:

1. a collection of linear maps $\ell_i: S_{\mathbb{K}}^i(\mathcal{E}) \rightarrow \mathcal{E}$ of degree $+1$ called *i -ary brackets*
2. a \mathcal{O} -linear map $\mathcal{E}_{-1} \rightarrow \text{Der}(\mathcal{O})$ called *anchor map*

satisfying the following axioms :

(i) the *higher Jacobi identity*:

$$\sum_{i=1}^n \sum_{\sigma \in \mathfrak{S}_{i, n-i}} \epsilon(\sigma) \ell_{n-i+1}(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0, \quad (2.3)$$

for all $n \geq 1$ and homogeneous elements $x_1, \dots, x_n \in \mathcal{E}$,

(ii) for $i \neq 2$, the bracket ℓ_i is \mathcal{O} -linear, while for $i = 2$,

$$\ell_2(x, fy) = \rho(x)[f]y + f\ell_2(x, y) \text{ for all } x, y \in \mathcal{E}, f \in \mathcal{O},$$

where, by convention, $\rho_{\mathcal{E}}$ is extended by zero on \mathcal{E}_{-i} for all $i \geq 2$,

(iii) $\rho \circ \ell_1 = 0$ on \mathcal{E}_{-2} ,

(iv) ρ is a morphism of brackets, i.e., $\rho(\ell_2(x, y)) = [\rho(x), \rho(y)]$ for all $x, y \in \mathcal{E}_{-1}$.

We denote it by $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$.

Convention 2.1.9. From now on, we simply say “Lie ∞ -algebroid” for “negatively graded Lie ∞ -algebroid”.

Let us explain these axioms.

Remark 2.1.10. It follows from Definition 2.1.8 that:

1. The higher Jacobi identity is equivalent to

$$\sum_{i=1}^n [\ell_i, \ell_{n+1-i}]_{\text{RN}} = 0,$$

for all positive integer n . See Definition 1.2.19 a definition of $[\cdot, \cdot]_{\text{RN}}$.

2. For $n = 1$, higher Jacobi identity yields $\ell_1 \circ \ell_1 = 0$. Thus,

$$\dots \xrightarrow{\ell_1} \mathcal{E}_{-3} \xrightarrow{\ell_1} \mathcal{E}_{-2} \xrightarrow{\ell_1} \mathcal{E}_{-1} \quad (2.4)$$

is a complex of projective \mathcal{O} -modules.

3. Higher Jacobi identity for $n = 2$ reads $\ell_1(\ell_2(x, y)) + \ell_2(\ell_1(x), y) + (-1)^{|x|}\ell_2(x, \ell_1(y)) = 0$.

4. The third and the fourth axiom are consequences of item (i), and (ii) if \mathcal{O} has no zero divisors¹ on \mathcal{E}_{-1} . Indeed, for all $x \in \mathcal{E}_{-2}$ and $y \in \mathcal{E}_{-1}$ and $f \in \mathcal{O}$. Higher Jacobi identity specialized at $n = 2$ and Leibniz identity read

$$\begin{aligned} \ell_1(\ell_2(x, fy)) &= \ell_2(\ell_1(x), fy) \\ &= f\ell_2(\ell_1(x), y) + \rho(\ell_1(x))[f]y \end{aligned}$$

Using Leibniz identity again, the left-hand side of the equation above also reads $\ell_1(\ell_2(x, fy)) = f\ell_1(\ell_2(x, y))$. Hence:

$$\rho(\ell_1(x))[f]y = 0.$$

If \mathcal{O} has no zero divisors on \mathcal{E}_{-1} , then $\rho(\ell_1(x))[f] = 0$. Since x and f are arbitrary, we have $\rho \circ \ell_1|_{\mathcal{E}_{-2}} = 0$.

Also, by writing higher Jacobi for $n = 3$ on elements $x, y, z \in \mathcal{E}_{-1}$ and $f \in \mathcal{O}$ while using Leibniz identity we get

$$\begin{aligned} 0 &= \ell_1(\ell_3(x, y, fz)) + \ell_2(\ell_2(x, y), fz) - \ell_2(\ell_2(x, fz), y) + \ell_2(\ell_2(y, fz), x) \\ &= f\ell_1(\ell_3(x, y, z)) + f\ell_2(\ell_2(x, y), z) + \rho(\ell_2(x, y))[f]z - \ell_2(f\ell_2(x, z), y) - \ell_2(\rho(x)[f]z, y) \\ &\quad + \ell_2(f\ell_2(y, z), x) + \ell_2(\rho(y)[f]z, x) \\ &= f \underbrace{(\ell_1(\ell_3(x, y, fz)) + \ell_2(\ell_2(x, y), z) - \ell_2(\ell_2(x, z), y) + \ell_2(\ell_2(y, z), x))}_{=0} + \rho(\ell_2(x, y))[f]z \\ &\quad + \rho(y)[\rho(x)[f]]z - \rho(x)[\rho(y)[f]]z + \underline{\rho(y)[f]\ell_2(x, z)} + \underline{\rho(y)[f]\ell_2(z, x)} - \underline{\rho(x)[f]\ell_2(z, y)} \\ &\quad - \underline{\rho(x)[f]\ell_2(y, z)}. \end{aligned}$$

This implies, $(\rho(\ell_2(x, y))[f] - [\rho(x), \rho(y)][f])z = 0$. Therefore, when \mathcal{O} has no zero divisors on \mathcal{E}_{-1} , we obtain that

$$\rho(\ell_2(x, y)) = [\rho(x), \rho(y)].$$

Definition 2.1.11. A Lie ∞ -algebroid is said to be *acyclic* if the complex (2.4) has no cohomology in degree ≤ -2 .

¹i.e. for every $f \in \mathcal{O}$ the linear map $\mathcal{E}_{-1} \xrightarrow{f} \mathcal{E}_{-1}$ given by multiplication by f , is injective.

Example 2.1.12. When $\mathcal{O} = \mathbb{K}$ we have only the axiom (i) of 2.1.8, since the other axioms are trivial. Therefore, we recover the axioms that define Lie ∞ -algebras [Sta92, LS93].

Remark 2.1.13. Geometrically, we are in the case where the projective modules $(\mathcal{E}_{-i})_{i \geq 1}$ of Definition 2.1.8 are finitely generated. Serre-Swan theorem [Swa62] assures for every $i \geq 1$, $\mathcal{E}_{-i} = \Gamma(E_{-i})$ for some vector bundle $E_{-i} \rightarrow M$ over a manifold M . Hence, Definition 2.1.8 is rewritten word by word as follows: A (finitely generated) negatively graded Lie ∞ -algebroid $(E, (\ell_k)_{k \geq 1}, \rho)$ over a manifold M is

1. a collection of vector bundles $E = (E_{-i})_{i \geq 1}$ over M
2. together with a sheaf of Lie ∞ -algebra structures $(\ell_k)_{k \geq 1}$ over the sheaf of sections of E
3. that comes with a vector bundle morphism $\rho: E_{-1} \rightarrow TM$, called the *anchor*,
4. such that the k -ary-brackets are all \mathcal{O} -multilinear except when $k = 2$ and at least one of the arguments is of degree -1 . The 2-ary bracket satisfies the Leibniz identity

$$\ell_2(x, fy) = \rho(x)[f]y + f\ell_2(x, y), x \in \Gamma(E_{-1}), y \in \Gamma(E). \quad (2.5)$$

2.2 Lie ∞ -algebroids as homological co-derivations

When it comes to manipulating morphisms of Lie ∞ -algebroids, it quickly becomes quite tedious. Therefore, it is very useful to dualize in order to make the notion of morphisms clearer.

In the finite dimensional case [Vor10], it is usual to see it as a derivation of the symmetric algebra of the dual, i.e. as a Q -manifold (see Chapter 6.1). The duality finite rank Lie ∞ -algebroids and Q -manifolds is especially efficient to deal with morphisms.

In infinite dimension case, we cannot dualize Lie ∞ -algebroids in the sense of T. Voronov [Vor10, BP13] anymore, since the identification of $S(V)^*$ with $S(V^*)$ does not hold. What I do to deal with this problem in the infinite case, is to stay in the world of squared to zero co-derivations and impose some particular additionnal properties on their Taylor coefficients, related to \mathcal{O} -linearity and the Leibniz rule.

Now we give an alternative description of Lie ∞ -algebroids in terms of co-derivation (extending the usual [Vor05, Vor10, BP13] correspondence between Lie ∞ -algebroids and Q -manifolds in the finite rank case).

Definition 2.2.1. A co-derivation $Q \in \text{CoDer}(S_{\mathbb{K}}(\mathcal{E}))$ of degree $+1$ is said to be an *homological co-derivation* or *co-differential* when $Q^2 = 0$.

Given such a co-derivation Q , the triplet $(S_{\mathbb{K}}(\mathcal{E}), \Delta, Q)$ is then called a *differential graded co-algebra*.

The following lemma is important.

Lemma 2.2.2. A co-derivation $Q \in \text{CoDer}(S_{\mathbb{K}}(\mathcal{E}))$ of degree $+1$ is an homological co-derivation if and only if $\text{pr} \circ Q^2 = 0$.

Proof. By Proposition 1.2.10, the bracket

$$[Q, Q] = Q \circ Q - (-1)^{|Q||Q|} Q \circ Q = 2Q^2$$

is a co-derivation, since co-derivations of a graded co-algebra is closed under the graded commutator bracket $[\cdot, \cdot]$. Therefore, $Q^2 = \frac{1}{2}[Q, Q]$ is a co-derivation, and it is completely determined by $\text{pr} \circ Q^2$. \square

Remark 2.2.3. The composition of two co-derivations is not a co-derivation in general. In particular, the composition of a co-derivation with itself is not a co-derivation unless it is of odd degree.

For a good understanding of the next proposition, see notations of Taylor coefficients in Equation (1.8).

Proposition 2.2.4. *Given a collection $\mathcal{E} = (\mathcal{E}_{-i})_{i \geq 1}$ of projective \mathcal{O} -modules, there is a one-to-one correspondence between Lie ∞ -algebroid structures $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ on \mathcal{E} and pairs $(Q_\mathcal{E}, \rho)$ made of an homological co-derivation $Q_\mathcal{E}: S_{\mathbb{K}}^\bullet(\mathcal{E}) \rightarrow S_{\mathbb{K}}^\bullet(\mathcal{E})$ and a \mathcal{O} -linear morphism, $\rho: \mathcal{E}_{-1} \rightarrow \text{Der}(\mathcal{O})$ called the anchor, such that*

1. for $k \neq 1$ the k -th Taylor coefficient $Q_\mathcal{E}^{(k)}: S_{\mathbb{K}}^{k+1}(\mathcal{E}') \rightarrow \mathcal{E}$ of $Q_\mathcal{E}$ is \mathcal{O} -multilinear,
2. for all $x, y \in \mathcal{E}$ and $f \in \mathcal{O}$, we have, $Q_\mathcal{E}^{(1)}(x \cdot fy) = fQ_\mathcal{E}^{(1)}(x \cdot y) + \rho(x)[f]y$,
3. $\rho \circ Q_\mathcal{E}^{(0)} = 0$ on \mathcal{E}_{-2} ,
4. $\rho \circ Q_\mathcal{E}^{(1)}(x \cdot y) = [\rho(x), \rho(y)]$, for all $x, y \in \mathcal{E}_{-1}$.

The correspondence consists in assigning to a Lie ∞ -algebroid $(\mathcal{E}_\bullet, \ell_\bullet, \rho) = (\ell_1, \ell_2, \ell_3, \dots)$ the co-derivation $Q_\mathcal{E}$ whose k -th Taylor coefficient is the k -ary bracket ℓ_{k+1} for all $k \in \mathbb{N}_0$.

Proof. Take the i -th Taylor coefficient of co-derivation $Q_\mathcal{E}$ that satisfies the requirements 1. 2. 3. and 4. of Proposition 2.2.4 as, $\text{pr} \circ Q_\mathcal{E}^{(i)} = \ell_{i+1}: S_{\mathbb{K}}^{i+1}(\mathcal{E}) \rightarrow \mathcal{E}$. By Lemma 1.2.22 we have

$$2Q_\mathcal{E}^2 = [Q_\mathcal{E}, Q_\mathcal{E}] = \sum_{n \geq 1} \sum_{i+j=n+1} \overline{[\ell_i, \ell_j]}_{\text{RN}}.$$

Thus, by uniqueness of the extension as co-derivation, $Q_\mathcal{E}^2 = 0$ is equivalent to

$$0 = \sum_{i=1}^n [\ell_i, \ell_{n+1-i}]_{\text{RN}} \in \text{Hom}_{\mathbb{K}}^2(S_{\mathbb{K}}^n(\mathcal{E}), \mathcal{E}).$$

For all integer $n \geq 1$. \square

Remark 2.2.5. Note that, e.g. if $\mathcal{O} = \mathbb{K}$, we recover the equivalence between Lie ∞ -algebras and co-differentials.

Convention 2.2.6. *From now, when relevant, we sometime denote an underlying structure of Lie ∞ -algebroid $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ on \mathcal{E} by $(\mathcal{E}, Q_\mathcal{E}, \rho)$ instead.*

Remark 2.2.7. Notice that $Q_\mathcal{E}: S_{\mathbb{K}}^\bullet(\mathcal{E}) \rightarrow S_{\mathbb{K}}^\bullet(\mathcal{E})$ does not induce a co-derivation on $\odot^\bullet \mathcal{E}$ unless $\rho = 0$.

Let us make the correspondence given by Proposition 2.2.4 explicit on the following examples.

Example 2.2.8. *Differential Graded Lie Algebras.* We come back to Example 2.1.6. Any differential graded Lie algebra $(\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, [\cdot, \cdot], d)$ is Lie ∞ -algebroid with trivial anchor, where the unary bracket ℓ_1 and the binary bracket ℓ_2 are obtained by adding a sign to the differential map d and the graded skew-symmetric Lie bracket $[\cdot, \cdot]$ respectively, and the other brackets ℓ_k for $k \geq 3$ vanish. For $k \geq 3$, the k -th Taylor coefficient of the corresponding co-derivation $Q_{\mathfrak{g}}$ is zero, hence it can be written as $Q_{\mathfrak{g}} = Q_{\mathfrak{g}}^{(0)} + Q_{\mathfrak{g}}^{(1)}$. For every homogeneous monomial $x_1 \cdots x_k \in S^k \mathfrak{g}$

$$Q_{\mathfrak{g}}^{(0)}(x_1 \cdots x_k) = \sum_{i=1}^k (-1)^{(|x_1| + \cdots + |x_{i-1}|)|x_i|} \ell_1(x_i) \cdot x_1 \cdots \hat{x}_i \cdots x_k,$$

$$Q_{\mathfrak{g}}^{(1)}(x_1 \cdots x_k) = \sum_{1 \leq i < j \leq k} (-1)^{(|x_1| + \cdots + |x_{j-1}|)|x_j| + (|x_1| + \cdots + |x_{i-1}|)|x_i|} \ell_2(x_i, x_j) \cdot x_1 \cdots \tilde{x}_i \cdots \hat{x}_j \cdots x_k.$$

Proposition 2.2.4 and Remark 2.2.5 say that $(\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, [\cdot, \cdot], d)$ is a DGLA if and only if

- $Q_{\mathfrak{g}}^2 = 0$;
- or
- $(Q_{\mathfrak{g}}^{(0)})^2 = (Q_{\mathfrak{g}}^{(1)})^2 = 0$ and $Q_{\mathfrak{g}}^{(0)} \circ Q_{\mathfrak{g}}^{(1)} + Q_{\mathfrak{g}}^{(1)} \circ Q_{\mathfrak{g}}^{(0)} = 0$.

2.3 Morphisms of Lie ∞ -algebroids and homotopies

This section extends Section 3.4 of [LLS20] to the infinite dimensional setting.

2.3.1 Morphisms of Lie ∞ -algebroids

Definition 2.3.1. A Lie ∞ -algebroid morphism (or Lie ∞ -morphism) from a Lie ∞ -algebroid $(\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ to a Lie ∞ -algebroid $(\mathcal{E}, Q_{\mathcal{E}}, \rho)$, is a co-morphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ such that

$$\Phi \circ Q_{\mathcal{E}'} = Q_{\mathcal{E}} \circ \Phi \tag{2.6}$$

and

1. all Taylor coefficients are \mathcal{O} -multilinear,
2. which satisfies $\rho \circ \Phi_0 = \rho'$ on \mathcal{E}'_{-1} .

Above, $\Phi_0 = \Phi|_{\mathcal{E}'}$: $\mathcal{E}' \rightarrow \mathcal{E}$ is the 0-th Taylor coefficient of Φ . Notice that item 1. is equivalent to saying Φ is \mathcal{O} -multilinear in the sense of Lemma 1.2.17.

Remark 2.3.2. Recall from [LS93] that Lie ∞ -algebra morphisms from $(S_{\mathbb{K}}^{\bullet}(\mathcal{E}), Q_{\mathcal{E}})$ to $(S_{\mathbb{K}}^{\bullet}(\mathcal{E}'), Q_{\mathcal{E}'})$ are defined to be co-algebra morphisms $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ that satisfy (2.6). Definition 2.3.1 adds two additional assumptions to turn a Lie ∞ -algebra morphism into a Lie ∞ -algebroid morphism.

Remark 2.3.3. For a Lie ∞ -algebroid morphism $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$, the 0-th Taylor coefficient $\Phi_0: (\mathcal{E}', \ell'_1) \rightarrow (\mathcal{E}, \ell_1)$ of Φ is the chain map, that is, the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{E}'_{-3} & \xrightarrow{\ell'_1} & \mathcal{E}'_{-2} & \xrightarrow{\ell'_1} & \mathcal{E}'_{-1} & \xrightarrow{\rho'} & \text{Der}(\mathcal{O}) \\ & & \downarrow \Phi_0 & & \downarrow \Phi_0 & & \downarrow \Phi_0 & & \downarrow \text{id} \\ \cdots & \longrightarrow & \mathcal{E}_{-3} & \xrightarrow{\ell_1} & \mathcal{E}_{-2} & \xrightarrow{\ell_1} & \mathcal{E}_{-1} & \xrightarrow{\rho} & \text{Der}(\mathcal{O}) \end{array} \tag{2.7}$$

namely, $\Phi_0 \circ \ell'_1 = \ell_1 \circ \Phi_0$.

Also, by going a bit further, it satisfies for all $x, y \in \mathcal{E}'$

$$\Phi_0 \circ \ell'_2(x, y) + \Phi_1 \circ \ell'_1(x \odot y) = \ell_1 \circ \Phi_1(x \odot y) + \ell_2(\Phi_0(x), \Phi_0(y)). \quad (2.8)$$

Example 2.3.4. Let $(A, [\cdot, \cdot]_A, \rho_A)$ and $(B, [\cdot, \cdot]_B, \rho_B)$ be two Lie algebroids over a manifold M . A Lie algebroid morphism $\phi: A \rightarrow B$ over the identity [Mac05] induces a Lie ∞ -algebroid morphism $S^\bullet(\phi): S^\bullet(\Gamma(A)) \rightarrow S^\bullet(\Gamma(B))$ in the section level: where corresponding co-algebra morphism $S^\bullet(\phi)$ is defined as

$$a_1 \cdots a_n \mapsto \phi(a_1) \cdots \phi(a_n),$$

for all $a_1, \dots, a_n \in A$.

Example 2.3.5. *Lie ∞ -morphisms of Differential Graded Lie Algebras.* Let us consider $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}, d_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}}, d_{\mathfrak{h}})$ two differential graded Lie algebras (see Example 2.1.6), where $Q_{\mathfrak{g}}$ and $Q_{\mathfrak{h}}$ denote their respective associated co-derivations. Let $\Phi: (S^\bullet(\mathfrak{g}[1]), Q_{\mathfrak{g}}) \rightarrow (S^\bullet(\mathfrak{h}[1]), Q_{\mathfrak{h}})$ be a Lie ∞ -morphism, i.e. one has

$$\Phi \circ Q_{\mathfrak{g}} = Q_{\mathfrak{h}} \circ \Phi. \quad (2.9)$$

In particular, $\Phi_0 \circ d_{\mathfrak{g}} = d_{\mathfrak{h}} \circ \Phi_0$. Let us write down what the restriction of Equation (2.9) to low Taylor coefficients: Let $x, y \in \mathfrak{g}$ be two homogeneous elements. One has,

$$\begin{aligned} \Phi(x \cdot y) &= \Phi_1(x \cdot y) + \Phi_0(x) \cdot \Phi_0(y), \\ \Phi(x) &= \Phi_0(x). \end{aligned}$$

A direct computation of the LHS of Equation (2.9) applied to $x \cdot y$ gives,

$$\begin{aligned} \Phi \circ Q_{\mathfrak{g}}(x \cdot y) &= \Phi \left(-d_{\mathfrak{g}}(x) \cdot y - (-1)^{(|x|-1)(|y|-1)} d_{\mathfrak{g}}(y) \cdot x + (-1)^{|x|} [x, y]_{\mathfrak{g}} \right) \\ &= -\Phi_1(d_{\mathfrak{g}}(x) \cdot y) - \Phi_0(d_{\mathfrak{g}}(x)) \cdot \Phi_0(y) - (-1)^{(|x|-1)(|y|-1)} (\Phi_1(d_{\mathfrak{g}}(y) \cdot x) - \Phi_0(d_{\mathfrak{g}}(y)) \cdot \Phi_0(x)) \\ &\quad + (-1)^{|x|} \Phi_0([x, y]_{\mathfrak{g}}) \end{aligned}$$

Also, the RHS of Equation (2.9) applied to $x \cdot y$ gives,

$$\begin{aligned} Q_{\mathfrak{h}} \circ \Phi(x \cdot y) &= Q_{\mathfrak{h}}(\Phi_1(x \cdot y) + \Phi_0(x) \cdot \Phi_0(y)) \\ &= -d_{\mathfrak{h}}(\Phi_1(x)) \cdot y - d_{\mathfrak{h}}(\Phi_0(x)) \cdot \Phi_0(y) - (-1)^{(|x|-1)(|y|-1)} d_{\mathfrak{h}}(\Phi_0(y)) \cdot \Phi_0(x) \\ &\quad + (-1)^{|x|} [\Phi_0(x), \Phi_0(y)]_{\mathfrak{h}} \end{aligned}$$

Since both sides are equal, and $\Phi_0 \circ d_{\mathfrak{g}} = d_{\mathfrak{h}} \circ \Phi_0$ we obtain,

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)]_{\mathfrak{h}} = (-1)^{(|x|-1)|y|} \Phi_1(d_{\mathfrak{g}}(y) \cdot x) - (-1)^{|x|} (\Phi_1(d_{\mathfrak{g}}(x) \cdot y) - d_{\mathfrak{h}}(\Phi_1(x \cdot y))).$$

2.3.2 Homotopies

In this section, we define homotopy between Lie ∞ -morphisms, and between Lie ∞ -algebroids. This extends [LLS20] from finite dimensional Q -manifolds to arbitrary Lie ∞ -algebroids.

A practical definition

Let us consider $(\Omega^\bullet([a, b]), \wedge, d_{dR})$ the differential graded algebra made of the forms on $[a, b]$ together with the wedge product and the Rham differential. If we denote by t the coordinate on $[a, b]$, we have $\Omega^\bullet([a, b]) = C^\infty([a, b]) \oplus C^\infty([a, b]) dt$. We equip $\Omega^\bullet([a, b])$ with a co-associative co-algebra structure $\Delta: \Omega^\bullet([a, b]) \rightarrow \Omega^\bullet([a, b]) \otimes_{\Omega^\bullet([a, b])} \Omega^\bullet([a, b])$, given by $\Delta_\Omega(1) = 1 \otimes 1$, where we extend by $\Omega^\bullet([a, b])$ -linearity.

Let $(\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ and $(\mathcal{E}, Q_{\mathcal{E}}, \rho)$ be Lie ∞ -algebroids over \mathcal{O} .

Lemma 2.3.6. *The triplet*

$$(S_{\mathbb{K}}^\bullet(\mathcal{E}) \otimes_{\mathbb{K}} \Omega^\bullet([a, b]), \bar{\Delta} = (\text{id} \otimes \tau \otimes \text{id}) \circ \Delta \boxtimes \Delta_\Omega, \partial = Q_{\mathcal{E}'} \otimes \text{id} + \text{id} \otimes d_{dR})$$

is a differential graded co-algebra.

Proof. One should understand that for $\alpha \in C^\infty([a, b])$ and for an homogeneous element $v \in S_{\mathbb{K}}^\bullet(\mathcal{E})$,

$$(Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes d_{dR})(v \otimes \alpha) = Q_{\mathcal{E}}(v) \otimes \alpha + (-1)^{|v|} v \otimes \alpha' dt$$

also,

$$\begin{aligned} (Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes d_{dR})(v \otimes \alpha dt) &= Q_{\mathcal{E}}(v) \otimes \alpha dt + v \otimes \underbrace{d_{dR}(\alpha dt)}_{=0} \\ &= Q_{\mathcal{E}}(v) \otimes \alpha dt. \end{aligned}$$

Clearly, $\partial^2 = 0$, see the definition of tensor product of complexes, Appendix B. Let us check that ∂ is indeed a co-derivation w.r.t the co-product $\bar{\Delta}$. Take $\alpha \in C^\infty([a, b])$ and a homogeneous element $v \in S_{\mathbb{K}}^\bullet(\mathcal{E})$, we will use the Sweedler notation, $\Delta(v) = v_{(1)} \otimes v_{(2)}$, to avoid a long useless text. On one hand,

$$\begin{aligned} \bar{\Delta} \circ \partial(v \otimes \alpha) &= (\text{id} \otimes \tau \otimes \text{id}) \circ \Delta \boxtimes \Delta_\Omega(Q_{\mathcal{E}}(v) \otimes \alpha + (-1)^{|v|} v \otimes \alpha' dt) \\ &= \text{id} \otimes \tau \otimes \text{id} \left(\Delta \circ Q_{\mathcal{E}}(v) \boxtimes \alpha \otimes 1 + (-1)^{|v|} \Delta(v) \boxtimes \alpha' dt \otimes 1 \right) \\ &= (\text{id} \otimes \tau \otimes \text{id}) \circ \left((Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes Q_{\mathcal{E}}) \circ \Delta(v) \boxtimes \alpha \otimes 1 + (-1)^{|v|} \Delta(v) \boxtimes \alpha' dt \otimes 1 \right) \\ &= Q_{\mathcal{E}}(v_{(1)}) \otimes \alpha \boxtimes v_{(2)} \otimes 1 + (-1)^{|v_{(1)}|} v_{(1)} \otimes \alpha \boxtimes Q_{\mathcal{E}}(v_{(2)}) \otimes 1 + \\ &\quad (-1)^{|v|+|v_{(2)}|} v_{(1)} \otimes (\alpha' dt) \boxtimes v_{(2)} \otimes 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} (\partial \boxtimes \text{id} + \text{id} \boxtimes \partial) \circ \bar{\Delta}(v \otimes \alpha) &= (\partial \boxtimes \text{id} + \text{id} \boxtimes \partial) \circ (\text{id} \otimes \tau \otimes \text{id})(\Delta(v) \boxtimes \alpha \otimes 1) \\ &= (\partial \boxtimes \text{id} + \text{id} \boxtimes \partial)(v_{(1)} \otimes \alpha \boxtimes v_{(2)} \otimes 1) \\ &= \partial(v_{(1)} \otimes \alpha) \boxtimes v_{(1)} \otimes 1 + (-1)^{|v_{(1)}|} v_{(1)} \otimes \alpha \boxtimes \partial(v_{(2)} \otimes 1) \\ &= \left(Q_{\mathcal{E}}(v_{(1)} \otimes \alpha + (-1)^{|v_{(1)}|} v_{(1)} \otimes \alpha' dt) \right) \boxtimes v_{(2)} \otimes 1 + \\ &\quad (-1)^{|v_{(1)}|} v_{(1)} \otimes \alpha \boxtimes Q_{\mathcal{E}}(v_{(2)}) \otimes 1. \end{aligned}$$

Both sides are obviously equal. Likewise, a straightforward computation shows that $\bar{\Delta} \circ \partial(v \otimes \alpha dt) = (\partial \boxtimes \text{id} + \text{id} \boxtimes \partial) \circ \bar{\Delta}(v \otimes \alpha dt)$. Hence

$$\bar{\Delta} \circ \partial = (\partial \boxtimes \text{id} + \text{id} \boxtimes \partial) \circ \bar{\Delta}$$

on $S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \otimes_{\mathbb{K}} \Omega^{\bullet}([a, b])$. \square

Remark 2.3.7. Elements of degree k of $S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \otimes_{\mathbb{K}} \Omega^{\bullet}([a, b])$ are \mathbb{K} -linear combinations of elements of the form $v \otimes \alpha$ and $w \otimes \beta dt$, with $\alpha, \beta \in C^{\infty}([a, b])$ and $v \in S_{\mathbb{K}}^{\bullet}(\mathcal{E})|_k$ and $w \in S_{\mathbb{K}}^{\bullet}(\mathcal{E})|_{k-1}$. The latter can be seen as maps, $t \in [a, b] \mapsto \alpha(t)v \otimes 1$ and $t \in [a, b] \mapsto \beta(t)w \otimes dt$. Hence, elements of degree k of this complex can be considered as element of $S_{\mathbb{K}}(\mathcal{E}) \otimes_{\mathbb{K}} \Omega^{\bullet}([a, b])$ that depend on t , i.e. as $t \in [a, b] \mapsto v_t \otimes 1 + w_t \otimes dt$, with $v_t \in S_{\mathbb{K}}^{\bullet}(\mathcal{E}')|_k$ and $w_t \in S_{\mathbb{K}}^{\bullet}(\mathcal{E}')|_{k-1}$.

The following is a temporary definition. It will be generalized later to another more practical for gluing homotopies.

Definition 2.3.8. A homotopy that joins two Lie ∞ -algebroid morphisms $\Phi, \Psi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ is the datum made of an interval $[a, b] \subset \mathbb{R}$ and a chain map

$$\begin{aligned} (S_{\mathbb{K}}^{\bullet}(\mathcal{E}'), Q_{\mathcal{E}'}) &\xrightarrow{\mathcal{H}} (S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \otimes_{\mathbb{K}} \Omega^{\bullet}([a, b]), Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes d_{dR}) \\ v &\longmapsto \left(t \in [a, b] \mapsto J_t(v) \otimes 1 - (-1)^{|v|} H_t(v) \otimes dt \right) \end{aligned}$$

1. which is a co-algebra morphism,
2. and that coincides with Φ and Ψ at $t = a$ and b respectively, i.e. for all $v \in S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$, one has $\mathcal{H}(v)(a) = \Phi(v) \otimes 1$ and $\mathcal{H}(v)(b) = \Psi(v) \otimes 1$.

Remark 2.3.9. In the definition above, the map \mathcal{H} induces for every $t \in [a, b]$ two different \mathcal{O} -multilinear maps

$$\begin{cases} J_t: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \longrightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \\ H_t: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \longrightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E}). \end{cases}$$

Since \mathcal{H} is of degree 0, one of the maps is of degree zero and the other one of degree -1 respectively. By using respectively the property of co-algebra morphisms and chain map property, we obtain the following for every $t \in [a, b]$:

1. Let $\bar{\Delta}$ be the co-product on $S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \otimes_{\mathbb{K}} \Omega^{\bullet}([a, b])$ like in Lemma 2.3.6. We have,

$$\bar{\Delta} \circ \mathcal{H}(v) = \mathcal{H} \boxtimes \mathcal{H} \circ \Delta'(v).$$

Let $v = v_1 \cdots v_n \in S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$, a direct computation of gives us

$$\Delta \boxtimes \Delta_{\Omega} \circ \mathcal{H}(v)(t) = \Delta \circ J_t(v) \boxtimes 1 \otimes 1 - (-1)^{|v|} \Delta \circ H_t(v) \boxtimes dt \otimes 1.$$

Let us use the Sweedler notation just like in Lemma 2.3.6 to compute $\mathcal{H} \boxtimes \mathcal{H} \circ \Delta'(v)(t)$. We have,

$$\begin{aligned} \mathcal{H} \boxtimes \mathcal{H} \circ \Delta'(v)(t) &= \mathcal{H} \boxtimes \mathcal{H}(v_{(1)} \otimes v_{(2)})(t) \\ &= \mathcal{H}(t)(v_{(1)}) \boxtimes \mathcal{H}(v_{(1)})(t) \\ &= \left(J_t(v_{(1)}) \otimes 1 - (-1)^{|v_{(1)}} H_t(v_{(1)}) \otimes dt \right) \boxtimes \left(J_t(v_{(2)}) \otimes 1 - (-1)^{|v_{(2)}} H_t(v_{(2)}) \otimes dt \right) \end{aligned}$$

Hence,

$$\begin{aligned} (\text{id} \otimes \tau \otimes \text{id})^{-1} \circ (\mathcal{H} \otimes \mathcal{H}) \circ \Delta'(v) = \\ (J_t \otimes J_t) \circ \Delta'(v) \boxtimes 1 \otimes 1 + \cancel{H_t \otimes H_t \circ \Delta'(v) \boxtimes dt \otimes dt} \\ + (-1)^{|v|} (H_t \otimes J_t \circ \Delta'(v) + J_t \otimes H_t \circ \Delta'(v)) \boxtimes dt \times 1 \end{aligned}$$

By equating both sides, we obtain equations that say that J_t is a co-morphism and H_t a J_t -co-derivation.

2. and for any homogeneous element $v \in S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$ we have: in one side

$$\mathcal{H} \circ Q_{\mathcal{E}'}(v)(t) = J_t \circ Q_{\mathcal{E}'}(v) \otimes 1 - (-1)^{|v|+1} H_t(v) \circ Q_{\mathcal{E}'}(v) \otimes dt.$$

to the other side²,

$$\begin{aligned} (Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes d_{dR}) \circ \mathcal{H}(v)(t) &= (Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes d_{dR}) \circ \left(J_t(v) \otimes 1 - (-1)^{|v|} H_t(v) \otimes dt \right) \\ &= Q_{\mathcal{E}} \circ J_t(v) \otimes 1 + (-1)^{|v|} \frac{dJ_t}{dt}(v) \otimes dt - (-1)^{|v|} Q_{\mathcal{E}} \circ H_t(v) \otimes dt \end{aligned}$$

By equating both sides we see that J_t and H_t satisfy the following condition

$$\begin{cases} J_t \circ Q_{\mathcal{E}'}(v) = Q_{\mathcal{E}} \circ J_t(v) \\ \frac{dJ_t}{dt}(v) = Q_{\mathcal{E}} \circ H_t(v) + H_t \circ Q_{\mathcal{E}'}(v). \end{cases} \quad (2.10)$$

In addition, we have for $v \in \mathcal{E}'_{-1}$ that

$$\frac{dJ_t^{(0)}}{dt}(v) = Q_{\mathcal{E}}^{(0)} \circ H_t^{(0)}(v) + \underbrace{H_t^{(0)} \circ Q_{\mathcal{E}'}^{(0)}(v)}_{=0}$$

and

$$\begin{aligned} J_t^{(0)}(v) &= \Phi^{(0)}(v) + \int_a^t \ell_1 \circ H_s^{(0)}(v) ds \\ &= \Phi^{(0)}(v) + \ell_1 \circ \int_a^t H_s^{(0)}(v) ds \end{aligned} \quad (2.11)$$

This implies that $\rho \circ J_t^{(0)}(v) = \rho \circ \Phi^{(0)}(v) = \rho'(v)$, since $\rho \circ \ell_1 = 0$.

Gluing homotopies as defined in Definition 2.3.8 is however not so easy. We can reformulate the definition of homotopies between Lie ∞ -morphisms in the following manner. Before we do this, we would like to fix some vocabulary.

Let us recall some facts on vector-valued functions. Let V be a vector space. Unless a topology on V is chosen, the notion of V -valued continuous or differentiable or smooth function, and the concept of a limit on an interval $I = [a, b] \subset \mathbb{R}$ do not make any sense. However, we can always define the following notion

²It should be understood that e.g. $(Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes d_{dR})(\alpha(t)v \otimes 1) = (Q_{\mathcal{E}}(v) \otimes \alpha + v \otimes d_{dR}\alpha)(t)$.

Definition 2.3.10. A vector-valued map $\gamma: I \rightarrow V$ is said to be a *piecewise rational path* on I if there exists a finite increasing sequence $a = t_0 \leq \dots \leq t_N = b$ of *gluing points*, such that for all $i = 0, \dots, N - 1$ the restriction γ^i of γ to $[t_i, t_{i+1}]$ is of the form

$$\gamma^i(t) = \sum_{j=1}^n \beta_j^i(t) v_j, \quad \text{for some } n \in \mathbb{N}$$

where for every $j = 1, \dots, n$, $v_j \in V$ and $\beta_j^i: I \rightarrow \mathbb{R}$ a real rational function on $[t_i, t_{i+1}]$ that has no pole on $[t_i, t_{i+1}]$.

1. We say that γ is *continuous* on I , if for all $i = 1, \dots, N - 1$, γ_i and γ_{i+1} coincide at the gluing point t_{i+1} .
2. When V is a space of linear maps between the vector spaces S and T , we shall say that a V -valued map $\Xi: I \rightarrow V$ is a *piecewise rational (continuous)* if the map $(t \in I \mapsto \Xi_t(s))$ is a piecewise rational (continuous) T -valued path for all $s \in S$.
3. We define the limit of γ at $u \in [t_i, t_{i+1}]$ to be equal to

$$\sum_{j=1}^n \lim_{t \rightarrow u} \beta_j^i(t) v_j$$

when, for every $j = 1, \dots, n$, β_j^i admits a limit at u . One can assume that $\{v_1, v_2, v_3 \dots\}$ is a basis of V .

Let us recall the following.

Remark 2.3.11. It is a very classical fact that the integral of a piecewise- C^1 function $\beta: I \rightarrow \mathbb{R}$ on a compact interval $I = [a, b] \subset \mathbb{R}$ which is subordinate to a subdivision $a = t_0 < \dots < t_N = b$ admits primitives which are piecewise- C^1 on the same subdivision $a = t_0 < \dots < t_N = b$. An important fact is that continuous piecewise- C^1 functions $\beta: I \rightarrow \mathbb{R}$, i.e. piecewise- C^1 functions which are also continuous (even at the junction points) admit piecewise continuous derivatives $\beta'(t)$, and $\beta(b) - \beta(a) = \int_a^b \beta'(t) dt$.

Definition 2.3.10 has this important consequence.

Lemma 2.3.12. *A piecewise rational continuous function $\gamma: I \rightarrow V$ is differentiable at every point which is not a gluing point, and the latter is again piecewise rational on I .*

Conversely, every piecewise rational functions admit a piecewise rational continuous primitive, unique up to a constant.

Proof. The derivative of γ can be defined in the usual way using the item 3 of Definition 2.3.10. Likewise, for their primitives. The proof is then immediate. \square

We can now give the following definition.

Definition 2.3.13. A family $(J_t)_{t \in I}$ of co-algebra morphisms are said to be *piecewise rational continuous* if its Taylor coefficients $J_t^{(n)}$ are piecewise rational continuous for all $n \in \mathbb{N}$.

For such a family $(J_t)_{t \in I}$, a family $(H_t)_{t \in I}$ made of J_t -co-derivations is said to be *piecewise rational* if all its Taylor coefficients are.

We are now ready to define formulate the definition of homotopies as follows and extend Definition 3.53 in [LLS20] to the infinite rank case.

Definition 2.3.14. Let Φ and Ψ be Lie ∞ -algebroid morphisms from $(\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ to $(\mathcal{E}, Q_{\mathcal{E}}, \rho)$. A *homotopy between* or that *joins* Φ and Ψ is a pair $(J_t, H_t)_{t \in [a, b]}$ consisting of:

1. a piecewise rational continuous path $t \mapsto J_t$ valued in Lie ∞ -algebroid morphisms between $S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$ and $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ satisfying the boundary condition:

$$\Phi_a = \Phi \quad \text{and} \quad \Phi_b = \Psi,$$

2. a piecewise rational path $t \mapsto H_t$, with H_t a J_t -co-derivations of degree -1 from $S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$ to $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$, such that the following equation:

$$\frac{dJ_t}{dt} = Q_{\mathcal{E}} \circ H_t + H_t \circ Q_{\mathcal{E}'} \quad (2.12)$$

holds for every $t \in]a, b[$ where it is defined (that is, not a gluing point for the Taylor coefficients). More precisely, for every $v \in S_{\mathbb{K}}^{\leq n}(\mathcal{E}')$,

$$\frac{dJ_t}{dt}(v) = Q_{\mathcal{E}} \circ H_t(v) + H_t \circ Q_{\mathcal{E}'}(v) \quad (2.13)$$

for all t which is not a gluing point of the Taylor coefficient of $J_t^{(k)}, H_t^{(k)}$ for $k = 0, \dots, n$.

When these conditions are satisfied, we say that Φ and Ψ are *homotopy equivalent*, and we write $\Phi \sim \Psi$.

Remark 2.3.15. This clearly extends (2.10).

Remark 2.3.16. In the above definitions, it is not required that the gluing points of the various Taylor coefficients $J_t^{(n)}$ or $H_t^{(n)}$ to be the same for all $n \in \mathbb{N}_0$.

The following Proposition shows that the notion of homotopy given in Definition 2.3.14 implies the usual notion of homotopy between chain maps (see Appendix B).

Proposition 2.3.17. *Let Φ and Ψ be Lie ∞ -algebroid morphisms from $(\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ to $(\mathcal{E}, Q_{\mathcal{E}}, \rho)$ which are homotopic. Then,*

$$\Psi - \Phi = Q_{\mathcal{E}} \circ H + H \circ Q_{\mathcal{E}'} \quad (2.14)$$

for some \mathcal{O} -linear map $H: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \longrightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ of degree -1 .

Proof. Take $(J_t, H_t)_{t \in [a, b]}$ a homotopy between Φ and Ψ as in Definition 2.3.14. By definition, $t \mapsto J_t$ is piecewise rational continuous, therefore it is continuous on $[a, b]$ (even at the junctions points), we can use Remark 2.3.11 and write

$$\begin{aligned} \Phi_b - \Phi_a &= \int_a^b \frac{dJ_t}{dt} dt \\ \Psi - \Phi &= \int_a^b (Q_{\mathcal{E}} \circ H_t + H_t \circ Q_{\mathcal{E}'}) dt \\ &= Q_{\mathcal{E}} \circ \left(\int_a^b H_t dt \right) + \left(\int_a^b H_t dt \right) \circ Q_{\mathcal{E}'}. \end{aligned}$$

Thus, the \mathcal{O} -multilinear map $H := \int_a^b H_t dt$ satisfies Equation (2.14). \square

Definition 2.3.18. We say that two Lie ∞ -algebroids $(\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ to $(\mathcal{E}, Q_{\mathcal{E}}, \rho)$ over \mathcal{O} are *homotopy equivalent* or *homotopic* if there exists two Lie ∞ -algebroid morphisms

$$(\mathcal{E}', Q_{\mathcal{E}'}, \rho') \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{array} (\mathcal{E}, Q_{\mathcal{E}}, \rho)$$

such that $\Phi \circ \Psi: (\mathcal{E}, Q_{\mathcal{E}}, \rho) \rightarrow (\mathcal{E}, Q_{\mathcal{E}}, \rho)$ and $\Psi \circ \Phi: (\mathcal{E}', Q_{\mathcal{E}'}, \rho') \rightarrow (\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ are homotopy equivalent to the identity map of respective space.

The following proposition justifies Definition 2.3.14.

Proposition 2.3.19. *Let Φ be a Lie ∞ -algebroid morphism from $(\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ to $(\mathcal{E}, Q_{\mathcal{E}}, \rho)$. For all $t \in [a, b]$, let $H_t^{(n)}: S_{\mathbb{K}}^{n+1}(\mathcal{E}') \rightarrow \mathcal{E}$ be a family \mathcal{O} -multilinear piecewise rational maps indexed by $n \in \mathbb{N}_0$. Then,*

1. *There exists a unique piecewise rational continuous family of co-algebra morphisms J_t such that*

$$(a) \quad J_a = \Phi$$

(b) *(J_t, H_t) is a solution of the differential equation (2.12), where H_t is the J_t -co-derivation whose n -th Taylor coefficient is $H_t^{(n)}$ for all $n \geq 0$.*

2. *Moreover, for all $t \in [a, b]$, $(J_s, H_s)_{s \in [a, t]}$ is a homotopy that joins Φ and J_t .*

Proof. Let us show item 1. We claim that equation (2.12) is a differential equation that can be solved recursively. In polynomial-degree zero, it reads,

$$\frac{dJ_t^{(0)}}{dt} = Q_{\mathcal{E}}^{(0)} \circ H_t^{(0)} + H_t^{(0)} \circ Q_{\mathcal{E}'}^{(0)} \quad (2.15)$$

and

$$J_t^{(0)} = \Phi^{(0)} + \int_a^t \left(Q_{\mathcal{E}}^{(0)} \circ H_s^{(0)} + H_s^{(0)} \circ Q_{\mathcal{E}'}^{(0)} \right) ds \quad (2.16)$$

is defined for all $t \in [a, b]$. Also, $\frac{d}{dt} J_t^{(n+1)}: S^{n+2}(\mathcal{E}') \rightarrow \mathcal{E}$ is an algebraic expression of $Q_{\mathcal{E}}^{(0)}, \dots, Q_{\mathcal{E}}^{(n+1)}, Q_{\mathcal{E}'}^{(0)}, \dots, Q_{\mathcal{E}'}^{(n+1)}, J_t^{(0)}, \dots, J_t^{(n)}, H_t^{(0)}, \dots, H_t^{(n+1)}$. But $J_t^{(n+1)}$ does not appear in the $(n+1)$ -th Taylor coefficient of $Q_{\mathcal{E}} \circ H_t + H_t \circ Q_{\mathcal{E}'}$ by Equation (1.2.16). By Lemma 2.3.12, there exists a unique piecewise rational continuous solution $J_t^{(n+1)}$ such that $J_a^{(n+1)} = \Phi^{(n+1)}$. The construction of the Taylor coefficients of the co-algebra morphisms J_t then goes by recursion. Recursion formulas also show that J_t is unique.

Let us show item 2. i.e. that J_t is a \mathcal{O} -multilinear chain map for all $t \in [a, b]$: For the same reason as in Equation (2.11), Equation (2.16) implies that $\rho \circ J_t^{(0)}|_{\mathcal{E}'} = \rho'$. The function given by

$$\Lambda_k(t) = (Q_{\mathcal{E}} \circ J_t - J_t \circ Q_{\mathcal{E}'})^{(k)} \quad \text{for all } t \in [a, b], k \in \mathbb{N}_0$$

are differentiable at all points t except for a finitely many $t \in [a, b]$ and are piecewise rational continuous. The map $\frac{dJ_t}{dt}$ is a Lie ∞ -morphism because $Q_{\mathcal{E}}^2 = 0$ and $Q_{\mathcal{E}'}^2 = 0$, hence $\Lambda'(t) = 0$. By continuity, $\Lambda_k(t)$ is constant over the interval $[a, b]$. Since $J_a = \Phi$ is a Lie ∞ -algebroid morphism, we have $\Lambda_k(a) = 0$. Thus, $\Lambda_k(t) = 0$ and,

$$Q_{\mathcal{E}} \circ J_t = J_t \circ Q_{\mathcal{E}'}, \quad \text{for all } t \in [a, b].$$

□

Let us show that homotopy in the sense above defines an equivalence relation \sim between Lie ∞ -morphisms. We have the following lemma.

Lemma 2.3.20. *A pair (J_t, H_t) is a homotopy between Lie ∞ -algebroid morphisms J_a and J_b if and only if for all rational function, $g: [a, b] \rightarrow [c, d]$ without poles on $[a, b]$, the pair $(J_{g(t)}, g'(t)H_{g(t)})$ is a homotopy between $J_{g(a)}$ and $J_{g(b)}$.*

Proof. Let $g: [a, b] \rightarrow [c, d]$ be a rational function without poles on $[a, b]$. A straightforward computation gives:

$$\begin{aligned} \frac{dJ_t}{dt} &= Q_{\mathcal{E}} \circ H_t + H_t \circ Q_{\mathcal{E}'} && \text{(by definition)} \\ \Rightarrow \frac{dJ}{dt}(g(t)) &= Q_{\mathcal{E}} \circ H_{g(t)} + H_{g(t)} \circ Q_{\mathcal{E}'} && \text{(by replacing } t \text{ by } g(t)) \\ \Rightarrow \frac{dJ_{g(t)}}{dt} &= Q_{\mathcal{E}} \circ (g'(t)H_{g(t)}) + (g'(t)H_{g(t)}) \circ Q_{\mathcal{E}'} && \text{(by multiplying both sides by } g'(t)). \end{aligned}$$

The last equation means that $(J_{g(t)}, g'(t)H_{g(t)})$ is a homotopy between $J_{g(a)}$ and $J_{g(b)}$. The backward implication is obvious, it suffices to consider $a = c$, $b = d$ and $g = \text{id}$. \square

Proposition 2.3.21. *Homotopy between Lie ∞ -morphisms is an equivalence relation. In addition, it is compatible with composition, that is, if $\Phi, \Psi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ are homotopic Lie ∞ -algebroid morphisms and $\hat{\Phi}, \hat{\Psi}: S_{\mathbb{K}}^{\bullet}(\mathcal{E}) \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E}'')$ are homotopic Lie ∞ -algebroid morphisms, then, so are their compositions $\hat{\Phi} \circ \Phi$ and $\hat{\Psi} \circ \Psi$.*

Proof. We first show that this notion of homotopy is an equivalence relation. Let Φ, Ψ and $\Xi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ be three Lie ∞ -morphisms of algebroids.

- Reflexivity: The pair $(J_t = \Phi, H_t = 0)_{t \in [0,1]}$ defines a homotopy between Φ and Φ .
- Symmetry: Let $(J_t, H_t)_{t \in [0,1]}$ be a homotopy between Φ to Ψ . By applying Lemma 2.3.20 with $g(t) = 1 - t$, we obtain a homotopy between Ψ and Φ via the pair $(\Phi_{1-t}, -H_{1-t})_{t \in [0,1]}$.
- Transitivity: Assume $\Phi \sim \Psi$ and $\Psi \sim \Xi$ and let $(J_t, H_{1,t})_{t \in [0, \frac{1}{2}]}$ be a homotopy between Φ and Ψ and let $(\bar{J}_t, H_{2,t})_{t \in [\frac{1}{2}, 1]}$ be a homotopy between Ψ and Ξ . By gluing J_t and \bar{J}_t , respectively $H_{1,t}$ and $H_{2,t}$ we obtain a homotopy $(\tilde{J}_t, H_t)_{t \in [0,1]}$ between Φ and Ξ .

We then show it is compatible with composition. Let us denote by (J_t, H_t) the homotopy between Φ and Ψ , and (\hat{J}_t, \hat{H}_t) the homotopy between $\hat{\Phi}$ and $\hat{\Psi}$. We obtain,

$$\begin{aligned} \frac{d\hat{J}_t \circ J_t}{dt} &= \frac{d\hat{J}_t}{dt} \circ J_t + \hat{J}_t \circ \frac{dJ_t}{dt} \\ &= Q_{\mathcal{E}''} \circ \left(\hat{H}_t \circ J_t + \hat{J}_t \circ H_t \right) + \left(\hat{H}_t \circ J_t + \hat{J}_t \circ H_t \right) \circ Q_{\mathcal{E}'}. \end{aligned}$$

Hence, $\hat{\Phi} \circ \Phi$ and $\hat{\Psi} \circ \Psi$ are homotopic via the pair $(\hat{J}_t \circ J_t, \hat{H}_t \circ J_t + \hat{J}_t \circ H_t)$ which is easily checked to satisfy all axioms. This concludes the proof. \square

We conclude this section with a lemma that will be useful in the sequel. Notice that this is the lemma that forces to extend Definition 2.3.8, for it would not be true anymore in the smooth setting.

Lemma 2.3.22. *Let $(J_t, H_t)_{t \in [c, +\infty[}$ be a homotopy such that for all $n \in \mathbb{N}_0$ and for every $t \geq n$, $H_t^{(n)} = 0$. Then the n -th Taylor coefficient $J_t^{(n)}$ is constant on $[n, +\infty[$ and the co-algebra morphism J_{∞} whose n -th Taylor coefficient is $J_t^{(n)}$ for any $n \in \mathbb{N}_0$ and $t \in [n, +\infty[$ is a Lie ∞ -algebroid morphism.*

Moreover, for $g: [a, b] \rightarrow [c, +\infty[$ a rational function with no pole on $[a, b[$ and such that $\lim_{t \rightarrow b} g(t) = +\infty$, the pair $(J_{g(t)}, g'(t)H_{g(t)})$ is a homotopy between J_c and J_{∞} .

Proof. Since the n -th Taylor coefficient of the J_t -co-derivation $\frac{dJ_t^{(n)}}{dt} = (Q_{\mathcal{E}} \circ H_t + H_t \circ Q_{\mathcal{E}'})^{(n)}$ depends only on $H_t^{(i)}$ for $i = 0, \dots, n-1$, we have by assumption $\frac{dJ_t^{(n)}}{dt} = 0$ for all $t \geq n$. As a consequence, $J_t^{(n)}$ is constant on $[n, +\infty[$. It follows from Proposition 2.3.19 that J_∞ is a Lie ∞ -algebroid morphism since for every $n \in \mathbb{N}_0$ and $t \in [n, +\infty[$

$$\begin{aligned} (Q_{\mathcal{E}} \circ J_\infty - J_\infty \circ Q_{\mathcal{E}'})^{(n)} &= \sum_{i+j=n} (Q_{\mathcal{E}}^{(i)} \circ J_t^{(j)} - J_t^{(j)} \circ Q_{\mathcal{E}'}^{(i)}) \\ &= 0 \quad (\text{since } J_t \text{ is a Lie } \infty\text{-algebroid morphism}). \end{aligned}$$

Let us prove the last part of the statement. By assumption, there exists $a \leq b_n \leq b$ such that for all $t \in [b_n, b]$, we have $g(t) \geq n$, so that $J_{g(t)}^{(n)} = J_\infty^{(n)}$ and $g'(t)H_{g(t)}^{(n)} = 0$ on $[b_n, b]$. The function $J_t^{(n)}$ (resp. $H_t^{(n)}$) being piecewise rational continuous (resp. piecewise rational) on $[c, n]$, the same holds for $J_{g(t)}^{(n)}$ (resp. $g'(t)H_{g(t)}^{(n)}$) on $[a, b_n]$. By gluing with a constant function J_∞ (resp. with 0), we see that all Taylor coefficients of $J_{g(t)}$ (resp. $g'(t)H_{g(t)}$) are piecewise rational continuous (resp. piecewise rational) with finitely many gluing points. This completes the proof. \square

Remark 2.3.23. Lemma 2.3.22 explains how to glue infinitely many homotopies, at least when for a given $n \in \mathbb{N}_0$, only finitely of them affects the n -th Taylor coefficient. This would not be possible using only Definition 2.3.8.

2.3.3 More technical lemmas and propositions

Let us state and prove these technical assertions for later use.

Let $(\mathcal{E}', Q_{\mathcal{E}'}, \rho')$ and $(\mathcal{E}, Q_{\mathcal{E}}, \rho)$ be a Lie ∞ -algebroid over \mathcal{O} .

Proposition 2.3.24. *Let $\Phi: S_{\mathbb{K}}^\bullet(\mathcal{E}') \rightarrow S_{\mathbb{K}}^\bullet(\mathcal{E})$ be a Lie ∞ -algebroid morphism. For $\mathcal{H}: S_{\mathbb{K}}^\bullet(\mathcal{E}') \rightarrow S_{\mathbb{K}}^\bullet(\mathcal{E})$ a \mathcal{O} -multilinear Φ -co-derivation of degree $k \leq -1$, $\mathcal{H} \circ Q_{\mathcal{E}'} - (-1)^k Q_{\mathcal{E}} \circ \mathcal{H}$ is a \mathcal{O} -multilinear Φ -co-derivation of degree $k+1$.*

Proof. We first check that $\mathcal{H} \circ Q_{\mathcal{E}} - (-1)^k Q_{\mathcal{E}'} \circ \mathcal{H}$ is a Φ -co-derivation:

$$\begin{aligned} \Delta \circ \mathcal{H} \circ Q_{\mathcal{E}'} &= (\mathcal{H} \otimes \Phi + \Phi \otimes \mathcal{H}) \circ \Delta' \circ Q_{\mathcal{E}'}, \text{ by definition of } \mathcal{H} \\ &= (\mathcal{H} \otimes \Phi + \Phi \otimes \mathcal{H}) \circ (Q_{\mathcal{E}'} \otimes \text{id} + \text{id} \otimes Q_{\mathcal{E}'}) \circ \Delta', \text{ by definition of } Q_{\mathcal{E}'} \\ &= (\mathcal{H} \circ Q_{\mathcal{E}'} \otimes \Phi + (-1)^k \Phi \circ Q_{\mathcal{E}'} \otimes \mathcal{H} + \mathcal{H} \otimes \Phi \circ Q_{\mathcal{E}'} + \Phi \otimes \mathcal{H} \circ Q_{\mathcal{E}'}) \circ \Delta' \end{aligned}$$

Subtracting a similar equation for $(-1)^k \Delta \circ Q_{\mathcal{E}} \circ \mathcal{H}$ and using (2.6), one obtains the Φ -co-derivation property for $\mathcal{H} \circ Q_{\mathcal{E}'} - (-1)^k Q_{\mathcal{E}'} \circ \mathcal{H}$. We now check that $\mathcal{H} \circ Q_{\mathcal{E}'} - (-1)^k Q_{\mathcal{E}} \circ \mathcal{H}$ is \mathcal{O} -multilinear, for which it suffices to check that its Taylor coefficients are \mathcal{O} -multilinear by Lemma 1.2.17. Let $x_1, \dots, x_n \in \mathcal{E}'$ be homogeneous elements. Assume $x_i \in \mathcal{E}'_{-1}$ (if we have more elements of degree -1 the same reasoning holds). To verify \mathcal{O} -multilinearity it suffices to check that for all $f \in \mathcal{O}$:

$$\begin{aligned} \text{pr} \circ (\mathcal{H} \circ Q_{\mathcal{E}'} - (-1)^k Q_{\mathcal{E}} \circ \mathcal{H})(x_1, \dots, x_i, \dots, f x_j \dots, x_n) &= \\ f \text{pr} \circ (\mathcal{H} \circ Q_{\mathcal{E}'} - (-1)^k Q_{\mathcal{E}} \circ \mathcal{H})(x_1, \dots, x_i, \dots, x_j \dots, x_n). \end{aligned}$$

Only the terms where the 2-ary bracket with a degree -1 element on one-side and f on the other side may forbid f to go in front. There are two such terms:

$$\epsilon(x_i, x_j, x_{I'ij}) \mathcal{H}^{(n-1)}(\ell'_2(x_i, f x_j), x_{I'ij}) \text{ and } -(-1)^k \epsilon(x_i, x_{I'i}) \ell_2(\Phi_0(x_i), f \mathcal{H}^{(n-1)}(x_{I'i}))$$

where x_{I^i} and $x_{I^{ij}}$ stand for the list x_1, \dots, x_n where x_i and x_i, x_j are missing, respectively, and $\mathcal{H}^{(n-1)}$ is the $(n-1)$ -th Taylor coefficient of \mathcal{H} . Since $\rho \circ \Phi_0 = \rho'$, in both terms $\rho(x_i)[f]$ appears, and these two terms add up to zero. \square

Remark 2.3.25. If the degree of \mathcal{H} is non-negative, then $\mathcal{H} \circ Q_{\mathcal{E}'} - (-1)^k Q_{\mathcal{E}} \circ \mathcal{H}$ may not be \mathcal{O} -multilinear anymore, since there may exist extra terms where the anchor map appears, e.g. terms of the form $\ell_2(\mathcal{H}(x_{I^j}), f\Phi^{(0)}(x_j))$.

Lemma 2.3.26. *Let $\Phi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ be a co-algebra morphism such that*

1. Φ is \mathcal{O} -multilinear,
2. $\rho \circ \Phi_0 = \rho'$ on \mathcal{E}' ,

If for every $n \in \mathbb{N}_0$, $(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(i)} = 0$ for each $0 \leq i \leq n$, then the map $S_{\mathbb{K}}(\mathcal{E}') \rightarrow S_{\mathbb{K}}(\mathcal{E})$ given by:

$$(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)}$$

1. is a $\Phi^{(0)}$ -co-derivation of degree +1,
2. is \mathcal{O} -multilinear,
3. and the induced $\Phi^{(0)}$ -co-derivation $(\odot \bullet \mathcal{E}', Q_{\mathcal{E}'}^{(0)}) \rightarrow (\odot \bullet \mathcal{E}, Q_{\mathcal{E}}^{(0)})$ satisfies:

$$Q_{\mathcal{E}}^{(0)} \circ (\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)} = (Q_{\mathcal{E}} \circ \Phi - \Phi \circ Q_{\mathcal{E}'})^{(n+1)} \circ Q_{\mathcal{E}'}^{(0)}.$$

Proof. A straightforward computation yields:

$$\begin{aligned} \Delta(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi) &= (\Phi \otimes \Phi) \circ \Delta' \circ Q_{\mathcal{E}'} - (Q_{\mathcal{E}} \otimes \text{id} + \text{id} \otimes Q_{\mathcal{E}}) \circ \Delta \circ \Phi \\ &= ((\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi) \otimes \Phi + \Phi \otimes (\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)) \circ \Delta'. \end{aligned}$$

Now, Δ preserves polynomial-degree, i.e. $\Delta: S_{\mathbb{K}}^n(\mathcal{E}) \rightarrow \bigoplus_{i+j=n} S_{\mathbb{K}}^i(\mathcal{E}) \otimes S_{\mathbb{K}}^j(\mathcal{E})$ and so does Δ' . Taking into account the assumption $(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(i)} = 0$ for every $0 \leq i \leq n$, we obtain:

$$\begin{aligned} &\Delta \circ (\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)} \\ &= \left((\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)} \otimes \Phi^{(0)} + \Phi^{(0)} \otimes (\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)} \right) \circ \Delta'. \end{aligned}$$

All the other terms disappear for polynomial-degree reasons. Hence, $(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)}$ is a $\Phi^{(0)}$ -co-derivation.

Let us prove that it is \mathcal{O} -linear. It suffices to check \mathcal{O} -linearity of $T_{\Phi} := \Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi$. Let us choose homogeneous elements $x_1, \dots, x_N \in \mathcal{E}'$ and let us assume that $x_i \in \mathcal{E}'_{-1}$ is the only term of degree -1 : The proof in the case where there is more than one such an homogeneous element of degree -1 is identical. We choose $j \neq i$ and we compute $T_{\Phi}(x_1, \dots, x_i, \dots, fx_j, \dots, x_N)$ for some $f \in \mathcal{O}$. The only terms in the previous expression which are maybe non-linear in f are those for which the 2-ary brackets of a term containing fx_j with x_i or $\Phi_0(x_i)$ appear (since Φ and all other brackets

³ $\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi$ being a Φ -co-derivation, its component of polynomial-degree i is zero for $0 \leq i \leq n$ if only if its i -th Taylor coefficient is zero for $0 \leq i \leq n$.

are \mathcal{O} -linear). There are two such terms. The first one appears when we apply $Q_{\mathcal{E}'}$ first, and then Φ : this forces $\Phi(\ell'_2(x_i, fx_j), x_{I^ij})$ to appear, and the non-linear term is then:

$$\epsilon(\sigma_i)\rho'(x_i)[f]\Phi(x_{I^i}) \quad (2.17)$$

with σ_i the permutation that let i goes in front and leave the remaining terms unchanged. There is a second term that appears when one applies Φ first, then $Q_{\mathcal{E}}$. Since it is a co-morphism, $\Phi(x_1 \dots x_i \dots, fx_j \dots x_N)$ is the product of several terms among which only one is of degree -1 , namely the term

$$\epsilon(x, \sigma_i)\Phi_0(x_i)\Phi(fx_{I^i}).$$

Applying $Q_{\mathcal{E}}$ to this term yields the non-linear term

$$\epsilon(\sigma_i)\rho(\Phi_0(x_i))[f]\Phi(x_{I^i}), \quad (2.18)$$

where I^i and I^{ij} are as in Proposition 2.3.24. Since $\rho \circ \Phi_0 = \rho'$, we see that the terms (2.17) and (2.18) containing an anchor add up to zero.

Let us check that $(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)}$ is a chain map, in the sense that it satisfies item 3). Considering again $T_{\Phi} := \Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi$, we have that $T_{\Phi}^{(k)} = 0$, for all $k = 0, \dots, n$. Since $T_{\Phi} \circ Q_{\mathcal{E}'} = Q_{\mathcal{E}} \circ T_{\Phi}$, one has

$$0 = (T_{\Phi} \circ Q_{\mathcal{E}'} + Q_{\mathcal{E}} \circ T_{\Phi})^{(n+1)} = T_{\Phi}^{(n+1)} \circ Q_{\mathcal{E}'}^{(0)} + Q_{\mathcal{E}}^{(0)} \circ T_{\Phi}^{(n+1)} + \sum_{\substack{i+j=n+1 \\ i,j \geq 1}} \underbrace{\left(T_{\Phi}^{(i)} \circ Q_{\mathcal{E}'}^{(j)} + Q_{\mathcal{E}}^{(j)} \circ T_{\Phi}^{(i)} \right)}_0$$

By consequent, the \mathcal{O} -linear map $(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)}$ satisfies item 3). \square

Lemma 2.3.27. *Let $\Phi, \Xi : S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ be \mathcal{O} -linear Lie ∞ -algebroid morphisms. Let $n \in \mathbb{N}_0$. If $\Xi^{(i)} = \Phi^{(i)}$ for every $0 \leq i \leq n$, then $(\Xi - \Phi)^{(n+1)} : S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$*

1. is a $\Phi^{(0)}$ -co-derivation

2. is \mathcal{O} -multilinear

3. and the induced $\Phi^{(0)}$ -co-derivation $\left(\odot \bullet \mathcal{E}', Q_{\mathcal{E}'}^{(0)} \right) \rightarrow \left(\odot \bullet \mathcal{E}, Q_{\mathcal{E}}^{(0)} \right)$ satisfies:

$$Q_{\mathcal{E}}^{(0)} \circ (\Xi - \Phi)^{(n+1)} = (\Xi - \Phi)^{(n+1)} \circ Q_{\mathcal{E}'}^{(0)}.$$

Proof. For all $x_1, \dots, x_k \in \mathcal{E}'$, one has:

$$\begin{aligned} \Delta(\Xi - \Phi)(x_1 \odot \dots \odot x_k) &= \sum_{j=1}^k \sum_{\sigma \in \mathfrak{S}_{(j, k-j)}} \epsilon(\sigma) \Xi(x_{\sigma(1)} \odot \dots \odot x_{\sigma(j)}) \otimes \Xi(x_{\sigma(j+1)} \odot \dots \odot x_{\sigma(k)}) \\ &\quad - \sum_{j=1}^k \sum_{\sigma \in \mathfrak{S}_{(j, k-j)}} \epsilon(\sigma) (\Phi(x_{\sigma(1)} \odot \dots \odot x_{\sigma(j)}) \otimes \Phi(x_{\sigma(j+1)} \odot \dots \odot x_{\sigma(k)})) \\ &= ((\Xi - \Phi) \otimes \Phi + \Xi \otimes (\Xi - \Phi)) \circ \Delta'(x_1 \odot \dots \odot x_k). \end{aligned}$$

Since Δ has polynomial-degree 0 and $(\Xi - \Phi)^{(i)} = 0$ for all $0 \leq i \leq n$, we obtain

$$\Delta(\Xi - \Phi)^{(n+1)}(x_1 \odot \dots \odot x_k) = \left((\Xi - \Phi)^{(n+1)} \otimes \Phi^{(0)} + \Phi^{(0)} \otimes (\Xi - \Phi)^{(n+1)} \right) \circ \Delta'(x_1 \odot \dots \odot x_k). \quad (2.19)$$

This proves the first item. Since both Φ and Ξ are \mathcal{O} -multilinear, $(\Xi - \Phi)^{(n+1)}$ is \mathcal{O} -multilinear. This proves the second item. Since, Φ and Ξ are Lie ∞ -morphisms:

$$(\Xi - \Phi) \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ (\Xi - \Phi) = 0. \quad (2.20)$$

By looking at the component of polynomial-degree $n + 1$, one obtains, $(\Xi - \Phi)^{(n+1)} \circ Q_{\mathcal{E}'}^{(0)} - Q_{\mathcal{E}}^{(0)} \circ (\Xi - \Phi)^{(n+1)} = 0$. This proves the third item. \square

Conclusion:

This chapter recapitulates classical definition of Lie ∞ -algebroids, and in particular describes it as co-derivation, which is not usual. Morphisms and homotopies are described. About homotopies, two versions are given: one uses smooth maps, and is way easier (see Definition 2.3.8). Unfortunately, to allow infinitely many gluings, we have to introduce a more complicated notion of homotopy, which uses continuous locally C^1 -maps (see Definition 2.3.14).

Lie-Rinehart algebras and their morphisms

Except for Remark 3.2.1, this section is essentially a review of the literature on the subject, see, e.g. [Hue04, Hue98].

3.1 Definitions

Lie-Rinehart algebras are the algebraic encoding of the notion of Lie algebroids over a manifold. We owe this concept to [Rin63].

Definition 3.1.1. A *Lie-Rinehart algebra* over an algebra \mathcal{O} is a triple $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ with \mathcal{A} an \mathcal{O} -module, $[\cdot, \cdot]_{\mathcal{A}}$ a Lie algebra bracket on \mathcal{A} , and $\rho_{\mathcal{A}}: \mathcal{A} \rightarrow \text{Der}(\mathcal{O})$ an \mathcal{O} -linear Lie algebra morphism called *anchor map*, satisfying the so-called *Leibniz identity*:

$$[a, fb]_{\mathcal{A}} = \rho_{\mathcal{A}}(a)[f]b + f[a, b]_{\mathcal{A}} \text{ for all } a, b \in \mathcal{A}, f \in \mathcal{O}.$$

1. Let $\eta: \mathcal{O} \rightarrow \mathcal{O}'$ be an algebra morphism. A *Lie-Rinehart algebra morphism over η* is a Lie algebra morphism $\phi: \mathcal{A} \rightarrow \mathcal{A}'$ such that for every $a \in \mathcal{A}$ and $f \in \mathcal{O}$:

- (a) $\phi(fa) = \eta(f)\phi(a)$
- (b) $\eta(\rho_{\mathcal{A}}(a)[f]) = \rho_{\mathcal{A}'}(\phi(a)[\eta(f)])$,

When $\mathcal{O} = \mathcal{O}'$ and $\eta = \text{id}$, we say that ϕ is a *Lie-Rinehart algebra morphism over \mathcal{O}* .

2. A submodule $\mathcal{B} \subseteq \mathcal{A}$ is said to be a *Lie-Rinehart subalgebra of \mathcal{A}* if it carries a Lie-Rinehart algebra structure over \mathcal{O} whose Lie bracket and anchor map are the restriction of the bracket $[\cdot, \cdot]_{\mathcal{A}}$ and the anchor $\rho_{\mathcal{A}}$ respectively to \mathcal{B} .

Lie-Rinehart algebras over \mathcal{O} form a category that we denote by **Lie-Rhart- alg/\mathcal{O}**

Remark 3.1.2. A Lie-Rinehart algebra is said to be a *Lie algebroid* if \mathcal{A} is a projective \mathcal{O} -module. See Example 3.2.2 for the relation with usual Lie algebroid as vector bundles.

Remark 3.1.3. For every Lie ∞ -algebroid $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ over \mathcal{O} , the quotient space $\frac{\mathcal{E}_{-1}}{\ell_1(\mathcal{E}_{-2})}$ comes equipped with a natural Lie-Rinehart algebra over \mathcal{O} . The 2-ary bracket ℓ_2 goes to quotient to $\frac{\mathcal{E}_{-1}}{\ell_1(\mathcal{E}_{-2})}$ to define a Lie algebra since for all $x \in \mathcal{E}_{-2}$ and $y \in \mathcal{E}_{-1}$ we have

$$\ell_2(\ell_1(x), y) = \ell_1(\ell_2(x, y)).$$

Also, the Jacobi identity holds, since

$$\ell_2(\ell_2(x, y), y) + \mathcal{O}(x, y, z) = -\ell_1(\ell_3(x, y, z)) \quad \forall x, y, z \in \mathcal{E}_{-1}.$$

In addition, the anchor map ρ goes to quotient to a Lie algebra morphism $\frac{\mathcal{E}_{-1}}{\ell_1(\mathcal{E}_{-2})} \rightarrow \text{Der}(\mathcal{O})$, since $\rho \circ \ell_1 = 0$. This Lie-Rinehart algebra is called the *basic* Lie-Rinehart algebra of $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$.

To summarize, every Lie ∞ -algebroid induces a Lie-Rinehart algebra. The opposite direction, that is, wondering whether a Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$ over \mathcal{O} is the basic Lie-Rinehart algebra of some almost differential graded Lie algebroid or more generally a Lie ∞ -algebroid over \mathcal{O} is part of the questions that I discussed in this thesis (see Chapter 4). This extends the main results of [LLS20] from locally real analytic finitely generated singular foliations (see Example 3.2.4) to arbitrary Lie-Rinehart algebras.

Let us fix some vocabulary that will be used in the sequel.

Definition 3.1.4. We say that an almost differential graded Lie algebroid $(\mathcal{E}_\bullet, \ell_1, \ell_2, \rho)$ or a Lie ∞ -algebroid $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ over \mathcal{O}

1. *covers (through a hook π)* a Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$, if there exists a morphism of brackets $\pi: \mathcal{E}_{-1} \rightarrow \mathcal{A}$ such that $\pi \circ \rho_{\mathcal{A}} = \rho$ and $\pi(\mathcal{E}_{-1}) = \mathcal{A}$,
2. *terminates in \mathcal{A} through the hook π* , when $\pi(\mathcal{E}_{-1}) \subseteq \mathcal{A}$.

According to Remark 3.1.3, any Lie ∞ -algebroid $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ covers its basic Lie-Rinehart algebra through the hook π which is given by the projection $\pi: \mathcal{E}_{-1} \rightarrow \mathcal{E}_{-1}/\ell_1(\mathcal{E}_{-2})$, since π respects the brackets and $\pi(\mathcal{E}_{-1}) = \mathcal{E}_{-1}/\ell_1(\mathcal{E}_{-2})$.

Definition 3.1.5. Let $(\mathcal{E}'_\bullet, \ell'_\bullet, \rho')$ and $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ be Lie ∞ -algebroids over \mathcal{O} that terminate in a Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ through the hooks π' and π respectively. We say that a Lie ∞ -algebroid morphism $\Phi: S_{\mathbb{K}}^\bullet(\mathcal{E}') \rightarrow S_{\mathbb{K}}^\bullet(\mathcal{E})$ is *over \mathcal{A}* , if $\pi \circ \Phi_0 = \pi'$.

We will need the following lemma.

Lemma 3.1.6. *Let $(\mathcal{E}'_\bullet, \ell'_\bullet, \rho')$ and $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ be Lie ∞ -Lie algebroids that terminate in some Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ through hooks π' and π . Let $(J_t, H_t)_{t \in [a, b]}$ be a homotopy that joins J_a and J_b . If J_a is Lie ∞ -algebroid morphism that terminates at \mathcal{A} (i.e., $\pi \circ J_a^{(0)}|_{\mathcal{E}'} = \pi'$), then so is the ∞ -algebroid morphism J_t for all $t \in [a, b]$.*

Proof. This is a direct consequence of Equation (2.16), since $Q_{\mathcal{E}}^{(0)}|_{\mathcal{E}} = \ell_1: \mathcal{E}_{-2} \rightarrow \mathcal{E}_{-1}$ and $Q_{\mathcal{E}'}^{(0)}|_{\mathcal{E}'} = \ell'_1: \mathcal{E}'_{-2} \rightarrow \mathcal{E}'_{-1}$ are valued in the kernels of π and π' respectively.

Last, \mathcal{O} -multilinearity of J_t follows from the \mathcal{O} -multilinearity of $Q_{\mathcal{E}} \circ H_t + H_t \circ Q_{\mathcal{E}'}$, which is granted by Proposition 2.3.24. This completes the proof. \square

3.2 Algebraic and geometric examples

Example 3.2.1. For every commutative \mathbb{K} -algebra \mathcal{O} , the Lie algebra $\mathcal{A} = \text{Der}(\mathcal{O})$ of derivations of a commutative algebra \mathcal{O} is a Lie-Rinehart algebra over \mathcal{O} , with the identity as an anchor map. Notice that for every Lie-Rinehart $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ the anchor $\mathcal{A} \xrightarrow{\rho_{\mathcal{A}}} \text{Der}(\mathcal{O})$ is obviously a Lie-Rinehart algebra morphism over \mathcal{O} .

In particular, vector fields on a smooth or Stein manifold or an affine variety are examples of Lie-Rinehart algebras over their respective natural algebras of functions.

Example 3.2.2. Let M be a smooth manifold. By Serre-Swan Theorem, Lie algebroids over M are precisely Lie-Rinehart algebras over $C^\infty(M)$ of the form $(\Gamma(A), [\cdot, \cdot], \rho)$ where A is a vector bundle over M and $\rho : A \rightarrow TM$ is a vector bundle morphism.

Example 3.2.3. Sections of a Lie algebroid that have values in the kernel of the anchor map form a Lie-Rinehart algebra $\text{Ker}(\rho_A)$ for which the anchor map is zero.

The next two examples are part of our the main source of motivation to study Lie-Rinehart algebras. The second is more general than the first one.

Example 3.2.4. *Singular foliations.* [AS09, AZ14, Cer79, Deb01, LLS20, LGLR22] A *singular foliation* on a smooth, real analytic, or complex manifold M or Zariski open subset $\mathcal{U} \subseteq \mathbb{C}^d$ is a subsheaf $\mathfrak{F} \subseteq \mathfrak{X}(M)$ that fulfills the following conditions

- **Stability under Lie bracket:** $[\mathfrak{F}, \mathfrak{F}] \subseteq \mathfrak{F}$.
- **Locally finitely generateness:** every $m \in M$ admits an open neighborhood \mathcal{U} together with a finite number of vector fields $X_1, \dots, X_r \in \mathfrak{X}(\mathcal{U})$ such that for every open subset $\mathcal{V} \subseteq \mathcal{U}$ the vector fields $X_1|_{\mathcal{V}}, \dots, X_r|_{\mathcal{V}}$ generates \mathfrak{F} on \mathcal{V} as a $C^\infty(\mathcal{V})$ -module.

There are several other ways to define singular foliations on a manifold M [AZ13, Cer79, Daz85, Deb01]. All these definitions have in common to define then as Lie-Rinehart sub-algebras \mathfrak{F} of the Lie-Rinehart algebra $\mathfrak{X}(M)$ of vector fields on M (or compactly supported vector fields $\mathfrak{X}_c(M)$ on M). Here are some important consequences of the above definition.

1. Singular foliation admits leaves: there exists a partition of M into submanifolds called leaves such that for all $m \in M$, the image of the evaluation map $\mathfrak{F} \rightarrow T_m M$ is the tangent space of the leaf through m . When \mathfrak{F} coincides with the space of vector fields tangent to all leaves at all points, we shall speak of a “*Stefan-Sussman singular foliation*”.
2. *Singular foliations are self-preserving* : the flow of vector fields in \mathfrak{F} , whenever defined, preserves \mathfrak{F} [AS09, GY18].

Notice that in Remark 2.1.13, $\mathfrak{F} := \rho(\Gamma(E_{-1}))$ is a singular foliation in the sense above, called the *basic singular foliation* of $(E, (\ell_k)_{k \geq 1}, \rho)$. We say, then the Lie ∞ -algebroid $(E, (\ell_k)_{k \geq 1}, \rho)$ is *over* \mathfrak{F} .

Example 3.2.5. [AZ18, Zam18, ZA18] Let $(A, [\cdot, \cdot]_A, \rho_A)$ be a Lie algebroid over a manifold M . A *singular subalgebroid* \mathcal{B} of A , is a $C^\infty(M)$ -submodule of $\Gamma(A)$ that is locally finitely generated and involutive i.e. stable under the Lie bracket $[\cdot, \cdot]_A$. This notion is a generalization of singular foliations in the sense of Example 3.2.4, since singular foliations on M are singular subalgebroids of TM .

A singular subalgebroid \mathcal{B} is an example of Lie-Rinehart algebra: its bracket and anchor are the restrictions of $[\cdot, \cdot]_A$ and ρ_A to \mathcal{B} respectively.

Example 3.2.6. For a singular foliation \mathfrak{F} on a manifold M , consider $\mathcal{S} := \{X \in \mathfrak{X}(M) \mid [X, \mathfrak{F}] \subseteq \mathfrak{F}\}$ (i.e. infinitesimal symmetries of \mathfrak{F}) and

$$\mathcal{C} := \{f \in C^\infty(M) \mid Y[f] = 0, \text{ for all } Y \in \mathfrak{F}\}$$

(that can be thought of as functions constant along the leaves of \mathfrak{F}). The quotient $\frac{\mathcal{S}}{\mathfrak{F}}$ is Lie-Rinehart algebra over \mathcal{C} .

Example 3.2.7 (*Poisson manifold*). [LGPV13] We recall that a *Poisson manifold* is a manifold M together with a biderivation $\{\cdot, \cdot\}$ on its algebra of smooth functions $C^\infty(M)$ that satisfies Jacobi's identity, i.e.

1. $(C^\infty(M), \{\cdot, \cdot\})$ is Lie algebra.
2. The biderivation $\{\cdot, \cdot\}$ is compatible with the product of functions in the following sense: for all $f, g, h \in C^\infty(M)$

$$\{f \cdot g, h\} = f \cdot \{g, h\} + g \cdot \{f, h\}.$$

This is equivalent to giving a bivector field $\pi \in \Gamma(\wedge^2 TM)$ such that the Schouten-Nijenhuis bracket with itself vanishes, that is, $[\pi, \pi]_{SN} = 0$.

For every a Poisson manifold (M, π) , the operator $\delta_\pi: \Gamma(\wedge^\bullet TM) \rightarrow \Gamma(\wedge^{\bullet+1} TM), P \mapsto -[P, \pi]_{SN}$ defines a complex

$$\dots \longrightarrow \Gamma(\wedge^{p-1} TM) \xrightarrow{\delta_\pi^{p-1}} \Gamma(\wedge^p TM) \xrightarrow{\delta_\pi^p} \Gamma(\wedge^{p+1} TM) \longrightarrow \dots,$$

since $[\pi, \pi]_{SN} = 0$ implies $\delta_\pi \circ \delta_\pi = 0$. For every $p \in \mathbb{N}_0$, the quotient $H_\pi^p(M) := \frac{\ker \delta_\pi^p}{\text{Im} \delta_\pi^{p-1}}$ is called the p -th *Poisson cohomology of (M, π)* . We define $\mathcal{A} := H_\pi^1(M)$ to be the first Poisson cohomology of π and $\mathcal{O} := H_\pi^0(M) = \text{Cas}(\pi)$ to be the algebra of Casimir functions. The bracket of vector fields makes \mathcal{A} a Lie-Rinehart algebra over $\text{Cas}(\pi)$.

3.2.1 Basic constructions

Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ a Lie-Rinehart algebra over an algebra \mathcal{O} and $\mathcal{I} \subset \mathcal{O}$ be an ideal. The submodule $\mathcal{I}\mathcal{A}$ is a Lie-Rinehart subalgebra of \mathcal{A} and its anchor is given by the restriction of $\rho_{\mathcal{A}}$ over $\mathcal{I}\mathcal{A} \subset \mathcal{A}$. This follows easily from

$$[fa, gb]_{\mathcal{A}} = fg[a, b]_{\mathcal{A}} + f\rho_{\mathcal{A}}(a)[g]b - g\rho_{\mathcal{A}}(b)[f]a \text{ for all } a, b \in \mathcal{A}, f, g \in \mathcal{I}.$$

1. *Restriction.* Consider a Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ over \mathcal{O} . For every *Lie-Rinehart ideal* $\mathcal{I} \subset \mathcal{O}$, i.e. any ideal such that

$$\rho_{\mathcal{A}}(\mathcal{A})[\mathcal{I}] \subset \mathcal{I}$$

the quotient space $\mathcal{A}/\mathcal{I}\mathcal{A}$ inherits a natural Lie-Rinehart algebra structure over \mathcal{O}/\mathcal{I} . We call this Lie-Rinehart algebra the *restriction w.r.t the Lie-Rinehart ideal \mathcal{I}* . In the context of affine varieties, when \mathcal{I} is the ideal of functions vanishing on an affine sub-variety W , we shall denote $\frac{\mathcal{A}}{\mathcal{I}\mathcal{A}}$ by $\mathfrak{i}_W^* \mathcal{A}$.

2. *Localization.* Localizing w.r.t a multiplicative subset $S \subset \mathcal{O}$ (i.e. $S \subseteq \mathcal{O} \setminus \{0\}$ is closed under multiplication and $1 \in S$) is a very powerful tool in commutative algebra and in algebraic geometry to deal with "global" problems by reducing them to "local" ones. Let us recall the construction (e.g. see [Sta22], Section 10.9 or [And] for details).

(a) Let \mathcal{V} be a \mathcal{O} -module. The *localization of \mathcal{V} at S* is defined as follows: consider the equivalence relation on $S \times \mathcal{V}$ that is given by

$$(s, v) \sim (s', v') \iff \exists u \in S, u(s'v - sv') = 0.$$

The class of an element $(s, v) \in S \times \mathcal{V}$ by $\frac{v}{s}$, and by $S^{-1}\mathcal{V} := S \times \mathcal{V} / \sim$ the set of equivalence classes.

In particular, the *localization of \mathcal{O} at S* is the localization of \mathcal{O} as a \mathcal{O} -module. In that case, $S^{-1}\mathcal{O}$ is a \mathbb{K} -algebra together with the addition and multiplication defined by

$$\frac{f}{s} + \frac{g}{s'} := \frac{fs' + sg}{ss'} \quad \text{and} \quad \frac{f}{s} \frac{g}{s'} := \frac{fg}{ss'}.$$

Similarly, $S^{-1}\mathcal{V}$ is a $S^{-1}\mathcal{O}$ -module with addition and scalar product multiplication defined in an obvious way. Also, we have $S^{-1}\mathcal{V} \simeq S^{-1}\mathcal{O} \otimes_{\mathcal{O}} \mathcal{V}$.

(b) A derivation $D \in \text{Der}(\mathcal{O})$ admits a unique extension to a derivation $S^{-1}D \in \text{Der}(S^{-1}\mathcal{O})$ which given for $(s, f) \in S \times \mathcal{O}$ by the classical formula

$$(S^{-1}D) \left(\frac{f}{s} \right) := \frac{D(f)s - D(s)f}{s^2}.$$

Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho)$ be a Lie-Rinehart algebra over \mathcal{O} . The localization module $S^{-1}\mathcal{A} = S^{-1}\mathcal{O} \otimes_{\mathcal{O}} \mathcal{A}$ comes equipped with a natural structure of Lie-Rinehart algebra over the localization algebra $S^{-1}\mathcal{O}$. The new anchor map is defined by

$$\frac{a}{s} \in S^{-1}\mathcal{A} \mapsto \frac{1}{s} S^{-1}(\rho_{\mathcal{A}}(a)) \in \text{Der}(S^{-1}\mathcal{O}),$$

and the new Lie algebra bracket is given by

$$\left[\frac{1}{s}a, \frac{1}{u}b \right]_{S^{-1}\mathcal{A}} = \frac{1}{su} [a, b]_{\mathcal{A}} - \frac{\rho_{\mathcal{A}}(a)[u]}{su^2} b + \frac{\rho_{\mathcal{A}}(b)[s]}{s^2u} a$$

for $a, b \in \mathcal{A}, (s, u) \in S^2$. The localization map $\mathcal{A} \hookrightarrow S^{-1}\mathcal{A}$ is a Lie-Rinehart algebra morphism over the localization map $\mathcal{O} \hookrightarrow S^{-1}\mathcal{O}$. Since localization exists, the notion of sheaf of Lie-Rinehart algebras [Vil20] over a projective variety, or a scheme, makes sense.

3. *Algebra extension.* Assume that the algebra \mathcal{O} has no zero divisor, and let \mathbb{O} be its field of fraction. For any subalgebra $\tilde{\mathcal{O}}$ with $\mathcal{O} \subset \tilde{\mathcal{O}} \subset \mathbb{O}$ such that $\rho(a)$ is for any $a \in \mathcal{A}$ valued in derivations of \mathbb{O} that preserves $\tilde{\mathcal{O}}$, there is a natural Lie-Rinehart algebra structure over $\tilde{\mathcal{O}}$ on the space $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{A}$.
4. *Blow-up at the origin.* Let us consider a particular case of the previous construction, when \mathcal{O} is the algebra $\mathbb{C}[x_1, \dots, x_N]$. If the anchor map of a Lie-Rinehart algebra \mathcal{A} over \mathcal{O} takes values in

vector fields on \mathbb{C}^N vanishing at the origin, then for all $i = 1, \dots, N$, the polynomial algebra \mathcal{O}_{U_i} generated by $\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_N}{x_i}$ satisfies the previous condition, and $\mathcal{O}_{U_i} \otimes_{\mathcal{O}} \mathcal{A}$ comes equipped with a Lie-Rinehart algebra. Geometrically, this operation corresponds to taking the blow-up of \mathbb{C}^N at the origin, then looking at the i -th natural chart U_i on this blow-up: \mathcal{O}_{U_i} are the polynomial functions on U_i . The family $\mathcal{O}_{U_i} \otimes_{\mathcal{O}} \mathcal{A}$ (for $i = 1, \dots, N$) is therefore an atlas for a sheaf of Lie-Rinehart algebras (in the sense of [Vil20]) on the blow-up of \mathbb{C}^N at the origin, referred to as the *blow-up of \mathcal{A} at the origin*.

3.2.2 On free resolutions of length ≤ 2 and Lie-Rinehart algebras

In this section, we discuss the case when a Lie-Rinehart algebra \mathcal{A} admits a free resolution of length 1 and 2. In those cases, we claim that there are Lie algebra-like structures on them. See B.1 for the notion of free resolution of modules. These results will be soon generalized.

I start with the following remark (owed to Marco Zambon).

Remark 3.2.8. Any Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ is the image of an almost differential graded Lie algebroid (\mathcal{E}_{-1}, ρ) concentrated in degree -1 : to see this, let $\{a_i \in \mathcal{A} \mid i \in I\}$ be a set of generators of \mathcal{A} (take all elements of \mathcal{A} if necessary). There exists elements $u_{ij}^k \in \mathcal{O}$, such that for given indices i, j , the coefficient u_{ij}^k is zero except for finitely many indices k , together with

$$[a_i, a_j]_{\mathcal{A}} = \sum_{k \in I} u_{ij}^k a_k \quad \forall i, j \in I. \quad (3.1)$$

The coefficients u_{ij}^k can be chosen to satisfy the skew-symmetry condition $u_{ij}^k = -u_{ji}^k$: by skew-symmetry of the bracket $[\cdot, \cdot]_{\mathcal{A}}$ on has

$$\begin{aligned} [a_i, a_j]_{\mathcal{A}} &= \frac{1}{2} ([a_i, a_j]_{\mathcal{A}} - [a_j, a_i]_{\mathcal{A}}) \\ &= \sum_{k \in I} \frac{1}{2} (u_{ij}^k - u_{ji}^k) a_k. \end{aligned}$$

Thus, one can replace u_{ij}^k by $\frac{1}{2}(u_{ij}^k - u_{ji}^k)$ if necessary.

Now, choose \mathcal{E}_{-1} to be the free \mathcal{O} -module generated by the symbols $(e_i)_{i \in I}$ together with the surjective map

$$\pi: \mathcal{E}_{-1} \rightarrow \mathcal{A}, \quad e_i \mapsto a_i.$$

We now define:

1. an anchor map by $\rho(e_i) := \rho_{\mathcal{A}}(a_i)$, for all $i \in I$, i.e. $\rho = \rho_{\mathcal{A}} \circ \pi$,
2. a skew-symmetric operation $[\cdot, \cdot]_{\mathcal{E}_{-1}}$ on \mathcal{E} as follows:

$$[e_i, e_j]_{\mathcal{E}_{-1}} = \sum_{k \in I} u_{ij}^k e_k \quad \text{for all } i, j \in I.$$

We extend by \mathcal{O} -linearity and Leibniz identity. By construction $(\mathcal{E}_{-1}, [\cdot, \cdot]_{\mathcal{E}_{-1}}, \rho)$ is an almost Lie algebroid over \mathcal{O} whose image through the anchor map is \mathcal{A} .

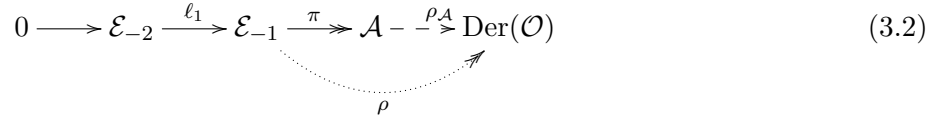
Remark 3.2.9. In general, this bracket does not satisfy the Jacobi identity. If $\mathcal{E}_{-1} \simeq \mathcal{A}$, this bracket is a Lie algebroid.

On free resolutions of length 1

Lie-Rinehart algebra \mathcal{A} admits a free resolution of length 1 if and only if \mathcal{A} is free. In that case, the almost Lie algebroid bracket $[\cdot, \cdot]_{\mathcal{E}_{-1}}$ is a Lie algebroid bracket. In conclusion: free resolutions of length 1 admit a Lie algebroid structure.

Free resolutions of length 2

Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra that admits a free resolution of length 2, namely an exact sequence of the form

$$0 \longrightarrow \mathcal{E}_{-2} \xrightarrow{\ell_1} \mathcal{E}_{-1} \xrightarrow{\pi} \mathcal{A} \xrightarrow{-\rho_{\mathcal{A}}} \text{Der}(\mathcal{O}) \quad (3.2)$$


with $\mathcal{E}_{-1}, \mathcal{E}_{-2}$ free modules.

Since Equation (3.2) is a free resolution of \mathcal{A} , the map π is in particular surjective. By Remark 3.2.8, \mathcal{E}_{-1} can be endowed with an almost Lie algebroid bracket, such that π is a morphism of brackets. We can extend this bracket to sections of degree -2 to obtain an almost differential graded Lie algebroid. Let us compute it:

Let $(e'_i)_{i \in I'}$ and $(e_j)_{j \in I}$ be a basis of \mathcal{E}_{-2} and \mathcal{E}_{-1} , respectively. For all $i, j, k \in I \cup I'$ we have

1.

$$\pi([\ell_1 e'_i, e_j]_{\mathcal{E}_{-1}}) = [\pi \circ \ell_1(e'_i), \pi(e_j)] = 0, \text{ (since } \pi \circ \ell_1 \equiv 0\text{)}.$$

In other words, $[\ell_1(e'_i), e_j]_{\mathcal{E}_{-1}} \in \ker \pi$. By exactness of the complex (3.2) there exists an element $\nabla_{e'_i} e_j \in \mathcal{E}_{-2}$ such that

$$\pi(\nabla_{e'_i} e_j) = [\ell_1(e'_i), e_j]_{\mathcal{E}_{-1}}. \quad (3.3)$$

Equation (3.3) allows defining a bilinear map:

$$\begin{aligned} \mathcal{E}_{-1} \otimes \mathcal{E}_{-2} &\rightarrow \mathcal{E}_{-2} \\ (x, y) &\mapsto \nabla_x y \end{aligned}$$

by extending the $\nabla_{e'_i} e_j$'s by linearity and Leibniz identity with the understanding that the anchor map ρ vanishes on \mathcal{E}_{-2} in order to have

$$(a) \quad \ell_1(\nabla_x y) = [\ell_1(x), y]_{\mathcal{E}_{-1}}, \forall x \in \mathcal{E}_{-2}, y \in \mathcal{E}_{-1},$$

$$(b) \quad \text{for all } f \in \mathcal{O}: \nabla_x f y = f \nabla_x y + \rho(x)[f] y \text{ and } \nabla_{f x} y = f \nabla_x y, \text{ for all } x \in \mathcal{E}_{-1}, y \in \mathcal{E}_{-2},$$

2. Remember that

$$\text{Jac}(e_i, e_j, e_k) := [e_i, [e_j, e_k]_2]_2 + [e_j, [e_k, e_i]_2]_2 + [e_k, [e_i, e_j]_2]_2 \in \ker \pi.$$

By using again exactness of the complex (3.2) there is an element that we denote by $[e_i, e_j, e_k]_{\mathcal{E}_{-1}} \in \mathcal{E}_{-2}$ that satisfies

$$\ell_1([e_i, e_j, e_k]_{\mathcal{E}_{-1}}) = \text{Jac}(e_i, e_j, e_k). \quad (3.4)$$

Thus, we can define a skew-symmetric trilinear map:

$$[\cdot, \cdot, \cdot]_{\mathcal{E}_{-1}}: \mathcal{E}_{-1} \wedge \mathcal{E}_{-1} \wedge \mathcal{E}_{-1} \longrightarrow \mathcal{E}_{-2}$$

such that

$$\ell_1([x, y, z]_{\mathcal{E}_{-1}}) = [x, [y, z]_2]_2 + [y, [z, x]_2]_2 + [z, [x, y]_2]_2, \quad \forall x, y, z \in \mathcal{E}_{-1}.$$

The following Proposition concludes the discussion above.

Proposition 3.2.10. *Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra that admits a free resolution of length 2 as in (3.2)*

1. \mathcal{E}_{-1} admits an almost Lie algebroid structure $(\mathcal{E}_{-1}, [\cdot, \cdot]_{\mathcal{E}_{-1}}, \rho)$.

2. There is a bilinear map:

$$\begin{aligned} \mathcal{E}_{-1} \otimes \mathcal{E}_{-2} &\rightarrow \mathcal{E}_{-2} \\ (x, y) &\mapsto \nabla_x y \end{aligned}$$

and a skew-symmetric trilinear map:

$$[\cdot, \cdot, \cdot]_{\mathcal{E}_{-1}}: \mathcal{E}_{-1} \wedge \mathcal{E}_{-1} \wedge \mathcal{E}_{-1} \longrightarrow \mathcal{E}_{-2}$$

such that for all $f \in \mathcal{O}$:

(a) $\nabla_x f y = f \nabla_x y + \rho(x)[f]y$ and $\nabla_{fx} y = f \nabla_x y$, for all $x \in \mathcal{E}_{-1}, y \in \mathcal{E}_{-2}$,

(b) $[fx, y, z]_{\mathcal{E}_{-1}} = f[x, y, z]_{\mathcal{E}_{-1}}$ for all $x, y, z \in \mathcal{E}_{-1}$,

such that the 2-ary bracket on $\mathcal{E}_{-1} \oplus \mathcal{E}_{-2}$ defined by:

$$[x, y]_2 = \begin{cases} [x, y]_{\mathcal{E}_{-1}} & \text{for } x, y \in \mathcal{E}_{-1} \\ \nabla_x y & \text{for } x \in \mathcal{E}_{-1}, y \in \mathcal{E}_{-2} \\ \nabla_y x & \text{for } x \in \mathcal{E}_{-2}, y \in \mathcal{E}_{-1} \\ 0 & \text{for } x, y \in \mathcal{E}_{-2} \end{cases}$$

together with the 3-ary bracket on $\mathcal{E}_{-1} \oplus \mathcal{E}_{-2}$ defined by $[x, y, z]_3 = [x, y, z]_{\mathcal{E}_{-1}}$ if $x, y, z \in \mathcal{E}_{-1}$ and zero otherwise, satisfies

(a) for all $x \in \mathcal{E}_{-2}, y \in \mathcal{E}_{-1}$,

$$\ell_1([x, y]_2) + [\ell_1(x), y]_2 = 0, \tag{3.5}$$

(b) for all $x, y, z \in \mathcal{E}_{-1}$

$$\ell_1([x, y, z]_3) + [x, [y, z]_2]_2 + [y, [z, x]_2]_2 + [z, [x, y]_2]_2 = 0$$

(c) for all $x, y \in \mathcal{E}_{-1}$ and $z \in \mathcal{E}_{-2}$

$$[x, y, \ell_1(z)]_3 + [x, [y, z]_2]_2 + [y, [z, x]_2]_2 + [z, [x, y]_2]_2 = 0.$$

The structure $(\mathcal{E}_\bullet, \ell_1, \rho, [\cdot, \cdot]_2, [\cdot, \cdot, \cdot]_3)$ described in Proposition 3.2.10 is called *Lie 2-algebroid* [BC03, Lea17, SZ11], i.e. a Lie ∞ -algebroid with $\mathcal{E}_{-i} = 0$ for $i \geq 3$.

A generalization of this construction on free resolutions of higher (even infinite) length is the object of the next Chapter.

Conclusion:

We describe Lie-Rinehart algebras, give examples and several constructions. Section 3.2.2 is a pedagogical section, which presents an elementary case of the general case that we will study.

4.1 Presentation of the problem

In order to understand the geometry of affine varieties or some similar problems related to singular foliations using the Lie algebra of their vector fields which is in fact Lie-Rinehart algebras, have motivated us to understand Lie-Rinehart algebras in general from another point of view.

We have seen in the Chapter 3, Remark 3.1.3 that any Lie ∞ -algebroid over \mathcal{O} induces a Lie-Rinehart algebra which we call its basic Lie-Rinehart algebra. In this chapter, we are investigating the opposite direction, i.e., we study the following questions: given a Lie-Rinehart algebra \mathcal{A} over \mathcal{O} , can we find a Lie ∞ -algebroid over \mathcal{O} whose basic Lie-Rinehart algebra is \mathcal{A} ? Also, in case where it exists, do we have uniqueness?

Now let us formalize that in a categorical language.

We denote by $\mathbf{Lie}\text{-}\infty\text{-alg}\text{-oids}/\mathcal{O}$ the quotient category where

1. the objects are Lie ∞ -algebroids over \mathcal{O} ,
2. arrows are homotopy equivalence classes of morphisms of Lie ∞ -algebroids over \mathcal{O} .

and by $\mathbf{Lie}\text{-Rhrt}\text{-alg}/\mathcal{O}$ the category of Lie-Rinehart algebra over \mathcal{O} . We want to study the functor,

$$\left| \mathbf{Lie}\text{-}\infty\text{-alg}\text{-oids}/\mathcal{O} \right| \begin{array}{c} \xrightarrow{\mathfrak{F}} \\ \xleftarrow{\exists! ?} \end{array} \left| \mathbf{Lie}\text{-Rhrt}\text{-alg}/\mathcal{O} \right|$$

The question now turn out to be: when does \mathfrak{F} admit a left/right inverse?

In the next section, we present in detail the main results related to this question.

4.2 Main results

An existence Theorem

The results below appeared in my first article [LGL22b] entitled "Lie-Rinehart algebra \simeq acyclic Lie ∞ -algebroid" co-written with my supervisor C. Laurent-Gengoux.

This section extends the main results of [LLS20] from locally real analytic finitely generated singular foliations to arbitrary Lie-Rinehart algebras.

Here is our first main result. It states that universal Lie ∞ -algebroids over a given Lie-Rinehart algebra exist. It extends Theorem 2.8 in [LLS20]. We are convinced that it may be deduced using the methods of semi-models categories as in Theorem 4.2 in [FJO18], but does not follow from a simple homotopy transfer argument.

Theorem 4.2.1. *Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra over \mathcal{O} . Any resolution of \mathcal{A} by free \mathcal{O} -modules*

$$\cdots \xrightarrow{\ell_1} \mathcal{E}_{-3} \xrightarrow{\ell_1} \mathcal{E}_{-2} \xrightarrow{\ell_1} \mathcal{E}_{-1} \xrightarrow{\pi} \mathcal{A} \quad (4.1)$$

comes equipped with a Lie ∞ -algebroid structure whose unary bracket is ℓ_1 and that covers \mathcal{A} through the hook π .

Since any module admits free resolutions (see Proposition B.2.5), Theorem 4.2.1 implies that:

Corollary 4.2.2. *Any Lie-Rinehart algebra \mathcal{A} over \mathcal{O} is the basic Lie-Rinehart algebra of an acyclic Lie ∞ -algebroid over \mathcal{O} .*

While proving Theorem 4.2.1, we will see that if \mathcal{E}_{-1} can be equipped with a Lie algebroid bracket (i.e. a bracket whose Jacobiator is zero), then all k -ary brackets of the universal Lie ∞ -algebroid structure may be chosen to be zero on \mathcal{E}_{-1} :

Proposition 4.2.3. *Let $(\mathcal{E}, \ell_1, \pi)$ be a free resolution of a Lie-Rinehart algebra \mathcal{A} . If \mathcal{E}_{-1} admits a Lie algebroid bracket $[\cdot, \cdot]$ such that $\pi : \mathcal{E}_{-1} \rightarrow \mathcal{A}$ is a Lie-Rinehart morphism, then there exists a structure of universal Lie ∞ -algebroid $(\mathcal{E}_{\bullet}, \ell_{\bullet}, \rho)$ that covers \mathcal{A} whose 2-ary bracket coincides with $[\cdot, \cdot]$ on \mathcal{E}_{-1} and such that for every $k \geq 3$ the k -ary bracket ℓ_k vanishes on $\odot^k \mathcal{E}_{-1}$.*

Universality of Theorem 4.2.1 and its corollaries

Here is our second main result. It is related to Proposition 2.1.4 in [FJO18] (but morphisms are not the same), and extends Theorem 2.9 in [LLS20].

Theorem 4.2.4. *Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra over \mathcal{O} . Given,*

- a) *a Lie ∞ -algebroid $(\mathcal{E}'_{\bullet}, \ell'_{\bullet}, \rho')$ that covers \mathcal{A} through a hook π' , and*
- b) *any acyclic Lie ∞ -algebroid $(\mathcal{E}_{\bullet}, \ell_{\bullet}, \rho)$ that covers \mathcal{A} through a hook π ,*

then

1. *there exists a morphism of Lie ∞ -algebroids from $(\mathcal{E}'_{\bullet}, \ell'_{\bullet}, \rho')$ to $(\mathcal{E}_{\bullet}, \ell_{\bullet}, \rho)$ over \mathcal{A} .*

2. and any two such morphisms are homotopic.

Here is an immediate corollary of Theorem 4.2.4.

Corollary 4.2.5. *Any two acyclic Lie ∞ -algebroids that cover a given Lie-Rinehart algebra are homotopy equivalent. This homotopy equivalence, moreover, is unique up to homotopy.*

We will prove that the morphism that appears in Theorem 4.2.4 can be made trivial upon choosing a “big enough” universal Lie ∞ -algebroid:

Proposition 4.2.6. *Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra over \mathcal{O} . Given a Lie ∞ -algebroid structure $(\mathcal{E}', \ell', \rho')$ that terminates in \mathcal{A} through a hook π' , then there exist an acyclic Lie ∞ -algebroid $(\mathcal{E}, \ell, \rho)$ that covers \mathcal{A} through a hook π such that*

1. \mathcal{E} contains \mathcal{E}' as a subcomplex,
2. the Lie ∞ -algebroid morphism from \mathcal{E}' to \mathcal{E} announced in Theorem 4.2.4 can be chosen to be the inclusion map $\mathcal{E}' \hookrightarrow \mathcal{E}$ (i.e. a Lie ∞ -morphism where the only non-vanishing Taylor coefficient is the inclusion $\mathcal{E}' \hookrightarrow \mathcal{E}$).

The following Corollary follows immediately from Proposition 4.2.6:

Corollary 4.2.7. *Let \mathcal{A} be a Lie-Rinehart algebra over \mathcal{O} and \mathcal{B} be a Lie-Rinehart subalgebra of \mathcal{A} . Any universal Lie ∞ -algebroid of \mathcal{B} can be contained in a universal Lie ∞ -algebroid of \mathcal{A} .*

Induced Lie ∞ -algebroids structures on $\mathrm{Tor}_{\mathcal{O}}(\mathcal{A}, \mathcal{O}/\mathcal{I})$.

Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra over \mathcal{O} .

Definition 4.2.8. We say that an ideal $\mathcal{I} \subset \mathcal{O}$ is a *Lie-Rinehart ideal of \mathcal{A}* if $\rho_{\mathcal{A}}(a)[\mathcal{I}] \subset \mathcal{I}$ for all $a \in \mathcal{A}$.

Remark 4.2.9. For any Lie-Rinehart ideal $\mathcal{I}\mathcal{O}$ of \mathcal{A} ,

1. we have $[\mathcal{I}\mathcal{A}, \mathcal{A}]_{\mathcal{A}} \subset \mathcal{I}\mathcal{A}$. Therefore, the quotient space $\mathcal{A}/\mathcal{I}\mathcal{A}$ comes equipped with a natural Lie-Rinehart algebra structure over \mathcal{O}/\mathcal{I} .
2. For $(\mathcal{E}, \ell, \rho)$ an acyclic Lie ∞ -algebroid that covers \mathcal{A} , the quotient space $\mathcal{E}/\mathcal{I} \simeq \mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} \mathcal{E}$ comes equipped with an induced Lie ∞ -algebroid structure: the n -ary brackets for $n \neq 2$ go to quotient by linearity, while for $n = 2$, the 2-ary bracket goes to the quotient in view of the relation $\rho_{\mathcal{E}}(\mathcal{E}_{-1})[\mathcal{I}] \subset \mathcal{I}$. Also, π goes to the quotient to a Lie-Rinehart algebra morphism $\mathcal{E}_{-1}/\mathcal{I} \rightarrow \mathcal{A}/\mathcal{I}\mathcal{A}$.

Definition 4.2.10. Let \mathcal{A} be a Lie-Rinehart algebra over \mathcal{O} . For every Lie-Rinehart ideal $\mathcal{I} \subset \mathcal{O}$, we call *Lie ∞ -algebroid of \mathcal{I}* the quotient Lie ∞ -algebroid $\mathcal{E}_{\bullet}/\mathcal{I}$, with $(\mathcal{E}_{\bullet}, \ell_{\bullet}, \rho)$ a universal Lie ∞ -algebroid that covers \mathcal{A} .

Remark 4.2.11. The complexes on which the Lie ∞ -algebroids of the ideal \mathcal{I} are defined compute $\mathrm{Tor}_{\mathcal{O}}(\mathcal{A}, \mathcal{O}/\mathcal{I})$ by construction (see B.2).

Moreover, for any two universal Lie ∞ -algebroids of \mathcal{A} , defined on $\mathcal{E}, \mathcal{E}'$ the homotopy equivalences $\Phi : \mathcal{E}' \rightarrow \mathcal{E}$ and $\Psi : \mathcal{E} \rightarrow \mathcal{E}'$, whose existence is granted by Corollary 4.2.5, go to the quotient and induce an homotopy equivalences between $\mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} \mathcal{E}_{\bullet} \simeq \mathcal{E}_{\bullet}/\mathcal{I}$ and $\mathcal{O}/\mathcal{I} \otimes_{\mathcal{O}} \mathcal{E}'_{\bullet} \simeq \mathcal{E}'_{\bullet}/\mathcal{I}$. The following corollary is then an obvious consequence of Theorem 4.2.4.

Corollary 4.2.12. *Let \mathcal{A} be a Lie-Rinehart algebra over \mathcal{O} . Let $\mathcal{I} \subset \mathcal{O}$ be a Lie-Rinehart ideal. Then any two Lie ∞ -algebroids of \mathcal{I} are homotopy equivalent, and there is a distinguished class of homotopy equivalences between them.*

Taking under account Remark 4.2.11, here is an alternative manner to restate this corollary.

Corollary 4.2.13. *Let \mathcal{A} be a Lie-Rinehart algebra over \mathcal{O} . Let $\mathcal{I} \subset \mathcal{O}$ be a Lie-Rinehart ideal. Then the complex computing $\mathrm{Tor}_{\mathcal{O}}^{\bullet}(\mathcal{A}, \mathcal{O}/\mathcal{I})$ comes equipped with a natural Lie ∞ -algebroid structure over \mathcal{O}/\mathcal{I} , and any two such structures are homotopy equivalent in a unique up to homotopy manner.*

When, in addition to being a Lie-Rinehart ideal, \mathcal{I} is a maximal ideal, then $\mathbb{K} := \mathcal{O}/\mathcal{I}$ is a field and Lie ∞ -algebroids of \mathcal{I} are a homotopy equivalence class of Lie ∞ -algebras. In particular their common cohomologies, which is easily seen to be identified to $\mathrm{Tor}_{\mathcal{O}}^{\bullet}(\mathcal{A}, \mathbb{K})$ comes equipped with a graded Lie algebra structure. In particular, $\mathrm{Tor}_{\mathcal{O}}^{-1}(\mathcal{A}, \mathbb{K})$ is a Lie algebra, $\mathrm{Tor}_{\mathcal{O}}^{-2}(\mathcal{A}, \mathbb{K})$ is a representation of this algebra, and the 3-ary bracket defines a class in the third Chevalley-Eilenberg cohomology of $\mathrm{Tor}_{\mathcal{O}}^{-1}(\mathcal{A}, \mathbb{K})$ valued in $\mathrm{Tor}_{\mathcal{O}}^{-2}(\mathcal{A}, \mathbb{K})$. This class does not depend on any choice made in its construction by the previous corollaries, and trivially extends the class called NMRLA-class in [LLS20]. If it is not zero, then there is no Lie algebroid of rank r equipped with a surjective Lie-Rinehart algebra morphism onto \mathcal{A} , where r is the rank of \mathcal{A} as a module over \mathcal{O} . All these considerations can be obtained by repeating verbatim Section 4.5.1 in [LLS20] (where non-trivial examples are given).

A categorical approach of the results

Theorem 4.2.4 means that acyclic Lie ∞ -algebroids that covers Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ are terminal objects in the subcategory of Lie- ∞ -alg-oids/ \mathcal{O} whose objects are Lie ∞ -algebroids that terminate in \mathcal{A} . Whence, acyclic Lie ∞ -algebroids over \mathcal{O} that cover $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ are deserved to be called "*universal ∞ -algebroids of \mathcal{A}* ". From now on, we call them by this name.

Let us re-state Corollary 4.2.5 differently. By associating to any Lie ∞ -algebroid its basic Lie-Rinehart algebra one obtains therefore a natural functor:

- from the category **Lie- ∞ -alg-oids**/ \mathcal{O} ,
- to the category **Lie-Rhart- ∞ -alg**/ \mathcal{O}

Theorem 4.2.1 gives a right inverse of this functor. In particular, this functor becomes an equivalence of categories when restricted to homotopy equivalence classes of acyclic Lie ∞ -algebroids over \mathcal{O} , i.e:

Corollary 4.2.14. *There is an equivalence of categories between:*

- (i) *Lie-Rinehart algebras over \mathcal{O} ,*
- (ii) *acyclic Lie ∞ -algebroids over \mathcal{O} .*

This corollary justifies the title of the first part of the thesis.

Remark 4.2.15. In the language of categories, Corollary 4.2.12 means that there exists a functor from Lie-Rinehart ideals of a Lie-Rinehart algebra over \mathcal{O} , to the category of Lie ∞ -algebroids, mapping a Lie-Rinehart ideal \mathcal{I} to an equivalence class of Lie ∞ -algebroids over \mathcal{O}/\mathcal{I} .

4.3 Proof of main results

4.3.1 A crucial bi-complex: $\mathfrak{Page}^{(n)}(\mathcal{E}', \mathcal{E})$

Description of $\mathfrak{Page}^{(n)}(\mathcal{E}', \mathcal{E})$

Let \mathcal{V} be an \mathcal{O} -module, and let (\mathcal{E}, d, π) and (\mathcal{E}', d', π') be complexes of projective \mathcal{O} -modules that terminates at \mathcal{V} :

$$\dots \xrightarrow{d} \mathcal{E}_{-2} \xrightarrow{d} \mathcal{E}_{-1} \xrightarrow{\pi} \mathcal{V}, \quad \dots \xrightarrow{d'} \mathcal{E}'_{-2} \xrightarrow{d'} \mathcal{E}'_{-1} \xrightarrow{\pi'} \mathcal{V}. \quad (4.2)$$

For every $k \geq 1$, the $(k+1)$ -th graded symmetric power $\odot^{k+1} \mathcal{E}'$ of \mathcal{E}' over \mathcal{O} is a projective \mathcal{O} -module, and comes with a natural grading induced by the grading on \mathcal{E}' .

Definition 4.3.1. Let $k \in \mathbb{N}_0$. We call *page number k* of (\mathcal{E}, d, π) and (\mathcal{E}', d', π') the bicomplex of \mathcal{O} -modules on the upper left quadrant $\mathbb{Z}^- \times \mathbb{N}_0$ defined by:

$$\text{Page}^{(k)}(\mathcal{E}', \mathcal{E})_{j,m} := \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{E}'|_{-k-m-1}, \mathcal{E}_j \right), \quad \text{for } m \geq 0 \text{ and } j \leq -1 \quad (4.3)$$

$$\text{Page}^{(k)}(\mathcal{E}', \mathcal{E})_{0,m} := \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{E}'|_{-k-m-1}, \mathcal{V} \right), \quad \text{for } m \geq 0, \quad (4.4)$$

together with the vertical differential D^v defined for any one of the two \mathcal{O} -modules (4.3) or (4.4) by

$$D^v : \text{Page}^{(k)}(\mathcal{E}', \mathcal{E})_{j,m} \longrightarrow \text{Page}^{(k)}(\mathcal{E}', \mathcal{E})_{j,m+1}$$

$$\Phi \longmapsto D^v(\Phi) : \odot^{k+1} \mathcal{E}' \longrightarrow \mathcal{E}_j$$

$$x_1 \odot \dots \odot x_{k+1} \longmapsto \Phi \circ d'(x_1 \odot \dots \odot x_{k+1})$$

where d' acts as an \mathcal{O} -derivation on $x_1 \odot \dots \odot x_{k+1} \in \odot^{k+1} \mathcal{E}'$ (and is 0 on \mathcal{E}'_{-1}). The horizontal differential, $D^h : \text{Page}^{(k)}(\mathcal{E}', \mathcal{E})_{j,m} \longrightarrow \text{Page}^{(k)}(\mathcal{E}', \mathcal{E})_{j+1,m}$, is given by

$$\Phi \mapsto d \circ \Phi \quad \text{or} \quad \Phi \mapsto \pi \circ \Phi$$

depending on whether Φ is of type (4.3) with $j \leq -2$ or the type (4.3) with $j = -1$. It is zero on elements of type (4.4). We denote by $\left(\mathfrak{Page}_{\bullet}^{(k)}(\mathcal{E}', \mathcal{E}), D \right)$ its associated total complex. When $\mathcal{E}' = \mathcal{E}$ we shall write $\mathfrak{Page}_{\bullet}^{(k)}(\mathcal{E})$ instead of $\mathfrak{Page}_{\bullet}^{(k)}(\mathcal{E}, \mathcal{E})$.

The following diagram recapitulates the whole picture of $\mathfrak{Page}_{\bullet}^{(k)}(\mathcal{E}', \mathcal{E})$:

$$\begin{array}{ccccc}
& \vdots & & \vdots & & \vdots \\
& \uparrow & & \uparrow & & \uparrow \\
\cdots & \rightarrow & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-3}, \mathcal{E}_{-2}\right) & \xrightarrow{D^h} & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-3}, \mathcal{E}_{-1}\right) & \xrightarrow{D^h} & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-3}, \mathcal{V}\right) & \rightarrow & 0 \\
& & D^v \uparrow & & D^v \uparrow & & D^v \uparrow & & \\
\cdots & \rightarrow & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-2}, \mathcal{E}_{-2}\right) & \xrightarrow{D^h} & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-2}, \mathcal{E}_{-1}\right) & \xrightarrow{D^h} & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-2}, \mathcal{V}\right) & \rightarrow & 0 \\
& & D^v \uparrow & & D^v \uparrow & & D^v \uparrow & & \\
\cdots & \rightarrow & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-1}, \mathcal{E}_{-2}\right) & \xrightarrow{D^h} & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-1}, \mathcal{E}_{-1}\right) & \xrightarrow{D^h} & \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-k-1}, \mathcal{V}\right) & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & & \\
& & \text{"-2 column"} & & \text{"-1 column"} & & \text{"last column"} & &
\end{array} \tag{4.5}$$

For later use, we spell out the meaning of being D -closed.

Remark 4.3.2. An element $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E}', \mathcal{E})$ in $\bigoplus_{i \geq 1} \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-j-i}, \mathcal{E}_{-i}\right)$ is D -closed if and only if:

1. the component $P_{-1}: \odot^{k+1} \mathcal{E}'|_{-j-1} \rightarrow \mathcal{E}_{-1}$ is valued in the kernel of $\pi: \mathcal{E}_{-1} \rightarrow \mathcal{V}$,
2. the following diagram commutes:

$$\begin{array}{ccc}
\odot^{k+1} \mathcal{E}'|_{-j-i} & \xrightarrow{(-1)^j d'} & \odot^{k+1} \mathcal{E}'|_{-j-i+1} \\
P_{-i} \downarrow & & \downarrow P_{-i+1} \\
\mathcal{E}_{-i} & \xrightarrow{d} & \mathcal{E}_{-i+1}
\end{array}$$

with P_{-i} being the component of P in $\mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-j-i}, \mathcal{E}_{-i}\right)$.

For $(\mathcal{E}, d) = (\mathcal{E}', d')$, the second condition above also reads $[d, P]_{\mathrm{RN}} = 0$. Here is an important technical result.

Proposition 4.3.3. *Let (\mathcal{E}, d, π) be a resolution of \mathcal{V} in the category of \mathcal{O} -modules. Then, for every $k \geq 0$,*

1. *the cohomology of the complex $(\mathfrak{Page}_{\bullet}^{(k)}(\mathcal{E}), D)$ for the total differential $D_{\bullet} := D^h - (-1)^{\bullet} D^v$ is zero in all degrees;*
2. *Moreover, a D -closed element whose component on the “last column” of the diagram above is zero is the image through D of some element whose two last components are also zero.*
3. *More generally, for all $n \geq 1$, for a D -closed element $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E})$ of the form*

$$\bigoplus_{i \geq n} \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-j-i}, \mathcal{E}_{-i}\right),$$

one has $P = D(R)$ and $R \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{E})$ can be chosen in $\bigoplus_{i \geq n+1} \mathrm{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{E}'|_{-j-i+1}, \mathcal{E}_{-i}\right)$.

Proof. Since $\odot^k \mathcal{E}'|_{j+m-k}$ is a projective \mathcal{O} -module for all $(j, m) \in \mathbb{Z}^- \times \mathbb{N}_0$, and (\mathcal{E}, d, π) is a resolution, all the lines of the above bicomplex are exact. This proves the first item. The second and the third are obtained by diagram chasing (see Appendix B.2 for more details). \square

Interpretation of $\mathfrak{Page}^{(n)}(\mathcal{E}', \mathcal{E})$ in our context

We denote by $Q_{\mathcal{E}}^{(0)}$ and $Q_{\mathcal{E}'}^{(0)}$ the differentials of polynomial-degree 0 on $\odot^{\bullet} \mathcal{E}$ and $\odot^{\bullet} \mathcal{E}'$ induced by d and d' . Whence, $(\text{Hom}_{\mathcal{O}}(\odot^{\bullet} \mathcal{E}', \odot^{\bullet} \mathcal{E}), \partial)$ with

$$\partial: H \mapsto Q_{\mathcal{E}}^{(0)} \circ H - (-1)^{|H|} H \circ Q_{\mathcal{E}'}^{(0)}$$

is a complex of \mathcal{O} -modules (see Lemma B.1.13).

Lemma 4.3.4. *Let $\phi: (\mathcal{E}', d') \rightarrow (\mathcal{E}, d)$ be a chain map, and let $\Phi^{(0)}: \odot^{\bullet} \mathcal{E}' \rightarrow \odot^{\bullet} \mathcal{E}$ be its extension to a co-algebra morphism, namely:*

$$\Phi^{(0)}(x_1 \cdots x_n) := \phi(x_1) \cdots \phi(x_n).$$

1. We have, $\Phi^{(0)} \circ Q_{\mathcal{E}'}^{(0)} = Q_{\mathcal{E}}^{(0)} \circ \Phi^{(0)}$.
2. $\Phi^{(0)}$ -co-derivations form a sub-complex of $(\text{Hom}_{\mathcal{O}}(\odot^{\bullet} \mathcal{E}', \odot^{\bullet} \mathcal{E}), \partial)$.

Proof. The first item can be easily checked. The second item follows exactly the same pattern as Lemma 2.3.24. \square

The following proposition is very important.

Proposition 4.3.5. *For every $k \in \mathbb{N}_0$, and $\phi: (\mathcal{E}', d') \rightarrow (\mathcal{E}, d)$ be a chain map as in Lemma 4.3.4. The sub-complex of $\Phi^{(0)}$ -co-derivations of polynomial-degree k is isomorphic to the complex $\widehat{\mathfrak{Page}}^{(k)}(\mathcal{E}', \mathcal{E})$ obtained from $\mathfrak{Page}^{(k)}(\mathcal{E}', \mathcal{E})$ by crossing its “last column”, see diagram (6.16).*

Proof. The chain isomorphism $\delta: \text{Hom}_{\mathcal{O}}(\odot^{\bullet} \mathcal{E}', \odot^{\bullet} \mathcal{E}) \rightarrow \text{Hom}_{\mathcal{O}}(\odot^{\bullet} \mathcal{E}', \mathcal{E})$ consists in mapping a $\Phi^{(0)}$ -co-derivation H of polynomial-degree k and degree j to its unique Taylor coefficient $H_k \in \mathfrak{Page}_j^{(k)}(\mathcal{E}', \mathcal{E})$, i.e. $\delta(H) = \text{pr} \circ H$. Let us check that this map is indeed a chain map: for every $x = x_1 \odot \cdots \odot x_{k+1} \in \odot^{k+1} \mathcal{E}'$ one has,

$$\begin{aligned} \delta \circ (Q_{\mathcal{E}}^{(0)} \circ H - (-1)^{|H|} H \circ Q_{\mathcal{E}'}^{(0)})(x) &= \ell_1 \circ H_k(x) (-1)^{|H|} H_k \left(\sum_{i=1}^{k+1} (-1)^{(|x_1| + \cdots + |x_{i-1}|) |x_i|} \ell'_1(x_i) \odot x_1 \odot \cdots \odot x_{k+1} \right) \\ &= D^h(H_k)(x) - (-1)^{|H_k|} D^v(H_k)(x), \quad \text{by definition of } D^h \text{ and } D^v \\ &= D(H_k)(x) \\ &= D \circ \delta(H)(x), \quad \text{by definition of } \delta. \end{aligned}$$

\square

Here is another type of interpretation for $\widehat{\mathfrak{Page}}^{\bullet}(\mathcal{E}', \mathcal{E})$ involving the Richardson-Nijenhuis bracket.

Proposition 4.3.6. *[FN56] When $\mathcal{E}' = \mathcal{E}$, $\widehat{\mathfrak{Page}}^{\bullet}(\mathcal{E}', \mathcal{E})$ with no 0-column is the bi-graded complex of exterior forms on \mathcal{E} and the differential D of $\widehat{\mathfrak{Page}}^{\bullet}(\mathcal{E}', \mathcal{E})$ is $D(\cdot) = [d, \cdot]_{\text{RN}}$.*

We finish the section with the following lemma that will be important to prove Proposition 4.2.6. It uses the consequence of the cone construction (see Appendix B).

Lemma 4.3.7. *Let $(\mathcal{R}, d^{\mathcal{R}}, \pi^{\mathcal{R}})$ be an arbitrary complex of projective \mathcal{O} -module that terminates in a \mathcal{O} -module \mathcal{V} . There exists a projective resolution $(\mathcal{E}, d^{\mathcal{E}}, \pi^{\mathcal{E}})$ of \mathcal{V} , which contains $(\mathcal{R}, d^{\mathcal{R}}, \pi^{\mathcal{R}})$ as a sub-complex. Moreover, we can assume that \mathcal{R} admits a projective submodule in \mathcal{E} in direct sum.*

Proof. Resolutions of an \mathcal{O} -modules \mathcal{V} are universal objects in the category of complexes of projective \mathcal{O} -modules. In particular, for every projective resolution $(\mathfrak{F}, d^{\mathfrak{F}}, \pi^{\mathfrak{F}})$ of \mathcal{V} , there exist a (unique up to homotopy) chain map:

$$\phi: (\mathcal{R}, d^{\mathcal{R}}, \pi^{\mathcal{R}}) \rightarrow (\mathfrak{F}, d^{\mathfrak{F}}, \pi^{\mathfrak{F}}).$$

We apply the cone construction (see, e.g. [Cha14], Section 1.5) to:

1. the complex $(\mathcal{R}, d^{\mathcal{R}}, \pi^{\mathcal{R}})$
2. the direct sum of the complexes $(\mathcal{R}, d^{\mathcal{R}})$ and $(\mathfrak{F}, d^{\mathfrak{F}})$ namely, $(\mathcal{R} \oplus \mathfrak{F}, d^{\mathcal{R}} \oplus d^{\mathfrak{F}}, \pi^{\mathcal{R}} \oplus \pi^{\mathfrak{F}})$
3. the chain map obtained by mapping any $x \in \mathcal{R}$ to $(x, \phi(x)) \in \mathcal{R} \oplus \mathfrak{F}$.

The differential is given by

$$d^{\mathcal{E}}(x, y, z) = (-d^{\mathcal{R}}x, d^{\mathcal{R}}y - x, d^{\mathfrak{F}}z - \varphi(x)) \quad (4.6)$$

for all $(x, y, z) \in \mathcal{E}_{-i} = \mathcal{R}_{-i+1} \oplus \mathcal{R}_{-i} \oplus \mathfrak{F}_{-i}$, $i \geq 2$. Since the chain given in item 3 is a quasi-isomorphism, its cone is an exact complex. We truncate the latter at degree -1 without destroying its exactness by replacing the cone differential at degree -1 as follows: $\pi^{\mathcal{E}}: \mathcal{R}_{-1} \oplus \mathfrak{F}_{-1} \rightarrow \mathcal{V}$, $(r, e) \mapsto \pi^{\mathfrak{F}}(e) - \pi^{\mathcal{R}}(r)$. For a visual description, see Equation (4.7) below: the resolution of \mathcal{V} described in Lemma 4.3.7 is defined by:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathfrak{F}_{-3} & \xrightarrow{d^{\mathfrak{F}}} & \mathfrak{F}_{-2} & \xrightarrow{d^{\mathfrak{F}}} & \mathfrak{F}_{-1} & \xrightarrow{\pi^{\mathfrak{F}}} & \mathcal{V} \\ & & & \nearrow & & \nearrow & & \nearrow & \\ \cdots & \longrightarrow & \mathcal{R}_{-3} & \xrightarrow{d^{\mathcal{R}}} & \mathcal{R}_{-2} & \xrightarrow{d^{\mathcal{R}}} & \mathcal{R}_{-1} & \xrightarrow{\pi^{\mathcal{R}}} & \\ & & \phi & \nearrow & \phi & \nearrow & & \nearrow & \\ \cdots & \longrightarrow & \mathcal{R}_{-2} & \xrightarrow{d^{\mathcal{R}}} & \mathcal{R}_{-1} & & & & \end{array} \quad (4.7)$$

The proof of the exactness of this complex is left to the reader.

The henceforth defined complex $(\mathcal{E}, d^{\mathcal{E}}, \pi^{\mathcal{E}})$ is a resolution of \mathcal{V} , and obviously contains $(\mathcal{R}, d^{\mathcal{R}}, \pi^{\mathcal{R}})$ as a sub-chain complex of \mathcal{O} -modules. \square

Let $(\mathcal{E}, d^{\mathcal{E}}, \pi^{\mathcal{E}})$ be a free resolution of \mathcal{V} and $(\mathcal{R}, d^{\mathcal{R}}, \pi^{\mathcal{R}})$ a subcomplex of projective \mathcal{O} -modules, as in Lemma 4.3.7. We say that $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E})$ of the form $\bigoplus_{i \geq n} \text{Hom}_{\mathcal{O}}(\odot^{k+1} \mathcal{E}_{|-j-i}, \mathcal{E}_{-i})$ preserves \mathcal{R} if $\odot^{k+1} \mathcal{R}_{|-j-i}$ is mapped by P to \mathcal{R}_{-i} for all possible indices. In such case, it defines by restriction to $\odot^{\bullet} \mathcal{R}$ an element $\iota_{\mathcal{R}}^* P$ in the graded \mathcal{O} -module $\mathfrak{Page}_j^{(k)}(\mathcal{R}) := \bigoplus_{i \geq n} \text{Hom}_{\mathcal{O}}(\odot^{k+1} \mathcal{R}_{|-j-i}, \mathcal{R}_{-i})$. For the sake of clarity, let us denote by $D^{\mathcal{E}}$ and $D^{\mathcal{R}}$ the respective differentials of the bi-complexes $\mathfrak{Page}_j^{(k)}(\mathcal{E})$ and $\mathfrak{Page}_j^{(k)}(\mathcal{R})$ and by $D_h^{\mathcal{E}}$, $D_h^{\mathcal{R}}$ and $D_v^{\mathcal{E}}$, $D_v^{\mathcal{R}}$ the horizontal differential resp. vertical differential, of their associated bi-complexes. Also, $\iota_{\mathcal{R}}^* P$ stands for the restriction of $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E})$ to $\odot^{\bullet} \mathcal{R}$ (a priori it is not valued in \mathcal{R} but in \mathcal{E}).

Lemma 4.3.8. *Let $(\mathcal{E}, d^{\mathcal{E}}, \pi^{\mathcal{E}})$ be a free resolution of \mathcal{V} . Let $\mathcal{R} \subset \mathcal{E}$ be a subcomplex made of free sub- \mathcal{O} -modules such that there exists a graded free \mathcal{O} -module \mathcal{V} such that $\mathcal{E} = \mathcal{R} \oplus \mathcal{V}$.*

1. For every $k \geq 0$, a $D^\mathcal{E}$ -cocycle $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E})$ which preserves \mathcal{R} is the image through $D^\mathcal{E}$ of some element $Q \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{E})$ which preserves \mathcal{R} if and only if its restriction $\iota_{\mathcal{R}}^* P \in \mathfrak{Page}_j^{(k)}(\mathcal{R})$ is a $D^\mathcal{R}$ -coboundary.
2. In particular, if the restriction of $d^\mathcal{E}$ and $\pi^\mathcal{E}$ to \mathcal{R} makes it a resolution of $\pi^\mathcal{E}(\mathcal{R}_{-1}) \subset \mathcal{V}$, then any $D^\mathcal{E}$ -cocycle $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E})$ which preserves \mathcal{R} is the image through $D^\mathcal{E}$ of some element $Q \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{E})$ which preserves \mathcal{R} .

Proof. Let us decompose the element $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E})$ as $P = \sum_{i \geq 1} P_i$ with, for all $i \geq 1$, P_i in $\text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{E}_{|-j-i}, \mathcal{E}_{-i} \right)$. Assume $P \in \mathfrak{Page}_j^{(k)}(\mathcal{E})$ is a $D^\mathcal{E}$ -cocycle which preserves \mathcal{R} .

Let us prove one direction of item 1. If P is the image through $D^\mathcal{E}$ of some element $Q \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{E})$ which preserves \mathcal{R} , then $D^\mathcal{R}(\iota_{\mathcal{R}}^* Q) = \iota_{\mathcal{R}}^* D^\mathcal{E}(Q) = \iota_{\mathcal{R}}^* P$, with $\iota_{\mathcal{R}}^* Q \in \mathfrak{Page}_j^{(k)}(\mathcal{R})$. Thus, the restriction $\iota_{\mathcal{R}}^* P \in \mathfrak{Page}_j^{(k)}(\mathcal{R})$ of P is a $D^\mathcal{R}$ -coboundary.

Conversely, let us assume that $\iota_{\mathcal{R}}^* P \in \mathfrak{Page}_j^{(k)}(\mathcal{R})$ is a $D^\mathcal{R}$ -coboundary, i.e. $\iota_{\mathcal{R}}^* P = D^\mathcal{R} Q_{\mathcal{R}}$ for some $Q_{\mathcal{R}} \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{R})$. Take $\hat{Q} \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{E})$ any extension of $Q_{\mathcal{R}}$ (e.g. define \hat{Q} to be 0 as soon as one element in \mathcal{V} is applied to it). Then $P - D^\mathcal{E}(\hat{Q}) : \odot^{k+1} \mathcal{E} \rightarrow \mathcal{E}$ is zero on $\odot^{k+1} \mathcal{R}$. We have to check that it is a $D^\mathcal{E}$ -coboundary of a map with the same property. Put $\kappa = P - D^\mathcal{E}(\hat{Q})$. By Proposition 4.3.3, item 1, there exists $\tau \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{E})$ such that $D^\mathcal{E}(\tau) = \kappa$. The equation $D^\mathcal{E}(\tau) = \kappa$ is equivalent to the datum of a collection of equations

$$D_v^\mathcal{E}(\tau_i) + D_h^\mathcal{E}(\tau_{i+1}) = \kappa_{i+1}, i \geq 1, \quad \text{and} \quad D_h^\mathcal{E}(\tau_1) = \kappa_1, \quad (4.8)$$

with, $\tau_i \in \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{E}_{|-j-i+1}, \mathcal{E}_{-i} \right)$ and $\kappa_i \in \text{Hom}_{\mathcal{O}} \left(\odot^{k+1} \mathcal{E}_{|-j-i}, \mathcal{E}_{-i} \right)$ for every $i \geq 1$. Since $\iota_{\mathcal{R}}^* \kappa_1 = 0$, we have that $D_h^\mathcal{E}(\iota_{\mathcal{R}}^* \tau_1) = \iota_{\mathcal{R}}^* (D_h^\mathcal{E}(\tau_1)) = 0$, (with the understanding that $\iota_{\mathcal{R}}^* \tau_1|_{\mathcal{V}} \equiv 0$). Using the exactness of the horizontal differential $D_h^\mathcal{E}$, there exists $C_1 \in \mathfrak{Page}^{(k)}(\mathcal{E})$ such that $D_h^\mathcal{E}(C_1) = \iota_{\mathcal{R}}^* \tau_1$. We now change τ_1 to τ'_1 and τ_2 to τ'_2 by putting $\tau'_1 := \tau_1 - \iota_{\mathcal{R}}^* \tau_1$ and $\tau'_2 := \tau_2 + D_v^\mathcal{E}(C_1)$. One can easily check that Equation (4.8) still holds under these changes, i.e.,

$$D_v^\mathcal{E}(\tau'_1) + D_h^\mathcal{E}(\tau'_2) = \kappa_2 \quad \text{and} \quad D_h^\mathcal{E}(\tau'_2) + D_v^\mathcal{E}(\tau_3) = \kappa_3.$$

We can therefore choose τ such that $\iota_{\mathcal{R}}^* \tau_1 = 0$. We then iterate this procedure, which allows us to choose $\tau \in \mathfrak{Page}_{j-1}^{(k)}(\mathcal{E})$ such that $\iota_{\mathcal{R}}^* \tau = 0$ and $D^\mathcal{E}(\tau) = \kappa$. By construction, $Q := \tau + \hat{Q}$ preserves \mathcal{R} , while $\iota_{\mathcal{R}}^* Q = Q_{\mathcal{R}}$, and $D^\mathcal{E}(Q) = P$. The second item follows from the first one. \square

4.3.2 Proof on the existence

In this section, we prove Theorem 4.2.1.

Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra. Consider $(\mathcal{E}, d = \ell_1, \pi)$ a resolution of \mathcal{A} by free \mathcal{O} -modules: such resolutions always exist, see Proposition B.2.5. To start, we define a binary bracket ℓ_2 . The pair (d, ℓ_2) will obey the axioms of the object that we now introduce.

Definition 4.3.9. [Lav17] An almost differential graded Lie algebroid of a Lie-Rinehart algebra $(\mathcal{A}, \rho_{\mathcal{A}}, [\cdot, \cdot]_{\mathcal{A}})$ is a complex

$$\dots \xrightarrow{d} \mathcal{E}_{-3} \xrightarrow{d} \mathcal{E}_{-2} \xrightarrow{d} \mathcal{E}_{-1} \xrightarrow{\pi} \mathcal{A}$$

of projective \mathcal{O} -modules equipped a graded almost differential graded Lie algebroid $(\mathcal{E}_{\bullet}, \ell_1, \ell_2, \rho)$ over \mathcal{O} such that

1. $\rho = \rho_{\mathcal{A}} \circ \pi: \mathcal{E}_{-1} \longrightarrow \text{Der}(\mathcal{O})$,
2. π is a morphism, i.e. for all $x, y \in \mathcal{E}_{-1}$

$$\pi(\ell_2(x, y)) = [\pi(x), \pi(y)]_{\mathcal{A}}.$$

We start by proving this lemma.

Lemma 4.3.10. *Every free resolution (\mathcal{E}, d, π) of a Lie-Rinehart algebra $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ comes equipped with a binary bracket ℓ_2 that makes it an almost differential graded Lie algebroid of \mathcal{A} .*

Proof. For all $k \geq 1$, let us denote by $(e_i^{(-k)})_{i \in I_k}$ a family of generators of the free \mathcal{O} -module \mathcal{E}_{-k} . By construction $\{a_i = \pi(e_i^{(-1)}) \in \mathcal{A} \mid i \in I_1\}$ is a set of generators of \mathcal{A} . In particular, there exists elements $u_{ij}^k \in \mathcal{O}$, such that for given indices i, j , the coefficient u_{ij}^k is zero except for finitely many indices k , and satisfying the skew-symmetry condition $u_{ij}^k = -u_{ji}^k$ together with

$$[a_i, a_j]_{\mathcal{A}} = \sum_{k \in I} u_{ij}^k a_k \quad \forall i, j \in I_1 \quad (4.9)$$

We now define:

1. an anchor map by $\rho(e_i^{(-1)}) = \rho_{\mathcal{A}}(a_i)$ for all $i \in I$,
2. a degree +1 graded symmetric operation $\tilde{\ell}_2$ on \mathcal{E} as follows:
 - (a) $\tilde{\ell}_2(e_i^{(-1)}, e_j^{(-1)}) = \sum_{k \in I} u_{ij}^k e_k^{(-1)}$ for all $i, j \in I_{-1}$.
 - (b) $\tilde{\ell}_2(e_i^{(-k)}, e_j^{(-l)}) = 0$ for all $i \in I_k, j \in I_l$ with $k \geq 2$ or $l \geq 2$.
 - (c) we extend $\tilde{\ell}_2$ to \mathcal{E} using \mathcal{O} -bilinearity and Leibniz identity with respect to the anchor ρ .

By construction, $\tilde{\ell}_2$ satisfies the Leibniz identity with respect to the anchor $\rho_{\mathcal{E}}$. Also, $\rho \circ d = 0$ on \mathcal{E}_{-2} . The map defined for all homogeneous $x, y \in \mathcal{E}$ by

$$[d, \tilde{\ell}_2]_{\text{RN}}(x, y) = d \circ \tilde{\ell}_2(x, y) + \tilde{\ell}_2(dx, y) + (-1)^{|x|} \tilde{\ell}_2(x, dy),$$

is a graded symmetric degree +2 operation $(\mathcal{E} \otimes \mathcal{E})_{\bullet} \longrightarrow \mathcal{E}_{\bullet+2}$, and $[d, \tilde{\ell}_2]_{\text{RN}|_{\mathcal{E}_{-1}}} = 0$. Let us check that it is \mathcal{O} -bilinear, i.e. for all $f \in \mathcal{O}, x, y \in \mathcal{E}$:

$$[d, \tilde{\ell}_2]_{\text{RN}}(x, fy) - f[d, \tilde{\ell}_2]_{\text{RN}}(x, y) = 0.$$

1. if $x \in \mathcal{E}_{-1}$, this quantity is zero in view of

$$[d, \tilde{\ell}_2]_{\text{RN}}(x, fy) = f[d, \tilde{\ell}_2](x, y) + \underbrace{d\rho(x)[f]y - \rho(x)[f]dy}_{=0}$$

2. if $x \in \mathcal{E}_{-2}$, one has

$$[d, \tilde{\ell}_2]_{\text{RN}}(x, fy) - f[d, \tilde{\ell}_2](x, y) = \tilde{\ell}_2(dx, fy) - f\tilde{\ell}_2(dx, y) = \rho(dx)(f)y = 0$$

since $\rho \circ d = \rho_{\mathcal{A}} \circ \pi \circ d = 0$,

3. if $x \in \mathcal{E}_{-i}$ with $i \geq 3$, it is obvious by \mathcal{O} -linearity of $\tilde{\ell}_2$ on the involved spaces.

As a consequence, $[d, \tilde{\ell}_2]_{\text{RN}}$ is a degree +2 element in the total complex $\mathfrak{Page}^{(1)}(\mathcal{E})$. By construction $[d, \tilde{\ell}_2]_{\text{RN}}$ has no component on the last column. Since $\pi([d, \tilde{\ell}_2]_{\text{RN}|_{\mathcal{E}_{-1}}}) = 0$ and also $[d, [d, \tilde{\ell}_2]_{\text{RN}}]_{\text{RN}|_{\mathcal{E}_{\leq -2}}} = 0$, the \mathcal{O} -bilinear operator $[d, \tilde{\ell}_2]_{\text{RN}}$ is D -closed in $\mathfrak{Page}^{(1)}(\mathcal{E})$.

By virtue of the first item of Proposition 4.3.3, the operator $[d, \tilde{\ell}_2]_{\text{RN}}$ is then a D -coboundary, so there exists $\tau_2 \in \bigoplus_{j \geq 2} \text{Hom}_{\mathcal{O}}(\odot^2 \mathcal{E}_{|-j-1}, \mathcal{E}_{-j})$ such as $D(\tau_2) = -[d, \tilde{\ell}_2]_{\text{RN}}$. Upon replacing $\tilde{\ell}_2$ by $\tilde{\ell}_2 + \tau_2$ we obtain a 2-ary bracket ℓ_2 of degree +1 which satisfies all items of Definition 4.3.9. \square

Proof (of Theorem 4.2.1). Lemma 4.3.10 gives the existence of an almost differential graded Lie algebroid with differential $\ell_1 = d$ and binary bracket ℓ_2 . We have to construct now the higher brackets ℓ_k for $k \geq 3$.

Step 1: Construction of the 3-ary bracket ℓ_3 . (Its construction being different from the one of the higher brackets, we put it apart). We first notice that the graded Jacobiator defined for all $x, y, z \in \mathcal{E}$ by

$$\text{Jac}(x, y, z) := \ell_2(\ell_2(x, y), z) + (-1)^{|y||z|} \ell_2(\ell_2(x, z), y) + (-1)^{|x||y|+|x||z|} \ell_2(\ell_2(y, z), x)$$

is \mathcal{O} -linear in each variable, hence is a degree +2 element in $\bigoplus_{j \geq 1} \text{Hom}_{\mathcal{O}}(\odot^3 \mathcal{E}_{|-j-2}, \mathcal{E}_{-j}) \subset \mathfrak{Page}^{(2)}(\mathcal{E})$. For degree reason, its component on the last column of diagram (6.16) is zero, i.e. it belongs to $\widehat{\mathfrak{Page}}^{(1)}(\mathcal{E})$.

Let us check that it is D -closed: for this purpose we have to check that both conditions in Lemma 4.3.2 hold:

1. Since π is a morphism from $(\mathcal{E}_{-1}, \ell_2)$ to $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$, and since $[\cdot, \cdot]_{\mathcal{A}}$ satisfies the Jacobi identity, one has for all $x, y, z \in \mathcal{E}_{-1}$:

$$\text{Jac}(x, y, z) \in \ker \pi.$$

2. Furthermore, a direct computation of $[d, \text{Jac}]_{\text{RN}}$ gives in view of item 2 of Definition 4.3.9:

$$d\text{Jac}(x, y, z) = \text{Jac}(dx, y, z) + (-1)^{|x|} \text{Jac}(x, dy, z) + (-1)^{|x|+|y|} \text{Jac}(x, y, dz)$$

for all $x, y, z \in \mathcal{E}$.

Thus, $D(\text{Jac}) = 0$. By Proposition 4.3.3, item 2, Jac is a D -coboundary, and, more precisely, there exists an element $\ell_3 = \sum_{j \geq 2} \ell_3^j \in \widehat{\mathfrak{Page}}^{(2)}(\mathcal{E})$ with $\ell_3^j \in \text{Hom}(\odot^3 \mathcal{E}_{|-j-1}, \mathcal{E}_{-j})$ such that

$$D(\ell_3) = -\text{Jac} \quad \text{i.e.} \quad [d, \ell_3]_{\text{RN}} = -\text{Jac}. \quad (4.10)$$

We choose the 3-ary bracket to be ℓ_3 .

Step 2: Recursive construction of the k -ary brackets ℓ_k for $k \geq 4$. Let us recapitulate: $\ell_1 = d$, ℓ_2 and ℓ_3 are constructed and the lowest polynomial-degree terms of $[\ell_1 + \ell_2 + \ell_3, \ell_1 + \ell_2 + \ell_3]_{\text{RN}}$ satisfy

1. $[\ell_1, \ell_1]_{\text{RN}} = 0$ (since $d^2 = 0$),
2. $[\ell_1, \ell_2]_{\text{RN}} = 0$ (since $d = \ell_1$ and ℓ_2 define an almost Lie algebroid structure).
3. $[\ell_2, \ell_2]_{\text{RN}} + 2[\ell_3, \ell_1]_{\text{RN}} = 2(\text{Jac} + [\ell_3, \ell_1]_{\text{RN}}) = 0$ by definition of ℓ_3 , and because $[\ell_2, \ell_2]_{\text{RN}} = 2\text{Jac}$.

However, the following term of degree +2 and polynomial-degree 3 may not be equal to zero:

$$[\ell_3, \ell_2]_{\text{RN}} \in \bigoplus \text{Hom}_{\mathcal{O}} \left(\bigodot^4 \mathcal{E}_{j+1}, \mathcal{E}_{-j} \right) = \widehat{\mathfrak{Page}}_1^{(3)}(\mathcal{E}). \quad (4.11)$$

Let us check that this term is indeed a \mathcal{O} -multilinear map: For $x_1 \in \mathcal{E}_{-1}$, $x_2, x_3, x_4 \in \mathcal{E}_{\leq -2}$ and $f \in \mathcal{O}$, the only terms of $(\ell_3 \circ \ell_2 + \ell_2 \circ \ell_3)(x_1, fx_2, x_3, x_4)$ where the anchor shows up are:

$$\begin{cases} \ell_3(\ell_2(x_1, fx_2), x_3, x_4) & = \rho(x_1)[f]\ell_3(x_2, x_3, x_4) + f(\ell_3(\ell_2(x_1, x_2), x_3, x_4)) \\ (-1)^{|x_2|+|x_3|+|x_4|}\ell_2(f\ell_3(x_2, x_3, x_4), x_1) & = -\rho(x_1)[f]\ell_3(x_2, x_3, x_4) \\ & + f((-1)^{|x_2|+|x_3|+|x_4|}\ell_2(f\ell_3(x_2, x_3, x_4), x_1)) \end{cases}$$

The terms containing the anchor map add up to zero. When there are more elements in \mathcal{E}_{-1} , the computation follows the same line. Moreover, by graded Jacobi identity of the Richardson-Nijenhuis bracket:

$$[[\ell_1 + \ell_2 + \ell_3, \ell_1 + \ell_2 + \ell_3]_{\text{RN}}, \ell_1 + \ell_2 + \ell_3]_{\text{RN}} = 0$$

The term of polynomial-degree 4 in the previous expression gives $[[\ell_3, \ell_2]_{\text{RN}}, \ell_1]_{\text{RN}} = 0$. Hence, by Proposition 4.3.6, $[\ell_3, \ell_2]_{\text{RN}}$ is a D -cocycle in the complex $\widehat{\mathfrak{Page}}^{(3)}(\mathcal{E})$, whose components on the last column and the column -1 are zero. It is therefore a coboundary by Proposition 4.3.3 item 3: we can continue a step further and define $\ell_4 \in \bigoplus_{j \geq 3} \text{Hom} \left(\bigodot^4 \mathcal{E}_{|-j-1}, \mathcal{E}_{-j} \right)$ such that:

$$-[\ell_2, \ell_3]_{\text{RN}} = [\ell_1, \ell_4]_{\text{RN}} = [d, \ell_4]_{\text{RN}}. \quad (4.12)$$

We choose the 4-ary bracket to be ℓ_4 . We now proceed by recursion. We assume that we have constructed all the k -ary brackets, ℓ_k such as :

$$[d, \ell_k]_{\text{RN}} = - \sum_{\substack{i+j=k+1 \\ i \leq j}} [\ell_i, \ell_j]_{\text{RN}} = -\frac{1}{2} \sum_{\substack{i+j=k+1 \\ i, j \geq 1}} [\ell_i, \ell_j]_{\text{RN}} \quad (4.13)$$

for every $k = 1, \dots, n$ with $n \geq 4$. The $(n+1)$ -ary bracket is constructed as follows. First, the operator $\sum_{\substack{i+j=k+1 \\ i, j \geq 1}} [\ell_i, \ell_j]_{\text{RN}}$ is checked to be \mathcal{O} -linear as before. Now, we have

$$\sum_{\substack{i+j=n+2 \\ i, j \geq 1}} [d, [\ell_i, \ell_j]_{\text{RN}}]_{\text{RN}} = -2 \sum_{\substack{i+j=n+2 \\ i, j \geq 1}} [\ell_i, [d, \ell_j]_{\text{RN}}]_{\text{RN}} \quad (\text{by graded Jacobi identity}).$$

Since ℓ_j satisfies Equation (4.13) up to order n , we obtain

$$\sum_{\substack{i+j=n+2 \\ i, j \geq 1}} [d, [\ell_i, \ell_j]_{\text{RN}}]_{\text{RN}} = \sum_{\substack{i+j+k=n+3 \\ i, j, k \geq 1}} [\ell_i, [\ell_j, \ell_k]_{\text{RN}}]_{\text{RN}} = 0,$$

where we used the graded Jacobi identity of the Nijenhuis-Richardson bracket in the last step. Therefore, $\sum_{\substack{i+j=n+2 \\ i, j \geq 1}} [\ell_i, \ell_j]_{\text{RN}}$, seen as an element in $\widehat{\mathfrak{Page}}^{(i+j-2)}(\mathcal{E})$ by Remark 4.3.6, is a cocycle and for degree reason it has no element on the last column, and the columns $-1, \dots, 3-n$ in 6.16. The third item of Proposition 4.3.3 gives the existence of an $(n+1)$ -ary bracket ℓ_{n+1} such as

$$[d, \ell_{n+1}]_{\text{RN}} = - \sum_{\substack{i+j=n+2 \\ i \leq j}} [\ell_i, \ell_j]_{\text{RN}}.$$

This completes the proof. \square

Proof of Proposition 4.2.3 and Proposition 4.2.6

Proof (of Proposition 4.2.3). This is a consequence of Proposition 4.2.1 and the third item of the Proposition 4.3.3: If the component of Jac on the column -1 is zero, we can choose ℓ_3 with no component on the last column and in column -1 (see Proposition 4.3.3), i.e. the restriction of ℓ_3 to $\odot^3 \mathcal{E}_{-1}$ is zero. Then ℓ_3 has no component on the last column, the column -1 and the column -2 . so $[\ell_2, \ell_3]_{\text{RN}}$ has no component in the last column, -1 and -2 columns as well. Hence, ℓ_4 can be chosen with no component on column -1 , -2 and -3 by the third item of Proposition 4.3.3. The proof continues by recursion. \square

We finish this section with a proof of Proposition 4.2.6.

Proof (of Proposition 4.2.6). We prove this Proposition in two steps.

1. Lemma 4.3.7 guarantees the existence a free resolution (\mathcal{E}, d, π) of the Lie-Rinehart algebra \mathcal{A} such that \mathcal{E} contains \mathcal{E}' and such that there exists a graded free module \mathcal{V} with $\mathcal{E}' \oplus \mathcal{V} = \mathcal{E}$.
2. Let $D^{\mathcal{E}}$ and $D^{\mathcal{E}'}$ be as in the proof of Lemma 4.3.8. We construct the n -ary brackets on \mathcal{E} by extending the ones of $(\mathcal{E}', (\ell'_k)_{k \geq 1}, \rho_{\mathcal{E}'}, \pi')$ in the following way:

- (a) We first construct an almost Lie algebroid bracket $\tilde{\ell}_2$ on \mathcal{E}_{-1} that extends the 2-ary bracket of \mathcal{E}'_{-1} . Since the 2-ary bracket is determined by its value on a basis, the existence of a free module \mathcal{V}_{-1} such that $\mathcal{E}'_{-1} \oplus \mathcal{V}_{-1} = \mathcal{E}_{-1}$ allows to construct $\tilde{\ell}_2$ on \mathcal{E} such that its restriction to \mathcal{E}' is ℓ'_2 and such that it satisfies the Leibniz identity.

As in the proof of Theorem 4.2.1 (to be more precise: Lemma 4.3.10), we see that $[\tilde{\ell}_2, d^{\mathcal{E}}]_{\text{RN}}$ is \mathcal{O} -linear, hence belongs to $\mathfrak{Page}_2^{(2)}(\mathcal{E})$ and is a $D^{\mathcal{E}}$ -cocycle. Since \mathcal{E}' is a Lie ∞ -algebroid, its restriction to $\odot^2 \mathcal{E}'$ is zero. Lemma 4.3.8 allows to change $\tilde{\ell}_2$ to an 2-ary bracket $\ell_2 := \tilde{\ell}_2 + \tau_2$ with $\tau_2 = 0$ on $\odot^2 \mathcal{E}'$. Hence, ℓ_2 defines a graded almost Lie algebroid bracket, whose restriction to \mathcal{E}' is still ℓ'_2 .

- (b) Since ℓ_2 is an extension of ℓ'_2 , its Jacobiator $\text{Jac} \in \mathfrak{Page}_2^{(2)}(\mathcal{E})$ of the 2-ary bracket ℓ_2 preserves \mathcal{E}' . Also, its restriction $\iota_{\mathcal{E}'}^* \text{Jac} \in \mathfrak{Page}_2^{(2)}(\mathcal{E}')$ is the Jacobiator of ℓ'_2 , and the latter is the $D^{\mathcal{E}'}$ -coboundary of ℓ'_3 in view of the higher Jacobi identity of \mathcal{E}' . Since $\text{Jac} \in \mathfrak{Page}_2^{(2)}(\mathcal{E})$ is a $D^{\mathcal{E}}$ -cocycle, Lemma 4.3.8 assures that Jac is the image through $D^{\mathcal{E}}$ of some element $\ell_3 \in \mathfrak{Page}_1^{(2)}(\mathcal{E})$ which preserves \mathcal{E}' and whose restriction to $\odot^3 \mathcal{E}'$ is ℓ'_3 . The proof continues by recursion: at the n -th step, we use Lemma 4.3.8 to construct an n -ary bracket for \mathcal{E} that extends the n -ary bracket of \mathcal{E}' .

By construction, the inclusion map $\iota: \mathcal{E}' \hookrightarrow \mathcal{E}$ is a morphism for the n -ary brackets for all $n \geq 1$. \square

4.3.3 Proof of universality

Before proving the universal character of the construction, we need to do some preparations.

Let us prove the following lemma,

Lemma 4.3.11. *Let $\Psi, \Xi: S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ be \mathcal{O} -linear Lie ∞ -algebroid morphisms. Let $n \in \mathbb{N}_0$. If $\Xi^{(i)} = \Psi^{(i)}$ for every $0 \leq i \leq n$, there exists*

1. a Lie ∞ -morphism of algebroids $J_1: S_{\mathbb{K}}(\mathcal{E}') \rightarrow S_{\mathbb{K}}(\mathcal{E})$

2. and a homotopy $(J_t, H_t)_{[0,1]}$ joining Ψ and J_1 ,

such that

1. the components of polynomial-degree less or equal to n of H_t vanish,

2. $J_1^{(i)} = \Xi^{(i)}$ for every $0 \leq i \leq n + 1$.

Proof. Let us consider $(\Xi - \Psi)^{(n+1)} : \odot^\bullet \mathcal{E}' \longrightarrow \odot^\bullet \mathcal{E}$. By assumption, $(\Xi - \Psi)^{(i)} = 0$ for all $i \leq n$, so that in view of Lemma 2.3.27

1. $(\Xi - \Psi)^{(n+1)}$ is a $\Psi^{(0)}$ -co-derivation.

2. Proposition 4.3.5 means that the map the restriction of the map $(\Xi - \Psi)^{(n+1)}$ to $\odot^{n+2} \mathcal{E}'$ corresponds to a closed element of degree 0 in $\mathfrak{Page}^{(n+1)}(\mathcal{E}', \mathcal{E})$ equipped with differentials ℓ_1, ℓ'_1 .

Proposition 4.3.3 implies that there exists \mathcal{O} -linear map $H_{n+1} : \odot^{n+2}(\mathcal{E}') \longrightarrow \mathcal{E}$, a degree -1 and of polynomial-degree $n + 1$, i.e. an element in $\mathfrak{Page}^{(n+1)}(\mathcal{E}', \mathcal{E})$, such that,

$$(\Xi - \Psi)^{(n+1)} = Q_{\mathcal{E}}^{(0)} \circ H_{n+1} + H_{n+1} \circ Q_{\mathcal{E}'}^{(0)}. \quad (4.14)$$

We denote its extension to a $\Psi^{(0)}$ -co-derivation of degree -1 by $H^{(n+1)}$. We now consider the following differential equation for $t \in [0, 1]$:

$$\frac{dJ_t}{dt} = Q_{\mathcal{E}} \circ H_t + H_t \circ Q_{\mathcal{E}'}, \quad \text{and} \quad J_0 = \Psi, \quad (4.15)$$

where H_t is the unique J_t -co-derivation of degree -1 whose unique non-zero Taylor coefficient is H_{n+1} . The existence of a solution for the differential equation (4.15) is granted by Proposition 2.3.19. By considering the component of polynomial-degree $1, \dots, n, n + 1$ in Equation (4.15), we find

$$\begin{cases} \frac{dJ_t^{(i)}}{dt} = 0 & \text{for } i = 0, \dots, n, \\ \frac{dJ_t^{(n+1)}}{dt} = Q_{\mathcal{E}}^{(0)} \circ H^{(n+1)} + H^{(n+1)} \circ Q_{\mathcal{E}'}^{(0)} = (\Xi - \Psi)^{(n+1)} \end{cases}$$

Hence:

$$\begin{cases} J_t^{(i)} = \Psi^{(i)} & \text{for } i = 0, \dots, n, \\ J_t^{(n+1)} = \Phi^{(n+1)} + t(\Xi - \Psi)^{(n+1)}. \end{cases}$$

Therefore, applying $t = 1$ to the previous relation, one finds

$$\begin{cases} J_1^{(i)} = \Psi^{(i)} & \text{for } i = 0, \dots, n, \\ J_1^{(n+1)} = \Psi^{(n+1)} + (\Xi - \Psi)^{(n+1)} = \Xi^{(n+1)} \end{cases}$$

This completes the proof. \square

Construction of the Lie ∞ -morphism

Proof (of Theorem 4.2.4). Let us prove item 1. We construct the Taylor coefficients of the Lie ∞ -algebroid Φ by recursion.

The Taylor coefficient of polynomial-degree 0 is obtained out of classical properties of projective resolutions of \mathcal{O} -modules. Given any complex $(\mathcal{E}', \rho', \ell'_1, \pi')$ which terminates in \mathcal{A} through π , for

every free resolution $(\mathcal{E}, \rho, \ell_1, \pi)$ of \mathcal{A} , there exists a chain map $\Phi^{(0)}: (\mathcal{E}', \ell'_1) \rightarrow (\mathcal{E}, \ell_1)$ as in Equation (4.2), and any two such chain maps are homotopic. We still denote by $\Phi^{(0)}$ its extension to an polynomial-degree 0 co-morphism $\odot^\bullet \mathcal{E}' \rightarrow \odot^\bullet \mathcal{E}$.

To construct the second Taylor coefficient, let us consider the map:

$$\begin{aligned} S_{\mathbb{K}}^2(\mathcal{E}') &\rightarrow \mathcal{E} \\ (x, y) &\mapsto \Phi^{(0)} \circ \ell'_2(x, y) - \ell_2(\Phi^{(0)}(x), \Phi^{(0)}(y)). \end{aligned} \quad (4.16)$$

This map is in fact \mathcal{O} -bilinear, i.e. belongs to $\text{Hom}_{\mathcal{O}}(\odot^2 \mathcal{E}', \mathcal{E})$, hence to $\mathfrak{Page}^{(1)}(\mathcal{E}', \mathcal{E})$, see Equation (6.16). Let us check that it is a D -cocycle:

A. If either one of the homogeneous elements $x \in \mathcal{E}'$ or $y \in \mathcal{E}'$ is not of degree -1 , a straightforward computation gives:

$$\begin{aligned} \ell_1 \circ \left(\Phi^{(0)} \circ \ell'_2(x, y) - \ell_2 \left(\Phi^{(0)}(x), \Phi^{(0)}(y) \right) \right) &= \Phi^{(0)} \circ \ell'_1 \circ \ell'_2(x, y) + \ell_2 \left(\Phi^{(0)} \circ \ell'_1(x), \Phi^{(0)}(y) \right) \\ &\quad + (-1)^{|x|} \ell_2 \left(\Phi^{(0)}(x), \Phi^{(0)} \circ \ell'_1(y) \right) \\ &= \left(\Phi^{(0)} \circ \ell'_2 - \ell_2 \left(\Phi^{(0)}, \Phi^{(0)} \right) \right) \circ \ell'_1(x \odot y). \end{aligned}$$

B. If both $x, y \in \mathcal{E}'$ are of degree -1 :

$$\begin{aligned} \pi \left(\Phi^{(0)} \ell'_2(x, y) - \ell_2(\Phi^{(0)}x, \Phi^{(0)}y) \right) &= \pi' \circ \ell'_2(x, y) - \pi \circ \ell_2 \left(\Phi^{(0)}x, \Phi^{(0)}y \right) \\ &= [\pi'(x), \pi'(y)] - \left[\pi \left(\Phi^{(0)}x \right), \pi \left(\Phi^{(0)}y \right) \right] \\ &= [\pi'(x), \pi'(y)] - [\pi'(x), \pi'(y)] \\ &= 0. \end{aligned}$$

By Proposition 4.3.3 item 2), there exists $\Phi^{(1)} \in \text{Hom}_{\mathcal{O}} \left(\odot^2 \mathcal{E}', \mathcal{E} \right)$, of degree 0, so that

$$\Phi^{(0)} \circ \ell'_2(x, y) + \Phi^{(1)} \circ \ell'_1(x \odot y) = \ell_1 \circ \Phi^{(1)}(x, y) + \ell_2(\Phi^{(0)}(x), \Phi^{(0)}(y)) \quad \text{for all } x, y \in \mathcal{E}'. \quad (4.17)$$

$\Phi^{(0)}$ is a chain map and Relation (4.17) can be rewritten in terms of $Q_{\mathcal{E}}$ and $Q_{\mathcal{E}'}$ as follows

$$\begin{cases} Q_{\mathcal{E}}^{(0)} \circ \Phi^{(0)} &= \Phi^{(0)} \circ Q_{\mathcal{E}'}^{(0)} \\ Q_{\mathcal{E}}^{(0)} \circ \Phi^{(1)} - \Phi^{(1)} \circ Q_{\mathcal{E}'}^{(0)} &= \Phi^{(0)} \circ Q_{\mathcal{E}'}^{(1)} - Q_{\mathcal{E}}^{(1)} \circ \Phi^{(0)} \end{cases} \quad (4.18)$$

The construction of the morphism Φ announced in Theorem 4.2.4 is then done by recursion. The recursion assumption is that we have already defined a \mathcal{O} -multilinear co-morphism $\Phi: S_{\mathbb{K}}^\bullet(\mathcal{E}') \rightarrow S_{\mathbb{K}}^\bullet(\mathcal{E})$ with

$$(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(k)} = 0 \quad \text{for all } 0 \leq k \leq n.$$

The co-morphism $\Phi: S_{\mathbb{K}}^\bullet(\mathcal{E}') \rightarrow S_{\mathbb{K}}^\bullet(\mathcal{E})$ with Taylor coefficients $\Phi^{(0)}$ and $\Phi^{(1)}$ satisfies the recursion assumption for $n = 1$.

Assume now that we have a co-morphism Φ that satisfies this assumption for some $n \in \mathbb{N}$, and consider the map $T_{\Phi} := \Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi$. Lemma 2.3.26 implies that $T_{\Phi}^{(n+1)}$ is a \mathcal{O} -multilinear $\Phi^{(0)}$ -co-derivation, and that it corresponds to a D -closed element¹ in $\mathfrak{Page}^{(n+1)}(\mathcal{E}', \mathcal{E})$. Since it has no

¹The following remark is crucial. Under the assumptions of Lemma 2.3.26, $(\Phi \circ Q_{\mathcal{E}'} - Q_{\mathcal{E}} \circ \Phi)^{(n+1)}$ corresponds to a D -closed element of degree $+1$ in the bi-complex $\mathfrak{Page}^{(n+1)}(\mathcal{E}', \mathcal{E})$ through the chain isomorphism described in Proposition 4.3.5. Here, $\mathcal{E}, \mathcal{E}'$ are equipped with the differentials ℓ_1, ℓ'_1 which are the restriction of the components $Q_{\mathcal{E}}^{(0)}, Q_{\mathcal{E}'}^{(0)}$.

component on the last column for degree reason, Proposition 4.3.3 implies that $T_{\Phi}^{(n+1)}$ is a coboundary: That is to say that there is a $\Phi^{(0)}$ -co-derivation $\Theta \in \mathfrak{P}\mathfrak{a}\mathfrak{g}\mathfrak{e}^{(n+1)}(\mathcal{E}', \mathcal{E})$ (of polynomial-degree $n+1$ and degree 0) which can be seen as a map $\Theta : \odot^{n+2}(\mathcal{E}') \rightarrow \mathcal{E}$ such that:

$$T_{\Phi}^{(n+1)} = Q_{\mathcal{E}}^{(0)} \circ \Theta - \Theta \circ Q_{\mathcal{E}'}^{(0)}.$$

Consider now the co-morphism $\tilde{\Phi}$ whose Taylor coefficients are those of Φ in polynomial-degree $0, \dots, n$ and $\Phi^{(n+1)} + \Theta$ in polynomial-degree $n+1$:

$$\tilde{\Phi}^{(i)} := \begin{cases} \Phi^{(i)} & \text{if } 0 \leq i \leq n, \\ \Phi^{(n+1)} + \Theta & \text{if } i = n+1 \end{cases} \quad (4.19)$$

This is easily seen to satisfy the recursion relation for $n+1$. This concludes the recursion. The Taylor coefficients obtained by recursion define a Lie ∞ -algebroid $\Phi : S_{\mathbb{K}}^{\bullet}(\mathcal{E}') \rightarrow S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ which is compatible by construction with the hooks π, π' .

By continuing this procedure, we construct a Lie ∞ -morphism from $S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$ to $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$. This proves the first item of Theorem 4.2.4.

Construction of a homotopy that joins two such morphisms

Let us prove the second item in Theorem 4.2.4. Notice that in the proof of the existence of the Lie ∞ -morphism between $S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$ and $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$ obtained in the first item, we made many choices, since we have chosen a coboundary at each step of the recursion.

Let Φ, Ψ be two such Lie ∞ -morphisms between $S_{\mathbb{K}}^{\bullet}(\mathcal{E}')$ and $S_{\mathbb{K}}^{\bullet}(\mathcal{E})$. The polynomial-degree 0 component of the co-morphisms Φ and Ψ restricted to \mathcal{E}' are chain maps:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{E}'_{-2} & \longrightarrow & \mathcal{E}'_{-1} & & \\ & & \searrow & & \searrow & \nearrow & \pi' \\ & & \Psi^{(0)} & & \Psi^{(0)} & & \mathcal{A} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \Phi^{(0)} & & \Phi^{(0)} & & \pi \\ & & \swarrow & & \swarrow & & \\ \cdots & \longrightarrow & \mathcal{E}_{-2} & \longrightarrow & \mathcal{E}_{-1} & & \end{array}$$

(Note: Dashed arrows labeled h connect $\mathcal{E}'_{-2} \rightarrow \mathcal{E}_{-2}$ and $\mathcal{E}'_{-1} \rightarrow \mathcal{E}_{-1}$.)

which are homotopy equivalent in the usual sense because (\mathcal{E}, ℓ_1) is a projective resolution of \mathcal{A} : said differently, there exists a degree -1 \mathcal{O} -linear map $h : \mathcal{E}' \rightarrow \mathcal{E}$ such that

$$\Psi^{(0)} - \Phi^{(0)} = \ell_1 \circ h + h \circ \ell'_1 \quad \text{on } \mathcal{E}'. \quad (4.20)$$

Let us consider the following differential equation:

$$\begin{cases} \frac{dJ_t}{dt} = Q_{\mathcal{E}} \circ H_t(J_t) + H_t(J_t) \circ Q_{\mathcal{E}'}, & \text{for } t \in [0, 1] \\ J_0 = \Phi. \end{cases} \quad (4.21)$$

with $H_t(J_t)$ being a J_t -co-derivation of degree -1 whose Taylor coefficient of polynomial-degree 0 is h . This equation does admit solutions in view of Proposition 2.3.19.

By looking at the component polynomial-degree 0 of Equation (4.21) on \mathcal{E}' , one has:

$$\begin{aligned} \frac{dJ_t^{(0)}}{dt} &= \ell_1 \circ h + h \circ \ell'_1 \\ &= \Psi^{(0)} - \Phi^{(0)}. \end{aligned}$$

Hence, $J_t^{(0)} = \Phi^{(0)} + t(\Psi^{(0)} - \Phi^{(0)})$ is a solution such that $J_1^{(0)} = \Psi^{(0)}$. By construction, J_1 is homotopic to Φ via the pair (J_t, H_t) over $[0, 1]$, and its polynomial-degree 0 Taylor coefficient coincides with the Taylor coefficient of Ψ .

From there, the construction goes by recursion using Lemma 4.3.11. Indeed, this lemma allows constructing recursively a sequence of Lie ∞ -algebroids morphism $(\Psi_n)_{n \geq 0}$ and homotopies $(J_{n,t}, H_{n,t})$ (with $t \in [n, n+1]$) between Ψ_n and Ψ_{n+1} such that: $H_{n,t}^{(i)}$ is zero for $t \geq n$ and $i \neq n+1$. By Lemma 4.3.11, all these homotopies are compatible with the hooks. These homotopies are glued in a homotopy $(J_t, H_t)_{[0, +\infty[}$ such that for every $n \in \mathbb{N}_0$, the components of polynomial-degree n of the Lie ∞ -algebroids morphism $J_t^{(n)}$ are constant and equal to $\Psi^{(n)}$ for $t \geq n$. By Lemma 2.3.22, these homotopies can be glued to a homotopy on $[0, 1]$. Explicitly, since $t \mapsto \frac{t}{1-t}$ maps $[0, 1[$ to $[0, +\infty[$ and by Lemma 2.3.22, the pair $\left(J_{\frac{t}{1-t}}, \frac{1}{(1-t)^2} H_{k, \frac{t}{1-t}} \right)$ is a homotopy between Φ and Ψ . This proves the second item of the Theorem 4.2.4. \square

4.4 Examples of universal Lie ∞ -algebroids of Lie-Rinehart algebras

4.4.1 New constructions from old ones

In this section, we explain how to construct universal Lie ∞ -algebroids of some Lie-Rinehart algebra which is derived from a second one through one of natural constructions as in Section 3.2.1 (localization, germification, restriction), when a universal Lie ∞ -algebroid of the latter is already known.

Localization

Localization is an useful algebraic tool, specially in algebraic geometry. When \mathcal{O} is an algebra of functions, it corresponds to study local properties of a space, or germs of functions.

Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra over \mathcal{O} . Let $S \subset \mathcal{O}$ be a multiplicative-closed subset containing no zero divisor. We recall from item 2, Example 3.2.1 that the localization $S^{-1}\mathcal{A} \cong \mathcal{A} \otimes_{\mathcal{O}} S^{-1}\mathcal{O}$ of \mathcal{A} at S comes equipped with a natural structure of Lie-Rinehart algebra over the localization algebra $S^{-1}\mathcal{O}$. Recall that for $\varphi: \mathcal{E} \rightarrow \mathcal{T}$ a homomorphism of \mathcal{O} -modules, there is a well-defined homomorphism of \mathcal{O} -modules,

$$\varphi \otimes \text{id}: \mathcal{E} \otimes_{\mathcal{O}} S^{-1}\mathcal{O} \rightarrow \mathcal{T} \otimes_{\mathcal{O}} S^{-1}\mathcal{O}, \quad \varphi \otimes \text{id} \left(x \otimes \frac{f}{s} \right) := \varphi(x) \otimes \frac{f}{s}$$

that can be considered as a $S^{-1}\mathcal{O}$ -module homomorphism

$$S^{-1}\varphi: S^{-1}\mathcal{E} \rightarrow S^{-1}\mathcal{T} \quad \text{with} \quad S^{-1}\varphi \left(\frac{x}{s} \right) := \frac{\varphi(x)}{s}, \quad x \in \mathcal{E}, (f, s) \in \mathcal{O} \times S,$$

called the *localization of φ* .

Given a Lie ∞ -algebroid structure $(\mathcal{E}_{\bullet}, \ell_{\bullet}, \rho)$ that covers \mathcal{A} through π . The triplet $(\mathcal{E}'_{\bullet}, \ell'_{\bullet}, \rho')$ is a Lie ∞ -algebroid structure that covers $S^{-1}\mathcal{A}$ through the hook π' where

1. $\mathcal{E}' = S^{-1}\mathcal{E}$;

2. The anchor map ρ' is defined by

$$\begin{aligned} \rho': S^{-1}\mathcal{E}_{-1} &\longrightarrow \text{Der}(S^{-1}\mathcal{O}) \\ \frac{x}{s} &\longmapsto \rho'\left(\frac{x}{s}\right): S^{-1}\mathcal{O} \longrightarrow S^{-1}\mathcal{O} \\ \frac{f}{u} &\longmapsto \frac{1}{s} \cdot \left(\frac{\rho(x)[f]u - f\rho(x)[u]}{u^2} \right) \end{aligned}$$

for $x \in \mathcal{E}, f \in \mathcal{O}, (s, u) \in S \times S$;

3. $\ell'_k = S^{-1}\ell_k$, for all $k \in \mathbb{N} \setminus \{2\}$;

4. The binary bracket is more complicated because of the anchor map: we set

$$\ell'_2\left(\frac{1}{s}x, \frac{1}{u}y\right) = \frac{1}{su}\ell_2(x, y) - \frac{\rho(x)[u]}{su^2}y + \frac{\rho(y)[s]}{s^2u}x$$

for $x, y \in \mathcal{E}, (s, u) \in S^2$ (with the understanding that $\rho \equiv 0$ on \mathcal{E}_{-i} with $i \geq 2$);

5. $\pi' = S^{-1}\pi$.

One can check that these operations above are well-defined and for all $z \in S^{-1}\mathcal{E}_{-1}$ the map $\rho'(z)$ is indeed a derivation on $S^{-1}\mathcal{O}$. The previously defined structure is also a Lie ∞ -algebroid that we call *localization of the Lie ∞ -algebroid $(\mathcal{E}_\bullet, \ell_\bullet, \rho)$ with respect to S* .

Proposition 4.4.1. *Let $S \subset \mathcal{O}$ be a multiplicative subset containing no zero divisor. The localization of a universal Lie ∞ -algebroid of a Lie-Rinehart algebra \mathcal{A} is a universal Lie ∞ -algebroid of $S^{-1}\mathcal{A}$.*

Proof. The object $((\mathcal{E}'_\bullet, \ell'_\bullet, \rho'))$ described above is also a Lie ∞ -algebroid terminating in $S^{-1}\mathcal{A}$ through π' . It is universal because localization preserves exact sequences [And]. \square

Restriction

When \mathcal{O}_Y is the ring of functions of an affine variety Y (see Section 5.1), to every subvariety $X \subset Y$ corresponds its zero locus, which is an ideal $\mathcal{I}_X \subset \mathcal{O}_Y$. A Lie ∞ -algebroid or a Lie-Rinehart algebra over \mathcal{O}_Y may not restrict to a Lie-Rinehart algebra over \mathcal{O}_X : it only does so when one can quotient all brackets by \mathcal{I}_X , which geometrically means that the anchor map takes values in vector fields tangent to X . We can then “restrict”, i.e. replace \mathcal{O}_Y by $\mathcal{O}_Y/\mathcal{I}_X$. This operation has already been defined in Section 4.2, and here is an immediate consequence of Corollary 4.2.13:

Proposition 4.4.2. *Let $\mathcal{I} \subset \mathcal{O}$ be a Lie-Rinehart ideal, i.e. an ideal such that $\rho_{\mathcal{A}}(\mathcal{A})[\mathcal{I}] \subset \mathcal{I}$. The quotient of a universal Lie ∞ -algebroid of \mathcal{A} with respect to an ideal \mathcal{I} is a Lie ∞ -algebroid that terminates in $\mathcal{A}/\mathcal{I}\mathcal{A}$. It is universal if and only if $((\frac{\mathcal{E}_{-i}}{\mathcal{I}\mathcal{E}_{-i}})_{i \geq 1}, \bar{\ell}_1, \bar{\pi})$ is exact, i.e. if $\text{Tor}_{\mathcal{O}}^\bullet(\mathcal{A}, \mathcal{O}/\mathcal{I}) = 0$.*

Remark 4.4.3. Note that the anchor map $\rho: \mathcal{E}_{-1} \rightarrow \text{Der}(\mathcal{O})$ goes to quotient to $\frac{\mathcal{E}_{-1}}{\mathcal{I}\mathcal{E}_{-1}} \rightarrow \text{Der}(\mathcal{O})$ as an \mathcal{O} -linear map, but needs the extra condition $\rho(\mathcal{E}_{-1})[\mathcal{I}] \subset \mathcal{I}$ to induce an \mathcal{O}/\mathcal{I} -linear map $\frac{\mathcal{E}_{-1}}{\mathcal{I}\mathcal{E}_{-1}} \rightarrow \text{Der}(\mathcal{O}/\mathcal{I})$.

Germification

Let $W \subseteq \mathbb{C}^N$ be an affine variety and \mathcal{O}_W its coordinates ring (see Section 5.1). For $a \in W$, consider $\mathcal{O}_{W,a}$ the local ring at a . Note that $\mathcal{O}_{W,a} \simeq (\mathcal{O}_W)_{\mathfrak{m}_a}$ [Har77], where $\mathfrak{m}_a = \{f \in \mathcal{O}_W \mid f(a) = 0\}$ and $(\mathcal{O}_W)_{\mathfrak{m}_a}$ is the localization w.r.t the complement of \mathfrak{m}_a , Proposition 4.4.1 implies the following statement:

Proposition 4.4.4. *Let W be an affine variety with functions \mathcal{O}_W . For every point $a \in W$ and any Lie-Rinehart \mathcal{A} over \mathcal{O}_W , the germ at a of the universal Lie ∞ -algebroid of \mathcal{A} is the universal Lie ∞ -algebroid of the germ of \mathcal{A} at a .*

Therefore, the germ at a of a Lie-Rinehart algebra or a Lie ∞ -algebroid is simply its localization w.r.t the complement of \mathfrak{m}_a .

4.4.2 Sections vanishing on a codimension 1 subvariety

Let $(\mathcal{A}, [\cdot, \cdot], \rho_{\mathcal{A}})$ be an arbitrary Lie-Rinehart algebra over \mathcal{O} . For any ideal $\mathcal{I} \subset \mathcal{O}$, $\mathcal{I}\mathcal{A}$ is also a Lie-Rinehart algebra (see Example 3.2.1). When \mathcal{O} are functions on a variety M , \mathcal{I} are functions vanishing on a subvariety X and \mathcal{A} is a \mathcal{O} -module of sections over M , $\mathcal{I}\mathcal{A}$ corresponds geometrically to sections vanishing along X . It is not an easy task. In codimension 1, i.e. when \mathcal{I} is generated by one element, the construction can be done by hand.

Proposition 4.4.5. *Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be a Lie-Rinehart algebra over a commutative algebra \mathcal{O} . Let $(\mathcal{E}, \ell_k = \{\dots\}_{k \geq 1}, \rho)$ be a Lie ∞ -algebroid that terminates in \mathcal{A} through a hook π . For any element $\chi \in \mathcal{O}$, the \mathcal{O} -module $\mathcal{A}' = \chi\mathcal{A} \subseteq \mathcal{A}$ is closed under the Lie bracket, so the triple $(\chi\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ is a Lie-Rinehart algebra over \mathcal{O} . A Lie ∞ -algebroid $(\mathcal{E}' = \mathcal{E}, \ell'_k = \{\dots\}'_{k \geq 1}, \rho')$ hooked in $\chi\mathcal{A}$ through π' can be defined as follows:*

1. The brackets are given by

$$(a) \{\cdot\}'_1 = \{\cdot\}_1,$$

(b) the 2-ary bracket:

$$\{x, y\}'_2 := \chi\{x, y\}_2 + \rho(x)[\chi]y + (-1)^{|x||y|}\rho(y)[\chi]x, \quad (4.22)$$

for all $x, y \in \mathcal{E}_{\bullet}$, with the understanding that $\rho = 0$ on $\mathcal{E}_{\leq -2}$,

$$(c) \{\dots\}'_k = \chi^{k-1}\{\dots\}_k \text{ for all } k \geq 3,$$

2. $\rho_{\mathcal{E}'} = \chi\rho$,

3. $\pi' = \chi\pi$.

Proof. We leave it to the reader. □

Proposition 4.4.6. *If χ is not a zero-divisor in \mathcal{O} , and $(\mathcal{E}, \{\dots\}_{k \geq 1}, \rho, \pi)$ is a universal Lie ∞ -algebroid of \mathcal{A} , then the Lie ∞ -structure described in the four items of 4.4.1 is the universal Lie ∞ -algebroid of $\chi\mathcal{A}$.*

Proof. If χ is not a zero-divisor in \mathcal{A} (i.e. if $a \mapsto \chi a$ is an injective endomorphism of \mathcal{A}), then the kernel of π' coincides with the kernel of π , i.e. with the image of $\{\cdot\}'_1 = \{\cdot\}_1$, so that $(\mathcal{E}, \ell_1, \chi\pi)$ is a resolution of $\chi\mathcal{A}$. □

4.4.3 Algebra extension

Recall that for \mathcal{O} a unital algebra with no zero divisor, derivations of \mathcal{O} induce derivations of its field of fractions \mathbb{O} .

Proposition 4.4.7. *Let \mathcal{O} be an unital algebra with no zero divisor, \mathbb{O} its field of fractions, and $\tilde{\mathcal{O}}$ an algebra with $\mathcal{O} \subset \tilde{\mathcal{O}} \subset \mathbb{O}$. For every Lie-Rinehart algebra \mathcal{A} over \mathcal{O} whose anchor map takes values in derivations of \mathcal{O} preserving $\tilde{\mathcal{O}}$, then*

1. *any Lie ∞ -algebroid structure $(\mathcal{E}, (\ell_k)_{k \geq 1}, \rho, \pi)$ that terminates at \mathcal{A} extends for all $i = 0, \dots, n$ to a Lie ∞ -algebroid structure on $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{E}$,*
2. *and this extension $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{E}$ is a Lie ∞ -algebroid that terminates at the Lie-Rinehart algebra $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{A}$.*

Proof. Since they are \mathcal{O} -linear, the hook π , the anchor ρ , and the brackets ℓ_k for $k \neq 2$ are extended to $\tilde{\mathbb{O}}$ -linear maps. Since the image of ρ is the image of $\rho_{\mathcal{A}}$, it is made of derivations preserving \mathbb{O} , which is easily seen to allow an extension of ℓ_2 to $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{E}$ using the Leibniz identity. \square

Remark 4.4.8. Of course, the Lie ∞ -algebroid structure obtained on $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{E}$ is not in general the universal Lie ∞ -algebroid of $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{A}$, because the complex $(\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{E}, \ell_1, \pi)$ may not be a resolution of $\tilde{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{A}$ (see Example 4.4.10).

Remark 4.4.9. Since any module over a field is projective, any Lie-Rinehart algebra over a field is a Lie algebroid. If we choose $\tilde{\mathcal{O}} = \mathbb{O}$ therefore, the Lie-Rinehart algebra $\mathbb{O} \otimes_{\mathcal{O}} \mathcal{A}$ is a Lie algebroid, so is homotopy equivalent to any of its universal Lie ∞ -algebroid. Unless \mathcal{A} is a Lie algebroid itself, the Lie ∞ -algebroid in Proposition 4.4.7 will not be homotopy equivalent to a Lie ∞ -algebroid whose underlying complex is of length one, and is therefore not a universal Lie ∞ -algebroid of $\mathbb{O} \otimes_{\mathcal{O}} \mathcal{A}$.

4.4.4 Blow-up

Let us investigate the behavior of the universal Lie ∞ -algebroids under blow-up.

We recall that the blowup of \mathbb{C}^{N+1} at the origin is given by, $B_0(\mathbb{C}^{N+1}) = \{(x, \ell) \in \mathbb{C}^{N+1} \times \mathbb{P}^N \mid x \in \ell\}$ together with the map

$$\begin{aligned} \pi: B_0(\mathbb{C}^{N+1}) &\longrightarrow \mathbb{C}^{N+1} \\ (x, \ell) &\mapsto x. \end{aligned}$$

For $\mathcal{O} = \mathbb{C}[z_0, \dots, z_N]$ the coordinate ring of \mathbb{C}^{N+1} , the blow-up of \mathbb{C}^{N+1} at the origin is covered by affine charts: in the i -th affine chart U_i , the coordinate ring is

$$\mathcal{O}_{U_i} = \mathbb{C}[z_0/z_i, \dots, z_i, \dots, z_N/z_i].$$

By Remark 3.2.1, and Proposition 4.4.7 for any Lie-Rinehart algebra \mathcal{A} whose anchor map takes values in vector fields vanishing at $0 \in \mathbb{C}^{N+1}$, we obtain a Lie ∞ -algebroid of $\mathcal{O}_{U_i} \otimes_{\mathcal{O}} \mathcal{A}$ that we call *blow-up of at 0 in the chart U_i* . Proposition 4.4.7 then says that the blow-up at 0 of the universal Lie ∞ -algebroid of \mathcal{A} , in each chart, is a Lie ∞ -algebroid that terminates in the blow-up of \mathcal{A} (as defined in remark 3.2.1). It may not be the universal one, see Example 4.4.10.

Example 4.4.10 (Universal Lie- ∞ -algebroids and blow-up: a counter example). Consider the polynomial function in $N + 1$ variables $\varphi = \sum_{i=0}^N z_i^3$. Let us consider the singular foliation $\mathfrak{F}_\varphi \subset \mathfrak{X}(\mathbb{C}^N)$ as in Example 4.4.6. Its generators are $\Delta_{ij} := z_i^2 \frac{\partial}{\partial z_j} - z_j^2 \frac{\partial}{\partial z_i}$, for $0 \leq i < j \leq N$.

Let us consider its blow-up in the chart U_N . Geometrically speaking, $\widetilde{\mathfrak{F}}_\varphi = \mathcal{O}_{U_N} \otimes_{\mathcal{O}} \mathfrak{F}_\varphi$ is the \mathcal{O}_{U_N} -module generated by the blown-up vector fields $\widetilde{\Delta}_{ij} = z_N \left(z_i^2 \frac{\partial}{\partial z_j} - z_j^2 \frac{\partial}{\partial z_i} \right)$, $j \neq N$, and $\widetilde{\Delta}_{iN} = z_N \left(z_N z_i^2 \frac{\partial}{\partial z_N} - \frac{\partial}{\partial z_i} - z_i^2 \sum_{j=0}^{N-1} z_j \frac{\partial}{\partial z_j} \right)$ of the vector fields Δ_{ij} , for $i, j \in \{0, \dots, N\}$, w.r.t affine chart U_N . The vector fields $\widetilde{\Delta}_{ij}$, $j \neq N$, belong to the \mathcal{O}_{U_N} -module generated by the vector fields $\widetilde{\Delta}_{iN}$, (explicitly $\widetilde{\Delta}_{ij} = z_j^2 \widetilde{\Delta}_{iN} - z_i^2 \widetilde{\Delta}_{jN}$). Said differently, the vector fields $\widetilde{\Delta}_{iN}$, $i = 0, \dots, N-1$ are generators of $\widetilde{\mathfrak{F}}_\varphi$. Since they are independent, the singular foliation $\widetilde{\mathfrak{F}}_\varphi$ is a free \mathcal{O}_{U_N} -module. A universal Lie ∞ -algebroid for it is therefore concentrated in degree -1 and is given by the \mathcal{O}_{U_N} -module generated by some set $(e_i, i = 0 \dots N-1)$, equipped with the Lie bracket:

$$[e_i, e_j] := 2z_N (z_i^2 e_j - z_j^2 e_i) \quad (4.23)$$

together with the anchor map ρ which assigns e_i to $\widetilde{\Delta}_{iN}$, for $i = 1 \dots N-1$.

On the other hand, the blow-up of the universal Lie ∞ -algebroid of \mathfrak{F}_φ is not homotopy equivalent to a Lie algebroid. Let us show this point. In this case, the Lie ∞ -algebroid is given in Section 4.4.6: notice that $E_{-i} \neq 0$ for $i = 1, \dots, N$ and that $\ell_1|_0 = 0$. The pull-back Lie ∞ -algebroid $(\widetilde{E}, (\widetilde{\ell}_k)_{k \geq 1}, \widetilde{\rho}, \widetilde{\pi})$ verifies by construction that $\widetilde{E}_{-i} \neq 0$ for $i = 1, \dots, N$ and that $\widetilde{\ell}_1|_x = 0$ for every x in the inverse image of zero, and such a complex can not be homotopy equivalent to a complex of length 1. In other words, $\mathcal{O}_{U_i} \otimes_{\mathcal{O}} \cdot$ is not an exact functor in this case (which is a classical fact in algebraic geometry).

This example tells us that the blow-up of the universal Lie ∞ -algebroid of an affine variety W may not be the universal Lie ∞ -algebroid of its blow-up, (even locally).

4.4.5 Universal Lie ∞ -algebroids of some singular foliations

Singular foliation is defined in Section 3.2. The coming example uses the notion "multi-derivation", it is useful to recall the definition and write down some basic operations on them. We refer the reader to Section 3.1 of [LGPV13] for more details on this notion.

Definition 4.4.11. A skew symmetric k -multilinear map $P \in \text{Hom}_{\mathbb{K}}(\mathcal{O}^k, \mathcal{O})$ is said to a k -multi-derivation of \mathcal{O} if P is a derivation in each of its argument, i.e., for every $i \in \{1, \dots, k\}$ and for all $f, g \in \mathcal{O}$ we have

$$P[f_1, \dots, \underbrace{fg}_{i\text{-th slot}}, \dots, f_k] = P[f_1, \dots, \underbrace{f}_{i\text{-th slot}}, \dots, f_k] g + f P[f_1, \dots, \underbrace{g}_{i\text{-th slot}}, \dots, f_k] \quad (4.24)$$

When \mathcal{O} happen to be the coordinate ring of some affine variety W , k -multi-derivations of \mathcal{O} are called k -multi-vector fields on W , and denote by $\mathfrak{X}^k(W)$ the module of k -multi-vector fields on W . In general, it is denoted by $\mathfrak{X}^k(\mathcal{O})$.

Remark 4.4.12. By skew-symmetry argument, it is enough that the relation (4.24) holds for the first slot, i.e. for $i = 1$.

Lemma 4.4.13. *The graded module $\mathfrak{X}^\bullet(\mathcal{O}) := \bigoplus_{k \geq 0} \mathfrak{X}^k(\mathcal{O})$ comes equipped with graded algebra structure whose product \wedge is defined as follows,*

$$(P \wedge R)[f_1, \dots, f_{k+l}] := \sum_{\sigma \in \mathfrak{S}(k,l)} \epsilon(\sigma) P[f_{\sigma(1)}, \dots, f_{\sigma(k)}] R[f_{\sigma(k+1)}, \dots, f_{\sigma(k+l)}],$$

for $P \in \mathfrak{X}^k(\mathcal{O})$, $R \in \mathfrak{X}^l(\mathcal{O})$ and $f_1, \dots, f_{k+l} \in \mathcal{O}$. By convention $\mathfrak{X}^0(\mathcal{O}) = \mathcal{O}$. Also, $f \wedge X := fX$ for all $f \in \mathcal{O}$.

Lemma 4.4.14. *For every $f \in \mathcal{O}$, the contraction by f , $\iota_f: \mathfrak{X}^k(\mathcal{O}) \rightarrow \mathfrak{X}^{k-1}(\mathcal{O}), \forall k \geq 1$, which is defined as follows²*

$$P \mapsto \iota_f(P)[f_1, \dots, f_{k-1}] := P[f, f_1, \dots, f_{k-1}], \quad \text{for all } f_1, \dots, f_{k-1} \in \mathcal{O} \quad (4.25)$$

satisfies

- $\iota_f \circ \iota_f = 0$,
- for all $P \in \mathfrak{X}^k(\mathcal{O})$ and $R \in \mathfrak{X}^l(\mathcal{O})$,

$$\iota_f(P \wedge R) = \iota_f(P) \wedge R + (-1)^k P \wedge \iota_f(R). \quad (4.26)$$

Proof. The first item is true by skew-symmetry of multi-derivations. Now let us prove the second item. For $f_1, \dots, f_{k+l-1} \in \mathcal{O}$, we have

$$\begin{aligned} \iota_f(P \wedge R)[f_1, \dots, f_{k+l-1}] &= (P \wedge R)[f, f_1, \dots, f_{k+l-1}] \\ &= \sum_{\sigma \in \mathfrak{S}(k-1,l)} \epsilon(\sigma) P[f, f_{\sigma(1)}, \dots, f_{\sigma(k-1)}] R[f_{\sigma(k)}, \dots, f_{\sigma(k+l-1)}] + \\ &\quad (-1)^k \sum_{\sigma \in \mathfrak{S}(k,l-1)} \epsilon(\sigma) \underbrace{P[f_{\sigma(1)}, \dots, f_{\sigma(k)}] R[f, f_{\sigma(k+1)}, \dots, f_{\sigma(k+l-1)}]}_{\text{here, the inversion number with } f \text{ is } k} \\ &= \iota_f(P) \wedge R + (-1)^k P \wedge \iota_f(R). \end{aligned}$$

We have used the fact that for every $\sigma \in \mathfrak{S}(k, l)$, one has that either $\sigma(1) = 1$ or $\sigma(k+1) = 1$. \square

Remark 4.4.15. Notice the following:

- For all $f \in \mathcal{O}$ and $X \in \text{Der}(\mathcal{O})$, one has $\iota_f(X) = X[f]$.
- For any two derivation $X, Y \in \text{Der}(\mathcal{O})$ we have

$$(X \wedge Y)[f, g] = X[f]Y[g] - Y[g]X[f].$$

- For a family of derivations $X_1, \dots, X_k \in \text{Der}(\mathcal{O})$, the k -multi-derivation $X_1 \wedge \dots \wedge X_k$ applied to $f_1, \dots, f_k \in \mathcal{O}$ is equal to the determinant

$$\begin{vmatrix} X_1[f_1] & \cdots & X_1[f_k] \\ \vdots & & \vdots \\ X_k[f_1] & \cdots & X_k[f_k] \end{vmatrix}.$$

- When \mathcal{O} is the algebra of polynomial functions in d variables $\mathbb{K}[x_1, \dots, x_d]$, a k -multi-derivation P admits the coordinate expression

$$P = \sum_{1 \leq i_1 < \dots < i_k \leq d} P[x_{i_1}, \dots, x_{i_k}] \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}.$$

²It should be understood that $\iota_f(\mathcal{O}) := 0$.

Lie derivative

For $P \in \mathfrak{X}^k(\mathcal{O})$ and $X \in \text{Der}(\mathcal{O})$, the *Lie derivative* $\mathcal{L}_X P \in \mathfrak{X}^k(\mathcal{O})$ of P along X is defined as

$$(\mathcal{L}_X P)[f_1, \dots, f_k] := X[P[f_1, \dots, f_k]] - \sum_{j=1}^k P[f_1, \dots, X[f_j], \dots, f_k]. \quad (4.27)$$

There is another important operation on multi-derivations, the so-called "Schouten bracket" which is a generalization of the commutator of derivations, also Lie derivative of multi-derivation along a vector field. We will not recall how it is defined here, since we do not really make use of it. We refer the reader to, e.g. [LGPV13] for this notion.

4.4.6 Vector fields annihilating a Koszul function φ

This universal Lie ∞ -algebroid was already described in Section 3.7 of [LLS20], where the brackets were simply checked to satisfy the higher Jacobi identities - with many computations left to the reader. Here, we give a theoretical explanation of the construction presented in [LLS20].

Let \mathcal{O} be the algebra of all polynomials on $V := \mathbb{C}^d$. A function $\varphi \in \mathcal{O}$ is said to be a *Koszul polynomial*, if the *Koszul complex*

$$\dots \xrightarrow{\iota_\varphi} \mathfrak{X}^3(V) \xrightarrow{\iota_\varphi} \mathfrak{X}^2(V) \xrightarrow{\iota_\varphi} \mathfrak{X}(V) \xrightarrow{\iota_\varphi} \mathcal{O} \longrightarrow 0 \quad (4.28)$$

is exact in all degree, except in degree 0. By virtue of a theorem of Koszul [Eis95], see [Hid89] Theorem 16.5 (i), φ is *Koszul* if $\left(\frac{\partial \varphi}{\partial x_1}, \dots, \frac{\partial \varphi}{\partial x_d}\right)$ is a regular sequence.

From now on, we choose φ a Koszul function, and consider the Lie-Rinehart algebra (which is a singular foliation)

$$\mathfrak{F}_\varphi := \{X \in \mathfrak{X}(V) : X[\varphi] = 0\} = \text{Ker}(\iota_\varphi) : \mathfrak{X}(V) \xrightarrow{\iota_\varphi} \mathcal{O}. \quad (4.29)$$

The Koszul complex (4.28) truncated of its degree 0 term gives a free resolution (\mathcal{E}, d, ρ) of \mathfrak{F}_φ , with $\mathcal{E}_{-i} := \mathfrak{X}^{i+1}(V)$, $d := \iota_\varphi$, and $\rho := -\iota_\varphi$.

Remark 4.4.16. Exactness of the Koszul complex implies in particular that \mathfrak{F}_φ is generated by the vector fields:

$$\left\{ \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_i}, \mid 1 \leq i < j \leq d \right\}. \quad (4.30)$$

In [LLS20], this resolution is equipped with a Lie ∞ -algebroid structure, whose brackets we now recall.

Proposition 4.4.17. *A universal Lie ∞ -algebroid of $\mathfrak{F}_\varphi \subset \mathfrak{X}(V)$ is given on the free resolution $(\mathcal{E}_{-\bullet} = \mathfrak{X}^{\bullet+1}(V), d = \iota_\varphi, \rho = -\iota_\varphi)$ by defining the following n -ary brackets:*

$$\{\partial_{I_1}, \dots, \partial_{I_n}\}_n := \sum_{i_1 \in I_1, \dots, i_n \in I_n} \epsilon(i_1, \dots, i_n) \varphi_{i_1 \dots i_n} \partial_{I_1^{i_1} \bullet \dots \bullet I_n^{i_n}}; \quad (4.31)$$

and the anchor map given for all $i, j \in \{1, \dots, n\}$ by

$$\rho \left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) := \frac{\partial \varphi}{\partial x_j} \frac{\partial}{\partial x_i} - \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_j}. \quad (4.32)$$

Above, for every multi-index $J = \{j_1, \dots, j_n\} \subseteq \{1, \dots, d\}$ of length n , ∂_J stands for the n -vector field $\frac{\partial}{\partial x_{j_1}} \wedge \dots \wedge \frac{\partial}{\partial x_{j_n}}$ and $\varphi_{j_1 \dots j_n} := \frac{\partial^n \varphi}{\partial x_{j_1} \dots \partial x_{j_n}}$. Also, $I_1 \bullet \dots \bullet I_n$ is a multi-index obtained by

concatenation of n multi-indices I_1, \dots, I_n . For every $i_1 \in I_1, \dots, i_n \in I_n$, $\epsilon(i_1, \dots, i_n)$ is the signature of the permutation which brings i_1, \dots, i_n to the first n slots of $I_1 \bullet \dots \bullet I_n$. Last, for $i_s \in I_s$, we define $I_s^{i_s} := I_s \setminus i_s$.

To understand this structure, let us first define a sequence of degree +1 graded symmetric poly-derivations on $\mathfrak{X}^\bullet(V)$ (by convention, i -vector fields are of degree $-i + 1$) by:

$$\{\partial_{i_1}, \dots, \partial_{i_k}\}'_k := \frac{\partial^k \varphi}{\partial x_{i_1} \dots \partial x_{i_k}}. \quad (4.33)$$

We extend them to a graded poly-derivation of $\mathfrak{X}^\bullet(V)$.

Lemma 4.4.18. *The poly-derivations $(\{\dots\}'_k)_{k \geq 1}$ are \mathcal{O} -multilinear and equip $\mathfrak{X}^\bullet(V)$ with a (graded symmetric) Poisson ∞ -algebra structure. Also, $\{\cdot\}'_1 = \iota_\varphi$.*

Proof. For degree reason, $\{F, X_1, \dots, X_{k-1}\}'_k = 0$ for all $X_1, \dots, X_{k-1} \in \mathfrak{X}^\bullet(V)$ and all $F \in \mathfrak{X}^0(V) = \mathcal{O}$. This implies the required \mathcal{O} -multilinearity. It is clear that the higher Jacobi identities hold since brackets of generators $\{\delta_{i_1}, \dots, \delta_{i_n}\}'$ are elements in \mathcal{O} , and all brackets are zero when applied an element in \mathcal{O} . \square

Proof (of Proposition 4.4.17). The brackets introduced in Proposition 4.4.17 are modifications of the Poisson ∞ -algebra described in Lemma 4.4.18. By construction, $\{\dots\}'_n = \{\dots\}_n$ when all arguments are generators of the form ∂_I for some $I \subset \{1, \dots, n\}$ of cardinal ≥ 2 . By \mathcal{O} -multilinearity, this implies $\{\dots\}'_n = \{\dots\}_n$ when $n \geq 3$, or when $n = 2$ and no argument is a bivector-field, or when $n = 1$ and the argument is not a bivector field. As a consequence, all higher Jacobi identities hold when applied to n -vector fields with $n \geq 3$.

Let us see what happens when one of the arguments is a bivector field, i.e. in the case where we deal with at least an element of degree -1 . Let us assume that there is one such element, i.e. $Q_1 = \partial_i \wedge \partial_j$, $Q_2 = \partial_{I_2}, \dots, Q_n = \partial_{I_n}$ with $|I_j| \geq 3$, $j = 2, \dots, n$. Then, in view of the higher Jacobi identity for the Poisson ∞ -brackets $(\{\dots\}'_k)_{k \geq 1}$ gives:

$$0 = \sum_{2 \leq k \leq n-2} \sum_{\sigma \in S_{k, n-k}} \epsilon(\sigma) \left\{ \{Q_{\sigma(1)}, \dots, Q_{\sigma(k)}\}'_k, Q_{\sigma(k+1)}, \dots, Q_{\sigma(n)} \right\}'_{n-k+1} \quad (4.34)$$

$$+ \sum_{\sigma \in S_{n-1, 1}, \sigma(n) \neq 1} \epsilon(\sigma) \left\{ \{Q_{\sigma(1)}, \dots, Q_{\sigma(n-1)}\}'_{n-1}, Q_{\sigma(n)} \right\}'_2 \quad (4.35)$$

$$+ \sum_{\sigma \in S_{1, n-1}, \sigma(1) \neq 1} \epsilon(\sigma) \left\{ \{Q_{\sigma(1)}\}'_1, Q_{\sigma(2)} \dots, Q_{\sigma(n)} \right\}'_n \quad (4.36)$$

$$+ (-1)^{\sum_{k=2}^n |\partial_{I_k}|} \left\{ \{Q_2, \dots, Q_n\}'_{n-1}, Q_1 \right\}'_2 \quad (4.37)$$

$$+ \left\{ \{Q_1\}'_1, Q_{\sigma(2)} \dots, Q_{\sigma(n)} \right\}'_n. \quad (4.38)$$

In lines (4.34)-(4.35)-(4.36) above, we have $\{\dots\}' = \{\dots\}$ for all the terms involved. This is not the case for (4.37)-(4.38). Indeed:

$$\begin{aligned} \left\{ \{\partial_{I_2}, \dots, \partial_{I_n}\}'_{n-1}, \partial_i \wedge \partial_j \right\}'_2 &= \left\{ \{\partial_{I_2}, \dots, \partial_{I_n}\}_{n-1}, \partial_i \wedge \partial_j \right\}_2 \\ &\quad - \sum_{i_2 \in I_2, \dots, i_n \in I_n} \epsilon(i_2, \dots, i_n) \rho(\partial_i \wedge \partial_j) [\varphi_{i_2 \dots i_n}] \partial_{I_2^{i_2} \dots I_n^{i_n}} \end{aligned}$$

and

$$\begin{aligned} \{\{\partial_i \wedge \partial_j\}'_1, \partial_{I_2} \dots, \partial_{I_n}\}'_n &= (-1)^{\sum_{k=2}^n |\partial_{I_k}| + 1} (\varphi_i \{\partial_j, \partial_{I_2} \dots, \partial_{I_n}\}'_n - \varphi_j \{\partial_i, \partial_{I_2} \dots, \partial_{I_n}\}'_n) \\ &= -(-1)^{\sum_{k=2}^n |\partial_{I_k}|} \sum_{i_2 \in I_2, \dots, i_n \in I_n} \epsilon(i_2, \dots, i_n) \iota_\varphi(\partial_i \wedge \partial_j) [\varphi_{i_2 \dots i_n}] \partial_{I_2^{i_2} \dots I_n^{i_n}} \\ &= (-1)^{\sum_{k=2}^n |\partial_{I_k}|} \sum_{i_2 \in I_2, \dots, i_n \in I_n} \epsilon(i_2, \dots, i_n) \rho(\partial_i \wedge \partial_j) [\varphi_{i_2 \dots i_n}] \partial_{I_2^{i_2} \dots I_n^{i_n}} \end{aligned}$$

since $\rho = -\iota_\varphi$. Hence, the quantities in lines (4.38) and (4.37) add up, when we re-write them in terms of the new brackets $\{\dots\}_k$, to yield precisely the higher Jacobi identity for this new bracket. It is then not difficult to see this is still the case if there is more than one bivector field, by using many times the same computations. \square

4.4.7 Restriction to $\varphi = 0$ of vector fields annihilating φ

We keep the convention and notations of the previous section. Let us consider the restriction $\mathfrak{i}_W^* \mathfrak{F}_\varphi$ of the Lie-Rinehart algebra \mathfrak{F}_φ to the zero-locus W of a Koszul polynomial φ . Since all vector fields in \mathfrak{F}_φ are tangent to W , this restriction is now a Lie-Rinehart algebra over $\mathcal{O}_W = \frac{\mathcal{O}}{\varphi}$, see Section 3.2.1 (1).

Proposition 4.4.19. *Let φ be a Koszul Polynomial. The restriction of the universal Lie ∞ -algebroid of Proposition 4.4.17 to the zero-locus W of φ is a universal Lie ∞ -algebroid of the Lie-Rinehart algebra $\mathfrak{i}_W^* \mathfrak{F}_\varphi$.*

Since the image of its anchor map are vector fields tangent to W , it is clear that the universal Lie ∞ -algebroid of Proposition 4.4.17 restricts to W . To prove Proposition 4.4.19, it suffices to check that the restriction $\mathfrak{i}_W^* \mathfrak{X}(V)$ to W of the Koszul complex is still exact, except in degree 0.

Lemma 4.4.20. *The restriction to the zero locus W of φ of the Koszul complex (4.28), namely the complex,*

$$\dots \xrightarrow{\iota_\varphi} \mathfrak{i}_W^* \mathfrak{X}^3(V) \xrightarrow{\iota_\varphi} \mathfrak{i}_W^* \mathfrak{X}^2(V) \xrightarrow{\iota_\varphi} \mathfrak{i}_W^* \mathfrak{F}$$

is a free resolution of $\mathfrak{i}_W^ \mathfrak{F}$ in the category of \mathcal{O}_W -modules.*

Proof. Let $k \geq 2$ be an integer. For a given $P \in \mathfrak{i}_W^* \mathfrak{X}^k(V)$, the relation $\iota_\varphi P = 0$ means that for any $\tilde{P} \in \mathfrak{X}^k(V)$ extending P , there exists $U \in \mathfrak{X}^{k-1}(V)$ such that $\iota_\varphi \tilde{P} = \varphi U$. By exactness of the Koszul complex (4.28), one has, $U = \iota_\varphi \tilde{Q}$ for some $\tilde{Q} \in \mathfrak{X}^k(V)$. Hence, $\tilde{P} - \varphi \tilde{Q}$ is an extension of the bivector field P such that $\iota_\varphi(\tilde{P} - \varphi \tilde{Q}) = 0$. Using one more time exactness of the Koszul complex (4.28), we construct $\tilde{R} \in \mathfrak{X}^{k+1}(V)$ such that $\tilde{P} - \varphi \tilde{Q} + \iota_\varphi \tilde{R} = 0$. Thus, $P = \mathfrak{i}_W^* \tilde{P} = \iota_\varphi \mathfrak{i}_W^* \tilde{R} = \iota_\varphi R$. \square

4.4.8 Vector fields vanishing on subsets of a vector space

Let \mathcal{O} be the algebra of smooth or holomorphic or polynomial or formal functions on \mathbb{K}^d , and $\mathcal{I} \subset \mathcal{O}$ be an ideal. Then $\mathcal{I}\text{Der}(\mathcal{O})$, i.e. vector fields of the form: $\sum_{i=1}^d f_i \frac{\partial}{\partial x_i}$, with $f_1, \dots, f_d \in \mathcal{I}$, is a Lie-Rinehart algebra (It is also a singular foliation).

Remark 4.4.21. Geometrically, when \mathcal{I} corresponds to functions vanishing on a sub-variety $N \subset \mathbb{K}^n$, $\mathcal{I}\text{Der}(\mathcal{O})$ must be interpreted as vector fields vanishing along N .

Let us describe a Lie ∞ -algebroid that terminates at $\mathcal{I}\text{Der}(\mathcal{O})$, then discuss when it is universal. Let $(\varphi_i)_{i \in I}$ be generators of \mathcal{I} . Consider the free graded algebra $\mathcal{K} = \mathcal{O}[(\mu_i)_{i \in I}]$ generated by variables $(\mu_i)_{i \in I}$ of degree -1 . The degree -1 derivation $\partial := \sum_{i \in I} \varphi_i \frac{\partial}{\partial \mu_i}$ squares to zero. The \mathcal{O} -module \mathcal{K}_{-j} of elements degree j in \mathcal{K}_\bullet is made of all sums $\sum_{i_1, \dots, i_j \in I} f_{i_1 \dots i_j} \mu_{i_1} \cdots \mu_{i_j}$ with $f_{i_1 \dots i_j} \in \mathcal{O}$. Consider the complex of free \mathcal{O} -modules

$$\dots \xrightarrow{\partial \otimes \text{id}} \mathcal{K}_{-2} \otimes_{\mathcal{O}} \text{Der}(\mathcal{O}) \xrightarrow{\partial \otimes \text{id}} \mathcal{K}_{-1} \otimes_{\mathcal{O}} \text{Der}(\mathcal{O}) \quad (4.39)$$

Proposition 4.4.22. *The complex (4.39) comes equipped with a Lie ∞ -algebroid structure that terminates in $\mathcal{I}\text{Der}(\mathcal{O})$ through the anchor map given by $\mu_i \frac{\partial}{\partial x_j} \mapsto \varphi_i \frac{\partial}{\partial x_j}$ for all $i \in I$, and $j \in 1, \dots, d$.*

Proof. First, one defines a \mathcal{O} -linear Poisson- ∞ -algebra structure on the free algebra generated by $(\mu_i)_{i \in I}$ (in degree -1) and $\left(\frac{\partial}{\partial x_j}\right)_{j=1}^d$ (in degree 0) and 1 by:

$$\left\{ \mu_i, \frac{\partial}{\partial x_{j_1}}, \dots, \frac{\partial}{\partial x_{j_r}} \right\}'_{r+1} := \frac{\partial^r \varphi_i}{\partial x_{i_1} \dots \partial x_{i_r}} \quad (4.40)$$

all other brackets of generators being equal to 0. Since the brackets of generators take values in \mathcal{O} , and since an n -ary bracket where an element of \mathcal{O} appears is zero, this is easily seen to be a Poisson ∞ -structure. The general formula is

$$\left\{ \mu_{I_1} \otimes_{\mathcal{O}} \partial_{x_{a_1}}, \dots, \mu_{I_n} \otimes_{\mathcal{O}} \partial_{x_{a_n}} \right\}'_n := \sum_{\substack{j=1, \dots, n \\ i_j \in I_j}} \epsilon \frac{\partial^{n-1} \varphi_{i_j}}{\partial x_{a_1} \cdots \widehat{\partial x_{a_j}} \cdots \partial x_{a_n}} \mu_{I_1} \cdots \mu_{I_j}^{i_j} \cdots \mu_{I_n} \otimes_{\mathcal{O}} \partial_{x_{a_j}}, \quad (4.41)$$

where $\mu_J = \mu_{j_1} \dots \mu_{j_s}$ for every list $J = \{j_1, \dots, j_s\}$, where ϵ is the Koszul sign, and where for a list J containing j , J^j stands for the list J from which the element j is crossed out, as in Equation (4.31).

The \mathcal{O} -module generated by $\mu_{i_1} \cdots \mu_{i_k} \otimes_{\mathcal{O}} \partial_{x_a}$, i.e. the complex (4.39) is easily seen to be stable under the brackets $\{\cdots\}'_k$ for all $k \geq 1$, so that we can define on $\mathcal{K} \otimes_{\mathcal{O}} \text{Der}(\mathcal{O})$ a sequence of brackets $(\ell_k = \{\cdots\}'_k)_{k \geq 1}$ by letting them coincide with the previous brackets on the generators, i.e. $\{\cdots\}'_n$ is given by Equation (4.41) for all $n \geq 1$. The brackets are then extended by derivation, \mathcal{O} -linearity or Leibniz identity with respect to the given anchor map, depending on the degree. In particular, $\{\cdots\}'_k = \{\cdots\}'_k$ for $k \geq 2$. For $k = 1$, $\{\cdots\}'_1 = \{\cdots\}'_1$ on $\oplus_{i \geq 2} \mathcal{K}_{-i} \otimes_{\mathcal{O}} \text{Der}(\mathcal{O})$. For $k = 2$, we still have $\{\cdots\}'_2 = \{\cdots\}'_2$ on $\oplus_{i, j \geq 2} \mathcal{K}_{-i} \otimes_{\mathcal{O}} \mathcal{K}_{-j} \otimes_{\mathcal{O}} \text{Der}(\mathcal{O})$.

Let us verify that all required axioms are satisfied. For $n = 2$, Equation (4.41) specializes to:

$$\ell_2(\mu_i \otimes_{\mathcal{O}} \partial_{x_a}, \mu_j \otimes_{\mathcal{O}} \partial_{x_b}) = \frac{\partial \varphi_j}{\partial x_a} \mu_i \otimes_{\mathcal{O}} \partial_{x_b} - \frac{\partial \varphi_i}{\partial x_b} \mu_j \otimes_{\mathcal{O}} \partial_{x_a}$$

which proves that the anchor map is a morphism when compared with the relation:

$$[\varphi_i \partial_{x_a}, \varphi_j \partial_{x_b}] = \frac{\partial \varphi_j}{\partial x_a} \varphi_i \partial_{x_b} - \frac{\partial \varphi_i}{\partial x_b} \varphi_j \partial_{x_a}.$$

The higher Jacobi identities are checked on generators as follows:

1. When there are no degree -1 generators, it follows from the higher Jacobi identities of the Poisson ∞ -structure (4.40) and the \mathcal{O} -multilinearity of all Lie ∞ -algebroid brackets involved.

2. When generators of degree -1 are involved, the higher Jacobi identities are obtained by doing the same procedure as in the proof of Proposition 4.4.18, that is, we first consider the higher Jacobi identities for the Poisson ∞ -structure (4.40), and we put aside the terms where $\{\cdot\}_1$ is applied to these degree -1 generators. We then check that the latter terms are exactly the terms coming from an anchor map when the 2-ary bracket is applied to generators of degree -1 and the $(n-1)$ -ary brackets of the remaining generators.

□

4.4.9 Vector fields vanishing on a complete intersection

Proposition 4.4.23. *Let $W \subset \mathbb{C}^n$ be an affine variety defined by a regular sequence $\varphi_1, \dots, \varphi_k \in \mathcal{O}$. Then, the Lie ∞ -algebroid described in Proposition 4.4.22 is the universal Lie ∞ -algebroid of the singular foliation of vector fields vanishing along W .*

Proof. In the notation of the proof of Proposition 4.4.22, \mathcal{K}_\bullet equipped with the derivation $\partial = \sum_{i=1}^k \varphi_i \frac{\partial}{\partial \mu_k}$ is a free \mathcal{O} -resolution of the ideal \mathcal{I}_W of functions vanishing along W , since $\varphi_1, \dots, \varphi_k$ is a regular sequence. Since $\mathfrak{X}(\mathbb{C}^d)$ is a flat \mathcal{O} -module, the sequence

$$\dots \xrightarrow{\partial \otimes_{\mathcal{O}} \text{id}} \mathcal{K}_{-2} \otimes_{\mathcal{O}} \mathfrak{X}(\mathbb{C}^d) \xrightarrow{\partial \otimes_{\mathcal{O}} \text{id}} \mathcal{K}_{-1} \otimes_{\mathcal{O}} \mathfrak{X}(\mathbb{C}^d) \xrightarrow{\partial \otimes_{\mathcal{O}} \text{id}} \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d). \quad (4.42)$$

is a free \mathcal{O} -resolution of the singular foliation $\mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$. The Lie ∞ -algebroid structure of Proposition 4.4.22 is therefore universal. □

Example 4.4.24. As a special case of the Proposition 4.4.23, let us consider a complete intersection defined by *one* function, i.e. an affine variety W whose ideal $\langle \varphi \rangle$ is generated by a regular polynomial $\varphi \in \mathbb{C}[X_1, \dots, X_d]$. One has a free resolution of the space of vector fields vanishing on W given as follows:

$$0 \longrightarrow \mathcal{O}\mu \otimes_{\mathcal{O}} \mathfrak{X}(\mathbb{C}^d) \xrightarrow{\varphi \frac{\partial}{\partial \mu} \otimes_{\mathcal{O}} \text{id}} \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d),$$

where μ is a degree -1 variable, so that $\mu^2 = 0$. The universal Lie ∞ -algebroid structure over that resolution is given on the set of generators by :

$$\{\mu \otimes_{\mathcal{O}} \partial_{x_a}, \mu \otimes_{\mathcal{O}} \partial_{x_b}\}_2 := \frac{\partial \varphi}{\partial x_a} \mu \otimes_{\mathcal{O}} \partial_{x_b} - \frac{\partial \varphi}{\partial x_b} \mu \otimes_{\mathcal{O}} \partial_{x_a}$$

and $\{\dots\}_k := 0$ for every $k \geq 3$. It is a Lie algebroid structure. Notice that this construction could be also be recovered using Section 4.4.2.

Conclusion:

This chapter described the 1-1 correspondence "Lie-Rinehart algebras \longleftrightarrow Lie ∞ -algebroids on acyclic complexes". It extends greatly [LLS20] for singular foliations. The functor " \longleftarrow " consists in the 1-truncation of the Lie ∞ -algebroid structure. The converse functor consists in taking any free resolution, and constructing the brackets by recursion.

We prove that it is unique by proving it is universal. Notice that we need the "complicated" notion of homotopy given in Definition 2.3.14.

Last, some examples of [LLS20] are conceptually understood, and new examples are given. Some algebraic constructions (blow-up, localization, germs, quotient) are also given.

Obstruction classes to the existence of a Lie algebroid with a surjective morphism onto the Lie-Rinehart algebra are also described.

We never assume finite rank here!

Part II

Geometric Applications

 Universal Lie ∞ -algebroids of affine varieties

In this chapter, we apply the results of Chapter 4 to answer some elementary but open questions that have to do with algebraic geometry, such as the interaction between the singularities of an affine variety and its Lie algebra of vector fields.

Notice that the Theorem 2.1 of Chapter 4.2 only says that it is possible to associate a Lie ∞ -algebroid structure to an affine variety by considering the Lie-Rinehart algebra made of its vector fields. But the construction can be extremely complicated, see e.g. Section 5.3. We only have an existence theorem. This leads to the natural question:

Question. *How is the geometry of an affine variety related to its universal Lie ∞ -algebroid?*

For instance, in view of the construction of Section 3.10. It is also relevant to ask about the effect of blow-ups on this construction. We will see in Example 4.4.10 that blow-ups may change a universal Lie ∞ -algebroid to a Lie algebroid. But this example does not tell us really how the higher brackets disappear under the effect of blow-ups. One of the important question we may ask is about the description of “the big theorem” of Hironaka [Já07] in terms of the universal Lie ∞ -algebroids obtained at each step while resolving singularities. This question remains open, but there are several other problems about which we are able to make some progresses. Those are quite modest, but, at least, we want to have the question clarified. Application to "blowup-up" will be discusses in Section 7.2.

5.1 Background on affine varieties and some constructions

We recall definitions and some main properties of the notion of affine variety in order to fix notations. Our main references for this chapter are [Har77, Eis95, LB15].

In this chapter we will sometimes see \mathbb{C}^d as the d -dimensional affine space which is commonly denoted by $\mathbb{A}_{\mathbb{C}}^d$, forget about its vector space structure, but here we will not make any notational distinctions. The latter is equipped with the Zariski topology that is, the topology whose closed subsets are the zero set of some ideal $\mathcal{I} \subseteq \mathbb{C}[x_1, \dots, x_d] =: \mathcal{O}$, i.e. which are of the form $\{a =$

$(a_1, \dots, a_d) \in \mathbb{C}^d \mid f(a) = 0, \forall f \in \mathcal{I}$. One can check that these subsets indeed define a topology on \mathbb{C}^d [KES00, Har77, LB15].

Definition 5.1.1. An affine variety $W \subseteq \mathbb{C}^d$ is the zero locus of an ideal $\mathcal{I} \subseteq \mathcal{O}$, i.e., $W := Z(\mathcal{I}) := \{a = (a_1, \dots, a_d) \in \mathbb{C}^d \mid f(a) = 0, \forall f \in \mathcal{I}\}$. It admits a topology, induced by the Zariski topology in \mathbb{C}^d .

Remark 5.1.2. In Definition 5.1.1, we do not exclude irreducible varieties, e.g. $W = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$ is an affine variety.

The following facts and remarks are important. For $W \subseteq \mathbb{C}^d$ an affine variety in the notation of Definition 5.1.1,

1. we denote by \mathcal{I}_W the vanishing ideal of W , namely,

$$\mathcal{I}_W = \{f \in \mathcal{O} \mid f(x) = 0, \forall x \in W\}.$$

In general, we have $\mathcal{I}_W \neq \mathcal{I}$ because the vanishing ideal of the affine variety $W = \{x^2 = 0\}$ is the ideal $\mathcal{I}_W = \langle x \rangle \neq \langle x^2 \rangle$. Notice that \mathcal{I}_W can be defined for any arbitrary subset $W \subset \mathbb{C}^d$.

It is easy to check that

- (a) for $S \subseteq T \subseteq \mathcal{O}$, one has $Z(S) \supseteq Z(T)$,
- (b) for $U \subseteq V \subseteq \mathbb{C}^d$, one has $\mathcal{I}_U \supseteq \mathcal{I}_V$.

2. The ideal \mathcal{I}_W is larger than \mathcal{I} . Hilbert's Nullstellensatz theorem (e.g. see Theorem 1.6 of [Eis95]) claims that $\mathcal{I}_W = \sqrt{\mathcal{I}} := \{f \in \mathcal{O} \mid f^N \in \mathcal{I}, \text{ for some } N \in \mathbb{N}\}$.
3. We have, $W = Z(\mathcal{I}_W)$: if $x \in W$, then by definition of \mathcal{I}_W , one has $f(x) = 0$ for all $f \in \mathcal{I}_W$. Whence, $W \subseteq Z(\mathcal{I}_W)$. Conversely, it is clear that $\mathcal{I} \subseteq \mathcal{I}_W$. This fact proves the other inclusion.
4. Also, by Noetheriality of the polynomial ring \mathcal{O} , the ideal $\mathcal{I}_W \subset \mathcal{O}$ is generated by a finite number of generators. In the sequel, we shall define an affine variety W as the zero locus of an ideal generated by a finite set of polynomials $\varphi_1, \dots, \varphi_r \in \mathcal{O}$.
5. The Zariski closure \bar{V} of a subset $V \subseteq \mathbb{C}^d$ is equal to $Z(\mathcal{I}_V)$: by definition of \mathcal{I}_V , one has $V \subseteq Z(\mathcal{I}_V)$. Conversely, $V \subseteq \bar{V} = Z(\mathcal{I})$ for some ideal $\mathcal{I} \subseteq \mathcal{O}$. By item 1.(b), $\mathcal{I}_{\bar{V}} \subseteq \mathcal{I}_V$. In particular, $\mathcal{I} \subseteq \mathcal{I}_V$. By item 1.(a), this implies $Z(\mathcal{I}_V) \subseteq Z(\mathcal{I}) = \bar{V}$.

Definition 5.1.3. Let $W \subseteq \mathbb{C}^d$ be an affine variety.

1. A function $F: W \rightarrow \mathbb{C}$ is said to be a *polynomial* if $F(x) = f(x), \forall x \in W$, for some element $f \in \mathcal{O}$. The set $\mathbb{C}[W]$ of polynomial functions on W is, under the restriction map $f \in \mathcal{O} \mapsto f|_W$, isomorphic to the quotient $\mathcal{O}/\mathcal{I}_W =: \mathcal{O}_W$, called the *coordinate ring* of W .
2. Elements of the Lie algebra of \mathbb{C} -linear derivations, $\text{Der}(\mathcal{O}_W) =: \mathfrak{X}(W)$, of \mathcal{O}_W are called *vector fields* on W .

Remark 5.1.4. Notice that

1. the coordinate ring \mathcal{O}_W of an affine variety besides being a ring is also a vector space over \mathbb{C} , hence it is a \mathbb{C} -algebra. This algebra is generated by the images $\bar{x}_1, \dots, \bar{x}_d$ in \mathcal{O}_W through the projection map of the coordinate functions $x_1, \dots, x_d \in \mathcal{O}$.

For each $a \in W$, the kernel of the *evaluation map* $ev_a: \mathcal{O}_W \rightarrow \mathbb{C}, F \mapsto F(a)$, is the maximal ideal $\mathfrak{m}_a := \ker(ev_a)$ made of all polynomial functions on W that vanish at a .

2. $W = \mathbb{C}^d$ is an affine variety with $\mathcal{I}_W = \{0\}$, and $\mathcal{O}_W = \mathcal{O}$.

3. $W = \{a\} \subseteq \mathbb{C}^d$ is an affine variety with $\mathcal{I}_W = (x_1 - x_1(a), \dots, x_d - x_d(a))$ the maximal ideal of a , and $\mathcal{O}_W = \mathbb{C}$.

4. For an affine variety $W \subset \mathbb{C}^d$ with corresponding ideal \mathcal{I}_W , we have

$$\text{Der}(\mathcal{O}_W) \simeq \frac{\{X \in \mathfrak{X}(\mathbb{C}^d) \mid X[\mathcal{I}_W] \subset \mathcal{I}_W\}}{\mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)}.$$

Lemma 5.1.5. *For every affine variety $W \subseteq \mathbb{C}^d$, the ring \mathcal{O}_W is Noetherian.*

Proof. Let \mathcal{I} be an ideal of \mathcal{O}_W . Denote by $p: \mathcal{O} \rightarrow \mathcal{O}/\mathcal{I}_W$ be the quotient map. Then $p^{-1}(\mathcal{I})$ is also an ideal of \mathcal{O} . By Noetheriality of \mathcal{O} , $p^{-1}(\mathcal{I})$ is finitely generated. In particular, $\mathcal{I} = p(p^{-1}(\mathcal{I}))$ is finitely generated. This shows that \mathcal{O}_W is Noetherian. \square

Germ

Here we mention the notion of local rings. We refer the reader to [Cha14, Hid89, Eis95] for more details.

Let $W \subseteq \mathbb{C}^N$ be an affine variety and \mathcal{O}_W its coordinates ring. We recall for $U \subseteq W$ an open subset, a function $f: U \rightarrow \mathbb{C}$ is said to be *regular* at $a \in U$ if there exists $g, h \in \mathcal{O}_W$ with $h(a) \neq 0$ such that $f = \frac{g}{h}$ in a neighborhood of a , namely there exists an open set $V \subset U$ that contains a such that $f|_V = \frac{g}{h}|_V$. A *function germ* at a point $a \in W$ is an equivalence class $(f)_a$ of pairs (U, f) with $a \in U \subset W$ an open subset containing a , and $f: U \rightarrow \mathbb{C}$ is regular at a , under the relation equivalence: $(U, f) \sim (V, g)$ if $f|_W = g|_W$ on an open subset $W \subseteq U \cap V$. The set of equivalence classes of the above equivalence relation inherits naturally an associative \mathbb{C} -algebra, that is called *germs of regular functions at a* and is denoted by $\mathcal{O}_{W,a}$. Also, a function germ $(f)_a$ at $a \in W$ has a well-defined value at a , given by the image of any representative (U, f) at a , namely $(f)_a(a) := f(a)$. Since the map

$$\mathcal{O}_{W,a} \longrightarrow (\mathcal{O}_W)_{\mathfrak{m}_{W,a}}, (U, f) \mapsto f|_U = \frac{g}{h}$$

with $g, h \in \mathcal{O}_W$ and h does not vanish on U , is a bijection. One has, $\mathcal{O}_{W,a} \simeq (\mathcal{O}_W)_{\mathfrak{m}_{W,a}}$ [Har77]. Here $\mathfrak{m}_{W,a} = \{f \in \mathcal{O}_W \mid f(a) = 0\}$ and $(\mathcal{O}_W)_{\mathfrak{m}_{W,a}}$ is the localization w.r.t the complement of $\mathfrak{m}_{W,a}$. It is important to notice that $\mathcal{O}_{W,a}$ is a local ring. We denote the unique maximal ideal of $\mathcal{O}_{W,a}$ again by $\mathfrak{m}_{W,a}$. Also, It is worth it to notice that $\mathcal{O}_{W,a}$ is isomorphic to the quotient $\mathcal{O}_a/\mathcal{I}_a$ of the local ring \mathcal{O}_a of \mathbb{C}^d at a by the ideal \mathcal{I}_a which is spanned by the ideal \mathcal{I}_W in \mathcal{O}_a .

The Zariski tangent space

Let $a \in W \subseteq \mathbb{C}^d$ a point of an affine variety W . There are several equivalent descriptions of the tangent space of W variety at the point a . Here we define it as pointwise derivations of the local ring

$\mathcal{O}_{W,a}$, see [Har77, R94] or Appendix B.2 of [LGPV13], for more details on this topic.

A *pointwise derivation* of \mathcal{O}_W at a is a \mathbb{C} -linear map

$$\delta_a: \mathcal{O}_{W,a} \rightarrow \mathbb{C}$$

satisfying the following Leibniz identity, $\delta_a(FG) = \delta_a(F)G(a) + F(a)\delta_a(G)$. In particular, if a regular function f is constant in a neighborhood of a then its germ $(f)_a$ at a satisfies $\delta_a((f)_a) = 0$, since

$$\begin{aligned} \delta_a(1 \cdot 1) &= \delta_a(1) \cdot 1 + 1 \cdot \delta_a(1) = 2\delta_a(1) \\ \delta_a(1) &= 0. \end{aligned}$$

It is not hard to check that the set of all pointwise derivations of \mathcal{O}_W at a is a \mathbb{C} -vector space.

Remark 5.1.6. Assume now that $W = \mathbb{C}^d$. Denote by (e_1, \dots, e_d) the canonical basis of \mathbb{C}^d . For every $i \in \{1, \dots, d\}$

$$(e_i)_a: \mathcal{O}_{W,a} \rightarrow \mathbb{C}, (f)_a \mapsto \lim_{t \rightarrow 0} \frac{f(a + te_i) - f(a)}{t} =: \frac{\partial f}{\partial x_i}(a),$$

is a well-defined pointwise derivation of \mathcal{O} at a . One can show that (see e.g [LGPV13], Appendix B.2) any pointwise derivation δ_a of \mathcal{O} at $a \in \mathbb{C}^d$ has the form

$$\delta_a = \sum_{i=1}^d \delta_a((x_i)_a) (e_i)_a, \quad (5.1)$$

i.e.,

$$\delta_a(f)_a = \sum_{i=1}^d \frac{\partial f}{\partial x_i}(a) \delta_a(x_i)_a. \quad (5.2)$$

Hence, pointwise derivations $(e_i)_a$, $i = 1, \dots, d$ form a basis for the vector space of pointwise derivation of \mathcal{O} at a .

Definition 5.1.7. The *Zariski tangent space* $T_a W$ of $W \subseteq \mathbb{C}^d$ at $a \in W$ is the vector space of all pointwise derivations of \mathcal{O}_W at a .

Proposition 5.1.8. [R94] For $a \in W$, one has $T_a W \simeq (\mathfrak{m}_{W,a}/\mathfrak{m}_{W,a}^2)^*$.

The vector space $\mathfrak{m}_{W,a}/\mathfrak{m}_{W,a}^2$ is called the *cotangent space* to W at a .

Remark 5.1.9. Notice that,

1. by Remark 5.1.6, we have $T_a \mathbb{C}^d \simeq \mathbb{C}^d$.
2. the tangent space $T_a W$ of $W \subseteq \mathbb{C}^d$ at $a \in W$ can be seen as pointwise derivations δ_a of \mathcal{O} at a of the form (5.1) such that $\delta_a((f)_a) = 0$ for all $f \in \mathcal{I}_W$. From this point of view, one has

$$T_a W \simeq \left\{ (v_1, \dots, v_d) \in \mathbb{C}^d \mid \sum_{i=1}^d v_i \frac{\partial f}{\partial x_i}(a) = 0, \forall f \in \mathcal{I}_W \right\}.$$

Singularities of an affine variety W

In this section we recall some definitions and some facts on singularities on affine varieties and fix some notations. We refer the reader to [Har77, R94] for the full theory.

There are various equivalent ways to define the dimension of an affine variety W , we refer the reader to Page 4 of [Har77] also to the Chapter 11 of [Joe92] for more details. The *dimension* $\dim W$ of W is defined to be the maximal length d of the chains $W_0 \subset W_1 \subset \dots \subset W_d$ of distinct nonempty irreducible sub-varieties of W . Notice that a chain of irreducible sub-varieties corresponds to a chain of prime ideals in \mathcal{O}_W , by Noetheriality it must be of finite length.

Definition 5.1.10. A point $a \in W$ is said to be *regular* if $\dim T_a W = \dim W$. Otherwise, we say that a is *singular*. The set of regular points of W is denoted by W_{reg} , and the singular ones by W_{sing} .

We say that W is *regular* if $W_{sing} = \emptyset$.

Proposition 5.1.11. [Har77, R94] We have the following

1. For every $a \in W$, $\dim T_a W \geq \dim W$
2. There is an open dense subset of W such that the map $a \mapsto \dim T_a W$ is constant. In particular,
 - regular points of W form an open dense subset of W ,
 - and singular points a (closed) proper sub-variety of W .

Remark 5.1.12. In particular, $a \in W$ is a singular point of W if only if $\dim T_a W > \dim W$. That is,

$$W_{sing} = \{a \in W \mid \dim T_a W > \dim W\}.$$

Local coordinates at a point

Let $a \in W \subseteq \mathbb{C}^d$. We recall that (see e.g [R94]) that a family of elements $t_1, \dots, t_r \in \mathcal{O}_{W,a}$ are called *local coordinates* of W at a , if they vanish at a (i.e. $t_i \in \mathfrak{m}_{W,a}$ for $i = 1, \dots, r$), and if the classes of $\bar{t}_1, \dots, \bar{t}_r \in \mathfrak{m}_{W,a}/\mathfrak{m}_{W,a}^2$ form a basis.

Example 5.1.13. If $a = (a_1, \dots, a_d) \in \mathbb{C}^d$, then $x_1 - a_1, \dots, x_d - a_d$ are local coordinates of \mathbb{C}^d at a .

Remark 5.1.14. Notice that:

Local coordinates at $a \in \mathbb{C}^d$ generate the maximal ideal \mathfrak{m}_a of \mathcal{O}_a . Indeed, let $t_1, \dots, t_d \in \mathcal{O}_a$ be local coordinates at a . By applying Nakayama Lemma (B.2.11) to $R = \mathcal{O}_a \supseteq \mathfrak{m}_a$ and $\mathcal{V} = \mathfrak{m}_a$: the basis $\bar{t}_1, \dots, \bar{t}_d \in \mathfrak{m}_a/\mathfrak{m}_a^2$ lifts to a (minimal) generating set t_1, \dots, t_d for \mathfrak{m}_a .

The following proposition explains how an affine variety looks around a regular point (see [Hau14] Proposition 3.5, also [dJP00]).

Proposition 5.1.15. Let $W \subseteq \mathbb{C}^d$ be affine variety of codimension k , (i.e., $k = d - \dim W$). Then, W is regular at a point $a \in W$ if and only if there exist local coordinates y_1, \dots, y_d of \mathbb{C}^d at a such that W is locally of the form

$$y_1 = \dots = y_k = 0.$$

5.1.1 Three main constructions

Consider an affine variety W which is given by an ideal $\mathcal{I}_W \subset \mathcal{O}$, with \mathcal{O} the algebra of polynomials in d -variables, and $\mathcal{O}_W = \mathcal{O}/\mathcal{I}_W$ the algebra of functions on W . There are three natural Lie-Rinehart algebras associated to W :

1. The \mathcal{O}_W -module $\mathfrak{X}(W)$ of vector fields on W (i.e. derivations of \mathcal{O}_W) is a Lie-Rinehart algebra over \mathcal{O}_W ; its anchor map is the identity.
2. The \mathcal{O} -module $\mathfrak{X}_W(\mathbb{C}^d)$ of vector fields on \mathbb{C}^d tangent to W (i.e. derivations of \mathcal{O} preserving \mathcal{I}_W) is a Lie-Rinehart algebra with respect to the \mathcal{O} -module structure; its anchor map is the inclusion $\mathfrak{X}_W(\mathbb{C}^d) \hookrightarrow \mathfrak{X}(\mathbb{C}^d)$.
3. The \mathcal{O} -module $\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d)$ of vector fields on \mathbb{C}^d vanishing at every point of W (i.e. \mathcal{I}_W -valued derivations of \mathcal{O}) is a Lie-Rinehart algebra with respect to the \mathcal{O} -module structure; its anchor map is again the inclusion $\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d) \hookrightarrow \mathfrak{X}(\mathbb{C}^d)$.

These three Lie-Rinehart algebras are related:

1. There is an inclusion $\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d) \subset \mathfrak{X}_W(\mathbb{C}^d)$
2. the restriction of $\mathfrak{X}_W(\mathbb{C}^d)$ to W coincides with $\mathfrak{X}(W)$. Let us justify this. Every vector field on W extends to \mathbb{C}^d : to see that, let $\delta \in \mathfrak{X}(W)$, we have $\delta(x_i + \mathcal{I}_W) = f_i + \mathcal{I}_W$ for some $f_i \in \mathbb{C}[x_1, \dots, x_d], i = 1, \dots, d$. We define the vector field

$$\tilde{\delta} := \sum_{i=1}^d f_i \frac{\partial}{\partial x_i}$$

on \mathbb{C}^d . The vector field $\tilde{\delta}$ restricts to δ on W , since for every $f \in \mathbb{C}[x_1, \dots, x_n]$,

$$\tilde{\delta}(f) + \mathcal{I}_W = \delta(f + \mathcal{I}_W).$$

In particular, $\tilde{\delta}(\mathcal{I}_W) \subset \mathcal{I}_W$.

Note that the Lie-Rinehart algebras $\mathfrak{X}_W(\mathbb{C}^d), \mathcal{I}_W\mathfrak{X}(\mathbb{C}^d) \subset \mathfrak{X}(\mathbb{C}^d)$ are finitely generated as \mathcal{O} -modules, since \mathcal{O} is Noetherian (Proposition B.2.7). Whence, these Lie-Rinehart algebras are singular foliations on the complex manifold \mathbb{C}^d in the sense of Example 3.2.4.

Remark 5.1.16. What happens if we take a look at the evaluation map at some point $a \in \mathbb{C}^d$? Any vector field $X = \sum_{i=1}^d X[x_i] \frac{\partial}{\partial x_i} \in \mathfrak{X}_W(\mathbb{C}^d)$ tangent to W induces a pointwise derivation of \mathcal{O}_W at $a \in W$ as follows:

1. We extend X by localization at the maximal ideal \mathfrak{m}_a to a derivation $(X) \in \text{Der}(\mathcal{O}_{W,a})$.
2. We define $X|_a((f)_a) := \sum_{i=1}^d X[x_i](a)(e_i)_a((f)_a) \in T_a W$.

$X|_a$ is well-defined, since $X[\mathcal{I}_W] \subset \mathcal{I}_W$ for all $f \in \mathcal{I}_W$, in particular $X|_a((f)_a) = 0$ for all $f \in \mathcal{I}_W$.

In fact, we do not need to localize X to define $X|_a := (X[x_1](a), \dots, X[x_d](a)) \in T_a W \hookrightarrow \mathbb{C}^d$.

Lemma 5.1.17. *The image of the map*

$$\text{Ev}_a: \mathfrak{X}_W(\mathbb{C}^d) \rightarrow \mathbb{C}^d, X \mapsto X|_a \quad (5.3)$$

is denoted by $T_a\mathfrak{X}_W(\mathbb{C}^d)$

1. if $a \in W$, then $T_a\mathfrak{X}_W(\mathbb{C}^d) \subseteq T_aW$.
2. If $a \notin W$, then $T_a\mathfrak{X}_W(\mathbb{C}^d) = \mathbb{C}^d$.
3. If W is a complete intersection (i.e. $\mathcal{I}_W = (\varphi_1, \dots, \varphi_r)$ and $\dim W = d - r$), then $T_a\mathfrak{X}_W(\mathbb{C}^d) = T_aW$ for all $a \in W_{\text{reg}}$.
4. More generally, for an arbitrary affine variety W , $T_a\mathfrak{X}_W(\mathbb{C}^d) = T_aW$ for all $a \in W_{\text{reg}}$.

Proof. Item 1. is given by the construction in item 2. of Remark 5.1.16. Let us show item 2: let $(v_1, \dots, v_d) \in \mathbb{C}^d$. For $a \notin W$, there exists $\varphi \in \mathcal{I}_W$ such that $\varphi(a) \neq 0$. The vector field

$$X = \frac{\varphi}{\varphi(a)} \sum_{i=1}^d v_i \frac{\partial}{\partial x_i}$$

belongs to $\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d) \subset \mathfrak{X}_W(\mathbb{C}^d) = \mathbb{C}^d$ and $X(a) = (v_1, \dots, v_d)$.

Now we prove item 3. Let $(v_1, \dots, v_d) \in T_aW = \ker(J(a))$, with $J := \left(\frac{\partial \varphi_i}{\partial x_j} \right)_{i,j}$. By assumption, we have $\text{rk}(J(a)) = r$. Thus, J admits (r, r) -minor μ such that $\mu(a) \neq 0$. We can assume that μ is the determinant of the first r -columns of J . Consider the vector fields

$$H_j := \begin{vmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_r} & \frac{\partial}{\partial x_j} \\ \frac{\partial \varphi_1}{\partial x_1} & \cdots & \frac{\partial \varphi_1}{\partial x_r} & \frac{\partial \varphi_1}{\partial x_j} \\ \frac{\partial \varphi_2}{\partial x_1} & \cdots & \frac{\partial \varphi_2}{\partial x_r} & \frac{\partial \varphi_2}{\partial x_j} \\ \vdots & & \vdots & \vdots \\ \frac{\partial \varphi_r}{\partial x_1} & \cdots & \frac{\partial \varphi_r}{\partial x_r} & \frac{\partial \varphi_r}{\partial x_j} \end{vmatrix}, \quad \text{for } j \in \{r+1, \dots, d\},$$

understood as the cofactor expansion along the first row. Since for each $i \in \{1, \dots, r\}$, $H_j[\varphi_i]$ has two repetitive lines, therefore it vanishes. Therefore H_j 's are tangent to W . We claim that the following vector fields does the job, namely

$$X = (-1)^r \sum_{j=r+1}^d \frac{v_j}{\mu(a)} H_j. \quad (5.4)$$

Indeed, if we denote by μ_1, \dots, μ_r the minors associated to the partials $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_r}$, respectively, then for every j the decomposition of H_j reads

$$H_j = \sum_{i=1}^r (-1)^{i+1} \mu_i \frac{\partial}{\partial x_i} + (-1)^r \mu \frac{\partial}{\partial x_j}.$$

Hence,

$$X = \sum_{j=r+1}^d \frac{v_j}{\mu(a)} \mu \frac{\partial}{\partial x_j} + \frac{(-1)^r}{\mu(a)} \left(\sum_{i=1}^r \sum_{j=1}^d (-1)^{i+1} v_j \mu_i \frac{\partial}{\partial x_i} \right)$$

But, by developing the minors $\mu_i(a)$'s along their last columns, it takes the form,

$$\mu_i(a) = \pm \frac{\partial \varphi_1}{\partial x_j}(a) C_1 \pm \cdots \pm \frac{\partial \varphi_{r-1}}{\partial x_j}(a) C_{r-1}.$$

Notice that the determinants C_1, \dots, C_{r-1} are the same for each H_j . Thus,

$$\begin{aligned} \sum_{j=1}^d v_j \mu_i(a) &= \sum_{s=1}^{r-1} \pm \underbrace{\left(\sum_{j=1}^d v_j \frac{\partial \varphi_s}{\partial x_j}(a) \right)}_{=0} C_s \\ &= 0. \end{aligned}$$

Item 4. is obtained as follows: the local ring at a is by definition the localization \mathcal{O}_a of $\mathbb{C}[x_1, \dots, x_d]$ with respect to the multiplicative set of all polynomials that do not vanish at a . By Proposition 5.1.15 $a \in W$ is a regular point if and only if there exists "local coordinates" $y_1, \dots, y_d \in \mathcal{O}_a$ such that W is locally of the form

$$y_1 = \cdots = y_k = 0,$$

i.e. the localization of \mathcal{I}_W is generated by these variables. Hence, the tangent space at m is the vector space, $\text{span}\left\{\frac{\partial}{\partial y_i}|_m, i \geq k+1\right\}$. Therefore, for $v \in T_a W$ the local vector field

$$X = \sum_{i=1}^{\dim W} v_i \frac{\partial}{\partial y_{k+i}}$$

maps \mathcal{O}_a to \mathcal{O}_a , in particular it maps \mathcal{O} to \mathcal{O}_a and we have $X[\mathcal{I}_W] \subset (\mathcal{I}_W)_{m_a}$. Therefore, for every, $i \in \{1, \dots, d\}$ there exists a polynomial function g_i that does not vanish at a such that $g_i Y[x_i] \in \mathbb{C}[x_1, \dots, x_d]$. Hence, the vector field $\hat{X} = \frac{g_1 \cdots g_r}{g_1(a) \cdots g_r(a)} X$ is tangent to W satisfies $\hat{X}(a) = v$ and $\hat{X}[\mathcal{I}_W] \subset \mathcal{I}_W$.

This concludes the proof. □

Remark 5.1.18. We may not have equality in item 1. in Lemma 5.1.17. To see this, consider the cups,

$$W = \{(x, y) \mid \mathbb{C}^2 \mid x^3 - y^2 = 0\}.$$

It is clear that the tangent space $T_0 W$ of W at $0 \in W$ is the whole space \mathbb{C}^2 . But the vector fields in $\mathfrak{X}_W(\mathbb{C}^2)$ vanish at zero, since it is spanned as a $\mathbb{C}[x, y]$ -module by the Hamiltonian $2y \frac{\partial}{\partial x} + 3x^2 \frac{\partial}{\partial y}$ and the weighted Euler vector field $2x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y}$ (see Proposition 5.3.1).

The following lemma shows that the vector fields that are tangent to W are also tangent to every strata of the stratification that consists of by taking the singular locus W_{sing} of the singular locus of W then the singular locus $(W_{\text{sing}})_{\text{sing}}$ of the singular locus W_{sing} and so on.... We obtain a sequence of inclusions of the form

$$W \supset \underbrace{(W_{\text{sing}})_{\text{sing}}}_{=: W_1} \supset \underbrace{((W_{\text{sing}})_{\text{sing}})_{\text{sing}}}_{=: W_2} \supset \cdots \quad (5.5)$$

Lemma 5.1.19. *We have the following inclusions*

$$\mathfrak{X}_W(\mathbb{C}^d) \subseteq \mathfrak{X}_{W_1}(\mathbb{C}^d) \subseteq \cdots \subseteq \mathfrak{X}_{W_i}(\mathbb{C}^d) \subseteq \cdots \quad (5.6)$$

Proof. Let us prove that if $X \in \mathfrak{X}(\mathbb{C}^d)$ is such that $X[\mathcal{I}_W] \subset \mathcal{I}_W$ then $X[\mathcal{I}_{W_{sing}}] \subset \mathcal{I}_{W_{sing}}$, where $\mathcal{I}_{W_{sing}}$ is the ideal of functions on the singular part of W . Since $\mathcal{I}_{W_{sing}}$ is obtained by considering the minors of order $k = d - \dim W$ of k elements chosen into the generators $\varphi_1, \dots, \varphi_r$. That is, W_{sing} is given the ideal

$$\left\langle \varphi_1, \dots, \varphi_r, P[\varphi_{i_1}, \dots, \varphi_{i_k}], P \in \mathfrak{X}^k(\mathbb{C}^d), \text{ for integers } 1 \leq i_1 < \dots < i_k \leq r \right\rangle \quad (5.7)$$

Let us explain why the vector fields that tangent to W are also tangent to its singular locus.

For a vector field $X \in \mathfrak{X}_W(\mathbb{C}^d)$ one has by Formula (4.27) that,

$$X [P[\varphi_{i_1}, \dots, \varphi_{i_k}]] = (\mathcal{L}_X P)[\varphi_{i_1}, \dots, \varphi_{i_k}] + \sum_{j=1}^k P[\varphi_{i_1}, \dots, X[\varphi_{i_j}], \dots, \varphi_{i_k}]. \quad (5.8)$$

Notice that $(\mathcal{L}_X P)[\varphi_{i_1}, \dots, \varphi_{i_k}] \in \mathcal{I}_{sing}$ since $(\mathcal{L}_X P) \in \mathfrak{X}^k(\mathbb{C}^d)$. On the other hand, for every j there exists polynomial functions f_1, \dots, f_r such that $X[\varphi_{i_j}] = \sum_{l=1}^r f_l \varphi_l$. Since P is a multi-derivation, one has,

$$P[\varphi_{i_1}, \dots, X[\varphi_{i_j}], \dots, \varphi_{i_k}] = \sum_{l=1}^r \varphi_l P[\varphi_{i_1}, \dots, f_l, \dots, \varphi_{i_k}] + \sum_{i=1}^r f_i P[\varphi_{i_1}, \dots, \varphi_i, \dots, \varphi_{i_k}]$$

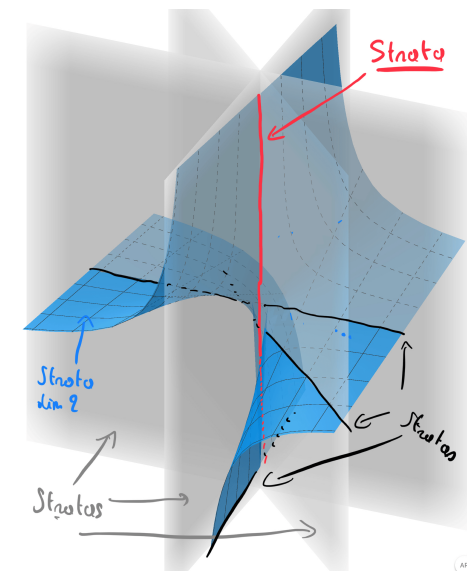
It is now clear that the RHS of the equation (5.8) is in the ideal \mathcal{I}_{sing} . The proof goes by recursion. \square

Here is a direct consequence of Lemma 5.1.19.

Theorem 5.1.20. *Every vector field $X \in \mathfrak{X}(W)$ is tangent to the stratification (5.5), i.e. $X \in \mathfrak{X}(W_i)$ for each $i \geq 1$.*

The coming example shows that the inclusions (5.6) may be strict. This Example can also be found in the problem list [LLG22] of the lecture on singular foliations, Poisson 2022 [LGLR22].

Example 5.1.21. Let $W = \{(x, y, z) \in \mathbb{C}^3 \mid xy(x + y)(x + yz) = 0\} \subset \mathbb{C}^3$.



The straight line $x = y = 0$ is a strata of the previous affine variety W . Any vector field tangent to W is tangent to this straight line. Let us show that it has to vanish at every point of this straight line. If not, its flow at time t would map a point $(0, 0, z_0)$ to a point $(0, 0, z_1)$ with $z_1 \neq z_0$. Its differential then induce a linear automorphism of the normal bundle of that straight line that has to preserve the straight lines $x = 0, y = 0, x + y = 0$. Since a linear endomorphism of \mathbb{C}^2 preserving three straight lines has to be a multiple of the identity map, this differential cannot map the straight line $x + z_0y$ to the straight line $x + z_1y$.

On their universal Lie ∞ -algebroids

Let W be an affine variety. The following is a direct consequence of the results of Chapter 4.

Proposition 5.1.22. *Let W be an affine variety as above.*

1. $\mathfrak{X}(W)$ admits a universal Lie ∞ -algebroid made of free \mathcal{O}_W -modules of finite rank.
2. $\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d)$ and $\mathfrak{X}_W(\mathbb{C}^d)$ admit universal Lie ∞ -algebroids made of finitely many free \mathcal{O} -modules of finite rank.

Proof. Classical theorems of commutative algebras allows equipping the three Lie-Rinehart algebras above with resolutions of a certain type:

1. Since \mathcal{O} is a Noetherian regular ring, $\mathfrak{X}_W(\mathbb{C}^d)$ and $\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d)$ admit free resolutions by finitely generated \mathcal{O} -modules. By Hilbert Syzygy Theorem B.2.8, those can be chosen to be of finite length.
2. Since \mathcal{O}_W is a Noetherian ring, $\mathfrak{X}(W)$ admits a free resolution by finitely generated \mathcal{O}_W -modules (by Proposition B.2.7).

In view of Theorem 4.2.1, these three resolutions do admit Lie ∞ -algebroid structures over their respective algebras, and those are universal Lie ∞ -algebroids. We denote by $\mathbb{U}_{\mathcal{O}_W}(\mathfrak{X}(W))$, $\mathbb{U}_{\mathcal{O}}(\mathfrak{X}_W(\mathbb{C}^d))$ and $\mathbb{U}_{\mathcal{O}}(\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d))$ the universal Lie ∞ -algebroids associated to the three Lie-Rinehart algebras above. \square

Remark 5.1.23. There exist natural Lie ∞ -algebroid morphisms between these structures:

1. Since $\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d) \subset \mathfrak{X}_W(\mathbb{C}^d)$, Theorem 4.2.4 implies the existence of a unique up to homotopy Lie ∞ -algebroid morphism $\Psi: \mathbb{U}_{\mathcal{O}}(\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d)) \longrightarrow \mathbb{U}_{\mathcal{O}}(\mathfrak{X}_W(\mathbb{C}^d))$. In view of Proposition 4.2.6, the morphism Ψ may be represented by the inclusion map for well-chosen representation of $\mathbb{U}_{\mathcal{O}}(\mathcal{I}_W\mathfrak{X}(\mathbb{C}^d))$ and $\mathbb{U}_{\mathcal{O}}(\mathfrak{X}_W(\mathbb{C}^d))$.
2. The anchor map of $\mathbb{U}_{\mathcal{O}}(\mathfrak{X}_W(\mathbb{C}^d))$ is tangent to W , hence the restriction $i_W^*\mathbb{U}_{\mathcal{O}}(\mathfrak{X}_W(\mathbb{C}^d))$ to W of $\mathbb{U}_{\mathcal{O}}(\mathfrak{X}_W(\mathbb{C}^d))$ exists, and is a Lie ∞ -algebroid over $\mathfrak{X}(W)$. In general, it does not need to be a universal one, but Theorem 4.2.4 implies the existence of a unique up to homotopy Lie ∞ -algebroid morphism:

$$\Phi: i_W^*\mathbb{U}_{\mathcal{O}}(\mathfrak{X}_W(\mathbb{C}^d)) \longrightarrow \mathbb{U}_{\mathcal{O}_W}(\mathfrak{X}(W)).$$

Notice that this morphism is in general not a Lie ∞ -quasi-isomorphism.

5.2 Universal Lie ∞ -algebroid of an affine variety

We give the following definition,

Definition 5.2.1. A Lie ∞ -algebroid of an affine variety W is the homotopy class of Lie- ∞ -algebroid associated to the Lie-Rinehart algebra $\mathfrak{X}(W)$ over \mathcal{O}_W .

Example 5.2.2 (Vector fields on hyperelliptic curves). We follow the notations of [BF18]. Let us consider on \mathbb{C}^2 the hyperelliptic curve \mathcal{H} given by the equation $y^2 = 2h(x)$, where h is a monic polynomial of odd degree $2\nu + 1 \geq 3$. Let $\mathcal{O}_{\mathcal{H}} = \frac{\mathbb{C}[x,y]}{\langle y^2 - 2h(x) \rangle}$ be the coordinate ring of \mathcal{H} . Let $\mathfrak{X}(\mathcal{H}) = \text{Der}(\mathcal{O}_{\mathcal{H}})$ be the Lie-Rinehart algebra of derivations of $\mathcal{O}_{\mathcal{H}}$, i.e. of vector fields on \mathcal{H} . As an $\mathcal{O}_{\mathcal{H}}$ -module, $\mathfrak{X}(\mathcal{H})$ is the submodule

$$\left\{ f \frac{\partial}{\partial x} + g \frac{\partial}{\partial y} \mid f, g \in \mathcal{O}_{\mathcal{H}}, yg - h'(x)f = 0 \right\} \subset \mathcal{O}_{\mathcal{H}} \frac{\partial}{\partial x} \oplus \mathcal{O}_{\mathcal{H}} \frac{\partial}{\partial y}.$$

The curve \mathcal{H} is non-singular if and only if $\text{gcd}(h(x), h'(x)) = 1$. In that case, the Lie algebra of vector fields $\mathfrak{X}(\mathcal{H})$ is a free $\mathcal{O}_{\mathcal{H}}$ -module of rank 1 generated by the vector field $X = y \frac{\partial}{\partial x} + h'(x) \frac{\partial}{\partial y}$. A universal Lie ∞ -algebroid is given by the Lie-Rinehart algebra $\mathcal{E}_{-1} = \mathcal{O}_{\mathcal{H}} X \subset \mathcal{O}_{\mathcal{H}} \frac{\partial}{\partial x} \oplus \mathcal{O}_{\mathcal{H}} \frac{\partial}{\partial y}$. In the singular case, i.e., $\text{gcd}(h(x), h'(x)) = d(x) \neq 1$, the $\mathcal{O}_{\mathcal{H}}$ -module $\mathfrak{X}(\mathcal{H})$ is not free and has two generators, $X = y \frac{\partial}{\partial x} + h'(x) \frac{\partial}{\partial y}$ and $Y = 2 \frac{h(x)}{d(x)} \frac{\partial}{\partial x} + y \frac{h'(x)}{d(x)} \frac{\partial}{\partial y}$ with a relation $yX = d(x)Y$ (see [BF18] for more details). A free resolution is described as follows: \mathcal{E}_{-1} is the $\mathcal{O}_{\mathcal{H}}$ -module, generated by two elements that we denote by τ, μ . The anchor is defined then by

$$\rho(\tau) = X, \quad \rho(\mu) = Y.$$

Then we choose \mathcal{E}_{-2} to be the $\mathcal{O}_{\mathcal{H}}$ -module given by the generator η , and we set $\mathcal{E}_{-i} = 0$ for $i \geq 3$. The differential map $\ell_1 = d$ is chosen to be zero, except on degree -2 where it is the $\mathcal{O}_{\mathcal{H}}$ -linear map $d: \mathcal{E}_{-2} \rightarrow \mathcal{E}_{-1}$ given by

$$d(\eta) = y\tau - d(x)\mu.$$

Let us now describe a universal Lie ∞ -algebroid structure: The 2-ary bracket is defined on generators of \mathcal{E}_{-1} by

$$\{\tau, \mu\}_2 = \frac{h'(x)}{d(x)}\tau - y \frac{d'(x)}{d(x)}\mu.$$

Then we extend this bracket to the whole space \mathcal{E}_{-1} by $\mathcal{O}_{\mathcal{H}}$ -linearity, skew-symmetric and Leibniz identity. Notice that $\{d\eta, \tau\}_2 = \{d\eta, \mu\}_2 = 0$, thus, one can define the 2-bracket by $\{\eta, \tau\}_2 = \{\eta, \mu\}_2 = 0$. We extend all brackets using Leibniz identity. All k -ary brackets are zero for $k \geq 3$.

Lie ∞ -algebra of an affine variety at a point: Let $\mathbb{U}_{\mathcal{O}_W}(\mathfrak{X}(W)) = (\mathcal{E}, \ell_{\bullet}, \rho)$ be a universal Lie algebroid of W . Let us choose $a \in W$. As stated in Section 4.4.1, the Lie ∞ -algebroid structure of W restricts at a (i.e. goes to the quotient with respect to the maximal ideal \mathfrak{m}_a) if and only if $\rho[\mathfrak{m}_a] \subseteq \mathfrak{m}_a$, (i.e. \mathfrak{m}_a is a Lie-Rinehart ideal).

Define the \mathcal{O}_W -submodule

$$\text{Skera}_a(\rho) := \{e \in \mathcal{E}_{-1} \mid \rho(e)[\mathcal{O}] \subseteq \mathfrak{m}_a\} \subseteq \mathcal{E}_{-1}.$$

The k -ary bracket $\ell_k, k \geq 1$ and ρ restrict to the exact complex

$$\cdots \longrightarrow \mathcal{E}_{-3} \xrightarrow{\ell_1} \mathcal{E}_{-2} \xrightarrow{\ell_1} \text{Sk}_a(\rho) \xrightarrow{\rho} \text{Der}(\mathcal{O}_W). \quad (5.9)$$

Let us check that ℓ_2 is well-defined: for all $e, e' \in \text{Sk}_a(\rho)$,

$$\begin{aligned} \rho(\ell_2(e, e'))[\mathcal{O}] &= [\rho(e), \rho(e')](\mathcal{O}), & (\text{since } \rho \text{ is morphism of brackets}) \\ &\subset \mathfrak{m}_a, & (\text{by definition of the commutator } [\cdot, \cdot]). \end{aligned}$$

The latter Lie ∞ -algebroid goes to quotient to a Lie ∞ -algebroid

$$\left(\left(\bigoplus_{i \geq 2} \frac{\mathcal{E}_{-i}}{\mathfrak{m}_a \mathcal{E}_{-i}} \right) \oplus \ker_a(\rho), \bar{\ell}_\bullet, \bar{\rho} \right) \quad (5.10)$$

over $\mathcal{O}_W/\mathfrak{m}_a$, where $\ker_a(\rho) := \frac{\text{Sk}_a(\rho)}{\mathfrak{m}_a \text{Sk}_a(\rho)}$. Since this quotient is the base field \mathbb{C} , we obtain in fact a Lie ∞ -algebra.

Remark 5.2.3. In particular, if $a \in W$ is an isolated singular point, then $\mathcal{E}_{-1} = \text{Sk}_a(\rho)$. In that case, $\mathbb{U}_{\mathcal{O}_W}(\mathfrak{X}(W))$ is a universal of the Lie-Rinehart algebra $\mathcal{A}_a = \{\delta \in \mathfrak{X}(W) \mid \delta[\mathfrak{m}_a] \subset \mathfrak{m}_a\} = \mathfrak{X}(W)$. By Section 4.4.1 again, (5.10) is a Lie ∞ -algebra on $\text{Tor}_{\mathcal{O}_W}(\mathfrak{X}(W), \mathbb{C})$.

Lie ∞ -algebroids on minimal resolutions

The germ at a of the Lie-Rinehart algebra of vector fields on W is easily checked to coincide with the Lie-Rinehart algebra of derivations of $\mathcal{O}_{W,a}$.

Here is an immediate consequence of Proposition 4.4.4.

Proposition 5.2.4. *Let W be an affine variety. For every $a \in W$, the germ at a of the universal Lie ∞ -algebroid of W is the universal Lie ∞ -algebroid of $\text{Der}(\mathcal{O}_{W,a})$.*

To describe this structure, let us start with the following Lemma.

Lemma 5.2.5. *Let a be a point of an affine variety W . The universal Lie ∞ -algebroid of $\text{Der}(\mathcal{O}_{W,a})$ can be constructed on a resolution $((\mathcal{E}_{-i}^a)_{i \geq 1}, \ell_1, \pi)$, with \mathcal{E}_{-i}^a free $\mathcal{O}_{W,a}$ -modules of finite rank for all $i \geq 1$, which is minimal in the sense that $\ell_1(\mathcal{E}_{-i-1}) \subset \mathfrak{m}_a \mathcal{E}_{-i}$ for all $i \geq 1$.*

Proof. Since Noetherian property is stable by localization, the ring $\mathcal{O}_{W,a}$ is a Noetherian local ring. Proposition 8.2 in [May] assures that $\mathcal{O}_{W,a} \otimes_{\mathcal{O}_W} \text{Der}(\mathcal{O}_W)$ admits a free minimal resolution by free finitely generated $\mathcal{O}_{W,a}$ -modules. Since $\mathcal{O}_{W,a}$ is a local ring with maximal idea \mathfrak{m}_a , we can assume that this resolution is minimal. In view of Theorem 4.2.1, there exists a Lie ∞ -algebroid structure over this resolution, and the latter is an universal of $\text{Der}(\mathcal{O}_{W,a})$. \square

By Theorem 4.2.1, a resolution of $\text{Der}(\mathcal{O}_{W,a})$ as in Lemma 5.2.5 comes equipped with a universal Lie ∞ -algebroid structure for $\text{Der}(\mathcal{O}_{W,a})$. The quotient with respect to \mathfrak{m}_a is a Lie ∞ -algebra of the isolated singular point a with trivial 1-ary bracket. Using Corollary 4.2.13 and its subsequent discussion, we can prove the next statement.

Proposition 5.2.6. *For any universal Lie ∞ -algebroid structure on a resolution of $\text{Der}(\mathcal{O}_{W,a})$ as in Lemma 5.2.5, the quotient with respect to the ideal \mathfrak{m}_a is a representative of the Lie ∞ -algebra of the*

isolated singular point a , with trivial 1-ary bracket, on a graded vector space canonically isomorphic to $\mathrm{Tor}_{\mathcal{O}_W}(\mathfrak{X}_W, \mathbb{C})$ (\mathbb{C} being a \mathcal{O}_W -module through evaluation at a).

In particular, its 2-ary bracket is a graded Lie bracket on $\mathrm{Tor}_{\mathcal{O}_W}(\mathfrak{X}_W, \mathbb{C})$ which does not depend on any choice made in the construction, and its 3-ary bracket is a Chevalley-Eilenberg cocycle whose class is also canonical.

5.3 Some examples of universal Lie-algebroids over an affine variety

5.3.1 Vector fields tangent to W : a codimension one example

The zero-set in $W \subset V = \mathbb{C}^d$ of a weight homogeneous polynomial function $\varphi \in \mathbb{C}[X_1, \dots, X_d]$ admitting only an isolated singularity at the origin is one of the simplest possible example of a non-smooth affine variety $W \subset V = \mathbb{C}^d$. We give in this section a description of $\mathbb{U}_{\mathcal{O}}(\mathfrak{X}(W))$. Although our description is not complete, it will show how complex the universal Lie ∞ -algebroids may be, even for simple objects.

First, let us describe $\mathfrak{X}(W) = \mathrm{Der}(\mathcal{O}_W)$. We denote by $\omega_1, \dots, \omega_d$ the weights of the variables x_1, \dots, x_d and by $|\varphi|$ the weighted degree of φ . Recall that, by definition, $\mathcal{O}_W := \frac{\mathbb{C}[X_1, \dots, X_d]}{\langle \varphi \rangle}$.

Proposition 5.3.1. *As a \mathcal{O}_W -module, $\mathfrak{X}(W) := \mathrm{Der}(\mathcal{O}_W)$ is generated by the restrictions to W of the following vector fields in \mathbb{C}^d :*

1. the weighted Euler vector field $\vec{E} := \sum_{i=1}^d \omega_i x_i \partial_{x_i}$
2. the $d(d-1)/2$ vector fields given by:

$$X_{ij} := \frac{\partial \varphi}{\partial x_i} \partial_{x_j} - \frac{\partial \varphi}{\partial x_j} \partial_{x_i} \text{ with } 1 \leq i < j \leq d.$$

We start with a lemma. Recall that a homogenous function φ with an isolated singularity at 0 is a Koszul function (see Example 4.4.6), so that the Koszul complex, i.e. the complex of poly-vector fields on $V = \mathbb{C}^d$, equipped with ι_φ :

$$\dots \xrightarrow{\iota_\varphi} \mathfrak{X}^2(\mathbb{C}^d) \xrightarrow{\iota_\varphi} \mathfrak{X}^1(\mathbb{C}^d) \xrightarrow{\iota_\varphi} \mathcal{O}$$

has no cohomology except in degree 0. We denote it by $(\mathfrak{X}^\bullet, \iota_\varphi)$.

Lemma 5.3.2. *If $P \in \mathfrak{X}^{i+1}(\mathbb{C}^d)$, $Q \in \mathfrak{X}^i(\mathbb{C}^d)$ satisfy $\iota_\varphi(P) = \varphi Q$, then there exists $R \in \mathfrak{X}^{i+2}(\mathbb{C}^d)$ such that*

$$P = \frac{1}{|\varphi|} \vec{E} \wedge Q + \iota_\varphi(R)$$

Proof. This follows from the easily checked fact that $\iota_\varphi(Q) = 0$, so that

$$\iota_\varphi \left(\frac{1}{|\varphi|} \vec{E} \wedge Q \right) = \varphi Q.$$

This implies that

$$\iota_\varphi \left(P - \frac{1}{|\varphi|} \vec{E} \wedge Q \right) = 0$$

and the existence of R now follows from the exactness of the Koszul complex. \square

Proof (of Proposition 5.3.1). Any $X \in \mathfrak{X}(W)$ is the restriction to W of a vector field \tilde{X} in V tangent to W , i.e. that satisfies $\tilde{X}[\varphi] = f\varphi$ for some $f \in \mathcal{O}$. By Lemma 5.3.2 the vector field \tilde{X} can be written as $\tilde{X} = \frac{f}{|\varphi|} \vec{E} + \iota_\varphi(P)$, for some bivector field $P \in \mathfrak{X}^2(\mathbb{C}^d)$. The restriction to W of the first (resp the second) term is tangent to W and is of the first (resp. second) type described in Proposition 5.3.1. This completes the proof. \square

We now intend to construct a free resolution of $\mathfrak{X}(W)$ in the category of \mathcal{O}_W . Let us denote by $\wedge \vec{E}: \mathfrak{X}^i(V) \rightarrow \mathfrak{X}^{i+1}(V)$ the map $\omega \mapsto \omega \wedge \vec{E}$. The map $\wedge \vec{E}$ is a chain map from the restriction $i_W^* \mathfrak{X}^\bullet$ to W of the Koszul complex, shifted by one to itself. More precisely, $\wedge \vec{E}$ in the diagram below is a chain map:

$$\begin{array}{ccccccc} \dots & \xrightarrow{\iota_\varphi} & i_W^* \mathfrak{X}^2(V) & \xrightarrow{\iota_\varphi} & i_W^* \mathfrak{X}(V) & \xrightarrow{i_\varphi} & \mathcal{O}_W \\ & & \downarrow \wedge \vec{E} & & \downarrow \wedge \vec{E} & & \downarrow \wedge \vec{E} \\ \dots & \xrightarrow{\iota_\varphi} & i_W^* \mathfrak{X}^3(V) & \xrightarrow{\iota_\varphi} & i_W^* \mathfrak{X}^2(V) & \xrightarrow{\iota_\varphi} & \mathfrak{X}(W) \end{array}$$

Lemma 5.3.3. *The chain map $\cdot \wedge \vec{E}$ is a quasi-isomorphism.*

Proof. It is quite clear for $i \geq 1$ since the complex $(\mathcal{E}'[1], \iota_\varphi)$ has no cohomology. Let us check bijectivity in the last column. Surjectivity follows from Proposition 5.3.1. To check injectivity consider a function $f \in \mathcal{O}$ such that

$$f \vec{E} = \iota_\varphi(\pi) + \varphi Q, \quad (5.11)$$

for some bivector field $\pi \in \mathfrak{X}^2(V)$ and vector field $Q \in \mathfrak{X}(V)$. Upon taking ι_φ to both sides, Equation (5.11) implies,

$$f|\varphi|\varphi = \varphi \iota_\varphi(Q).$$

Hence, we obtain $f = \frac{1}{|\varphi|} \iota_\varphi(Q)$. \square

The lemma 5.3.3 implies that the mapping cone construction provides a free resolution of the \mathcal{O}_W -module of vector fields of $\mathfrak{X}(W)$ on W namely:

$$\left(\mathcal{E}_{-i} = i_W^* \mathfrak{X}^{i-1}(V) \oplus i_W^* \mathfrak{X}^{i+1}(V), i \geq 1; D = \begin{pmatrix} -\iota_\varphi & 0 \\ -\wedge \vec{E} & \iota_\varphi \end{pmatrix}, \pi \right). \quad (5.12)$$

In degree -1 we read $\mathcal{E}_{-1} = \mathcal{O}_W \oplus i_W^* \mathfrak{X}^2(V)$ and π is defined on the generators of \mathcal{E}_{-1} as $\pi(1 \oplus 0) := \vec{E}$ and

$$\pi(0 \oplus \partial_{x_i} \wedge \partial_{x_j}) := \frac{\partial \varphi}{\partial x_i} \partial_{x_j} - \frac{\partial \varphi}{\partial x_j} \partial_{x_i}, \text{ for all, } 1 \leq i < j \leq d.$$

Let us now describe some of the k -ary brackets:

Proposition 5.3.4. *The Lie-Rinehart algebra $\mathfrak{X}(W)$ of vector fields on W admits a universal Lie ∞ -algebroid whose*

1. underlying complex is (5.12) (which is a free resolution of $\mathfrak{X}(W)$),
2. 1-ary bracket is given by the resolution (5.12).
3. anchor map $\rho := \pi$.

4. In the case where the generator $1 \oplus 0$ appears together with a bivector field on W , one can define the 2-ary bracket on elements of degree -1 which makes the anchor map a morphism as follows,

$$\{1 \oplus 0, 0 \oplus \hat{\partial}_{x_i} \wedge \hat{\partial}_{x_j}\}_2 := (|\varphi| - 2)0 \oplus \hat{\partial}_{x_i} \wedge \hat{\partial}_{x_j}.$$

5. The k -ary brackets can be chosen to be the same as in the structure given by Proposition 4.4.17 on $i_W^* \mathfrak{X}^{i+1}(V)$ for $i \geq 1$.
6. the 3-ary bracket can be chosen to be zero when evaluated at $1 \oplus 0$ and with two other elements of $i_W^* \mathfrak{X}^2(V)$.

Corollary 5.3.5. $\mathfrak{X}(W)$ does not come from a Lie algebroid of rank $1 + \frac{d(d+1)}{2}$.

Remark 5.3.6. In general, it is hard to compute the generators of $\mathfrak{X}(W)$. But, there is a case where we know all generators: it is when W is a complete intersection with isolated singularity at zero such that \mathcal{I}_W is generated by weight homogeneous polynomials $\varphi_1, \dots, \varphi_r$. In that case, $\mathfrak{X}_W(\mathbb{C}^d)$ is generated by $\mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$, the Euler vector field, and the Hamiltonian vector fields (see [HM93, Sie96]).

The computation of the Lie ∞ -algebroid remains complicated.

Conclusion:

As a particular case of Section 4, we notice that a Lie ∞ -algebroid can be associated to any affine variety. Explicit computations are difficult. Some applications to Mohsen's resolution of singularities will be given in Section 7.2

There are still many open questions on the geometric meaning of this Lie ∞ -algebroid structure. Although, we have not answered many questions, we state concepts, lemmas, and counterexamples that we hope to be able to use in the future.

 Universal Q -manifolds of a singular foliation

The aim of this chapter is to lay the ground for the subsequent results. More precisely, to explain how Theorem 4.2.1 extends the results on singular foliations of Sylvain Lavau’s PhD [Lav17] followed by a referred version by C. Laurent-Gengoux, S. Lavau and T. Strobl in [LLS20]. The results of Section Chapter 4 extend the latter for arbitrary Lie-Rinehart algebras and also to the infinite case, and it still holds even when we do not have a geometric resolution. We introduce the notion of longitudinal vector fields on a NQ -manifold and prove a new result on their cohomology.

This chapter is taken from the textbook [LGLR22] in which I am co-author. We refer the reader to [Vor10, BP13, LMP20, LLS20] for more details on this topic.

Throughout of this chapter M is a smooth, real analytic or complex manifold and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We denote by \mathcal{O} the sheaf of functions on M .

6.1 Q -manifolds

Let us first define \mathbb{N} -graded manifolds.

6.1.1 Graded manifolds

In words, as the name suggests graded manifolds are for graded vector spaces what manifolds are for vector spaces, in the sense that roughly speaking manifolds look locally like \mathbb{R}^n and graded manifolds are locally like $\mathbb{R}^n \times V$ for some graded vector space V . In this chapter we introduce the notion of graded manifolds, their vector fields, their morphisms, etc.

Definition 6.1.1. A (positively) graded manifold over the base manifold M is a sheaf

$$\mathcal{E}: \mathcal{U} \mapsto \mathcal{E}(\mathcal{U})$$

of graded commutative algebras over \mathbb{K} such that every $m \in M$ admits an open neighborhood $\mathcal{U} \subset M$ on which the sheaf structure takes the form

$$\mathcal{E}(\mathcal{U}) = \mathcal{O}_{\mathcal{U}} \otimes_{\mathbb{K}} S(E_{-\bullet}^*)$$

for some graded vector space $E = \bigoplus_{i=1}^{\infty} E_{-i}$. Sections of the sheaf \mathcal{E} are called *functions on E* . It is convenient to denote a graded manifold as a pair (M, \mathcal{E}) where E is implicit.

Remark 6.1.2. A function $\xi \in \mathcal{E}_j$ is a formal sum

$$\xi = \sum_{i=0}^{+\infty} \xi^{(i)} \quad (6.1)$$

with $\xi^{(i)} \in \mathcal{E}$ an element of polynomial-degree i and degree j . For degree reasons, the sum must be finite.

Local coordinates of a graded manifold

Recall that for $\mathcal{U} \subset M$ an open set, one has $(E_i^*)_{\mathcal{U}} \xrightarrow{\sim} \mathcal{U} \times \mathbb{K}^{\text{rk}(E_i^*)}$ for every $i \geq 1$. Hence, the *graded coordinates* on the graded manifold (M, \mathcal{E}) is the data made of:

In degree 0: a system of coordinates (x_1, \dots, x_n) of M on \mathcal{U}

In degree $i \geq 1$: a local trivialization $(\xi_i^1, \dots, \xi_i^{\text{rk}(E_i^*)})$ of E_i^* on \mathcal{U} .

That is, a system of graded coordinates of (M, \mathcal{E}) on \mathcal{U} is

$$(x_1, \dots, x_n, \xi_1^1, \dots, \xi_1^{\text{rk}(E_1^*)}, \dots, \xi_i^1, \dots, \xi_i^{\text{rk}(E_i^*)}, \dots).$$

Elements of $\mathcal{E}(\mathcal{U})$ are "polynomials" in $\{(\xi_i^j)_{j=1, \dots, \text{rk}(E_i^*)}, i \geq 1\}$ with coefficients in $\mathcal{O}(\mathcal{U})$.

Example 6.1.3. The sheaf of differential forms $(M, \mathcal{E} = \Omega(M))$ on a manifold M is a graded manifold since for every point $m \in M$, it takes the form $\mathcal{O}_{\mathcal{U}} \otimes_{\mathbb{K}} \wedge^{\bullet} T_m^* M$ where \mathcal{U} is an open neighborhood of m . Exterior forms can be seen as sections on the graded vector bundle $E_{-1} = TM$.

Example 6.1.4. Let k be a positive integer. A finite dimensional vector space E and its dual E^* can be seen as graded vector bundles of respective degree $-k$ and k over a point. E is a graded manifold over $M = \{pt\}$, with functions isomorphic to $\wedge E^*$ for k odd and $S(E^*)$ for k even.

Definition 6.1.5. A *morphism of graded manifolds* between the two graded manifolds (M, \mathcal{E}) and (M', \mathcal{E}') with respective base manifolds M and M' is a pair made of a smooth or real analytic or holomorphic map $\phi: M \rightarrow M'$ called the *base map* and a sheaf morphism over it, i.e. a family of graded algebra morphisms:

$$\mathcal{E}'(\mathcal{U}') \rightarrow \mathcal{E}(\phi^{-1}(\mathcal{U}')),$$

compatible with the restriction maps, such that

$$\Phi(f\alpha) = \phi^*(f)\Phi(\alpha). \quad (6.2)$$

for all $f \in \mathcal{O}'_{\mathcal{U}'}$ and $\alpha \in \mathcal{E}'(\mathcal{U}')$.

A *homotopy* between two morphisms of graded manifolds $\Phi, \Psi: (M, \mathcal{E}) \longrightarrow (N, \mathcal{E}')$ is a morphism of graded manifold

$$(M, \mathcal{E}) \times ([0, 1], \Omega([0, 1])) \longrightarrow (M', \mathcal{E}')$$

whose restrictions to $t = 0$ resp. $t = 1$ coincide with Φ and Ψ respectively.

Vector fields on graded manifolds

Vector fields on manifolds are derivations of its algebra of functions. For a graded manifold, the analog of functions are the sheaf of sections $\Gamma(S^\bullet(\mathcal{E}^*))$. Since it is not commutative but graded commutative, one has to consider graded derivations. A *graded derivation of degree k* of \mathcal{E} is the data, for every $\mathcal{U} \subset M$ of a linear map

$$Q: \mathcal{E}_\bullet(\mathcal{U}) \longrightarrow \mathcal{E}_{\bullet+k}(\mathcal{U}),$$

compatible with all restriction maps, that increases the degree by $+k$ and satisfies:

$$Q[FG] = Q[F]G + (-1)^{ki} FQ[G]$$

for every $F \in \mathcal{E}_i(\mathcal{U}), G \in \mathcal{E}(\mathcal{U})$. Since we think geometrically, we say "*vector fields of degree k* " instead of graded derivation.

Definition 6.1.6. Let (M, \mathcal{E}) be a graded manifold. For $\mathcal{U} \subset M$ and $k \in \mathbb{Z}$ let

$$\mathfrak{X}_k(E)(\mathcal{U}) := \text{Der}_k(\mathcal{E}(\mathcal{U}))$$

be the $\mathcal{E}(\mathcal{U})$ -module of derivation of degree k on $\mathcal{E}(\mathcal{U})$. The correspondence $\mathcal{U} \longmapsto \mathfrak{X}_\bullet(E)(\mathcal{U})$ is a sheaf of \mathcal{E} -module. Its sections are called *vector fields on E* .

Let us list some important facts on vector fields on E :

1. the \mathcal{E} -module $\mathfrak{X}_\bullet(E) := \bigoplus_{k \in \mathbb{Z}} \mathfrak{X}_k(E)$ of vector fields on E is naturally graded. The \mathcal{E} -module $\mathfrak{X}_\bullet(E)$ of vector fields on E is a graded Lie subalgebra of the graded Lie algebra $\text{Hom}_{\mathbb{K}}(\mathcal{E}, \mathcal{E})$ whose graded Lie bracket is the graded commutator. Precisely, the graded Lie bracket

$$[P, Q] = P \circ Q - (-1)^{kl} Q \circ P \tag{6.3}$$

of two vector fields P, Q of degree k, l respectively is a vector field of degree $k + l$. It is easily checked that the bracket (6.3) fulfills

- (a) $[P, Q] = -(-1)^{jk}[Q, P]$, (graded skew-symmetry)
- (b) $(-1)^{jl}[P, [Q, R]] + (-1)^{jk}[Q, [R, P]] + (-1)^{kl}[R, [P, Q]] = 0$, (graded Jacobi identity)
- (c) $[P, fQ] = P[f]Q + f[P, Q]$, (Leibniz identity)

for $f \in \mathcal{O}$ and P, Q, R are vector fields of degree j, k and l respectively.

2. Their description in local coordinates: notice that any homogeneous element $e \in \Gamma(E_{-k})$ corresponds to a vertical¹ vector field $\iota_e \in \mathfrak{X}_{-k}(E)$ (i.e. it is \mathcal{O} -linear) of degree $-k$ defined by contraction with e

$$\iota_e(\xi) := \langle \xi, e \rangle, \quad \xi \in \Gamma(E^*) \quad (6.4)$$

and we extend by \mathcal{O} -linear derivation, where $\langle \cdot, \cdot \rangle$ is the dual pairing between E^* and E . Notice that ι_e is by construction of polynomial-degree -1 .

Let $(\mathcal{U}, x_1, \dots, x_n)$ be a coordinate chart of M and $(\xi_i^j)_{j=1, \dots, \text{rk}(E_i^*)}$ with $i \geq 1$ be a homogeneous local trivialization of E_{-i}^* , it should be understood that ξ_i^j is the j -th elements of the local frame in $\Gamma(E_{-i}^*)$. Let $(e_i^j)_{j=1, \dots, \text{rk}(E_{-i})}$, $i \geq 1$ be the dual basis of $(\xi_i^j)_{j=1, \dots, \text{rk}(E_{-i}^*)}$, $i \geq 1$, then for every pair i, j , $\iota_{e_i^j} = \frac{\partial}{\partial \xi_i^j}$ is the partial derivative with respect to $\xi_i^j \in \Gamma(E_{-i}^*)$. By choosing a TM -connection on E , it is easy to check that for any $k \in \mathbb{Z}$ the family

$$\left(\xi_{i_1}^{j_1} \odot \dots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial x_j} \right)_{\substack{l \geq 0 \\ i_1 \dots i_l = k \\ j_1, \dots, j_l \\ j = 1, \dots, n}} \cup \left(\xi_{i_1}^{j_1} \odot \dots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial \xi_i^j} \right)_{\substack{l \geq 0 \\ i_1 \dots i_l - i = k \\ j_1, \dots, j_l \\ i \geq 1, j = 1, \dots, \text{rk}(E_{-i})}}$$

form a basis for $\mathfrak{X}_k(E)(\mathcal{U})$ up to permutations of the $\xi_{i_1}^{j_1} \odot \dots \odot \xi_{i_l}^{j_l}$'s. Here we adopt the convention $i_0 = j_0 = 0$ and $\xi^0 = 1 \in \Gamma(S^0(E^*)) \simeq \mathcal{O}$. Whence, any vector field $Q \in \mathfrak{X}_k(E)(U)$ admits coordinates decomposition as follows

$$Q = \sum_{\substack{l \geq 0 \\ i_1 \dots i_l = k \\ j_1, \dots, j_l \\ j = 1, \dots, n}} \frac{1}{l!} Q_{i_1 \dots i_l}^{j_1 \dots j_l} \xi_{i_1}^{j_1} \odot \dots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial x_j} + \sum_{\substack{i \geq 1, l \geq 0 \\ i_1 \dots i_l - i = k \\ j_1, \dots, j_l \\ j = 1, \dots, \text{rk}(E_{-i})}} \frac{1}{l!} Q_{i_1 \dots i_l}^{j_1 \dots j_l} \xi_{i_1}^{j_1} \odot \dots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial \xi_i^j}.$$

for some functions $Q_{i_1 \dots i_l}^{j_1 \dots j_l} \in \mathcal{O}$. These functions can be chosen in a unique manner to satisfy, e.g. ${}^{ij}Q_{i_{\sigma(1)} \dots i_{\sigma(l)}}^{j_{\sigma(1)} \dots j_{\sigma(l)}} = \epsilon(\sigma) Q_{i_1 \dots i_l}^{j_1 \dots j_l}$ for any permutation σ of $\{1, \dots, l\}$.

For example, if Q is of degree $+1$, then it can be written in these notations as

$$Q = \sum_{\substack{1 \leq u \leq \text{rk}(E_{-1}) \\ j = 1, \dots, n}} Q_1^u \xi_1^u \frac{\partial}{\partial x_j} + \sum_{\substack{i \geq 1, l \geq 0 \\ i_1 \dots i_l - i = 1 \\ j_1, \dots, j_l \\ j = 1, \dots, n}} \frac{1}{l!} {}^{ij}Q_{i_1 \dots i_l}^{j_1 \dots j_l} \xi_{i_1}^{j_1} \odot \dots \odot \xi_{i_l}^{j_l} \frac{\partial}{\partial \xi_i^j}.$$

This following lemma says that vector fields of polynomial-degree -1 are all the types (6.4).

Lemma 6.1.7. *Let (M, \mathcal{E}) be a graded manifold. For $i \geq 1$, a vector field $P \in \mathfrak{X}_{-i}(E)$ of polynomial-degree -1 and of degree $-i$, is of the form ι_e for some section $e \in \Gamma(E_{-i})$.*

Proof. Note that a vector field $P \in \mathfrak{X}_{-i}(E)$ of polynomial-degree -1 is vertical, i.e. $P(f) = 0$, since functions of M are of polynomial-degree zero. Hence, in local coordinates $(\mathcal{U}, x_1, \dots, x_n)$ M and $(\xi_i^j)_{j=1, \dots, \text{rk}(E_{-i}^*)}$ with be a homogeneous local trivialization of E_{-i}^* and $(e_i^j)_{j=1, \dots, \text{rk}(E_{-i})}$ be the dual

¹A vector field $P \in \mathfrak{X}(E)$ is said to be *vertical* if it is linear with respect to functions on M , in other words if $P[f] = 0$ for all $f \in \mathcal{O}$.

basis of $(\xi_i^j)_{j=1, \dots, \text{rk}(E_{-i}^*)}$; a polynomial-degree -1 vector field $P \in \mathfrak{X}_{-i}(E)$ of degree $-i$ is forced to be of the form

$$P|_{\mathcal{U}} = \sum_{j=1}^{\text{rk}(E_{-i})} f^j(x) \frac{\partial}{\partial \xi_i^j} \quad (6.5)$$

with $f^j \in C^\infty(U)$. Now, choose a local section $e_U \in \Gamma_{\mathcal{U}}(E_{-i})$ of the form,

$$e_U := \sum_{j \geq 1} f^j e_i^j. \quad (6.6)$$

It is then clear that $\iota_{e_U} = P|_{\mathcal{U}}$. Since local sections of E_{-i} given on different coordinates chart domains \mathcal{U}_a and \mathcal{U}_b coincide on $\mathcal{U}_a \cap \mathcal{U}_b$ and then lift to a global section of E_{-i} . Thus, $\iota_e = P$ for some $e \in \Gamma(E_{-i})$. \square

6.1.2 NQ-manifolds

Definition 6.1.8. A *dg-manifold* or *NQ-manifold* is a positively graded manifold (M, \mathcal{E}) endowed with a degree $+1$ homological vector field Q on E , i.e., $Q \in \mathfrak{X}_1(E)$ is such that $Q^2 = 0$.

They shall be denoted as a triple (M, \mathcal{E}, Q) .

Example 6.1.9. Given a finite dimension Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ of dimension d . We assume that \mathfrak{g} is concentrated in degree -1 . It is clear that $(M = \{pt\}, \mathcal{E} = \wedge^\bullet \mathfrak{g}^*)$ is a graded manifold over $M = \{pt\}$. This graded manifold carries a *dg-manifold* structure. Precisely, we define the corresponding homological vector field as follows: fix a basis $(e_i)_{i=1, \dots, n}$ of \mathfrak{g} and let these global coordinate functions $(\xi^i)_{i=1, \dots, n}$ on \mathfrak{g} be its dual. We have

$$[e_i, e_j] = \sum_{l=1}^n \lambda_{ij}^l e_l$$

for some coefficients $\lambda_{ij}^l \in \mathbb{K}$. One can check that the degree $+1$ vector field

$$Q = \frac{1}{2} \sum_{i,j,l=1}^n \lambda_{ij}^l \xi^i \wedge \xi^j \frac{\partial}{\partial \xi^l}$$

corresponds to the Chevalley-Eilenberg differential $(d^{\text{CE}}, \wedge^\bullet \mathfrak{g}^*)$. Therefore, $Q^2 = 0$. Notice that $Q^2 = 0$ is equivalent to the Jacobi identity for $[\cdot, \cdot]$.

Example 6.1.10. Given a differential graded vector bundle $((E_{-i})_{i \geq 1}, d)$ over M . There is a natural *dg* manifold given by its sheaf of sections $(M, \mathcal{E} = \Gamma(S(E^*)))$. In particular, the differential map $d: E \rightarrow E$ is dualized as a degree $+1$ map $S^1(E^*) \rightarrow S^1(E^*)$ that we extend to a $C^\infty(M)$ -linear derivation on \mathcal{E} squared to zero.

Example 6.1.11. Let $E = T[1]M$ be the shifted bundle of M . It induces a graded manifold structure $(M, \mathcal{E} = \Omega(M))$ over M . This graded manifold carries a *dg-manifold* structure Q that corresponds to the de Rham differential on $\Omega(M)$. In terms of coordinates, the homological vector field Q reads

$$\sum_{i=1}^n dx_i \frac{\partial}{\partial x_i}.$$

Let us introduce some vocabularies that will need to use.

Definition 6.1.12. Let (M, \mathcal{E}', Q') and (M, \mathcal{E}, Q) be two NQ -manifolds.

1. A linear map $\Phi: \mathcal{E} \rightarrow \mathcal{E}'$ is said to be of *polynomial-degree/degree* $j \in \mathbb{Z}$ provided that, for all function $\alpha \in \mathcal{E}$ of polynomial-degree/degree i , $\Phi(\alpha)$ is of polynomial-degree/degree $i + j$. Any map $\Phi: \mathcal{E} \rightarrow \mathcal{E}'$ of degree i decomposes w.r.t the polynomial-degree as follows:

$$\Phi = \sum_{r \in \mathbb{Z}} \Phi^{(r)}$$

with $\Phi^{(r)}: \mathcal{E} \rightarrow \mathcal{E}'$ a map of polynomial-degree r .

Remark 6.1.13. When $\Phi: \mathcal{E} \rightarrow \mathcal{E}'$ is a graded morphism of algebras, necessarily one has $\Phi^{(r)} = 0$ for all $r < 0$. Furthermore, for all $n, r \in \mathbb{N}$ and all $\xi_1, \dots, \xi_n \in \Gamma(V)$ one has:

$$\Phi^{(r)}(\xi_1 \odot \dots \odot \xi_n) = \sum_{i_1 + \dots + i_n = r} \Phi^{(i_1)}(\xi_1) \odot \dots \odot \Phi^{(i_n)}(\xi_n). \quad (6.7)$$

Obviously, in this case Φ is determined uniquely by the image of $\Gamma(V)$.

Definition 6.1.14. Let (M, \mathcal{E}, Q) and (M, \mathcal{E}', Q') be two NQ -manifolds over M with sheaves of functions \mathcal{E} and \mathcal{E}' respectively. A *morphism of NQ -manifold over M* from (M, \mathcal{E}', Q') to (M, \mathcal{E}, Q) is a morphism of graded manifolds $\Phi: \mathcal{E} \rightarrow \mathcal{E}'$ (of degree 0) over the identity map which intertwines Q and Q' , i.e.,

$$\Phi \circ Q = Q' \circ \Phi. \quad (6.8)$$

Remark 6.1.15. It is important to notice that

- morphisms of NQ -manifolds over M are by definition \mathcal{O} -linear, since they are defined over the identity map
- the component $\Phi^{(r)}$ of polynomial-degree $r \geq 0$ of any \mathcal{O} -linear map $\Phi: \mathcal{E} \rightarrow \mathcal{E}'$ maps $\Gamma(E^*)$ to $\Gamma(S^{r+1}(E'^*) \otimes E)$. By \mathcal{O} -linearity, it gives rise to a section $\phi_r \in \Gamma(S^{r+1}(E'^*) \otimes E)$. Therefore, one has

$$\Phi^{(r)}(\xi) = \langle \phi_r, \xi \rangle \quad (6.9)$$

for all $\xi \in \Gamma(E^*)$. It follows that Φ is entirely determined by the collection $(\phi_r \in \Gamma(S^{k+1}(E'^*) \otimes E))_{r \geq 0}$ when Φ is an algebra morphism or a Ξ -derivation for some map $\Xi: \mathcal{E} \rightarrow \mathcal{E}'$. In such case, for $r \geq 0$, $\phi_r \in \Gamma(S^{r+1}(E'^*) \otimes E)$ is then called the *r -th Taylor coefficient* of Φ . We also call the 0-th Taylor coefficient $\phi_0: E' \rightarrow E$ the *linear part* of Φ . The latter is a chain map

$$\begin{array}{ccccccc} \dots & \longrightarrow & E'_{-3} & \xrightarrow{d'^{(3)}} & E'_{-2} & \xrightarrow{d'^{(2)}} & E'_{-1} & \xrightarrow{\rho'} & TM \\ & & \downarrow \phi_0 & & \downarrow \phi_0 & & \downarrow \phi_0 & & \downarrow \text{id} \\ \dots & \longrightarrow & E_{-3} & \xrightarrow{d^{(3)}} & E_{-2} & \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM \end{array} \quad (6.10)$$

6.1.3 NQ-manifolds - Lie ∞ -algebroids

We have seen in Section 2.2 that Lie ∞ -algebroids (possibly infinite dimension) over \mathcal{O} are one-to-one with co-differentials of the graded symmetric algebra, which are compatible with the action of \mathcal{O} . This correspondence provides a simple characterization of Lie ∞ -algebroids over an arbitrary commutative unital algebra \mathcal{O} . In the finite dimensional case, we can work without co-differentials by using T. Voronov's higher derived brackets construction [Vor10]. Roughly speaking, it is shown in [Vor10] that (finite dimensional) Lie ∞ -algebroids as in Remark 2.1.13 are the same as NQ-manifolds. However, it is important to note the latter correspondence fails in infinite dimension case, since the identification $\Gamma(S^\bullet(E^*)) \simeq \Gamma(S^\bullet(E)^*)$ fails. We will not explain entirely the construction here, but we refer the reader to [Vor04, Vor05, Vor10] also to [BP13, LMP20] for more details on the construction.

The following statement is similar to our Proposition 2.2.4 the difference lies in the fact that ours remains valid in infinite dimension.

Proposition 6.1.16 (T. Voronov). *Let (M, \mathcal{E}) be a graded manifold of finite dimension in each degree. There is a one-to-one correspondence between:*

1. (finitely generated) negatively graded Lie ∞ -algebroids $(E, (\ell_k)_{k \geq 1}, \rho)$ over M ,

and

2. homological vector fields $Q \in \mathfrak{X}_{+1}(E)$.

The relation stated in Theorem 6.1.16 can be enlightened in terms of the k -ary brackets (ℓ_k) as follows:

$$[\cdots [[Q, \iota_{e_1}], \iota_{e_2}], \dots, \iota_{e_k}]^{(-1)} = \iota_{\ell_k(e_1, \dots, e_k)}, \quad \text{for all } e_1, \dots, e_k \in \Gamma(E) \quad (6.11)$$

$$\rho(e)[f] = [Q, \iota_e]^{(0)}(f), \quad \text{for all } f \in \mathcal{O}, e \in \Gamma(E_{-1}) \quad (6.12)$$

In particular²,

1. for all $f \in \mathcal{O}, e \in \Gamma(E_{-1})$

$$\langle Q^{(1)}[f], e \rangle = \rho(e)[f],$$

2. for all $\alpha \in \Gamma(E^*)$ and $e \in \Gamma(E)$:

$$\langle Q^{(0)}[\alpha], e \rangle = \langle \alpha, \ell_1(e) \rangle,$$

3. for all homogeneous elements $e_1, e_2 \in \Gamma(E)$ and $\alpha \in \Gamma(E^*)$

$$\langle Q^{(1)}[\alpha], e_1 \odot e_2 \rangle = \rho(e_1)[\langle \alpha, e_2 \rangle] - \rho(e_2)[\langle \alpha, e_1 \rangle] - \langle \alpha, \ell_2(e_1, e_2) \rangle,$$

with the understanding that the anchor ρ vanishes on E_{-i} when $i \geq 2$.

²Our sign's convention are those of [LLS20].

4. for every $k \geq 3$, the k -ary brackets $\ell_k: \Gamma(S_{\mathbb{K}}^k(E)) \rightarrow \Gamma(E)$ and the polynomial-degree $k-1$ component $Q^{(k-1)}: \Gamma(E^*) \rightarrow \Gamma(S_{\mathbb{K}}^k(E^*))$ of Q are dual to each other, i.e.

$$\left\langle Q^{(k-1)}[\alpha], e_1 \odot \cdots \odot e_k \right\rangle = \langle \alpha, \ell_k(e_1, \dots, e_k) \rangle, \quad \text{for } \alpha \in \Gamma(E^*) \text{ and } e_1, \dots, e_k \in \Gamma(E).$$

Convention 6.1.17. *From now on, unless otherwise mentioned, we shall simply say "Lie ∞ -algebroids over M " for "finitely generated ∞ -algebroids over M ". Also, Lie ∞ -algebroids $(E, (\ell_k)_{k \geq 1}, \rho)$ over M shall be denoted as (E, Q) .*

Remark 6.1.18. Whether we see Lie ∞ -algebroids as NQ -manifolds or as co-differentials, these two approaches have in common to give the same notion of Lie ∞ -morphism of ∞ -algebroids. This can be seen directly by writing the conditions (2.3.1) and (6.8) in terms of the k -ary brackets.

6.2 Universal Q -manifolds

This section can be understood as a consequence of the main results of the first part of the thesis. We recover the result on universal Q -manifolds of a singular foliation [LLS20, Lav17], whose existence was proved under the condition that geometric resolutions exist. We recall that the existence of such a resolution is not guaranteed, unlike the case of resolution by free modules. We refer the reader to Appendix B and B.3 for more details on resolutions of modules and geometric resolutions of singular foliations.

Theorem 6.2.1. *Let \mathfrak{F} be a singular foliation on a manifold M .*

1. *Any resolution of \mathfrak{F} by free \mathcal{O} -modules (which may not be a geometric resolution)*

$$\cdots \xrightarrow{d} P_{-3} \xrightarrow{d} P_{-2} \xrightarrow{d} P_{-1} \xrightarrow{\rho} \mathfrak{F} \longrightarrow 0 \quad (6.13)$$

carries a Lie ∞ -algebroid structure over \mathfrak{F} whose unary bracket is $\ell_1 := d$.

2. *In particular, when \mathfrak{F} admits a geometric resolution (E, d, ρ) , there exists a Lie ∞ -algebroid (E, Q) over \mathfrak{F} whose linear part is (E, d, ρ) .*

Proof. Apply Theorem 4.2.1 to \mathfrak{F} seen as a Lie-Rinehart algebra over \mathcal{O} . □

Proposition 6.2.2. *Let \mathfrak{F} be a singular foliation over M . Given,*

- a) *a Lie ∞ -algebroid (M, \mathcal{E}', Q') that terminates in \mathfrak{F} , i.e., $\rho'(\Gamma(E_{-1})) \subseteq \mathfrak{F}$,*
b) *a universal Lie ∞ -algebroid (M, \mathcal{E}, Q) of \mathfrak{F} ,*

then

1. *there exists a Lie ∞ -morphism from (M, \mathcal{E}', Q') to (M, \mathcal{E}, Q) .*
2. *and any two such morphisms are homotopic.*

Proof. Apply Theorem 4.2.4. □

Corollary 6.2.3. *Two universal Lie ∞ -algebroid of a singular foliation are homotopy equivalent. Moreover, the homotopy equivalence between them is unique up to homotopy.*

Proof. Apply Corollary 4.2.5. □

6.2.1 The complex defined by $\text{ad}_{Q^{(0)}}$

Let \mathfrak{F} be a singular foliation on M that admits a universal algebroid (E, Q) . For a fixed $k \in \mathbb{N}_0 \cup \{-1\}$ consider the bicomplex defined on $\text{Hom}_{\mathcal{O}}^{\bullet} \left(\Gamma(E^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right) \right)$ where the horizontal differential ∂^h (resp. vertical differential ∂^v) are given by left composition (resp. right composition with $Q^{(0)}$), namely, for all $\Psi^{(k)} \in \text{Hom}_{\mathcal{O}}^{\bullet} \left(\Gamma(E^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right) \right)$ one has,

$$\partial^h \left(\Psi^{(k)} \right) := \begin{cases} \Psi^{(k)} \circ Q^{(0)} & \text{when } \Psi^{(k)} \text{ restricts to } \Gamma(E_{-i}^*), \text{ with } i \neq 1, \\ \Psi^{(k)} \circ \rho^* & \text{when } \Psi^{(k)} \text{ restricts to } \Gamma(E_{-1}^*), \end{cases} \quad (6.14)$$

and

$$\partial^v \left(\Psi^{(k)} \right) := \begin{cases} \rho^* \circ \Psi^{(k)} & \text{when } \Psi^{(k)} \text{ is of degree } i \text{ and restricts to } \Gamma(E_{-i-1}^*), \\ Q^{(0)} \circ \Psi^{(k)} & \text{otherwise.} \end{cases} \quad (6.15)$$

With the understanding that $\Gamma \left(S_{\mathbb{K}}^0(E^*) \right) \simeq \mathcal{O}$. The total differential is given by the formula,

$$\partial \left(\Psi^{(k)} \right) = \partial^h \left(\Psi^{(k)} \right) - (-1)^{|\Psi^{(k)}|} \partial^v \left(\Psi^{(k)} \right) \quad \text{for all } \Psi^{(k)} \in \text{Hom}_{\mathcal{O}}^{\bullet} \left(\Gamma(E^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right) \right).$$

The following diagram pictures the idea of the bicomplex. The total degree is given by the anti-diagonals lines.

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow \\ \rightarrow & \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-2}^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right)_{k+3} \right) & \xrightarrow{\partial^h} & \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-1}^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right)_{k+3} \right) & \xrightarrow{\circ \rho^*} & \text{Hom}_{\mathcal{O}} \left(\mathfrak{F}^*, \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*)_{k+3} \right) \right) & \rightarrow 0 \\ & \partial^v \uparrow & & \partial^v \uparrow & & D^v \uparrow & \\ \rightarrow & \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-2}^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right)_{k+2} \right) & \xrightarrow{\partial^h} & \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-1}^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right)_{k+2} \right) & \xrightarrow{\circ \rho^*} & \text{Hom}_{\mathcal{O}} \left(\mathfrak{F}^*, \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*)_{k+2} \right) \right) & \rightarrow 0 \\ & \partial^v \uparrow & & \partial^v \uparrow & & \partial^v \uparrow & \\ \rightarrow & \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-2}^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right)_{k+1} \right) & \xrightarrow{\partial^h} & \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-1}^*), \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*) \right)_{k+1} \right) & \xrightarrow{\circ \rho^*} & \text{Hom}_{\mathcal{O}} \left(\mathfrak{F}^*, \Gamma \left(S_{\mathbb{K}}^{k+1}(E^*)_{k+1} \right) \right) & \rightarrow 0 \\ & \uparrow & & \uparrow & & \uparrow & \\ & 0 & & 0 & & 0 & \end{array} \quad (6.16)$$

There is no cohomology for the total complex which is governed by $\partial \equiv [\cdot, Q^{(0)}]$ since the lines are exact (see Proposition B.1.16). When $k = -1$ the bicomplex (6.16) is just the line complex:

$$\cdots \xrightarrow{\partial^h} \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-2}^*), \mathcal{O} \right) \xrightarrow{\partial^h} \text{Hom}_{\mathcal{O}} \left(\Gamma(E_{-1}^*), \mathcal{O} \right) \xrightarrow{\partial^h} \text{Hom}_{\mathcal{O}} \left(\mathfrak{F}^*, \mathcal{O} \right) \longrightarrow 0 \quad (6.17)$$

which is also exact.

6.2.2 A result on longitudinal vector fields and examples

In this section, we study the cohomology of longitudinal vector fields, which will help in proving the main results stated in the beginning of Chapter 8.2.

Let \mathfrak{F} be a singular foliation over M .

Definition 6.2.4. Let E be a splitted graded manifold over M with sheaf of function \mathcal{E} . A vector field $L \in \mathfrak{X}(E)$ is said to be a *longitudinal vector field* for \mathfrak{F} if there exists vector fields $X_1, \dots, X_k \in \mathfrak{F}$ and functions $\Theta_1, \dots, \Theta_k \in \mathcal{E}$ such that

$$L(f) = \sum_{i=1}^k X_i[f] \Theta_i, \quad \forall f \in \mathcal{O}. \quad (6.18)$$

We also need to define the following

Definition 6.2.5. A vector field $X \in \mathfrak{X}(M)$ is said to be an *infinitesimal symmetry* of \mathfrak{F} , if $[X, \mathfrak{F}] \subset \mathfrak{F}$. The Lie algebra of infinitesimal symmetries of \mathfrak{F} is denoted by, $\mathfrak{s}(\mathfrak{F})$

Example 6.2.6. Here are some examples.

1. Vertical vector fields on a graded manifold are longitudinal for any singular foliation.
2. For any Q -manifold (E, Q) over a manifold M . The homological vector field $Q \in \mathfrak{X}(E)$ is a longitudinal vector field for $\mathfrak{F} := \rho(\Gamma(E_{-1}))$: in local coordinates $(\mathcal{U}, x_1, \dots, x_n)$ on M and a local trivialization ξ^1, ξ^2, \dots of graded sections in $\Gamma(E^*)$. The vector fields Q are of the form:

$$\begin{aligned} Q &= \sum_j \sum_{k, |\xi^k|=1} Q_k^j(x) \xi^k \frac{\partial}{\partial x_j} + \sum_j \sum_{k, \iota_1, \dots, \iota_k} \frac{1}{k!} Q_{\iota_1, \dots, \iota_k}^j(x) \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j} \\ &= \sum_{k, |\xi^k|=1} \xi^k \left(\sum_j Q_k^j(x) \frac{\partial}{\partial x_j} \right) + \dots \end{aligned} \quad (6.19)$$

Take $X_k := \sum_j Q_k^j(x) \frac{\partial}{\partial x_j} \in \mathfrak{F}(\mathcal{U})$ for every k . One has for all $f \in C^\infty(\mathcal{U})$, $Q(f) = \sum_k \xi^k X_k[f]$.

3. For (E, Q) a Q -manifold and $\mathfrak{F} := \rho(\Gamma(E_{-1}))$ its basic singular foliation. For any extension of a symmetry $X \in \mathfrak{s}(\mathfrak{F})$ of \mathfrak{F} to a degree zero vector field $\hat{X} \in \mathfrak{X}(E)$, the degree +1 vector field $[Q, \hat{X}]$ is longitudinal for \mathfrak{F} .

Let us show this last point using local coordinates (x_1, \dots, x_n) on M and a local trivialization ξ^1, ξ^2, \dots of graded sections in $\Gamma(E^*)$. The vector fields Q and \hat{X} take the form:

$$\begin{aligned} Q &= \sum_j \sum_{k, |\xi^k|=1} Q_k^j(x) \xi^k \frac{\partial}{\partial x_j} + \sum_j \sum_{k, \iota_1, \dots, \iota_k} \frac{1}{k!} Q_{\iota_1, \dots, \iota_k}^j(x) \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j} \\ \hat{X} &= X + \sum_j \sum_{k, \iota_1, \dots, \iota_k} \frac{1}{k!} X_{\iota_1, \dots, \iota_k}^j(x) \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j} \end{aligned} \quad (6.20)$$

where $X = \sum_{i=1}^n X_i(x) \frac{\partial}{\partial x_i}$. By using Equation (6.20) we note that all the terms of $[Q, \hat{X}]$ are vertical except maybe for the ones where the vector field X appears. For $k \geq 1$, the vector field $[Q_{\iota_1, \dots, \iota_k}^j \xi^1 \odot \dots \odot \xi^k \frac{\partial}{\partial \xi^j}, X]$ is vertical; and for every fix k , one has

$$\left[\sum_{j=1}^n Q_k^j \xi^k \frac{\partial}{\partial x_j}, X \right] = \xi^k \left[\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j}, X \right].$$

Thus, $\left[\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j}, X \right] \in \mathfrak{F}$, since X is a symmetry for \mathfrak{F} and $\sum_{j=1}^n Q_k^j \frac{\partial}{\partial x_j} \in \mathfrak{F}$.

Remark 6.2.7. Longitudinal vector fields are stable under the graded Lie bracket. We denote by $\mathfrak{L}(E)$ the graded Lie algebra of longitudinal vector fields for \mathfrak{F} .

Remark 6.2.8. Let us study vector fields on E .

1. Sections of E are identified with derivations under the isomorphism mapping $e \in \Gamma(E) \mapsto \iota_e \in \mathfrak{X}(E)$. This allows us to identify a vertical vector field with (maybe infinite) sums of tensor products of the form $\Theta \otimes e$ with $\Theta \in \mathcal{E}$, $e \in \Gamma(E)$.

2. Any connection on $\Gamma(E^*)$ induces a vector field of degree zero $\tilde{\nabla}_X \in \mathfrak{X}(E)$ by setting for $f \in \mathcal{O}$, $\tilde{\nabla}_X(f) := X[f]$. Once a connection is chosen, we have for all $k \in \mathbb{Z}$

$$\mathfrak{X}_k(E) \simeq \bigoplus_{j \geq 1} \mathcal{E}_{k+j} \otimes_{\mathcal{O}} \Gamma(E_{-j}) \oplus \mathcal{E}_k \otimes_{\mathcal{O}} \mathfrak{X}(M) \simeq \bigoplus_{j \geq 1} \Gamma(S(E^*)_{k+j} \otimes E_{-j}) \oplus \Gamma(S(E^*)_k \otimes TM). \quad (6.21)$$

Thus, one can realize a vector field $P \in \mathfrak{X}_k(E)$ as a sequence $P = (p_0, p_1, \dots)$, where $p_0 \in \Gamma(S(E^*)_k \otimes TM)$ and $p_i \in \Gamma(S(E^*)_{k+i} \otimes E_{-i})$ for $i \geq 1$ are called *components* of P . In the diagram (6.23), $P = (p_0, p_1, \dots)$ is represented as an element of the anti-diagonal and p_i is on column i . We say that P is of *depth* $n \in \mathbb{N}$ if $p_i = 0$ for all $i < n$. In particular, vector fields of depth greater or equal to 1 are vertical. Under the decomposition (6.21), the differential map ad_Q takes the form

$$D = D^h + \sum_{s \geq 0} D^{v_s} \quad (6.22)$$

with $D^2 = 0$. Here $D^h = \text{id} \otimes d$ or $\text{id} \otimes \rho$, and $D^{v_s} : \Gamma(S(E^*)_k \otimes E_{-i}) \rightarrow \Gamma(S(E^*)_{k+s+1} \otimes E_{-i-s})$ for $i \geq 0, s \geq 0$, we shall denote $E_0 := TM$. We denote the latter complex by (\mathfrak{X}, D) . They can be represented as anti-diagonal lines in the following commutative diagram, whose lines are complexes of \mathcal{O} -modules

$$\begin{array}{ccccc}
 & \vdots & & \vdots & & \vdots \\
 & \uparrow & & \uparrow & & \uparrow \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_{k+2} \otimes E_{-2}) & \xrightarrow{\text{id} \otimes d} & \Gamma(S(E^*)_{k+2} \otimes E_{-1}) & \xrightarrow{\text{id} \otimes \rho} & \Gamma(S(E^*)_{k+2} \otimes TM) \\
 & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_{k+1} \otimes E_{-2}) & \xrightarrow{\text{id} \otimes d} & \Gamma(S(E^*)_{k+1} \otimes E_{-1}) & \xrightarrow{\text{id} \otimes \rho} & \Gamma(S(E^*)_{k+1} \otimes TM) \\
 & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots & & \uparrow Q \otimes \text{id} + \cdots \\
 \cdots & \longrightarrow & \Gamma(S(E^*)_k \otimes E_{-2}) & \xrightarrow{\text{id} \otimes d} & \Gamma(S(E^*)_k \otimes E_{-1}) & \xrightarrow{\text{id} \otimes \rho} & \Gamma(S(E^*)_k \otimes TM) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots & & \vdots \\
 & & \text{column 2} & & \text{column 1} & & \text{column 0} \\
 & & & & & & (6.23)
 \end{array}$$

Under this correspondence, we understand longitudinal vector fields as the following.

Lemma 6.2.9. *A graded vector field $P = (p_0, p_1, \dots) \in \mathfrak{X}$ is longitudinal if $p_0 \in \mathcal{E} \otimes_{\mathcal{O}} \mathfrak{F}$.*

The following theorem is crucial for Chapter 8.

Theorem 6.2.10. *Let (E, Q) be a universal Q -manifold of \mathfrak{F} .*

1. *Longitudinal vector fields form an acyclic complex.*

More precisely, any longitudinal vector field on E which is a ad_Q -cocycle is the image through ad_Q of some vertical vector field on E .

2. More generally, if a vector field on E of depth n is a ad_Q -cocycle, then it is the image through ad_Q of some vector field on E of depth $n + 1$.

Proof. Since (E, Q) is an universal Q -manifold of \mathfrak{F} , lines in (6.23) are exact. It is now a diagram chasing phenomena. Let $P = (p_0, p_1, \dots) \in \mathfrak{X}$ be a longitudinal element which is a D -cocycle. By longitudinarity there exists an element $b_1 \in \Gamma(S(E^*) \otimes E_{-1})$ such that $(\text{id} \otimes \rho)(b_1) = p_0$. Set $P_1 = (0, b_1, 0, \dots)$, that is we extend b_1 by zero on $\Gamma(S(E^*) \otimes E_{\leq -2})$ and $\Gamma(S(E^*) \otimes TM)$. It is clear that $P - D(P_1) = (0, p'_1, p'_2, \dots)$ is also a D -cocycle. In particular, we have $D^h(p'_1) = 0$ by exactness there exists $b_2 \in \Gamma(S(E^*) \otimes E_{-2})$ such that $D^h(b_2) = p'_1$. As before put $P_2 = (0, 0, b_2, 0, \dots)$. Similarly, $P - D(P_1) - D(P_2) = (0, 0, p''_2, p''_3, \dots)$ is a D -cocycle. By recursion, we end up to construct P_1, P_2, \dots that satisfy $P - D(P_1) - D(P_2) + \dots = 0$, that is, there exists an element $B = (0, b_1, b_2, \dots) \in \mathfrak{X}$ such that $D(B) = P$. This proves item 1.

To prove item 2 it suffices to cross out in the diagram (6.23) the columns number $0, \dots, n - 1$, which does not break exactness. The proof now follows as for item 1. \square

In particular, we deduce from Theorem 6.2.10 the following exact subcomplex.

Corollary 6.2.11. *Let (E, Q) be a universal Q -manifold of \mathfrak{F} . The subcomplex \mathfrak{V}_Q of $(\mathfrak{X}(E), \text{ad}_Q)$ made of vertical vector fields $P \in \mathfrak{X}(E)$ that satisfy $P \circ Q(f) = 0$ for all $f \in \mathcal{O}$ is exact.*

Proof. Let $P \in \mathfrak{X}(E)$ be a vertical vector field which is an ad_Q -cocycle (note that we have automatically $P \circ Q(f) = 0$ for all $f \in \mathcal{O}$). By Theorem 6.2.10 there exists a vertical vector field $\tilde{P} \in \mathfrak{X}(E)$ such that $[Q, \tilde{P}] = P$. Moreover, $\tilde{P} \in \mathfrak{V}_Q$, since for all $f \in \mathcal{O}$,

$$0 = [Q, \tilde{P}](f) = (-1)^{|\tilde{P}|} \tilde{P} \circ Q(f).$$

\square

The following remark will be used especially for the proof of Theorem 8.3.1 of Chapter 8.

Remark 6.2.12. For a cocycle $P \in \mathfrak{V}_Q$ of degree 0 one has $P^{(-1)} = 0$ (for degree reason). By Corollary 6.2.11, P is the image by ad_Q of an element $\tilde{P} \in \mathfrak{V}_Q$ i.e. such that $[Q, \tilde{P}] = P$. Also, one can choose $\tilde{P}^{(-1)} = 0$: we have

$$[Q, \tilde{P}]^{(-1)} = [Q^{(0)}, \tilde{P}^{(-1)}] = P^{(-1)} = 0.$$

By exactness of $\text{ad}_{Q^{(0)}}$ (see Section 6.2.1), we have $P^{(-1)} = [Q^{(0)}, \vartheta]$ for some \mathcal{O} -linear map $\vartheta \in \text{Hom}(\Gamma(E^*), \Gamma(S^0(E^*)))$ of degree -2 . Put $\bar{P} := \tilde{P} - [Q, \vartheta]$, where ϑ is extended to a derivation of polynomial-degree -1 . Clearly,

$$[Q, \bar{P}] = P \quad \text{and} \quad \bar{P}^{(-1)} = \tilde{P}^{(-1)} - [Q, \vartheta]^{(-1)} = \tilde{P}^{(-1)} - [Q^{(0)}, \vartheta] = 0.$$

Therefore, $P = \text{ad}_Q(\bar{P})$ with $\bar{P}^{(-1)} = 0$.

Here is direct consequence of Theorem 6.2.10 and Proposition B.1.7.

Corollary 6.2.13. *Let \mathfrak{F} a singular foliation on M . Let (E, Q) be a universal Q -manifold of \mathfrak{F} . Then,*

$$H^\bullet(\mathfrak{X}(E), \text{ad}_Q) \simeq H^\bullet\left(\frac{\mathfrak{X}(E)}{\mathfrak{L}(E)}, \overline{\text{ad}_Q}\right). \quad (6.24)$$

where $\overline{\text{ad}_Q}$ is induced by ad_Q on the quotient space.

Since all vertical vector fields are longitudinal, there is an isomorphism of \mathcal{O} -modules

$$\frac{\mathfrak{X}(E)}{\mathfrak{L}(E)} \simeq \Gamma(S^\bullet(E^*)) \otimes_{\mathcal{O}} \left(\frac{\mathfrak{X}(M)}{\mathfrak{F}} \right). \quad (6.25)$$

Let us describe the differential map $\overline{\text{ad}}_Q$ using this isomorphism: let $(\mathcal{U}, x_1, \dots, x_n)$ be local coordinates on M and take local trivialisations ξ^1, ξ^2, \dots of graded sections in $\Gamma(E^*)$, also let ξ_1, ξ_2, \dots the corresponding dual sections in $\Gamma(E)$. Notice that locally, any $\overline{P} \in \frac{\mathfrak{X}(E)}{\mathfrak{L}(E)}$ is represented by a vector field of the form

$$\sum_{j, i_1, \dots, i_k} f_{i_1, \dots, i_k}^j(x) \xi^{i_1} \odot \dots \odot \xi^{i_k} \otimes \frac{\partial}{\partial x_j}, \quad f_{i_1, \dots, i_k}^j \in C^\infty(\mathcal{U})$$

since the vector fields in $\frac{\partial}{\partial \theta^i}$ are vertical vector fields. That is to say, \overline{P} can be represented by an element of the form $P = \Theta \otimes X$ with $\Theta \in \Gamma_{\mathcal{U}}(S^\bullet(E^*))$ and $X \in \mathfrak{X}(\mathcal{U})$, using the decomposition (6.25). Since

$$\begin{aligned} [Q, P] &= Q[\Theta]X + (-1)^{|\Theta|} \Theta \cdot [Q, X] \\ &= Q[\Theta]X + (-1)^{|\Theta|} \Theta \odot \sum_{k, |\xi^k|=1} \xi^k \left[\sum_j Q_k^j(x) \frac{\partial}{\partial x_j}, X \right] + \text{vertical vector fields} \\ &= Q[\Theta]X + (-1)^{|\Theta|} \sum_{k, |\xi^k|=1} \Theta \odot \xi^k [\rho(\xi_k), X] + \text{vertical vector fields} \end{aligned}$$

we have

$$\overline{\text{ad}}_Q(\overline{P}) = Q[\Theta] \otimes \overline{X} + (-1)^{|\Theta|} \sum_{k, |\xi^k|=1} \Theta \odot \xi^k \otimes \overline{[\rho(\xi_k), X]}, \quad (6.26)$$

where \overline{X} stands for the class of a vector field $X \in \mathfrak{X}(M)$ in $\frac{\mathfrak{X}(M)}{\mathfrak{F}}$. Therefore, locally

$$\overline{\text{ad}}_Q = \overline{Q} \otimes \text{id} + \sum_{k, |\xi^k|=1} \text{id} \odot \xi^k \otimes \nabla_{\xi_k}^{\text{Bott}}$$

where $\nabla^{\text{Bott}}: \Gamma(E_{-1}) \times \frac{\mathfrak{X}(M)}{\mathfrak{F}} \longrightarrow \frac{\mathfrak{X}(M)}{\mathfrak{F}}$, is the Bott connection associated to \mathfrak{F} , i.e. $\nabla_e^{\text{Bott}} \overline{X} := \overline{[\rho(e), X]}$.

Remark 6.2.14. Assume that $M = \mathbb{R}^d$ and \mathfrak{F} is a singular foliation that admits two leaves: $\{0\}$ and $\mathbb{R}^d \setminus \{0\}$. Then

$$\frac{\mathfrak{X}(E)}{\mathfrak{L}(E)} \simeq \Gamma(S^\bullet(E^*))|_0 \otimes_{\mathbb{R}} \left(\mathbb{R}^d \oplus \frac{\mathcal{I}_0 \mathfrak{X}(M)}{\mathfrak{F}} \right).$$

The differential may be complicated to describe.

Example 6.2.15. Let $\mathfrak{F} = \mathcal{I}_0 \mathfrak{X}(\mathbb{R}^d)$ and (E, Q) one of its universal Lie algebroid such that $E_{-1} \simeq \mathfrak{gl}(\mathbb{R}^d)$ is the transformation Lie algebroid for the action of $\mathfrak{gl}(\mathbb{R}^d)$ on \mathbb{R}^d . For this singular foliation, $\frac{\mathfrak{X}(\mathbb{R}^d)}{\mathfrak{F}} \simeq \mathbb{R}^d$. Thus, we obtain

$$\frac{\mathfrak{X}(E)}{\mathfrak{L}(E)} \simeq \Gamma(S^\bullet(E^*))|_0 \otimes_{C^\infty(\mathbb{R}^d)} \mathbb{R}^d.$$

and the Bott connection is simply the Chevalley-Eilenberg differential for the action of $\mathfrak{gl}(\mathbb{R}^d)$ on \mathbb{R}^d . In particular, it is easy to see that

$$H^1(\mathfrak{X}(E)) = H^1 \left(\frac{\mathfrak{X}(E)}{\mathfrak{L}(E)} \right) = H_{CE}^1(\mathfrak{gl}(\mathbb{R}^d), \mathbb{R}^d) = 0$$

In other words, the universal Lie ∞ -algebroid is, in that case, rigid, i.e. any formal deformation $Q + \epsilon Q_1 + \epsilon^2 Q_2 + \dots$ of its derivation Q is formally trivial [LGL22a].

Here is a second example where the universal Lie ∞ -algebroid is rigid.

Example 6.2.16. Let \mathfrak{F} be the singular foliation given by the action of Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ on \mathbb{R}^2 through the vector fields:

$$\underline{e} = x \frac{\partial}{\partial y}, \quad \underline{f} = y \frac{\partial}{\partial x}, \quad \underline{h} = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y},$$

where e, f, h are the standard basis elements of $\mathfrak{sl}(2, \mathbb{R})$. The singular foliation \mathfrak{F} admits a universal Lie ∞ -algebroid (E, Q) built on a geometric resolution ([LLS20], Example 3.31) of length 2

$$0 \longrightarrow E_{-2} \xrightarrow{d} E_{-1} \xrightarrow{\rho} TM,$$

where $E_{-1} \simeq \mathfrak{sl}(2, \mathbb{R}) \times \mathbb{R}^2$ is a trivial vector bundle of rank 3, and E_{-2} a trivial vector bundle of rank 1. Here, the bracket between two constant sections of E_{-1} is defined as being their bracket in $\mathfrak{sl}(2, \mathbb{R})$ and all other brackets between generators is trivial. Also, Q takes the form

$$Q = xe^* \frac{\partial}{\partial y} + yf^* \frac{\partial}{\partial x} + h^* \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) + e^* \odot f^* \frac{\partial}{\partial h^*} - 2e^* \odot h^* \frac{\partial}{\partial e^*} + 2f^* \odot h^* \frac{\partial}{\partial f^*} + (x^2 f^* + y^2 e^* - xy h^*) \frac{\partial}{\partial \mu^*}$$

where μ is the generator of E_{-2} , and (e^*, f^*, h^*, μ^*) is the dual basis of $((e, f, h, \mu))$.

In this case, one has

$$\frac{\mathfrak{X}(\mathbb{R}^2)}{\mathfrak{F}} \simeq \mathbb{R}^2 \oplus \mathbb{R}$$

is generated by the classes of

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right\rangle.$$

Hence, as a graded vector space:

$$\frac{\mathfrak{X}(E)}{\mathfrak{L}(E)} \simeq (\wedge^\bullet \mathfrak{sl}_2^* \oplus \mathbb{R}[2]) \otimes_{\mathbb{R}} (\mathbb{R}^2 \oplus \mathbb{R}).$$

The differential induced by Q is easily checked to be the Chevalley-Eilenberg differential for the natural $\mathfrak{sl}(2, \mathbb{R})$ -action on these modules. In particular,

$$H^1(\mathfrak{X}(E)) = H^1 \left(\frac{\mathfrak{X}(E)}{\mathfrak{L}(E)} \right) = H^1 \left(\mathfrak{sl}(2, \mathbb{R}), \frac{\mathfrak{X}(\mathbb{R}^2)}{\mathfrak{F}} \right) = 0,$$

since the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is semisimple. This implies that the universal Lie ∞ -algebroid (E, Q) is rigid in the sense of [LGL22a].

Conclusion:

We recalled the duality "Lie ∞ -algebroid \longleftrightarrow Q -manifolds of finite rank in all degree" so that the universal Lie ∞ -algebroid becomes a universal Q -manifold. We recover [LLS20] the notion of universal Q -manifold (E, Q) of a singular foliation. We prove that ad_Q has^a no longitudinal cohomology.

^aWhich squares to zero on vector fields on E .

Isotropy Lie algebras of a singular foliation

In this chapter, we look at an extension of the Androulidakis and Skandalis isotropy Lie algebra [AZ13] of a singular foliation at a point.

Let us recall some definitions.

Definition 7.0.1. Let \mathfrak{F} be a singular foliation on a manifold M . Let $\mathcal{I}_m = \{f \in C^\infty(M) \mid f(m) = 0\}$.

1. The *tangent space* of \mathfrak{F} at a point $m \in M$ is the vector space $T_m\mathfrak{F} := \{X|_m \mid X \in \mathfrak{F}\}$ whose elements are evaluations of the vector fields of \mathfrak{F} at m . In other words, $T_m\mathfrak{F}$ is the image of the evaluation map $\text{ev}_m: \mathfrak{F} \rightarrow T_mM$, $X \mapsto X|_m$.
2. The set $\left\{m \in M \mid \frac{\mathfrak{F}}{\mathcal{I}_m\mathfrak{F}} \longrightarrow T_m\mathfrak{F} \text{ is bijective}\right\}$ is open and dense in M [AS09]. It is the set of *regular points* of (M, \mathfrak{F}) and is denoted by M_{reg} .

For any point $m \in M$, the \mathcal{O} -module $\mathfrak{F}(m) := \{X \in \mathfrak{F} \mid X(m) = 0\} = \ker(\text{ev}_m)$ is a Lie subalgebra of \mathfrak{F} . The evaluation map goes to quotient and induces the following exact sequence,

$$0 \longrightarrow \frac{\mathfrak{F}(m)}{\mathcal{I}_m\mathfrak{F}} \longrightarrow \frac{\mathfrak{F}}{\mathcal{I}_m\mathfrak{F}} \longrightarrow T_m\mathfrak{F} \longrightarrow 0.$$

Since $\mathcal{I}_m\mathfrak{F} \subseteq \mathfrak{F}(m)$ is a Lie ideal,

Definition 7.0.2. the quotient space $\mathfrak{g}_m = \frac{\mathfrak{F}(m)}{\mathcal{I}_m\mathfrak{F}}$ is a Lie algebra, which is called the *isotropy Lie algebra* of \mathfrak{F} at m .

Remark 7.0.3. The isotropy Lie algebras detect singular and regular points of (M, \mathfrak{F}) . It measures how far \mathfrak{F} is of being regular in neighborhood of a point. This can be stated as follows (see Lemma 1.1 of [AZ13]).

Proposition 7.0.4. For any point $m \in L$ of a leaf $L \subset M$, $\mathfrak{g}_m = \{0\}$ if and only if, L is a regular leaf. Also, the \mathfrak{g}_m 's are all isomorphic as Lie algebras while $m \in L$.

For \mathcal{A} a Lie-Rinehart algebra over \mathcal{O} , the same construction can be done for any maximal ideal \mathfrak{m} . Let

$$\mathcal{A}|_{\mathfrak{m}} := \{a \in \mathcal{A} \mid \rho(a)[\mathcal{O}] \subseteq \mathfrak{m}\}.$$

The quotient $\frac{\mathcal{A}|_{\mathfrak{m}}}{\mathfrak{m}\mathcal{A}}$ is a Lie algebra over \mathcal{O}/\mathfrak{m} . For $\mathcal{O} = C^\infty(M)$ and $\mathfrak{m} = \mathcal{I}_m$ for a point $m \in M$, the following sequence

$$0 \longrightarrow \frac{\mathcal{A}|_{\mathcal{I}_m}}{\mathcal{I}_m\mathcal{A}} \hookrightarrow \frac{\mathcal{A}}{\mathcal{I}_m\mathcal{A}} \twoheadrightarrow T_m\rho(\mathcal{A}) \longrightarrow 0 \quad (7.1)$$

is exact, where $T_m\rho(\mathcal{A}) \subset \text{Hom}(\mathfrak{m}/\mathfrak{m}^2, \mathcal{O}/\mathfrak{m})$ is the image of $\mathcal{A} \rightarrow \text{Hom}(\mathfrak{m}/\mathfrak{m}^2, \mathcal{O}/\mathfrak{m})$, $a \mapsto \overline{\rho(a)}$. Proposition 7.0.4 is however subtle to extend to this setting.

Proposition 7.0.5. *For a finitely generated Lie-Rinehart \mathcal{A} over $C^\infty(M)$, there exists a Lie algebroid $(\mathcal{A}, \rho_{\mathcal{A}}, [\cdot, \cdot]_{\mathcal{A}})$ such that $\mathcal{A} \simeq \Gamma(A)$ for some vector bundle $A \rightarrow M$ if and only if $\frac{\mathcal{A}}{\mathcal{I}_m\mathcal{A}}$, as a vector space, has constant rank at all points. In that case $\frac{\mathcal{A}^{(m)}}{\mathcal{I}_m\mathcal{A}}$ is $\ker(\rho_{\mathcal{A}}|_m)$ for all $m \in M$. In particular, it is the case in a neighborhood of every point where the dimension of $\frac{\mathcal{A}}{\mathcal{I}_m\mathcal{A}}$ is minimal.*

Proof. This is a consequence of Nakayama Lemma B.2.11. The germ \mathcal{O}_m of functions on a neighborhood of $\mathcal{U} \subseteq M$ of m is a local ring [RS94] with maximal ideal still denoted by \mathcal{I}_m . The tensor product $\mathcal{O}_m \otimes_{\mathcal{O}} \mathcal{A}$ is a module over \mathcal{O}_m . Let $r = \dim(\frac{\mathcal{A}}{\mathcal{I}_m\mathcal{A}})$. By Nakayama Lemma B.2.11 any basis $(\bar{e}_1, \dots, \bar{e}_r)$ of $\frac{\mathcal{A}}{\mathcal{I}_m\mathcal{A}}$ lifts to minimal generators on $\mathcal{O}_m \otimes_{\mathcal{O}} \mathcal{A}$ in a neighborhood of m . Since $r = \dim(\frac{\mathcal{A}}{\mathcal{I}_{m'}\mathcal{A}})$ for all points $m' \in \mathcal{U}$, again Nakayama Lemma B.2.11 implies (e_1, \dots, e_r) are minimal generators on a neighborhood of m' . Any representative (e_1, \dots, e_r) of $(\bar{e}_1, \dots, \bar{e}_r)$ are local sections of \mathcal{A} in a neighborhood \mathcal{U} of m that span \mathcal{A} on \mathcal{U} so that $\mathcal{A}|_{\mathcal{U}}$ is a free module. This applies that \mathcal{A} is a $C^\infty(\mathcal{U})$ -module, which is projective. By Serre-Swan theorem, $\mathcal{A} = \Gamma(A)$ for some vector bundle A . \square

7.1 Specialization of a Lie ∞ -algebroid at a point

- Let $(M, \mathcal{E}, Q) = (E_\bullet, \ell_\bullet, \rho)$ be a Lie ∞ -algebroid over a manifold M with anchor ρ . For every point $m \in M$, the k -ary brackets restrict to the graded vector space

$$ev(E, m) := \left(\bigoplus_{i \geq 2} E_{-i}|_m \right) \oplus \ker(\rho_m)$$

and equipped the latter with a Lie ∞ -algebra structure that we denote by $Istropy_m(\mathcal{E}, Q)$. For every $k \geq 1$, the restriction goes as follows:

$$\{x_1, \dots, x_k\}_k := \ell_k(s_1, \dots, s_k)|_m \quad (7.2)$$

for all $x_1, \dots, x_k \in ev(E, m)$ and $s_1, \dots, s_k \in \Gamma(E)$ sections of E such that $s_i(m) = x_i$ with $i = 1, \dots, k$. These brackets are well-defined. It is clear that for $k \neq 2$, since ℓ_k is linear over functions. But it is not immediate that the 2-ary bracket is well-defined as well. Let us check that.

On one hand, the new brackets $\{\dots\}_k$ have values in $ev(E, m)$ for degree reason, except may be for the 2-ary bracket when applied to elements of degree -1 (i.e. elements of the kernel of ρ_m)

but in that case it is in the kernel of ρ_m since

$$\begin{aligned}\rho_m(\{x_1, x_2\}_2) &= \rho_m(\ell_2(s_1, s_2)|_m) \\ &= \rho(\ell_2(s_1, s_2))|_m \\ &= [\rho(s_1), \rho(s_2)]|_m = 0\end{aligned}$$

In the last line we have used the fact that the Lie bracket of two vector fields that vanish at m is a vector field that vanishes again at m .

On the other hand, the 2-ary bracket $\{\cdot, \cdot\}_2$ is also well-defined when applied to elements of degree less or equal to -2 , we need to verify when we take the bracket with at least an element of degree -1 . Let $(e_1^i, \dots, e_{\text{rk}(E_{-i})}^i)$ be a local trivialization of E_{-i} on a neighborhood \mathcal{U} of the point $m \in M$. For $x_1 \in \ker(\rho_m)$ and $x_2 \in E_{-i}|_m$ write

$$x_1 = \sum_{k=1}^{\text{rk}(E_{-1})} \lambda_k e_k^1(m), \quad x_2 = \sum_{k=1}^{\text{rk}(E_{-i})} \mu_k e_k^i(m)$$

for some scalars (λ_i) in \mathbb{K} . The scalars (λ_k) , (μ_k) extend to functions (f_k) , (g_k) on \mathcal{U} . Therefore, we have

$$\{x_1, x_2\}_2 = \ell_2(s_1, s_2)|_m$$

with

$$s_1 = \sum_{k=1}^{\text{rk}(E_{-1})} f_k e_k^1, \quad s_2 = \sum_{k=1}^{\text{rk}(E_{-i})} g_k e_k^i.$$

If \tilde{s}_2 is another extension of x_2 , then $(s_2 - \tilde{s}_2)(m) = 0$ and this is equivalent to $(g_k - \tilde{g}_k)(m) = 0$ for $k = 1, \dots, \text{rk}(E_{-i})$. It follows that

$$\begin{aligned}\ell_2(s_1, s_2 - \tilde{s}_2)|_m &= \sum_{k=1}^{\text{rk}(E_{-i})} \ell_2(s_1, (f_k - \tilde{g}_k)e_k^i)|_m \\ &= \sum_{k=1}^{\text{rk}(E_{-i})} \underbrace{(f_k - \tilde{g}_k)(m)}_{=0} \ell_2(s_1, e_k^i)|_m + \underbrace{\rho(s_1)|_m}_{=0} [f_k - \tilde{g}_k] e_k^i \\ &= 0.\end{aligned}$$

Hence,

$$ev(E, m) : \dots \xrightarrow{\{\cdot\}_1 = \ell_1|_m} E_{-3}|_m \xrightarrow{\{\cdot\}_1 = \ell_1|_m} E_{-2}|_m \xrightarrow{\{\cdot\}_1 = \ell_1|_m} \ker(\rho_m)$$

comes equipped with a Lie ∞ -algebra whose brackets are $(\{\dots\}_k)_{k \geq 1}$.

- Any Lie ∞ -morphism of algebroids $\Phi: (M, \mathcal{E}', Q') \rightarrow (M, \mathcal{E}, Q)$ induces a Lie ∞ -algebra morphism $\Phi|_m: S^\bullet(V'_m) \rightarrow S^\bullet(V_m)$ since it is \mathcal{O} -linear.
- We define the graded vector space

$$\bigoplus_{i \geq 1} H^{-i}(E_\bullet, m) \tag{7.3}$$

as the cohomology group of the complex

$$\dots \xrightarrow{\ell_1|_m} E_{-3}|_m \xrightarrow{\ell_1|_m} E_{-2}|_m \xrightarrow{\ell_1|_m} \ker(\rho_m) \longrightarrow 0.$$

One can check that when (M, \mathcal{E}, Q) is universal for a singular foliation \mathfrak{F} , the graded space (7.3) does not depend on the underlying geometric resolution of \mathfrak{F} and is denoted $H(\mathfrak{F}, m)$.

This construction can be extended to any Lie ∞ -algebroid over \mathcal{O} for any maximal ideal \mathcal{I} . Let $(\mathcal{E}, \ell_\bullet, \rho)$ be a Lie ∞ -algebroid over \mathcal{O} . Define

$$\mathcal{E}_{-i}|_{\mathcal{I}} = \begin{cases} \{e \in \mathcal{E}_{-1} \mid \rho(e)[\mathcal{O}] \subseteq \mathcal{I}\} & \text{for } i = 1 \\ \frac{\mathcal{E}_{-i}}{\mathcal{I}\mathcal{E}_{-i}} & \text{for } i \geq 1 \end{cases} \quad (7.4)$$

The construction is purely formal. Also, the cohomology of the complex

$$\dots \xrightarrow{\ell_1|_{\mathcal{I}}} \mathcal{E}_{-3}|_{\mathcal{I}} \xrightarrow{\ell_1|_{\mathcal{I}}} \mathcal{E}_{-2}|_{\mathcal{I}} \xrightarrow{\ell_1|_{\mathcal{I}}} \mathcal{E}_{-1}|_{\mathcal{I}}$$

denoted by $H^\bullet(\mathcal{E}, \mathcal{I})$ does not depend on the choice of a free resolution of the Lie-Rinehart algebra \mathcal{A} , we denote it by $H^\bullet(\mathcal{E}, \mathcal{I})$.

The isotropy Lie ∞ -algebra of a singular foliation

We assume that (M, \mathcal{E}, Q) is universal for \mathfrak{F} . Note that the Lie ∞ -algebroid obtained by specialising at some point $m \in M$ does not induce directly a Lie ∞ -algebroid on the graded space $H(\mathfrak{F}, m)$ but the 2-ary bracket $\{\cdot, \cdot\}_2$ goes to quotient directly on elements of degree -1 i.e. to $H^{-1}(\mathfrak{F}, m)$, because

$$\{d_m^{(2)}(x_1), x_2\}_2 = d_m^{(2)}(\{x_1, x_2\}_2)$$

for all $x_1 \in E_{-2}|_m$ and $x_2 \in \ker(\rho_m)$. That endows $H^{-1}(\mathfrak{F}, m)$ with Lie algebra structure.

Proposition 7.1.1. *The Androuridakis and Skandalis isotropy Lie algebra $\mathfrak{g}_m = \frac{\mathfrak{F}(m)}{\mathcal{I}_m \mathfrak{F}}$ of the singular foliation \mathfrak{F} at a point $m \in M$, is isomorphic to $H^{-1}(\mathfrak{F}, m)$ equipped with the induced Lie algebra structure.*

Proof. For $m \in M$, we construct a Lie algebra isomorphism $\zeta: \frac{\ker(\rho_m)}{\text{im}(d_m^{(2)})} \rightarrow \mathfrak{g}_m$ as follows: For an element $u \in \ker(\rho_m)$, let \tilde{u} be an extension of u to a local section on E_{-1} . By construction, one has $\rho(\tilde{u}) \in \mathfrak{F}(m)$. Let $\tilde{\rho}_m$ be the surjective linear map defined by

$$\tilde{\rho}_m: \ker(\rho_m) \longrightarrow \mathfrak{g}_m, u \longmapsto [\rho(\tilde{u})].$$

Since any other extension \tilde{u} for u differs from the first one by a section in $\mathcal{I}_m \Gamma(E_{-1})$, the map $\tilde{\rho}_m$ is well-defined. Surjectivity is due to the fact that every vector field of \mathfrak{F} vanishing at $m \in M$ is of the form $\rho(e)$ with e a (local) section of E_{-1} whose value at m belongs to $\ker(\rho_m)$. In addition, it is not hard to see that $\tilde{\rho}_m$ is a morphism of brackets.

It remains to show that $\ker(\tilde{\rho}_m) = \text{im}(d_m^{(2)})$: let $u \in \ker(\tilde{\rho}_m) \subset \ker(\rho_m)$ and \tilde{u} be a local section of E_{-1} that extends u . By definition of u , the class of $\rho(\tilde{u})$ is zero in \mathfrak{g}_m , therefore, there exists some functions $f_i \in \mathcal{I}_m$ and $X_i \in \mathfrak{F}, i = 1, \dots, k$, local generators such that $\rho(\tilde{u}) = \sum_{i=1}^k f_i X_i$. This implies that,

$$\rho(\tilde{u} - \sum_{i=1}^k f_i e_i) = 0.$$

where for $i = 1, \dots, k$, e_i is a (local) section of E_{-1} whose image through ρ is X_i . Since $(E_\bullet, d^\bullet, \rho)$ is a geometric resolution, there exists a (local) section $q \in \Gamma(E_{-2})$ such that

$$\tilde{u} = \sum_{i=1}^k f_i e_i + d^{(2)}q \quad (7.5)$$

By evaluating Equation (7.5) at m , we find out that $u \in \text{im}(d_m^{(2)})$. Conversely, for $v \in E_{-2}|_m$, choose a (local) section q of E_{-2} through v . Therefore, $d^{(2)}q \in \ker \rho$, is a (local) extension of $d_m^{(2)}v \in \text{im}(d_m^{(2)})$. The image of $d_m^{(2)}v$ through $\tilde{\rho}_m$ is obviously zero. This proves that $\ker(\tilde{\rho}_m) = \text{im}(d_m^{(2)})$. \square

However, if the underlying complex of (M, \mathcal{E}, Q) is minimal at m then, for every $i \geq 2$, the vector space $H^{-i}(\mathfrak{F}, m)$ is canonically isomorphic to $E_{-i}|_m$. Also, $H^{-1}(\mathfrak{F}, m)$ is canonically isomorphic to $\ker(\rho_m)$.

Definition 7.1.2. Let (M, \mathcal{E}, Q) be a universal Lie ∞ -algebroid of a singular foliation \mathfrak{F} whose underlying complex is minimal at m . Then, $H(\mathfrak{F}, m)$ carries a Lie ∞ -algebra structure given by $Istropy_m(\mathcal{E}, Q)$ called the *isotropy Lie ∞ -algebra of the singular foliation \mathfrak{F} at m* .

One can show that this definition is independent of any choices made in the construction.

Remark 7.1.3. By Proposition 7.1.1, the isotropy Lie algebra of the singular foliation \mathfrak{F} at a point $m \in M$ in the sense of Androulidakis and Skandalis, is isomorphic to the degree minus one component $H^{-1}(\mathfrak{F}, m) \simeq \ker(\rho_m)$ of the isotropy Lie ∞ -algebra of \mathfrak{F} at m .

Remark 7.1.4. For an arbitrary Lie-Rinehart algebra \mathcal{A} and a maximal ideal \mathcal{I} , it is still true that $(H^\bullet(\mathcal{A}, \mathcal{I}), d = 0)$ is quasi-isomorphic to $(\mathcal{E}|_{\mathcal{I}}, \ell_1|_{\mathcal{I}})$ and it is still true that the bracket of the universal Lie ∞ -algebroid over \mathcal{O} constructed in Section 4.2 restricts to a ∞ -algebra structure on $(\mathcal{E}|_{\mathcal{I}}, \ell_1|_{\mathcal{I}})$. By Homotopy transfer, this Lie ∞ -algebra can be transferred to $(H^\bullet(\mathcal{A}, \mathcal{I}), d = 0)$. It is not clear whether Proposition 7.1.1 holds true or not.

Lemma 7.1.5. *Let (M, \mathfrak{F}) be a singular foliation. Let (E, Q) be a universal Lie ∞ -algebroid of \mathfrak{F} and let*

$$(E, d, \rho) : \quad \cdots \xrightarrow{d^{(4)}} E_{-3} \xrightarrow{d^{(3)}} E_{-2} \xrightarrow{d^{(2)}} E_{-1} \xrightarrow{\rho} TM \quad (7.6)$$

be its linear part.

1. For all $m \in M$, we have $\frac{\ker(\rho_m)}{\text{im}(d_m^{(2)})} \simeq \mathfrak{g}_m$ as Lie algebras.
2. The subset of regular points of \mathfrak{F} in M satisfies

$$M_{\text{reg}} = \{m \in M \mid \text{rk}(d_m^{(2)}) = \dim(\ker \rho_m)\}.$$

It is open and dense in M .

3. The restriction of the foliation \mathfrak{F} to M_{reg} is the set of sections of a subbundle of TM , i.e., is a regular foliation.
4. If (E, d, ρ) is of finite length, then the regular leaves have the same dimension.

Proof. Item 1. is proved in Proposition 7.1.1 (see also Proposition 4.14 [LLS20]). For items 2. and 3. we refer the reader to Proposition 1.5 of [AS09]. \square

7.2 A blow-up procedure for a singular foliation

We present in this section an interpretation of a blow-up procedure invented by Mohsen in [Moh21].

7.2.1 Grassmann bundle

For E a finite dimensional vector space, we denote by $\text{Gr}_{-\ell}(E)$ the set of all codimension $\ell \in \mathbb{N}$ vector subspaces in E . Let us recall a few facts on $\text{Gr}_{-\ell}(E)$ see [KES00, Joe92, LB15], our main reference is [R94]:

1. It is a metric space for the distance $d(V, V') = \|P_V - P_{V'}\|$, where P_V stands for the orthogonal projection of E onto $V \subset E$ with respect to an arbitrary metric on E .
2. The topology does not depend on the metric and makes $\text{Gr}_{-\ell}(E)$ a manifold.
3. It is moreover a compact manifold and an projective variety¹.

Affine coordinates charts: One of the ways of defining the standard affine coordinates on the Grassmanian $\text{Gr}_{-\ell}(E)$ is as follows (Example 1.24 of [R94]): Fix a basis $e_1, \dots, e_{d=\dim E}$ for E . Any element $V \in \text{Gr}_{-\ell}(E)$, i.e. a vector subspace $V \subset E$ of codimension ℓ , can be viewed as a $d \times (d - \ell)$ matrix $\mathfrak{M}_V = (C_1, \dots, C_{d-\ell})$ whose columns are formed by linearly independent column vectors obtained by choosing a basis for V . The *homogeneous coordinates* of V in $\text{Gr}_{-\ell}(E)$ are the components of the $n \times (d - \ell)$ matrix \mathfrak{M}_V . Any other choice of basis for V gives another maximal rank matrix \mathfrak{M}'_V and an invertible $(d - \ell) \times (d - \ell)$ -matrix $P \in GL(d - \ell, \mathbb{K})$ such that $\mathfrak{M}_V = \mathfrak{M}'_V \circ P$. In particular, since \mathfrak{M}_V has full rank, there exists a family of integers $1 \leq i_1 < \dots < i_{d-\ell} \leq d$, such that \mathfrak{M}_V is equivalent to a matrix \mathfrak{M}'_V whose submatrix made of the rows $i_1, \dots, i_{d-\ell}$ is the identity matrix.

For example, if the first $d - \ell$ rows of \mathfrak{M}_V are linearly independent, then the matrix is equivalent to the matrix

$$\begin{pmatrix} I_{d-\ell} \\ A \end{pmatrix}$$

where $A = (a_{ij})$ is a $\ell \times (d - \ell)$ -matrix. In that case, V admits a basis of the form

$$v_j := e_j + \sum_{k=1}^{\ell} a_{kj} e_k, \quad j = 1, \dots, d - \ell. \quad (7.7)$$

V is completely determined by A .

One define an atlas on $\text{Gr}_{-\ell}(E)$ as follows: Consider the map

$$\begin{aligned} \psi_{1, \dots, d}: M_{\ell, d-\ell}(\mathbb{K}) &\longrightarrow M_{d, d-\ell}(\mathbb{K}) \\ A &\longmapsto \begin{pmatrix} I_{d-\ell} \\ A \end{pmatrix}. \end{aligned}$$

¹For the notion of projective variety and notations, we refer the reader e.g. to the book [R94].

For a permutation $\sigma \in \mathfrak{S}_d$, we define the map $\psi_{\sigma(1), \dots, \sigma(d)}$ that associates every $A \in M_{l, d-l}(\mathbb{K})$ the matrix in $M_{d, d-l}(\mathbb{K})$ that permutes the lines of $\begin{pmatrix} I_{d-l} \\ A \end{pmatrix}$ w.r.t σ , that is to say

$$\psi_{\sigma(1), \dots, \sigma(d)}(A) := P(\sigma) \circ \psi_{1, \dots, d}(A)$$

where $P(\sigma)$ is the permutation matrix associated to σ .

Statement: For every ordered integers $1 \leq i_1 < \dots < i_{d-\ell} \leq d$, the coordinates chart on $\text{Gr}_{-\ell}(E)$ is the open set $\mathcal{U}_{i_1, \dots, i_{d-\ell}} \subset \text{Gr}_{-\ell}(E)$ consisting of all sub-vector space of E such that for every basis the submatrix which is made of the rows $i_1, \dots, i_{d-\ell}$ is invertible, and the coordinate map is $\psi_{\sigma(1), \dots, \sigma(d)}$ with σ is a permutation sending $1, \dots, d - \ell$ on $i_1, \dots, i_{d-\ell}$.

Grassmann bundle: For $E_{-1} \rightarrow M$ a vector bundle of rank d , the disjoint union:

$$\text{Gr}_{-\ell}(E_{-1}) := \coprod_{m \in M} \text{Gr}_{-\ell}(E_{-1}|_m)$$

comes equipped with a natural manifold structure and

$$\Pi: \text{Gr}_{-\ell}(E_{-1}) \longrightarrow M$$

is a fibration. It is called *the Grassmann $(d - \ell)$ -plane bundle*. To fix some notations, the fiber at $m \in M$ is

$$\Pi^{-1}(m) = \{(V, m) \mid V \in \text{Gr}_{-\ell}(E_{-1}|_m)\}.$$

For every open subset $\mathcal{U} \subset M$ on which E_{-1} is trivial, $\Pi^{-1}(\mathcal{U}) \simeq \text{Gr}_{-\ell}(\mathbb{R}^d) \times \mathcal{U}$.

7.2.2 A blow-up procedure

Settings: In what follows, we are given a foliated manifold (M, \mathfrak{F}) with M connected. We assume that a geometric resolution of finite length exists. Under these assumptions, all the regular leaves have the same dimension. Let M_{reg} the set of the regular points of (M, \mathfrak{F}) . Denote by ℓ the common dimension of the regular leaves.

Remark 7.2.1. For most of the present discussion, the whole geometric resolution is not needed: is it sufficient to assume that there exists vector bundles E_{-1}, E_{-2} and that all regular leaves have the same dimension.

Let $(E_{\bullet}, \ell_{\bullet}, \rho)$ be a universal Lie ∞ -algebroid of \mathfrak{F} . Consider the underlying geometric resolution

$$(E, d, \rho): \quad \dots \xrightarrow{\ell_1=d(4)} E_{-3} \xrightarrow{\ell_1=d(3)} E_{-2} \xrightarrow{\ell_1=d(2)} E_{-1} \xrightarrow{\rho} TM. \quad (7.8)$$

Notice that for every point $m \in M_{reg}$,

$$\dim(d_m^{(2)}) = \dim \ker(\rho_m) = \text{rk}(E_{-1}) - \ell.$$

Therefore, there exists a natural section of Π on M_{reg} defined by:

$$\sigma: M_{reg} \longrightarrow \text{Gr}_{-\ell}(E_{-1}), \quad m \longmapsto \text{im}(d_m^{(2)}) \quad (7.9)$$

We consider $\widetilde{M} := \overline{\sigma(M_{\text{reg}})}$ the closure of the graph of σ in $\text{Gr}_{-\ell}(E_{-1})$, and denote by π the restriction of Π to \widetilde{M} .

Recall from Section 7.1 that

$$\text{ev}(E, m) : \cdots \longrightarrow E_{-3}|_m \xrightarrow{d_m^{(3)}} E_{-2}|_m \xrightarrow{d_m^{(2)}} \ker(\rho_m)$$

comes equipped with a Lie ∞ -algebra $(\{\cdots\}_k)_{k \geq 1}$ whose unary bracket is $d_m^{(\bullet)}$.

Theorem 7.2.2. *The projection map*

$$\pi : \widetilde{M} \subset \text{Gr}_{-\ell}(E_{-1}) \longrightarrow M, (V, m) \longrightarrow m \quad (7.10)$$

fulfills the following properties

1. For each point $m \in M$, the set

$$\pi^{-1}(m) = \left\{ V \subset E_{-1}|_m \mid \exists (m_n) \in M_{\text{reg}}^{\mathbb{N}}, \text{ such that, } \text{im}(d_{m_n}^{(2)}) \xrightarrow{n \rightarrow +\infty} V \text{ as } m_n \xrightarrow{n \rightarrow +\infty} m \right\}$$

is non-empty.

2. For all $m \in M$, and $V \in \pi^{-1}(m)$ one has,

$$(a) \text{im}(d_m^{(2)}) \subseteq V \subseteq \ker(\rho_m).$$

(b) The 2-ary bracket $\{\cdot, \cdot\}_2$ on $\ker \rho_m$ restricts to V and the image of V in $\frac{\ker(\rho_m)}{\text{im}(d_m^{(2)})} \simeq \mathfrak{g}_m$, is a Lie subalgebra of codimension $\ell - \dim(L_m)$, where L_m is the leaf through m .

3. For all $m \in M_{\text{reg}}$, $\pi^{-1}(m) = \ker(\rho_m) = \text{im}(d_m^{(2)})$ is reduced to a point in $\text{Gr}_{-\ell}(\mathbb{R}^{\text{rk}(E_{-1})})$.

4. \widetilde{M} does not depend on the choice of a geometric resolution.

5. The projection $\pi : \widetilde{M} \rightarrow M$ is proper and onto.

Proof. Let us prove item 1 for $m \in M$. By compactness of the Grassmanian manifold $\text{Gr}_{-\ell}(E_{-1}|_m) \simeq \text{Gr}_{-\ell}(\mathbb{R}^{\text{rk}(E_{-1})})$, out of any sequence in $M_{\text{reg}}^{\mathbb{N}}$ that converges to m , we can extract a sequence $n \mapsto \kappa(\text{im}(d_{m_n}^{(2)})) \in \text{Gr}_{-\ell}(E_{-1}|_m)$ that has a limit, where κ is a local trivialization of the vector bundle E_{-1} which identifies the fibers $E_{-1}|_m$ and $E_{-1}|_{m_n}$. This proves item 1.

Let us show item 2.(a): Let $V \in \pi^{-1}(m)$ and $(m_n) \in M_{\text{reg}}^{\mathbb{N}}$ such that $m_n \xrightarrow{n \rightarrow +\infty} m$ and $\text{im}(d_{m_n}^{(2)}) \xrightarrow{n \rightarrow +\infty} V$. Let $v \in \text{im}(d_m^{(2)})$. We have $v = d_m^{(2)}u$ for some $u \in E_{-2}|_m$. Choose a (local) section \tilde{u} of E_{-2} through u . It implies that $d_{m_n}^{(2)}\tilde{u}(m_n) \xrightarrow{n \rightarrow +\infty} d_m^{(2)}u$, hence $d_{m_n}^{(2)}\tilde{u}(m_n) \in V$. Thus, $\text{im}(d_{m_n}^{(2)}) \subseteq V$. For any element $v \in V$, there exists a sequence $v_n \in \ker(\rho_{m_n}) = \text{im}(d_{m_n}^{(2)})$, $n \in \mathbb{N}$ that converges to v . In particular, $\rho_{m_n}(v_n) = 0$ for all n . Hence, by continuity, one has $v \in \ker(\rho_m)$.

To prove item (b), choose a local frame $e_1, \dots, e_r, \dots, e_{\ell+r}$ of E_{-1} such that $e_1(m), \dots, e_r(m)$ is an orthogonal basis of V for an arbitrary Hermitian structure on E_{-1} . For $i, j \in \{1, \dots, \ell+r\}$, let $(c_{ij}^k) \in \mathcal{O}(\mathcal{U})$ be a family of functions over \mathcal{U} such that

$$\ell_2(e_i, e_j) = \sum_{k=1}^{\ell+r} c_{ij}^k e_k.$$

In particular, for every $\{i, j\} \subset \{1, \dots, l+r\}$,

$$\{e_i(m), e_j(m)\}_2 = \sum_{k=1}^{l+r} c_{ij}^k(m) e_k(m).$$

Let $v_1, v_2 \in V$ with $v_1 = \sum_{i=1}^r \alpha^i e_i(m)$ and $v_2 = \sum_{i=j}^r \beta^i e_i(m)$. There exists sequences

$$v_1^n = \sum_{i=1}^{r+l} \alpha_n^i e_i(m) \xrightarrow{n \rightarrow +\infty} v_1 \quad \text{and} \quad v_2^n = \sum_{i=1}^{r+l} \beta_n^i e_i(m) \xrightarrow{n \rightarrow +\infty} v_2$$

with $v_1^n, v_2^n \in \ker \rho_{x_n}$, for all $n \in \mathbb{N}$. In particular, the sequences $(\alpha_n^i), (\beta_n^i) \in \mathbb{K}^{\mathbb{N}}$ with $i \in \{1, \dots, r+l\}$ satisfy $\alpha_n^i \xrightarrow{n \rightarrow +\infty} \alpha^i$, $\beta_n^i \xrightarrow{n \rightarrow +\infty} \beta^i$ ($\alpha^i = \beta^i = 0$ for $i \geq r+1$). For every $n \in \mathbb{N}$ we have,

$$\sum_{i,j,k=1}^{r+l} \alpha_n^i \beta_n^j c_{ij}^k(m_n) e_k(m_n) = \{v_1^n, v_2^n\}_2 \in \ker \rho_{x_n} = \text{im}(d_{m_n}^{(2)}). \quad (7.11)$$

Since

$$\sum_{i,j,k=1}^{r+l} \alpha_n^i \beta_n^j c_{ij}^k(m_n) e_k(m_n) \xrightarrow{n \rightarrow +\infty} \sum_{i,j,k=1}^{r+l} \alpha^i \beta^j c_{ij}^k(m) e_k(m) = \{v_1, v_2\}_2,$$

one has, $\{v_1, v_2\}_2 \in V$.

Item 3. is a consequence of 2.(a), since $m \in M_{\text{reg}}$ if and only if $\ker(\rho_m) = \text{im}(d_m^{(2)})$ (by item 1. of Lemma 7.1.5). Item 4. follows from the existence of homotopy equivalence between any two geometric resolutions. Item 5. follows from the fact that the projection Π is with compact fibers. This concludes the proof. \square

The corollary below is a direct consequence of item 2.(b) of Theorem 7.2.2.

Corollary 7.2.3. *There is a natural inclusion*

$$\widetilde{M} \hookrightarrow \coprod_{m \in M} \text{Gr}_{\ell - \dim(L_m)}(\mathfrak{g}_m). \quad (7.12)$$

Proof. Let $m \in M$. Since elements $V \in \pi^{-1}(m)$ satisfy $\text{im}(d_m^{(2)}) \subseteq V \subseteq \ker(\rho_m)$, they correspond injectively to a (unique) sub-vector space of codimension $\ell - \dim L_m$ in \mathfrak{g}_m . In particular, this implies $\pi^{-1}(m) \hookrightarrow \text{Gr}_{\ell - \dim(L_m)}(\mathfrak{g}_m)$. \square

Lemma 7.2.4. *For every $m \in M$, the set $K_m = \{V \in \text{Gr}_{-\ell}(E_{-1}|_m) \mid V \subseteq \ker \rho_m\} \subset \text{Gr}_{-\ell}(E_{-1}|_m)$ is locally an affine variety.*

Proof. Let e_1, \dots, e_d be a local frame of E_{-1} on an open subset $\mathcal{U} \subset M$. One has,

$$\rho(e_i) = \sum_{k=1}^{\dim M} f_i^k \frac{\partial}{\partial x_k} \in \mathfrak{F}, \quad (7.13)$$

for some local functions $f_i^k \in C^\infty(\mathcal{U})$. Without any loss of generality, consider for example the standard coordinates chart $\mathcal{U}_{1, \dots, d-\ell}$ for the grassmanian $\text{Gr}_{-\ell}(E_{-1}|_m)$. Let $V \in \mathcal{U}_{1, \dots, d-\ell}$ and let (a_{ij}) be the homogeneous coordinates of V . Define the sections

$$\tilde{v}_j := e_j + \sum_{k=1}^{\ell} a_{kj} e_k, \quad j = 1, \dots, d - \ell \quad (7.14)$$

By construction, the \tilde{v}_j 's, evaluated at m , form a basis for V . We have,

$$\rho(\tilde{v}_j) = \sum_{k=1}^{\dim M} \left(f_j^k + \sum_{s=1}^{\ell} a_{sj} f_s^k \right) \frac{\partial}{\partial x_k}.$$

$V \subseteq \ker \rho_m$ if and only if

$$\begin{aligned} f_j^k(m) + \sum_{s=1}^{\ell} a_{sj} f_s^k(m) &= 0, & j &= 1, \dots, d - \ell \\ & & k &= 1, \dots, \dim M. \end{aligned} \quad (7.15)$$

Therefore, K_m is defined by the polynomial Equations (7.15). \square

Theorem 7.2.5. *If \mathfrak{F} is a polynomial singular foliation on $M \in \{\mathbb{C}^N, \mathbb{R}^N\}$, then \widetilde{M} is a locally affine variety.*

Proof. We can choose a polynomial geometric resolution. Denote by $x = (x_1, \dots, x_N)$ the local coordinates on $\mathcal{W} \subseteq M$. We are using notations of Lemma 7.2.4. In Equation (7.13), the functions f_i^k are polynomial in (x_1, \dots, x_N) . By item 1, Theorem 7.2.2, every element V of $\widetilde{M} = \bigcup_{x \in M} \pi^{-1}(x)$ is obtained as a limit

$$\text{im}(d_{x_n}^{(2)}) \xrightarrow{n \rightarrow +\infty} V \in \pi^{-1}(x)$$

with $(x_n) \in M_{\text{reg}}^{\mathbb{N}}$, such that, $x_n \xrightarrow{n \rightarrow +\infty} x$. Fix $n \in \mathbb{N}_0$. In the notations of Lemma 7.2.4, take $V = \text{im}(d_{x_n}^{(2)})$. Thus, in the coordinate chart $\mathcal{U}_{1, \dots, d-\ell}$ (see Section 7.2.1), the coordinates (a_{ij}^n) of $\text{im}(d_{x_n}^{(2)})$ satisfy the polynomial equations

$$\begin{aligned} f_j^k(x_n) + \sum_{s=1}^{\ell} a_{sj}^n f_s^k(x_n) &= 0, & j &= 1, \dots, d - \ell \\ & & k &= 1, \dots, N. \end{aligned} \quad (7.16)$$

One has,

$$f_j^k(x_n) + \sum_{s=1}^{\ell} a_{sj}^n f_s^k(x_n) = 0 \xrightarrow{n \rightarrow +\infty} f_j^k(x) + \sum_{s=1}^{\ell} a_{sj} f_s^k(x) = 0,$$

where for all s, j , $a_{sj}^n \xrightarrow{n \rightarrow +\infty} a_{sj}$. Therefore, \widetilde{M} is given in local coordinates $\mathcal{W} \times \mathcal{U}_{1, \dots, d-\ell}$ by elements that satisfy

1. Equation (7.15)
2. and that are limit of elements of (7.15) in nearby regular points, elements which are unique. Hence, it is, on this affine variety, the irreducible components of (7.15) that projects onto M .

\square

The pull-back of \mathfrak{F} to \widetilde{M} : Let $X \in \mathfrak{F}$. There exists a section v of the vector bundle $p: E_{-1} \rightarrow M$ such that $\rho(v) = X$. Consider the linear vector field $\widetilde{X} \in \mathfrak{X}(E_{-1})$ defined as follows

$$\widetilde{X}[p^*f] := p^*(\rho(v)[f]), \quad \forall f \in C^\infty(M),$$

$$\langle \widetilde{X}[\alpha], e \rangle := \rho(v)[\langle \alpha, e \rangle] - \langle \alpha, \ell_2(v, e) \rangle, \quad \forall \alpha \in \Gamma(E^*), \quad e \in \Gamma(E_{-1}).$$

Notice that \widetilde{X} depends on the choice of the almost Lie algebroid bracket ℓ_2 . The following items hold (see [LGLR22], Lemma 2.1.19. p. 61).

1. The flow $\phi_t^{\widetilde{X}}: E_{-1} \rightarrow E_{-1}$ of \widetilde{X} when it is defined, is a vector bundle isomorphism over ϕ_t^X .
2. The diagram below commutes,

$$\begin{array}{ccc} E_{-1} & \xrightarrow{\phi_t^{\widetilde{X}}} & E_{-1} \\ \rho \downarrow & & \downarrow \rho \\ TM & \xrightarrow{d\phi_t^X} & TM \end{array} \quad (7.17)$$

where ϕ_t^X is the flow of X .

In particular, $\phi_t^{\widetilde{X}}$ preserve the grassmanian $\text{Gr}_{-\ell}(E_{-1})$.

Proposition 7.2.6. *For every $X \in \mathfrak{F}$, choose $v \in \Gamma(E_{-1})$ such that $\rho(v) = X$. The vector field \widetilde{X} induces a vector field $\widetilde{\widetilde{X}}$ on $\text{Gr}_{-\ell}(E_{-1})$ such that*

1. $\widetilde{\widetilde{X}}$ is tangent to \widetilde{M} .
2. $\widetilde{\widetilde{X}}$ projects onto X .

Proof. Since for every $x \in M$, $\phi_t^{\widetilde{X}}|_x$ is an isomorphism of $E_{-1}|_x$, therefore $\phi_t^{\widetilde{X}}|_x$ preserves $\Pi^{-1}(x) = \text{Gr}_{-\ell}(E_{-1}|_x)$. We define $\widetilde{\widetilde{X}}$ for $(V, x) \in \text{Gr}_{-\ell}(E_{-1})$ by

$$\widetilde{\widetilde{X}}(V) := \frac{d}{dt}\Big|_{t=0} \left(\phi_t^{\widetilde{X}}|_x \right) \Big|_V \in T_V \text{Gr}_{-\ell}(E_{-1}|_x). \quad (7.18)$$

This shows item 1.

Let us show item 1, $\phi_t^{\widetilde{X}}$ preserves \widetilde{M} : to see this take $(V, x) \in \widetilde{M}$, let $x_n \xrightarrow{n \rightarrow +\infty} x$ be such that $\text{imd}_{x_n}^{(2)} \xrightarrow{n \rightarrow +\infty} V$ with $(x_n) \subset M_{reg}$. For every $n \in \mathbb{N}_0$, one has

$$\phi_t^{\widetilde{X}}|_{x_n} \left(\text{imd}_{x_n}^{(2)} \right) = \text{imd}_{\phi_t^X(x_n)}^{(2)},$$

since $\rho \circ \phi_t^{\widetilde{X}} = d\phi_t^X \circ \rho$. Thus,

$$\begin{aligned} \phi_t^{\widetilde{X}}|_x(V) &= \lim_{n \rightarrow +\infty} \phi_t^{\widetilde{X}}|_{x_n} \left(\text{imd}_{x_n}^{(2)} \right) \\ &= \lim_{n \rightarrow +\infty} \left(\text{imd}_{\phi_t^X(x_n)}^{(2)} \right) \in \pi^{-1} \left(\phi_t^X(x) \right). \end{aligned}$$

By consequence, for $V \in \widetilde{M}$, we have $\widetilde{\widetilde{X}}(V) \in T_V \widetilde{M}$, by Equation (7.18). \square

Corollary 7.2.7. \widetilde{X} does not depend on the choice of the almost bracket ℓ_2 .

Proof. This is an obvious consequence of item 3. in Proposition 7.2.6, since $\pi: \widetilde{M} \rightarrow M$ is invertible on an open (dense) subset. \square

Denote by $\widetilde{\mathfrak{F}}$ the module generated by the \widetilde{X} 's, with $X \in \mathfrak{F}$. For \mathfrak{F} polynomial, $\widetilde{\mathfrak{F}}$ is a singular foliation on the locally affine variety \widetilde{M} , because \widetilde{X} is still polynomial.

Corollary 7.2.8. If \mathfrak{F} is a polynomial singular foliation, then it is lifted to a projective singular foliation $\widetilde{\mathfrak{F}}$ on the locally affine variety \widetilde{M} .

Generalization. The same construction would apply if, instead of considering $\text{im}(d^{(2)}) \subseteq E_{-1}$, we consider $\text{im}(d^{(i+1)}) \subseteq E_{-i}$ or $\text{im}(\rho) \subseteq TM$ for $i \geq 1$. Denote by \widetilde{M}_i the transformation that we would obtain by such a construction. A lift $\widetilde{\mathfrak{F}}_i$ of \mathfrak{F} can be defined exactly in the same manner by considering a lift \widetilde{X} of $X \in \mathfrak{F}$ on E_{-i} (or TM) associated to $\ell_2: \Gamma(E_{-1}) \times \Gamma(E_{-i}) \rightarrow \Gamma(E_{-i})$ or to $[X, \cdot]$.

More precisely, consider for $i \geq 0$, the Grassmann bundle $Gr_{-\ell^i}(E_{-i})$, with $E_0 := TM$, where $\ell^i := \text{rk}(d_m^{i+1})$ (with the understanding that $d^{(1)} = \rho$ for $i = 0$) at regular points. These bundles are manifolds that project to M through a proper map. Over M_{reg} , there exists a unique $V \in Gr_{-\ell^i}(E_{-i})$ such that $d_m^{(i)}(V) = 0$. These define maps

$$\sigma_i: M_{reg} \hookrightarrow Gr_{-\ell^i}(E_{-i}).$$

We define $\widetilde{M}_i := \overline{\sigma_i(M_{reg})}$. This is not a manifold in general. However, it is a locally affine variety if \mathfrak{F} is a polynomial singular foliation on \mathbb{R}^N or \mathbb{C}^N as in Theorem 7.2.5. Moreover,

1. independent of the choice of a geometric resolution, as in Theorem 7.2.2,
2. the projection $\widetilde{M}_i \rightarrow M$ is proper and onto, as in Theorem 7.2.2,
3. \mathfrak{F} lifts to a singular foliation on \widetilde{M}_i , and it is one-to-one to M_{reg} , as in Proposition 7.2.6.

For $i = 0$ the same conclusion holds for

$$\begin{aligned} \sigma_0: M &\longrightarrow Gr_{\ell^0}(TM) \\ m &\longmapsto T_m \mathfrak{F}. \end{aligned}$$

Let us give some examples.

Example 7.2.9. Notice that $Gr_{-\ell}(\mathbb{C}^\ell) = \{pt\}$, so that if $d^{(i)}$ is into on an open dense subset, the construction degenerates, in this case $Gr_{-\ell}(\mathbb{C}^\ell) \times M \simeq M$

1. If \mathfrak{F} is a projective singular foliation, then $\widetilde{M}_1 \simeq M$: because $\Gamma(E_{-1}) \simeq \mathfrak{F}$ and $E_{-1} \xrightarrow{\rho} TM$ is injective on the open dense subset M_{reg} , i.e. $\ell^0 = \text{rk}(E_{-1})$. Hence $Gr_{-\ell^0}(E_{-1}) \simeq M$.
2. If \mathfrak{F} admits open dense orbits, $\widetilde{M}_0 \simeq M$, since $Gr_{-\dim M}(TM) \simeq M$.
3. Let $W \subseteq M = \mathbb{C}^d$ be an affine variety generated by (independent functions) $\varphi, \psi \in \mathbb{C}[x_1, \dots, x_d]$, i.e. $\mathcal{I}_W = \langle \varphi, \psi \rangle$. Let \mathfrak{F}_W be the singular foliation of (polynomial) vector fields vanishing on W . At regular points $x \in M_{reg}$, $T_x \mathfrak{F}_W = d$, hence

(a) $\widetilde{M}_0 \simeq M$.

(b) Also,

- $E_{-1} = (\mathbb{C}^d \oplus \mathbb{C}^d) \times M$, $\rho: e_i \oplus 0 \mapsto \varphi \frac{\partial}{\partial x_i}$ and $0 \oplus e_i \mapsto -\psi \frac{\partial}{\partial x_i}$, for all $i = 1, \dots, d$.
- $E_{-2} = \mathbb{C}^d \times M$, and $d^{(2)}: e_i \mapsto \frac{\psi}{K} e_i \oplus \frac{\varphi}{K} e_i$ with, $K = \gcd(\varphi, \psi)$

$$\text{im}(d_x^{(2)}) = \ker \rho_x = \left\langle \frac{\psi(x)}{K(x)} u \oplus \frac{\varphi(x)}{K(x)} u, \text{ with } u \in \mathbb{C}^d \right\rangle$$

For a convergent sequence (y_n) in $M \setminus W$, $(\ker \rho_{y_n})$ converges if and only if $[\frac{\psi(y_n)}{K(y_n)} : \frac{\varphi(y_n)}{K(y_n)}]$ converges in $\mathbb{P}^1(\mathbb{C})$. In that case, \widetilde{M}_1 is the closure of the graph $\{(y, [\psi(y) : \varphi(y)]), y \in M \setminus W\}$, which is the blow up of \mathbb{C}^d along W .

- $\widetilde{M}_2 \simeq M$ since $d_x^{(2)}$ is injective at regular points, so that $Gr_{-d}(E_{-2}) \simeq M$.

4. Let W be the affine variety defined by $\varphi \in \mathbb{C}[x_1, \dots, x_d]$. Consider the singular foliation $\mathfrak{F}_\varphi = \{X \in \mathfrak{X}(\mathbb{C}^d) \mid X[\varphi] = 0\}$. For every $y \in \mathbb{C}^d$, $(T_y \mathfrak{F}_\varphi)^\perp = \langle \nabla_y \varphi \rangle$. For convergent sequence $y_n \xrightarrow{n \rightarrow +\infty} y \in W$. The sequence $\text{im}(\rho_{y_n})$ converges if and only if $\nabla_{y_n} \varphi$ converges in $Gr_{-(d-1)}(\mathbb{C}^d)$, that is $[\frac{\partial \varphi}{\partial x_1}(y_n) : \dots : \frac{\partial \varphi}{\partial x_d}(y_n)]$ converges in the projective space $\mathbb{P}^{d-1}(\mathbb{C})$. Therefore, \widetilde{M}_0 is the closure of the graph of the map, $y \mapsto (y, [\frac{\partial \varphi}{\partial x_1}(y) : \dots : \frac{\partial \varphi}{\partial x_d}(y)])$ which is the blow up of \mathbb{C}^d along the singular locus of W .

Let us conclude this chapter by an open question. Can we desingularize a singular affine variety $W \subseteq \mathbb{C}^d$ by applying the constructions above to the singular foliation $\mathfrak{F} = \mathfrak{X}(W)$ of vector fields tangent to W ? We should then understand \widetilde{W}_0 as the Nash modification of W [LU81]. The meaning of \widetilde{W}_1 is unclear. We would like to relate the \widetilde{W}_i 's and the $\widetilde{\mathfrak{F}}_i$ together and go further in the universal Lie ∞ -algebroid, e.g. to understand the role of the 3-ary bracket in this procedure.

Conclusion:

In this chapter, we recall the notion of isotropy Lie (∞) -algebra of a singular foliation (and of a Lie-Rinehart algebra).

Then, we use the geometric resolution (i.e. the resolution on which the universal Lie ∞ -algebroid is built) to recover several notions of resolution of singularities: one being due to Nash and a second one to Mohsen.

Symmetries of singular foliations through Lie ∞ -algebroids

This chapter is one of the main application of results in Chapter 8.3.1 and Section 6.2. These results are taken from my article [Lou22].

We introduce the notion of weak symmetry actions of a lie algebra \mathfrak{g} on a singular foliation \mathfrak{F} and study the interaction of those on the universal Lie ∞ -algebroids of \mathfrak{F} . Also, in the subsequent chapter, we apply these results to the problem of extending a strict Lie algebra action on a sub-affine variety on the ambient space.

Convention 8.0.1. *Throughout this chapter, M stands for a smooth or complex manifold, or an affine variety over \mathbb{C} . We denote the sheaf of smooth or complex, or regular functions on M by \mathcal{O} and the sheaf of vector fields on M by $\mathfrak{X}(M)$, and $X[f]$ stands for a vector field $X \in \mathfrak{X}(M)$ applied to $f \in \mathcal{O}$. Also, \mathbb{K} stands for \mathbb{R} or \mathbb{C} .*

8.1 Definitions and examples

Definition 8.1.1. Let $\mathfrak{F} \subset \mathfrak{X}(M)$ be a singular foliation over M .

- A diffeomorphism $\phi: M \rightarrow M$ is said to be a *symmetry* of \mathfrak{F} , if $\phi_*(\mathfrak{F}) = \mathfrak{F}$.
- A vector field $X \in \mathfrak{X}(M)$ is said to be an *infinitesimal symmetry* of \mathfrak{F} , if $[X, \mathfrak{F}] \subset \mathfrak{F}$. The Lie algebra of infinitesimal symmetries of \mathfrak{F} is denoted by $\mathfrak{s}(\mathfrak{F})$.

In particular, $\mathfrak{F} \subset \mathfrak{s}(\mathfrak{F})$, since $[\mathfrak{F}, \mathfrak{F}] \subset \mathfrak{F}$.

Proposition 8.1.2. [AS09, GY18] *Let M be a smooth or complex manifold. The flow of an infinitesimal symmetry of \mathfrak{F} , if it exists, is a symmetry of \mathfrak{F} .*

As we will see later, one of the consequences of our future results is that any symmetry $X \in \mathfrak{s}(\mathfrak{F})$ of a singular foliation \mathfrak{F} admits a lift to a degree zero vector field on any universal NQ -manifold over \mathfrak{F} that commutes with the homological vector field Q . This allows us to have an alternative proof and

interpretation of Proposition 8.1.2 (see Section 8.2).

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} , depending on the context. From now on and in the sequel \mathfrak{g} is concentrated in degree -1 .

Definition 8.1.3. A *weak symmetry action* of the Lie algebra \mathfrak{g} on a singular foliation \mathfrak{F} on M is a \mathbb{K} -linear map $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ that satisfies:

- $\forall x \in \mathfrak{g}, [\varrho(x), \mathfrak{F}] \subseteq \mathfrak{F}$,
- $\forall x, y \in \mathfrak{g}, \varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathfrak{F}$.

When $x \mapsto \varrho(x)$ is a Lie algebra morphism, we speak of *strict symmetry action* of \mathfrak{g} on \mathfrak{F} . There is an equivalence relation on the set of weak symmetry actions which is defined as follows: two weak symmetry actions, $\varrho, \varrho': \mathfrak{g} \rightarrow \mathfrak{X}(M)$ are said to be *equivalent* if there exists a linear map $\varphi: \mathfrak{g} \rightarrow \mathfrak{F}$ such that $\varrho - \varrho' = \varphi$.

Remark 8.1.4. It is important to notice that when \mathfrak{F} is a regular foliation and M/\mathfrak{F} is a manifold, any weak symmetry action of a Lie algebra \mathfrak{g} on \mathfrak{F} induces a strict action of \mathfrak{g} over M/\mathfrak{F} . Definition 8.1.3 is a way of extending this idea to all singular foliations.

Here is a list of some examples.

Example 8.1.5. Let $\pi: M \rightarrow N$ be a submersion. Since any vector field on N comes from a π -projectable vector field on M , therefore any Lie algebra morphism $\mathfrak{g} \rightarrow \mathfrak{X}(N)$ can be lifted to a weak symmetry action $\mathfrak{g} \rightarrow \mathfrak{X}(M)$ on the regular foliation $\Gamma(\ker d\pi)$, and any two such lifts are equivalent.

Furthermore, any weak action of a Lie algebra \mathfrak{g} on a singular foliation \mathfrak{F} on N can be lifted to a class of weak symmetry actions on the pull-back foliation $\pi^{-1}(\mathfrak{F})$, (see Definition 1.9 in [AS09]).

Example 8.1.6. Let \mathfrak{F} be a singular foliation on M . For any point $m \in M$, consider $\mathfrak{g}_m = \frac{\mathfrak{F}(m)}{\mathcal{I}_m \mathfrak{F}}$ the isotropy Lie algebra of \mathfrak{F} at m (see Definition 7.0.2). Let us denote its Lie bracket by $[\cdot, \cdot]_{\mathfrak{g}_m}$.

1. Consider $\varrho: \mathfrak{g}_m \rightarrow \mathfrak{F}(m) \subset \mathfrak{X}(M)$ a section of the projection map,

$$\mathcal{I}_m \mathfrak{F} \hookrightarrow \mathfrak{F}(m) \xleftarrow{\varrho} \mathfrak{g}_m \quad (8.1)$$

Then, $[\varrho(x), \mathcal{I}_m \mathfrak{F}] \subset \mathcal{I}_m \mathfrak{F}$ and $\varrho([x, y]_{\mathfrak{g}_m}) - [\varrho(x), \varrho(y)] \in \mathcal{I}_m \mathfrak{F}$. Hence, the map $\varrho: \mathfrak{g}_m \rightarrow \mathfrak{X}(M)$ is a weak symmetry action of the singular foliation $\mathcal{I}_m \mathfrak{F}$. A different section ϱ' of the projection map yields an equivalent weak symmetry action of \mathfrak{g}_m on $\mathcal{I}_m \mathfrak{F}$. An obstruction class for having a strict symmetry action equivalent to ϱ will be given later in Section 9.

2. In particular, for $k \geq 1$, let us denote by \mathfrak{g}_m^k the isotropy Lie algebra of the singular foliation $\mathcal{I}_m^k \mathfrak{F}$ at m . Any section $\varrho_k: \mathfrak{g}_m^k \rightarrow \mathfrak{X}(M)$ of the projection map

$$\mathcal{I}_m^{k+1} \mathfrak{F} \hookrightarrow \mathcal{I}_m^k \mathfrak{F} \xleftarrow{\varrho_k} \mathfrak{g}_m^k \quad (8.2)$$

is a weak symmetry action of the Lie algebra \mathfrak{g}_m^k on the singular foliation $\mathcal{I}_m^{k+1} \mathfrak{F}$.

Example 8.1.7. Let $(A, [\cdot, \cdot]_A, \rho)$ be an almost Lie algebroid on a smooth manifold M , and let $\mathfrak{F} = \rho(\Gamma(A))$. Assume there exists $p \in M$ such that $A|_p$ is a Lie algebra and $\rho|_p = 0$. The question of finding a map $A|_p \rightarrow \Gamma(A)$ such that the composition

$$A|_p \longrightarrow \Gamma(A) \xrightarrow{\rho} \mathfrak{X}(M) \quad (8.3)$$

be a Lie algebra morphism, i.e., to ask whether the singular foliation \mathfrak{F} comes from the transformation Lie algebroid of $A|_p$, can be formulated as a weak symmetry action of $A|_p$ on the singular foliation $\mathcal{I}_p\mathfrak{F}$ as follows: Consider a linear map

$$\begin{aligned} A|_p &\longrightarrow \Gamma(A) \\ a &\longmapsto \tilde{a} \end{aligned} \quad (8.4)$$

such that for all $a \in A|_p$ and \tilde{a} a section of A such that $\tilde{a}|_p = a$. It is easily checked that the map in (8.4) is not a Lie algebra morphism, but it satisfies

$$\widetilde{[a, b]}_{A|_p} - [\tilde{a}, \tilde{b}]_A \in \mathcal{I}_p\Gamma(A) \quad \text{and} \quad \left[\widetilde{A|_p}, \mathcal{I}_p\Gamma(A) \right] \subset \mathcal{I}_p\Gamma(A).$$

Therefore, the map $a \mapsto \rho(\tilde{a})$ is a weak symmetry action of $A|_p$ on $\mathcal{I}_p\mathfrak{F}$. Notice that the isotropy Lie algebra $\mathfrak{g}_p^1 = \frac{\mathcal{I}_p\mathfrak{F}}{\mathcal{I}_p^2\mathfrak{F}}$ of the singular foliation $\mathcal{I}_p\mathfrak{F}$, is Abelian. In Chapter 9, we show that in this case, the obstruction of having a Lie algebra morphism in (8.3) is a cocycle of Chevalley-Eilenberg of $A|_p$ valued in \mathfrak{g}_p^1 .

The next example comes from [LGR22], and follows the same patterns as in Examples 8.1.5 and 8.1.6. It is based on the notion of Ehresmann connection. Let us first recall quickly this concept for the sake of completeness and clarity. There are several equivalent manners of viewing Ehresmann connections (see [Iva93] Section 9, Page 76), the most relevant approach in this context is the following: An *Ehresmann connection* on a smooth fiber bundle¹ $\pi: E \rightarrow M$ is a vector subbundle H of TE , such that $TE = H \oplus V$, where $V := \{\xi \in TE \mid \pi_*(\xi) = 0\}$ is called the "*vertical bundle*" whose fiber at $e \in E$ is $V_e = T_e(E_{\pi(e)})$. The subbundle H is called the "*horizontal bundle*". Given a connection as defined above, for every $e \in E$, the linear map $\pi_*: T_eE \rightarrow T_{\pi(e)}M$ restricts to an isomorphism, $H_e \rightarrow T_{\pi(e)}M$, whose inverse $T_{\pi(e)}M \rightarrow H_e$ is called the *horizontal lift*.

Example 8.1.8. Let (M, \mathfrak{F}) be a singular foliation on a smooth manifold M and $L \subset M$ a leaf. Let $[L, M]$ be a neighborhood of L in M equipped with some projection² $\pi: [L, M] \rightarrow L$. According to [LGR22], upon replacing $[L, M]$ be a smaller neighborhood of L if necessary, there exists an Ehresmann connection (that is a vector sub-bundle $H \subset T[L, M]$ with $H \oplus \ker(\pi_*) = T[L, M]$) which satisfies that $\Gamma(H) \subset \mathfrak{F}$. Such an Ehresmann connection is called an *Ehresmann \mathfrak{F} -connection* and induces a $C^\infty(L)$ -linear section $\varrho_H: \mathfrak{X}(L) \rightarrow \mathfrak{F}^{\text{proj}}$ of the surjection $\mathfrak{F}^{\text{proj}} \rightarrow \mathfrak{X}(L)$, where $\mathfrak{F}^{\text{proj}}$ stands for vector fields of \mathfrak{F} π -projectable on elements of $\mathfrak{X}(L)$. The section ϱ_H is a weak symmetry action of $\mathfrak{X}(L)$ on the *transverse* foliation $\mathcal{T} := \Gamma(\ker \pi_*) \cap \mathfrak{F}$. When the Ehresmann connection H is flat, ϱ_H is bracket-preserving, and defines a strict symmetry of $\mathfrak{X}(L)$ on the transverse foliation \mathcal{T} .

¹this generalizes connections to arbitrary fiber bundle $\pi: E \rightarrow M$, here the total space E is itself a smooth manifold, and has its own tangent bundle $TE \rightarrow E$.

²such projection comes with the Tubular neighborhood theorem [dS01].

Example 8.1.9. Consider, for a fixed k , the singular foliation $\mathfrak{F}_k := \mathcal{I}_0^k \mathfrak{X}(\mathbb{R}^d)$ generated by all vector fields vanishing to order k at the origin. The action of the Lie algebra $\mathfrak{gl}(\mathbb{R}^d)$ on \mathbb{R}^d which is given by,

$$\mathfrak{gl}(\mathbb{R}^d) \longrightarrow \mathfrak{X}(\mathbb{R}^d), (a_{ij})_{1 \leq i, j \leq d} \longmapsto \sum_{1 \leq i, j \leq d} a_{ij} x_i \frac{\partial}{\partial x_j}$$

is a strict symmetry of \mathfrak{F}_k .

Example 8.1.10. Let $\varphi := (\varphi_1, \dots, \varphi_r)$ be a r -tuple of homogeneous polynomial functions in d variables over \mathbb{K} . Consider the singular foliation \mathfrak{F}_φ (see [LGL22b] Section 3.2.1) which is generated by all polynomial vector fields $X \in \mathfrak{X}(\mathbb{K}^d)$ that satisfy $X[\varphi_i] = 0$ for all $i \in \{1, \dots, r\}$. The action $\mathbb{K} \rightarrow \mathfrak{X}(\mathbb{K}^d)$, $\lambda \mapsto \lambda \vec{E}$, is a strict symmetry of \mathfrak{F}_φ . Here \vec{E} stands for the Euler vector field.

Example 8.1.11. Let $W \subset \mathbb{C}^d$ be an affine variety and $\mathcal{I}_W \subset \mathbb{C}[x_1, \dots, x_d]$ its corresponding ideal. Let us denote by $\mathfrak{X}(W) := \text{Der}(\mathbb{C}[x_1, \dots, x_d]/\mathcal{I}_W)$ the Lie algebra of vector field of W . Any vector field on W can be extended (not unique) to a vector field on \mathbb{C}^d (see Section 5.1.1).

Let $\mathfrak{F}_W := \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ the singular foliation made of vector fields vanishing on W . Since every vector field on W can be extended to a vector field on \mathbb{C}^d tangent to W . Any Lie algebra morphism $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ extends to a linear map $\tilde{\varrho}: \mathfrak{g} \rightarrow \mathfrak{X}(\mathbb{C}^d)$ that makes this diagram commutes

$$\begin{array}{ccc} & & \mathfrak{X}(\mathbb{C}^d) \\ & \nearrow \tilde{\varrho} & \downarrow \\ \mathfrak{g} & \xrightarrow{\varrho} & \mathfrak{X}(W) \end{array}$$

This extension $\tilde{\varrho}$ is a weak symmetry action of \mathfrak{g} on \mathfrak{F}_W over the ambient space \mathbb{C}^d . Two different extensions yield equivalent symmetry actions.

8.2 A Lie ∞ -morphism lifting a weak symmetry of a foliation

We recall that \mathcal{O} is the sheaf of smooth/complex functions on a smooth/complex manifold M , or the algebra of regular functions on an affine variety over \mathbb{C} . We refer the reader to the Chapters 6 and 4 for the notion of (universal) Lie ∞ -algebroid of a singular foliation. We denote them by (E, Q) and their functions by \mathcal{E} .

For the sake of clarity, let put this chapter in context, and fix some notations.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ a Lie algebra and (E, Q) a Lie ∞ -algebroid over M . In the sequel, the Lie algebra \mathfrak{g} is concentrated in degree -1 . The differential graded Lie algebra $(\mathfrak{X}(E), [\cdot, \cdot], \text{ad}_Q)$ of vector fields on E is shifted by 1, i.e. a derivation of degree k in $\mathfrak{X}_k(E)$ is of degree $k - 1$ as an element of the shifted space $\mathfrak{X}_k(E)[1]$. The graded symmetric Lie bracket on $\mathfrak{X}(E)[1]$ is of degree $+1$ and given on homogeneous elements $u, v \in \mathfrak{X}(E)[1]$ as

$$\{u, v\} := (-1)^{|v|} [u, v].$$

In the sequel, we write $(\mathfrak{X}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ instead of $(\mathfrak{X}(E)[1], \{\cdot, \cdot\}, \text{ad}_Q)$.

Let $(S_{\mathbb{K}}^{\bullet}\mathfrak{g}, Q_{\mathfrak{g}})$ respectively $(S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ be the corresponding formulations in terms of co-derivations of the differential graded Lie algebras $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ and $(\mathfrak{X}(E)[1], [\cdot, \cdot], \text{ad}_Q)$. Precisely, $Q_{\mathfrak{g}}$ is the co-derivation defined by putting for every homogeneous monomial $x_1 \wedge \cdots \wedge x_k \in S_{\mathbb{K}}^k \mathfrak{g}$,

$$Q_{\mathfrak{g}}(x_1 \wedge \cdots \wedge x_k) := \sum_{1 \leq i < j \leq k} (-1)^{i+j-1} [x_i, x_j]_{\mathfrak{g}} \wedge x_1 \wedge \cdots \wedge \hat{x}_i \cdots \wedge \hat{x}_j \cdots \wedge x_k, \quad (8.5)$$

and $\bar{Q} = \bar{Q}^{(0)} + \bar{Q}^{(1)}$ is the co-derivation of degree +1 where the only non-zero Taylor coefficients are, $\bar{Q}^{(0)}: S_{\mathbb{K}}^1(\mathfrak{X}(E)[1]) \xrightarrow{\text{ad}_Q} \mathfrak{X}(E)[1]$ and $\bar{Q}^{(1)}: S_{\mathbb{K}}^2(\mathfrak{X}(E)[1]) \xrightarrow{\{\cdot, \cdot\}} \mathfrak{X}(E)[1]$.

The following is a particular case of Example 2.3.5, see also [LM94].

Definition 8.2.1. A Lie ∞ -morphism³ $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ is a graded coalgebra morphism $\bar{\Phi}: (S_{\mathbb{K}}^{\bullet}\mathfrak{g}, Q_{\mathfrak{g}}) \longrightarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ of degree zero which satisfies,

$$\bar{\Phi} \circ Q_{\mathfrak{g}} = \bar{Q} \circ \bar{\Phi}. \quad (8.6)$$

In other words, it is the datum of degree zero linear maps $(\bar{\Phi}_k: S_{\mathbb{K}}^{k+1}\mathfrak{g} \longrightarrow \mathfrak{X}_{-k}(E)[1])_{k \geq 0}$ that satisfies

$$\begin{aligned} \sum_{1 \leq i < j \leq n+2} (-1)^{i+j-1} \bar{\Phi}_n([x_i, x_j]_{\mathfrak{g}}, x_1, \dots, \hat{x}_{ij}, \dots, x_{n+2}) &= [Q, \bar{\Phi}_{n+1}(x_1, \dots, x_{n+2})] + \\ &\sum_{\substack{i+j=n \\ i \leq j \\ \sigma \in \mathfrak{S}_{i+1, j+1}}} \epsilon(\sigma) [\bar{\Phi}_i(x_{\sigma(1)}, \dots, x_{\sigma(i+1)}), \bar{\Phi}_j(x_{\sigma(i+2)}, \dots, x_{\sigma(n+2)})] \end{aligned} \quad (8.7)$$

where \hat{x}_{ij} means that we take x_i, x_j out of the list. When there is no risk of confusion, we write Φ for $\bar{\Phi}$.

Remark 8.2.2. It is important to notice that:

1. Definition 8.2.1 and Definition 6.1.14 are compatible when $M = \{\text{pt}\}$. Therefore, morphisms in both sense match.
2. In [MZ12], Definition 8.2.1 corresponds to the definition of actions of a Lie ∞ -algebras of finite dimension on Lie ∞ -algebroids of finite rank. Here we only have a Lie algebra. In contrast to theirs, we do not assume that \mathfrak{g} is finite dimensional.

Remark 8.2.3. It follows from the axioms (8.7) that for all $x, y \in \mathfrak{g}$, $[Q, \Phi_0(x)] = 0$ and

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)]. \quad (8.8)$$

In particular, if the homological vector field Q vanishes at some point $m \in M$, then the map $x \longmapsto (P \in \mathfrak{X}(E), P|_m \mapsto [\Phi_0(x), P]|_m)$ endows the vector space $\mathfrak{X}(E)|_m \simeq (S(E^*) \otimes E)|_m$ with a \mathfrak{g} -module structure. Moreover, the restriction of the map $\Phi_1: \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{X}_{-1}(E)|_m$ at m is a 2-cocycle of Chevalley-Eilenberg.

Remark 8.2.4. Let (E, Q) be a Lie ∞ -algebroid and \mathfrak{F} its basic singular foliation. Any Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \longrightarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ gives a weak symmetry action of \mathfrak{g} on \mathfrak{F} . If $Q|_m = 0$ for some point $m \in M$, the \mathfrak{g} -action defined in Remark 8.2.3, is independent of the equivalence class of the weak symmetry action.

³Here, we use " \rightsquigarrow " to emphasize that Φ is not a DGLA morphism.

The following lemma explains what the 0-Taylor coefficient of a Lie ∞ -morphism as in Definition 8.2.1 induces on the linear part of (E, Q) . More details will be given in Proposition 8.2.10 and Remark 9.1.1.

Lemma 8.2.5. *The 0-th Taylor coefficient $\Phi_0: \mathfrak{g} \longrightarrow \mathfrak{X}_0(E)$ induces*

1. *a linear map $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$, $x \longmapsto (\varrho(x)[f] := \Phi_0(x)[f])$, $f \in \mathcal{O}$ and*
2. *a linear map $x \in \mathfrak{g} \longmapsto \nabla_x \in \text{Der}_0(E)$, i.e. for each $x \in \mathfrak{g}$, $\nabla_x: E \longrightarrow E$ is a degree zero map that satisfies*

$$\nabla_x(fe) = f\nabla_x(e) + \varrho(x)[f]e, \text{ for } f \in \mathcal{O}, e \in \Gamma(E).$$

such that

$$\langle \Phi_0(x)^{(0)}(\alpha), e \rangle = \varrho(x)[\langle \alpha, e \rangle] - \langle \alpha, \nabla_x(e) \rangle, \text{ for all } \alpha \in \Gamma(E^*), e \in \Gamma(E). \quad (8.9)$$

$\Phi_0(x)^{(0)}$ stands for the polynomial-degree zero of $\Phi_0(x)$.

Proof. Using Lemma 6.1.7, we have for every $x \in \mathfrak{g}$, and $e \in \Gamma(E)$,

$$[\Phi_0(x), \iota_e]^{(-1)} = \iota_{\nabla_x e},$$

for some \mathbb{K} -bilinear map $\nabla_x: \Gamma(E_{-\bullet}) \longrightarrow \Gamma(E_{-\bullet})$ that depends linearly on $x \in \mathfrak{g}$ and that satisfies

$$\nabla_x(fe) = f\nabla_x(e) + \varrho(x)[f]e, \text{ for } f \in \mathcal{O}, e \in \Gamma(E). \quad (8.10)$$

To see (8.10), compute $[\Phi_0(x), \iota_{fe}]^{(-1)}$:

$$\begin{aligned} \iota_{\nabla_x(fe)} &= [\Phi_0(x), \iota_{fe}]^{(-1)} \\ &= \Phi_0(x)[f]\iota_e + f[\Phi_0(x), \iota_e]^{(-1)} \\ &= \iota_{\varrho(x)[f]e + \nabla_x e}. \end{aligned}$$

In particular, one has for all $\alpha \in \Gamma(E^*)$, $e \in \Gamma(E)$,

$$\begin{aligned} \langle \Phi_0(x)^{(0)}(\alpha), e \rangle &= \Phi_0(x)^{(0)}[\langle \alpha, e \rangle] - [\Phi_0(x)^{(0)}, \iota_e]^{(-1)}(\alpha) \\ &= \varrho(x)[\langle \alpha, e \rangle] - \langle \alpha, \nabla_x(e) \rangle. \end{aligned}$$

□

8.2.1 Homotopies

The following is a particular case of Definition 2.3.14 of Section 2.3.2. We rewrite it in this special context for the sake of clarity.

Definition 8.2.6. Let $\bar{\Phi}, \bar{\Psi}: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \rightsquigarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ be Lie ∞ -morphisms. We say $\bar{\Phi}, \bar{\Psi}$ are *homotopic over the identity of M* if the following conditions hold:

1. there a piecewise rational continuous path $t \in [a, b] \mapsto \Xi_t: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \rightsquigarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ made of Lie ∞ -morphisms that coincide with $\bar{\Phi}$ and $\bar{\Psi}$ at $t = a$ and b , respectively,

2. and a piecewise rational path $t \in [a, b] \mapsto H_t$ of Ξ_t -co-derivations of degree -1 such that

$$\frac{d\Xi_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}}. \quad (8.11)$$

Remark 8.2.7. Homotopy equivalence in the sense of the Definition 8.2.6 is an equivalence relation, and it is compatible with composition of Lie ∞ -morphisms. Also, we "glue" infinitely many equivalences, as in Lemma 2.3.22.

Convention 8.2.8. *In the sequel, $Q_{\mathfrak{g}}$ and \bar{Q} will be in implicit.*

Remark 8.2.9. Definition 8.2.6 is slightly more general than the equivalence relation [MZ12]. In [MZ12], it is explained that Lie ∞ -oid morphisms are Maurer-Cartan elements in some Lie ∞ -algebroid $\mathfrak{g} \oplus E$ of certain form, and they define equivalence as gauge-equivalence of the Maurer-Cartan elements. This gauge equivalence corresponds to homotopies as above for which all functions are smooth. Also, we do not require nilpotence unlike in Definition 5.1 of [MZ12]. Last, we do not assume \mathfrak{g} to be of finite dimension.

Proposition 8.2.10. *Let \mathfrak{g} be a Lie algebra and (E, Q) a Lie ∞ -algebroid over M .*

1. *Any Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induces a weak symmetry action of \mathfrak{g} on the basic singular foliation $\mathfrak{F} = \rho(\Gamma(E_{-1}))$ of (E, Q) .*
2. *Homotopic Lie ∞ -morphisms $\Phi, \Psi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induce equivalent weak symmetry actions ϱ_a, ϱ_b of \mathfrak{g} on the basic singular foliation \mathfrak{F} .*

Proof. Item 1. is a consequence of Remark 8.2.3. Indeed, take $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ as in Lemma 8.2.5 (1.) We claim that ϱ is a weak symmetry action of \mathfrak{g} on \mathfrak{F} : Let $x, y \in \mathfrak{g}$, and $e \in \Gamma(E_{-1})$ and $f \in \mathcal{O}$.

- $[\Phi_0(x), Q] = 0$ entails,

$$\left\langle \Phi_0(x)^{(0)} \left[Q^{(1)}(f) \right], e \right\rangle = \left\langle Q^{(1)} \left(\Phi_0(x)^{(0)}[f] \right), e \right\rangle$$

$$\varrho(x)[\langle Q[f], e \rangle] - \langle Q[f], \nabla_x(e) \rangle = \rho(e)[\varrho(x)], \quad (\text{by Lemma 8.2.5 (2.)})$$

$$\varrho(x)[\rho(e)][f] - \rho(\nabla_x(e))[f] = \rho(e)[\varrho(x)]$$

By consequence, $[\varrho(x), \rho(e)] = \rho(\nabla_x(e)) \in \mathfrak{F}$. Therefore, $[\varrho(x), \mathfrak{F}] \subseteq \mathfrak{F}$.

- By Lemma 6.1.7, there exists a skew-symmetric linear map $\eta: \wedge^2 \mathfrak{g} \rightarrow \Gamma(E_{-1})$ such that $\Phi_1(x, y)^{(-1)} = \iota_{\eta(x, y)}$. Therefore, the polynomial-degree zero of Equation (8.8) evaluated at an arbitrary function $f \in \mathcal{O}$ yields:

$$\Phi_0([x, y]_{\mathfrak{g}})^{(0)}(f) - [\Phi_0(x), \Phi_0(y)]^{(0)}(f) = [Q, \Phi_1(x, y)]^{(0)}(f)$$

$$\implies \Phi_0([x, y]_{\mathfrak{g}})(f) - \left[\Phi_0(x)^{(0)}, \Phi_0(y)^{(0)} \right](f) = \left[Q^{(1)}, \Phi_1(x, y)^{(-1)} \right](f)$$

$$\implies \varrho([x, y]_{\mathfrak{g}})[f] - [\varrho(x), \varrho(y)][f] = \left[Q^{(1)}, \iota_{\eta(x, y)} \right](f)$$

$$= \rho(\eta(x, y))[f].$$

Since f is arbitrary, this proves item 1. Using Proposition 2.3.17, $\Phi \sim \Psi$ implies for $x \in \mathfrak{g}$ that

$$\begin{aligned}\Psi(x) - \Phi(x) &= \bar{Q} \circ H(x) + \cancel{H \circ Q_{\mathfrak{g}}(x)} \\ &= [Q, H(x)]\end{aligned}\tag{8.12}$$

with $H: \mathfrak{g} \rightarrow \mathfrak{X}_{-1}(E)$ a linear map. Let $\beta: \mathfrak{g} \rightarrow \Gamma(E_{-1})$ be a linear map such that $H(x)^{(-1)} = \iota_{\beta(x)}$. Taking the polynomial-degree zero of both sides in Equation (8.12) and evaluating at $f \in \mathcal{O}$ we obtain that

$$(\varrho_a(x) - \varrho_b(x)) [f] = [Q^{(1)}, H(x)^{(-1)}] [f] = [Q^{(1)}, \iota_{\beta(x)}] [f] = \rho(\beta(x)) [f].$$

Since f is arbitrary, this proves item 2. □

Proposition 8.2.10 tells us that Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ induces weak symmetry action on the base manifold M . The aim of the next section is to look at the opposite direction. It responds to the following question: Do any weak symmetry action of a Lie algebra on a singular foliation comes from a Lie ∞ -morphism? If so, can we extend uniquely?

Now we define what we call "lift" of a weak symmetry action of a Lie algebra \mathfrak{g} on a singular foliation \mathfrak{F} to a Lie ∞ -algebroid (E, Q) over \mathfrak{F} .

Definition 8.2.11. Let \mathfrak{F} be a singular foliation over M and (E, Q) a Lie ∞ -algebroid over \mathfrak{F} . Consider a weak symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of \mathfrak{g} on \mathfrak{F} .

- We say that a Lie ∞ -morphism of differential graded Lie algebras

$$\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$$

lifts the weak symmetry action ϱ to (E, Q) if for all $x \in \mathfrak{g}$, $f \in \mathcal{O}$, $\Phi_0(x)(f) = \varrho(x)[f]$.

- When Φ exists, we say then Φ is a *lift* of ϱ on (E, Q) .

8.3 Main statements

We now state the main theorem of this chapter. Proposition 8.2.10 showed that a Lie ∞ -morphism between a Lie algebra \mathfrak{g} and a Lie ∞ -algebroid (E, Q) induces a weak action of \mathfrak{g} on the basic foliation. In this section, we show that any weak symmetry action of a Lie algebra \mathfrak{g} on a singular foliation \mathfrak{F} arises this way.

More precisely,

Theorem 8.3.1. *Let \mathfrak{F} a be a singular foliation over a smooth manifold (or an affine variety) M and \mathfrak{g} a Lie algebra. Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a weak symmetry action of \mathfrak{g} on \mathfrak{F} . The following assertions hold:*

1. *for any universal Lie ∞ -algebroid (E, Q) of the singular foliation \mathfrak{F} , there exists a Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ that lifts ϱ to (E, Q) ,*

2. any two such Lie ∞ -morphisms are homotopy equivalent over the identity of M ,
3. any two such lifts of any two equivalent weak symmetry actions of \mathfrak{g} on \mathfrak{F} are homotopy equivalent.

Remark 8.3.2. Again, Lie ∞ -morphisms in item 1 of Theorem 8.3.1 are \mathfrak{g} -actions on (E, Q) in [MZ12].

Remark 8.3.3. The item 1 in Theorem 8.3.1 means that there exists a linear map $\Phi_0: \mathfrak{g} \rightarrow \mathfrak{X}_0(E)$ such that

$$\Phi_0(x)[f] = \varrho(x)[f], \text{ and } [Q, \Phi_0(x)] = 0, \quad \forall x \in \mathfrak{g}, f \in \mathcal{O}. \quad (8.13)$$

This morphism is not a graded Lie algebra morphism, but there exist a linear map $\Phi_1: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{X}_{-1}(E)$ such that for all $x, y, z \in \mathfrak{g}$,

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)]. \quad (8.14)$$

Also,

$$\Phi_1([x, y]_{\mathfrak{g}}, z) - [\Phi_0(x), \Phi_1(y, z)] + \mathcal{C}(x, y, z) = [Q, \Phi_2(x, y, z)] \quad (8.15)$$

for some linear map $\wedge^3 \mathfrak{g} \rightarrow \mathfrak{X}_{-2}(E)$. These sets of compatibility conditions continue to higher multilinear maps.

Corollary 8.3.4. Any symmetry $X \in \mathfrak{X}(M)$ of the singular foliation \mathfrak{F} can be lifted to a degree zero vector field $Z \in \mathfrak{X}_0(E)$ that commutes with Q , i.e. such that $[Z, Q] = 0$.

Proof. To construct Z , it suffices to apply Theorem 8.3.1 for $\mathfrak{g} = \mathbb{R}$ and take Z to be the image of 1 through $\Phi_0: \mathbb{R} \rightarrow \mathfrak{X}_0(E)$. \square

Remark 8.3.5. In particular, Corollary 8.3.4 has the following consequences:

1. for any admissible t , the flow $\Phi_t^Z: \mathcal{E} \rightarrow \mathcal{E}$ of Z induces an isomorphism of vector bundles $E_{-1} \rightarrow E_{-1}$. Since $[Q, Z] = 0$, the following diagram commutes,

$$\begin{array}{ccc} \Gamma(E_{-1}) & \xrightarrow{(\Phi_t^Z)^{(0)}} & \Gamma(E_{-1}) \\ \downarrow \rho & & \downarrow \rho \\ \mathfrak{X}(M) & \xrightarrow{(\phi_t^X)_*} & \mathfrak{X}(M) \end{array}$$

where ϕ_t^X is the flow of X at t .

2. Consequently, for any open set $U \subset M$ which is stable under φ_t^X , there exists an invertible matrix \mathfrak{M}_X^t with coefficients in $\mathcal{O}(U)$ that satisfies

$$(\phi_t^X)_* \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \mathfrak{M}_X^t \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix},$$

for some generators X_1, \dots, X_n of \mathfrak{F} over U . As announced earlier, we recover Proposition 8.1.2, that is, $(\phi_t^X)_*(\mathfrak{F}) = \mathfrak{F}$.

Let (E, Q) and (E', Q') be two universal Lie ∞ -algebroids of \mathfrak{F} . A direct consequence of Ricardo Campos's Theorem 4.1 in [Cam19] is that the differential graded Lie algebras $(\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ and $(\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'})$ are homotopy equivalent over the identity of M . This leads to the following statement.

Corollary 8.3.6. *Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a weak symmetry action of a Lie algebra \mathfrak{g} on \mathfrak{F} . Then, there exist Lie ∞ -morphisms, $\Phi: \mathfrak{g} \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ and $\Psi: \mathfrak{g} \rightsquigarrow (\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'})$ that lift ϱ , and Φ, Ψ make the following diagram commute up to homotopy*

$$\begin{array}{ccc} & \mathfrak{g} & \\ \Phi \swarrow & & \searrow \Psi \\ (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q) & \xleftarrow{\sim} & (\mathfrak{X}_\bullet(E')[1], [\cdot, \cdot], \text{ad}_{Q'}) \end{array} \quad (8.16)$$

Proof. The composition of Φ with the horizontal map in the diagram (8.16) is a lift of the action ϱ . It is necessarily homotopy equivalent to Ψ by item (2) in Theorem 8.3.1. \square

8.3.1 Proof of 8.3.1

This section is devoted to the proof of the main results, i.e. Theorem 8.3.1.

Let \mathfrak{F} be a singular foliation, and (E, Q) a universal Lie ∞ -algebroid of \mathfrak{F} . We start with the following lemma.

Lemma 8.3.7. *For every weak symmetry Lie algebra action of \mathfrak{g} on \mathfrak{F} there exists a linear map, $\Phi_0: \mathfrak{g} \rightarrow \mathfrak{X}_0(E)$, such that $[Q, \Phi_0(x)] = 0$ and $\Phi_0(x)[f] = \varrho(x)[f]$ for all $x \in \mathfrak{g}$, $f \in \mathcal{O}$.*

Proof. For $x \in \mathfrak{g}$, let $\widehat{\varrho(x)} \in \mathfrak{X}_0(E)$ be any arbitrary extension of $\varrho(x) \in \mathfrak{s}(\mathfrak{F})$ to a degree zero vector field on E . Since $\varrho(x)$ is a symmetry of \mathfrak{F} , the degree +1 vector field $[\widehat{\varrho(x)}, Q]$ is also a longitudinal vector field on E , see Example 6.2.6 item 3. In addition, $[\widehat{\varrho(x)}, Q]$ is a ad_Q -cocycle. By item 1 of Theorem 6.2.10, there exists a vertical vector field $Y(x) \in \mathfrak{a}(E)$ of degree zero such that

$$[Q, Y(x) + \widehat{\varrho(x)}] = 0. \quad (8.17)$$

Let us set for $x \in \mathfrak{g}$, $\Phi_0(x) := Y(x) + \widehat{\varrho(x)}$. By construction, we have, $[Q, \Phi_0(x)] = 0$ and $\Phi_0(x)[f] = \varrho(x)[f]$ for all $x \in \mathfrak{g}$, $f \in \mathcal{O}$. \square

We will need the following lemma.

Lemma 8.3.8. *Assume (E, Q) is a universal Lie ∞ -algebroid over M . Let $\bar{\Phi}: (S_{\mathbb{K}}^\bullet \mathfrak{g}, Q_{\mathfrak{g}}) \rightarrow (S_{\mathbb{K}}^\bullet \mathfrak{X}(E)[1], \bar{Q})$ be a coalgebra morphism which is a Lie ∞ -morphism up to polynomial-degree $n \geq 0$, i.e.*

$$(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(i)} = 0 \text{ for all integer } i \in \{0, \dots, n\}.$$

Then, $\bar{\Phi}$ can be lengthened to a ∞ -morphism up to polynomial-degree $n + 1$.

Proof. For convenience, we omit the variables $x \in S_{\mathbb{K}}^\bullet \mathfrak{g}$. The identity,

$$\bar{Q} \circ (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}) + (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}) \circ Q_{\mathfrak{g}} = 0$$

taken in polynomial-degree $n + 1$ yields,

$$0 = (\bar{Q} \circ (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi}))^{(n+1)} = [Q, (\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(n+1)}],$$

since $Q_{\mathfrak{g}}^{(0)} = 0$ and $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(i)} = 0$ for $i \in \{0, \dots, n\}$. It is clear that for all $n \geq 0$ the map $(\bar{\Phi} \circ Q_{\mathfrak{g}} - \bar{Q} \circ \bar{\Phi})^{(n+1)} : S_{\mathbb{K}}^{n+2} \mathfrak{g} \rightarrow \mathfrak{X}_{-n}(E)[1]$ take value in vertical vector fields on E . By virtue of Lemma 6.2.11 there exists a map $\zeta : S_{\mathbb{K}}^{n+2} \mathfrak{g} \rightarrow \mathfrak{X}_{-n-1}(E)[1]$ such that

$$[Q, \bar{\Phi}^{(n+1)} + \zeta] = \bar{\Phi}^{(n)} \circ Q_{\mathfrak{g}}^{(1)} - \bar{Q}^{(1)} \circ \bar{\Phi}^{(n)}. \quad (8.18)$$

By redefining the polynomial-degree $n+1$ of $\bar{\Phi}$ as $\bar{\Phi}^{(n+1)} := \bar{\Phi}^{(n+1)} + \zeta$. One obtains a Lie ∞ -morphism up to polynomial-degree $n + 1$. The proof continues by recursion. \square

Proof of Theorem 8.3.1. Let us show Item 1. Note that Lemma 8.3.7 gives the existence of a linear map $\Phi_0 : \mathfrak{g} \rightarrow \mathfrak{X}_0(E)$ such that, $[Q, \Phi_0(x)] = 0$ for all $x \in \mathfrak{g}$. For $x, y \in \mathfrak{g}$, consider

$$\Lambda(x, y) = \Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)]. \quad (8.19)$$

Since $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathfrak{F}$ for all $x, y \in \mathfrak{g}$, and since $\rho : \Gamma(E_{-1}) \rightarrow \mathfrak{F}$ surjective, we have $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y))$ for some element $\eta(x, y) \in \Gamma(E_{-1})$ depending linearly on x and y . Now we consider the vertical vector field of degree -1 , $\iota_{\eta(x, y)} \in \mathfrak{X}_{-1}(\mathbb{U}^{\mathfrak{F}})$ which is defined on $\Gamma(E^*)$ as:

$$\iota_{\eta(x, y)}(\alpha) := \langle \alpha, \eta(x, y) \rangle \text{ for all } \alpha \in \Gamma(E^*),$$

and extended it by derivation on the whole space. For every $f \in \mathcal{O}$,

$$\begin{aligned} (\Lambda(x, y) - [Q, \iota_{\eta(x, y)}])(f) &= (\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] - \rho(\eta(x, y)))(f) && \text{(by definition of } \Phi_0) \\ &= 0 && \text{(by definition of } \eta) \end{aligned}$$

It is clear that $\Lambda(x, y) + [Q, \iota_{\eta(x, y)}]$ is a ad_Q -cocycle. Also, $(\Lambda(x, y) + [Q, \iota_{\eta(x, y)}])^{(-1)}$: for every $\alpha \in \Gamma(E^*)$,

$$[Q, \iota_{\eta(x, y)}]^{(-1)}(\alpha) = [Q^{(0)}, \iota_{\eta(x, y)}](\alpha) = \cancel{Q^{(0)}[\langle \alpha, \eta(x, y) \rangle]} + \langle Q^{(0)}[\alpha], \eta(x, y) \rangle = 0,$$

where the first term (resp. the second term) is cancelled by \mathcal{O} -linearity of $Q^{(0)}$ (resp. for degree reason). Hence, by Corollary 6.2.11 and Remark 6.2.12, the degree zero vector field $\Lambda(x, y) + [Q, \iota_{\eta(x, y)}]$ is of the form $[Q, \Upsilon(x, y)]$ for some vertical vector field $\Upsilon(x, y) \in \mathfrak{X}_{-1}(E)$ of degree -1 with $\Upsilon(x, y)^{(-1)} = 0$. For all $x, y \in \mathfrak{g}$, we define the Taylor coefficient $\Phi_1 : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{X}(E)$ as $\Phi_1(x, y) := \Upsilon(x, y) + \iota_{\eta(x, y)}$. By construction, we have the following relation,

$$\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)] = [Q, \Phi_1(x, y)], \quad \forall x, y \in \mathfrak{g} \quad (8.20)$$

Consider for $x, y, z \in \mathfrak{g}$,

$$\vartheta(x, y, z) = \Phi_1([x, y]_{\mathfrak{g}}, z) - [\Phi_0(x), \Phi_1(y, z)] + \mathcal{C}(x, y, z). \quad (8.21)$$

Here, $\mathcal{C}(x, y, z)$ stands for circular permutation of x, y and z with Koszul sign. For degree reason $\vartheta(x, y, z)$ is \mathcal{O} -linear. Moreover, $\vartheta(x, y, z)$ is a ad_Q -cocycle:

$$\begin{aligned} [Q, \Phi_1([x, y]_{\mathfrak{g}}, z)] + \mathcal{C}(x, y, z) &= -[\Phi_0([x, y]_{\mathfrak{g}}), \Phi_0(z)] + \mathcal{C}(x, y, z) \\ &= [[\Phi_0(z), Q], \Phi_1(x, y)] - [[\Phi_1(x, y), \Phi_0(z)], Q] + \mathcal{C}(x, y, z) \\ &= [Q, [\Phi_0(x), \Phi_1(y, z)]] + \mathcal{C}(x, y, z). \end{aligned}$$

Here, we have used the fact that $[Q, \Phi_0(x)] = 0$ for all $x \in \mathfrak{g}$, and the Jacobi identity for the Lie brackets $[\cdot, \cdot]_{\mathfrak{g}}$ and $[\cdot, \cdot]$. By Corollary 6.2.11, there exists a derivation of degree -2 denoted by $\Phi_2(x, y, z) \in \mathfrak{X}_{-2}(E)[1]$ that satisfies,

$$\vartheta(x, y, z) = [Q, \Phi_2(x, y, z)]. \quad (8.22)$$

So far, in the construction of the Lie ∞ -morphism, we have shown the existence of a Lie ∞ -morphism $\bar{\Phi}: S_{\mathbb{K}}^{\bullet} \mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1])$ up to polynomial-degree 2 that is $(\bar{\Phi} \circ Q_{\mathfrak{g}})^{(i)} = (\bar{Q} \circ \bar{\Phi})^{(i)}$ with $i = 0, 1, 2$. The proof continues by recursion or by applying directly Lemma 8.3.8. This proves the part 1. of the theorem. \square

Before proving item 3 of Theorem 8.3.1 we will need the following lemma. For convenience, we sometimes omit the variables in \mathfrak{g} .

Lemma 8.3.9. *For any two Lie ∞ -morphisms $\Gamma, \Omega: (S_{\mathbb{K}}^{\bullet} \mathfrak{g}, Q_{\mathfrak{g}}) \rightsquigarrow (S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1]), \bar{Q})$ which coincide up to polynomial-degree $n \geq 1$, i.e. $\Gamma^{(i)} = \Omega^{(i)}$, for $0 \leq i \leq n$, their difference in polynomial-degree $n + 1$, namely,*

$$\Gamma^{(n+1)} - \Omega^{(n+1)}: S_{\mathbb{K}}^{n+2} \mathfrak{g} \rightarrow \mathfrak{X}_{-n-1}(E)[1]$$

is valued in ad_Q -coboundary.

Proof. Indeed, a direct computation yields

$$\begin{aligned} \bar{Q} \circ (\Gamma - \Omega) &= (\Gamma - \Omega) \circ Q_{\mathfrak{g}} \implies \bar{Q}^{(0)} \circ (\Gamma - \Omega)^{(n+1)} - \underbrace{((\Gamma - \Omega) \circ Q_{\mathfrak{g}})^{(n+1)}}_{=0} = 0 \\ &\implies [Q, \Gamma^{(n+1)} - \Omega^{(n+1)}] = 0 \\ &\implies \Gamma^{(n+1)} - \Omega^{(n+1)} = [Q, H^{(n+1)}] \quad (\text{by item 1 of Theorem 6.2.10}) \end{aligned}$$

for some linear map $H^{(n+1)}: S_{\mathbb{K}}^{n+2} \mathfrak{g} \rightarrow \mathfrak{X}_{-n-2}(E)[1]$. \square

Let us show item 2 of Theorem 8.3.1. Let $\Phi, \Psi: \mathfrak{g} \rightarrow \mathfrak{X}(E)[1]$ be two different lifts of the action $\mathfrak{g} \rightarrow \mathfrak{X}(M)$. We denote by $\bar{\Phi}, \bar{\Psi}: S_{\mathbb{K}}^{\bullet} \mathfrak{g} \rightarrow S_{\mathbb{K}}^{\bullet}(\mathfrak{X}(E)[1])$ the unique comorphisms given respectively by the Taylor coefficients

$$\begin{cases} \bar{\Phi}^{(r)}: S_{\mathbb{K}}^{r+1} \mathfrak{g} \xrightarrow{\Phi_r} \mathfrak{X}_{-r}(E)[1] \\ \bar{\Psi}^{(r)}: S_{\mathbb{K}}^{r+1} \mathfrak{g} \xrightarrow{\Psi_r} \mathfrak{X}_{-r}(E)[1] \end{cases}, \text{ for } r \geq 0. \quad (8.23)$$

For any $x \in \mathfrak{g}$, the degree zero vector field $\Phi_0(x) - \Psi_0(x) \in \mathfrak{X}_0(E)$ is vertical. Moreover, we have, $[Q, \Phi_0(x) - \Psi_0(x)] = 0$. By Corollary 6.2.11 there exists a vector field $H_0 \in \mathfrak{X}_{-1}(E)$ of degree -1 , such that $\Psi_0(x) - \Phi_0(x) = [Q, H_0(x)]$

$$\begin{array}{ccc} & & \mathfrak{g} \\ & \swarrow H_0 & \downarrow \Psi_0 - \Phi_0 \\ \mathfrak{X}_{-1}(E)[1] & \xrightarrow{\text{ad}_Q} & \mathfrak{X}_0(E)[1] \end{array} \quad (8.24)$$

Consider the following differential equation

$$\begin{cases} \frac{d\Xi_t}{dt} = \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}}, & t \in [0, 1] \\ \Xi_0 = \bar{\Phi} \end{cases} \quad (8.25)$$

where $(\Xi_t)_{t \in [0,1]}$ is as in Definition 8.2.6, and for $t \in [0, 1]$, H_t is the unique Ξ_t -co-derivation where the only non-zero polynomial-degree is $H^{(0)} = H_0$. Equation (8.25) gives a homotopy between $\bar{\Phi}$ and Ξ_1 . When we consider the polynomial-degree zero component in Equation (8.25), one obtains

$$\begin{aligned} \frac{d\Xi_t^{(0)}}{dt} &= \bar{Q}^{(0)} \circ H_t^{(0)} + H_t^{(0)} \circ Q_{\mathfrak{g}}^{(0)} \\ &= [Q, H_0] \\ &= \Psi_0 - \Phi_0 = \bar{\Psi}^{(0)} - \bar{\Phi}^{(0)}. \end{aligned}$$

Therefore, $\Xi_t^{(0)} = \bar{\Phi}^{(0)} + t(\bar{\Psi}^{(0)} - \bar{\Phi}^{(0)})$, and $\bar{\Phi} \sim \Xi_1$ with $\bar{\Psi}^{(0)} = \Xi_1^{(0)}$. Using Lemma 8.3.9, the image of any element through the map $\bar{\Psi}^{(1)} - \Xi_1^{(1)}: S_{\mathbb{K}}^2 \mathfrak{g} \longrightarrow \mathfrak{X}_{-1}(E)[1]$ is a ad_Q -coboundary. Thus, $\bar{\Psi}^{(1)} - \Xi_1^{(1)}$ can be written as

$$\bar{\Psi}^{(1)} - \Xi_1^{(1)} = [Q, H^{(1)}], \quad \text{with } H^{(1)}: S_{\mathbb{K}}^2 \mathfrak{g} \longrightarrow \mathfrak{X}_{-2}(E)[1]. \quad (8.26)$$

Let us go one step further by considering the differential equation on $[0, 1]$ given by

$$\begin{cases} \frac{d\Theta_t}{dt} &= \bar{Q} \circ H_t + H_t \circ Q_{\mathfrak{g}} \\ \Xi_0 &= \bar{\Xi}_1 \end{cases} \quad (8.27)$$

Here H_t is the extension of $H^{(1)}$ as the unique Θ_t -co-derivation where all its arities vanish except the polynomial-degree 1 which is given by $H^{(1)}$. In polynomial-degree zero, $(\Theta_t^{(0)})_{t \in [0,1]}$ is constant and has value $\Theta_1^{(0)} = \bar{\Psi}^{(0)}$. In polynomial-degree one, we have,

$$\begin{aligned} \frac{d\Theta_t^{(1)}}{dt} &= \bar{Q}^{(0)} \circ H_t^{(1)} \\ &= [Q, H^{(1)}] = \bar{\Psi}^{(1)} - \Xi_1^{(1)}. \end{aligned}$$

Hence, $\Theta_t^{(1)} = \bar{\Phi}^{(1)} + t(\bar{\Psi}^{(1)} - \Xi_1^{(1)})$ with $\bar{\Psi}^{(i)} = \Theta_1^{(i)}$ for $i = 0, 1$. We then continue this procedure by gluing all these homotopies as in the proof of item 2 of Theorem 4.2.4. We obtain at last a Lie ∞ -morphism Ω such that $\bar{\Phi} \sim \Omega$ and $\Omega^{(i)} = \bar{\Psi}^{(i)}$ for $i \geq 0$. That means $\Omega = \bar{\Psi}$, therefore $\bar{\Phi} \sim \bar{\Psi}$. This proves item 2. of Theorem 8.3.1.

Let us prove item 3 of Theorem 8.3.1. Given two equivalent weak symmetry actions ϱ, ϱ' of \mathfrak{g} on a singular foliation \mathfrak{F} , i.e. ϱ, ϱ' differ by a linear map $\mathfrak{g} \longrightarrow \mathfrak{X}(M)$ of the form $x \mapsto \rho(\beta(x))$ for some linear map $\beta: \mathfrak{g} \longrightarrow \Gamma(E_{-1})$. Let $\Phi, \Phi': \mathfrak{g} \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$ be a lift into a Lie ∞ -morphism of the action ϱ and ϱ' respectively. One has for all $x \in \mathfrak{g}$ and $f \in \mathcal{O}$,

$$\begin{aligned} (\Phi_0(x) - \Psi_0(x) - [Q, \iota_{\varphi(x)}])(f) &= \rho(\varphi(x))[f] - \langle Q(f), \varphi(x) \rangle \\ &= 0. \end{aligned}$$

Since $[Q, \Phi_0(x) - \Psi_0(x) - [Q, \iota_{\varphi(x)}]] = 0$, by Corollary 6.2.11 there exists a vertical derivation $\hat{H}(x) \in \mathfrak{X}_{-1}(E)$ of degree -1 depending linearly on $x \in \mathfrak{g}$ such that

$$\Phi_0(x) - \Psi_0(x) = [Q, \hat{H}(x) + \iota_{\varphi(x)}].$$

Let $H(x) := \hat{H}(x) + \iota_{\varphi(x)}$, for $x \in \mathfrak{g}$. The proof continues the same as for item 2 of Theorem 8.3.1

8.3.2 Particular examples

We recall that for a regular foliation \mathfrak{F} on a manifold M , the Lie algebroid $TF \subset TM$, whose sections form \mathfrak{F} , is a universal Lie ∞ -algebroid of \mathfrak{F} . Its corresponding Q -manifold is given by the leafwise De Rham differential on $\Gamma(\wedge^\bullet T^*F)$.

Example 8.3.10. Let \mathfrak{F} be a regular foliation on a manifold M . Any weak symmetry action $\mathfrak{g} \longrightarrow \mathfrak{X}(M)$, $x \longmapsto \varrho(x)$, of \mathfrak{F} , can be lifted to Lie ∞ -morphism $\Phi: \mathfrak{g} \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ given explicitly as follows:

$$x \in \mathfrak{g} \longmapsto \Phi_0(x) = \mathcal{L}_{\varrho(x)} \in \mathfrak{X}_0(\wedge^\bullet T^*F) \quad (8.28)$$

$$x \wedge y \in \wedge^2 \mathfrak{g} \longmapsto \Phi_1(x, y) = \iota_{\chi(x, y)} \in \mathfrak{X}_{-1}(\wedge^\bullet T^*F) \quad (8.29)$$

and $(\Phi_i: \wedge^{i+1} \mathfrak{g} \longrightarrow \mathfrak{X}_{-i}(\wedge^\bullet T^*F)) \equiv 0$, for all $i \geq 2$, where $\chi(x, y) := \varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)]$ for $x, y \in \mathfrak{g}$. Also, \mathcal{L}_X stands for the Lie derivative on multi-forms w.r.t $X \in \mathfrak{X}(M)$, and ι_X is the internal product.

Example 8.3.11. Let \mathfrak{F} be a singular foliation on a manifold M together with a strict symmetry action $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ such that $\underline{\mathfrak{g}} \subset \mathfrak{F}$. Hence, $C^\infty(M)\underline{\mathfrak{g}}$ is a singular foliation which is the image of the transformation Lie algebroid $\mathfrak{g} \times M$. The universality theorem (see [LLS20, LGL22b]) provides the existence of a Lie ∞ -morphism $\nu: \mathfrak{g} \longrightarrow \mathbb{U}^{\mathfrak{F}}$. Let us call its Taylor coefficients $\nu_n: \wedge^{n+1} \mathfrak{g} \longrightarrow E_{-n-1}$, $n \geq 0$. We may take for example the 0-th and 1-th Taylor coefficients of a Lie ∞ -morphism that lifts ϱ as:

$$\begin{aligned} \Phi_0(x) &:= [Q, \iota_{\nu_0(x)}] \in \mathfrak{X}_0(\mathbb{U}^{\mathfrak{F}}), \text{ for } x \in \mathfrak{g}. \\ \Phi_1(x, y) &:= [Q, \iota_{\nu_1(x, y)}]^{(-1)} - \sum_{k \geq 0} [[Q, \iota_{\nu_0(x)}], \iota_{\nu_0(y)}]^{(k)} \in \mathfrak{X}_{-1}(\mathbb{U}^{\mathfrak{F}}), \text{ for } x, y \in \mathfrak{g}. \end{aligned}$$

Note that in this case the action ϱ is equivalent to zero, therefore by item 3 of Theorem 8.3.1 the Lie ∞ -morphism Φ is homotopic to zero.

8.4 Lifts of weak symmetry actions and Lie ∞ -algebroids

In this section, \mathfrak{g} is a finite dimensional Lie algebra that we see as the trivial vector bundle over M with fiber \mathfrak{g} .

The following theorem says that any lift of strict symmetry action of \mathfrak{g} on a singular foliation \mathfrak{F} induces a Lie ∞ -algebroids with some special properties and vice versa. See [MZ12], Prop. 3.3, for a proof of the following statement.

Proposition 8.4.1. *Let (E, Q) be a Lie ∞ -algebroid over a singular foliation \mathfrak{F} . Any Lie ∞ -morphism $\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_\bullet(E)[1], [\cdot, \cdot], \text{ad}_Q)$ with \mathfrak{g} of finite dimension induces a Lie ∞ -algebroid $(E \oplus \mathfrak{g}, Q')$ with*

$$Q' := d^{CE} + Q + \sum_{k \geq 1, i_1, \dots, i_k = 1, \dots, \dim(\mathfrak{g})} \frac{1}{k!} \xi^{i_1} \odot \dots \odot \xi^{i_k} \Phi_{k-1}(\xi_{i_1}, \dots, \xi_{i_k}), \quad (8.30)$$

where d^{CE} is the Chevalley-Eilenberg complex of \mathfrak{g} , and $\xi^1, \dots, \xi^{\dim(\mathfrak{g})} \in \mathfrak{g}^*$ is the dual basis of some basis $\xi_1, \dots, \xi_{\dim(\mathfrak{g})} \in \mathfrak{g}$ and for all $k \geq 0$, $\Phi_k: S^{k+1} \mathfrak{g} \longrightarrow \mathfrak{X}_{-k}(E)[1]$ is the k -th Taylor coefficients of Φ .

In the dual point of view, (8.30) corresponds to a Lie ∞ -algebroid over the complex

$$\cdots \xrightarrow{\ell_1} E_{-3} \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} \mathfrak{g} \oplus E_{-1} \xrightarrow{\rho'} TM \quad (8.31)$$

whose brackets satisfy

1. the anchor map ρ' sends an element $x \oplus e \in \mathfrak{g} \oplus E_{-1}$ to $\varrho(x) + \rho(e) \in \varrho(\mathfrak{g}) + T\mathfrak{F}$,
2. the binary bracket satisfies

$$\ell_2(\Gamma(E_{-1}), \Gamma(E_{-1})) \subset \Gamma(E_{-1}) \quad \text{and} \quad \ell_2(\Gamma(E_{-1}), x) \subset \Gamma(E_{-1}), \quad \forall x \in \mathfrak{g}$$

3. the \mathfrak{g} -component of the binary bracket on constant sections of $\mathfrak{g} \times M$ is the Lie bracket of \mathfrak{g} .

Conversely, if there exists a Lie ∞ -algebroid (E', Q') whose underlying complex of vector bundles is of the form (8.31) and that satisfies item 1, 2 and 3, then there is a Lie ∞ -morphism

$$\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q)$$

which is defined on a given basis ξ_1, \dots, ξ_d of \mathfrak{g} by:

$$\Phi_{k-1}(\xi_{i_1}, \dots, \xi_{i_k}) = \text{pr} \circ [\cdots [[Q', \iota_{\xi_{i_1}}], \iota_{\xi_{i_2}}], \dots, \iota_{\xi_{i_k}}] \in \mathfrak{X}(E)[1], \quad k \in \mathbb{N}, \quad (8.32)$$

where pr stands for the projection map $\mathfrak{X}(E')[1] \longrightarrow \mathfrak{X}(E)[1]$.

Proof. We explain the idea of the proof. A direct computation gives the first implication. Conversely, let us denote by Q' the homological vector fields of Lie ∞ -algebroid whose underlying complex of vector bundles is of the form (8.31). The map defined in Equation (8.32) is indeed a lift into a Lie ∞ -morphism of the weak symmetry action ϱ :

- It is not difficult to check that, for any $\xi \in \mathfrak{g}$, one has $[Q, \Phi_0(\xi)] = 0$.
- The fact that Φ defines a Lie ∞ -morphism can be found using Voronov trick [Vor04, Vor05], i.e., doing Jacobi's identity inside the null derivation

$$0 = \text{pr} \circ [\cdots [[Q', Q'], \iota_{\xi_{i_1}}], \iota_{\xi_{i_2}}], \dots, \iota_{\xi_{i_k}}]. \quad (8.33)$$

A direct computation of Equation (8.33) falls exactly on the requirements of Definition 8.2.1. \square

Remark 8.4.2. Let us compute Equation (8.33) for a small number of generators (e.g $k = 2, 3$) in order to show how it works: from the identity

$$\left[\left[[Q', Q'], \iota_{\xi_{i_1}} \right], \iota_{\xi_{i_2}} \right] = 0,$$

one obtains by using twice the Jacobi identity the following relation,

$$[Q', [[Q', \xi_{i_1}], \xi_{i_2}]] - [[Q', \xi_{i_1}], [Q', \xi_{i_2}]] = 0. \quad (8.34)$$

One should notice that $[[Q', \xi_{i_1}], \xi_{i_2}]$ splits into two parts. One part where the Chevalley-Eilenberg acts to give $[[d^{\text{CE}}, \xi_{i_1}], \xi_{i_2}] = \iota_{[\xi_{i_1}, \xi_{i_2}]_{\mathfrak{g}}}$, while the other part is $[[Q' - d^{\text{CE}}, \xi_{i_1}], \xi_{i_2}]$. Hence, by putting them in Equation (8.34), afterwards projecting on $\mathfrak{X}(S^{\bullet}(E^*))$, we get

$$\text{pr} \circ [Q', \iota_{[\xi_{i_1}, \xi_{i_2}]_{\mathfrak{g}}}] + \text{pr} \circ [Q', [[Q' - d^{\text{CE}}, \xi_{i_1}], \xi_{i_2}]] - \text{pr} \circ [[Q', \xi_{i_1}], [Q', \xi_{i_2}]] = 0.$$

From here, we deduce that

$$\Phi_0([\xi_{i_1}, \xi_{i_2}]_{\mathfrak{g}}) = [Q, \Phi_1(\xi_{i_1}, \xi_{i_2})] + [\Phi_0(\xi_{i_1}), \Phi_0(\xi_{i_2})].$$

Here is an application of Theorem 8.4.1.

Corollary 8.4.3.

1. Let \mathfrak{g} be a Lie algebra and G its Lie group.
2. Let (M, \mathfrak{F}) a singular foliation together with a weak symmetry action $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$.

The following assertions hold:

1. On $G \times M$ the $C^\infty(G \times M)$ -module generated by

$$\begin{cases} (\overleftarrow{u}, \varrho(u)) & u \in \mathfrak{g} \\ (0, X) & X \in \mathfrak{F} \end{cases} \quad (8.35)$$

is a singular foliation that we denote by $(\mathfrak{X}(G) \times_{\varrho} \mathfrak{F})$.

2. If (E, d, ϱ) is a geometric resolution of (M, \mathfrak{F}) , then

$$\cdots \longrightarrow p^*E_{-3} \xrightarrow{d} p^*E_{-2} \xrightarrow{d} \mathfrak{g} \oplus p^*E_{-1} \xrightarrow{\rho'} T(G \times M) \quad (8.36)$$

with $\rho'(e, u) = (\overleftarrow{u}, \varrho(u) + \rho(e))$, is a geometric resolution of $(\mathfrak{X}(G) \times_{\varrho} \mathfrak{F})$. Here, $p: G \times M \longrightarrow M$ is the projection on M .

Let (E, Q) be a universal Lie ∞ -algebroid structure of (M, \mathfrak{F}) . Let $\Phi_k: \wedge^{k+1} \mathfrak{g} \longrightarrow \mathfrak{X}_{-k}(E)$ be the Taylor coefficients of a Lie ∞ -morphism $\mathfrak{g} \rightsquigarrow \mathfrak{X}(E)$ that lifts ϱ . Then

$$Q' := d_{dR}^G + Q + \sum_{k \geq 1, i_1, \dots, i_k = 1, \dots, \dim(\mathfrak{g})} \frac{1}{k!} \xi^{i_1} \odot \cdots \odot \xi^{i_k} \Phi_{k-1}(\xi_{i_1}, \dots, \xi_{i_k}), \quad (8.37)$$

with $(\xi_1, \dots, \xi_{\dim(\mathfrak{g})})$, $(\xi^1, \dots, \xi^{\dim(\mathfrak{g})})$ be dual basis of \mathfrak{g} and \mathfrak{g}^*

1. is a universal Lie ∞ -algebroid of $(\mathfrak{X}(G) \times_{\varrho} \mathfrak{F})$
2. whose coefficients are left invariant for the action of G on $G \times M$ given by $g \cdot (h, m) := (gh, m)$.

Conversely, a left invariant Lie ∞ -algebroid on (8.36) can be interpreted as Taylor coefficients of a lift of ϱ .

8.4.1 A more general statement of Proposition 8.4.1

We end the chapter with a generalization of Proposition 8.4.1.

In the previous section, Proposition 8.4.1 is stated in the finite dimensional context, i.e. it needs \mathfrak{g} to be finite dimensional and the existence of a geometric resolution for the singular foliation \mathfrak{F} . In this section we prove that given a weak symmetry action of a Lie algebra \mathfrak{g} (may be of infinite dimensional) on a Lie-Rinehart algebra $\mathfrak{F} \subset \mathfrak{X}(M)$ (we do not require \mathfrak{F} being locally finitely generated), such Lie ∞ -algebroid described at the sections level of the complex (8.31) as Proposition 8.4.1 exists.

We state the following theorem in the context of singular foliations, but the same statement and the same proof are valid word-by-word by replacing \mathfrak{F} by a Lie-Rinehart algebra.

Theorem 8.4.4. *Let \mathfrak{g} be a (possibly infinite dimensional) Lie \mathbb{K} -algebra and let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a weak symmetry action of \mathfrak{g} on a singular foliation \mathfrak{F} . Let $((\mathcal{K}_{-i})_{i \geq 1}, d, \rho)$ be a free resolution of the singular foliation \mathfrak{F} over \mathcal{O} . The complex of trivial vector bundles over M*

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} \mathfrak{g} \oplus E_{-1} \xrightarrow{\rho'} TM \quad (8.38)$$

where $\Gamma(E_{-1}) = \mathcal{K}_{-1}$, comes equipped with a Lie ∞ -algebroid structure

1. whose unary bracket is d and whose anchor map ρ' , sends an element $x \oplus e \in \mathfrak{g} \oplus E_{-1}$ to $\varrho(x) + \rho(e) \in \varrho(\mathfrak{g}) + T\mathfrak{F}$,
2. the binary bracket satisfies

$$\ell_2(\Gamma(E_{-1}), \Gamma(E_{-1})) \subset \Gamma(E_{-1}) \quad \text{and} \quad \ell_2(\Gamma(E_{-1}), \Gamma(\mathfrak{g})) \subset \Gamma(E_{-1}),$$

3. the \mathfrak{g} -component of the binary bracket on constant sections of $\mathfrak{g} \times M$ is the Lie bracket of \mathfrak{g} .

Remark 8.4.5. When we have $\varrho(\mathfrak{g}) \cap T_m\mathfrak{F} = \{0\}$ for all m in M , Equation (8.38) is a free resolution of the singular foliation $C^\infty(M)\varrho(\mathfrak{g}) + \mathfrak{F}$ and we can apply directly the Theorem 4.2.1. Otherwise, we need to show there is no obstruction in degree -1 while doing the construction of the brackets if the result still needs to hold.

Proof. (of Theorem 8.4.4) The complex of Equation (8.38) being exact everywhere except in degree -1 we cannot apply directly Theorem 2.1 in [LGL22b] but we can mimic the proof given for Theorem 2.1 in [LGL22b] to construct the higher brackets when there is no obstruction in degree -1 . For convenience, let us denote $\mathcal{R}_{-1} := \Gamma(\mathfrak{g}) \oplus \Gamma(E_{-1})$ and $\mathcal{R}_{-i} := \Gamma(E_{-i})$ for $i \geq 2$. Given a natural number $k \geq 0$, we consider the total complex $\left(\widehat{\mathfrak{Page}}_\bullet^{(k)}(\mathcal{R}), D = [d, \cdot]_{\text{RN}} \right)$ of the following bicomplex

$$\begin{array}{ccccc} & \vdots & & \vdots & \\ & \uparrow & & \uparrow & \\ \cdots & \rightarrow & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-3}, \mathcal{R}_{-3}\right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-3}, \mathcal{R}_{-2}\right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-3}, d\mathcal{R}_{-2}\right) & \rightarrow & 0 \\ & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\ \cdots & \rightarrow & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-2}, \mathcal{R}_{-3}\right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-2}, \mathcal{R}_{-2}\right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-2}, d\mathcal{R}_{-2}\right) & \rightarrow & 0 \\ & & \delta \uparrow & & \delta \uparrow & & \delta \uparrow & & \\ \cdots & \rightarrow & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-1}, \mathcal{R}_{-3}\right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-1}, \mathcal{R}_{-2}\right) & \xrightarrow{d} & \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}_{|-k-1}, d\mathcal{R}_{-2}\right) & \rightarrow & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & 0 & & \\ & & \text{"-3 column"} & & \text{"-2 column"} & & \text{"-1 column"} & & \end{array} \quad (8.39)$$

The map δ stands for the vertical differential which is defined for all $\Phi \in \text{Hom}_{\mathcal{O}}\left(\odot^{k+1} \mathcal{R}, \mathcal{R}\right)$ by

$$\delta(\Phi)(r_1, \dots, r_{k+1}) := \Phi \circ d(r_1 \odot \dots \odot r_{k+1}), \quad \forall r_1, \dots, r_{k+1} \in \mathcal{R},$$

where here d acts as an \mathcal{O} -derivation on $r_1 \odot \dots \odot r_{k+1} \in \odot^k \mathcal{R}$ and the horizontal differential given by

$$\Phi \mapsto d \circ \Phi.$$

Since the line the bicomplex is exact, the total complex $\left(\widehat{\mathfrak{Page}}_\bullet^{(k)}(\mathcal{R}), D = [d, \cdot]_{\text{RN}} \right)$ is also exact.

Construction of the 2-ary bracket: its construction is almost the same as in [LGL22b] we adapt what has been done to our case. We first construct a 2-ary bracket on \mathcal{R}_{-1} to extend on every degree. For all $k \geq 1$, let us denote by $(e_i^{(-k)})_{i \in I_k}$ a basis of $\Gamma(E_{-k})$. The set $\{X_i = \rho(e_i^{(-1)}) \in \mathfrak{F} \mid i \in I_1\}$ is a set of generators of \mathfrak{F} . In particular, there exists elements $c_{ij}^k \in \mathcal{O}$ and satisfying the skew-symmetry condition $c_{ij}^k = -c_{ji}^k$ together with

$$[X_i, X_j] = \sum_{k \in I} c_{ij}^k X_k \quad \forall i, j \in I_1. \quad (8.40)$$

By definition of weak symmetry one has

$$[\varrho(\xi_i), \rho(e_j^{(-1)})] \in \mathfrak{F} \quad \text{and} \quad \varrho([\xi_i, \xi_j]_{\mathfrak{g}}) - [\varrho(\xi_i), \varrho(\xi_j)] \in \mathfrak{F} \quad \text{for all } (i, j) \in I_{\mathfrak{g}} \times I_{-1}. \quad (8.41)$$

Here, $(\xi_i)_{i \in I_{\mathfrak{g}}}$ is a basis for \mathfrak{g} . Since $((\mathcal{K}_{-i})_{i \geq 1}, d, \rho)$ is a free resolution of \mathfrak{F} , there exists two \mathcal{O} -bilinear maps $\chi: \Gamma(\mathfrak{g}) \times \Gamma(E_{-1}) \rightarrow \Gamma(E_{-1})$, $\eta: \Gamma(\mathfrak{g}) \times \Gamma(\mathfrak{g}) \rightarrow \Gamma(E_{-1})$ defined on generators $\xi_i, e_j^{(-1)}$ by the relations

$$[\varrho(\xi_i), \rho(e_j^{(-1)})] = \rho(\chi(\xi_i, e_j^{(-1)})) \quad \text{and} \quad \varrho([\xi_i, \xi_j]_{\mathfrak{g}}) - [\varrho(\xi_i), \varrho(\xi_j)] = \rho(\eta(\xi_i, \xi_j)).$$

We first define a naive 2-ary bracket on $\Gamma(E_{-1})$ as follows:

1. an anchor map by $\rho'(e_i^{(-1)}) = X_i$, and $\rho'(\xi_i) = \varrho(\xi_i)$, for all $i \in I, I_{\mathfrak{g}}$,
2. a degree +1 graded symmetric operation $\tilde{\ell}_2$ on \mathcal{R}_{\bullet} as follows:

- (a) $\tilde{\ell}_2(e_i^{(-1)}, e_j^{(-1)}) = \sum_{k \in I} c_{ij}^k e_k^{(-1)}$ for all $i, j \in I_{-1}$,
- (b) $\tilde{\ell}_2(\xi_i, e_j^{(-1)}) = \chi(\xi_i, e_j^{(-1)})$,
- (c) $\tilde{\ell}_2(\xi_i, \xi_j) = [\xi_i, \xi_j]_{\mathfrak{g}} + \eta(\xi_i, \xi_j)$,
- (d) $\tilde{\ell}_2$ is zero on the other generators,
- (e) we extend $\tilde{\ell}_2$ to \mathcal{R} using \mathcal{O} -bilinearity and Leibniz identity with respect to the anchor ρ' .

By (a), (b), (c), (d), (e), $\tilde{\ell}_2$ satisfies the Leibniz identity with respect to the anchor $\tilde{\rho}$ and (a), (b), (c) makes the latter a bracket morphism. The map defined for all homogeneous $r_1, r_2 \in \mathcal{R}_{\bullet}$ by

$$[d, \tilde{\ell}_2]_{\text{RN}}(r_1, r_2) = d \circ \tilde{\ell}_2(r_1, r_2) + \tilde{\ell}_2(dr_1, r_2) + (-1)^{|r_1|} \tilde{\ell}_2(r_1, dr_2), \quad (8.42)$$

is a graded symmetric degree +2 operation $(\mathcal{R} \otimes \mathcal{R})_{\bullet} \rightarrow \mathcal{R}_{\bullet+2}$, and $[d, \tilde{\ell}_2]_{\text{RN}}|_{\mathcal{R}_{-1}} = 0$. It is \mathcal{O} -bilinear, i.e. for all $f \in \mathcal{O}, r_1, r_2 \in \mathcal{R}$

$$[d, \tilde{\ell}_2]_{\text{RN}}(r_1, fr_2) - f[d, \tilde{\ell}_2]_{\text{RN}}(r_1, r_2) = 0.$$

We also have that $\rho([d, \tilde{\ell}_2]_{\text{RN}}(r_1, fr_2)) = \rho(\tilde{\ell}_2(dr_1, r_2)) = 0$, for all $r_1 \in \mathcal{R}_{-2}, r_2 \in \mathcal{R}_{-1}$, since $\rho \circ d = 0$. Thus, $[d, \tilde{\ell}_2]_{\text{RN}}|_{\mathcal{R}_{-2} \times \mathcal{R}_{-1}} \in d\mathcal{R}_{-2}$, because $((E_{-i})_{i \geq 1}, d, \rho)$ is a geometric resolution.

Therefore, $[d, \tilde{\ell}_2]_{\text{RN}}$ is a degree +2 element in the total complex $\widehat{\mathfrak{Page}}^{(1)}(\mathcal{R})$. The \mathcal{O} -bilinear operator $[d, \tilde{\ell}_2]_{\text{RN}}$ is D -closed in $\widehat{\mathfrak{Page}}^{(1)}(\mathcal{R})$, since $[d, [d, \tilde{\ell}_2]_{\text{RN}}]_{\text{RN}}|_{\mathcal{R}_{\leq -2}} = 0$. So there exists $\tau_2 \in \bigoplus_{j \geq 2} \text{Hom}_{\mathcal{O}}(\odot^2 \mathcal{R}_{-j-1}, \mathcal{R}_{-j})$ such as $D(\tau_2) = -[d, \tilde{\ell}_2]_{\text{RN}}$. By replacing $\tilde{\ell}_2$ by $\tilde{\ell}_2 + \tau_2$ we get a 2-ary bracket ℓ_2 of degree +1 which is compatible with the differential map d and the anchor map $\tilde{\rho}$.

Construction of higher brackets: notice that by construction of the 2-ary bracket ℓ_2 one has, $\text{Jac}(r_1, r_2, r_3) \in d\mathcal{R}_{-2}$ for all $r_1, r_2, r_3 \in \mathcal{R}_{-1}$. In other words, $\text{Jac} \in \text{Hom}_{\mathcal{O}}(\odot^3 \mathcal{R}_{-1}, d\mathcal{R}_{-2})$. A direct computation shows

$$d\text{Jac}(r_1, r_2, r_3) = \text{Jac}(dr_1, r_2, r_3) + (-1)^{|r_1|} \text{Jac}(r_1, dr_2, r_3) + (-1)^{|r_1|+|r_2|} \text{Jac}(r_1, r_2, dr_3)$$

for all $r_1, r_2, r_3 \in \mathcal{R}$. Which means, $[\text{Jac}, d]_{\text{RN}}(r_1, r_2, r_3) = 0$ for all $r_1, r_2, r_3 \in \mathcal{R}$.

Thus, $D(\text{Jac}) = 0$. It follows that, Jac is a D -coboundary, there exists an element $\ell_3 = \sum_{j \geq 2} \ell_3^j \in \widehat{\mathfrak{Page}}_1^{(2)}(\mathcal{R})$ with $\ell_3^j \in \text{Hom}(\odot^3 \mathcal{R}_{|-j-1}, \mathcal{R}_{-j})$ such that

$$D(\ell_3) = -\text{Jac}. \quad (8.43)$$

We choose the 3-ary bracket to be ℓ_3 . For degree reason, the remaining terms of the k -ary brackets for $k \geq 3$ have trivial components on the column -1 of the bicomplex (8.39). From this point, the proof continues exactly as in Section 4.3.2. \square

Example 8.4.6. We return to Example 8.1.8. In that case, the Lie algebra $\mathfrak{g} = \mathfrak{X}(L)$ that acts on the singular foliation \mathcal{T} . In that case, we have $\varrho(\mathfrak{g}) \cap T_m \mathcal{T} = \{0\}$ for all m in $[L, M]$. We can apply directly Theorem 4.2.1, to obtain a Lie ∞ -algebroid structure on the complex

$$\dots \xrightarrow{d} \Gamma(E_{-3}) \xrightarrow{d} \Gamma(E_{-2}) \xrightarrow{d} \mathfrak{g} \oplus \Gamma(E_{-1}) \xrightarrow{\rho'} \mathfrak{X}([L, M]). \quad (8.44)$$

Notice that here the Lie algebra \mathfrak{g} is infinite dimensional, therefore we are not allowed to use the duality between Lie ∞ -algebroid and Q -manifold. Therefore, we cannot use the explicit Formula (8.32) to define at lift Φ . However, we have to rely on the existence theorem 8.3.1 to assure the existence of a lift Φ .

Conclusion:

We show that actions of a Lie algebra \mathfrak{g} on the leaf space M/\mathfrak{F} i.e. weak symmetry actions, lift to Lie ∞ -algebra morphisms $\mathfrak{g} \rightsquigarrow \mathfrak{X}(E)$ on the DGLA of vector fields on an universal Lie ∞ -algebroid (E, Q) , provided it exists, in a unique up to homotopy manner.

We explain how to use chapter 4 to get rid of all finite ranks/dimensions assumptions, using the universal Lie ∞ -algebroids of Lie-Rinehart algebras.

On weak and strict symmetries: an obstruction theory

In this chapter, we apply the main theorems of 8.2 of Chapter 8 to define a class obstructing the existence of strict symmetry action equivalent to a given weak symmetry action.

9.1 Introduction

Recall that Theorem 8.3.1 assures that any weak symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of a Lie algebra \mathfrak{g} on a singular foliation \mathfrak{F} admits a lift to a Lie ∞ -morphism

$$\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q). \quad (9.1)$$

To understand the compatibility conditions between the low terms of Φ see Remark 8.3.3. By playing with the definition of Φ we make the following observation.

Remark 9.1.1. What does Φ induces on the linear part of (E, Q) ? We have seen in Lemma 8.2.5 and Proposition 8.2.10 that the 0-th Taylor coefficient of the lift Φ induces a linear map $x \in \mathfrak{g} \mapsto (\nabla_x: E_{-i} \rightarrow E_{-i})$ for every $i \geq 1$ that satisfies

$$\nabla_x(fe) = f\nabla_x(e) + \varrho(x)[f]e, \text{ for } f \in \mathcal{O}, e \in \Gamma(E).$$

1. Also, for every $x \in \mathfrak{g}$ and $e \in \Gamma(E_{-1})$,

$$\rho(\nabla_x(e)) = [\varrho(x), \rho(e)].$$

2. A direct computation gives

$$\begin{aligned}
0 &= [[Q, \Phi_0(x)], \iota_e]^{(-1)} = [Q, [\Phi_0(x), \iota_e]]^{(-1)} - [\Phi_0(x), [Q, \iota_e]]^{(-1)} \\
&= \left[Q^{(0)}, [\Phi_0(x), \iota_e]^{(-1)} \right] - \left[\Phi_0(x)^{(0)}, [Q, \iota_e]^{(-1)} \right] \\
&= \left[Q^{(0)}, \iota_{\nabla_x(e)} \right] - \left[\Phi_0(x)^{(0)}, \iota_{\ell_1(e)} \right] \\
&= \iota_{\ell_1 \circ \nabla_x(e)} - \iota_{\nabla_x \circ \ell_1(e)}
\end{aligned}$$

We have used graded Jacobi identity and the dual correspondence between Lie ∞ -algebroids and NQ -manifolds (see Proposition 6.1.16). We recapitulate in the following commutative diagram:

$$\begin{array}{ccccc}
\cdots & \xrightarrow{d} & \Gamma(E_{-2}) & \xrightarrow{d} & \Gamma(E_{-1}) & \xrightarrow{\rho} & \mathfrak{F} \\
& & \downarrow \nabla_x & & \downarrow \nabla_x & & \downarrow \text{ad}_{\varrho(x)} \\
\cdots & \xrightarrow{d} & \Gamma(E_{-2}) & \xrightarrow{d} & \Gamma(E_{-1}) & \xrightarrow{\rho} & \mathfrak{F}
\end{array} \tag{9.2}$$

which means,

$$\ell_1 \circ \nabla_x = \nabla_x \circ \ell_1 \quad \text{and} \quad \rho \circ \nabla_x = \text{ad}_{\varrho(x)} \circ \rho.$$

Here, ℓ_1 stands for the corresponding unary bracket of (E, Q) . Also, for $X \in \mathfrak{X}(M)$, $\text{ad}_X := [X, \cdot]$.

3. For $x, y \in \mathfrak{g}$, and $e \in \Gamma(E)$, the relation (8.14) yields

$$\begin{aligned}
&[\Phi_0([x, y]_{\mathfrak{g}}) - [\Phi_0(x), \Phi_0(y)], \iota_e]^{(-1)} = [[Q, \Phi_1(x, y)], \iota_e]^{(-1)} \\
&\iota_{\nabla_{[x, y]_{\mathfrak{g}}}(e)} - [\Phi_0(x), [\Phi_0(y), \iota_e]]^{(-1)} + [\Phi_0(y), [\Phi_0(x), \iota_e]]^{(-1)} = \quad " \\
&\iota_{\nabla_{[x, y]_{\mathfrak{g}}}(e)} - \left[\Phi_0(x)^{(0)}, \iota_{\nabla_y(e)} \right] + \left[\Phi_0(y)^{(0)}, \iota_{\nabla_x(e)} \right] = [[Q, \Phi_1(x, y)], \iota_e]^{(-1)}.
\end{aligned}$$

Thus

$$[[Q, \Phi_1(x, y)], \iota_e]^{(-1)} = \iota_{\nabla_{[x, y]_{\mathfrak{g}}}(e)} - \iota_{\nabla_x \circ \nabla_y(e)} + \iota_{\nabla_y \circ \nabla_x(e)}. \tag{9.3}$$

On the other hand, $\Phi_1(x, y)$ admits a polynomial decomposition

$$\Phi_1(x, y)^{(-1)} + \sum_{i \geq 0} \Phi_1(x, y)^{(i)} = \iota_{\eta(x, y)} + \sum_{i \geq 0} \Phi_1(x, y)^{(i)}$$

and that

$$\left[\sum_{i \geq 0} \Phi_1(x, y)^{(i)}, \iota_e \right]^{(-1)} = \left[\Phi_1(x, y)^{(0)}, \iota_e \right] = \iota_{\gamma(x, y)(e)},$$

for some linear map $\gamma(x, y): \Gamma(E_{-\bullet}) \longrightarrow \Gamma(E_{|e|_1})$ depending linearly on x, y . Therefore,

$$\begin{aligned}
[[Q, \Phi_1(x, y)], \iota_e]^{(-1)} &= [[Q, \iota_{\eta(x, y)}], \iota_e]^{(-1)} + \left[\left[Q, \sum_{i \geq 0} \Phi_1(x, y)^{(i)} \right], \iota_e \right]^{(-1)} \\
&= \iota_{\ell_2(\eta(x, y), e)} + \left[Q, \left[\sum_{i \geq 0} \Phi_1(x, y)^{(i)}, \iota_e \right] \right]^{(-1)} - \left[\sum_{i \geq 0} \Phi_1(x, y)^{(i)}, [Q, \iota_e] \right]^{(-1)} \\
&= \iota_{\ell_2(\eta(x, y), e)} + \left[Q^{(0)}, \left[\sum_{i \geq 0} \Phi_1(x, y)^{(i)}, \iota_e \right] \right]^{(-1)} - [\Phi_1(x, y)^{(0)}, [Q, \iota_e]^{(-1)}] \\
&= \iota_{\ell_2(\eta(x, y), e)} + [Q^{(0)}, \iota_{\gamma(x, y)(e)}] - [\Phi_1(x, y)^{(0)}, \iota_{\ell_1(e)}] \\
&= \iota_{\ell_2(\eta(x, y), e)} + \iota_{\ell_1(\gamma(x, y)(e))} - \iota_{\gamma(x, y)(\ell_1(e))}
\end{aligned}$$

By equating the latter with (9.3) we can recapitulate as follows:

Conclusion: In general, the map $\mathfrak{g} \longrightarrow \text{Der}(E)$, $x \mapsto \nabla_x$ is not a Lie algebra morphism even when the action ϱ is strict. In fact, there exists a bilinear map $\gamma: \wedge^2 \mathfrak{g} \longrightarrow \text{End}(E)[1]$ of degree 0 that satisfies

$$\nabla_{[x, y]_{\mathfrak{g}}} - [\nabla_x, \nabla_y] = \gamma(x, y) \circ \ell_1 - \ell_1 \circ \gamma(x, y) + \ell_2(\eta(x, y), \cdot),$$

here ℓ_2 is the corresponding 2-ary bracket of (E, Q) , and $\eta: \wedge^2 \mathfrak{g} \longrightarrow \Gamma(E_{-1})$ is such that $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y))$.

4. Also, Equation (8.15) taken in polynomial-degree -1 implies, that, for all $\alpha \in \Gamma(E_{-1}^*)$

$$\Phi_1([x, y]_{\mathfrak{g}}, z)^{(-1)} - [\Phi_0(x)^{(0)}, \Phi_1(y, z)^{(-1)}] + \mathfrak{O}(x, y, z) = [Q^{(0)}, \Phi_2(x, y, z)^{(-1)}].$$

$$\implies \iota_{\eta([x, y]_{\mathfrak{g}}, z)} - [\Phi_0(x)^{(0)}, \iota_{\eta(y, z)}] + \mathfrak{O}(x, y, z) = [Q^{(0)}, \iota_{\zeta(x, y, z)}], \quad \text{with } \zeta: \wedge^3 \mathfrak{g} \rightarrow \Gamma(E_{-2})$$

$$\implies \langle \alpha, \eta([x, y]_{\mathfrak{g}}, z) \rangle - \left(\Phi_0(x)^{(0)} [\langle \alpha, \eta(y, z) \rangle] - \langle \Phi_0(x)^{(0)}(\alpha), \eta(y, z) \rangle \right) + \mathfrak{O} = \langle Q^{(0)}[\alpha], \zeta(x, y, z) \rangle,$$

\implies

$$\langle \alpha, \eta([x, y]_{\mathfrak{g}}, z) \rangle - \left(\underline{\varrho(x)}[\langle \alpha, \eta(y, z) \rangle] - \underline{\varrho(x)}[\langle \alpha, \eta(y, z) \rangle] + \langle \alpha, \nabla_x \eta(y, z) \rangle \right) + \mathfrak{O}(x, y, z) = \langle \alpha, \ell_1(\zeta(x, y, z)) \rangle.$$

We have used Equations (8.13) and (8.9) in the last line. Hence,

$$\langle \alpha, \eta([x, y]_{\mathfrak{g}}, z) - \nabla_x \eta(y, z) \rangle + \mathfrak{O}(x, y, z) = \langle \alpha, \ell_1(\zeta(x, y, z)) \rangle.$$

Since α is arbitrary, one obtains

$$\nabla_x \eta(y, z) - \eta([x, y]_{\mathfrak{g}}, z) + \mathfrak{O}(x, y, z) = \ell_1(\zeta(x, y, z)). \quad (9.4)$$

In particular, if $m \in M$ is such that $\ell_1|_m = 0$ and $\ell_2|_m = 0$, then the map $x \mapsto \nabla_x$ defines an action on the isotropy Lie algebra \mathfrak{g}_m of \mathfrak{F} , since ∇_x preserves the kernel of ρ . If in addition $\eta(x, y) \in \ker \rho_m$, then $\eta|_m: \wedge^2 \mathfrak{g} \longrightarrow \mathfrak{g}_m$ is a cocycle of Chevalley-Eilenberg.

Remark 9.1.2. Notice that by using the duality which is given in Proposition 8.4.1, the linear map $x \mapsto \nabla_x$ is simply

$$x \mapsto \ell'_2(x, \cdot),$$

where ℓ'_2 is the 2-ary bracket between sections of \mathfrak{g} and E of the Lie ∞ -algebroid $(Q', E \oplus \mathfrak{g})$ over the complex

$$\cdots \xrightarrow{\ell_1} E_{-3} \xrightarrow{\ell_1} E_{-2} \xrightarrow{\ell_1} \mathfrak{g} \oplus E_{-1} \xrightarrow{\rho'} TM \quad (9.5)$$

like in Proposition 8.4.1. Indeed, for $\alpha \in \Gamma(E_{-1}^*)$ and $e \in \Gamma(E_{-1})$,

$$\begin{aligned} \Phi_0(x)^{(0)} &= \text{pr} \circ [Q', \iota_x]^{(0)} \\ &= \text{pr} \circ [Q'^{(1)}, \iota_x]. \end{aligned}$$

This implies that:

$$\begin{aligned} \langle \Phi_0(x)^{(0)} \alpha, e \rangle &= \langle Q'^{(1)} \alpha, x \odot e \rangle \\ &= \rho'(x)[\langle \alpha, e \rangle] - \cancel{\rho'(e)[\langle \alpha, x \rangle]} - \langle \alpha, \ell'_2(x, e) \rangle \\ &= \varrho(x)[\langle \alpha, e \rangle] - \langle \alpha, \ell'_2(x, e) \rangle. \end{aligned}$$

9.2 An obstruction theory

Let us start with some generalities. Assume we are given

- a Lie algebra \mathfrak{g} ,
- a weak symmetry action $\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(M)$ of \mathfrak{g} on a singular foliation \mathfrak{F} , together with $\eta: \wedge^2 \mathfrak{g} \longrightarrow \Gamma(E_{-1})$ such that $x, y \in \mathfrak{g}$

$$\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] = \rho(\eta(x, y)). \quad (9.6)$$

- an universal Lie ∞ -algebroid (E, Q_E) of \mathfrak{F} ,

Theorem 8.3.1 assures $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ admits a lift to a Lie ∞ -morphism

$$\Phi: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \rightsquigarrow (\mathfrak{X}_{\bullet}(E)[1], [\cdot, \cdot], \text{ad}_Q). \quad (9.7)$$

Equivalently, if \mathfrak{g} is of finite dimension, (9.7) corresponds (by Proposition 8.4.1) to a Lie ∞ -algebroid (E', Q') over M such that

- (E, Q_E) is included as a sub-Lie ∞ -algebroid in a Lie algebroid (E', Q) over M ,
- its underlying complex is, $E'_{-1} := \mathfrak{g} \oplus E_{-1}$, and for any $i \geq 2$, $E'_{-i} = E_{-i}$, namely

$$\cdots \xrightarrow{d} E_{-3} \xrightarrow{d} E_{-2} \xrightarrow{d} \mathfrak{g} \oplus E_{-1} \xrightarrow{\rho'} TM, \quad (9.8)$$

- we have,

$$\ell'_2(x \oplus 0, y \oplus 0) = [x, y]_{\mathfrak{g}} \oplus \eta(x, y)$$

and

$$\ell'_2(x, \Gamma(E_{-1})) \subset \Gamma(E_{-1})$$

for all $x \in \mathfrak{g}$.

Remark 9.2.1. It is important to note that the Lie ∞ -algebroid (E', Q') can be constructed directly out of the weak symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$, even if \mathfrak{g} is of infinite dimension (see Theorem 8.4.4).

Remark 9.2.2. In Equation (9.8), the complex (E, ℓ_1) can be chosen to be minimal at a point $m \in M$, i.e., $\ell_1|_m = 0$, provided that a geometric resolution of \mathcal{F} exists. By Proposition 4.14 in [LLS20] the isotropy Lie algebra $\mathfrak{g}_m = \frac{\mathcal{F}(m)}{\mathcal{I}_m \mathcal{F}}$ of the singular foliation \mathcal{F} at the point $m \in M$ is isomorphic to $\ker(\rho_m)$. This allows to denote the latter space also by \mathfrak{g}_m .

In what follows, we will use the ∞ -algebroid which is described on the complex (9.8).

Lemma 9.2.3. *Let $m \in M$ be a fixed point of the \mathfrak{g} -action ϱ . Assume that the underlying complex (E, ℓ_1) is minimal at a point m , i.e. $\ell_1|_m = 0$. The map*

$$\nu: \mathfrak{g} \rightarrow \text{End}(\mathfrak{g}_m), \quad x \mapsto \ell'_2(x, \cdot)|_m$$

satisfies

$$(a) \quad \nu([x, y]_{\mathfrak{g}}) - [\nu(x), \nu(y)] + \ell_2(\cdot, \eta(x, y))|_m = 0,$$

$$(b) \quad \nu(z) (\eta(x, y)|_m) - \eta([x, y]_{\mathfrak{g}}, z)|_m + \mathfrak{O}(x, y, z) = 0.$$

Proof. The map ν is well-defined. The Jacobi identity on elements $x, y \in \mathfrak{g}$, $e \in \Gamma(E_{-1})$, evaluated at the point m , implies that

$$\nu([x, y]_{\mathfrak{g}})(e|_m) - [\nu(x), \nu(y)](e|_m) + \ell_2(\eta(x, y), e)|_m = 0.$$

This proves item (a). Likewise, Jacobi identity on elements $x, y, z \in \mathfrak{g}$ and since $\ell_1|_m = 0$ give:

$$\begin{aligned} \ell'_2(\ell'_2(x, y), z)|_m + \mathfrak{O}(x, y, z) = 0 &\implies \ell'_2([x, y]_{\mathfrak{g}}, z)|_m + \ell'_2(\eta(x, y), z)|_m + \mathfrak{O}(x, y, z) = 0, \\ &\implies \nu(z) (\eta(x, y)|_m) - \eta([x, y]_{\mathfrak{g}}, z)|_m + \mathfrak{O}(x, y, z) = 0. \end{aligned}$$

Here we have used the definition of ℓ'_2 on degree -1 elements and Jacobi identity for the bracket $[\cdot, \cdot]_{\mathfrak{g}}$. This proves item (b). \square

By Lemma 9.2.3, \mathfrak{g}_m is equipped with a \mathfrak{g} -module structure when $\eta(x, y)|_m$ is for all $x, y \in \mathfrak{g}$ valued in the center the Lie algebra \mathfrak{g}_m . Recall that η is defined by Equation (9.6). Notice that if $m \in M$ is a fixed point of the \mathfrak{g} -action ϱ , then this implies in particular that $\eta(x, y)|_m \in \ker \rho_m = \mathfrak{g}_m$.

Proposition 9.2.4. *Let $m \in M$ be a fixed point of the \mathfrak{g} -action ϱ . Assume that*

- the underlying complex (E, ℓ_1) of (E, Q) is minimal at m ,
- for all $x, y \in \mathfrak{g}$, $\eta(x, y)|_m$ is valued in the center¹ $Z(\mathfrak{g}_m)$ of \mathfrak{g}_m .

¹In particular, when the 2-ary bracket ℓ_2 is zero at m , on elements of degree -1 we have, $Z(\mathfrak{g}_m) = \mathfrak{g}_m$.

Then,

1. the restriction of the 2-ary bracket

$$\ell'_2: \mathfrak{g} \otimes Z(\mathfrak{g}_m) \longrightarrow Z(\mathfrak{g}_m)$$

endows $Z(\mathfrak{g}_m)$ with a \mathfrak{g} -module structure which does not depend neither on the choices of weak symmetry action ϱ , a universal Lie ∞ -algebroid of \mathfrak{F} , nor of the Lie ∞ -morphism $\Phi: \mathfrak{g} \longrightarrow \mathfrak{X}(E)$.

2. the restriction of the map $\eta: \wedge^2 \mathfrak{g} \longrightarrow \Gamma(E_{-1})$ at m

$$\eta|_m: \wedge^2 \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$$

is a 2-cocycle for the Chevalley-Eilenberg complex of \mathfrak{g} valued in $Z(\mathfrak{g}_m)$,

3. the cohomology class of this cocycle does not depend on the representatives of the equivalence class of ϱ ,
4. if ϱ is equivalent to a strict symmetry action, then $\eta|_m$ is exact.

Proof. (E, ℓ_1) being minimal at m , $\ell'_2|_m$ satisfies the Jacobi identity. In particular, for every $x \in \mathfrak{g}[1]$, $\ell'_2(x, \cdot)|_m$ preserves $Z(\mathfrak{g}_m)$. By item (a) of Lemma 9.2.3, the restriction of the 2-ary bracket

$$\ell'_2: \mathfrak{g} \otimes Z(\mathfrak{g}_m) \longrightarrow Z(\mathfrak{g}_m)$$

endows $Z(\mathfrak{g}_m)$ with a \mathfrak{g} -module, since $\ell_2(\cdot, \eta(x, y))|_m = 0$ by assumption. It is easy to see that if we change the action ϱ to $\varrho + \rho \circ \beta$ for some vector bundle morphism $\beta: \mathfrak{g} \longrightarrow E_{-1}$ such that $\beta|_m: \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$, the new 2-ary bracket between sections of $\mathfrak{g}[1]$ and E_{-1} constructed as in the proof of Theorem 8.4.4 is modified by $(x, e) \mapsto \ell'_2(x, e) + \ell_2(\beta(x), e)$. The second term of the latter vanishes at m , by definition of $\beta|_m$. As a result, the action of ℓ'_2 on $Z(\mathfrak{g}_m)$ does not depend on the choices made in the construction. This proves item 1.

Item 2 follows from item (b) of Lemma 9.2.3 that tells that $\eta|_m: \wedge^2 \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$ is a 2-cocycle for the Chevalley-Eilenberg complex of \mathfrak{g} valued in $Z(\mathfrak{g}_m)$.

Let ϱ' be a weak symmetry action of \mathfrak{g} on \mathcal{F} which is equivalent to ϱ , i.e., there exists a vector bundle morphism $\beta: \mathfrak{g} \longrightarrow E_{-1}$ (with $\beta|_m: \mathfrak{g} \longrightarrow Z(\mathfrak{g}_m)$) such that $\varrho'(x) = \varrho(x) + \rho(\beta(x))$ for all $x \in \mathfrak{g}$. Let $\eta': \wedge^2 \mathfrak{g} \longrightarrow E_{-1}$ be such that $\varrho'([x, y]_{\mathfrak{g}}) - [\varrho'(x), \varrho'(y)] = \rho(\eta'(x, y))$ for all $x, y \in \mathfrak{g}$. Following the constructions in the proof of Theorem 8.4.4, this implies that

$$\eta'(x, y) = \eta(x, y) + \beta([x, y]_{\mathfrak{g}}) - \ell'_2(x, \beta(y)) + \ell'_2(y, \beta(x)) - \ell_2(\beta(x), \beta(y)), \quad \text{for all } x, y \in \mathfrak{g}. \quad (9.9)$$

Equation (9.9) implies that $\eta'(x, y)|_m - \eta(x, y)|_m = d^{CE}(\beta|_m)(x, y)$, where d^{CE} stands for the Chevalley-Eilenberg differential. As a consequence, $\eta'|_m$ and $\eta|_m$ define the same class in the Chevalley-Eilenberg complex of \mathfrak{g} valued in $Z(\mathfrak{g}_m)$. This proves item 3 and 4. \square

Remark 9.2.5. When $\ell_2|_m \neq 0$. The weak symmetry action ϱ is equivalent to strict one if the Maurer-Cartan-like equation (9.9) has no solution with $\eta'|_m = 0$.

Let \mathfrak{F} be a singular foliation. Let us choose a universal Lie ∞ -algebroid (E, Q) such that (E, ℓ_1) is minimal at a point $m \in M$. Such a structure always exists (see Proposition B.2.14). By Proposition 4.14 in [LLS20] the isotropy Lie algebra \mathfrak{g}_m of the singular foliation \mathfrak{F} at the point $m \in M$ is isomorphic to $\ker(\rho_m)$. The following is a direct consequence of Proposition 9.2.4.

Corollary 9.2.6. *Let $m \in M$ be a fixed point of the \mathfrak{g} -action ϱ . Assume that the isotropy Lie algebra \mathfrak{g}_m of \mathfrak{F} at m is Abelian. Then, for any weak symmetry action ϱ of a Lie algebra action \mathfrak{g} on \mathfrak{F} such that $\varrho([x, y]_{\mathfrak{g}}) - [\varrho(x), \varrho(y)] \in \mathfrak{F}(m)$ for all $x, y \in \mathfrak{g}$*

1. \mathfrak{g}_m is a \mathfrak{g} -module.
2. The bilinear map, $\eta|_m : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}_m$, is a Chevalley-Eilenberg 2-cocycle of \mathfrak{g} valued in \mathfrak{g}_m .
3. Its class $\text{cl}(\eta) \in H^2(\mathfrak{g}, \mathfrak{g}_m)$ does not depend on the choices made in the construction.
4. Furthermore, $\text{cl}(\eta)$ is an obstruction of having a strict symmetry action equivalent to ϱ .

Example 9.2.7. We return to Example 8.1.6 with $m \in M$ a leaf of \mathfrak{F} . Since the isotropy Lie algebra \mathfrak{g}_m^k is Abelian for every $k \geq 2$ the following assertions hold by Corollary 9.2.6:

1. For each $k \geq 1$, the vector space \mathfrak{g}_m^{k+1} is a \mathfrak{g}_m^k -module.
2. The obstruction of having a strict symmetry action equivalent to ϱ_k is a Chevalley-Eilenberg cocycle valued in \mathfrak{g}_m^{k+1} .

Here is a particular case of this example.

Example 9.2.8. Let $\mathfrak{F} := \mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)$ be the singular foliation generated by vector fields vanishing to order 3 at the origin. The quotient $\mathfrak{g} := \frac{\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)}{\mathcal{I}_0^3 \mathfrak{X}(\mathbb{R}^n)}$ is a trivial Lie algebra. There is a weak symmetry action of \mathfrak{g} on \mathfrak{F} which assigns to an element in \mathfrak{g} a representative in $\mathcal{I}_0^2 \mathfrak{X}(\mathbb{R}^n)$. In this case, the isotropy Lie algebra of \mathfrak{F} at zero is Abelian and $\ell'_2(\mathfrak{g}, \mathfrak{g}_0)|_0 = 0$. Thus, the action of \mathfrak{g} on \mathfrak{g}_0 is trivial. One can choose $\eta : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}_0$ such that $\eta \left(x_i^2 \frac{\partial}{\partial x_i}, x_j^2 \frac{\partial}{\partial x_j} \right) = 2e_{ij}$, with e_{ij} a constant section in a set of generators of degree -1 whose image by the anchor is $x_i^3 \frac{\partial}{\partial x_j}$. Therefore, $\eta|_0 \left(x_i^2 \frac{\partial}{\partial x_i}, x_j^2 \frac{\partial}{\partial x_j} \right) \neq 0$. This implies that the class of η is not zero at the origin. Therefore, by item 2 of Corollary 9.2.6 the weak symmetry action of \mathfrak{g} on \mathfrak{F} is not equivalent to a strict one.

Example 9.2.9. Consider again the Example 8.4.6 and let $(E, \ell_{\bullet}, \rho)$ be a universal Lie ∞ -algebroid of \mathcal{T} . Recall that a *flat Ehresmann* connection is a horizontal distribution whose sections are closed under the Lie bracket of vector fields. Let $m \in M$. Assume in Example 8.1.8 that the section $\varrho_H : \mathfrak{X}(L) \rightarrow \mathfrak{F}^{\text{proj}}$ satisfies

$$\varrho_H([a, b]) - [\varrho_H(a), \varrho_H(b)] = \rho(\eta(a, b)) \in \mathcal{T}(m)$$

for some bilinear map $\eta : \wedge^2 \mathfrak{X}(L) \rightarrow E_{-1}$. By Proposition 9.2.6, the isotropy Lie algebra $\mathfrak{g}_m^{\mathcal{T}}$ of \mathcal{T} at m is a $\mathfrak{X}(L)$ -module, and η is a cocycle of Chevalley-Eilenberg. This provides an obstruction for the \mathcal{F} -connection to be flat.

Also, we have the following consequence of Corollary 9.2.6 for Lie algebra actions on affine varieties, as in Example 8.1.11. Before going to Corollary 9.2.13 let us write definitions and some facts.

Settings: Let W be an affine variety realized as a subvariety of \mathbb{C}^d , and defined by some ideal $\mathcal{I}_W \subset \mathbb{C}[x_1, \dots, x_d]$. We denote by $\mathfrak{X}(W) := \text{Der}(\mathcal{O}_W)$ the Lie algebra of vector fields on W , where \mathcal{O}_W is coordinates ring of W .

Definition 9.2.10. A point $p \in W$ is said to be *strongly singular* if for all $f \in \mathcal{I}_W$, $d_p f \equiv 0$ or equivalently if for all $f \in \mathcal{I}_W$ and $X \in \mathfrak{X}(\mathbb{C}^d)$, one has $X[f](p) \in \mathcal{I}_p$.

Example 9.2.11. Any singular point of a hypersurface W defined by a polynomial $\varphi \in \mathbb{C}[x_1, \dots, x_d]$ is strongly singular.

The lemma below is immediate.

Lemma 9.2.12. *In a strongly singular point, the isotropy Lie algebra of the singular foliation $\mathfrak{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$ is Abelian.*

Corollary 9.2.13. *Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(W)$ be a Lie algebra morphism.*

1. Any extension $\tilde{\varrho}$ as in Example 8.1.11 is a weak symmetry action for the singular foliation $\mathfrak{F} = \mathcal{I}_W \mathfrak{X}(\mathbb{C}^d)$.
2. For any strongly singular point p in W if the class $cl(\eta)$ does not vanish the strict action $\varrho: \mathfrak{g} \rightarrow \text{Der}(\mathcal{O}_W)$ can not be extended to the ambient space.

Let us give an example of a Lie algebra action on an affine variety that do not extend to the ambient space.

We hope to construct an example as follows

Example 9.2.14. Let $W \subset \mathbb{C}^d$ be an affine variety generated by a regular homogeneous polynomial $\varphi \in \mathcal{O} = \mathbb{C}[x_1, \dots, x_d]$ of degree ≥ 2 . Assume there exists two vector fields $X, Y \in \mathfrak{X}(\mathbb{C}^d)$ that satisfy $X[\varphi] = f\varphi$, $Y[\varphi] = g\varphi$, with $f, g \in \mathcal{I}_0$, and such that $[X, Y] = \varphi Z$, for some $Z \in \mathfrak{X}(\mathbb{C}^d)$. Consider the action of the trivial $\mathfrak{g} = \mathbb{R}^2$ on W that sends its canonical basis to X , and Y respectively. It is a weak symmetry action on the singular foliation $\mathfrak{F}^\varphi := \langle \varphi \rangle \mathfrak{X}(\mathbb{C}^d)$ and induces a Lie algebra map, $\mathfrak{g} \rightarrow \mathfrak{X}(W)$.

Notice that the universal Lie algebroid of \mathfrak{F}^φ is a Lie algebroid (see Example of [LGL22b]) because,

$$0 \longrightarrow \mathcal{O}\mu \otimes_{\mathcal{O}} \mathfrak{X}(\mathbb{C}^d) \xrightarrow{\varphi \frac{\partial}{\partial \mu} \otimes_{\mathcal{O}} \text{id}} \mathfrak{F}^\varphi.$$

is a \mathcal{O} -module isomorphism. Here μ is a degree -1 variable, so that $\mu^2 = 0$. The universal algebroid structure over that resolution is given on the set of generators by:

$$\ell_2 \left(\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x_a}, \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x_b} \right) := \frac{\partial \varphi}{\partial x_a} \mu \otimes_{\mathcal{O}} \partial_{x_b} - \frac{\partial \varphi}{\partial x_b} \mu \otimes_{\mathcal{O}} \partial_{x_a}$$

and $\ell_k := 0$ for every $k \geq 3$. Write $Z = \sum_{i=1}^d f_i \frac{\partial}{\partial x_i}$, with $(f_i)_{i=1, \dots, d} \subset \mathcal{O}$. We have,

$$\eta(e_1, e_2) := \sum_{i=1}^d f_i \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x_i}.$$

where e_1, e_2 is the canonical basis of \mathbb{R}^2 .

This Lie ∞ -algebroid structure satisfies all the assumptions of Proposition 9.2.6. Assume that the vector field Z does not vanish at zero. Since the action is trivial at zero and $\eta|_0 \neq 0$, therefore its class is non-zero. By consequence, such action cannot be extended to ambient space.

Let us make it explicit

Example 9.2.15. Let $W \subset \mathbb{C}^2$ be the affine variety generated by the polynomial $\varphi = FG$ with $F, G \in \mathbb{C}[x, y] =: \mathcal{O}$. We consider the vector fields $U = F\mathcal{X}_G, V = G\mathcal{X}_F \in \mathfrak{X}(\mathbb{C}^2)$, where \mathcal{X}_F and \mathcal{X}_G are Hamiltonian vector fields w.r.t the Poisson structure $\{x, y\} := 1$. Note that U, V are tangent to W , i.e. $U[\varphi], V[\varphi] \in \langle \varphi \rangle$. It is easily checked that $[U, V] = \varphi\mathcal{X}_{\{F, G\}}$.

The action of the trivial Lie algebra $\mathfrak{g} = \mathbb{R}^2$ on W that sends its canonical basis (e_1, e_2) to U , and V respectively, is a weak symmetry action on the singular foliation $\mathfrak{F}^\varphi := \langle \varphi \rangle \mathfrak{X}(\mathbb{C}^2)$, and induces a Lie algebra map,

$$\varrho: \mathfrak{g} \longrightarrow \mathfrak{X}(W). \quad (9.10)$$

A universal Lie algebroid of \mathfrak{F}^φ is a Lie algebroid (see Example 3.19 of [LGL22b]) because,

$$0 \longrightarrow \mathcal{O}\mu \otimes_{\mathcal{O}} \mathfrak{X}(\mathbb{C}^2) \xrightarrow{\varphi \frac{\partial}{\partial \mu} \otimes \text{id}} \mathfrak{F}^\varphi$$

is a \mathcal{O} -module isomorphism. Here μ is a degree -1 variable, so that $\mu^2 = 0$. The universal algebroid structure over that resolution is given on the set of generators by:

$$\ell_2 \left(\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}, \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y} \right) := \frac{\partial \varphi}{\partial x} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y} - \frac{\partial \varphi}{\partial y} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x} \quad (9.11)$$

and $\ell_k := 0$ for every $k \geq 3$. Write $\mathcal{X}_{\{F, G\}} = \frac{\partial \{F, G\}}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \{F, G\}}{\partial x} \frac{\partial}{\partial y}$. Therefore, we can put

$$\eta(e_1, e_2) := \frac{\partial \{F, G\}}{\partial y} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x} - \frac{\partial \{F, G\}}{\partial x} \mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y}. \quad (9.12)$$

Take for example, $F(x, y) = y - x^2$ and $G(x, y) = y + x^2$. The isotropy Lie algebra $\mathfrak{g}_{(0,0)}$ of \mathfrak{F}^φ is Abelian, since zero is a strong singular point of W . By Corollary 9.2.6 (1), $\mathfrak{g}_{(0,0)}$ is a \mathbb{R}^2 -module. A direct computation shows that the action on $\mathfrak{g}_{(0,0)}$ is not trivial, but takes value in $\mathcal{O}\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}$. Besides, Equation (9.12) applied to $\{F, G\} = 4x$ gives

$$\eta(e_1, e_2) = -4\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial y}. \quad (9.13)$$

If $\eta|_{(0,0)}$ were a coboundary of Chevalley Eilenberg, we would have (in the notations of Proposition 9.2.4) that

$$\eta(x, y)|_{(0,0)} = \beta([x, y]_{\mathbb{R}^2}) - \ell'_2(x, \beta(y)) + \ell'_2(y, \beta(x)) \in \mathcal{O}\mu \otimes_{\mathcal{O}} \frac{\partial}{\partial x}, \quad \text{for all } x, y \in \mathfrak{g} \quad (9.14)$$

for some linear map $\beta: \mathfrak{g} \longrightarrow \mathfrak{g}_{(0,0)}$. Therefore, Equation (9.14) is impossible by Equation (9.13) and since $\eta|_{(0,0)} \neq 0$. In other words, its class $\text{cl}(\eta)$ does not vanish. By Corollary 9.2.13 (2), the action ϱ given in Equation (9.10) cannot be extended to ambient space.

Conclusion:

We applied the existence of a Lie ∞ -morphism to the question of lifting to M an \mathfrak{g} -action on M/\mathfrak{F} , i.e. of making strict a weak \mathfrak{g} -action.

We apply this question to several geometric issues, e.g. neighbourhood of leaves, Lie algebra actions on an affine variety.

10.1 Symmetries of bi-submersions

In this chapter, we introduce the notion “bi-submersion towers”. The work contained in this chapter is entirely original, except for the notion below that arose in a discussion between C. Laurent-Gengoux, L. Ryvkin, and I, and will be the object of a separate study.

Let us firstly recall the definition of bi-submersion. The concept of bi-submersion over singular foliations has been introduced in [AS09].

Definition 10.1.1. Let M be a manifold endowed with a singular foliation \mathfrak{F} . A *bi-submersion*

$B \begin{matrix} \xrightarrow{s} \\ \xleftarrow{t} \end{matrix} M$ over \mathfrak{F} is a triple (B, s, t) where:

- B is a manifold,
- $s, t: B \rightarrow M$ are submersions, respectively called *source* and *target*,

such that the pull-back singular foliations $s^{-1}\mathfrak{F}$ and $t^{-1}\mathfrak{F}$ are both equal to the space of vector fields of the form $\xi + \zeta$ with $\xi \in \Gamma(\ker(ds))$ and $\zeta \in \Gamma(\ker(dt))$. Namely,

$$s^{-1}\mathfrak{F} = t^{-1}\mathfrak{F} = \Gamma(\ker(ds)) + \Gamma(\ker(dt)). \quad (10.1)$$

In that case, we also say that (B, s, t) is a bi-submersion over (M, \mathfrak{F}) .

Example 10.1.2. Let \mathfrak{F} be a singular foliation over a manifold M . For $x \in M$ and $X_1, \dots, X_n \in \mathfrak{F}$ inducing generators for $\mathfrak{F}_x := \mathfrak{F}/\mathcal{I}_x\mathfrak{F}$. We know from [AS09] that there is an open neighborhood \mathcal{W} of $(x, 0) \in M \times \mathbb{R}^n$ such that (\mathcal{W}, t, s) is a bi-submersion over \mathfrak{F} , where

$$s(x, y) = x \quad \text{and} \quad t(x, y) = \exp_x \left(\sum_{i=1}^n y_i X_i \right) = \varphi_1^{\sum_{i=1}^n y_i X_i}(x), \quad (10.2)$$

where for $X \in \mathfrak{X}(M)$, φ_1^X denotes the time-1 flow of X . Such bi-submersions are called *path holonomy bi-submersions* [AZ13].

Now we can introduce the following definition.

Definition 10.1.3. A *bi-submersion tower* over a singular foliation \mathfrak{F} on M , is a (finite or infinite) sequence of manifolds and maps as follows

$$\mathcal{T}_B : \cdots \begin{array}{c} \xrightarrow{s_{i+1}} \\ \xrightarrow{t_{i+1}} \end{array} B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xrightarrow{t_i} \end{array} B_i \begin{array}{c} \xrightarrow{s_{i-1}} \\ \xrightarrow{t_{i-1}} \end{array} \cdots \begin{array}{c} \xrightarrow{s_1} \\ \xrightarrow{t_1} \end{array} B_1 \begin{array}{c} \xrightarrow{s_0} \\ \xrightarrow{t_0} \end{array} B_0, \quad (10.3)$$

together with a sequence \mathfrak{F}_i of singular foliations on B_i , with the convention that $B_0 = M$ and $\mathfrak{F}_0 = \mathfrak{F}$, such that

- for all $i \geq 1$, $\mathfrak{F}_i \subset \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$,
- for each $i \geq 1$, $B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xrightarrow{t_i} \end{array} B_i$ is a bi-submersion over \mathfrak{F}_i .

A bi-submersion tower over (M, \mathfrak{F}) shall be denoted as $(B_{i+1}, s_i, t_i, \mathfrak{F}_i)_{i \geq 0}$. The bi-submersion tower over \mathfrak{F} in (10.3) is said to be of *length* $n \in \mathbb{N}$ if $B_j = B_n, s_j = t_j = \text{id}$ and $\mathfrak{F}_j = \{0\}$ for all $j \geq n$.

Remark 10.1.4. Let us spell out some consequences of the axioms. For $i \geq 1$, two points $b, b' \in B_i$ of the same leaf of \mathfrak{F}_i satisfy $s_{i-1}(b) = s_{i-1}(b')$ and $t_{i-1}(b) = t_{i-1}(b')$. Also, for all $b \in B_i$, $T_b \mathfrak{F}_i \subset (\ker ds_{i-1})|_b \cap (\ker dt_{i-1})|_b$.

Let us explain how such towers can be constructed out of a singular foliation. Let \mathfrak{F} be a singular foliation on M . Then,

1. By Proposition 2.10 in [AS09], there always exists a bi-submersion $B_1 \begin{array}{c} \xrightarrow{s_0} \\ \xrightarrow{t_0} \end{array} M$ over \mathfrak{F} .
2. The $C^\infty(B_1)$ -module $\Gamma(\ker ds_0) \cap \Gamma(\ker dt_0)$ is closed under Lie bracket. When it is locally finitely generated, it is a singular foliation on B_1 . Then, it admits a bi-submersion $B_2 \begin{array}{c} \xrightarrow{s_1} \\ \xrightarrow{t_1} \end{array} B_1$. Therefore, we have obtained the two first terms of a bi-submersion tower.
3. We can then continue this construction provided that $\Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is locally finitely generated as a $C^\infty(B_2)$ -module, and that it is so at each step¹.

Definition 10.1.5. A bi-submersion tower $\mathcal{T}_B = (B_{i+1}, s_i, t_i, \mathfrak{F}_i)$ over (M, \mathfrak{F}) is called *exact bi-submersion tower* over (M, \mathfrak{F}) when $\mathfrak{F}_{i+1} = \Gamma(\ker(ds_i)) \cap \Gamma(\ker(dt_i))$ for all $i \geq 0$. It is called a *path holonomy bi-submersion tower* (resp. *path holonomy atlas bi-submersion tower*) if $B_{i+1} \begin{array}{c} \xrightarrow{s_i} \\ \xrightarrow{t_i} \end{array} B_i$ is a path holonomy bi-submersion (resp. a path holonomy atlas) for \mathfrak{F}_i for each $i \geq 0$. When a path holonomy bi-submersion tower is exact, we speak of *exact path holonomy bi-submersion tower*.

The following theorem gives a condition which is equivalent to the existence of a bi-submersion tower over a singular foliation. The proof uses Lemma 10.1.17 which is stated in the next section.

Theorem 10.1.6. *Let \mathfrak{F} be a singular foliation on M . The following items are equivalent:*

¹In real analytic case, the module $\Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is locally finitely generated because of the noetherianity of the ring of germs of real analytic functions [Fri67, Siu69].

1. \mathfrak{F} admits a geometric resolution.

2. There exists an exact path holonomy bi-submersion tower over (M, \mathfrak{F}) .

Convention 10.1.7. For a submersion $\phi: M \rightarrow N$ and a smooth map $\psi: M \rightarrow N$, we denote by $\psi\Gamma(\ker d\phi)$ the space of ψ -projectable vector fields in $\Gamma(\ker d\phi) \subset \mathfrak{X}(M)$.

Proof. (1) \Rightarrow (2) : Assume that \mathcal{F} admits a geometric resolution (E, d, ρ) . In particular, (E_{-1}, ρ) is an anchored bundle over \mathcal{F} . We need to show by recursion on $i \geq 0$ that $\Gamma(\ker ds_i) \cap \Gamma(\ker dt_i)$ is locally finitely generated because $\ker d^{(i+1)}$ or $\ker \rho$ is locally finitely generated. We actually repeat at each step $i \geq 0$, the general fact that the pull-back complex of vector bundle by the submersion $\varphi = t_0 \circ t_1 \circ \dots \circ t_i: B_{i+1} \rightarrow M$

$$\dots \longrightarrow \varphi^* E_{-i-3} \xrightarrow{\varphi^* d^{(i+3)}} \varphi^* E_{-i-2} \xrightarrow{\varphi^* d^{(i+2)}} \varphi^* E_{-i-1} \longrightarrow TB_{i+1} \quad (10.4)$$

remains exact at the sections level (at degree² ≤ -1), since $C^\infty(B_{i+1})$ is a flat $C^\infty(M)$ -module. In addition, for $i \geq 1$, the complex (10.4) defines a geometric resolution of \mathcal{F}_{i+1} .

Let (B_1, s_0, t_0) be a path holonomy bi-submersion over (M, \mathcal{F}) . Consider the map

$$\begin{aligned} R: \Gamma(t_0^* E_{-1}) &\longrightarrow {}^{t_0}\Gamma(\ker ds_0) \subset \mathfrak{X}(B_1) \\ t_0^* e &\longmapsto \vec{e} \end{aligned} \quad (10.5)$$

defined as in Proposition 10.1.16. By Lemma 10.1.17 (1), the map R in (10.5) comes from a vector bundle morphism $t_0^* E_{-1} \rightarrow \ker ds_0$ and is surjective on an open subset $V_1 \subset B_1$, by item 2 of Lemma 10.1.17. In particular the map R in (10.5) restricts to a surjective map

$$\ker(t_0^* \rho) \longrightarrow \ker(dt_0|_{\Gamma(\ker ds_0)}) = \Gamma(\ker ds_0) \cap \Gamma(\ker dt_0) \subset \mathfrak{X}(B_1) \quad (10.6)$$

By exactness in degree -1 , $\ker \rho = d^{(2)}(\Gamma(E_{-2}))$. Therefore, $\ker(t_0^* \rho)$ is locally finitely generated. By surjectivity of the map (10.6), $\Gamma(\ker ds_0) \cap \Gamma(\ker dt_0) =: \mathcal{F}_1$ is also locally finitely generated on V_1 , in particular \mathcal{F}_1 is a singular foliation on $V_1 \subset B_1$ (we may assume that $B_1 = V_1$). Thus, one can take a path holonomy bi-submersion (B_2, s_1, t_1) over (B_1, \mathcal{F}_1) . The proof continues the same as the previous step.

Let us make a step further for clarity. The composition

$$\Gamma(E_{-2}) \xrightarrow{d^{(2)}} \text{im}(d^{(2)}) = \ker \rho \twoheadrightarrow \mathcal{F}_1$$

together with E_{-2} is an anchored bundle over \mathcal{F}_1 . Let $\varphi = t_0 \circ t_1$. Just like in the first step, define the surjective $C^\infty(B_2)$ -linear map,

$$\begin{aligned} \Gamma(\varphi^* E_{-2}) &\longrightarrow {}^{t_1}\Gamma(\ker ds_1) \subset \mathfrak{X}(B_2) \\ \varphi^* e &\longmapsto \overleftarrow{e}. \end{aligned} \quad (10.7)$$

By Lemma 10.1.17 (2), the map 10.7 restricts (upon taking B_2 smaller) to a surjective map

$$\ker \left(\Gamma(\varphi^* E_{-2}) \xrightarrow{\varphi^* d^{(2)}} \Gamma(\varphi^* E_{-1}) \right) \longrightarrow \ker(dt_1|_{\Gamma(\ker ds_1)}) = \Gamma(\ker ds_1) \cap \Gamma(\ker dt_1) \subset \mathfrak{X}(B_2) \quad (10.8)$$

²We shall understand that the degree of elements of $\varphi^* E_{-i-j}$ is $-j$.

By exactness in degree -2 , the $C^\infty(M)$ -module

$$\ker \left(\Gamma(E_{-2}) \xrightarrow{d^{(2)}} \Gamma(E_{-1}) \right) = d^{(3)}(\Gamma(E_{-3}))$$

is (locally) finitely generated, hence $\mathcal{F}_2 := \Gamma(\ker ds_1) \cap \Gamma(\ker dt_1)$ is a singular foliation on B_2 . Thus, one can take a path holonomy bi-submersion (B_3, s_2, t_2) over (B_2, \mathcal{F}_2) . By recursion on $i \geq 1$, we use a path holonomy bi-submersion (B_{i+1}, s_i, t_i) over (B_i, \mathcal{F}_i) and construct an anchor bundle over \mathcal{F}_i by the composition

$$\Gamma(\varphi^* E_{-i-1}) \xrightarrow{\varphi^* d^{(i+1)}} \text{im}(\varphi^* d^{(i+1)}) = \ker(\varphi^* d^{(i)}) \twoheadrightarrow \mathcal{F}_i$$

with $\varphi = t_0 \circ t_1 \circ \cdots \circ t_{i-1}: B_i \rightarrow M$ and show as for $i = 0, 1$ that $\mathcal{F}_{i+1} := \Gamma(\ker ds_i) \cap \Gamma(\ker dt_i)$ is a singular foliation on B_{i+1} . The proof follows.

(2) \Rightarrow (1) is proven by Lemma 10.1.8 and Remark 10.1.9 below. \square

Lemma 10.1.8. *Let \mathfrak{F} be a singular foliation on M . Assume that there exists a bi-submersion tower $\mathcal{T}_B = (B_i, t_i, s_i, \mathfrak{F}_i)_{i \geq 0}$ over \mathfrak{F} . Then,*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \ker ds_2 & \xrightarrow{dt_2} & \ker ds_1 & \xrightarrow{dt_1} & \ker ds_0 & \xrightarrow{dt_0} & TM \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & B_3 & \xrightarrow{t_2} & B_2 & \xrightarrow{t_1} & B_1 & \xrightarrow{t_0} & M. \end{array} \quad (10.9)$$

is a complex of vector bundles, which is exact on the sections level³ if \mathcal{T}_B is an exact bi-submersion tower, i.e. if $\mathfrak{F}_i = \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$ for all $i \geq 1$.

Proof. For any element $b \in B_{i+1}$ and any vector $v \in \ker ds_i \subset T_b B_{i+1}$ one has

$$\begin{aligned} dt_i(v) &\in T_{t_i(b)} \mathfrak{F}_i, \quad (\text{since } \Gamma(\ker ds_i) \subset t_i^{-1}(\mathfrak{F}_i)). \\ \implies dt_i(v) &\in (\ker ds_{i-1} \cap \ker dt_{i-1})|_{t_i(b)} \quad \text{by Definition 10.1.3.} \\ \implies dt_i(v) &\in \ker ds_{i-1} \quad \text{and} \quad dt_{i-1} \circ dt_i(v) = 0, \quad \text{for all } i \geq 1. \end{aligned}$$

This shows the sequence (10.9) is a well-defined complex of vector bundles.

Let us prove that it is exact when $\mathfrak{F}_i = \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1})$ for all $i \geq 1$. Let $\xi \in \Gamma(\ker ds_{i-1})$ be a t_{i-1} -projectable vector field that projects to zero, i.e. $dt_{i-1}(\xi) = 0$. This implies that $\xi \in \Gamma(\ker ds_{i-1}) \cap \Gamma(\ker dt_{i-1}) = \mathfrak{F}_i$. Since t_i is a submersion, there exists a t_i -projectable vector field $\zeta \in t_i^{-1}(\mathfrak{F}_i)$ that satisfies $dt_i(\zeta) = \xi$. The vector field ζ can be written as $\zeta = \zeta_1 + \zeta_2$ with $\zeta_1 \in \Gamma(\ker dt_i)$ and $\zeta_2 \in \Gamma(\ker ds_i)$, because $t_i^{-1}(\mathfrak{F}_i) = \Gamma(\ker ds_i) + \Gamma(\ker dt_i)$. One has, $dt_i(\zeta_2) = \xi$. A similar argument shows that the map, $\Gamma(\ker ds_0) \xrightarrow{dt_0} t_0^* \mathfrak{F}$, is surjective. This proves exactness in all degree. \square

Remark 10.1.9. One of the consequence of Lemma 10.1.8 is that:

³Let us explain the notion of exactness at the level of sections when the base manifolds are not the same: what we mean is that for all $n \geq 0$, $\Gamma(\ker dt_n) \cap \Gamma(\ker ds_n) = (t_{n+1})_*(\Gamma(\ker ds_{n+1}))$.

Equivalently, it means that the pull-back of the vector bundles in (10.13) to any one of the manifold B_m with $m \geq n$ is exact at the level of sections, i.e

$$\Gamma(t_{n+1,m}^* \ker ds_{n+1}) \xrightarrow{dt_{n+1}} \Gamma(t_{n,m}^* \ker ds_n) \xrightarrow{dt_n} \Gamma(t_{n-1,m}^* \ker ds_{n-1}) \quad (10.10)$$

is a short exact sequence of $C^\infty(B_m)$ -modules, with $t_{n,m} = t_n \circ \cdots \circ t_m$ for all $m \geq n$.

1. If there exists a sequence of maps

$$M \xrightarrow{\varepsilon_0} B_1 \xrightarrow{\varepsilon_1} B_2 \xrightarrow{\varepsilon_2} \dots \quad (10.11)$$

where for all $i \geq 0$, ε_i is a section for both s_i and t_i then the pull-back of (10.9) on M through the sections $(\varepsilon_i)_{i \geq 0}$ i.e.

$$\dots \xrightarrow{dt_3} \varepsilon_{2,0}^* \ker ds_2 \xrightarrow{dt_2} \varepsilon_{1,0}^* \ker ds_1 \xrightarrow{dt_1} \varepsilon_0^* \ker ds_0 \xrightarrow{dt_0} TM \quad (10.12)$$

is a complex of vector bundles, with the convention $\varepsilon_{n,0} = \varepsilon_n \circ \dots \circ \varepsilon_0$. If \mathcal{T}_B is an exact bi-submersion tower then, (10.12) is a geometric resolution of \mathfrak{F} .

2. In case that \mathcal{T}_B is an exact path holonomy bi-submersion tower, such a sequence (1) always exists, since the bi-submersions (B_{i+1}, s_i, t_i) are as in Example 10.1.2. For such bi-submersions, the zero section $x \mapsto (x, 0)$ is a section for both s_i and t_i .

Corollary 10.1.10. *Under the assumptions of Lemma 10.1.8, assume the tower of bi-submersion \mathcal{T}_B is of length $n + 1$. Then, the pull-back of the sequence of vector bundles*

$$\ker ds_n \xrightarrow{dt_n} t_n^* \ker ds_{n-1} \longrightarrow \dots \xrightarrow{dt_2} t_{2,n}^* \ker ds_1 \xrightarrow{dt_1} TB_{n+1} \times_{TM} \ker ds_0 \xrightarrow{\text{pr}_1} TB_{n+1} \quad (10.13)$$

is a geometric resolution of the pull-back foliation $t_{0,n}^{-1}(\mathfrak{F}) \subset \mathfrak{X}(B_{n+1})$, where pr_1 is the projection on TB_{n+1} and for $i \geq 1$, $t_{i,j}$ is the composition $t_i \circ \dots \circ t_j: B_{j+1} \rightarrow B_i$.

Proof. By Lemma 10.1.8, the complex in Equation (10.13) is exact. By construction, the projection of the fiber product $TB_{n+1} \times_{TM} \ker ds_0$ to TB_{n+1} induces the singular foliation $t_{0,n}^{-1}(\mathfrak{F})$. \square

10.1.1 Lift of a symmetry to the bi-submersion tower

Let us investigate what an action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of a Lie algebra \mathfrak{g} on (M, \mathfrak{F}) would induce on a bi-submersion tower \mathcal{T}_B over \mathfrak{F} .

We start with some vocabulary and preliminary results.

Definition 10.1.11. Let (B, s, t) be a bi-submersion of a singular foliation \mathfrak{F} on a manifold M . We call *lift* of a vector field $X \in \mathfrak{X}(M)$ to the bi-submersion (B, s, t) a vector field $\tilde{X} \in \mathfrak{X}(B)$ which is both s -projectable on X and t -projectable on X .

The coming proposition means that the notion of lift to a bi-submersion only makes sense for symmetries of the singular foliation.

Proposition 10.1.12. *If a vector field on M admits a lift to a bi-submersion (B, s, t) , then it is a symmetry of \mathfrak{F} .*

Proof. Let $\tilde{X} \in \mathfrak{X}(B)$ be a lift of $X \in \mathfrak{X}(M)$. Since \tilde{X} is s -projectable, $[\tilde{X}, \Gamma(\ker ds)] \subset \Gamma(\ker ds)$. Since \tilde{X} is t -projectable, $[\tilde{X}, \Gamma(\ker dt)] \subset \Gamma(\ker dt)$. Hence:

$$\begin{aligned} [\tilde{X}, s^{-1}(\mathfrak{F})] &= [\tilde{X}, \Gamma(\ker ds) + \Gamma(\ker dt)] \\ &= [\tilde{X}, \Gamma(\ker(ds))] + [\tilde{X}, \Gamma(\ker dt)] \\ &\subset \Gamma(\ker ds) + \Gamma(\ker dt) = s^{-1}(\mathfrak{F}). \end{aligned}$$

In words, \tilde{X} is a symmetry of $s^{-1}\mathfrak{F}$. Since \tilde{X} projects through s to X , X is a symmetry of \mathfrak{F} . \square

We investigate the existence of lifts of symmetries of \mathfrak{F} to bi-submersions over \mathfrak{F} .

Remark 10.1.13. For a given $X \in \mathfrak{s}(\mathfrak{F})$,

1. the lift \tilde{X} to a given bi-submersion is not unique, even when it exists. However, two different lifts of a $X \in \mathfrak{s}(\mathfrak{F})$ to a bi-submersion (B, s, t) differ by an element of the intersection $\Gamma(\ker(ds)) \cap \Gamma(\ker(dt))$.
2. \tilde{X} is a symmetry of $\Gamma(\ker(ds)) \cap \Gamma(\ker(dt))$: $[\tilde{X}, \Gamma(\ker(ds)) \cap \Gamma(\ker(dt))] \subset \Gamma(\ker(ds)) \cap \Gamma(\ker(dt))$, since \tilde{X} is s -projectable and t -projectable.

As the following example shows, the lift of a symmetry to a bi-submersion may not exist.

Example 10.1.14. Consider the trivial foliation $\mathfrak{F} := \{0\}$ on M . For any diffeomorphism $\phi: M \rightarrow M$, (M, id, ϕ) is a bi-submersion over \mathfrak{F} . Every vector field $X \in \mathfrak{X}(M)$ is a symmetry of \mathfrak{F} . If it exists, its lift has to be given by, $\tilde{X} = X$ since the source map is the identity. But $\tilde{X} = X$ is t -projectable if and only if X is ϕ -invariant. A non ϕ -invariant vector field X therefore admits no lift to (M, id, ϕ) .

However, internal symmetries, i.e. elements in \mathfrak{F} admit lifts to any bi-submersion.

Proposition 10.1.15. *Let (B, s, t) be any bi-submersion of a singular foliation \mathfrak{F} on a manifold M . Every internal symmetry, i.e. every vector field in \mathfrak{F} , admits a lift to (B, s, t) .*

Proof. Let $X \in \mathfrak{F}$. Since $s: B \rightarrow M$ is a submersion, there exists $X^s \in \mathfrak{X}(B)$ s -projectable on X . Since t is a submersion, there exists $X^t \in \mathfrak{X}(B)$ t -projectable on X . By construction $X^s \in s^{-1}(\mathfrak{F})$ and $X^t \in t^{-1}(\mathfrak{F})$. Using the property (10.1) of the bi-submersion (B, s, t) , the vector fields X^s and X^t decompose as

$$\begin{cases} X^s = X_s^s + X_t^s & \text{with } X_s^s \in \Gamma(\ker(ds)), X_t^s \in \Gamma(\ker(dt)), \\ X^t = X_s^t + X_t^t & \text{with } X_s^t \in \Gamma(\ker(ds)), X_t^t \in \Gamma(\ker(dt)). \end{cases}$$

By construction, X_t^s is s -projectable to X and t -projectable to 0 while X_s^t is s -projectable to 0 and t -projectable to X . It follows that, $\tilde{X} := X_s^t + X_t^s$, is a lift of X to the bi-submersion (B, s, t) . \square

Let us make the Proposition 10.1.15 more precise. The proof we give for Theorem 10.1.6 uses the notion of left-invariant, resp. right-invariant, vector fields on a bi-submersion over a singular foliation. We define the latter in the next proposition. It uses the notion of anchored bundle over a singular foliation and almost Lie algebroid, see [LLS20, LGLR22] for more details.

Proposition 10.1.16. *Let (B, s, t) be a bi-submersion over a singular foliation \mathcal{F} on a manifold M . Let (A, ρ) be an anchored bundle over \mathcal{F} , i.e., $A \rightarrow M$ is a vector bundle and $\rho: A \rightarrow TM$ is a vector bundle morphism such that $\rho(\Gamma(A)) = \mathcal{F}$. There exists two maps*

$$\begin{aligned} \Gamma(A) &\longrightarrow \mathfrak{X}(B) \\ a &\longmapsto \overleftarrow{a} \\ a &\longmapsto \overrightarrow{a} \end{aligned} \tag{10.14}$$

fulfilling the following conditions:

1. the vector field $\overrightarrow{a} \in \mathfrak{X}(B)$ (resp. $\overleftarrow{a} \in \mathfrak{X}(B)$) is t -related (resp. s -related) with $\rho(a) \in \mathcal{F}$,
2. the vector field \overrightarrow{a} (resp. \overleftarrow{a}) is tangent to the fibers of s (resp. t),
3. $\overrightarrow{fa} = t^*(f)\overrightarrow{a}$ and $\overleftarrow{fa} = s^*(f)\overleftarrow{a}$ for all $a \in \Gamma(A)$, $f \in C^\infty(M)$.

The vector fields \overleftarrow{a} (resp. \overrightarrow{a}) for $a \in \Gamma(A)$ are called left-invariant (resp. right-invariant) vector fields of (B, s, t) .

Proof. By Proposition 10.1.15, given a section $a \in \Gamma(A)$ the vector field $\rho(a) \in \mathcal{F}$ admits a lift $\widetilde{\rho(a)} \in \mathfrak{X}(B)$ on (B, s, t) of the form

$$\widetilde{\rho(a)} := \rho(a)_s^t + \rho(a)_t^s$$

with $\rho(a)_s^t \in \Gamma(\ker(ds))$ and $\rho(a)_t^s \in \Gamma(\ker(dt))$. Also, $dt(\rho(a)_s^t) = \rho(a)$ and $ds(\rho(a)_t^s) = \rho(a)$. Let $b \in B$ and \mathcal{U}_b an open neighborhood of b . Let (a_1, \dots, a_r) be a local trivialization of A on the open subset $\mathcal{U} = s(\mathcal{U}_b) \subset M$. We define a map R_U on local generators by

$$\begin{aligned} R_U: \Gamma_{\mathcal{U}_b}(t^*A) &\longrightarrow \Gamma_{\mathcal{U}_b}(\ker(ds)) \\ t^*a_i &\longmapsto \rho(a_i)_s^t \end{aligned} \tag{10.15}$$

and extend by $C^\infty(\mathcal{U}_b)$ -linearity. These maps can be glued using partitions of unity. More precisely, let $(\chi_\lambda)_{\lambda \in \Lambda}$ be a partition of unity subordinate to an open cover $(\mathcal{U}_\lambda)_{\lambda \in \Lambda}$ by open sets that trivialize the vector bundle A . We define a map R on $\Gamma(t^*A)$ as

$$\sum_{\lambda \in \Lambda} \chi_\lambda R_{\mathcal{U}_\lambda}.$$

Now for $a \in \Gamma(A)$ we define $\overrightarrow{a} := R(s^*a)$. The map $\overleftarrow{\cdot}$ is defined similarly. Item 1, 2 and 3 hold by construction.

Assume that (A, ρ) is equipped with an almost Lie algebroid bracket $[\cdot, \cdot]_A$. For all $a, b \in \Gamma(A)$ one has

$$\begin{aligned} ds\left(\overleftarrow{[a, b]_A} - [\overleftarrow{a}, \overleftarrow{b}]\right) &= \rho([a, b]_A) - [\rho(a), \rho(b)] \\ &= 0, \end{aligned}$$

because \overleftarrow{a} is s -projectable to $\rho(a)$ and ρ is a morphism of brackets. Since left-invariant vector fields are tangent to the fibers of t , one has $dt\left(\overleftarrow{[a, b]_A} - [\overleftarrow{a}, \overleftarrow{b}]\right) = 0$. The proof is similar for $\overrightarrow{[a, b]_A} - [\overrightarrow{a}, \overrightarrow{b}]$. This ends the proof. \square

The following lemma is important in the proof of Theorem 10.1.6.

Lemma 10.1.17. *Let (B, s, t) be any bi-submersion over a singular foliation \mathcal{F} on a manifold M , and (A, ρ) an anchored bundle over \mathcal{F} .*

1. *There exists vector bundle morphisms $R: t^*A \rightarrow \ker ds$ and $L: s^*A \rightarrow \ker dt$ inducing (10.14).*
2. *Let $x \in M$. If (B, s, t) is a path holonomy bi-submersion over \mathcal{F} near $(x, 0)$ then, every $b \in B$ such that $t(b) = x$ admits a neighborhood V such that every t -projectable vector field of $\Gamma_V(\ker ds)$ is of the form $R(\xi)$ for some $\xi \in \Gamma_V(t^*A)$.*

Remark 10.1.18. In item 1 of Lemma 10.1.17, by “inducing (10.14)” we mean that for every $a \in \Gamma(A)$, $\overrightarrow{a} := R(s^*a)$ and $\overleftarrow{a} := L(t^*a)$.

Proof. Item 1 is an immediate consequence of Proposition 10.1.16 and is in fact used in its proof. Let us prove item 2. By assumption, B is a neighborhood of $(0, x)$ in $M \times \mathbb{R}^n$ with $n = \text{rk}_x(\mathcal{F}) = \dim(\mathcal{F}_x := \mathcal{F}/\mathcal{I}_x\mathcal{F})$ near $x \in M$ (see Example 10.1.2). Let $b \in B$ and \mathcal{U}_b an open neighborhood of b . Let (a_1, \dots, a_r) be a local trivialization of A on the open subset $\mathcal{U} = t(\mathcal{U}_b) \subset M$. One has by definition of right-invariant vector fields of (B, s, t) that $dt(\overrightarrow{a}_i) = t^*\rho(a_i)$ for $i = 1, \dots, \text{rk}(A)$. The vector fields $\rho(a_i)$ are generators of \mathcal{F} on \mathcal{U} . We necessarily have $n \leq \text{rk}(A)$. Since the $\rho(a_i)(x)$'s are generators of \mathcal{F}_x , without loss of generality we can assume that $\rho(a_1)(x), \dots, \rho(a_n)(x)$ is a basis of \mathcal{F}_x . Since $\text{rk}(\ker ds) = n$, $(\overrightarrow{a}_i(b))_{i=1, \dots, n}$ form a basis of $\ker ds|_x$. Therefore, the \overrightarrow{a}_i 's are independent at every point of some neighborhood $V \subset \mathcal{U}_b$ of b i.e., they form a local trivialization of the vector bundle $\ker ds \rightarrow B$. As a result, vector fields of $\Gamma_V(\ker ds)$ are of the form $\sum_i^n f_i \overrightarrow{a}_i$ with $f_i \in C^\infty(V)$ for $i = 1, \dots, n$. This ends the proof. \square

We can now state one of the important results of this section. It uses several concepts introduced in [AS09], which are recalled in the proof.

Proposition 10.1.19. *Let \mathfrak{F} be a singular foliation on a manifold M . Any symmetry $X \in \mathfrak{s}(\mathfrak{F})$ admits a lift*

1. *to any path holonomy bi-submersion (B, s, t) ,*
2. *to Androulidakis-Skandalis' path holonomy atlas,*
3. *to a neighborhood of any point in a bi-submersion through which there exists a local bisection that induces the identity.*

Remark 10.1.20. In cases (1) or (2) in Proposition 10.1.19, a linear lift

$$X \rightarrow \tilde{X}$$

can be defined on the whole space $\mathfrak{s}(\mathfrak{F})$ of symmetries of \mathfrak{F} . As an immediate consequence of Remark 10.1.13, we obtain that for all $X, Y \in \mathfrak{s}(\mathfrak{F})$,

$$\widetilde{[X, Y]} - [\tilde{X}, \tilde{Y}] \in \Gamma(\ker ds) \cap \Gamma(\ker dt). \quad (10.16)$$

Proof of (Proposition 10.1.19). Let $X \in \mathfrak{s}(\mathfrak{F})$. Assume that $(B, s, t) = (\mathcal{W}, s_0, t_0)$ is a path holonomy bi-submersion associated to some generators $X_1, \dots, X_n \in \mathfrak{F}$ as in Example 10.1.2. Fix $(u, y = (y_1, \dots, y_n)) \in \mathcal{W} \subset M \times \mathbb{R}^n$, set $Y := \sum_{i=1}^n y_i X_i$. Since $d\varphi_1^Y(X) = (\varphi_1^Y)_*(X) \in X + \mathfrak{F}$, there exists

$Z_y \in \mathfrak{F}$ depending smoothly on y such that $dt_0(X, 0) = X + Z_y$. Take $\tilde{Z}_y \in t_0^{-1}(\mathfrak{F})$ such that $dt_0(\tilde{Z}_y) = Z_y$. One has,

$$dt_0\left((X, 0) - \tilde{Z}_y\right) = X = ds_0(X, 0).$$

We can write $\tilde{Z}_y = \tilde{Z}_y^1 + \tilde{Z}_y^2$, with $\tilde{Z}_y^1 \in \Gamma(\ker ds_0)$, $\tilde{Z}_y^2 \in \Gamma(\ker dt_0)$. By construction, $\tilde{X} := (X, -\tilde{Z}_y^1)$ is a lift of X to the bi-submersion (\mathcal{W}, s_0, t_0) . This proves item 1.

If $X_B \in \mathfrak{X}(B)$ and $X_{B'} \in \mathfrak{X}(B')$ are two lifts of the symmetry X on the path holonomy bi-submersions (B, s, t) and (B', s', t') respectively, then $(X_B, X_{B'})$ is a lift of X on the composition bi-submersion $B \circ B'$. This proves item 2, since the path holonomy atlas is made of fibered products and inverse of holonomy path holonomy bi-submersions [AS09]. Also X_a is a symmetry of (B, t, s) .

Item 2 in Proposition 2.10 of [AS09] states that if the identity of M is carried by (B, s, t) at some point $v \in B$, then there exists an open neighborhood $V \subset B$ of v that satisfies $s|_V = s_0 \circ g$ and $t|_V = t_0 \circ g$, for some submersion $g: V \rightarrow \mathcal{W}$, for \mathcal{W} of the form as in item 1. Thus, for all $X \in \mathfrak{s}(\mathfrak{F})$ there exists a vector field $\tilde{X} \in \mathfrak{X}(V)$ fulfilling $ds|_V(\tilde{X}) = dt|_V(\tilde{X}) = X$. This proves item 3. \square

Definition 10.1.21. A symmetry of the tower of bi-submersion $\mathcal{T}_B = (B_{i+1}, s_i, t_i, \mathcal{F}_i)_{i \geq 0}$ is a family $X = (X_i)_{i \geq 0}$, with the i -th component X_i in $\mathfrak{s}(\mathcal{F}_i)$, such that $ds_{i-1}(X_i) = dt_{i-1}(X_i) = X_{i-1}$ for all $i \geq 1$. We denote by $\mathfrak{s}(\mathcal{T}_B)$ the Lie algebra of symmetries of \mathcal{T}_B .

The next theorem gives a class of bi-submersion tower to which any symmetry of the base singular foliation \mathfrak{F} lifts.

Theorem 10.1.22. Let \mathfrak{F} be a foliation. Let \mathcal{T}_B be a path holonomy bi-submersion tower (or an exact path holonomy atlas bi-submersion tower). A vector field $X \in \mathfrak{X}(M)$ is a symmetry of \mathfrak{F} , i.e. $[X, \mathfrak{F}] \subset \mathfrak{F}$, if and only if it is the component on M of a symmetry of \mathcal{T}_B .

Proof. It is a direct consequence of Proposition 10.1.12 and of item 1. resp. item 2. in Proposition 10.1.19. It is due to the fact that the tower \mathcal{T}_B is generated by path holonomy bi-submersions, and then we can lift symmetries at every stage of the tower \mathcal{T}_B . \square

Remark 10.1.23. Let $(X^i)_{i \geq 0}$ be a lift of $X_0 := X \in \mathfrak{s}(\mathfrak{F})$. For $i \geq 1$, $\nabla_X^i := \text{ad}_{X_i}$ preserves $\Gamma(\ker ds_{i-1})$, since X_i is s_{i-1} -projectable. Altogether, they define a chain map $(\nabla_X^i)_{i \geq 0}$ at the section level of the complex (10.9), on projectable vector fields in (10.10), since for every $i \geq 0$ and any t_i -projectable vector field $\xi \in \ker ds_i$,

$$\begin{aligned} dt_i([X_{i+1}, \xi]) &= [dt_i(X_{i+1}), dt_i(\xi)] \\ &= [X_i, dt_i(\xi)], \end{aligned}$$

that is $dt_i \circ \nabla_X^{i+1} = \nabla_X^i \circ dt_i$.

Remark 10.1.24. In [GZ21], under some assumptions, it is shown that if a Lie group G acts on a foliated manifold (M, \mathfrak{F}) , then it acts on its holonomy groupoid. It is likely that this result follows from Theorem 10.1.22, this will be addressed in another study.

10.1.2 Lifts of actions of a Lie algebra on a bi-submersion tower

We end the section with the following constructions and some natural questions.

Let $\mathcal{T}_B = (B_{i+1}, s_i, t_i, \mathfrak{F}_i)_{i \geq 0}$ be an exact path holonomy bi-submersion tower over a singular foliation (M, \mathfrak{F}) .

By Theorem 10.1.22, any vector field $X \in \mathfrak{s}(\mathfrak{F})$ lifts to a symmetry $(X^i)_{i \geq 0}$ of \mathcal{T}_B . Once a lift is chosen, we can define a linear map,

$$X \in \mathfrak{s}(\mathfrak{F}) \mapsto (X^i)_{i \geq 1} \in \mathfrak{s}(\mathcal{T}_B).$$

Let $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ be a strict symmetry action of a Lie algebra \mathfrak{g} on (M, \mathfrak{F}) . For $x \in \mathfrak{g}$, there exists $(\varrho(x)^i)_{i \geq 0}$, with $\varrho(x)^i \in \mathfrak{s}(\mathfrak{F}_i) \subset \mathfrak{X}(B_i)$ a symmetry of \mathcal{T}_B such that $\varrho(x)^0 = \varrho(x) \in \mathfrak{s}(\mathfrak{F})$, by Theorem 10.1.22. Consider the composition,

$$x \in \mathfrak{g} \longmapsto \varrho(x) \in \mathfrak{s}(\mathfrak{F}) \longmapsto (\varrho(x)^i)_{i \geq 0} \in \mathfrak{s}(\mathcal{T}_B) \mapsto \varrho(x)^1 \in \mathfrak{X}(B_1). \quad (10.17)$$

Lemma 10.1.25. *For all $x, y \in \mathfrak{g}$,*

$$[\varrho(x), \varrho(y)]^1 - [\varrho(x)^1, \varrho(y)^1] = dt_1(C_1(x, y))$$

with $C_1(x, y) \in \Gamma(\ker ds_1 \rightarrow B_2)$ a t_1 -projectable vector field, for some bilinear map

$$C_1: \wedge^2 \mathfrak{g} \longrightarrow \Gamma(\ker ds_1 \rightarrow B_2).$$

Proof. This follows from Lemma 10.1.8, because $[\varrho(x), \varrho(y)]^1 - [\varrho(x)^1, \varrho(y)^1] \in \Gamma(\ker ds_0) \cap \Gamma(\ker dt_0)$. \square

Theorem 10.1.26. *The map $C_1: \wedge^2 \mathfrak{g} \longrightarrow \Gamma(\ker ds_1 \rightarrow B_2)$ of Lemma 10.1.25 satisfies for all $x, y, z \in \mathfrak{g}$,*

$$C_1([x, y]_{\mathfrak{g}}, z) + \nabla_{\varrho(x)}^2(C_1(y, z)) + \mathfrak{O}(x, y, z) = dt_2(C_2(x, y, z)) \quad (10.18)$$

for some tri-linear map $C_2: \wedge^3 \mathfrak{g} \longrightarrow \Gamma(\ker ds_2 \rightarrow B_3)$. Here, ∇^2 is as in Remark 10.1.23.

Proof. For $x, y, z \in \mathfrak{g}$,

$$\begin{aligned} dt_1(C_1([x, y]_{\mathfrak{g}}, z)) + \mathfrak{O}(x, y, z) &= [\varrho([x, y]_{\mathfrak{g}}), \varrho(z)]^1 - [\varrho([x, y]_{\mathfrak{g}})^1, \varrho(z)^1] + \mathfrak{O}(x, y, z) \\ &= \underline{[[\varrho(x), \varrho(y)], \varrho(z)]^1} - [[\varrho(x), \varrho(y)]^1, \varrho(z)^1] + \mathfrak{O}(x, y, z) \\ &= -\underline{[[\varrho(x)^1, \varrho(y)^1], \varrho(z)^1]} + [dt_1(C_1(x, y)), \varrho(z)^1] + \mathfrak{O}(x, y, z) \\ &= dt_1([C_1(x, y), \varrho(z)^2]) + \mathfrak{O}(x, y, z). \end{aligned}$$

We have used Jacobi identity and $dt_1(\varrho(z)^2) = \varrho(z)^1$. This implies that

$$dt_1(C_1([x, y]_{\mathfrak{g}}, z) - [C_1(x, y), \varrho(z)^2]) + \mathfrak{O}(x, y, z) = 0. \quad (10.19)$$

Again Lemma 10.1.8 implies the result. \square

Here is a natural question:

Question: Can we continue the construction in Theorem 10.1.26 to a Lie ∞ -morphism?

There is another natural type of questions:

Question: Can a strict symmetry action $\varrho: \mathfrak{g} \rightarrow \mathfrak{X}(M)$ of a Lie algebra \mathfrak{g} on a singular foliation (M, \mathfrak{F}) lift to a given geometric resolution (E_\bullet, d, ρ) of \mathfrak{F} ?

Discussion

- Can we answer this question with what we have?
- We know by Theorem 8.3.1 that ϱ lifts on any universal Lie ∞ -algebroid (E, Q) of \mathfrak{F} to a Lie ∞ -morphism $\Phi: \mathfrak{g} \rightarrow \mathfrak{X}_\bullet(E)$. If we can choose at least the polynomial-degree zero of the Taylor coefficient $\Phi_1: \wedge^2 \mathfrak{g} \rightarrow \mathfrak{X}_{-1}(E)$ to be zero, then the answer is yes.
- If yes, can we assume that the previous action to preserve the dg -almost Lie algebroid? Again, if yes, then the polynomial-degree +1 can be chosen to be zero.

It seems that being able to suppress the higher Taylor coefficients has a strong geometric meaning. We intend to study that problem in a subsequent paper.

Question: Can a bi-submersion tower be equipped with a (local) Lie ∞ -groupoid structure?

Appendices

We have used almost everywhere in the thesis, modules or vector spaces that arise as the quotient of the tensor algebra. It is worth it to dedicate a section to recall the construction and some basic facts on tensor algebras. In this chapter we assume that the reader is familiar with the notion of \mathcal{O} -modules, graded \mathcal{O} -modules, and vector spaces.

It is known that many algebras such as the exterior algebra, symmetric algebra, Clifford algebras [Tod11], universal enveloping algebras [Bek] and many other algebras are the quotient of tensor algebras. These make the tensor algebra a fundamental and a very useful notion. In this thesis we have dealt a lot with \mathcal{O} -multilinear maps, it simply means that these maps are \mathcal{O} -linear w.r.t each argument while we fix the other arguments. The tensor algebra is used to characterize multilinear relations between algebraic objects related to modules or vector spaces. There are several ways to construct the tensor algebra, we refer the reader e.g. to chapter 16 of [Lan05] or [And, Kei05] for more details on the matter.

When $\mathcal{O} = \mathbb{K}$ the reader may replace " \mathcal{O} -module" by " \mathbb{K} -vector space" and the construction of the tensor algebra that we give below works the same.

A.1 Tensor product

Let \mathcal{V} and \mathcal{W} and \mathcal{Z} be \mathcal{O} -modules. The *tensor product* $\mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}$ of \mathcal{V} and \mathcal{W} over \mathcal{O} is the \mathcal{O} -module which is defined by the following universal property

Theorem A.1.1. *There exists a \mathcal{O} -bilinear map, $p: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}$, that satisfies the following properties:*

1. any \mathcal{O} -bilinear map $B: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{Z}$ admits a unique \mathcal{O} -linear map $\gamma: \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W} \rightarrow \mathcal{Z}$, such that

$\gamma \circ p = B$. In other words, such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{V} \times \mathcal{W} & \xrightarrow{p} & \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W} \\ & \searrow B & \swarrow \gamma \\ & & \mathcal{Z} \end{array} \quad (\text{A.1})$$

2. Furthermore, if there is another \mathcal{O} -module \mathcal{Q} and a \mathcal{O} -bilinear map, $p': \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{Q}$ with the property (A.1), then there exists a unique isomorphism $i: \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W} \rightarrow \mathcal{Q}$ such that $p' = i \circ p$.

Proof. We first show "uniqueness" when such \mathcal{O} -module exists, then show existence.

Uniqueness: Suppose that there exist two pairs $(\mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}, p: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W})$ and $(\mathcal{V} \tilde{\otimes}_{\mathcal{O}} \mathcal{W}, \tilde{p}: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \tilde{\otimes}_{\mathcal{O}} \mathcal{W})$ satisfying the property A.1.1. By using two times A.1.1, we deduce the existence of a unique \mathcal{O} -linear map $i: \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W} \rightarrow \mathcal{V} \tilde{\otimes}_{\mathcal{O}} \mathcal{W}$, and another one $j: \mathcal{V} \tilde{\otimes}_{\mathcal{O}} \mathcal{W} \rightarrow \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}$ such that $i \circ p = \tilde{p}$ and $j \circ \tilde{p} = p$. This implies that $j \circ i \circ p = p$ and $i \circ j \circ \tilde{p} = \tilde{p}$. By unicity of the factorization in A.1.1, we must have $i \circ j = \text{id}$ and $j \circ i = \text{id}$. This proves item 2.

Existence: Let us consider the free \mathcal{O} -module \mathcal{P} generated by elements of the Cartesian product $\mathcal{V} \times \mathcal{W}$, i.e. elements of \mathcal{P} are formal finite sums of elements of the form $f(v, w)$, with $f \in \mathcal{O}$ and $v \in \mathcal{V}, w \in \mathcal{W}$. Next, we divide by the submodule $\mathcal{R} \subset \mathcal{P}$ of \mathcal{P} generated by elements of the following types:

$$\begin{aligned} &(v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ &(v, w_1 + w_2) - (v, w_1) - (v, w_2), \\ &(fv, w) - f(v, w), \text{ and } (v, fw) - f(v, w) \end{aligned}$$

which are the relations that elements of the tensor product must satisfy with $v, v_1, v_2 \in \mathcal{V}$ and $w, w_1, w_2 \in \mathcal{W}$ and $f \in \mathcal{O}$. In those notations, the tensor product of \mathcal{V} and \mathcal{W} over \mathcal{O} is defined as the quotient

$$\mathcal{V} \otimes_{\mathcal{O}} \mathcal{W} := \mathcal{P}/\mathcal{R}.$$

For $v \in \mathcal{V}, w \in \mathcal{W}$, we denote the class of $(v, w) \in \mathcal{V} \times \mathcal{W}$ by $v \otimes w \in \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}$. This quotient comes with the natural map, $p: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}$, $(v, w) \mapsto v \otimes w$. For any \mathcal{O} -linear map $B: \mathcal{V} \times \mathcal{W} \rightarrow \mathcal{Z}$, the \mathcal{O} -linear map $q: \mathcal{P} \rightarrow \mathcal{Z}$ which is given on the basis of \mathcal{P} by $(x, y) \mapsto q((x, y)) := B(x, y)$ clearly goes to quotient to define a \mathcal{O} -linear map $\bar{q}: \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W} \rightarrow \mathcal{Z}$, since e.g.

$$\begin{aligned} q((v_1 + v_2, w) - (v_1, w) - (v_2, w)) &= q((v_1 + v_2, w)) - q((v_1, w)) - q((v_2, w)) \\ &= B(v_1 + v_2, w) - B(v_1, w) - B(v_2, w) \\ &= 0, \quad \text{by bilinearity of } B \end{aligned}$$

also,

$$\begin{aligned} q((fv, w) - f(v, w)) &= q((fv, w)) - fq((v, w)), \quad \text{by } \mathcal{O}\text{-linearity of } q \\ &= B(fv, w) - fB(v, w) = 0, \quad \text{by } \mathcal{O}\text{-bilinearity of } B. \end{aligned}$$

We did everything to get, $\bar{q} \circ p = B$. Therefore, we can take $\gamma = \bar{q}$ to satisfy (A.1).

□

Proposition A.1.2. *Let $\mathcal{V}, \mathcal{W}, \mathcal{Z}$ be \mathcal{O} -modules. We have the following isomorphisms*

1. $\mathcal{V} \otimes_{\mathcal{O}} \mathcal{W} \simeq \mathcal{W} \otimes_{\mathcal{O}} \mathcal{V}$.
2. $\mathcal{O} \otimes_{\mathcal{O}} \mathcal{V} \simeq \mathcal{V}$, and for any ideal $\mathcal{I} \subset \mathcal{O}$, Also, $\frac{\mathcal{O}}{\mathcal{I}} \otimes_{\mathcal{O}} \mathcal{V} \simeq \frac{\mathcal{V}}{\mathcal{I}\mathcal{V}}$
3. $(\mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}) \otimes_{\mathcal{O}} \mathcal{Z} \simeq \mathcal{V} \otimes_{\mathcal{O}} (\mathcal{W} \otimes_{\mathcal{O}} \mathcal{Z})$.

Proof. For item 1, the map $(v, w) \in \mathcal{V} \times \mathcal{W} \mapsto w \otimes v \in \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}$ goes to quotient to *the twist map* $v \otimes w \mapsto w \otimes v$ and give the isomorphism whose inverse is $w \otimes v \in \mathcal{W} \otimes_{\mathcal{O}} \mathcal{V} \mapsto v \otimes w \in \mathcal{V} \otimes_{\mathcal{O}} \mathcal{W}$.

For item 2, the map $(f, v) \in \mathcal{O} \times \mathcal{V} \rightarrow fv$ also induces an isomorphism whose inverse is $v \in \mathcal{V} \mapsto 1 \otimes v \in \mathcal{O} \otimes_{\mathcal{O}} \mathcal{V}$. A similar map gives the second clause.

For item 3, the isomorphism is, obviously, $(v_1 \otimes v_2) \otimes v_3 \mapsto v_1 \otimes (v_2 \otimes v_3)$. This is trivially extended to more \mathcal{O} -modules. \square

Remark A.1.3. A similar construction as in Theorem A.1.1 can be done for a finite family of \mathcal{O} -modules $\mathcal{V}_1, \dots, \mathcal{V}_r$, as in the case of bilinear maps. Then $\mathcal{V}_1 \otimes_{\mathcal{O}} \dots \otimes_{\mathcal{O}} \mathcal{V}_r$ is a universal object that factorizes r -multilinear maps, defined on $\mathcal{V}_1 \times \dots \times \mathcal{V}_r$. We naturally have,

$$(\mathcal{V}_1 \otimes_{\mathcal{O}} \mathcal{V}_2) \otimes_{\mathcal{O}} \mathcal{V}_3 \simeq \mathcal{V}_1 \otimes_{\mathcal{O}} (\mathcal{V}_2 \otimes_{\mathcal{O}} \mathcal{V}_3) \simeq \mathcal{V}_1 \otimes_{\mathcal{O}} \mathcal{V}_2 \otimes_{\mathcal{O}} \mathcal{V}_3.$$

So, in this thesis, we make no difference in how we denote the elements $(v_1 \otimes v_2) \otimes v_3$, $v_1 \otimes (v_2 \otimes v_3)$, or $v_1 \otimes v_2 \otimes v_3$.

Remark A.1.4. It is important to notice that if \mathcal{C} and \mathcal{C}' are graded \mathcal{O} -algebras, then $\mathcal{C} \otimes_{\mathcal{O}} \mathcal{C}'$ is a graded \mathcal{O} -algebra with product

$$(c_1 \otimes c'_1)(c_2 \otimes c'_2) := (-1)^{|c'_1||c_2|} c_1 c_2 \otimes c'_1 c'_2 \quad (\text{A.2})$$

for homogeneous elements $c_1, c_2 \in \mathcal{C}$ and $c'_1, c'_2 \in \mathcal{C}'$. Moreover, if \mathcal{C} and \mathcal{C}' are unitary with units $1_{\mathcal{C}}$ and $1_{\mathcal{C}'}$, respectively, then the algebra $\mathcal{C} \otimes_{\mathcal{O}} \mathcal{C}'$ is also unitary with unit $1_{\mathcal{C}} \otimes 1_{\mathcal{C}'}$. Here, to avoid explosion of notations, we denote the product of two elements a, b by ab . Also, we will not make notational distinctions between the unit elements.

Convention A.1.5. For $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ and $\Psi: \mathcal{C}'' \rightarrow \mathcal{C}'''$ two homogeneous morphisms of \mathbb{Z} -graded \mathcal{O} -modules, then $\Phi \otimes \Psi: \mathcal{C} \otimes_{\mathcal{O}} \mathcal{C}'' \rightarrow \mathcal{C}' \otimes_{\mathcal{O}} \mathcal{C}'''$ stands for the following morphism:

$$(\Phi \otimes \Psi)(x \otimes y) = (-1)^{|\Psi||x|} \Phi(x) \otimes \Psi(y), \text{ for all homogeneous } x \in \mathcal{C}, y \in \mathcal{C}''.$$

Remark A.1.6. In virtue of Remark A.1.4 and Convention A.1.5, if $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ and $\Psi: \mathcal{C}'' \rightarrow \mathcal{C}'''$ are graded algebra morphisms, then $\Phi \otimes \Psi$ is also a graded algebra morphism w.r.t to the product defined in (A.2):

$$\begin{aligned} \Phi \otimes \Psi((c_1 \otimes c'_1)(c_2 \otimes c'_2)) &= (-1)^{|c'_1||c_2|} \Phi \otimes \Psi(c_1 c_2 \otimes c'_1 c'_2) \\ &= (-1)^{|c'_1||c_2|} \Phi(c_1 c_2) \otimes \Psi(c'_1 c'_2) \\ &= (-1)^{|c'_1||c_2|} \Phi(c_1) \Phi(c_2) \otimes \Psi(c'_1) \Psi(c'_2) \\ &= (\Phi(c_1) \otimes \Phi(c'_1)) (\Psi(c_2) \otimes \Psi(c'_2)) \\ &= (\Phi \otimes \Psi)(c_1 \otimes c'_1) (\Phi \otimes \Psi)(c_2 \otimes c'_2), \quad \text{since } \Phi, \Psi \text{ are of degree 0.} \end{aligned}$$

A.2 The tensor algebra of a linear space

Let \mathcal{V} be an \mathcal{O} -module. For $k \in \mathbb{N}_0$, the k -th tensor power $T_{\mathcal{O}}^k \mathcal{V}$ over \mathcal{O} of \mathcal{V} (elements of polynomial-degree k) is the tensor product of \mathcal{V} with itself k times, namely

$$T_{\mathcal{O}}^k \mathcal{V} := \underbrace{\mathcal{V} \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} \mathcal{V}}_{k \text{ times}}$$

we also adopt the convention $T_{\mathcal{O}}^0 \mathcal{V} \simeq \mathcal{O}$, and $\mathcal{V} \simeq T_{\mathcal{O}}^1 \mathcal{V}$. This leads to consider the \mathcal{O} -module constructed as the direct sum of the tensor powers, i.e.

$$T_{\mathcal{O}}^{\bullet} \mathcal{V} := \bigoplus_{k=1}^{\infty} T_{\mathcal{O}}^k \mathcal{V} = \mathcal{O} \oplus \mathcal{V} \oplus (\mathcal{V} \otimes_{\mathcal{O}} \mathcal{V}) \oplus (\mathcal{V} \otimes_{\mathcal{O}} \mathcal{V} \otimes_{\mathcal{O}} \mathcal{V}) \oplus \cdots$$

Proposition A.2.1. *The \mathcal{O} -module $T_{\mathcal{O}}^{\bullet} \mathcal{V}$ comes equipped with a graded unital \mathcal{O} -algebra structure, which is induced by the canonical map*

$$T_{\mathcal{O}}^k \mathcal{V} \times T_{\mathcal{O}}^{\ell} \mathcal{V} \longrightarrow T_{\mathcal{O}}^k \mathcal{V} \otimes_{\mathcal{O}} T_{\mathcal{O}}^{\ell} \mathcal{V} \simeq T_{\mathcal{O}}^{k+\ell} \mathcal{V},$$

that is extended by bilinearity to all $T_{\mathcal{O}}^{\bullet} \mathcal{V}$. We denote this product by \otimes . The unit element is $1 \in \mathcal{O} \simeq T_{\mathcal{O}}^0 \mathcal{V}$, in particular, we have $1 \otimes v = 1 \cdot v = v$ for every $v \in \mathcal{V}$.

A.2.1 $T_{\mathcal{O}}^{\bullet} \mathcal{V}$ as a co-algebra

In this section, we consider an \mathcal{O} -module \mathcal{V} which can be possibly graded. However, the construction is independent whether the module is graded or not.

The notion of co-algebra structure is important in the context of the thesis, since it allows dealing with infinite dimension objects. It appears all along the thesis. For this concept, see Definition 1.2.1. A natural way to construct a co-product (see e.g [JLL, Kas12]) structure ¹ on $T_{\mathcal{O}}^{\bullet} \mathcal{V}$

$$\Delta: T_{\mathcal{O}}^{\bullet} \mathcal{V} \longrightarrow T_{\mathcal{O}}^{\bullet} \mathcal{V} \otimes T_{\mathcal{O}}^{\bullet} \mathcal{V}$$

is to define it on elements $v \in \mathcal{V} \simeq T_{\mathcal{O}}^1 \mathcal{V}$ of polynomial-degree 1, and on the unit element $1 \in \mathcal{O} \simeq T_{\mathcal{O}}^0 \mathcal{V}$ and extend it to a (degree 0) \mathcal{O} -algebra morphism to the whole $T_{\mathcal{O}}^{\bullet} \mathcal{V}$, namely for $v_1 \otimes \cdots \otimes v_k \in T_{\mathcal{O}}^k \mathcal{V}$,

$$\Delta(v_1 \otimes \cdots \otimes v_k) = \Delta(v_1) \cdots \Delta(v_k).$$

The one which is defined by

$$\begin{aligned} v &\mapsto v \otimes 1 + 1 \otimes v \\ 1 &\mapsto 1 \otimes 1 \end{aligned}$$

endows $T_{\mathcal{O}}^{\bullet} \mathcal{V}$ with a co-associative (co-commutative) co-algebra structure, that is, it satisfies the axioms of Definition 1.2.1. Indeed, the maps

$$\Delta \otimes \text{id}, \text{id} \otimes \Delta: T_{\mathcal{O}}^{\bullet} \mathcal{V} \otimes T_{\mathcal{O}}^{\bullet} \mathcal{V} \rightarrow T_{\mathcal{O}}^{\bullet} \mathcal{V} \otimes T_{\mathcal{O}}^{\bullet} \mathcal{V} \otimes T_{\mathcal{O}}^{\bullet} \mathcal{V}$$

¹here \otimes is an "outer" tensor product, it should not be confused with the internal tensor product in $T_{\mathcal{O}}^{\bullet} \mathcal{V}$ that denotes also its graded algebra structure.

are algebra morphisms, so are $(\Delta \otimes \text{id}) \circ \Delta$ and $(\text{id} \otimes \Delta) \circ \Delta$, therefore it suffices to check that they coincide on \mathcal{V} . Let us check that. For $v \in \mathcal{V}$,

$$\begin{aligned}
(\Delta \otimes \text{id}) \circ \Delta(v) &= \Delta \otimes \text{id}(v \otimes 1 + 1 \otimes v) \\
&= \Delta(v) \otimes 1 + \Delta(1) \otimes v \\
&= (v \otimes 1 + 1 \otimes v) \otimes 1 + (1 \otimes 1) \otimes v \\
&= (v \otimes 1) \otimes 1 + (1 \otimes v) \otimes 1 + (1 \otimes 1) \otimes v \\
&= (\text{id} \otimes \Delta) \circ \Delta(v), \quad \text{by associativity of } \otimes.
\end{aligned}$$

By extending Δ to an algebra morphism, one gets explicit expressions as follows. For $v \otimes w \in T_{\mathcal{O}}^2 \mathcal{V}$, one has

$$\begin{aligned}
\Delta(v \otimes w) &= \Delta(v) \Delta(w) \\
&= (v \otimes 1 + 1 \otimes v)(w \otimes 1 + 1 \otimes w) \\
&= (v \otimes w) \otimes 1 + v \otimes w + (-1)^{|v||w|} w \otimes v + 1 \otimes (v \otimes w).
\end{aligned}$$

We have used Formula (A.2) and $1 \otimes v = 1 \cdot v = v$, each time \otimes is the tensor symbol in $T_{\mathcal{O}} \mathcal{V}$. If \mathcal{V} is not graded, i.e., concentrated in degree zero, there is no sign $(-1)^{|v||w|}$. More generally, for every $v_1, \dots, v_n \in \mathcal{V}$,

$$\Delta(v_1 \otimes \dots \otimes v_n) = \sum_{i=1}^{n-1} \epsilon(\sigma) \sum_{\sigma \in \mathfrak{S}(i, n-i)} v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(i)} \otimes v_{\sigma(i+1)} \otimes \dots \otimes v_{\sigma(n)}, \quad (\text{A.3})$$

where $\sigma \in \mathfrak{S}(i, n-i)$ is a $(i, n-i)$ -shuffle and $\epsilon(\sigma)$ is the Koszul sign associated to the n -uplet v_1, \dots, v_n . See Section 1.1 for more details.

The goal of this chapter is to introduce some important results on homological algebra, which are used in this thesis. Although, these are classical notions in commutative algebra, I think it is important to make a brush-up on them for the readability of the thesis. Most of the notions of this chapter can be found in [Cha14, Eis95, Hid89, Mic07], I also have learned a lot from the Lecture notes [Las18].

B.1 Complexes of modules

We recall that a module over \mathcal{O} is like a vector space in the sense that all the axioms still hold, except that the underlying field is replaced by \mathcal{O} . In this section, when $\mathcal{O} = \mathbb{K}$, the reader may replace "module" by "vector space".

For us, a \mathbb{Z} -graded module over \mathcal{O} is a module \mathcal{V} endowed with a direct sum decomposition $\mathcal{V} = \bigoplus_{i \in \mathbb{Z}} \mathcal{V}_i$ of \mathcal{O} -modules. We simply say " \mathcal{O} -modules" for graded modules which are concentrated in degree zero. For every $i \in \mathbb{Z}$, elements of \mathcal{V}_i are said to be of *degree* i . Let \mathcal{V} and \mathcal{W} be graded \mathbb{Z} -modules over \mathcal{O} , a \mathcal{O} -linear map $L: \mathcal{V} \rightarrow \mathcal{W}$ is said to be *homogeneous* or a *morphism of \mathbb{Z} -graded \mathcal{O} -modules of degree* $|L| := \ell \in \mathbb{Z}$, if $L(\mathcal{V}_k) \subseteq \mathcal{V}_{k+\ell}$ for all $k \in \mathbb{Z}$. The set of all \mathcal{O} -linear maps of degree ℓ from \mathcal{V} to \mathcal{W} form an \mathcal{O} -module that we denote by $\text{Hom}_{\mathcal{O}}^{\ell}(\mathcal{V}, \mathcal{W})$. This implies that $\text{Hom}_{\mathcal{O}}(\mathcal{V}, \mathcal{W}) := \bigoplus_{\ell \in \mathbb{Z}} \text{Hom}_{\mathcal{O}}^{\ell}(\mathcal{V}, \mathcal{W})$ is a graded module. Also, this graded module comes equipped with natural graded Lie bracket given by the graded commutator, namely

$$[F, G] := F \circ G - (-1)^{|F||G|} G \circ F \tag{B.1}$$

for any homogeneous elements $F, G \in \text{Hom}_{\mathcal{O}}(\mathcal{V}, \mathcal{W})$. It is easily checked that the bracket satisfies

1. $[F, G] = -(-1)^{|F||G|} [G, F]$ (*graded skew-symmetry*)
2. $(-1)^{|F||H|} [F, [G, H]] + (-1)^{|H||G|} [H, [F, G]] + (-1)^{|G||F|} [G, [H, F]] = 0$, (*graded Jacobi identity*)

for homogeneous \mathcal{O} -linear maps $F, G, H \in \text{Hom}_{\mathcal{O}}(\mathcal{V}, \mathcal{W})$.

Definition B.1.1. A complex of \mathcal{O} -modules (\mathcal{V}_\bullet, d) is a graded module $\mathcal{V} = \bigoplus_{i \in \mathbb{Z}} \mathcal{V}_i$ together with a squared to zero \mathcal{O} -linear map $d: \mathcal{V} \rightarrow \mathcal{V}$ of degree $+1$ called the *differential map*. In other words, it is a sequence

$$\cdots \longrightarrow \mathcal{V}_{i-1} \xrightarrow{d} \mathcal{V}_i \xrightarrow{d} \mathcal{V}_{i+1} \longrightarrow \cdots \quad (\text{B.2})$$

\mathcal{O} -linear maps such that $d^2 = d \circ d = 0$.

1. For every $i \in \mathbb{Z}$, elements of \mathcal{V}_i are called *cochains of degree i* .
2. We say that (B.2) is *bounded below/above* if $\mathcal{V}_i = 0$ for $i \leq n/i \geq n$, for some $n \in \mathbb{Z}$.
3. A *subcomplex* of a complex (\mathcal{V}, d) a collection of \mathcal{O} -modules $(\mathcal{V}'_i \subseteq \mathcal{V}_i)_{i \in \mathbb{Z}}$ such that $d(\mathcal{V}'_i) \subseteq \mathcal{V}_{i+1}$ for each $i \in \mathbb{Z}$. In particular $(\mathcal{V}', d' = d|_{\mathcal{V}'})$ is a complex of \mathcal{O} -modules.

In particular, it induces a complex $(\mathcal{V}/\mathcal{V}', \bar{d})$ called the *quotient complex* where for $i \in \mathbb{Z}$, $(\mathcal{V}/\mathcal{V}')_i := \mathcal{V}_i/\mathcal{V}'_i$ and

$$d: \mathcal{V}_i/\mathcal{V}'_i \longrightarrow \mathcal{V}_{i+1}/\mathcal{V}'_{i+1}$$

is determined uniquely by the universal property of the quotient.

Remark B.1.2. Let (\mathcal{V}, d) be a complex of \mathcal{O} -modules. Denote by $d_i: \mathcal{V}_i \rightarrow \mathcal{V}_{i+1}$ the restriction of d to \mathcal{V}_i . Then $d^2 = 0$ means that $d_i \circ d_{i-1} = 0$ for each $i \in \mathbb{Z}$. In particular, $\text{Im} d_{i-1} \subseteq \ker d_i$ for every $i \in \mathbb{Z}$. This leads us to the next definition.

Definition B.1.3. Let (\mathcal{V}, d) be a complex of \mathcal{O} -modules. For each $i \in \mathbb{Z}$,

1. a *i -cocycle* of (\mathcal{V}, d) is an element of $\ker(\mathcal{V}_i \xrightarrow{d} \mathcal{V}_{i+1})$;
2. a *i -coboundary* of (\mathcal{V}, d) is an element of $\text{Im}(\mathcal{V}_{i-1} \xrightarrow{d} \mathcal{V}_i)$;
3. the *i -th cohomology group* of (\mathcal{V}, d) is the quotient $H^i(\mathcal{V}) := \frac{\ker(\mathcal{V}_i \xrightarrow{d} \mathcal{V}_{i+1})}{\text{Im}(\mathcal{V}_{i-1} \xrightarrow{d} \mathcal{V}_i)}$.

The complex (\mathcal{V}, d) is *exact at i* if $H^i(\mathcal{V}) = \{0\}$. It is said to be *exact* or *acyclic* if it is exact at every degree $i \in \mathbb{Z}$.

Definition B.1.4. Let $(\mathcal{V}, d^\mathcal{V})$ and $(\mathcal{W}, d^\mathcal{W})$ be complexes of \mathcal{O} -modules.

1. A *chain map* or *complex of \mathcal{O} -modules morphism* between the complexes (\mathcal{V}, d) and (\mathcal{W}, d) is a \mathcal{O} -linear map $L: \mathcal{V} \rightarrow \mathcal{W}$ of degree 0 , which commutes with the differentials, that is a collection of \mathcal{O} -linear map $L_\bullet: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$, such that the following diagram commutes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathcal{V}_i & \xrightarrow{d^\mathcal{V}} & \mathcal{V}_{i+1} & \longrightarrow & \cdots \\ & & \downarrow L_i & & \downarrow L_{i+1} & & \\ \cdots & \longrightarrow & \mathcal{W}_i & \xrightarrow{d^\mathcal{W}} & \mathcal{W}_{i+1} & \longrightarrow & \cdots \end{array} \quad (\text{B.3})$$

i.e. $d^\mathcal{W} \circ L_i = L_{i+1} \circ d^\mathcal{V}$ for every $i \in \mathbb{Z}$.

2. A *homotopy* between two chain maps $K_\bullet, L_\bullet: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$ is the datum $\{h_i: \mathcal{V}_i \rightarrow \mathcal{W}_{i-1}\}_{i \geq 1}$ of \mathcal{O} -linear maps, that satisfies for each $i \in \mathbb{Z}$, $K_i - L_i = d^{\mathcal{W}} \circ h_i + h_{i-1} \circ d^{\mathcal{V}}$. These maps are displayed in the following diagram as

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \mathcal{V}_{i-1} & \xrightarrow{d^{\mathcal{V}}} & \mathcal{V}_i & \xrightarrow{d^{\mathcal{V}}} & \mathcal{V}_{i+1} & \longrightarrow & \cdots \\
 & & \downarrow K_{i-1}-L_{i-1} & & \downarrow h_i & \swarrow K_i-L_i & \downarrow h_{i+1} & & \downarrow K_{i+1}-L_{i+1} \\
 \cdots & \longrightarrow & \mathcal{W}_{i-1} & \xrightarrow{d^{\mathcal{W}}} & \mathcal{W}_i & \xrightarrow{d^{\mathcal{W}}} & \mathcal{W}_{i+1} & \longrightarrow & \cdots
 \end{array} \tag{B.4}$$

- (a) When there is a homotopy between two chain maps, $L_\bullet, K_\bullet: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$, we often write $L \sim K$. One can check that \sim is indeed an equivalence relation.
- (b) Two complexes of \mathcal{O} -modules $(\mathcal{V}, d^{\mathcal{V}})$ and $(\mathcal{W}, d^{\mathcal{W}})$ are said to be *homotopy equivalent*, if there exist chain maps $L_\bullet: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$ and $K_\bullet: \mathcal{W}_\bullet \rightarrow \mathcal{V}_\bullet$ such that $L \circ K \sim \text{id}_{\mathcal{W}_\bullet}$ and $K \circ L \sim \text{id}_{\mathcal{V}_\bullet}$. Likewise, one can check that homotopy equivalence between complexes of \mathcal{O} -modules is an equivalence relation.

Remark B.1.5. Note that in particular, if $L_\bullet: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$ is a chain map that is an \mathcal{O} -linear isomorphism, then its inverse is also a chain map. Thus, they define a homotopy equivalence between $(\mathcal{V}, d^{\mathcal{V}})$ and $(\mathcal{W}, d^{\mathcal{W}})$.

Remark B.1.6. Let $L_\bullet: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$ be a chain map. For every $i \in \mathbb{Z}$, we have

$$L \left(\ker(\mathcal{V}_i \xrightarrow{d^{\mathcal{V}}} \mathcal{V}_{i+1}) \right) \subseteq \ker(\mathcal{W}_i \xrightarrow{d^{\mathcal{W}}} \mathcal{W}_{i+1}), \text{ and } L \left(\text{Im}(\mathcal{V}_{i-1} \xrightarrow{d^{\mathcal{V}}} \mathcal{V}_i) \right) \subseteq \text{Im}(\mathcal{W}_{i-1} \xrightarrow{d^{\mathcal{W}}} \mathcal{W}_i).$$

L induces naturally a well-defined \mathcal{O} -linear map $H(L): H(\mathcal{V}) \rightarrow H(\mathcal{W}), [v] \mapsto [L(v)]$: for every $v \in \ker(\mathcal{V}_i \xrightarrow{d^{\mathcal{V}}} \mathcal{V}_{i+1})$ and $v_0 \in \mathcal{V}_{i-1}$

$$H(L)([v + d^{\mathcal{V}}(v_0)]) = [L(v + d^{\mathcal{V}}(v_0))] = [L(v) + d^{\mathcal{W}} \circ L(v_0)] = [L(v)] = H(L)([v]).$$

Notice that

1. homotopic chain maps induce the same map on cohomology groups.
2. if the complexes of \mathcal{O} -modules $(\mathcal{V}, d^{\mathcal{V}})$ and $(\mathcal{W}, d^{\mathcal{W}})$ are homotopy equivalent through L , then $H(L): H(\mathcal{V}) \rightarrow H(\mathcal{W})$ is an isomorphism.

Proposition B.1.7. *If (\mathcal{V}', d') is an acyclic subcomplex of a complex (\mathcal{V}, d) , then*

$$H^\bullet(\mathcal{V}/\mathcal{V}') \simeq H^\bullet(\mathcal{V}).$$

Proof. The projection $p_\bullet: \mathcal{V} \rightarrow \mathcal{V}/\mathcal{V}'$ is a chain map from (\mathcal{V}, d) and $(\mathcal{V}/\mathcal{V}', \bar{d})$. We claim p_\bullet induces an isomorphism on the cohomology groups. To show injectivity of $H(p): H^\bullet(\mathcal{V}) \rightarrow H^\bullet(\mathcal{V}/\mathcal{V}')$, let $e \in \ker d$ such that $p(e) = \bar{e} \in \text{Im } \bar{d}$, i.e. there is $u \in \mathcal{V}_{|e|-1}$ such that $\bar{e} = \bar{d}(u)$. It follows that $e - du \in \mathcal{V}'_{|e|}$. This implies,

$$d'(e - du) = d(e - du) = 0. \tag{B.5}$$

By Exactness of (\mathcal{V}', d') ,

$$\begin{aligned}
 &\implies e - du = d'(v), && \text{(for some } v \in \mathcal{V}'_{|e|-1} \subset \mathcal{V}_{|e|-1}) \\
 &\implies e = d(u + v) \in \text{Im } d.
 \end{aligned}$$

This proves injectivity. Surjectivity goes as follows: let $e \in \mathcal{V}_i$ such that $\bar{d}(\bar{e}) = 0$.

$$\bar{d}(\bar{e}) = 0 \implies d(e) \in \mathcal{V}'_{i+1}$$

We have, $d'(d(e)) = d \circ d(e) = 0$. By exactness of (\mathcal{V}', d') , we can write $d(e) = d'(v)$ for some $v \in \mathcal{V}'_i$. This implies that, $e - v = u \in \ker d$. Therefore,

$$H(p)([u]) = [p(u)] = [p(e) - p(v)] = [p(e)] = [\bar{e}].$$

This completes the proof. □

The Chevalley-Eilenberg complex

The following example of complex is important. We have used it several times in this thesis. Especially in Chapter 9 to define obstruction classes. Let us recall the definition. We refer the reader e.g. to [Wag10] for more details.

Let $(\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}})$ be a Lie algebra and V a \mathbb{K} -vector space.

Definition B.1.8. A *representation or action* of \mathfrak{g} on V is a Lie algebra morphism

$$\nu: (\mathfrak{g}, [\cdot, \cdot]_{\mathfrak{g}}) \longrightarrow (\text{End}(V), [\cdot, \cdot]) \tag{B.6}$$

where $(\text{End}(V), [\cdot, \cdot])$ denotes the vector space $\text{End}(V)$ of endomorphisms of V together with the Lie bracket $[\cdot, \cdot]$ which is the commutator: $[\alpha, \beta] = \alpha \circ \beta - \beta \circ \alpha$, $\forall \alpha, \beta \in \text{End}(V)$. In this case, V is then called a \mathfrak{g} -*module* (w.r.t to ν). In the literature, the action ν is often denoted by \cdot .

Equation (B.6) means that for all $x, y \in \mathfrak{g}$,

$$\nu([x, y]_{\mathfrak{g}}) = [\nu(x), \nu(y)] = \nu(x) \circ \nu(y) - \nu(y) \circ \nu(x).$$

Example B.1.9. Here are two important examples of \mathfrak{g} -modules that we often use.

- *The adjoint action:* \mathfrak{g} acts on itself by the Lie bracket, i.e. $\nu(x)(y) := [x, y]_{\mathfrak{g}}$ for all $x, y \in \mathfrak{g}$. Indeed, $\nu: \mathfrak{g} \longrightarrow \text{End}(\mathfrak{g})$, $x \longmapsto [x, \cdot]_{\mathfrak{g}} =: \text{ad}_x$ is a Lie algebra morphism:

$$\begin{aligned} \text{ad}_{[x, y]_{\mathfrak{g}}} &= [[x, y]_{\mathfrak{g}}, \cdot]_{\mathfrak{g}} \\ &= -[[y, \cdot]_{\mathfrak{g}}, x]_{\mathfrak{g}} - [[\cdot, x]_{\mathfrak{g}}, y]_{\mathfrak{g}}, \quad (\text{by identity of Jacobi}) \\ &= [x, [y, \cdot]_{\mathfrak{g}}]_{\mathfrak{g}} - [y, [x, \cdot]_{\mathfrak{g}}]_{\mathfrak{g}} \\ &= \text{ad}_x \circ \text{ad}_y - \text{ad}_y \circ \text{ad}_x = [\text{ad}_x, \text{ad}_y]. \end{aligned}$$

- *The trivial action:* i.e. \mathbb{K} is a \mathfrak{g} -module through the action $\nu(x)(\lambda) := 0$ for all $x \in \mathfrak{g}$ and all $\lambda \in \mathbb{K}$.

Now let us recall the definition of the Chevalley-Eilenberg complex.

Definition B.1.10. Let $\nu: \mathfrak{g} \rightarrow \text{End}(V)$ be an action of \mathfrak{g} on V . The *Chevalley-Eilenberg complex of \mathfrak{g} valued in V* is the complex

$$\cdots \rightarrow \text{Hom}_{\mathbb{K}}(\wedge^{i-1} \mathfrak{g}, V) \xrightarrow{d^{CE}} \text{Hom}_{\mathbb{K}}(\wedge^i \mathfrak{g}, V) \xrightarrow{d^{CE}} \text{Hom}_{\mathbb{K}}(\wedge^{i+1} \mathfrak{g}, V) \rightarrow \cdots \quad (\text{B.7})$$

whose i -th cochains space is defined to be $\text{Hom}_{\mathbb{K}}(\wedge^i \mathfrak{g}, V)$, the vector space of i -linear skew-symmetric linear maps from $\underbrace{\mathfrak{g} \times \cdots \times \mathfrak{g}}_{i\text{-times}}$ to V , under the convention $\text{Hom}_{\mathbb{K}}(\wedge^0 \mathfrak{g}, V) \simeq V$. The differential map is defined for $\mu \in \text{Hom}_{\mathbb{K}}(\wedge^i \mathfrak{g}, V)$ by

$$\begin{aligned} (d^{CE} \mu)(x_1, \dots, x_{i+1}) &= \sum_{k=1}^{i+1} (-1)^{k-1} \nu(x_k)(\mu(x_1, \dots, \hat{x}_k, \dots, x_{i+1})) \\ &+ \sum_{1 \leq k < l \leq i+1} (-1)^{k+l} \mu([x_k, x_l]_{\mathfrak{g}}, x_1, \dots, \hat{x}_{kl}, \dots, x_{i+1}) \end{aligned}$$

where \hat{x}_k means x_k is missing in the k -th place also \hat{x}_{kl} means that x_k, x_l are missing in the k -th and l -th place respectively.

Remark B.1.11.

1. It follows from the identity of Jacobi that $d^{CE} \circ d^{CE} = 0$.

2. For all $x, y, z \in \mathfrak{g}$ we have,

- $d^{CE}(x) = \nu(x)$.
- For $\mu \in \text{Hom}_{\mathbb{K}}(\mathfrak{g}, V)$,

$$(d^{CE} \mu)(x, y) = \nu(x)(\mu(y)) - \nu(y)(\mu(x)) - \mu([x, y]_{\mathfrak{g}}).$$

- For $\eta \in \text{Hom}_{\mathbb{K}}(\wedge^2 \mathfrak{g}, V)$,

$$\begin{aligned} (d^{CE} \eta)(x, y, z) &= \nu(x)(\eta(y, z)) - \nu(y)(\eta(x, z)) + \nu(z)(\eta(x, y)) \\ &- \eta([x, y]_{\mathfrak{g}}, z) + \eta([x, z]_{\mathfrak{g}}, y) - \eta([y, z]_{\mathfrak{g}}, x). \end{aligned}$$

Remark B.1.12. When \mathfrak{g} is of finite dimension, the Chevalley-Eilenberg complex (B.7) is canonically isomorphic to the complex $(\wedge^{\bullet} \mathfrak{g}^* \otimes V, d)$ and for $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in \wedge^k \mathfrak{g}^*$,

$$d(\xi \otimes v) = \sum_{i=1}^n (\xi \wedge \xi_i) \otimes (\xi_i \cdot v) - d_{\mathfrak{g}}(\xi) \otimes v \quad (\text{B.8})$$

where ξ_1, \dots, ξ_n is a basis of \mathfrak{g} . In the formula (B.8), $d_{\mathfrak{g}}$ is the Chevalley-Eilenberg differential of \mathfrak{g} w.r.t the trivial action on \mathbb{K} .

B.1.1 Operations on complexes

We have used and adapted the following lemma many times in this thesis, e.g. Section 4.3.1.

Lemma B.1.13. *Let $(\mathcal{V}_\bullet, d^\mathcal{V})$ and $(\mathcal{W}_\bullet, d^\mathcal{W})$ be complexes of \mathcal{O} -modules. Then, the pair $(\text{Hom}_\mathcal{O}^\bullet(\mathcal{V}, \mathcal{W}), \partial)$ is a complex of \mathcal{O} -modules, where the differential map is given by*

$$\partial(F) := d^\mathcal{W} \circ F - (-1)^{|F|} F \circ d^\mathcal{V}, \tag{B.9}$$

for all homogeneous element $F \in \text{Hom}_\mathcal{O}(\mathcal{V}, \mathcal{W})$.

Proof. The \mathcal{O} -linear map, $\partial: \text{Hom}_\mathcal{O}(\mathcal{V}, \mathcal{W}) \rightarrow \text{Hom}_\mathcal{O}(\mathcal{V}, \mathcal{W})$, is clearly of degree $+1$, since the maps $d^\mathcal{V}$ and $d^\mathcal{W}$ are. Moreover, for every $F \in \text{Hom}_\mathcal{O}(\mathcal{V}, \mathcal{W})$

$$\begin{aligned} \partial^2(F) &= \partial(d^\mathcal{W} \circ F - (-1)^{|F|} F \circ d^\mathcal{V}) \\ &= \underbrace{d^\mathcal{W} \circ d^\mathcal{W} \circ F}_{=0} - (-1)^{|F|+1} \cancel{d^\mathcal{W} \circ F \circ d^\mathcal{V}} - (-1)^{|F|} \left(\cancel{d^\mathcal{W} \circ F \circ d^\mathcal{V}} - (-1)^{|F|+1} \underbrace{F \circ d^\mathcal{V} \circ d^\mathcal{V}}_{=0} \right) \\ &= 0. \end{aligned}$$

□

Remark B.1.14. It's worth it to notice that

1. the cocycles F of $(\text{Hom}_\mathcal{O}^\bullet(\mathcal{V}, \mathcal{W}), \partial)$ are those that satisfy $d^\mathcal{V} \circ F = (-1)^{|F|} F \circ d^\mathcal{W}$. In particular, the chain maps between $(\mathcal{V}, d^\mathcal{V})$ and $(\mathcal{W}, d^\mathcal{W})$ are the 0-cocycles of $(\text{Hom}_\mathcal{O}^\bullet(\mathcal{V}, \mathcal{W}), \partial)$.
2. two chain maps $F_\bullet, G_\bullet: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$ are homotopic if and only if $G - F$ is 0-coboundary of $(\text{Hom}_\mathcal{O}^\bullet(\mathcal{V}, \mathcal{W}), \partial)$, that is, there exists a \mathcal{O} -linear map $H \in \text{Hom}_\mathcal{O}^{-1}(\mathcal{V}, \mathcal{W})$ of degree -1 such that

$$F - G = d^\mathcal{W} \circ H + H \circ d^\mathcal{V}.$$

3. when $(\mathcal{V}_\bullet, d^\mathcal{V}) = (\mathcal{W}_\bullet, d^\mathcal{W})$, we have $(\text{Hom}_\mathcal{O}^\bullet(\mathcal{V}, \mathcal{V}), \partial = [d^\mathcal{V}, \cdot])$.

Direct sum and tensor product of complexes

Let $(\mathcal{V}_\bullet, d^\mathcal{V})$ and $(\mathcal{W}_\bullet, d^\mathcal{W})$ be complexes of \mathcal{O} -modules. Then, the tensor product $\mathcal{V}_\bullet \otimes_\mathcal{O} \mathcal{W}_\bullet$ together with the grading

$$(\mathcal{V} \otimes_\mathcal{O} \mathcal{W})_k = \bigoplus_{r+i=j=k} \mathcal{V}_i \otimes_\mathcal{O} \mathcal{W}_j$$

for $k \in \mathbb{Z}$, comes equipped with a differential map classically defined by

$$\partial = d^\mathcal{V} \otimes \text{id} + \text{id} \otimes d^\mathcal{W},$$

namely,

$$\partial(v \otimes w) = d^\mathcal{V}(v) \otimes w + (-1)^{|v|} v \otimes d^\mathcal{W}(w),$$

for all homogeneous elements $v, w \in \mathcal{V} \otimes_\mathcal{O} \mathcal{W}$, is complex of \mathcal{O} -modules. The operator ∂ is indeed of degree $+1$. It is easily checked that $\partial^2 = 0$.

Bi-complex

Definition B.1.15. A *bi-complex* or *double complex* is a collection of \mathcal{O} -modules $\mathcal{V} = (\mathcal{V}_{i,j})_{i,j \in \mathbb{Z}}$ together with two families of \mathcal{O} -linear maps

1. $d_{i,j}^h: \mathcal{V}_{i,j} \rightarrow \mathcal{V}_{i+1,j}$, such that $d_{i,j}^h \circ d_{i-1,j}^h = 0$, for $i, j \in \mathbb{Z}$ called *horizontal differential map*
2. $d_{i,j}^v: \mathcal{V}_{i,j} \rightarrow \mathcal{V}_{i,j+1}$, such that $d_{i,j}^v \circ d_{i,j-1}^v = 0$, for all $i, j \in \mathbb{Z}$ called *vertical differential map*

that obey for all $i, j \in \mathbb{Z}$ the identity

$$d_{i,j+1}^h \circ d_{i,j}^v = d_{i,j}^v \circ d_{i+1,j}^h.$$

In this thesis we only consider first quadrant bi-complexes that is, $\mathcal{V}_{i,j} = 0$ for all $i \in \mathbb{Z}_{\leq 0}$ and $j \in \mathbb{N}_0$. These can be represented as the commutative diagram

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & \uparrow \\
 \cdots & \rightarrow & \mathcal{V}_{i,j} & \xrightarrow{d^h} & \mathcal{V}_{i,j} & \xrightarrow{d^h} & \mathcal{V}_{i,j} & \rightarrow & 0 \\
 & & d^v \uparrow & & d^v \uparrow & & d^v \uparrow & & \\
 \cdots & \rightarrow & \mathcal{V}_{i,j} & \xrightarrow{d^h} & \mathcal{V}_{i,j} & \xrightarrow{d^h} & \mathcal{V}_{i,j} & \rightarrow & 0 \\
 & & d^v \uparrow & & d^v \uparrow & & d^v \uparrow & & \\
 \cdots & \rightarrow & \mathcal{V}_{i,j} & \xrightarrow{d^h} & \mathcal{V}_{i,j} & \xrightarrow{d^h} & \mathcal{V}_{i,j} & \rightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & \text{"-2 column"} & & \text{"-1 column"} & & \text{"last column"} & &
 \end{array} \tag{B.10}$$

One associate to the bi-complex (B.10) the so-called *total complex* which is defined by the anti-diagonals of (B.10), namely $(T_r := \bigoplus_{i+j=r} \mathcal{V}_{i,j})_r$ with *total differential* $D: T_r \rightarrow T_{r+1}$ defined by

$$D(\tau_{ij}) := d^h(\tau_{ij}) - (-1)^r d^v(\tau_{ij}), \text{ for } \tau_{ij} \in \mathcal{V}_{i,j}.$$

Indeed,

$$\begin{aligned}
 D^2 &= (d^h - (-1)^{r+1} d^v) \circ (d^h - (-1)^r d^v) \\
 &= \underbrace{(d^h)^2}_{=0} - \cancel{(-1)^r d^h \circ d^v} - \cancel{(-1)^{r+1} d^v \circ d^h} - \underbrace{(d^v)^2}_{=0} \\
 &= 0.
 \end{aligned}$$

Proposition B.1.16 (Acyclic Assembly Lemma [Cha14]). *Let $(\mathcal{V}_{i,j})_{i,j \in \mathbb{Z}}$ be a first quadrant bi-complex like in the notation above such that the rows are exact, then the total complex (T_\bullet, D) is exact.*

Mapping cone. Let $(\mathcal{V}, d^\mathcal{V})$ and $(\mathcal{W}, d^\mathcal{W})$ be two complexes and $L: \mathcal{V}_\bullet \rightarrow \mathcal{W}_\bullet$ a chain map. The *mapping cone* is the complex (\mathcal{C}, ∂) whose degree i is given by $\mathcal{V}_{i-1} \oplus \mathcal{W}_i$ and whose differential is defined as

$$\partial = \begin{pmatrix} -d^\mathcal{V} & 0 \\ -L & d^\mathcal{W} \end{pmatrix}$$

That is, the differential is given on elements $(v, w) \in \mathcal{V} \oplus \mathcal{W}$ by

$$\partial(v, w) = (-d^{\mathcal{V}}(v), d^{\mathcal{W}}(w) - L(v)).$$

The mapping cone (\mathcal{C}, ∂) is exact if and only if $L: \mathcal{V}_{\bullet} \rightarrow \mathcal{W}_{\bullet}$ is a quasi-isomorphism, [Cha14], Cor. 1.5.4 p.19.

B.2 Resolutions of a module

Definition B.2.1. An \mathcal{O} -module \mathcal{V} is said to be *projective* if it fulfills the following: given a \mathcal{O} -linear map $L: \mathcal{V} \rightarrow \mathcal{Z}$, every surjective \mathcal{O} -linear map $J: \mathcal{W} \rightarrow \mathcal{Z}$ admits a \mathcal{O} -linear map $\tilde{L}: \mathcal{V} \rightarrow \mathcal{W}$ such that the following diagram commutes

$$\begin{array}{ccc} & \mathcal{V} & \\ & \swarrow \tilde{L} & \downarrow L \\ \mathcal{W} & \longrightarrow & \mathcal{Z} \longrightarrow 0 \end{array} \quad (\text{B.11})$$

Example B.2.2. Free modules are projective modules (see e.g. Proposition 5.1.2 of [Mic07]).

Definition B.2.3. Let \mathcal{A} be a \mathcal{O} -module. A *free/projective resolution* (or *resolution by free/projective modules*) of \mathcal{A} is an exact complex $(\mathcal{V}_{\bullet}, d)$

$$\cdots \longrightarrow \mathcal{V}_{-i-1} \xrightarrow{d} \mathcal{V}_{-i} \xrightarrow{d} \mathcal{V}_{-i+1} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{V}_{-2} \xrightarrow{d} \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A} \longrightarrow 0 \quad (\text{B.12})$$

such that the \mathcal{V}_{-i} 's are free/projective modules.

Remark B.2.4. The complex (B.12) may be of infinite length. When the length is finite, we say that we have a *finite* free/projective resolution. In that case, the sequence

$$0 \longrightarrow \mathcal{V}_{-n} \xrightarrow{d} \mathcal{V}_{-n+1} \xrightarrow{d} \mathcal{V}_{-n+2} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{V}_{-2} \xrightarrow{d} \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A} \longrightarrow 0 \quad (\text{B.13})$$

is exact in every degree. In particular, the sequence $0 \rightarrow \mathcal{V}_{-n} \xrightarrow{d} \mathcal{V}_{-n+1}$ is exact at $-n$. For instance, the map $\mathcal{V}_{-n} \xrightarrow{d} \mathcal{V}_{-n+1}$ is injective. Thus, $\mathcal{V}_{-n} \simeq \text{Im}(\mathcal{V}_{-n} \xrightarrow{d} \mathcal{V}_{-n+1})$. By exactness, we have that $\mathcal{V}_{-n} \simeq \text{Im}(\mathcal{V}_{-n} \xrightarrow{d} \mathcal{V}_{-n+1}) = \ker(\mathcal{V}_{-n+1} \xrightarrow{d} \mathcal{V}_{-n})$.

Proposition B.2.5. *Every \mathcal{O} -module \mathcal{A} admits a free/projective resolution.*

Proof. Firstly, notice that every module is isomorphic to a quotient of a free module: to see this, choose a set of generators $\{v_i \in \mathcal{A} \mid i \in I\}$ of the module \mathcal{A} so that $\mathcal{V} = \sum_{i \in I} \mathcal{O}v_i$. The \mathcal{O} -linear map

$$\pi: \bigoplus_{i \in I} \mathcal{O} \rightarrow \mathcal{A}, (f_j)_{j \in J \subseteq I} \mapsto \sum_{j \in J} f_j v_j, J \text{ is finite}$$

is surjective. By the first isomorphism theorem, it follows that $\mathcal{A} \simeq (\bigoplus_{i \in I} \mathcal{O}) / \ker \pi$. Now put $\bigoplus_{i \in I} \mathcal{O} =: \mathcal{V}_{-1}$. This yields an exact sequence

$$0 \longrightarrow \ker \pi \hookrightarrow \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A} \longrightarrow 0.$$

But $\ker \pi$ does not need to be a free \mathcal{O} -module, but there is a free \mathcal{O} -module \mathcal{V}_{-2} together with a surjective \mathcal{O} -linear map $\mathcal{V}_{-2} \xrightarrow{\pi_1} \ker \pi$ such that $\mathcal{V}_{-2} / \ker \pi_1 \simeq \ker \pi$. This is added to the previous sequence as follows

$$0 \longrightarrow \ker d_1 \hookrightarrow \mathcal{V}_{-2} \xrightarrow{\pi_1} \ker \pi \hookrightarrow \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A} .$$

Once again, $\ker d_1$ does not need to be a free \mathcal{O} -module, therefore we can continue the procedure. Inductively, assume that we have constructed a \mathcal{O} -linear map, $d_n: \mathcal{V}_{-n-1} \rightarrow \mathcal{V}_{-n}$, for $n \geq 2$. Thus, there exists a free \mathcal{O} -module \mathcal{V}_{-n-2} together with a surjective \mathcal{O} -linear map $\pi_{n+1}: \mathcal{V}_{-n-2} \rightarrow \mathcal{V}_{-n-1}$ such that $\mathcal{V}_{-n-2}/\ker \pi_{n+1} \simeq \ker d_n$. Joining this to the previous sequence, one get

$$0 \longrightarrow \ker d_{n+1} \hookrightarrow \mathcal{V}_{-n-2} \xrightarrow{\pi_{n+1}} \ker d_n \hookrightarrow \mathcal{V}_{-n-1} \xrightarrow{d_n} \mathcal{V}_{-n} \longrightarrow \cdots \longrightarrow \mathcal{V}_{-2} \xrightarrow{d_1} \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A}$$

(B.14)

By construction, we have $\ker(d_n) = \text{Im}(d_{n+1})$. Therefore, we have built an exact sequence up to length $n + 1$. This completes the proof.

One shall notice that the process (B.14) may be continued forever without reaching a free kernel. □

Here is an important operation on complexes. The following Proposition states the localization in the sense of item 2 of Section 3.2.1 preserves exactness. More precisely,

Proposition B.2.6 ([Sta22], Section 10.9 or [And]). *Let \mathcal{A} be an \mathcal{O} -module and $S \subset \mathcal{O}$ a multiplicative subset. For any free resolution of \mathcal{A}*

$$\cdots \xrightarrow{d} \mathcal{V}_{-3} \xrightarrow{d} \mathcal{V}_{-2} \xrightarrow{d} \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A} ,$$

the complex

$$\cdots \xrightarrow{S^{-1}d} S^{-1}\mathcal{V}_{-3} \xrightarrow{S^{-1}d} S^{-1}\mathcal{V}_{-2} \xrightarrow{S^{-1}d} S^{-1}\mathcal{V}_{-1} \xrightarrow{S^{-1}\pi} S^{-1}\mathcal{A}$$

is a free resolution of $S^{-1}\mathcal{A}$ by $S^{-1}\mathcal{O}$ -modules.

It is well-known that a submodule of a finitely generated module is not finitely generated in general. The following assertion assures that for module over Noetherian rings (see Proposition 1.4 of [Eis95], Page 28).

Proposition B.2.7. *If \mathcal{O} is Noetherian as a ring, then all submodules of an \mathcal{O} -module \mathcal{V} are finitely generated if and only if \mathcal{V} is finitely generated.*

The following theorem assures existence of free resolution of finite length in a special case.

Theorem B.2.8 (Hilbert Syzygy Theorem). [Pee, Eis04] *Assume $\mathcal{O} = \mathbb{C}[x_1, \dots, x_d]$. Any finitely generated (graded) \mathcal{O} -module \mathcal{A} admits a finite graded free resolution by finitely generated \mathcal{O} -modules*

$$\cdots \xrightarrow{d} \mathcal{V}_{-3} \xrightarrow{d} \mathcal{V}_{-2} \xrightarrow{d} \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A}$$

of length $N \leq d + 1$.

Tor complex

Let $(\mathcal{V}_\bullet, d^\mathcal{V}, \pi)$ be a projective resolution of an \mathcal{O} -module \mathcal{A} . For any \mathcal{O} -module \mathcal{B} , we consider the complex

$$\cdots \longrightarrow \mathcal{V}_{-3} \otimes_{\mathcal{O}} \mathcal{B} \xrightarrow{d^\mathcal{V} \otimes \text{id}} \mathcal{V}_{-2} \otimes_{\mathcal{O}} \mathcal{B} \xrightarrow{d^\mathcal{V} \otimes \text{id}} \mathcal{V}_{-1} \otimes_{\mathcal{O}} \mathcal{B} \xrightarrow{d^\mathcal{V} \otimes \text{id}} \mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}.$$

We define for $i \geq 0$

$$\text{Tor}^{-i}(\mathcal{A}, \mathcal{B}) := H^{-i}(\mathcal{V}_\bullet \otimes_{\mathcal{O}} \mathcal{B}).$$

Here are some properties of Tor [Las18].

Proposition B.2.9. *The functor Tor satisfies*

1. $\text{Tor}^0(\mathcal{A}, \mathcal{B}) \simeq \mathcal{A} \otimes_{\mathcal{O}} \mathcal{B}$.
2. If \mathcal{A} is projective, then $\text{Tor}^{-i}(\mathcal{A}, \mathcal{B}) = 0$ for every $i \geq 1$.
3. If \mathcal{B} is flat, then $\text{Tor}^{-i}(\mathcal{A}, \mathcal{B}) = 0$ for every $i \geq 1$.
4. For all i , $\text{Tor}^{-i}(\mathcal{A}, \mathcal{B}) = \text{Tor}^{-i}(\mathcal{B}, \mathcal{A})$.
5. The construction of $\text{Tor}(\mathcal{A}, \mathcal{B})$ is independent of the choice of the resolution, i.e. any other projective resolution of \mathcal{A} or \mathcal{B} yields the same Tor groups.

Minimal resolutions

Detailed definitions on local rings can be found in [Mas62].

- Definition B.2.10.**
1. A ring R is said to be a *local ring* if it has a "unique maximal ideal", i.e., a proper ideal $\mathfrak{m} \subset R$ such that \mathfrak{m} contains every other ideal of R .
 2. A local ring R with maximal ideal \mathfrak{m} is called *regular* if \mathfrak{m} can be generated by n elements, where $n = \dim R$ is the Krull dimension¹ of R .

Assume now that R is a local ring with maximal ideal \mathfrak{m} , and let $\mathbb{K} := R/\mathfrak{m}$. Geometrically, local rings correspond to germs of functions on a manifold or affine variety at a point.

We assume that (R, \mathfrak{m}) is a local Noetherian commutative ring. Here is an important lemma that uses definition of local rings.

Lemma B.2.11 (Nakayama). *Let \mathcal{V} be a finitely generated R -module such that $r = \dim(\mathcal{V}/\mathfrak{m}\mathcal{V}) < \infty$. Then, any basis of the vector space $\mathcal{V}/\mathfrak{m}\mathcal{V}$ lifts to a (minimal) generating set for \mathcal{V} as a R -module. In particular, \mathcal{V} can be generated by r elements.*

Proof. [Eis04] Lemma 1.4, p. 6. □

Remark B.2.12. By Nakayama Lemma, a local ring R is regular if the dimension of the R/\mathfrak{m} -vector space $\mathfrak{m}/\mathfrak{m}^2$ is $\dim R$.

¹the Krull dimension of R is by definition the supremum of lengths of all chains of prime ideals in R [Hid89], p. 30

One can construct free resolutions of finitely generated modules over R like in the proof of Proposition B.2.5 by taking a minimal set of homogeneous generators at each step. This can be formalized as follows,

Definition B.2.13. A projective resolution

$$\cdots \xrightarrow{d} \mathcal{V}_{-3} \xrightarrow{d} \mathcal{V}_{-2} \xrightarrow{d} \mathcal{V}_{-1} \xrightarrow{\pi} \mathcal{A} \longrightarrow 0 \quad (\text{B.15})$$

of a finitely generated R -module \mathcal{A} is said to be *minimal* if the differential map satisfies $d(\mathcal{V}_{-i}) \subseteq \mathfrak{m}\mathcal{V}_{-i+1}$ for all $i \geq 2$.

Proposition B.2.14. [Eis04] *Every finitely generated R -module \mathcal{A} admits a minimal free resolution.*

Proof. The proof goes just like in Proposition B.2.5, one just need to take minimal set of generators in the construction. By doing so, it suffices to check that $d(\mathcal{V}_{-2}) \subseteq \mathcal{V}_{-1}$: Let $r = \dim(\mathcal{A}/\mathfrak{m}\mathcal{A})$. By Nakayama Lemma, we can take $\mathcal{V}_{-1} = R^r$. One has a short exact sequence,

$$0 \longrightarrow d(\mathcal{V}_{-2}) = \ker \pi \hookrightarrow \mathcal{V}_{-1} \longrightarrow \mathcal{A} \longrightarrow 0.$$

It induces an exact sequence

$$d(\mathcal{V}_{-2})/\mathfrak{m}d(\mathcal{V}_{-2}) \hookrightarrow \mathcal{V}_{-1}/\mathfrak{m}\mathcal{V}_{-1} \longrightarrow \mathcal{A}/\mathfrak{m}\mathcal{A} \longrightarrow 0,$$

by tensorizing with $\mathbb{K} = R/\mathfrak{m}$. Since by construction $\mathcal{V}_{-1}/\mathfrak{m}\mathcal{V}_{-1} \simeq \mathcal{A}/\mathfrak{m}\mathcal{A} \simeq \mathbb{K}^r$, the image of the map $d(\mathcal{V}_{-2})/\mathfrak{m}d(\mathcal{V}_{-2}) \hookrightarrow \mathcal{V}_{-1}/\mathfrak{m}\mathcal{V}_{-1}$ is zero, i.e., $d(\mathcal{V}_{-2}) \subseteq \mathfrak{m}\mathcal{V}_{-1}$. Likewise, by Noetheriality of R , $\ker \pi \subset \mathcal{V}_{-1}$ is finitely generated. One can repeat the procedure by starting with a minimal set of generators of $\ker \pi$ and so on. The proof continues by recursion. \square

Remark B.2.15. Minimal resolutions and the Tor complex: given a minimal projective resolution as in (B.15), its quotient by the maximal ideal \mathfrak{m} corresponds to the complex

$$\cdots \longrightarrow \mathcal{V}_{-3} \otimes_{\mathcal{O}} \mathbb{K} \xrightarrow{d \otimes \text{id}} \mathcal{V}_{-2} \otimes_{\mathcal{O}} \mathbb{K} \xrightarrow{d \otimes \text{id}} \mathcal{V}_{-1} \otimes_{\mathcal{O}} \mathbb{K} \xrightarrow{\pi \otimes \text{id}} \mathcal{A} \otimes_{\mathcal{O}} \mathbb{K} \longrightarrow 0$$

whose cohomology compute $\text{Tor}^{\bullet}(\mathcal{A}, \mathbb{K})$. By minimality of $(\mathcal{V}_{\bullet}, d, \pi)$ one has $d \otimes \text{id} \equiv 0$, therefore $\text{Tor}^{-i}(\mathcal{A}, \mathbb{K}) \simeq \mathcal{V}_{-i} \otimes_{\mathcal{O}} \mathbb{K}$ for every $i \geq 2$. In particular, the ranks of the \mathcal{V}_{-i} 's for $i \geq 2$ are independent of the choices made in the construction of the minimal projective resolution.

Koszul complex

Assume that \mathcal{V} is a free \mathcal{O} -module of finite rank n , which is concentrated in degree -1 . Here, we denote by $\bigwedge^{\bullet} \mathcal{V}$ the graded symmetric algebra of \mathcal{V} . Let $F: \mathcal{V} \rightarrow \mathcal{O}$ be a \mathcal{O} -linear map. Given a free basis e_1, \dots, e_n of \mathcal{V} , F is completely determined by n -uplet $(f_1, \dots, f_n) \subset \mathcal{O}$.

Definition B.2.16. [Eis95, Hid89] The *Koszul complex associated to (f_1, \dots, f_n)* is the complex

$$0 \longrightarrow \bigwedge^n \mathcal{V} \xrightarrow{d} \bigwedge^{n-1} \mathcal{V} \xrightarrow{d} \cdots \longrightarrow \bigwedge^2 \mathcal{V} \xrightarrow{d} \mathcal{V} \xrightarrow{F} \mathcal{O}. \quad (\text{B.16})$$

whose differential $d: \bigwedge^{\bullet} \mathcal{V} \xrightarrow{d} \bigwedge^{\bullet-1} \mathcal{V}$ is the unique derivation of $\bigwedge^{\bullet} \mathcal{V}$ such that, $d|_{\mathcal{V}} = F$. In other words,

$$d(e_{i_1} \wedge \cdots \wedge e_{i_k}) = \sum_{j=1}^k (-1)^{j-1} F(e_{i_j}) e_{i_1} \wedge \cdots \wedge \hat{e}_{i_j} \cdots \wedge e_{i_k}. \quad (\text{B.17})$$

It is easily checked that this gives a well-defined complex.

Definition B.2.17. An ordered sequence of elements $f_1, \dots, f_r \in \mathcal{O}$ is called a *regular sequence* on a module \mathcal{V} if the following conditions are satisfied

1. f_i is not a zero divisor on the quotient $\mathcal{V}/\langle f_1, \dots, f_{i-1} \rangle \mathcal{V}$,
2. $\langle f_1, \dots, f_r \rangle \mathcal{V} \neq \mathcal{V}$.

Proposition B.2.18. *If (f_1, \dots, f_k) is regular sequence on \mathcal{O} , then the Koszul complex (B.16) has no cohomology in degree less equal to -1 .*

Proof. Theorem 16.5. of [Hid89]. □

Example B.2.19. For $\mathcal{O} = \mathbb{K}[x_1, \dots, x_d]$ be the polynomial ring in d indeterminates, the Koszul complex which is associated to (x_1, \dots, x_d) induces a free resolution of $\mathbb{K} \simeq \mathcal{O}/\langle x_1, \dots, x_d \rangle$.

B.3 Geometric resolutions of a singular foliation

Let us start with this definition.

Definition B.3.1. Let \mathfrak{F} be a singular foliation on M . A *complex of vector bundles over \mathfrak{F}* consists of a triple $(E_\bullet, d^\bullet, \rho)$, where

1. $E_\bullet = (E_{-i})_{i \geq 1}$ is a family of vector bundles over M , indexed by negative integers.
2. $d^{(i+1)} \in \text{Hom}(E_{-i-1}, E_{-i})$ is a vector bundle morphism over the identity of M called the *differential map*
3. $\rho: E_{-1} \rightarrow TM$ is a vector bundle morphism over the identity of M called the *anchor map* with $\rho(\Gamma(E_{-1})) = \mathfrak{F}$.

such that

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & E_{-i-1} & \xrightarrow{d^{(i+1)}} & E_{-i} & \xrightarrow{d^{(i)}} & E_{-i-1} & \longrightarrow & \cdots & \xrightarrow{d^{(2)}} & E_{-1} & \xrightarrow{\rho} & TM \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
 & & M & & M & & M & & & & M & & M
 \end{array} \tag{B.18}$$

which form a (chain) complex, i.e.

$$d^{(i)} \circ d^{(i+1)} = 0 \text{ and } \rho \circ d^{(2)} = 0.$$

Remark B.3.2. Two main cohomology groups can be associated to a complex of vector bundles over \mathfrak{F} :

1. **Cohomology at the level of sections.** The complex of vector bundles (B.18) induces a complex of sheaves of modules over functions. More explicitly, for every open subset $\mathcal{U} \subset M$, there is a complex:

$$\cdots \longrightarrow \Gamma_{\mathcal{U}}(E_{-i-1}) \xrightarrow{d^{(i+1)}} \Gamma_{\mathcal{U}}(E_{-i}) \xrightarrow{d^{(i)}} \Gamma_{\mathcal{U}}(E_{-i+1}) \longrightarrow \cdots \xrightarrow{d^{(2)}} \Gamma_{\mathcal{U}}(E_{-1}) \xrightarrow{\rho} \mathfrak{F}_{\mathcal{U}} \subset \mathfrak{X}(\mathcal{U}).$$

Since $\text{Im}(d^{(i+1)}) \subseteq \ker d^{(i)}$ for every $i \in \mathbb{N}$, we are allowed to define the quotient spaces,

$$H^{-i}(E_{\bullet}, \mathcal{U}) = \begin{cases} \frac{\ker\left(\Gamma_{\mathcal{U}}(E_{-1}) \xrightarrow{\rho} \mathfrak{F}\right)}{\text{Im}\left(\Gamma_{\mathcal{U}}(E_{-2}) \xrightarrow{d^{(2)}} \Gamma_{\mathcal{U}}(E_{-1})\right)} & \text{for } i = 1 \\ \frac{\ker\left(\Gamma_{\mathcal{U}}(E_{-i}) \xrightarrow{d^{(i)}} \Gamma_{\mathcal{U}}(E_{i+1})\right)}{\text{Im}\left(\Gamma_{\mathcal{U}}(E_{-i-1}) \xrightarrow{d^{(i+1)}} \Gamma_{\mathcal{U}}(E_{-i})\right)} & \text{if } i \geq 2. \end{cases}$$

They are modules over functions on \mathcal{U} . For each $i \geq 1$, $H^{-i}(E_{\bullet}, \mathcal{U})$ is called the i -th cohomology of $(E_{\bullet}, d^{\bullet}, \rho)$ at the level of sections on \mathcal{U} .

- (a) We say then $(E_{\bullet}, d^{\bullet}, \rho)$ is a *geometric resolution* of \mathfrak{F} if for every open set $\mathcal{U} \subset M$ and $i \geq 1$, if it induces an exact complex on the level of sections, that is

$$H^{-i}(E_{\bullet}, \mathcal{U}') = \{0\}$$

for every open subset $\mathcal{U}' \subset \mathcal{U}$.

- (b) A geometric resolution $(E_{\bullet}, d^{\bullet}, \rho)$ of \mathfrak{F} is said to be *minimal* at a point $m \in M$ if for each $i \geq 2$ the linear map $d_{|m}^{(i)}: E_{-i} \rightarrow E_{-i+1}|_m$ vanishes.

2. **Cohomology at an arbitrary point** $m \in M$. Also, the complex of vector bundles (B.18), at an arbitrary point $m \in M$, restricts to a complex of vector spaces

$$\cdots \rightarrow E_{-i-1}|_m \xrightarrow{d_{|m}^{(i+1)}} E_{-i}|_m \xrightarrow{d_{|m}^{(i)}} E_{-i+1}|_m \rightarrow \cdots \xrightarrow{d_{|m}^{(2)}} E_{-1} \xrightarrow{\rho|_m} T_m M.$$

We can look at the quotient vector spaces:

$$H^{-i}(E_{\bullet}, m) = \begin{cases} \frac{\ker\left(E_{-1}|_m \xrightarrow{\rho|_m} T_m M\right)}{\text{Im}\left(E_{-2}|_m \xrightarrow{d_{|m}^{(2)}} E_{-1}|_m\right)} & \text{for } i = 1 \\ \frac{\ker\left(E_{-i}|_m \xrightarrow{d_{|m}^{(i)}} E_{i+1}|_m\right)}{\text{Im}\left(E_{-i-1}|_m \xrightarrow{d_{|m}^{(i+1)}} E_{-i}|_m\right)} & \text{if } i \geq 2. \end{cases}$$

Here we call $H^{-i}(E_{\bullet}, m)$ the i -th cohomology of $(E_{\bullet}, d^{\bullet}, \rho)$ at the point m .

It is important to notice the following: even if $(E_{\bullet}, d^{\bullet}, \rho)$ is a geometric resolution of \mathfrak{F} , there are no reasons for $(E_{\bullet}, d^{\bullet}, \rho)$ to be exact at m . For example, if the resolution is minimal at some point $m \in M$, then $H^{-i}(E_{\bullet}, m) \simeq E_{-i}|_m$ for each $i \geq 2$ and $H^{-1}(E_{\bullet}, m) \simeq \ker(\rho|_m)$.

Remark B.3.3. Most of the definitions and operations on chain complexes of modules are adapted in an obvious manner to complexes of vector bundles over M , both on the level of the sections or at a point. See also [LLS20, Lav17] for more.

On existence of geometric resolutions

Geometric resolutions of a singular foliation \mathfrak{F} as defined in Remark B.3.2(a) are not guaranteed in all contexts. However, there always exists a projective resolution of \mathfrak{F} as a \mathcal{O} -module (see B.2). But these resolutions do not induce always a geometric resolution of \mathfrak{F} , since the projective modules of a projective resolution may not correspond to vector bundles because they may not be locally finitely generated. When the latter condition is satisfied, the Serre-Swan theorem [Swa62, Mor13] states that there is a one-to-one correspondence between locally finitely generated projective modules and sections of vector bundles. Under the assumptions of the latter theorem, geometric resolutions of singular foliations are exactly projective resolutions at the sections level in the category of chain complexes by \mathcal{O} -modules, since sections of vector bundles over M are projective \mathcal{O} -modules.

The following proposition summarizes some contexts where geometric resolutions exist, see [LLS20] for their proofs.

Proposition B.3.4.

1. Any algebraic singular foliation² on \mathbb{K}^d admits geometric resolutions by trivial vector bundles and of length $\leq d + 1$.

The same holds for a real analytic or holomorphic singular foliation, but only in a neighborhood of a point.

2. A locally real analytic singular foliation on a manifold of dimension d admits a geometric resolution of length $\leq d + 1$ on any relatively compact open subset of M .

Here we have some examples of geometric resolutions of singular foliations.

Example B.3.5. Let $\mathfrak{F}_0 = \{X \in \mathfrak{X}(V) \mid X(0) = 0\}$ be the singular foliation made of all vector fields vanishing at the origin of a vector space V (e.g. think of \mathbb{C}^N or \mathbb{R}^N). The contraction by the Euler vector field $\vec{E} = \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}$ gives rise to a complex of trivial vector bundles

$$\dots \longrightarrow \wedge^3 T^*V \xrightarrow{\iota_{\vec{E}}} \wedge^2 T^*V \xrightarrow{\iota_{\vec{E}}} T^*V \xrightarrow{\iota_{\vec{E}}} \mathbb{C} \times V =: \underline{\mathbb{C}}, \quad (\text{B.19})$$

whose complex on the sections level is $(\Omega^\bullet(V), \iota_{\vec{E}})$. Here (x_1, \dots, x_N) are the canonical coordinates on V . The latter is the Koszul complex associated to the coordinate functions x_1, \dots, x_N of V . Since the x_i 's form a regular sequence, it is well known that $(\Omega^\bullet(V), \iota_{\vec{E}})$ is exact.

The following complex of vector bundles over V

$$\dots \longrightarrow \wedge^3 T^*V \otimes TV \xrightarrow{\iota_{\vec{E}}^{\otimes \text{id}}} \wedge^2 T^*V \otimes TV \xrightarrow{\iota_{\vec{E}}^{\otimes \text{id}}} T^*V \otimes TV \xrightarrow{\iota_{\vec{E}}^{\otimes \text{id}}} \underline{\mathbb{C}} \otimes TV. \quad (\text{B.20})$$

is a geometric resolution of \mathfrak{F}_0 since $(\Omega^\bullet(V) \otimes \mathfrak{X}(V), \iota_{\vec{E}} \otimes \text{id})$ is also exact (here $\Omega^i(V) := \Gamma(\wedge^i T^*V)$ stands for the sheaf of i -forms on V).

²A singular foliation which is generated by polynomial vector fields on \mathbb{K}^d .

More generally, the construction we have made in (B.20) is still possible by contracting with any vector field $X = \sum_{i=1}^N X^i \frac{\partial}{\partial x_i} \in \mathfrak{X}(V)$. The latter yields a complex of vector bundles that covers the singular foliation \mathfrak{F}_X generated by the $X^i \frac{\partial}{\partial x_j}$'s. For instance, if X is a polynomial vector field and (X^1, \dots, X^N) form a regular sequence, we get a geometric resolution of \mathfrak{F}_X .

Example B.3.6. Let $\mathfrak{F}_2 = \mathcal{I}_0^2 \mathfrak{X}(\mathbb{K}^2) \subset \mathfrak{F}_0$ be the sub-singular singular foliation made of vector fields vanishing at order 2 at the origin of \mathbb{K}^2 , where $\mathcal{I}_0^2 \subset \mathcal{O}(\mathbb{K}^2)$ is the ideal generated by the monomials x^2, xy, y^2 . Note that the ideal \mathcal{I}_0^2 admits a free resolution of the form

$$0 \longrightarrow \mathcal{O}(\mathbb{K}^2) \oplus \mathcal{O}(\mathbb{K}^2) \xrightarrow{\delta_1} \mathcal{O}(\mathbb{K}^2) \oplus \mathcal{O}(\mathbb{K}^2) \oplus \mathcal{O}(\mathbb{K}^2) \xrightarrow{\delta_0} \mathcal{I}_0^2 \longrightarrow 0, \quad (\text{B.21})$$

where for all $f, g, h \in \mathcal{O}(\mathbb{K}^2)$,

$$\delta_0(f, g, h) = x^2 f + xyg + y^2 h \text{ and } \delta_1(f, g) = (xf, xf - yg, xg).$$

The free resolution (B.21) has to take the form

$$0 \longrightarrow \Gamma(\mathcal{I}_{-2}) \xrightarrow{\delta_1} \Gamma(\mathcal{I}_{-1}) \xrightarrow{\delta_0} \mathcal{I}_0^2 \longrightarrow 0,$$

for sum trivial vector bundles $\mathcal{I}_{-1}, \mathcal{I}_{-2}$ on \mathbb{K}^2 . Thus, the following complex

$$0 \longrightarrow \mathcal{I}_{-2} \otimes T\mathbb{K}^2 \xrightarrow{\delta_1 \otimes \text{id}} \mathcal{I}_{-1} \otimes T\mathbb{K}^2 \xrightarrow{\delta_0 \otimes \text{id}} \mathcal{I}_0^2 \otimes T\mathbb{K}^2 \longrightarrow 0 \quad (\text{B.22})$$

is a geometric resolution of \mathfrak{F}_2 . Note that \mathcal{I}_{-1} can be identified with the trivial vector bundle $S^2((\mathbb{K}^2)^*)$.

More generally, let \mathfrak{F}_k be the singular foliation made of vector fields vanishing at order k at the origin of a vector space V of dimension N over \mathbb{R} or \mathbb{C} . The Hilbert's syzygy theorem assures the existence of a free resolution of length $N + 1$ of the ideal \mathcal{I}_0^k made of functions on V vanishing to order k at the origin. This resolution is of the form

$$\dots \longrightarrow \Gamma(\mathcal{I}_{-2}) \xrightarrow{\delta_1} \Gamma(\mathcal{I}_{-1}) \xrightarrow{\delta_0} \mathcal{I}_0^k \longrightarrow 0,$$

for some family of trivial vector bundles $(\mathcal{I}_{-i})_{i \geq 1}$ over V . We obtain a geometric resolution of \mathfrak{F}^k of the form

$$\dots \longrightarrow \Gamma(\mathcal{I}_{-2} \otimes TV) \xrightarrow{\delta_1 \otimes \text{id}} \Gamma(\mathcal{I}_{-1} \otimes TV) \xrightarrow{\delta_0 \otimes \text{id}} \mathcal{I}_0^k \otimes \mathfrak{X}(V) = \mathfrak{F}^k.$$

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