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Nicolas Frantz

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Théorie spectrale et théorie de la diffusion pour des opérateurs non auto-adjoints

THÈSE

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Doctorat de l'Université de Lorraine

(mention Mathématiques)

par

Nicolas FRANTZ

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Mis en page avec la classe thesul.

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Chapitre 1

Introduction

1.1 Motivations

Dans cette thèse nous nous intéressons à la théorie spectrale et la théorie de la diffusion pour des modèles non-auto-adjoints. Plus particulièrement nous souhaitons étudier l'équation de Schrödinger linéaire

$$i\partial_t\varphi = H\varphi, \tag{1.1}$$

sur un espace de Hilbert avec un Hamiltonien non-auto-adjoint. Dans cette première partie de l'introduction, nous donnons des motivations physiques justifiant cette étude. Nous présentons ici deux sujets différents. Le premier porte sur les systèmes dissipatifs, il s'agit de systèmes physiques dans lesquels il n'y a pas de conservation d'énergie des états du système au cours du temps ; on parle alors de systèmes non-conservatifs. Le second porte sur les opérateurs \mathcal{PT} -symétriques. Il s'agit d'opérateurs qui commutent avec l'opérateur de parité et avec celui d'inversion du temps. Pour d'autres exemples d'utilisations des opérateurs non-auto-adjoints en physique, le lecteur pourra consulter par exemple [4, 62] ainsi que les références mentionnées dans ces travaux.

1.1.1 Les systèmes dissipatifs

Lorsqu'un système quantique S interagit avec un autre système quantique S' , une partie de son énergie peut être transférée au système S' et ceci de façon irréversible. Cette perte irréversible d'énergie est un phénomène physique appelé dissipation quantique. En particulier l'interaction entre S et S' peut aboutir à l'absorption de S par le système S' . Un exemple célèbre de modèle dissipatif est le modèle optique nucléaire introduit par Feshbach, Porter et Weisskopf [37] en 1954. Le terme « optique » dans le nom du modèle est une référence au phénomène de réfraction et d'absorption de la lumière dans un milieu continu en optique. Ce modèle décrit l'interaction entre un neutron et un noyau. Lorsque le neutron est envoyé sur un noyau complexe cible, il peut-être diffusé élastiquement ou bien absorbé par le noyau. S'il y a absorption, le neutron et le noyau forment un nouveau système appelé noyau composite. Ce concept a été introduit par Bohr [7] en 1936. Dans

le modèle optique nucléaire la force exercée sur le neutron est décrite par un opérateur de la forme

$$H = -\Delta + V - iW, \quad (1.2)$$

agissant sur $L^2(\mathbb{R}^3, \mathbb{C})$, avec V réel et $\text{Im}(W) \geq 0$. Ici $-\Delta$ correspond à l'énergie cinétique et $V - iW$ est le potentiel associé à la force exercée par le noyau sur le neutron. Le caractère dissipatif du système se traduit dans le fait que $\text{Im}(W) \geq 0$. Comme le neutron et le noyau sont localisés en espace, V et W sont supposés à support compact ou décroissant rapidement à l'infini. Sous cette forme, $-iH$ engendre un semi-groupe fortement continu de contractions $\{e^{-itH}\}_{t \geq 0}$ (pour tout $t \geq 0$, $\|e^{-itH}\|_{B(L^2(\mathbb{R}^3, \mathbb{C}))} \leq 1$). En particulier, si à l'état initial, le neutron est dans un état normalisé φ_0 ($\|\varphi_0\|_{L^2(\mathbb{R}^3, \mathbb{C})} = 1$), alors à chaque instant $t \geq 0$ le neutron est dans l'état *non-normalisé* $\varphi_t = e^{-itH}\varphi_0$ solution de l'équation de Schrödinger (1.1). La propriété de contraction du semi-groupe d'évolution associé à $-iH$ fait que l'application $t \mapsto \|e^{-itH}\varphi_0\|_{L^2(\mathbb{R}^3, \mathbb{C})}$ est décroissante sur $[0, +\infty[$, la probabilité que le neutron dans l'état φ_0 soit absorbé par le noyau est alors

$$\rho_{\text{abs}}(\varphi_0) := 1 - \lim_{t \rightarrow +\infty} \|e^{-itH}\varphi_0\|_{L^2(\mathbb{R}^3, \mathbb{C})}^2,$$

et la probabilité que le neutron dans l'état φ_0 soit diffusé par le noyau est donnée par

$$\rho_{\text{scat}}(\varphi_0) := 1 - \rho_{\text{abs}}(\varphi_0).$$

Dans le cas où la probabilité d'absorption du neutron dans l'état φ_0 est strictement plus petite que 1, on s'attend à ce qu'il existe un état φ_+ tel que $\|\varphi_+\|_{L^2(\mathbb{R}^3, \mathbb{C})}^2 = \rho_{\text{scat}}(\varphi_0)$ et

$$\lim_{t \rightarrow +\infty} \|e^{-itH}\varphi_0 - e^{-itH_0}\varphi_+\|_{L^2(\mathbb{R}^3, \mathbb{C})} = 0. \quad (1.3)$$

Le modèle optique nucléaire a été obtenu empiriquement dans le sens où les potentiels V et W ont été déterminés grâce à des données expérimentales. Cependant sous sa forme actuelle, les quantités physiques mesurées grâce à l'expérience sont très proches des quantités physiques calculées grâce à ce modèle. Le problème décrit par l'équation (1.3) est une motivation à l'investigation d'une théorie de la diffusion pour des opérateurs dissipatifs. De nombreux auteurs ont étudié la théorie spectrale et la théorie de la diffusion pour des opérateurs dissipatifs ; non exhaustivement, on peut citer les travaux Davies [14, 16], Faupin et Fröhlich [35] et Martin [68] pour des opérateurs dissipatifs agissant sur des espaces de Hilbert abstraits, Faupin et Nicoleau [34] pour des liens avec la matrice de diffusion et Simon [85], Faupin [32], Wang et Zhu [92] pour les opérateurs de Schrödinger dissipatifs. D'autres travaux liés aux nôtres sur les opérateurs dissipatifs ont également été considérés, comme la théorie de la diffusion pour les opérateurs de Lindbald [17, 31], des estimation dispersives pour des opérateurs de Schrödinger dissipatifs [91], ou encore des méthodes de commutateurs de Mourre dans un cadre dissipatif [78, 79].

1.1.2 Les opérateurs \mathcal{PT} symétriques

Les opérateurs \mathcal{PT} -symétriques sont une classe d'opérateurs non-auto-adjoints très étudiés en mécanique quantique. En effet, comme l'ont observé Bender et Boettcher [5], une large classe d'Hamiltoniens \mathcal{PT} -invariants ont un spectre réel. Les opérateurs \mathcal{PT} -symétriques peuvent alors, dans certains cas, être pertinents en mécanique quantique. Pour des opérateurs de Schrödinger, $H = -\Delta + V(x)$ sur $L^2(\mathbb{R}^d, \mathbb{C})$, avec V un potentiel complexe, \mathcal{PT} -symétrique signifie que

$$[H, \mathcal{PT}] = 0,$$

où $[\cdot, \cdot]$ est le commutateur. \mathcal{P} est l'opérateur de parité, c'est à dire $(\mathcal{P}(\varphi))(x) = \varphi(-x)$ et \mathcal{T} est l'opérateur d'inversion du temps, $(\mathcal{T}(\varphi))(x) = \overline{\varphi(x)}$. Si en plus de la \mathcal{PT} -symétrie, l'opérateur étudié possède une certaine \mathcal{C} -symétrie, alors le spectre de celui-ci est réel. Cependant comme cela est expliqué dans [64], pour qu'un opérateur soit pertinent physiquement, il n'est pas suffisant que son spectre soit réel. En particulier, des cas mathématiquement pathologiques peuvent se produire si \mathcal{CPT} n'est pas un opérateur borné. Mathématiquement il est alors peut-être plus pertinent, plutôt que de parler de \mathcal{CPT} -symétrie, de parler de quasi-hermité dans le sens où il existe un opérateur positif Θ tel que $H\Theta = \Theta H^*$. Pour plus de détails sur la notion de quasi-hermité, nous renvoyons à [20, 83].

Bien sûr, de nombreux autres sujets concernant la théorie spectrale des opérateurs non-auto-adjoints ont été étudiés, comme par exemple l'estimation du nombre de valeurs propres et les estimations de Lieb-Thirring pour certains opérateurs non-auto-adjoints [11, 42, 43]. De tels résultats sont en particulier aussi pertinents dans notre contexte (voir l'hypothèse 1.2).

1.2 Présentation des problèmes étudiés

Dans la section 1.1, nous avons donné deux exemples différents d'applications des opérateurs non-auto-adjoints en mécanique quantique. Dans cette section nous présentons le modèle mathématique que nous avons étudié dans cette thèse ainsi que les différents problèmes auxquels nous nous sommes intéressés pour ce modèle.

1.2.1 Le modèle

Soit $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ un espace de Hilbert sur le corps des complexes. La convention que nous choisissons est que $\langle \cdot, \cdot \rangle$ est linéaire par rapport à la seconde variable et anti-linéaire par rapport à la première variable. De plus nous noterons $\mathcal{B}(\mathcal{H})$, l'espace vectoriel des opérateurs bornés sur \mathcal{H} pour la norme $\|\cdot\|_{\mathcal{H}}$. Nous considérons un opérateur H agissant sur \mathcal{H} de la forme

$$H := H_0 + V,$$

où H_0 est un opérateur auto-adjoint borné inférieurement de domaine $\mathcal{D}(H_0)$ et V est un opérateur borné. Rappelons que H_0 est borné inférieurement signifie qu'il existe $\delta \geq 0$ tel que pour tout $u \in \mathcal{D}(H_0)$,

$$\langle u, H_0 u \rangle_{\mathcal{H}} \geq \delta \|u\|_{\mathcal{H}}^2. \quad (1.4)$$

On définit la borne inférieure de H_0 comme le plus petit réel δ satisfaisant (1.4). En particulier, quitte à lui soustraire sa borne inférieure, on peut supposer que H_0 est borné inférieurement par 0. Pour l'opérateur V on suppose qu'il se factorise de la forme

$$V = CWC,$$

où C et W sont bornés. Ce type de décomposition a été introduit par Kato dans [55]. De plus on suppose que C est un opérateur *métrique*, relativement compact par rapport à H_0 . Un opérateur métrique est un opérateur positif et injectif. Dire que C est relativement compact par rapport à H_0 signifie qu'il existe $z \in \rho(H_0)$, tel que $C\mathcal{R}_0(z)$ est un opérateur compact, où $\rho(H_0)$ est l'ensemble résolvant de H_0 et $\mathcal{R}_0(z) := (H_0 - z)^{-1}$ est la résolvante de H_0 . Comme V est une perturbation bornée de l'opérateur auto-adjoint H_0 , alors H est un opérateur fermé et $-iH$ engendre un groupe d'évolution satisfaisant

$$\|e^{-itH}\|_{\mathcal{B}(\mathcal{H})} \leq e^{t\|V\|_{\mathcal{B}(\mathcal{H})}}, \forall t \in \mathbb{R}.$$

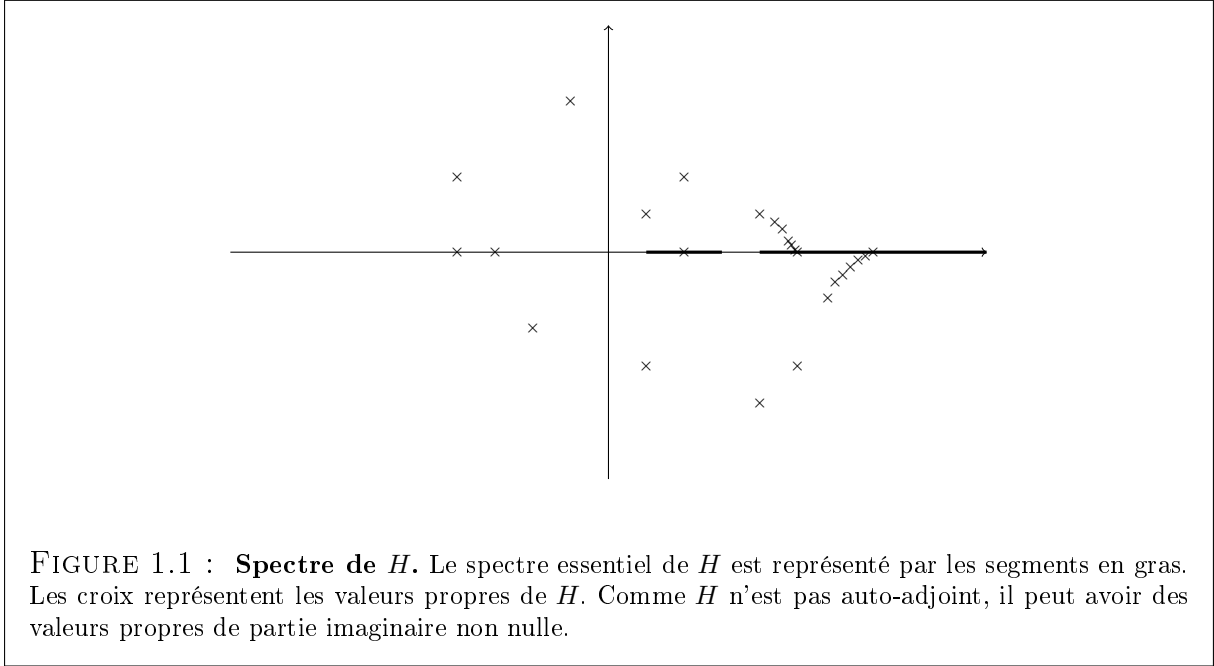
Nous définissons le spectre essentiel de H comme le complémentaire dans \mathbb{C} du spectre discret noté $\sigma_{\text{disc}}(H)$. Le spectre discret de H est défini comme l'ensemble des valeurs propres discrètes telles que le rang de la projection de Riesz est fini. Rappelons que si λ est une valeur propre discrète de H , alors la projection de Riesz de H associée à λ est définie par

$$\Pi_{\lambda}(H) := \frac{1}{2\pi i} \int_{\gamma} (z \text{Id} - H)^{-1} dz,$$

Où γ est un cercle de centre λ et de rayon suffisamment petit de sorte que l'intérieur du cercle γ ne contienne que λ comme élément du spectre de H . Comme C est relativement compact par rapport à H_0 et que l'ensemble des opérateurs compacts est un idéal bilatère de $\mathcal{B}(\mathcal{H})$, alors V est relativement compact par rapport à H_0 . Ainsi comme H_0 est auto-adjoint, le spectre essentiel de H_0 et celui de H coïncident : $\sigma_{\text{ess}}(H_0) = \sigma_{\text{ess}}(H)$ (voir par exemple [76, section XIII.4] pour plus de détails sur les perturbations relativement compactes). La figure 1.1 est un exemple d'illustration du spectre de H .

1.2.2 Description des problèmes étudiés

Dans cette sous-section nous présentons les problèmes que nous avons étudiés pour ce modèle. Plus précisément, nous avons voulu comprendre la structure spectrale et la dynamique des solutions de l'équation de Schrödinger pour l'Hamiltonien non-auto-adjoint introduit dans la section 1.2.1.



1.2.2.1 Espace des états qui disparaissent asymptotiquement

Nous définissons l'espace des états qui disparaissent asymptotiquement lorsque $t \rightarrow \pm\infty$ (ou solutions évanescents de l'équation de Schrödinger (1.1)), noté $\mathcal{H}_{\text{ads}}^{\pm}(H)$ dans la suite par

$$\mathcal{H}_{\text{ads}}^{\pm}(H) := \left\{ u \in \mathcal{H}, \lim_{t \rightarrow \pm\infty} \|e^{-itH}u\|_{\mathcal{H}} = 0 \right\}^{\text{cl}},$$

où A^{cl} désigne la fermeture d'un ensemble A . Pour le modèle optique nucléaire introduit à la section 1.1.1, les états qui disparaissent en plus infini sont les états du neutron qui sont absorbés par le noyau avec probabilité 1. L'objectif est de chercher une caractérisation de $\mathcal{H}_{\text{ads}}^{\pm}(H)$ en terme de sous-espace spectraux associés à l'opérateur H . Remarquons que si u est une combinaison linéaire finie de vecteurs propres de H associés à des valeurs propres de partie imaginaire strictement négative, alors en utilisant le développement en série de l'exponentielle, il n'est pas difficile de voir que $u \in \mathcal{H}_{\text{ads}}^{+}(H)$. En effet, si par exemple λ est une valeur propre de H de partie imaginaire strictement négative associée à un vecteur propre u , alors

$$\|e^{-itH}u\|_{\mathcal{H}} = e^{t\text{Im}(\lambda)} \|u\|_{\mathcal{H}} \xrightarrow{t \rightarrow +\infty} 0.$$

On note $\mathcal{H}_{\text{p}}^{\pm}(H)$ la fermeture de l'espace vectoriel engendré par les vecteurs propres généralisés de H associés à des valeurs propres de partie imaginaire strictement négative/positive,

$$\mathcal{H}_{\text{p}}^{\pm}(H) := (\text{Span} \{u \in \text{Ker}((H - \lambda)^{\nu_{\lambda}}), \lambda \in \sigma_{\text{disc}}(H), \pm \text{Im}(\lambda) < 0\})^{\text{cl}},$$

où ν_{λ} est la multiplicité algébrique de λ . Celle-ci est définie comme la dimension du sous-espace vectoriel limite de la suite des noyaux itérés $(\text{Ker}(H - \lambda)^k)_{k \in \mathbb{N}}$ associés à la valeur

propre λ . On a alors l'inclusion

$$\mathcal{H}_p^\pm(H) \subset \mathcal{H}_{\text{ads}}^\pm(H).$$

La question de déterminer des conditions sous lesquelles la réciproque de cette inclusion est vraie est un problème qui avait été soulevé par Davies dans [16] pour des opérateurs dissipatifs. Pour plus de détails concernant les états qui disparaissent à l'infini, nous renvoyons à la section 2.2.2.4.

1.2.2.2 Sous-espace spectral absolument continu

Nous définissons le sous-espace spectral absolument continu de H par

$$\mathcal{H}_{\text{ac}}(H) := \mathcal{M}(H)^{\text{cl}} := \left\{ u \in \mathcal{H}, \forall v \in \mathcal{H}, \int_{\mathbb{R}} |\langle e^{itH} u, v \rangle_{\mathcal{H}}|^2 dt \leq c_u \|v\|_{\mathcal{H}}^2 \right\}^{\text{cl}}.$$

Quelques commentaires s'imposent pour justifier cette définition. Si H est auto-adjoint, notre définition coïncide avec celle du sous-espace spectral absolument continu défini en terme de mesure spectrale voir par exemple [15, Théorème 6.24]. On peut remarquer que si u appartient à $\mathcal{M}(H)$ alors u est orthogonal à toute combinaison linéaire finie de vecteurs propres de H^* . En particulier, on a l'inclusion

$$\mathcal{H}_{\text{ac}}(H) \subset \mathcal{H}_p(H^*)^\perp,$$

où $\mathcal{H}_p(H^*)$ est l'espace vectoriel engendré par l'ensemble des vecteurs propres généralisés de H^* ,

$$\mathcal{H}_p(H^*) := (\text{Span} \{u \in \text{Ker}((H^* - \lambda)^{\nu_\lambda}), \lambda \in \sigma_p(H^*)\})^{\text{cl}},$$

où $\sigma_p(H)$ est l'ensemble des valeurs propres de H .

En effet, si par exemple $u \in \mathcal{M}(H)$ et v est un vecteur propre de H^* associé à une valeur propre λ réelle alors comme

$$|\langle e^{itH} u, v \rangle| = e^{t \text{Im}(\lambda)} |\langle u, v \rangle|,$$

nécessairement, u est orthogonal à v .

Une question naturelle est de déterminer des conditions sur le modèle pour obtenir l'inclusion réciproque

$$\mathcal{H}_p(H^*)^\perp \subset \mathcal{H}_{\text{ac}}(H).$$

Établir cette autre inclusion donne une autre justification de la pertinence de la définition du sous-espace spectral absolument continu. D'autres définitions de sous-espace spectral absolument continu pour des opérateurs non-auto-adjoints ont également été considérées dans la littérature, notamment grâce à la théorie des dilatations. Pour une exposition de la théorie des dilatation nous pouvons citer [15, chapitre 5] pour les opérateurs dissipatifs et [74] pour les opérateurs non-auto-adjoints. Pour plus de détails concernant notre définition de sous-espace spectral absolument continu, nous renvoyons à la section 2.2.2.5.

1.2.2.3 Théorie de la diffusion

Pour étudier le comportement asymptotique des solutions de l'équation de Schrödinger associée à notre opérateur H , nous voulons établir une théorie de la diffusion pour comparer le comportement asymptotique des groupes d'évolution associés à H et H_0 . Le problème est le suivant : pour tout $u \in \mathcal{H}$ existe-t-il $u_0 \in \mathcal{H}$ tel que

$$\lim_{t \rightarrow \infty} \|e^{-itH}u - e^{-itH_0}u_0\|_{\mathcal{H}} = 0 \quad ? \quad (1.5)$$

On remarque que si u est un vecteur propre de H associé à une valeur propre λ de partie imaginaire positive, alors comme

$$\|e^{-itH}u\|_{\mathcal{H}} = e^{t\text{Im}(\lambda)} \|u\|_{\mathcal{H}} \xrightarrow{t \rightarrow \infty} +\infty,$$

il n'existe pas de vecteur u_0 satisfaisant (1.5), car comme H_0 est auto-adjoint, le groupe d'évolution engendré par $-iH_0$ est unitaire. De la même façon, si u est un vecteur propre associé à une valeur propre de partie imaginaire négative, alors comme $u \in \mathcal{H}_{\text{ads}}^+(H)$, on a

$$\|e^{-itH}u\|_{\mathcal{H}} \xrightarrow{t \rightarrow \infty} 0.$$

Le vecteur u_0 répondant à (1.5) est le vecteur nul. Ces deux phénomènes peuvent se produire car H n'est pas auto-adjoint et n'a donc a priori pas que du spectre réel. Par ailleurs, si u_0 est un vecteur propre généralisé de H associé à une valeur propre λ réelle, alors en utilisant un développement en série entière, on observe que la norme de $e^{-itH}u_0$ explose polynomiale quand $t \rightarrow +\infty$. Ces exemples montrent donc qu'il est nécessaire de pouvoir projeter en dehors des vecteurs propres de H . On voit que résoudre le problème de théorie de la diffusion est lié à l'existence d'un sous-espace spectral absolument continu, à la caractérisation des états qui disparaissent à l'infini et plus généralement à l'existence d'un théorème de décomposition spectrale. Dans le cas où u n'est pas un vecteur propre généralisé de H , alors en utilisant que le groupe d'évolution associé à $-iH_0$ est unitaire, (1.5) se réécrit

$$\lim_{t \rightarrow \infty} \|e^{itH_0}e^{-itH}u - u_0\|_{\mathcal{H}} = 0.$$

En posant alors

$$W_+(H_0, H) = \text{s-lim}_{t \rightarrow +\infty} e^{itH_0}e^{-itH}(\text{Id} - \Pi_p(H))$$

où s-lim désigne la limite forte et $\Pi_p(H)$ désigne la projection sur le sous-espace spectral associé au spectre ponctuel de H (voir la sous-section 2.2.2.3 pour la construction de $\Pi_p(H)$), (1.5) revient à prouver l'existence de $W_+(H_0, H)$ et étudier son image et son noyau. On verra que sous nos hypothèses, la définition de $W_+(H_0, H)$ nécessite d'être modifiée afin de pouvoir montrer son existence. La théorie de la diffusion et en particulier l'étude des opérateurs d'onde a fait l'objet de très nombreux travaux dans le cas auto-adjoint, voir par exemple Agmon [2], Enss [29, 30], Kato [58, 59], Kuroda [65, 66], Simon [76, 85], Yafaev [94, 95]. Dans le cas non-auto-adjoint, c'est un peu moins le cas, cependant nous mentionnons les travaux de Davies [14, 16], Faupin, Fröhlich [35], Martin [68] et Faupin et Nicoleau [32] dans un cadre dissipatif abstrait et ceux de Kato [55], Kako et Yajima [54], Stepin [87, 88] dans le cas non dissipatif.

1.2.2.4 Singularités spectrales

L'une des spécificités de cette thèse est d'autoriser H à avoir des singularités spectrales. Les singularités spectrales sont des points du spectre essentiel de H qui ne satisfont pas un certain principe d'absorption limite. Pour commencer définissons la notion de point spectral régulier pour H .

Definition 1.2.1 (Point spectral régulier et singularité spectrale). *Soit $\lambda \in \sigma_{\text{ess}}(H)$.*

- i) On dit que λ est un point spectral régulier sortant/entrant si λ n'est pas un point d'accumulation de valeurs propres localisées en $\lambda \pm i]0, +\infty[$ et si la limite*

$$\lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\lambda \pm i\varepsilon)CW$$

existe dans $\mathcal{B}(\mathcal{H})$. Si λ n'est pas un point spectral régulier sortant/entrant de H , on dit que λ est une singularité spectrale sortante/entrante de H .

- ii) On dit que λ est un point spectral régulier de H si λ est un point spectral régulier entrant et sortant de H . Sinon on dit que λ est une singularité spectrale de H .*

- iii) On dit que l'infini est un point spectral régulier sortant/entrant de H s'il existe $m > 0$ tel que pour tout $\lambda > m$, λ est un point spectral régulier sortant/entrant et si l'application*

$$[m, +\infty[\ni \lambda \mapsto C\mathcal{R}_H(\lambda \pm i0^+)CW \in \mathcal{B}(\mathcal{H})$$

est bornée. Si l'infini n'est pas un point spectral régulier sortant/entrant de H on dit que H a une singularité spectrale sortante/entrante à l'infini.

Pour les opérateurs de Schrödinger les singularités spectrales correspondent aux résonances réelles (voir section 1.3.3). La notion de singularité spectrale que nous présentons ici est très proche de celle des articles [32, 34, 35]. Le lecteur pourra consulter la section 2.2.4 pour plus d'explications concernant les singularités spectrales.

1.3 Présentation des résultats principaux

Dans cette section nous présentons les hypothèses et les résultats principaux que nous avons obtenus dans cette thèse ainsi que quelques unes de leurs conséquences. Nous illustrerons ces résultats avec l'exemple des opérateurs de Schrödinger.

1.3.1 Hypothèses sur le modèle

Tout d'abord nous présentons les hypothèses principales et nous introduisons au fur et à mesure les notations et notions nécessaires. Pour plus de détails et de justifications concernant ces hypothèses nous renvoyons aux sections 2.3.1 et 3.3.1.

Nous supposons que la résolvante de H_0 satisfait un principe d'absorption limite avec le poids C :

Hypothèse 1.1. *On a*

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|C\mathcal{R}_0(z)C\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

Cette hypothèse assure en particulier que H_0 a un spectre purement absolument continu (voir [13, Proposition 4.1]).

Ensuite nous supposons que H a un nombre fini de valeurs propres de multiplicité algébrique finie.

Hypothèse 1.2. *H possède un nombre fini de valeurs propres et chaque valeur propre est de multiplicité algébrique finie.*

Rappelons que dans cette thèse, nous supposons que H peut avoir des singularités spectrales. Ainsi il n'est pas possible de supposer que la résolvante de H satisfait un principe d'absorption limite similaire à celui de l'hypothèse 1.1. Cependant nous supposons que les singularités de la résolvante à poids de H sur l'axe réel sont « d'ordre fini ». Tout d'abord définissons l'ordre d'une singularité spectrale.

Definition 1.3.1 (Ordre d'une singularité spectrale). *i) Soit $\lambda \in \sigma_{\text{ess}}(H)$ une singularité spectrale sortante/entrante. On dit que λ est une singularité spectrale d'ordre fini s'il existe un entier n et $\varepsilon > 0$ tels que*

$$\sup_{z \in D(\lambda, \varepsilon) \cap \mathbb{C}^\pm} |\lambda - z|^n \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (1.6)$$

Sinon on dit que λ est une singularité spectrale d'ordre infini. Si λ est une singularité spectrale sortante/entrante d'ordre fini, on définit son ordre comme le plus petit entier satisfaisant (1.6).

ii) Si H possède une singularité spectrale sortante/entrante à l'infini, on dit que l'infini est une singularité spectrale sortante/entrante d'ordre fini s'il existe un entier n , $\varepsilon_0 > 0$, $m > 0$ et $z_0 \in \rho(H) \setminus \mathbb{R}$ tels que

$$\sup_{\substack{\text{Re}(z) > m \\ \pm \text{Im}(z) < \varepsilon_0}} |z - z_0|^{-n} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (1.7)$$

Sinon on dit que l'infini est une singularité spectrale sortante/entrante d'ordre infini. Si l'infini est une singularité spectrale sortante/entrante d'ordre fini, on définit son ordre comme le plus petit entier satisfaisant (1.7).

L'hypothèse suivante nous permet de gérer les singularités :

Hypothèse 1.3. *H a un nombre fini de singularités spectrales et chaque singularité spectrale est d'ordre fini. De plus, pour tout intervalle fermé borné $I \subset \sigma_{\text{ess}}(H_0)$, ne contenant pas de singularités spectrales, il existe $\varepsilon_0 > 0$ tel que*

$$\sup_{\substack{\text{Re}(z) \in I \\ \pm \text{Im}(z) \in]0, \varepsilon_0[}} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

Enfin l'infini n'est pas une singularité spectrale, il existe $\varepsilon_0 > 0$, $m > 0$ tels que

$$\sup_{\substack{\operatorname{Re}(z) > m \\ \pm \operatorname{Im}(z) < \varepsilon_0}} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

Comme cela est expliqué à la section 2.4.4, les valeurs propres de H plongées dans le spectre essentiel sont des singularités spectrales. L'hypothèse 1.3 tient compte de cette spécificité particulière. Dans la suite nous posons $r_j(z) = \frac{(z-\lambda_j)^{\nu_j}}{(z-z_0)^{\nu_j}}$ la fonction définie sur $\mathbb{C} \setminus \{z_0\}$ avec $z_0 \in \rho(H) \setminus \mathbb{R}$ et $r_\infty(z) := \frac{1}{(z-z_0)^{\nu_\infty}}$. En posant alors

$$r(z) := \prod_{j=1}^{\infty} r_j(z),$$

l'hypothèse 1.3 implique qu'il existe $\varepsilon_0 > 0$ tel que

$$\sup_{\substack{\operatorname{Re}(z) \in \sigma_{\text{ess}}(H_0) \\ \pm \operatorname{Im}(z) \in]0, \varepsilon_0[}} |r(z)| \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

C'est cette fonction r qui permet de régulariser les singularités spectrales lorsque la résolvante à poids de H approche le spectre essentiel. De la même façon, on notera r_+ la fonction régularisant les singularités spectrales sortantes de H et r_- la fonction régularisant les singularités spectrales entrantes de H .

L'hypothèse suivante permet d'assurer l'existence des projections spectrales associées à des valeurs propres plongées dans le spectre essentiel.

Hypothèse 1.4 (Opérateur de conjugaison). *Il existe une application anti-unitaire continue $J : \mathcal{H} \rightarrow \mathcal{H}$ telle que*

- i) $J\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ et $\forall u \in \mathcal{D}(H_0), JH_0u = H_0Ju$.
- ii) $JC = CJ$ et $JW = W^*J$.

De plus pour chaque valeur propre plongée dans le spectre essentiel, $\lambda \in \sigma_{\text{ess}}(H)$, la forme bilinéaire symétrique

$$\operatorname{Ker}((H - \lambda)^{m_\lambda}) \ni (u, v) \mapsto \langle Ju, v \rangle_{\mathcal{H}} \quad \text{est non dégénérée.}$$

Enfin pour certains résultats nous aurons besoin de l'hypothèse technique suivante.

Hypothèse 1.5. *Le domaine de la restriction de H_0 à \mathcal{H}_C défini par*

$$\mathcal{D}(H_0|_{\mathcal{H}_C}) := \{u \in \mathcal{D}(H_0) \cap \mathcal{H}_C, H_0u \in \mathcal{H}_C\},$$

est dense dans \mathcal{H}_C pour la topologie de \mathcal{H}_C .

1.3.2 Résultats principaux

Dans cette sous-section, nous détaillons les résultats principaux que nous avons obtenus et nous donnons une idée des preuves et techniques que nous avons utilisées.

Nous introduisons d'abord quelques notations et remarques qui nous seront utiles dans la suite. Définissons $\mathcal{H}_C := \text{Ran}(C)$ et munissons le de la norme $\|\cdot\|_{\mathcal{H}_C}$ définie par

$$\forall u \in \mathcal{H}_C, \quad \|u\|_{\mathcal{H}_C} = \|C^{-1}u\|_{\mathcal{H}}.$$

Notons que $\|\cdot\|_{\mathcal{H}_C}$ est bien définie car C est injectif. Par ailleurs, munissons \mathcal{H} de la norme $\|\cdot\|_C$ définie par

$$\forall u \in \mathcal{H}, \quad \|u\|_C = \|Cu\|_{\mathcal{H}}.$$

Alors on peut montrer que le complété de \mathcal{H} pour la topologie de la norme $\|\cdot\|_C$ coïncide avec l'anti-dual (l'espace vectoriel des formes antilinéaires sur \mathcal{H}_C pour la norme $\|\cdot\|_{\mathcal{H}}$) de \mathcal{H}_C noté \mathcal{H}'_C . On obtient alors un triplet de Gelfand

$$\mathcal{H}_C \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}'_C.$$

Grâce à cette construction, la condition (1.8) est équivalente au fait que

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{R}_0(\lambda \pm i\varepsilon)$$

existe dans $\mathcal{B}(\mathcal{H}_C, \mathcal{H}'_C)$, l'ensemble des opérateurs bornés de \mathcal{H}_C dans \mathcal{H}'_C . Cette propriété sera utilisée dans la suite pour caractériser les singularités spectrales de H . Pour plus de détails concernant cette construction, le lecteur pourra consulter la section 2.2.3.

Théorème 1.3.1. *Supposons que l'hypothèse 1.5 est satisfaite. Soit $\lambda \in \sigma_{\text{ess}}(H)$ tel que la limite*

$$C\mathcal{R}_0(\lambda \pm i0^+)C := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_0(\lambda \pm i\varepsilon)C \tag{1.8}$$

existe pour la topologie de la norme de $\mathcal{B}(\mathcal{H})$. Alors les conditions suivantes sont équivalentes :

- (i) λ est une singularité spectrale sortante/entrante de H ,
- (ii) λ est une valeur propre de H' associée à un vecteur propre $\Psi \in \mathcal{H}'_C^\pm(\lambda)$.

Ici $\mathcal{H}'_C^\pm(\lambda)$ désigne l'espace des états résonants sortants/entrants. Il est défini par

$$\mathcal{H}'_C^\pm(\lambda) := \text{Ker}(\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V').$$

De plus, H' est le prolongement de H à \mathcal{H}'_C que l'on construit en considérant l'opérateur dual de H restreint à \mathcal{H}_C avec l'hypothèse 1.5, de même V' est le prolongement de V à \mathcal{H}'_C (voir 2.2.3.2). L'argument principal de la preuve du théorème 1.3.1 repose sur un principe de Birman-Schwinger, (voir section 2.4.1) dans le sens où λ est un point spectral régulier sortant/entrant si et seulement si $\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V'$ est inversible. Le théorème

1.3.1 a des conséquences nouvelles aussi pour les opérateurs dissipatifs (voir section 2.4.6).

Les deux théorèmes qui suivent répondent aux questions introduites précédemment concernant le sous-espace spectral des états qui disparaissent à l'infini et le sous-espace spectral absolument continu.

Théorème 1.3.2. *Supposons que les hypothèses 1.1-1.3 sont satisfaites. Alors*

$$\mathcal{H}_{\text{ads}}^{\pm}(H) = \mathcal{H}_{\text{p}}^{\pm}(H).$$

Pour des opérateurs dissipatifs, des conditions pour que ce résultat soit satisfait ont été données dans [35], en utilisant des résultats de théorie de la diffusion (en particulier l'existence et les propriétés des opérateurs d'onde). La preuve que nous donnons du théorème 1.3.2 ne repose pas sur les propriétés des opérateurs d'onde.

Théorème 1.3.3. *Supposons que les hypothèses 1.1-1.4 sont satisfaites. Alors*

$$\mathcal{H}_{\text{ac}}(H) = \text{Ran}(\text{Id} - \Pi_{\text{p}}(H)) = \mathcal{H}_{\text{p}}(H^*)^{\perp}.$$

Les preuves de ces deux théorèmes reposent sur une formule de résolution de l'identité « régularisée » qui tient en particulier compte du fait que H peut avoir un nombre fini de singularités spectrales dans son spectre essentiel. En effet, si les hypothèses 1.1 à 1.4 sont satisfaites, on peut montrer la formule suivante :

$$r(H) = r(H)\Pi_{\text{disc}}(H) + \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\sigma_{\text{ess}}(H)} r(\lambda) (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda,$$

où $\Pi_{\text{disc}}(H)$ désigne la projection sur le spectre discret de H définie comme la somme des projections de Riesz associées aux valeurs propres discrètes de H . En particulier cette formule généralise la formule de résolution de l'identité bien connue pour des opérateurs auto-adjoints. Si l'on enlève le facteur régularisant r dans le deuxième terme du membre de droite de la formule précédente, on retrouve la formule de Stone qui permet de définir des projections spectrales. Les projections spectrales (sans le facteur régularisant) dans le cas non auto-adjoint avaient été considérées plusieurs fois dans la littérature, par exemple dans [70, 71] pour des opérateurs différentiels et dans [53, 87] pour des modèles abstraits (voir section 2.5 pour plus de détails concernant cette formule). Finalement les théorèmes 1.3.2 et 1.3.3 ont pour conséquence la décomposition de \mathcal{H} en somme directe J -orthogonale suivante :

$$\mathcal{H} = \mathcal{H}_{\text{ac}}(H) \oplus \mathcal{H}_{\text{ads}}^+(H) \oplus \mathcal{H}_{\text{ads}}^-(H) \oplus \mathcal{H}_{\text{b}}(H),$$

où $\mathcal{H}_{\text{b}}(H)$ est l'espace des états « bornés », c'est à dire l'espace vectoriel engendré par tous les vecteurs propres généralisés de H associés à des valeurs propres réelles.

Pour étudier la théorie de la diffusion pour notre modèle, nous définissons les opérateurs d'onde associés à H et H_0 par

$$W_{\pm}(H, H_0) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \Pi_{\text{ac}}(H) r_{\mp}(H) e^{-itH_0}.$$

Donnons quelques explications ; comme mentionné précédemment il est nécessaire de pouvoir projeter en dehors des vecteurs propres généralisés associés à H et, d'après le théorème 1.3.3, ajouter la projection sur le sous-espace spectral absolument continu de H le permet. Ajouter la fonction $r_{\mp}(H)$ permet de régulariser les singularités spectrales entrantes/sortantes de H et ainsi prouver l'existence de ces opérateurs sous nos hypothèses. Ces deux justifications sont nécessaires pour prouver l'existence des opérateurs d'onde, comme cela est détaillé à la section 3.4.2.1. Par ailleurs l'étude de ces opérateurs d'onde nous conforte dans le bien-fondé de leur définition, puisqu'ils vérifient les propriétés standards des opérateurs d'onde associés à des opérateurs auto-adjoints. Le théorème suivant résume les propriétés que nous avons obtenues :

Théorème 1.3.4. *Supposons que les hypothèses 1.1-1.4 sont satisfaites. Alors les opérateurs d'onde $W_{\pm}(H, H_0)$ existent. De plus ils sont injectifs, leurs images sont denses dans $\mathcal{H}_{\text{ac}}(H)$ et ils satisfont la propriété d'entrelacement*

$$\forall u \in \mathcal{D}(H_0), \quad HW_{\pm}(H, H_0)u = W_{\pm}(H, H_0)H_0u.$$

Pour prouver que ces opérateurs existent nous utilisons la méthode de Cook. Nous en avons donné une version à la section 3.4.1. Pour utiliser cette méthode, l'argument principal repose sur l'estimation suivante :

$$\exists c > 0, \forall u \in \mathcal{H}_{\text{ac}}(H), \quad \int_0^{\infty} \|Cr_{\pm}(H)e^{\mp itH}u\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2.$$

Voir le lemme 3.4.5 pour la preuve de cette estimation.

Au vu des propriétés des opérateurs d'onde, il nous semble naturel de définir la complétude asymptotique des opérateurs d'onde comme le fait que leur image est fermée dans $\mathcal{H}_{\text{ac}}(H)$. En effet, avec la proposition 1.3.4 si leur image est fermée on obtient qu'ils sont inversibles dans $\mathcal{B}(\mathcal{H}, \mathcal{H}_{\text{ac}}(H))$. Cette notion est consistante avec celle qui existe déjà dans le cas auto-adjoint. Nous avons obtenu une condition suffisante de complétude asymptotique que nous résumons dans le théorème suivant :

Théorème 1.3.5. *Supposons que les hypothèses 1.1-1.4 sont satisfaites. Si H n'a pas de singularité spectrale alors les opérateurs d'onde sont asymptotiquement complets.*

La réciproque de ce théorème est toujours une question ouverte mais on s'attend à ce que les opérateurs d'onde ne soient pas asymptotiquement complets si H possède des singularités spectrales, comme cela a été prouvé dans le cas dissipatif dans [35]. Enfin le théorème 1.3.5 a comme conséquence le résultat suivant :

Théorème 1.3.6. *Supposons que les hypothèses 1.1-1.4 sont satisfaites. Si H n'a pas de singularité spectrale alors il existe $m_1 > 0, m_2 > 0$ tels que pour tout $u \in \mathcal{H}_{\text{ac}}(H)$,*

$$m_1 \|u\|_{\mathcal{H}} \leq \|e^{itH}u\|_{\mathcal{H}} \leq m_2 \|u\|_{\mathcal{H}}.$$

Ce théorème montre en particulier que pour une condition initiale dans le sous-espace spectral absolument continu de H , alors en tout temps la norme de la solution de l'équation de Schrödinger est contrôlée inférieurement et supérieurement par la norme de la condition initiale. Un résultat similaire a été démontré dans [46] pour des opérateurs de Schrödinger et dans [35] pour une classe abstraite d'opérateurs dissipatifs.

1.3.3 Exemple des opérateurs de Schrödinger

Dans cette section, nous montrons que nos résultats s'appliquent aux opérateurs de Schrödinger sur \mathbb{R}^d avec un potentiel satisfaisant une condition de courte portée. Plus spécifiquement, considérons

$$\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}), \quad d \geq 3 \text{ impair}, \quad H_0 = -\Delta, \quad \mathcal{D}(H_0) = H^2(\mathbb{R}^d, \mathbb{C})$$

$$V \text{ un potentiel à valeurs complexes;}$$

avec $H^2(\mathbb{R}^d)$ l'espace de Sobolev usuel.

1.3.3.1 Singularités spectrales pour des potentiels à courte portée

Dans cette partie, nous donnons les conséquences du théorème 1.3.1 pour l'opérateur $H_0 + V$ sous une conditions de courte portée sur V . Supposons que

$$x \mapsto \langle x \rangle^\sigma V(x) \in L^\infty(\mathbb{R}^d), \quad \sigma > 1. \quad (1.9)$$

En particulier, V décroît à l'infini et donc $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0)$. Par ailleurs, il est bien connu que pour tout $\lambda > 0$,

$$\langle x \rangle^{-\delta} \mathcal{R}_0(\lambda \pm i0^+) \langle x \rangle^{-\delta} \quad (1.10)$$

existe dans $\mathcal{B}(\mathcal{H})$ pour tout $\delta > \frac{1}{2}$, où $\langle x \rangle = (1+x^2)^{\frac{1}{2}}$, alors on peut décomposer $V = CWC$ en choisissant C l'opérateur de multiplication par $\langle x \rangle^{-\sigma/2}$. Notons que si V satisfait la condition de courte portée (1.9) avec $\sigma \geq 2$ alors le principe d'absorption limite (1.10) est satisfait pour tout $\lambda \geq 0$. Ici \mathcal{H}_C est

$$\mathcal{H}_C = L^2_{\sigma/2} := \{f : \mathbb{R}^d \rightarrow \mathbb{C}, x \mapsto \langle x \rangle^{\frac{\sigma}{2}} f(x) \in L^2(\mathbb{R}^d)\},$$

muni de la norme usuelle et son dual est donné par

$$\mathcal{H}'_C = L^2_{-\sigma/2} = \{f : \mathbb{R}^d \rightarrow \mathbb{C}, x \mapsto \langle x \rangle^{-\frac{\sigma}{2}} f(x) \in L^2(\mathbb{R}^d)\}.$$

Comme l'ensemble des fonctions lisses à support compact est contenu dans $\mathcal{D}(H_0) \cap \mathcal{H}_C$ alors l'hypothèse 1.5 est satisfaite. Ainsi le théorème 1.3.1 s'applique et se réécrit sous la forme suivante :

Proposition 1.3.2. *Supposons que V est un potentiel à valeurs complexes satisfaisant (1.9) avec $\sigma > 1$ (respectivement $\sigma \geq 2$). Soit $C(x) = \langle x \rangle^{-\sigma/2}$. Alors pour tout $\lambda > 0$ (respectivement $\lambda \geq 0$), les conditions suivantes sont équivalentes :*

(i) λ est une singularité spectrale sortante/entrante de H au sens de la définition 1.2.1,

(ii) Il existe $\Psi \in \mathcal{H}_C^{\pm}(\lambda) \subset L^2_{-\sigma/2}$, $\Psi \neq 0$, tel que

$$(-\Delta + V(x) - \lambda)\Psi = 0.$$

Les résultats de la proposition 1.3.2 avait déjà été obtenu par Wang dans [90, 91] avec des arguments différents.

Pour plus de détails concernant les applications de cette caractérisation des singularités spectrales, nous renvoyons à la sous-section 2.3.3.1.

1.3.3.2 Application pour des potentiels à support compact

Nous montrons ici que nous pouvons appliquer les théorèmes 1.3.2, 1.3.3, 1.3.4 et 1.3.5 à $H = -\Delta + V(x)$ si V est une fonction essentiellement bornée à support compact.

Il est bien connu que

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|\langle x \rangle^{-s} \mathcal{R}_0(z) \langle x \rangle^{-s}\|_{L^2(\mathbb{R}^3, \mathbb{C})} < +\infty$$

pour tout $s > 2$. En particulier, l'hypothèse 1.1 est satisfaite. De plus comme V est un opérateur à support compact, alors en appliquant les résultats de [43], l'hypothèse 1.2 est satisfaite. L'hypothèse 1.3 est liée à la théorie des résonances. Les résonances sont définies comme les pôles du prolongement méromorphe de la fonction

$$\{z \in \mathbb{C}, \text{Im}(z) > 0\} \ni z \mapsto (H - z^2)^{-1} : L^2(\mathbb{R}^d, \mathbb{C}) \rightarrow L^2(\mathbb{R}^d, \mathbb{C})$$

en la fonction

$$\mathbb{C} \ni z \mapsto (H - z^2) : L^2_c(\mathbb{R}^d, \mathbb{C}) \rightarrow L^2_{\text{loc}}(\mathbb{R}^d, \mathbb{C}), \quad (1.11)$$

où $L^2_c(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d), u \text{ à support compact}\}$ et $L^2_{\text{loc}}(\mathbb{R}^d) := \{u : \mathbb{R}^d \rightarrow \mathbb{C}, u \in L^2(K) \text{ pour tout ensemble compact } K \subset \mathbb{R}^d\}$. Les pôles de l'application (1.11) sont appelés résonances de H . On peut vérifier que les résonances réelles positives/négatives de H sont en bijection avec les singularités spectrales sortantes/entrantes de H . De plus on sait que H a un nombre fini de résonances et, pour chaque résonance, à l'ordre de son pôle correspond l'ordre de la singularité spectrale (voir [25]). Donc l'hypothèse 1.3 est satisfaite. On en déduit alors la proposition suivante :

Proposition 1.3.3. *Supposons que V soit un potentiel à valeurs complexes essentiellement borné et à support compact. Alors*

$$\mathcal{H}_{\text{ads}}^{\pm}(H) = \mathcal{H}_p^{\pm}(H).$$

Pour l'application du théorème 1.3.3, nous avons besoin de vérifier que l'hypothèse 1.4 est également satisfaite. Pour les opérateurs de Schrödinger, on peut choisir J comme l'opérateur de conjugaison complexe. On obtient alors la proposition suivante :

Proposition 1.3.4. *Supposons que V soit un potentiel à valeurs complexes essentiellement borné et à support compact. Si de plus pour chaque valeur propre λ plongée dans le spectre essentiel de H , la forme bilinéaire*

$$\text{Ker}((H - \lambda)^{\nu_\lambda}) \ni (u, v) \mapsto \int_{\mathbb{R}^d} u(x)v(x)dx$$

est non dégénérée, où ν_λ désigne la multiplicité algébrique de λ , alors

$$\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H^*)^\perp.$$

Pour ces deux propositions, le lecteur pourra consulter la section 2.3.3.2.

Enfin, sous les mêmes hypothèses que la proposition précédente, les résultats de théorie de diffusion (voir section 3.3.3) se réécrivent de la façon suivante :

Proposition 1.3.5. *Supposons que V soit un potentiel à valeurs complexes essentiellement borné et à support compact. Si de plus pour chaque valeur propre λ plongée dans le spectre essentiel de H , la forme bilinéaire*

$$\text{Ker}((H - \lambda)^{\nu_\lambda}) \ni (u, v) \mapsto \int_{\mathbb{R}^d} u(x)v(x)dx$$

est non dégénérée, alors $W_\pm(H, H_0)$ existent, sont injectifs et d'image dense dans $\mathcal{H}_{\text{ac}}(H)$. De plus, si H n'a pas de singularités spectrales, alors les opérateurs d'onde sont asymptotiquement complets et il existe $m_1 > 0$ et $m_2 > 0$ tels que pour tout $u \in \mathcal{H}_{\text{ac}}(H)$,

$$\forall t \in \mathbb{R}, \quad m_1 \|u\|_{\mathcal{H}} \leq \|e^{itH}u\|_{\mathcal{H}} \leq m_2 \|u\|_{\mathcal{H}}.$$

1.4 Organisation du manuscrit

Ce manuscrit de thèse est divisé en quatre parties.

Le premier chapitre est l'introduction et un résumé des travaux de cette thèse. En particulier, nous avons présenté les résultats principaux que nous avons obtenus et nous nous sommes efforcés de citer à quelle section ou sous-section du manuscrit des détails pouvaient être trouvés.

Le second chapitre est la version la plus à jour de l'article rédigé en anglais par Jérémie Faupin et l'auteur de cette thèse, intitulée « *Spectral decomposition of some non-self-adjoint operators* ». Il porte sur la partie décomposition spectrale, et plus spécifiquement sur les résultats des théorèmes 1.3.1, 1.3.2 et 1.3.3.

Le troisième chapitre est une version à jour de l'article intitulé « *Scattering theory for non-self-adjoint operators* » rédigé en anglais par l'auteur de cette thèse. Cette partie porte sur la théorie de la diffusion pour le modèle présenté dans cette introduction et plus particulièrement sur les résultats des théorèmes 1.3.4, 1.3.5 et 1.3.6.

Le quatrième chapitre, rédigé en français, est dans la continuité de ce qui est présenté aux chapitres 2 et 3. La différence majeure est l'hypothèse que nous faisons sur le potentiel

V : nous ne supposons pas que celui-ci est borné. L'une des difficultés est que dans ce cas $-iH$ n'engendre pas un groupe d'évolution et donc, a priori, l'équation de Schrödinger n'a pas de solution. Sous des hypothèses similaires à celles des chapitres 2 et 3, nous construisons un calcul fonctionnel qui permet de construire des solutions à l'équation de Schrödinger.

Chapter 2

Spectral decomposition of some non-self-adjoint operators

We consider non-self-adjoint operators in Hilbert spaces of the form $H = H_0 + CWC$, where H_0 is self-adjoint, W is bounded and C is bounded and relatively compact with respect to H_0 . We suppose that C is a metric operator and that $C(H_0 - z)^{-1}C$ is uniformly bounded in $z \in \mathbb{C} \setminus \mathbb{R}$. We define the spectral singularities of H as the points of the essential spectrum $\lambda \in \sigma_{\text{ess}}(H)$ such that $C(H - \lambda \pm i\varepsilon)^{-1}CW$ does not have a limit as $\varepsilon \rightarrow 0^+$. We prove that the spectral singularities of H are in one-to-one correspondence with the eigenvalues, associated to resonant states, of an extension of H to a larger Hilbert space. Next, we show that the asymptotically disappearing states for H , i.e. the vectors φ such that $e^{\pm itH}\varphi \rightarrow 0$ as $t \rightarrow \infty$, coincide with the finite linear combinations of generalized eigenstates of H corresponding to eigenvalues $\lambda \in \mathbb{C}$, $\mp \text{Im}(\lambda) > 0$. Finally, we define the absolutely continuous spectral subspace of H and show that it satisfies $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H^*)^\perp$, where $\mathcal{H}_{\text{p}}(H^*)$ stands for the point spectral subspace of H^* . We thus obtain a direct sum decomposition of the Hilbert spaces in terms of spectral subspaces of H . One of the main ingredients of our proofs is a spectral resolution formula for a bounded operator $r(H)$ regularizing the identity at spectral singularities. Our results apply to Schrödinger operators with complex potentials.

2.1 Introduction

We are interested in this paper in the evolution of a quantum system governed by the time-dependent Schrödinger equation

$$i\partial_t\varphi = H\varphi, \tag{2.1}$$

with a *non-self-adjoint* operator H .

Non-self-adjoint ‘Hamiltonians’ in Quantum Mechanics are considered in various contexts, see [4, 62] and references therein. We mention here two frameworks that are especially relevant for our study.

As effective or phenomenological operators, non-self-adjoint operators are used to describe non-conservative phenomena. A celebrated example is the *optical model* in nuclear physics describing both the elastic and inelastic scattering of a neutron – or a proton – at a nucleus. It was introduced by Feshbach, Porter and Weisskopf [37] as an empirical model allowing, in particular, the description of the formation of a compound nucleus [7]. In this model, the unnormalized state of the neutron at a positive time t is given by the solution $\varphi_t = e^{-itH}\varphi$ to (2.1), with $H = -\Delta + V(x)$ a dissipative Schrödinger operator in $L^2(\mathbb{R}^d)$, $\text{Im}(V(x)) \leq 0$. Part of the energy of the neutron may be transferred to the nucleus, possibly leading to the absorption, or capture, of the neutron by the nucleus. Mathematically, this is reflected in the dissipative nature of the equation since, in particular, given a normalized initial state $\varphi \in L^2(\mathbb{R}^d)$, we have that $\|e^{-itH}\varphi\|_{L^2} \leq 1$ for all $t \geq 0$. The probability of absorption

$$p_{\text{abs}}(\varphi) := 1 - \lim_{t \rightarrow \infty} \|e^{-itH}\varphi\|_{L^2}^2$$

does not vanish in general.

The nuclear optical model leads to predictions that correspond to experimental scattering data to a high precision. Theoretical justifications of the model have been given in [38–40] (see also [4, 41, 50]). The idea of the justification consists in projecting out, in the Schrödinger equation associated to the total system nucleus – neutron, the degrees of freedom corresponding to the nucleus. This can be performed using Schur’s complement formula and leads to a Schrödinger equation for the neutron which is non-linear in energy. The effective, non-self-adjoint Hamiltonian for the neutron is then obtained by averaging over energy.

\mathcal{PT} -symmetric operators constitute another widely used class of non-self-adjoint operators in Quantum Mechanics. It was observed by Bender and Boettcher [5] that a large class of ‘ \mathcal{PT} -invariant Hamiltonians’ have real spectra and can therefore be quantum mechanically relevant in many situations. For Schrödinger operators, $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$, with V a complex potential, \mathcal{PT} -symmetry means that

$$[H, \mathcal{PT}] = 0,$$

where \mathcal{P} is the parity operator, $(\mathcal{P}\varphi)(x) = \varphi(-x)$, and \mathcal{T} the time-reversal operator, $(\mathcal{T}\varphi)(x) = \overline{\varphi(x)}$.

In recent years, \mathcal{PT} -symmetric operators have attracted lots of attention, from theoretical studies showing that \mathcal{PT} -invariant operators have real spectra under suitable conditions [22, 61, 63, 64, 73, 84], to experimental studies revealing \mathcal{PT} -symmetry-like structures, in particular in optics [67, 77, 81]. See [26] for more references and more recent developments, [64] for results on the pseudo-spectra of \mathcal{PT} -symmetric operators, and [9, 10, 93] for the study of \mathcal{PT} -symmetric operators having continuous spectra. It may also be worth mentioning that a mathematically related and, perhaps, more relevant concept than \mathcal{PT} -symmetry, is that of *quasi-self-adjointness* [20, 83]. We refer to [64] for a relationship between quasi-self-adjointness and \mathcal{PT} -symmetry.

In this paper, we consider an abstract class of non-self-adjoint operators in a complex Hilbert space \mathcal{H} , of the form $H = H_0 + V$. We suppose that H_0 is a self-adjoint operator with purely absolutely continuous spectrum and that V is a relatively compact perturbation of H_0 . In particular, the essential spectra of H and H_0 coincide. We suppose furthermore that V admits a factorization as $V = CWC$, with W bounded and C a strictly positive operator such that

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|C(H_0 - z)^{-1}C\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

Such factorizations go back to the seminal work of Kato [55], see also [65].

We are interested in a *spectral decomposition* of the non-self-adjoint operator H , in relation with the long-time behavior of the solutions to (2.1). Note that H being a bounded perturbation of H_0 , the operator $-iH$ generates a strongly continuous group $\{e^{-itH}\}_{t \in \mathbb{R}}$ and hence, for any $\varphi \in \mathcal{H}$, (2.1) admits a global solution $\mathbb{R} \ni t \mapsto e^{-itH}\varphi \in \mathcal{H}$.

Roughly speaking, our main contributions can be summarized as follows. First, defining the spaces of *asymptotically disappearing states* as

$$\mathcal{H}_{\text{ads}}^{\pm}(H) := \left\{ \varphi \in \mathcal{H}, \lim_{t \rightarrow \pm\infty} \|e^{-itH}\varphi\|_{\mathcal{H}} = 0 \right\},$$

we will show that $\mathcal{H}_{\text{ads}}^{\pm}(H)$ coincide with the vector space spanned by all eigenvectors, or generalized eigenvectors, corresponding to eigenvalues λ of \mathcal{H} such that $\mp \text{Im}(\lambda) > 0$. Next, defining the *absolutely continuous spectral subspace* $\mathcal{H}_{\text{ac}}(H)$ of H as the closure of

$$\left\{ \varphi \in \mathcal{H}, \exists c_{\varphi} > 0, \forall \psi \in \mathcal{H}, \int_{\mathbb{R}} |\langle e^{-itH}\varphi, \psi \rangle_{\mathcal{H}}|^2 dt \leq c_{\varphi} \|\psi\|_{\mathcal{H}}^2 \right\},$$

we will prove that $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H^*)^{\perp}$, where H^* stands for the adjoint of H and $\mathcal{H}_{\text{p}}(H^*)$ is the point spectral subspace of H^* , i.e. the vector space spanned by all eigenstates or generalized eigenstates of H^* . These characterizations of $\mathcal{H}_{\text{ads}}^{\pm}(H)$ and $\mathcal{H}_{\text{ac}}(H)$ in turn imply a J -orthogonal decomposition of the Hilbert space (for some conjugation operator J , see Hypothesis 2.4 for precise conditions on J) given by

$$\mathcal{H} = \mathcal{H}_{\text{ac}}(H) \oplus \mathcal{H}_{\text{ads}}^{+}(H) \oplus \mathcal{H}_{\text{ads}}^{-}(H) \oplus \mathcal{H}_{\text{b}}(H), \quad (2.2)$$

where $\mathcal{H}_{\text{b}}(H)$ is the space of ‘*bound states*’, i.e. the closure of the vector space spanned by all generalized eigenvectors of H corresponding to real eigenvalues. Hence we obtain a decomposition of the Hilbert space into ‘*scattering states*’ (corresponding to elements of $\mathcal{H}_{\text{ac}}(H)$), asymptotically disappearing states and bound states. This generalizes the well-known spectral decomposition for self-adjoint operators without singular continuous spectrum (in the case of self-adjoint operators H , of course, we have $\mathcal{H}_{\text{ads}}^{\pm}(H) = \{0\}$).

To prove these results, we will require that H only have finitely many eigenvalues (counting algebraic multiplicities) and finitely many *spectral singularities*. As in previous works concerning dissipative operators in Hilbert spaces [32, 34, 35], the notion of spectral

singularities plays a central role in this paper. In our context, we will define a spectral singularity as a point λ of the essential spectrum of H such that one of the two limits

$$\lim_{\varepsilon \rightarrow 0^+} C(H - \lambda \pm i\varepsilon)^{-1}CW$$

does not exist in the norm topology of $\mathcal{B}(\mathcal{H})$. We will prove that λ is a spectral singularity of H if and only if it is an eigenvalue of an extension H' of H to a larger Hilbert space \mathcal{H}'_C , corresponding to an eigenstate (a ‘*resonant*’ state) belonging to a suitable subspace of \mathcal{H}'_C (the space \mathcal{H}'_C will be defined as the anti-dual of $\text{Ran}(C)$, equipped with a suitable norm).

Our results concerning the spaces of asymptotically disappearing states $\mathcal{H}_{\text{ads}}^\pm(H)$ generalize previous results for dissipative operators recently obtained in [35]. Our proof is more direct, in particular it does not rely on the scattering theory for the pair (H, H_0) , which was a crucial element of the proof in [35]. Our results showing that $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H^*)^\perp$ seem to be new. It is worth mentioning that, contrary to previous results on absolutely continuous spectral subspaces for non-self-adjoint operators (see [16] for dissipative operators and [60, 74] and references therein for a more general context), we do not use the theory of dilations of contractive semigroups.

The abstract theory developed in this paper applies to Schrödinger operators $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$, under suitable decay assumptions on the complex potential V . In this case, spectral singularities correspond to real resonances. Some of our results on spectral singularities may thus be seen as abstract versions of corresponding well-known properties in the theory of resonances for Schrödinger operators [25]. On the other hand, the characterizations of the subspaces $\mathcal{H}_{\text{ads}}^\pm(H)$ and $\mathcal{H}_{\text{ac}}(H)$, as well as the spectral decomposition formula (2.2), seem to give new results also in the context of Schrödinger operators.

Before stating our results in more precise terms in Section 2.3, we begin with describing in details the abstract setting studied in this paper in Section 2.2.

Notations. In what follows, given two Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$, the notation $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ stands for the set of bounded operators from \mathcal{H}_1 to \mathcal{H}_2 . If $\mathcal{H}_1 = \mathcal{H}_2$, we set $\mathcal{B}(\mathcal{H}_1) := \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$.

If E is a Banach space and E' its anti-dual, we denote by

$$\langle u, \Phi \rangle_{E; E'} := \Phi(u), \quad \Phi \in E', \quad u \in E,$$

the usual duality bracket.

The domain of an operator A on a Hilbert space \mathcal{H} is denoted by $\mathcal{D}(A)$. The spectrum and resolvent set of A are denoted by $\sigma(A)$ and $\rho(A)$, respectively. For $z \in \rho(A)$, we let

$$\mathcal{R}_A(z) := (A - z)^{-1}$$

be the resolvent of A . In the case where A is the unperturbed operator H_0 , we will also use the shorthand $\mathcal{R}_0 := \mathcal{R}_{H_0}$.

We let $\mathbb{C}^\pm := \{z \in \mathbb{C}, \pm \text{Im}(z) > 0\}$ and $\bar{\mathbb{C}}^\pm := \{z \in \mathbb{C}, \pm \text{Im}(z) \geq 0\}$. The complex open disc centered at λ and of radius r is denoted by

$$\mathring{D}(\lambda, r) := \{z \in \mathbb{C}, |z - \lambda| < r\}.$$

2.2 Abstract setting

2.2.1 The model

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a complex separable Hilbert space. On \mathcal{H} , we consider the operator

$$H := H_0 + V, \quad (2.3)$$

where H_0 is self-adjoint and semi-bounded from below and $V \in \mathcal{B}(\mathcal{H})$ is a bounded operator. In particular, H is a closed operator with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ and its adjoint is given by

$$H^* = H_0 + V^*, \quad \mathcal{D}(H^*) = \mathcal{D}(H_0).$$

Without loss of generality, we suppose that $H_0 \geq 0$. Since H is a perturbation of the self-adjoint operator H_0 by the bounded operator V , $-iH$ is the generator of a strongly continuous one-parameter group $\{e^{-itH}\}_{t \in \mathbb{R}}$ satisfying

$$\|e^{-itH}\|_{\mathcal{B}(\mathcal{H})} \leq e^{|t|\|V\|_{\mathcal{B}(\mathcal{H})}}, \quad t \in \mathbb{R},$$

(see e.g [19] or [28]).

We assume that there exists a *metric operator* $C \in \mathcal{B}(H)$ such that C is relatively compact with respect to H_0 and V is of the form

$$V = CWC, \quad (2.4)$$

with $W \in \mathcal{B}(\mathcal{H})$. We recall that a metric operator is a strictly positive operator, i.e. $C \geq 0$ and $\text{Ker}(C) = \{0\}$, see e.g. [3].

We mention that the semi-boundedness of H_0 , which we assume throughout the paper for simplicity, can be relaxed, assuming for instance that H_0 has a spectral gap.

2.2.2 Spectral subspaces, spectral projections

Recall that $\sigma(H)$ stands for the spectrum of H and $\rho(H) = \mathbb{C} \setminus \sigma(H)$ its resolvent set. As usual, the point spectrum of H is defined as the set of all eigenvalues of H ,

$$\sigma_p(H) := \{\lambda \in \mathbb{C}, \text{Ker}(H - \lambda) \neq \{0\}\}.$$

For each eigenvalue λ , we define its algebraic multiplicities $m_\lambda(H)$ as the dimension of the generalized eigenspace associated to λ ,

$$m_\lambda(H) := \dim \left(\bigcup_{k=1}^{\infty} \text{Ker}((H - \lambda)^k) \right).$$

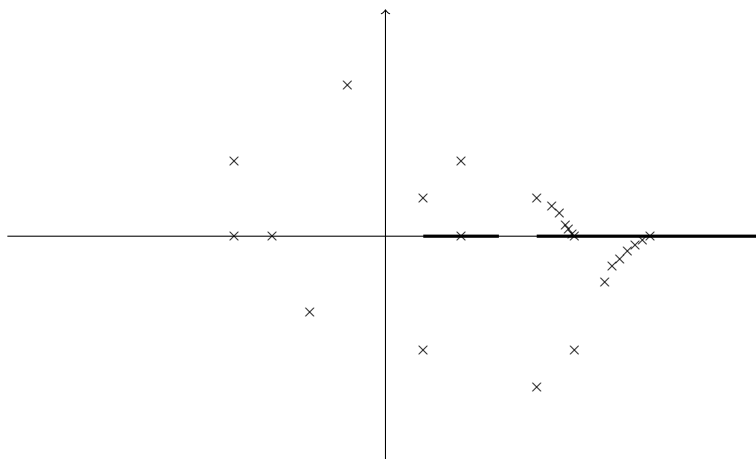


Figure 2.1: **Spectrum of H .** The essential spectrum of H , represented by thick lines, coincides with that of H_0 and is contained in $[0, \infty)$. Eigenvalues of H are represented by crosses. The discrete spectrum of H consists of isolated eigenvalues of finite algebraic multiplicities which may accumulate at any point of the essential spectrum. The point spectrum of H may also contain eigenvalues embedded in the essential spectrum.

If λ is an isolated eigenvalue of H , we denote by

$$\Pi_\lambda(H) := \frac{1}{2\pi i} \int_\gamma (z \text{Id} - H)^{-1} dz, \quad (2.5)$$

the usual Riesz projection, where γ is a circle oriented counterclockwise and centered at λ , of sufficiently small radius (so that λ is the only point of the spectrum of H contained in the interior of γ) is finite dimensional. The discrete spectrum of H , $\sigma_{\text{disc}}(H)$ is the set of all isolated eigenvalue λ such that the range of the associated Riesz projection is finite dimensional.

Under our assumptions, since V is a relatively compact perturbation of H_0 , the essential spectrum $\sigma_{\text{ess}}(H) := \sigma(H) \setminus \sigma_{\text{disc}}(H)$ coincides with the essential spectrum of H_0 and the discrete spectrum $\sigma_{\text{disc}}(H)$ is at most countable and can only accumulate at points of $\sigma_{\text{ess}}(H)$ (see [76, Corollary 2 of Theorem XIII.14]). See Figure 2.1.

It should be noted that there exist various definitions of essential spectra for non-self-adjoint operators [27, Chapter IX], but these definitions coincide in our context. This follows from the fact that H is a relatively compact perturbation of the self-adjoint operator H_0 together with [76, Corollary 2 of Theorem XIII.14] and [27, Theorems IX.1.6 and IX.2.1]. In turn, this shows that $\sigma_{\text{ess}}(H)$ coincides with the set of $\lambda \in \mathbb{C}$ such that $H - \lambda$ is not Fredholm.

We define in addition the set of all eigenvalues embedded in the essential spectrum of H :

$$\sigma_{\text{emb}}(H) := \sigma_{\text{p}}(H) \cap \sigma_{\text{ess}}(H).$$

2.2.2.1 Eigenspaces corresponding to isolated eigenvalues

For $\lambda \in \sigma_{\text{disc}}(H)$, as the range of the Riesz projection (2.5) is finite, it follows from [76, Theorem XII.5 (d)] that $\text{Ran}(\Pi_\lambda(H))$ is spanned by generalized eigenvectors of H associated to λ , i.e., by vectors $u \in \mathcal{D}(H^k)$ such that $(H - \lambda)^k u = 0$ for some $1 \leq k \leq m_\lambda(H)$. In particular we have $\text{Ran}(\Pi_\lambda(H)) = m_\lambda(H)$. In the case where H_0 does not have any point spectrum (which we will assume later on), as the essential spectra of H and H_0 coincide, the discrete spectrum of H coincides with the set of isolated eigenvalues of H with finite algebraic multiplicities.

We set

$$\mathcal{H}_{\text{disc}}(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_{\text{disc}}(H)\}^{\text{cl}},$$

where A^{cl} stands for the closure of a subset $A \subset \mathcal{H}$. We will sometimes assume that the discrete spectrum of H is finite. The spectral projection $\Pi_{\text{disc}}(H)$ onto $\mathcal{H}_{\text{disc}}(H)$ is then defined by

$$\Pi_{\text{disc}}(H) := \sum_{\lambda \in \sigma_{\text{disc}}(H)} \Pi_\lambda(H). \quad (2.6)$$

We will also consider the following three subspaces of $\mathcal{H}_{\text{disc}}(H)$:

$$\mathcal{H}_{\text{disc}}^+(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_{\text{disc}}(H), \text{Im}(\lambda) < 0\}^{\text{cl}}, \quad (2.7)$$

$$\mathcal{H}_{\text{disc}}^0(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_{\text{disc}}(H), \text{Im}(\lambda) = 0\}^{\text{cl}}, \quad (2.8)$$

$$\mathcal{H}_{\text{disc}}^-(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_{\text{disc}}(H), \text{Im}(\lambda) > 0\}^{\text{cl}}. \quad (2.9)$$

Observe that $\mathcal{H}_{\text{disc}}^+(H)$ is the closure of the vector space spanned by all generalized eigenvectors corresponding to eigenvalues with *negative* imaginary part, and likewise for $\mathcal{H}_{\text{disc}}^-(H)$. The reason for these conventions will be understood later (see Theorem 2.3.2). Clearly, if $\sigma_{\text{disc}}(H)$ is finite, we have the following direct sum decomposition:

$$\mathcal{H}_{\text{disc}}(H) = \mathcal{H}_{\text{disc}}^-(H) \oplus \mathcal{H}_{\text{disc}}^0(H) \oplus \mathcal{H}_{\text{disc}}^+(H).$$

The corresponding spectral projections $\Pi_{\text{disc}}^\sharp(H)$, where \sharp stands for $+$, $-$ or 0 , are defined as in (2.6).

2.2.2.2 Eigenspaces corresponding to embedded eigenvalues

Suppose now that λ is an eigenvalue of H embedded in its essential spectrum. The Riesz projection corresponding to λ is then ill-defined, but, under some further conditions, one can define the spectral projection $\Pi_\lambda(H)$ as follows.

In the following (see Hypothesis 2.4 below), we will suppose the existence of a *conjugation operator* J , i.e. an antiunitary operator satisfying

$$J\mathcal{D}(H_0) \subset \mathcal{D}(H_0) \quad \text{and} \quad \forall u \in \mathcal{D}(H_0), \quad JHu = H^*Ju. \quad (2.10)$$

We recall that J is called antiunitary if $J : \mathcal{H} \rightarrow \mathcal{H}$ is an antilinear, continuous, surjective map such that $J^*J = \text{Id}$, where J^* is defined by $\langle \varphi, J\psi \rangle = \langle \psi, J\varphi \rangle$ for all $\varphi, \psi \in \mathcal{H}$. Note

that the further property that J is involutive, $J^2 = \text{Id}$, is usually required in the definition of a conjugation operator. In some applications, the requirement that J is involutive may however be too restrictive, see [4] and references therein. We will therefore not impose this condition here.

Such an operator J establishes, in particular, a one-to-one correspondence between $\text{Ker}((H - \lambda)^k)$ and $\text{Ker}((H^* - \bar{\lambda})^k)$ for all $k \in \mathbb{N}$ and hence

$$m_\lambda(H) = m_{\bar{\lambda}}(H^*).$$

To shorten notation, let $m_\lambda = m_\lambda(H) = m_{\bar{\lambda}}(H^*)$. In order to study the absolutely continuous spectral subspace of H (see (2.17)–(2.18)), we will suppose that for each embedded eigenvalue $\lambda \in \sigma_{\text{ess}}(H)$, m_λ is finite and the symmetric bilinear form

$$\text{Ker}((H - \lambda)^{m_\lambda}) \times \text{Ker}((H - \lambda)^{m_\lambda}) \ni (u, v) \mapsto \langle Ju, v \rangle_{\mathcal{H}} \quad \text{is non-degenerate.} \quad (2.11)$$

This implies that there exists a basis $(\varphi_k)_{1 \leq k \leq m_\lambda}$ of $\text{Ker}((H - \lambda)^{m_\lambda})$ such that

$$\langle J\varphi_i, \varphi_j \rangle_{\mathcal{H}} = \delta_{ij}, \quad 1 \leq i, j \leq m_\lambda. \quad (2.12)$$

We can then define the spectral projection $\Pi_\lambda(H)$ onto the generalized eigenspace corresponding to λ as

$$\Pi_\lambda(H)u := \sum_{k=1}^{m_\lambda} \langle J\varphi_k, u \rangle_{\mathcal{H}} \varphi_k, \quad u \in \mathcal{H}. \quad (2.13)$$

It is not difficult to observe that $\Pi_\lambda(H)$ does not depend on the choice of the basis, is a projection commuting with H , such that $\Pi_\lambda(H) \in \mathcal{B}(\mathcal{H})$ and $\Pi_\lambda(H)^* = \Pi_\lambda(H^*)$ (see Proposition 2.2.1 below for a more precise statement), where

$$\Pi_\lambda(H^*)u := \sum_{k=1}^{m_\lambda} \langle \varphi_k, u \rangle_{\mathcal{H}} J\varphi_k, \quad u \in \mathcal{H}, \quad (2.14)$$

defines a projection onto the generalized eigenspace corresponding to λ for H^* . Note however that if (2.11) does not hold, it is not clear how to define such a projection. In fact, in the simplest case where $m_\lambda = 1$, one easily verifies that the condition $\langle J\varphi, \varphi \rangle \neq 0$ for any $\varphi \in \text{Ker}(H - \lambda) \setminus \{0\}$ is necessary to have the existence of a projection onto $\text{Ker}(H - \lambda)$ commuting with H .

The closure of the vector space spanned by all generalized eigenstates corresponding to embedded eigenvalues of H will be denoted by

$$\mathcal{H}_{\text{emb}}(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_{\text{emb}}(H)\}^{\text{cl}}.$$

If $\sigma_{\text{emb}}(H)$ is composed of finitely many eigenvalues with finite algebraic multiplicities and such that (2.11) holds, we will also use the notation

$$\Pi_{\text{emb}}(H) := \sum_{\lambda \in \sigma_{\text{emb}}(H)} \Pi_\lambda(H).$$

2.2.2.3 Point spectral subspace

With the definitions of the spectral projections $\Pi_\lambda(H)$ given in Sections 2.2.2.1 and 2.2.2.2, we have the following proposition, which covers both cases of isolated and embedded eigenvalues.

Proposition 2.2.1. *Suppose that there exists a conjugate operator J such that (2.10) holds. Let λ be an eigenvalue of H with finite algebraic multiplicity m_λ . If $\lambda \in \sigma_{\text{ess}}(H)$, suppose in addition that (2.11) holds. Then $\Pi_\lambda(H) \in \mathcal{B}(\mathcal{H})$, $\Pi_\lambda(H)$ is a projection which satisfies $\text{Ran}(\Pi_\lambda(H)) \subset \mathcal{D}(H)$ and commutes with H . Its range and adjoint are given respectively by*

$$\text{Ran}(\Pi_\lambda(H)) = \text{Ker}((H - \lambda)^{m_\lambda}), \quad \Pi_\lambda(H)^* = \Pi_{\bar{\lambda}}(H^*).$$

We have

$$(H\Pi_\lambda(H))^* = H^*\Pi_{\bar{\lambda}}(H^*),$$

and if λ, λ' are two distinct eigenvalues of H satisfying the assumptions above, then

$$\Pi_\lambda(H)\Pi_{\lambda'}(H) = 0.$$

In the case of isolated eigenvalues, Proposition 2.2.1 follows from the definition (2.5) of the Riesz projection $\Pi_\lambda(H)$ (see e.g. [76, Theorem XII.5]). In the general case, it suffices to observe that the restriction of H to each finite direct sum of ranges of such projections has a discrete spectrum (since each eigenvalue has finite algebraic multiplicity) and that the Riesz projections associated to these eigenvalues are given by the restrictions of (2.5) or (2.13) to $\mathcal{H}_p(H)$. For embedded eigenvalues, we prove Proposition 2.2.1 in appendix 2.6.1.

We will assume below that H only has a finite number of eigenvalues with finite algebraic multiplicities. Under this simplifying assumption, we set

$$\Pi_p(H) := \sum_{\lambda \in \sigma_p(H)} \Pi_\lambda(H),$$

and observe that

$$\Pi_p(H)^* = \sum_{\lambda \in \sigma_p(H)} \Pi_{\bar{\lambda}}(H^*) =: \Pi_p(H^*).$$

The point spectral subspace of H is then defined by

$$\mathcal{H}_p(H) := \text{Ran}(\Pi_p(H)),$$

and likewise for H^* . It satisfies

$$\mathcal{H}_p(H) = \mathcal{H}_{\text{disc}}(H) \oplus \mathcal{H}_{\text{emb}}(H).$$

We also observe that under our assumptions, the point spectral subspace corresponding to eigenvalues with positive/negative imaginary parts identifies to the corresponding discrete spectral subspace:

$$\mathcal{H}_p^+(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_p(H), \text{Im}(\lambda) < 0\}^{\text{cl}} = \mathcal{H}_{\text{disc}}^+(H), \quad (2.15)$$

$$\mathcal{H}_p^-(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_p(H), \text{Im}(\lambda) > 0\}^{\text{cl}} = \mathcal{H}_{\text{disc}}^-(H). \quad (2.16)$$

2.2.2.4 Subspaces of asymptotically disappearing states

We define the subspaces of *asymptotically disappearing states* as

$$\mathcal{H}_{\text{ads}}^{\pm}(H) := \left\{ u \in \mathcal{H}, \lim_{t \rightarrow \pm\infty} \|e^{-itH}u\|_{\mathcal{H}} = 0 \right\}^{\text{cl}}.$$

Note that $\mathcal{H}_{\text{ads}}^{+}(H)$ and $\mathcal{H}_{\text{ads}}^{-}(H)$ are closed. Using that for a generalized eigenvector $\varphi \in \mathcal{H}_{\text{p}}^{\pm}(H)$, the norm $\|e^{-itH}\varphi\|_{\mathcal{H}}$ decays exponentially as $t \rightarrow \pm\infty$, it is not difficult to verify that

$$\mathcal{H}_{\text{p}}^{\pm}(H) \subset \mathcal{H}_{\text{ads}}^{\pm}(H),$$

(see Proposition 2.5.6). In Theorem 2.3.2 below, we will give conditions under which this inclusion becomes an equality.

2.2.2.5 Absolutely continuous spectral subspace

Let

$$\mathcal{M}(H) := \left\{ u \in \mathcal{H}, \exists c_u > 0, \forall v \in \mathcal{H}, \int_{\mathbb{R}} |\langle e^{-itH}u, v \rangle_{\mathcal{H}}|^2 dt \leq c_u \|v\|_{\mathcal{H}}^2 \right\}. \quad (2.17)$$

We define the *absolutely continuous spectral subspace* of H , $\mathcal{H}_{\text{ac}}(H)$, as the closure of $\mathcal{M}(H)$ in \mathcal{H} ,

$$\mathcal{H}_{\text{ac}}(H) := \mathcal{M}(H)^{\text{cl}}. \quad (2.18)$$

Note that if H is self-adjoint, then $\mathcal{H}_{\text{ac}}(H)$ coincides with the usual absolutely continuous spectral subspace of H (see [15, Theorem 6.24]).

Such a definition of an absolutely continuous spectral subspace for non-self-adjoint operators goes back to [14], where dissipative operators, $\text{Im}(V) \leq 0$, are considered, and the integral in (2.17) is taken over $[0, \infty)$ instead of \mathbb{R} . For self-adjoint operators, our definition and that of [14] coincide; this is however not the case for non-self-adjoint operators. See Section 2.3.4 for a discussion comparing our results and those of [14, 16] for the absolutely continuous spectral subspace of dissipative operators. Note also that other definitions of an absolutely continuous spectral subspace for non-dissipative perturbations of a self-adjoint operator have been considered in the literature, using the theory of dilations. See [60, 74] and references therein.

Our definition of $\mathcal{H}_{\text{ac}}(H)$ is motivated by the following fact: it is not difficult to see that if v is a generalized eigenvector of H^* , then $t \mapsto \langle e^{-itH}u, v \rangle_{\mathcal{H}}$ cannot belong to $L^2(\mathbb{R})$ unless $\langle u, v \rangle = 0$ (see Section 2.5.5). In other words

$$\mathcal{H}_{\text{ac}}(H) \subset \mathcal{H}_{\text{p}}(H^*)^{\perp}.$$

Theorem 2.3.3 will show that, under suitable assumptions, $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H^*)^{\perp}$. This generalizes the equality $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H)^{\perp}$ which holds for self-adjoint operators without singular continuous spectrum.

2.2.3 Extension of the Hilbert space

In this section, we construct an extension of the Hilbert space \mathcal{H} containing the ‘outgoing and incoming resonant states’ that will be introduced below (see Definition 2.2.3). To this end we consider a Gelfand triple defined in terms of the metric operator C appearing in the definition (2.3)–(2.4) of H .

2.2.3.1 Gelfand triple

Recall that C is supposed to be relatively compact with respect to H_0 . Let

$$\mathcal{H}_C := \text{Ran}(C).$$

Since C is self-adjoint and injective, its inverse C^{-1} is a self-adjoint unbounded operator with dense domain $\mathcal{D}(C^{-1}) := \mathcal{H}_C$. We equip \mathcal{H}_C with the scalar product

$$\langle u, v \rangle_{\mathcal{H}_C} := \langle C^{-1}u, C^{-1}v \rangle_{\mathcal{H}}, \quad u, v \in \mathcal{H}_C.$$

It is not difficult to verify that the identity operator from \mathcal{H}_C to \mathcal{H} is a continuous embedding, $\mathcal{H}_C \hookrightarrow \mathcal{H}$.

Let \mathcal{H}'_C be the anti-dual of \mathcal{H}_C (the set of anti-linear continuous maps from $(\mathcal{H}_C, \|\cdot\|_{\mathcal{H}_C})$ to \mathbb{C}). Setting

$$\langle u, v \rangle_C := \langle Cu, Cv \rangle_{\mathcal{H}}, \quad u, v \in \mathcal{H},$$

one verifies that \mathcal{H}'_C identifies with the completion of \mathcal{H} for the norm $\|\cdot\|_C$ associated to $\langle \cdot, \cdot \rangle_C$. Thus we obtain the Gelfand triple

$$\mathcal{H}_C \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{H}'_C.$$

Now, given $A \in \mathcal{B}(\mathcal{H}_C)$ a bounded operator in \mathcal{H}_C , the anti-dual of A , denoted by $A' \in \mathcal{B}(\mathcal{H}'_C)$, is defined by

$$\langle u, A'\Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C} := \langle A^*u, \Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C}, \quad \Psi \in \mathcal{H}'_C, \quad u \in \mathcal{H}_C. \quad (2.19)$$

Since the restriction of C to \mathcal{H}_C belongs to $\mathcal{B}(\mathcal{H}_C)$, we can consider its anti-dual defined by (2.19); we still use the symbol C' to denote the anti-dual of $C|_{\mathcal{H}_C}$. It is not difficult to show that, for all $\Psi \in \mathcal{H}'_C$, $C'\Psi$ extends to an anti-linear continuous form on \mathcal{H} which identifies to an element of \mathcal{H} via the (anti-linear version of) the Riesz representation theorem. The map $C' : \mathcal{H}'_C \rightarrow \mathcal{H}$ then extends to a bounded operator $C' \in \mathcal{B}(\mathcal{H}'_C, \mathcal{H})$ satisfying

$$\|C'\|_{\mathcal{B}(\mathcal{H}'_C, \mathcal{H})} \leq 1.$$

Moreover, C' is an extension of C and for all $\Psi \in \mathcal{H}'_C$, there exists a sequence $(v_n)_{n \in \mathbb{N}} \subset \mathcal{H}$ such that

$$\|C'\Psi - Cv_n\|_{\mathcal{H}} \rightarrow 0, \quad n \rightarrow \infty.$$

Note that the anti-dual of $C^{-1} \in \mathcal{B}(\mathcal{H}_C, \mathcal{H})$, denoted by $(C^{-1})' \in \mathcal{B}(\mathcal{H}, \mathcal{H}'_C)$, satisfies $C'^{-1} = (C^{-1})'$. Note also that \mathcal{H}'_C is equipped with the scalar product

$$\langle \Psi, \Phi \rangle_{\mathcal{H}'_C} := \langle C'\Psi, C'\Phi \rangle_{\mathcal{H}}, \quad \Psi, \Phi \in \mathcal{H}'_C,$$

which is an extension of $\langle \cdot, \cdot \rangle_C$ to \mathcal{H}'_C .

2.2.3.2 Extension of H

Our next concern is to define the anti-dual of the operator

$$H = H_0 + V = H_0 + CWC.$$

First, we observe that since $V = CWC$ belongs to $\mathcal{B}(\mathcal{H}'_C, \mathcal{H}_C)$, its anti-dual $V' \in \mathcal{B}(\mathcal{H}'_C, \mathcal{H}_C)$ is well-defined and given by

$$V' = CWC'.$$

Now, in order for the anti-dual of the unbounded operator H_0 to be well-defined, we will assume (see Hypothesis 2.5 below) that

$$\mathcal{D}(H_0|_{\mathcal{H}_C}) := \{u \in \mathcal{D}(H_0) \cap \mathcal{H}_C, H_0u \in \mathcal{H}_C\}$$

is dense in \mathcal{H}_C for the topology of \mathcal{H}_C . The anti-dual H'_0 of H_0 is then defined by

$$\mathcal{D}(H'_0) := \{\Psi \in \mathcal{H}'_C, \exists \alpha > 0, \forall u \in \mathcal{D}(H_0|_{\mathcal{H}_C}), |\langle H_0u, \Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C}| \leq \alpha \|u\|_{\mathcal{H}_C}\},$$

and

$$\langle u, H'_0\Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C} := \langle H_0u, \Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C}, \quad \Psi \in \mathcal{D}(H'_0), \quad u \in \mathcal{D}(H_0|_{\mathcal{H}_C}).$$

In the same way, we can define

$$\mathcal{D}(H|_{\mathcal{H}_C}) := \{u \in \mathcal{D}(H_0) \cap \mathcal{H}_C, Hu \in \mathcal{H}_C\},$$

$$\langle u, H'\Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C} := \langle H^*u, \Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C}, \quad \Psi \in \mathcal{D}(H'), \quad u \in \mathcal{D}(H|_{\mathcal{H}_C}).$$

Then it is not difficult to see that $\mathcal{D}(H_0|_{\mathcal{H}_C}) = \mathcal{D}(H|_{\mathcal{H}_C})$, $\mathcal{D}(H'_0) = \mathcal{D}(H')$, and

$$H' = H'_0 + V' = H'_0 + CWC'.$$

Clearly, H' is an extension of H since for all $u \in \mathcal{D}(H_0)$, we have

$$\forall v \in \mathcal{D}(H_0|_{\mathcal{H}_C}), \quad |\langle u, H_0v \rangle_{\mathcal{H}}| \leq \|H_0u\|_{\mathcal{H}} \|C\|_{\mathcal{H}} \|v\|_{\mathcal{H}_C},$$

which implies that $u \in \mathcal{D}(H'_0)$ and that $H'u = Hu$, using the anti-linear version of the Riesz representation theorem.

2.2.3.3 The resolvent of H_0

One of our main hypotheses (see Hypothesis 2.1) will imply that the limits

$$C\mathcal{R}_0(\lambda \pm i0^+)C := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_0(\lambda \pm i\varepsilon)C \tag{2.20}$$

exist for a.e. $\lambda \in \sigma_{\text{ess}}(H_0)$, for the topology of $\mathcal{B}(\mathcal{H})$. In other words, the family of operators $(\mathcal{R}_0(\lambda \pm i\varepsilon))_{\varepsilon > 0}$ converges in $\mathcal{B}(\mathcal{H}_C, \mathcal{H}'_C)$ as $\varepsilon \rightarrow 0^+$ and its limit is denoted by

$$\mathcal{R}_0(\lambda \pm i0^+) \in \mathcal{B}(\mathcal{H}_C, \mathcal{H}'_C).$$

2.2.4 Regular spectral points and spectral singularities

In this section we define the notions of regular spectral points and spectral singularities that we consider in this paper.

Definition 2.2.2 (Regular spectral point and spectral singularity). *Let $\lambda \in \sigma_{\text{ess}}(H)$.*

- (i) *We say that λ is an outgoing/incoming regular spectral point of H if λ is not an accumulation point of eigenvalues located in $\lambda \pm i(0, \infty)$ and if the limit*

$$C\mathcal{R}_H(\lambda \pm i0^+)CW := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\lambda \pm i\varepsilon)CW \quad (2.21)$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$. If λ is not an outgoing/incoming regular spectral point, we say that λ is an outgoing/incoming spectral singularity of H .

- (ii) *We say that λ is a regular spectral point of H if it is both an incoming and an outgoing regular spectral point of H . If λ is not a regular spectral point, we say that λ is a spectral singularity of H .*

Our definition of a spectral singularity is related to that of [82] and to the notion of spectral projections for non-self-adjoint operators [23], which will also be an important tool in our paper. See (2.38) for the definition of the spectral projection $\mathbf{1}_I(H)$ corresponding to a spectral interval $I \subset \sigma_{\text{ess}}(H)$ without spectral singularities. In [82], a spectral singularity corresponds to an exceptional point λ_0 outside of which the ‘spectral resolution’ $I \mapsto \mathbf{1}_I(H)$ is countably additive and uniformly bounded. In our context, this is a weaker requirement than that of Definition 2.2.2, see Section 2.5.1.

Definition 2.2.2 generalizes the definition of spectral singularities considered in the context of dissipative operators in [32, 34, 35]. As we explain below (see Section 2.3.3), for Schrödinger operators, $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$ with V a complex, decaying potential, spectral singularities correspond to real resonances. Clearly, in our abstract setting, Definition 2.2.2 depends on the choice of the decomposition $V = CWC$. For Schrödinger operators, however, C is usually chosen as a multiplication operator $C = \langle x \rangle^{-\sigma/2}$ for σ larger than some critical exponent $\sigma_0 > 0$. It is then natural to consider the intersection over all $\sigma > \sigma_0$ to define a resonance. See Section 2.3.3 for more details.

Various characterizations of the notions introduced in Definition 2.2.2 will be given in Section 2.4. Note in particular that, under the assumption that the limits (2.20) exist, the operator W could have been put to the left of $C\mathcal{R}_H(\lambda \pm i0^+)C$ in (2.21), i.e. the limits in (2.21) exist if and only if the limits $W C\mathcal{R}_H(\lambda \pm i\varepsilon)C$, $\varepsilon \rightarrow 0^+$ exist. Moreover, under the same assumption, we will show that λ is an outgoing/incoming regular spectral point of H if and only if $\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V'$ is invertible in $\mathcal{B}(\mathcal{H}'_C)$. (Here it should be recalled that $V' \in \mathcal{B}(\mathcal{H}'_C, \mathcal{H}_C)$ and, assuming that the limits (2.20) exist, that $\mathcal{R}_0(\lambda \pm i0^+) \in \mathcal{B}(\mathcal{H}_C, \mathcal{H}'_C)$). Hence spectral singularities are naturally associated to resonant states defined as follows.

Definition 2.2.3 (Incoming/outgoing resonant states). *Let $\lambda \in \sigma_{\text{ess}}(H)$ be a spectral singularity of H . The space $\mathcal{H}'_C^+(\lambda) \subset \mathcal{H}'_C$ of outgoing resonant states corresponding to λ is defined by*

$$\mathcal{H}'_C^+(\lambda) := \text{Ker}(\text{Id} + \mathcal{R}_0(\lambda + i0^+)V').$$

The space $\mathcal{H}'_C^-(\lambda) \subset \mathcal{H}'_C$ of incoming resonant states is defined by

$$\mathcal{H}'_C^-(\lambda) := \text{Ker}(\text{Id} + \mathcal{R}_0(\lambda - i0^+)V').$$

As kernels of bounded operators, the vector spaces $\mathcal{H}'_C^\pm(\lambda)$ are closed. We will prove that eigenvectors associated to embedded eigenvalues of H belong to $\mathcal{H}'_C^\pm(\lambda)$.

Note that in the case of complex Schrödinger operators, $H = -\Delta + V(x)$ on $L^2(\mathbb{R}^d)$, λ is usually called an outgoing/incoming resonance if the quotient vector space

$$\text{Ker}((\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V')) / \text{Ker}((\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V')|_{\mathcal{H}}) \neq \{0\},$$

where $(\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V')|_{\mathcal{H}}$ stands for the restriction of $(\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V')$ to \mathcal{H} . An outgoing/incoming resonant state then corresponds to an element of $\text{Ker}((\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V'))$ which does not belong to \mathcal{H} . See Section 2.3.3 for more details.

2.3 Assumptions and main results

2.3.1 Hypotheses

In this section we detail our main abstract hypothesis. In Section 2.3.3 we will show that they are satisfied in the case of complex Schrödinger operators, under suitable assumptions on the potential.

In our first hypothesis, we require that H_0 satisfies a *limiting absorption principle* (with weight C) at any point of the essential spectrum.

Hypothesis 2.1 (Limiting absorption principle for H_0). *We have*

$$\sup_{z \in \mathbb{C}^\pm} \|C\mathcal{R}_0(z)C\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (2.22)$$

Note that (2.22) implies (see e.g. [13, Proposition 4.1]) that the spectrum of H_0 is purely absolutely continuous, i.e. that $\sigma_{\text{pp}}(H_0) = \emptyset$, $\sigma_{\text{ac}}(H_0) = \sigma(H_0)$, $\sigma_{\text{sc}}(H_0) = \emptyset$, where $\sigma_{\text{pp}}(H_0)$, $\sigma_{\text{ac}}(H_0)$, $\sigma_{\text{sc}}(H_0)$ stand for the usual pure point, absolutely continuous and singular continuous spectra of the self-adjoint operator H_0 .

By Fatou's Theorem, (2.22) yields that the limits $C\mathcal{R}_0(\lambda \pm i0^+)C$ exist for almost every $\lambda \in \sigma_{\text{ess}}(H)$, in the norm topology of $\mathcal{B}(\mathcal{H})$, and that the map $\mathbb{R} \ni \lambda \mapsto C\mathcal{R}_0(\lambda \pm i0^+)C \in \mathcal{B}(\mathcal{H})$ is bounded (observe that $C\mathcal{R}_0(\lambda \pm i0^+)C = C\mathcal{R}_0(\lambda)C$ if $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(H)$).

Note also that Hypothesis 2.1 implies (see [55] or [76, Theorem XIII.25 and its corollary]) that C is *relatively smooth* with respect to H_0 in the sense of Kato, i.e. that there exists a constant c_0 such that

$$\forall u \in \mathcal{H}, \quad \int_{\mathbb{R}} \|Ce^{-itH_0}u\|_{\mathcal{H}}^2 dt \leq c_0^2 \|u\|_{\mathcal{H}}^2. \quad (2.23)$$

Recall that (2.23) is equivalent to

$$\forall u \in \mathcal{H}, \quad \int_{\mathbb{R}} \left(\|C\mathcal{R}_0(\lambda - i0^+)u\|_{\mathcal{H}}^2 + \|C\mathcal{R}_0(\lambda + i0^+)u\|_{\mathcal{H}}^2 \right) d\lambda \leq 2\pi c_0^2 \|u\|_{\mathcal{H}}^2, \quad (2.24)$$

where $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i0^+)u$ denotes the limit of $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i\varepsilon)u$ in $L^2(\mathbb{R}; \mathcal{H})$ as $\varepsilon \rightarrow 0^+$ (see [55]).

Next we will assume that the point spectral subspace of H is finite.

Hypothesis 2.2 (Eigenvalues of H). *H has only a finite number of eigenvalues and they all have finite algebraic multiplicity.*

Hypothesis 2.2 prevents the essential spectrum of H from having an accumulation point of eigenvalues. It does not exclude, however, the presence of eigenvalues embedded in the essential spectrum of H .

Our next hypothesis concerns the spectral singularities of H . We will assume that H has finitely many spectral singularities with a ‘finite order’, in the sense that the map $z \mapsto C\mathcal{R}_H(z)CW$ blows up at most polynomially as z approaches any spectral singularity $\lambda \in \sigma_{\text{ess}}(H)$. We will also allow for singularities ‘at infinity’, in the sense that $z \mapsto C\mathcal{R}_H(z)C$ may blow up polynomially as z tends to ∞ (z close to the real axis).

Hypothesis 2.3 (Spectral singularities for H). *H only has a finite number of spectral singularities $\{\lambda_1, \dots, \lambda_n\} \subset \sigma_{\text{ess}}(H)$ and there exist $\varepsilon_0 > 0$ and integers $\nu_1, \dots, \nu_n, \nu_\infty \geq 0$ such that*

$$\sup_{\text{Re}(z) \in \sigma_{\text{ess}}(H), \pm \text{Im}(z) \in (0, \varepsilon_0)} \frac{1}{|z - z_0|^{\nu_\infty}} \left(\prod_{j=1}^n \frac{|z - \lambda_j|^{\nu_j}}{|z - z_0|^{\nu_j}} \right) \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty, \quad (2.25)$$

where z_0 is an arbitrary complex number such that $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$.

The factors $|z - \lambda_j|^{\nu_j}$ ‘regularize’ the singularities of $z \mapsto C\mathcal{R}_H(z)CW$ as z approaches λ_j . Dividing them by $|z - z_0|^{\nu_j}$ produces bounded terms. The factor $|z - z_0|^{-\nu_\infty}$ regularizes a possible singularity at ∞ .

Observe that since $\lambda_1, \dots, \lambda_n$ are the only spectral singularities of H , for all $\lambda \in \sigma_{\text{ess}}(H) \setminus \{\lambda_1, \dots, \lambda_n\}$, the limits $C\mathcal{R}_H(\lambda \pm i0^+)CW$ exist in the norm topology of $\mathcal{B}(\mathcal{H})$. The condition (2.25) then implies that the maps

$$\sigma_{\text{ess}}(H) \setminus \{\lambda_1, \dots, \lambda_n\} \ni \lambda \mapsto \frac{1}{|\lambda - z_0|^{\nu_\infty}} \left(\prod_{j=1}^n \frac{|\lambda - \lambda_j|^{\nu_j}}{|\lambda - z_0|^{\nu_j}} \right) C\mathcal{R}_H(\lambda \pm i0^+)CW \in \mathcal{B}(\mathcal{H}) \quad (2.26)$$

are bounded. Since, as we will show below (see Proposition 2.4.6), embedded eigenvalues are special spectral singularities, Hypothesis 2.3 is also a condition on embedded eigenvalues.

As mentioned above, to study the absolutely continuous spectral subspace of H , we require the existence of a ‘conjugation’ operator J satisfying, in particular, $JH = H^*J$. (Note that we do not require that J be involutive, see Section 2.2.2 above for more details).

Hypothesis 2.4 (Conjugation operator). *There exists an antiunitary operator $J : \mathcal{H} \rightarrow \mathcal{H}$ such that*

- (i) $J\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ and $\forall u \in \mathcal{D}(H_0)$, $JH_0u = H_0Ju$.
- (ii) $JC = CJ$ and $JW = W^*J$.

Moreover, for all embedded eigenvalues $\lambda \in \sigma_{\text{ess}}(H)$, the symmetric bilinear form

$$\text{Ker}((H - \lambda)^{m_\lambda}) \times \text{Ker}((H - \lambda)^{m_\lambda}) \ni (u, v) \mapsto \langle Ju, v \rangle \quad \text{is non-degenerate.} \quad (2.27)$$

Here it should be recalled (see Section 2.2.2) that the main purpose of (2.27) is to allow us to define suitable spectral projections for embedded eigenvalues.

Our last technical hypothesis is required in order for the anti-dual operators H'_0 , H' to be well-defined, see Section 2.2.3 for more details.

Hypothesis 2.5. *The domain of the restriction of H_0 to \mathcal{H}_C , defined as*

$$\mathcal{D}(H_0|_{\mathcal{H}_C}) := \{u \in \mathcal{D}(H_0) \cap \mathcal{H}_C, H_0u \in \mathcal{H}_C\},$$

is dense in \mathcal{H}_C for the topology of \mathcal{H}_C .

2.3.2 Main results

Now we can state our main results. First, we characterize the outgoing (respectively incoming) spectral singularities of H as eigenvalues of H' associated to eigenvectors belonging to the space of outgoing resonant states $\mathcal{H}_C^+(\lambda)$ (respectively $\mathcal{H}_C^-(\lambda)$).

Theorem 2.3.1. *Suppose that Hypothesis 2.5 holds. Let $\lambda \in \sigma_{\text{ess}}(H)$ and suppose that the limits*

$$C\mathcal{R}_0(\lambda \pm i0^+)C := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_0(\lambda \pm i\varepsilon)C$$

exist in the norm topology of $\mathcal{B}(\mathcal{H})$. The following conditions are equivalent:

- (i) λ is an outgoing/incoming spectral singularity of H ,
- (ii) λ is an eigenvalue of H' associated to an eigenvector $\Psi \in \mathcal{H}_C^\pm(\lambda)$.

Theorem 2.3.1 shows that λ is a spectral singularity of H if and only if the equation $H'\Psi = \lambda\Psi$ has a solution Ψ in $\mathcal{H}_C^\pm(\lambda)$ (recall that H' is an extension of H , acting in the Hilbert space \mathcal{H}'_C which contains the original Hilbert space \mathcal{H}). For non-self-adjoint Schrödinger operators, this corresponds to λ being a real resonance of $H = -\Delta + V(x)$ if and only if the equation $(-\Delta + V(x))\Psi = \lambda\Psi$ has a distributional solution such that Ψ belongs to suitable weighted L^2 -spaces. See Section 2.3.3 for a more detailed discussion.

Theorem 2.3.1 also has various consequences that we will detail in Section 2.4. Here we mention the following two consequences. We will show that the eigenvalues of H embedded in the essential spectrum are special spectral singularities, see Proposition

2.4.6. Moreover, in the particular case where H is dissipative, we will prove that H cannot have outgoing spectral singularities unless its self-adjoint part, $\operatorname{Re}(H)$, already has some, see Section 2.4.6.

Our next result shows that the subspace $\mathcal{H}_{\text{ads}}^\pm(H)$ of asymptotically disappearing states at $\pm\infty$ (recall that $\mathcal{H}_{\text{ads}}^\pm(H)$ has been defined in Section 2.2.2) coincides with the vector space spanned by all generalized eigenstates of H corresponding to eigenvalues λ of H such that $\pm\operatorname{Im}(\lambda) < 0$. In other words, the only solutions to (2.1) that vanish at as $t \rightarrow \pm\infty$ are linear combination of generalized eigenstates corresponding to non-real eigenvalues.

Theorem 2.3.2. *Suppose that Hypotheses 2.1-2.3 hold. Then*

$$\mathcal{H}_{\text{ads}}^\pm(H) = \mathcal{H}_p^\pm(H). \quad (2.28)$$

An analogous result was proven in [35] in the particular case of dissipative operators, answering a question left as an open problem in [16]. The proof in [35] relies in an essential way on the existence and properties of wave operators. Besides the fact that we are considering non-dissipative operators, our proof here is more direct – we do not use scattering theory – and allows for more general assumptions (compare Hypothesis 2.5 in [35] and Hypothesis 2.3 of the present paper, where a singularity at infinity of the weighted resolvent is allowed). The core of our argument is a suitable spectral decomposition formula, see Proposition 2.3.10 in the next subsection.

Next we will prove that the absolutely continuous spectral subspace of H defined in Section 2.2.2 coincides with the orthogonal complement of the point spectral subspace.

Theorem 2.3.3. *Suppose that Hypotheses 2.1-2.3 holds. If H has embedded eigenvalues, suppose in addition that Hypothesis 2.4 holds. Then*

$$\mathcal{H}_{\text{ac}}(H) = \operatorname{Ran}(\operatorname{Id} - \Pi_p(H)) = \mathcal{H}_p(H^*)^\perp.$$

As mentioned before, in the particular case where H is dissipative, Theorem 2.3.3 may be compared with a result of [16]. The absolutely continuous spectral subspace for dissipative operators is defined in [16] in the same way as in (2.17), but with the integral taken over $[0, \infty)$ instead of \mathbb{R} . Using the theory of dilations of dissipative operators, it is then proven in [16] that within such a definition, the absolutely continuous spectral subspace coincides with the orthogonal complement of ‘bound states’ (generalized eigenstates corresponding to real eigenvalues). In our context where H is not necessarily dissipative, the argument of [16] fails and there is no reason to choose positive times over negative times to define $\mathcal{H}_{\text{ac}}(H)$.

Our definition of $\mathcal{H}_{\text{ac}}(H)$ is also justified by the following two facts. First, Theorem 2.3.3 generalizes the well-known identity $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_p(H)^\perp$ which holds for self-adjoint operators without singular continuous spectrum. Another justification comes from dissipative scattering theory. Indeed, combined with the results of [14, 16, 35, 68], Theorem 2.3.3 implies that, in the particular case where H is dissipative, we have $\operatorname{Ran}(W_-(H, H_0))^{\text{cl}} = \mathcal{H}_{\text{ac}}(H)$, where $W_-(H, H_0) := \text{s-lim } e^{-itH} e^{itH_0}$, $t \rightarrow \infty$, is the usual

wave operator. This again generalizes the well-known relation which holds in the self-adjoint case.

We mention that if H has no embedded eigenvalues, Hypothesis 2.4 can be dropped in the statement of Theorem 2.3.3. This is also the case if H is supposed to be dissipative (see Proposition 2.5.10). In the general case, it is however clear that some assumption should be added to treat the pathological case where the map in (2.27) is degenerate. Indeed, considering the simplest case where $m_\lambda = 1$, if $\langle J\varphi, \varphi \rangle = 0$ for any $\varphi \in \text{Ker}(H - \lambda)$, then one can check that $\text{Ker}(H - \lambda) \subset \mathcal{H}_p(H^*)^\perp$. Therefore, since $\mathcal{H}_{ac}(H)$ should not contain eigenstates of H , we do not expect that the equality $\mathcal{H}_{ac}(H) = \mathcal{H}_p(H^*)^\perp$ holds. Another possibility to handle such pathological cases might be to suitably modify the definition of $\mathcal{H}_{ac}(H)$. We do not consider this possibility here.

To prove Theorem 2.3.3, we cannot rely on the theory of unitary dilations as in the dissipative case studied in [16]. In the same way as for Theorem 2.3.2, our proof of Theorem 2.3.3 relies on the spectral decomposition formula stated in Proposition 2.3.10.

The next remark gives the J -orthogonal spectral decomposition of the Hilbert space mentioned in the introduction.

Remark 2.3.4. *Combining Theorems 2.3.2 and 2.3.3, using in addition that $J : \mathcal{H}_p(H) \rightarrow \mathcal{H}_p(H^*)$ is bijective and that the map $\mathcal{H}_p(H) \ni (u, v) \mapsto \langle Ju, v \rangle$ is non-degenerate, we obtain the following J -orthogonal direct sum decompositions of the Hilbert space:*

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{ac}(H) \oplus \mathcal{H}_p(H) \\ &= \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{disc}(H) \oplus \mathcal{H}_{emb}(H) \\ &= \mathcal{H}_{ac}(H) \oplus \mathcal{H}_{ads}^+(H) \oplus \mathcal{H}_{ads}^-(H) \oplus \mathcal{H}_{disc}^0(H) \oplus \mathcal{H}_{emb}(H), \end{aligned}$$

the direct sum $\mathcal{H}_{disc}^0(H) \oplus \mathcal{H}_{emb}(H) =: \mathcal{H}_b(H)$ being the space of ‘bound states’, i.e. the closure of the vector space spanned by all generalized eigenvectors of H corresponding to real eigenvalues (either isolated or embedded).

2.3.3 Application to Schrödinger operators

We suppose in this section that

$$\mathcal{H} = L^2(\mathbb{R}^d), \quad H_0 = -\Delta \quad \text{and} \quad V \text{ is a complex-valued potential.}$$

For simplicity, we suppose that $d = 3$. (The following arguments easily generalize to any dimension $d \geq 3$ with d odd. In dimension $d = 1$ or if d is even the situation is more subtle, due to the singularity of the weighted resolvent of $-\Delta$ at 0).

2.3.3.1 Spectral singularities for short-range complex potentials

We begin with showing that Theorem 2.3.1 can be applied to $H = -\Delta + V(x)$, under a short-range condition on V . It is well-known that the limits

$$\langle x \rangle^{-s} \mathcal{R}_0(\lambda \pm i0^+) \langle x \rangle^{-s}, \quad \lambda > 0, \tag{2.29}$$

exist in the norm topology of $\mathcal{B}(\mathcal{H})$, for any $s > \frac{1}{2}$, where $\langle x \rangle := (1 + x^2)^{\frac{1}{2}}$. Hence, assuming that V satisfies the short-range condition

$$x \mapsto \langle x \rangle^\sigma V(x) \in L^\infty(\mathbb{R}^3), \quad (2.30)$$

with $\sigma > 1$, we can choose C to be the multiplication operator by $\langle x \rangle^{-\sigma/2}$. The Hilbert space \mathcal{H}_C then identifies with the weighted L^2 -space

$$\mathcal{H}_C = L^2_{\sigma/2} := \{f : \mathbb{R}^3 \rightarrow \mathbb{C}, x \mapsto \langle x \rangle^{\frac{\sigma}{2}} f(x) \in L^2(\mathbb{R}^3)\},$$

equipped with the usual norm, its dual being given by

$$\mathcal{H}'_C = L^2_{-\sigma/2} = \{f : \mathbb{R}^3 \rightarrow \mathbb{C}, x \mapsto \langle x \rangle^{-\frac{\sigma}{2}} f(x) \in L^2(\mathbb{R}^3)\}.$$

Since the set of smooth functions with compact supports is contained in $\mathcal{D}(H_0) \cap \mathcal{H}_C$, one easily deduces that Hypothesis 2.5 is satisfied. Applying Theorem 2.3.1, we thus obtain the following proposition.

Proposition 2.3.5. *Suppose that V is a complex-valued potential satisfying (2.30) with $\sigma > 1$. Let $C(x) = \langle x \rangle^{-\sigma/2}$. Then for all $\lambda > 0$, the following conditions are equivalent*

- (i) λ is an outgoing/incoming spectral singularity of H in the sense of Definition 2.2.2,
- (ii) There exists $\Psi \in \mathcal{H}'_C^\pm(\lambda) \subset L^2_{-\sigma/2}$, $\Psi \neq 0$, such that

$$(-\Delta + V(x) - \lambda)\Psi = 0. \quad (2.31)$$

If Ψ in (ii) belongs to $L^2(\mathbb{R}^3)$, then λ is an eigenvalue of H . Otherwise, λ is usually called a real resonance associated to a resonant state $\Psi \in \mathcal{H}'_C^\pm(\lambda) \setminus L^2(\mathbb{R}^3)$. Such a resonant state satisfies the outgoing/incoming Sommerfeld radiation condition

$$u(x) = |x|^{\frac{1}{2}} e^{\pm i\lambda^{\frac{1}{2}}|x|} \left(a\left(\frac{x}{|x|}\right) + o(1) \right), \quad |x| \rightarrow \infty,$$

with $a \in L^2(S^2)$, $a \neq 0$. By Proposition 2.3.5, one can choose any $\sigma > 1$. A resonant state is thus a distributional solution to (2.31) belonging to $L^2_{-\sigma/2}$.

Note that if V is real-valued, Agmon's fundamental work [2] shows that $H = -\Delta + V$ has no spectral singularities in $(0, \infty)$. Likewise, if H is dissipative, a simple argument combined with [2] proves that H cannot have outgoing spectral singularities in $(0, \infty)$ (see [91]). In general, however, spectral singularities cannot be excluded (see again [91] for an example showing that, for any $\lambda > 0$, there exists a smooth, compactly supported potential V such that λ is an incoming spectral singularity of H in the dissipative case).

At the threshold energy 0, the limiting absorption principle states that the limits

$$\langle x \rangle^{-s} \mathcal{R}_0(\pm i0^+) \langle x \rangle^{-s},$$

exist in the norm topology of $\mathcal{B}(\mathcal{H})$ for any $s \geq 1$. Note that, as we will argue in Section 2.4, the two limits above in fact coincide with the limit

$$\langle x \rangle^{-s} \mathcal{R}_0(0) \langle x \rangle^{-s} := \lim_{z \rightarrow 0, z \in \mathbb{C} \setminus \mathbb{R}_+} \langle x \rangle^{-s} \mathcal{R}_0(z) \langle x \rangle^{-s}.$$

(Note also that the limit $\langle x \rangle^{-s} \mathcal{R}_0(0) \langle x \rangle^{-s'}$ exists, more generally, provided that $s, s' > 1/2$ and $s + s' > 2$, see [53].) Assuming now that V satisfies the short-range condition (2.30) with $\sigma \geq 2$, we can choose C to be the multiplication operator by $\langle x \rangle^{-\sigma/2}$ and proceed as before. Since, as we will see in Section 2.4, 0 is an outgoing spectral singularity of H if and only if it is an incoming spectral singularity of H , this gives the following proposition.

Proposition 2.3.6. *Suppose that V is a complex-valued potential satisfying (2.30) with $\sigma \geq 2$. Let $C(x) = \langle x \rangle^{-\sigma/2}$. Then the following conditions are equivalent*

- (i) *0 is a spectral singularity of H in the sense of Definition 2.2.2,*
- (ii) *There exists $\Psi \in \mathcal{H}'_{\pm}(\lambda) \subset L^2_{-\sigma/2}$ such that*

$$(-\Delta + V(x))\Psi = 0.$$

Anticipating results that we will prove in the abstract setting in the case where H is dissipative (see Section 2.4.6), we also have the following proposition.

Proposition 2.3.7. *Suppose that V is a complex-valued potential such that $\text{Im}(V) \leq 0$ and $\text{Im}(V) < 0$ on a non-trivial open set. Suppose that (2.30) holds with $\sigma > 1$ and let $C(x) = \langle x \rangle^{-\sigma/2}$. Then*

H has no positive outgoing spectral singularities.

In particular, H has no positive embedded eigenvalues. Suppose that (2.30) holds with $\sigma \geq 2$. Then

0 is not a spectral singularity of H .

See Section 2.4.6 for a proof of Proposition 2.3.7. Note also that the results of Proposition 2.3.7 have been established in [90, 91], using different arguments.

2.3.3.2 Spectral decomposition for compactly supported complex potentials

We now show that Theorems 2.3.2 and 2.3.3 can be applied to $H = -\Delta + V(x)$ under the condition that

$$V \in L_c^\infty(\mathbb{R}^3) := \{u \in L^\infty(\mathbb{R}^3), u \text{ is compactly supported}\}. \quad (2.32)$$

Similarly as in (2.29), we have

$$\sup_{z \in \mathbb{C}^\pm} \left\| \langle x \rangle^{-s} \mathcal{R}_0(z) \langle x \rangle^{-s} \right\|_{\mathcal{B}(\mathcal{H})} < \infty,$$

for any $s > 1$. Hence, choosing $C(x) = \langle x \rangle^{-s}$, Hypothesis 2.1 is satisfied.

Assuming (2.32), it is known that H has only finitely many eigenvalues with finite algebraic multiplicities. See, e.g., [43] and references therein. In particular, Hypothesis 2.2 is satisfied.

To verify that Hypothesis 2.3 holds, we can rely on the theory of resonances, defined as poles of the meromorphic extension of the weighted resolvent, see e.g. [25]. Assuming (2.32), the map

$$\{z \in \mathbb{C}, \operatorname{Im}(z) > 0\} \ni z \mapsto (H - z^2)^{-1} : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$$

is meromorphic and extends to a meromorphic map

$$\mathbb{C} \ni z \mapsto \mathcal{R}(z^2) : L_c^2(\mathbb{R}^3) \rightarrow L_{\text{loc}}^2(\mathbb{R}^3), \quad (2.33)$$

where $L_c^2(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3), u \text{ is compactly supported}\}$ and $L_{\text{loc}}^2(\mathbb{R}^3) := \{u : \mathbb{R}^3 \rightarrow \mathbb{C}, u \in L^2(K) \text{ for all compact set } K \subset \mathbb{R}^3\}$. Poles of the map in (2.33) are called resonances of H . One then verifies that a real resonance $\pm\lambda_0$ of H , with $\lambda_0 \geq 0$, corresponds to an outgoing/incoming spectral singularity λ_0^2 in the sense of Definition 2.2.2. Moreover, H has only finitely many spectral singularities $\{\lambda_1, \dots, \lambda_n\}$ and Hypothesis 2.3 is satisfied, with ν_j the order of the corresponding resonances $\pm\sqrt{\lambda_j}$, and $\nu_\infty = 0$. See e.g. [25] and references therein for an exposition of the theory of resonances of Schrödinger operators, and [35, Section 6] for a more detailed comparison between the notions of resonances and spectral singularities considered in this paper.

Applying Theorem 2.3.2, we obtain the following result.

Proposition 2.3.8. *Suppose that V is a complex-valued potential such that $V \in L_c^\infty(\mathbb{R}^3)$. Then*

$$\mathcal{H}_{\text{ads}}^\pm(H) = \mathcal{H}_p^\pm(H).$$

To apply Theorem 2.3.3, we need to verify in addition that Hypothesis 2.4 holds. Clearly, we can take the conjugation operator J as the complex conjugation. We then obtain

Proposition 2.3.9. *Suppose that V is a complex-valued potential such that $V \in L_c^\infty(\mathbb{R}^3)$. Assume that, for all embedded eigenvalues $\lambda \in [0, \infty)$, the symmetric bilinear form*

$$\operatorname{Ker}((H - \lambda)^{m_\lambda}) \ni (u, v) \mapsto \int_{\mathbb{R}^3} u(x)v(x)dx \quad \text{is non-degenerate.} \quad (2.34)$$

Then

$$\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_p(H^*)^\perp. \quad (2.35)$$

In the case where H is dissipative, even if H has embedded eigenvalues, Proposition 2.5.10 below shows that condition (2.34) can be dropped. Moreover, under some ‘form-subordinate smallness conditions’, it is possible to show that H has no point spectrum (see [33]; see also [12] for similar results for Pauli and Dirac Hamiltonians). If H does have embedded eigenvalues, as explained before, (2.34) seems however necessary for (2.35) to hold. We mention that an assumption comparable to (2.34) has been used, for threshold eigenvalues, in the recent works [1, 89] to study the large-time behaviors of solutions to Schrödinger equations with a complex potentials.

2.3.4 Organisation of the paper and ingredients of the proof

The proof of Theorem 2.3.1 (given in Section 2.4.2) relies, in particular, on a ‘boundary value version’ of the *Birman-Schwinger principle* that we state and prove in Section 2.4.1, see Proposition 2.4.1. This proposition extends a related result for dissipative operators proven in [34, Lemma 4.1]. Further results concerning spectral singularities are proven in Section 2.4.

As mentioned in the previous section, the main ingredient in the proofs of Theorems 2.3.2 and 2.3.3 is a *spectral decomposition formula* suitably modified to take into account the spectral singularities $\{\lambda_j\}_{j=1}^n$ of H . It can be stated as follows. Assuming Hypothesis 2.3, with $\nu_1, \dots, \nu_n, \nu_\infty$ defined by this hypothesis, we set, for all $z \in \mathbb{C} \setminus \{z_0\}$,

$$r(z) := (z - z_0)^{-(\nu_1 + \dots + \nu_n + \nu_\infty)} \prod_{j=1}^n (z - \lambda_j)^{\nu_j}, \quad (2.36)$$

where we recall that $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$. We then write

$$r(H) = \mathcal{R}_H(z_0)^{\nu_1 + \dots + \nu_n + \nu_\infty} \prod_{j=1}^n (H - \lambda_j)^{\nu_j},$$

which defines a bounded operator in $\mathcal{B}(\mathcal{H})$. Note that if λ_j is an embedded eigenvalue of H , then for any generalized eigenstate φ_j corresponding to λ_j , we have $r(H)\varphi_j = 0$ (provided that ν_j is large enough). We will prove the following proposition.

Proposition 2.3.10. *Suppose that Hypotheses 2.1, 2.2 and 2.3 hold. Then*

$$r(H) = r(H)\Pi_{\text{disc}}(H) + \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\sigma_{\text{ess}}(H)} r(\lambda) (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (2.37)$$

Proposition 2.3.10 generalizes the well-known resolution of the identity formula for self-adjoint operators to a class of non self-adjoint operators with finitely many spectral singularities. In particular, if H has no spectral singularities and $\nu_\infty = 0$ in Hypothesis 2.3, then we can take $r = 1$ and (2.37) reduces to

$$\text{Id} = \Pi_{\text{disc}}(H) + \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\sigma_{\text{ess}}(H)} (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda,$$

which corresponds to Stone’s formula in the particular case where H is self-adjoint.

Equation (2.37) is also related to the notion of spectral projections for non-self-adjoint operators [23, 24, 82], defined by

$$\mathbf{1}_I(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2i\pi} \int_I (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda, \quad (2.38)$$

where $I \subset \sigma_{\text{ess}}(H)$ is a closed interval without spectral singularities. We mention that such spectral projections were used in a stationary approach to non-unitary scattering theory, for differential operators in [69, 70], and in an abstract setting in [47, 48, 52].

We will recall in Section 2.5 that the spectral projections (2.38) are well-defined on intervals without spectral singularities, and show that they induce a bounded Borel functional calculus. In intervals containing spectral singularities, we will construct a ‘regularized’ functional calculus, which in turn allows us to prove Proposition 2.3.10. Based on the latter, the proofs of Theorems 2.3.2 and 2.3.3 are given in Sections 2.5.4 and 2.5.5, respectively.

Some extensions of results already appearing in the literature are collected in appendices.

2.4 Spectral singularities

In this section we prove various characterizations of our definition of spectral singularities (see Definition 2.2.2). We will consider an arbitrary $\lambda \in \sigma_{\text{ess}}(H)$. Our main assumption will be that the limits

$$C\mathcal{R}_0(\lambda \pm i0^+)C := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_0(\lambda \pm i\varepsilon)C. \quad (2.39)$$

exist in the norm topology of $\mathcal{B}(\mathcal{H})$.

We begin in Section 2.4.1 with a characterization of spectral singularities analogous to the Birman-Schwinger principle for eigenvalues. Next we prove Theorem 2.3.1 in Section 2.4.2. In Section 2.4.3, we define the set of spectral singularities for the adjoint operator H^* and show that it coincides with the set of spectral singularities of H . Section 2.4.4 proves that eigenvalues embedded in the essential spectrum of H can be seen as particular spectral singularities. In Section 2.4.5, assuming that (2.39) is regular with respect to λ in a suitable sense, we show that the notion of spectral regularity introduced in Definition 2.2.2 is a local property. We also consider the special case of spectral singularities located at ‘endpoints’ of the essential spectrum, and show that in this case outgoing and incoming spectral singularities coincide. Finally, Section 2.4.6 is devoted to the particular case where H is a dissipative operator.

2.4.1 Birman-Schwinger principle for spectral singularities

Assuming that H_0 satisfies a limiting absorption principle at λ , as stated in (2.39), we have the following characterizations of the definition of a regular spectral point. Item (ii) can be seen as a ‘boundary value’ version of the Birman-Schwinger principle (see e.g. [6, 45, 49] and references therein). The proof of the next proposition is a quite straightforward extension to that of [34, Lemma 4.1], where the result is proven for dissipative operators. It is therefore deferred to Appendix 2.6.2.

Proposition 2.4.1. *Let $\lambda \in \sigma_{\text{ess}}(H)$ and suppose that the limits (2.39) exist in the norm topology of $\mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

- (i) λ is an outgoing/incoming regular spectral point of H ,

(ii) $\text{Id} + C\mathcal{R}_0(\lambda \pm i0^+)CW$ is invertible in $\mathcal{B}(\mathcal{H})$,

(iii) $\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V'$ is invertible in $\mathcal{B}(\mathcal{H}'_C)$.

Proof. See Appendix 2.6.2. □

It should be noted that, since C is relatively compact with respect to H_0 , the operator $C\mathcal{R}_0(\lambda \pm i\varepsilon)CW$ is compact in $\mathcal{B}(\mathcal{H})$, for all $\varepsilon > 0$. Hence $C\mathcal{R}_0(\lambda \pm i0^+)CW$ is also compact in $\mathcal{B}(\mathcal{H})$. By the Fredholm alternative, (ii) is then equivalent to $\text{Ker}(\text{Id} + C\mathcal{R}_0(\lambda \pm i0^+)CW) = \{0\}$. Likewise, (iii) is equivalent to $\text{Ker}(\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V') = \{0\}$ since $\mathcal{R}_0(\lambda \pm i0^+)V'$ is compact in $\mathcal{B}(\mathcal{H}'_C)$.

2.4.2 Proof of Theorem 2.3.1

Now we turn to the proof of Theorem 2.3.1, which characterizes outgoing/incoming spectral singularities as eigenvalues of the extended operator H' corresponding to eigenvectors belonging to the space $\mathcal{H}_C^\pm(\lambda)$ of outgoing/incoming resonant states.

Before proving Theorem 2.3.1, we need two preliminary lemmas. The first one is the following well-known estimate of the operator norm $\|\mathcal{R}_0(\lambda \pm i\varepsilon)C\|$, assuming that the limits (2.39) exist.

Lemma 2.4.2. *Let $\lambda \in \sigma_{\text{ess}}(H)$ and suppose that the limits (2.39) exist in the norm topology of $\mathcal{B}(\mathcal{H})$. There exists $c_0 > 0$ such that*

$$\forall \varepsilon > 0, \quad \|\mathcal{R}_0(\lambda \pm i\varepsilon)C\|_{\mathcal{B}(\mathcal{H})} \leq c_0 \varepsilon^{-\frac{1}{2}}. \quad (2.40)$$

Proof. See Appendix 2.6.2. □

Next we show that, under our assumptions, $\mathcal{R}_0(\lambda \pm i0^+)$ are right inverses of $H'_0 - \lambda$.

Lemma 2.4.3. *Suppose that Hypothesis 2.5 holds. Let $\lambda \in \sigma_{\text{ess}}(H)$ and suppose that the limits (2.39) exist in the norm topology of $\mathcal{B}(\mathcal{H})$. Then for all $v \in \mathcal{H}_C$, $\mathcal{R}_0(\lambda \pm i0^+)v \in \mathcal{D}(H'_0)$ and*

$$(H'_0 - \lambda)\mathcal{R}_0(\lambda \pm i0^+)v = v. \quad (2.41)$$

Proof. Let $v = C\varphi \in \mathcal{H}_C$. Since $\mathcal{R}_0(\lambda \pm i\varepsilon)$ converges to $\mathcal{R}_0(\lambda \pm i0^+)$ in $\mathcal{B}(\mathcal{H}_C, \mathcal{H}'_C)$, we have, for all $u \in \mathcal{D}(H_0|_{\mathcal{H}_C})$,

$$\begin{aligned} \langle (H_0 - \lambda)u, \mathcal{R}_0(\lambda \pm i0^+)v \rangle_{\mathcal{H}_C; \mathcal{H}'_C} &= \lim_{\varepsilon \rightarrow 0^+} \langle (H_0 - \lambda)u, \mathcal{R}_0(\lambda \pm i\varepsilon)C\varphi \rangle_{\mathcal{H}} \\ &= \langle u, C\varphi \rangle_{\mathcal{H}} \pm \lim_{\varepsilon \rightarrow 0^+} i\varepsilon \langle u, \mathcal{R}_0(\lambda \pm i\varepsilon)C\varphi \rangle_{\mathcal{H}}. \end{aligned}$$

It follows from (2.40) that

$$\|\mathcal{R}_0(\lambda \pm i\varepsilon)C\|_{\mathcal{H}} = \mathcal{O}(\varepsilon^{-\frac{1}{2}}), \quad \varepsilon \rightarrow 0^+.$$

Thus we obtain that, for all $u \in \mathcal{D}(H|_{\mathcal{H}_C})$,

$$\langle (H_0 - \lambda)u, \mathcal{R}_0(\lambda \pm i0^+)v \rangle_{\mathcal{H}_C; \mathcal{H}'_C} = \langle u, v \rangle_{\mathcal{H}}.$$

Since $|\langle u, v \rangle_{\mathcal{H}}| \leq \|u\|_{\mathcal{H}_C} \|v\|_{\mathcal{H}'_C}$, this shows that $\mathcal{R}_0(\lambda \pm i0^+)v \in \mathcal{D}(H'_0)$ and that (2.41) holds. \square

Now we are ready to prove Theorem 2.3.1

Proof of Theorem 2.3.1. (i) \Rightarrow (ii) Suppose for instance that λ is an outgoing spectral singularity. By Proposition 2.4.1, $\text{Id} + \mathcal{R}_0(\lambda + i0^+)V'$ is not invertible in $\mathcal{B}(\mathcal{H}'_C)$. Since $\mathcal{R}_0(\lambda + i0^+)V'$ is compact in $\mathcal{B}(\mathcal{H}'_C)$, it follows from the Fredholm alternative that there exists $\Psi \in \mathcal{H}'_C$, $\Psi \neq 0$, such that

$$-\mathcal{R}_0(\lambda + i0^+)V'\Psi = \Psi. \quad (2.42)$$

By Lemma 2.4.3, this implies that $\Psi \in \mathcal{D}(H'_0) = \mathcal{D}(H')$ and that

$$-V'\Psi = (H'_0 - \lambda)\Psi. \quad (2.43)$$

Since $H' = H'_0 + V'$, this proves (ii).

(ii) \Rightarrow (i) Suppose now that λ is an eigenvalue of H' associated to an eigenvector $\Psi \in \mathcal{H}'_C^+(\lambda)$, $\Psi \neq 0$. Then

$$(H' - \lambda)\Psi = 0 \quad \text{with} \quad \Psi = -\mathcal{R}_0(\lambda + i0^+)V'\Psi.$$

In particular,

$$(\text{Id} + \mathcal{R}_0(\lambda + i0^+)V')\Psi = 0,$$

and hence $\text{Id} + \mathcal{R}_0(\lambda + i0^+)V'$ is not invertible in $\mathcal{B}(\mathcal{H}'_C)$. By Proposition 2.4.1, this proves (i). \square

2.4.3 Spectral singularities of the adjoint operator

Recall that the regular spectral points and spectral singularities of H have been defined in Definition 2.2.2. The corresponding definition for the adjoint operator H^* is the following.

Definition 2.4.4 (Regular spectral point and spectral singularity for H^*). *Let $\lambda \in \sigma_{\text{ess}}(H)$.*

(i) *We say that λ is an outgoing/incoming regular spectral point of H^* if λ is not an accumulation point of eigenvalues located in $\lambda \mp i(0, \infty)$ and if the limit*

$$C\mathcal{R}_{H^*}(\lambda \mp i0^+)CW^* := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_{H^*}(\lambda \mp i\varepsilon)CW^* \quad (2.44)$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$. If λ is not an outgoing/incoming regular spectral point, we say that λ is an outgoing/incoming spectral singularity of H^ .*

(ii) We say that λ is a regular spectral point of H^* if it is both an incoming and an outgoing regular spectral point of H^* . If λ is not a regular spectral point, we say that λ is a spectral singularity of H^* .

The following proposition shows that, under our assumptions, λ is an outgoing/incoming regular spectral point of H if and only if λ is an outgoing/incoming regular spectral point of H^* .

Proposition 2.4.5. *Let $\lambda \in \sigma_{\text{ess}}(H)$ and suppose that the limits (2.39) exist in the norm topology of $\mathcal{B}(\mathcal{H})$. Then the following conditions are equivalent:*

(i) λ is a regular outgoing/incoming spectral point of H ,

(ii) λ is not an accumulation point of eigenvalues located in $\lambda \pm i(0, \infty)$ and

$$\lim_{\varepsilon \rightarrow 0^+} WCR_H(\lambda \pm i\varepsilon)C$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$,

(iii) λ is a regular outgoing/incoming spectral point of H^* .

Proof. Taking adjoints, it is clear that (ii) \Leftrightarrow (iii). We prove that (i) \Rightarrow (ii). Suppose for instance that λ is an outgoing regular spectral point of H . By Proposition 2.4.1, $\text{Id} + CR_0(\lambda + i0^+)CW$ is invertible in $\mathcal{B}(\mathcal{H})$. We claim that $\text{Id} + WCR_0(\lambda + i0^+)C$ is invertible in $\mathcal{B}(\mathcal{H})$. Indeed, for all $\varepsilon > 0$, a direct computation gives

$$\text{Id} = (\text{Id} - (W(\text{Id} + CR_0(\lambda + i\varepsilon)CW)^{-1}CR_0(\lambda + i\varepsilon)C)) (\text{Id} + WCR_0(\lambda + i\varepsilon)C).$$

Letting $\varepsilon \rightarrow 0^+$, we obtain that

$$\text{Id} = (\text{Id} - (W(\text{Id} + CR_0(\lambda + i0^+)CW)^{-1}CR_0(\lambda + i0^+)C)) (\text{Id} + WCR_0(\lambda + i0^+)C).$$

Thus $\text{Id} + WCR_0(\lambda + i0^+)C$ is injective. Since $WCR_0(\lambda + i0^+)C$ is compact, Fredholm's alternative implies that $\text{Id} + WCR_0(\lambda + i0^+)C$ is bijective in $\mathcal{B}(\mathcal{H})$. Now, writing

$$WCR_H(\lambda + i\varepsilon)C = (\text{Id} + WCR_0(\lambda + i\varepsilon)C)^{-1}WCR_0(\lambda + i\varepsilon)C,$$

for $\varepsilon > 0$ small enough and next letting $\varepsilon \rightarrow 0^+$, we deduce that $\lim_{\varepsilon \rightarrow 0^+} WCR_H(\lambda + i\varepsilon)C$ exists in $\mathcal{B}(\mathcal{H})$.

The proof of (i) \Rightarrow (ii) in the case of an incoming regular spectral point as well as the proof of (ii) \Rightarrow (i) are analogous. \square

2.4.4 Embedded eigenvalues

In this section, we prove that given our definition of spectral singularities (see Definition 2.2.2), an eigenvalue of H embedded in the essential spectrum is both an incoming and an outgoing spectral singularity.

Proposition 2.4.6. *Let $\lambda \in \sigma_{\text{ess}}(H)$ and suppose that the limits (2.39) exist in the norm topology of $\mathcal{B}(\mathcal{H})$. If λ is an eigenvalue of H , then λ is both an outgoing and an incoming spectral singularity of H .*

Proof. Let λ be an eigenvalue of H . There exists $u \in \mathcal{D}(H)$, $u \neq 0$, such that $(H - \lambda)u = 0$. Suppose by contradiction that λ is an outgoing regular spectral point of H . Since λ is not an accumulation point of eigenvalues located in $\lambda + i(0, \infty)$, we can write, for $\varepsilon > 0$ small enough,

$$0 = C\mathcal{R}_0(\lambda + i\varepsilon)(H - \lambda)u = (\text{Id} + C\mathcal{R}_0(\lambda + i\varepsilon)CW)Cu + i\varepsilon C\mathcal{R}_0(\lambda + i\varepsilon)u. \quad (2.45)$$

Lemma 2.4.2 yields

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon C\mathcal{R}_0(\lambda + i\varepsilon)u = 0.$$

Inserting this into (2.45), we obtain that

$$(\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW)Cu = 0,$$

which is impossible since $\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW$ is injective by Lemma 2.4.1.

Similarly, λ cannot be an incoming regular spectral point of H . This concludes the proof of the proposition. \square

It should also be noted that, by Theorem 2.3.1, the eigenvectors associated to an embedded eigenvalue λ belong to $\mathcal{H}_C^{\pm}(\lambda)$.

2.4.5 Local spectral regularity

In this section, we show that the notion of spectral regularity introduced in Definition 2.2.2 is a local property. We will need to distinguish the case of a spectral singularity embedded in the interior of the essential spectrum of H from the case of an ‘endpoint spectral singularity’. It should be noted that endpoints of the essential spectrum are sometimes called ‘thresholds’, but the notion of a spectral threshold also has different meanings in some contexts (see e.g. [51]). We mention that spectral singularities embedded in the essential spectrum, either in the interior or at endpoints, have been recently studied, in a Banach spaces setting, in [8].

Here we will say that a point $\lambda \in \sigma_{\text{ess}}(H)$ is a *spectral endpoint* of H if there exists $r > 0$ such that either

$$\sigma_{\text{ess}}(H) \cap \mathring{D}(\lambda, r) \subset [\lambda, \infty), \quad (2.46)$$

or

$$\sigma_{\text{ess}}(H) \cap \mathring{D}(\lambda, r) \subset (-\infty, \lambda]. \quad (2.47)$$

We say that λ is a *left spectral endpoint* if (2.46) holds and a *right spectral endpoint* if (2.47) holds.

If λ belongs to $\sigma_{\text{ess}}(H)$, we will in this section make the assumption that there exists $r > 0$ such that the maps

$$\mathring{D}(\lambda, r) \cap \mathbb{C}^{\pm} \ni z \mapsto C\mathcal{R}_0(z)C \quad (2.48)$$

extend by continuity to $\mathring{D}(\lambda, r) \cap \bar{\mathbb{C}}^{\pm}$.

The next proposition shows that the notion of spectral regularity is a local property. Note that Item (ii) of Proposition 2.4.7 corresponds to the definition of a regular spectral point in [34, 35], in the particular case where H is dissipative. The proof of the next result being a quite straightforward extension of the proof of [34, Lemma 4.1], it is deferred to Appendix 2.6.2.

Proposition 2.4.7. *Let $\lambda \in \sigma_{\text{ess}}(H)$. Suppose that there exists $r > 0$ such that the maps (2.48) extend by continuity to $\mathring{D}(\lambda, r) \cap \bar{\mathbb{C}}^{\pm}$. The following conditions are equivalent:*

- (i) λ is an outgoing/incoming regular spectral point,
- (ii) There exists a compact interval $K_{\lambda} \subset \mathbb{R}$ whose interior contains λ , such that K_{λ} does not have any accumulation point of eigenvalues of H located in \mathbb{C}^{\pm} , and such that the limit

$$C\mathcal{R}_H(\mu \pm i0^+)CW := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\mu \pm i\varepsilon)CW$$

exists uniformly in $\mu \in K_{\lambda}$ in the norm topology of $\mathcal{B}(\mathcal{H})$.

Proof. See Appendix 2.6.2. □

Proposition 2.4.7 has the following consequence.

Corollary 2.4.8. *Suppose that for all $\lambda \in \sigma_{\text{ess}}(H)$, there exists $r > 0$ such that the maps (2.48) extend by continuity to $\mathring{D}(\lambda, r) \cap \bar{\mathbb{C}}^{\pm}$. Then the set of spectral singularities of H is a closed set whose Lebesgue measure vanishes.*

Proof. Let $E := E^+ \cup E^-$, where

$$E^{\pm} := \{\lambda \in \sigma_{\text{ess}}(H_0), \lambda \text{ is an outgoing/incoming spectral singularity of } H\}.$$

It follows from Proposition 2.4.7 that E^+ and E^- are closed. Hence E is closed. Moreover, by the assumption that $z \mapsto C\mathcal{R}_0(z)CW$ extends by continuity to the real axis, we can apply [94, Theorem 1.8.3], which implies that $\text{Id} + C\mathcal{R}_0(\lambda \pm i0^+)CW$ is invertible in $\mathcal{B}(\mathcal{H})$ for a.e. $\lambda \in \sigma_{\text{ess}}(H)$. By Proposition 2.4.1, this proves that the Lebesgue measures of E^+ and E^- vanish. □

Our next concern is to characterize outgoing/incoming regular spectral points $\lambda \in \sigma_{\text{ess}}(H)$ as nontangential limits of the weighted resolvent $C\mathcal{R}_H(z)CW$, as $z \rightarrow \lambda$. If λ is a left spectral endpoint, we will assume that there exists $r > 0$ such that the map

$$\{\lambda + \nu, |\nu| < r, 0 < \arg(\nu) < 2\pi\} = \mathring{D}(\lambda, r) \setminus [\lambda, \infty) \ni z \mapsto C\mathcal{R}_0(z)C \quad (2.49)$$

extends by continuity to $\{\lambda + \nu, |\nu| < r, 0 \leq \arg(\nu) \leq 2\pi\}$. If λ is a right spectral endpoint, we will assume that there exists $r > 0$ such that the map

$$\{\lambda + \nu, |\nu| < r, -\pi < \arg(\nu) < \pi\} = \mathring{D}(\lambda, r) \setminus (-\infty, \lambda] \ni z \mapsto C\mathcal{R}_0(z)C \quad (2.50)$$

extends by continuity to $\{\lambda + \nu, |\nu| < r, -\pi \leq \arg(\nu) \leq \pi\}$. The next proposition proves, in particular, that under these assumptions, outgoing and incoming spectral singularities at endpoints coincide.

Proposition 2.4.9.

1. Let λ be in the interior of $\sigma_{\text{ess}}(H)$. Suppose that there exists $r > 0$ such that the maps (2.48) extend by continuity to $\mathring{D}(\lambda, r) \cap \mathbb{C}^\pm$. The following conditions are equivalent:

- (i) λ is an outgoing/incoming regular spectral point,
- (ii) There exist a complex neighborhood \mathcal{O}_λ of λ such that $\mathcal{O}_\lambda^\pm := \mathcal{O}_\lambda \cap \mathbb{C}^\pm \subset \rho(H)$ and a continuous map $\gamma : (0, 1] \rightarrow \mathcal{O}_\lambda^\pm$ such that

$$\lim_{\varepsilon \rightarrow 0^+} \gamma(\varepsilon) = \lambda \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\gamma(\varepsilon))CW$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$.

2. Let $\lambda \in \sigma_{\text{ess}}(H)$ be a spectral endpoint of H such that (2.46) holds. Suppose that there exists $r > 0$ such that the map (2.49) extends by continuity to $\{\lambda + \nu, |\nu| < r, 0 \leq \arg(\nu) \leq 2\pi\}$. The following conditions are equivalent:

- (i) λ is an outgoing regular spectral point of H ,
- (ii) λ is an incoming regular spectral point of H ,
- (iii) There exist a complex neighborhood \mathcal{O}_λ of λ such that $\mathcal{O}_\lambda \setminus [\lambda, \infty) \subset \rho(H)$ and a continuous map $\gamma : (0, 1] \rightarrow \mathcal{O}_\lambda \setminus [\lambda, \infty)$ such that

$$\lim_{\varepsilon \rightarrow 0^+} \gamma(\varepsilon) = \lambda \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\gamma(\varepsilon))CW \quad (2.51)$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$.

The same holds if (2.47) holds instead of (2.46), assuming that there exists $r > 0$ such that the map (2.50) extends by continuity to $\{\lambda + \nu, |\nu| < r, -\pi \leq \arg(\nu) \leq \pi\}$ and replacing $\mathcal{O}_\lambda \setminus [\lambda, \infty)$ by $\mathcal{O}_\lambda \setminus (-\infty, \lambda]$ in (iii).

Proof. Consider the most difficult case (2). We prove that (2)(i) \Rightarrow (2)(iii). Let $\lambda \in \sigma_{\text{ess}}(H)$ be an outgoing regular spectral point of H and suppose that (2.46) holds. By Proposition 2.4.1, we know that $\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW$ is invertible in $\mathcal{B}(\mathcal{H})$. Since, by assumption, the map (2.49) extends by continuity to $\{\lambda + \nu, |\nu| < r, 0 \leq \arg(\nu) \leq 2\pi\}$ for some $r > 0$, and since the set of invertible operators in $\mathcal{B}(\mathcal{H})$ is open, this implies that there exists a complex neighborhood $\mathcal{O}_\lambda \subset \mathring{D}(\lambda, r)$ of λ , such that the map

$$\mathcal{O}_\lambda \setminus [\lambda, \infty) \ni z \mapsto (\text{Id} + C\mathcal{R}_0(z)CW)^{-1} \quad (2.52)$$

is analytic. The usual Birman-Schwinger principle yields $\mathcal{O}_\lambda \setminus [\lambda, \infty) \subset \rho(H)$. It then suffices to take $\gamma(\varepsilon) = \lambda + i\delta\varepsilon$, with $\delta > 0$ small enough.

Next we prove that (2)(iii) \Rightarrow (2)(i). In the same way as in Proposition 2.4.1, the existence of the limit (2.51) is equivalent to the invertibility of $\text{Id} + C\mathcal{R}_0(\gamma(0^+))CW$ in $\mathcal{B}(\mathcal{H})$. Note that the limit $C\mathcal{R}_0(\gamma(0^+))C$ exists in the norm topology of $\mathcal{B}(\mathcal{H})$ since we have assumed that there exists $r > 0$ such that the map $z \mapsto C\mathcal{R}_0(z)C$ extends by continuity to $\{\lambda + \nu, |\nu| < r, 0 \leq \arg(\nu) \leq 2\pi\}$. We can then argue as before; this gives the existence of a complex neighborhood \mathcal{O}_λ of λ such that the map (2.52) is analytic and extends by continuity to $\{\lambda + \nu, |\nu| < r, 0 \leq \arg(\nu) \leq 2\pi\}$. In particular, $\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW$ is well-defined and invertible in $\mathcal{B}(\mathcal{H})$. Applying Proposition 2.4.1, this shows that λ is an incoming regular spectral point of H .

The proof of (2)(ii) \Leftrightarrow (2)(iii) is identical. One proceeds analogously to prove (2) in the case where (2.47) holds instead of (2.46).

Finally, the argument easily adapts to prove (1). \square

Note that in the case of an outgoing/incoming regular spectral point $\lambda \in \sigma_{\text{ess}}(H)$, our proof shows that $z \mapsto C\mathcal{R}_H(z)CW$ has a nontangential limit at λ , in the sense that the limit

$$\lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\gamma(\varepsilon))CW \quad (2.53)$$

does not depend on the continuous curve $\gamma : (0, 1] \rightarrow \mathbb{C}^\pm$ such that $\gamma(\varepsilon) \rightarrow \lambda$ as $\varepsilon \rightarrow 0^+$. See Figure 2.2. This property can also be deduced from Lindelöf's Theorem (see e.g. [80]) if the map $z \mapsto C\mathcal{R}_H(z)CW$ is known to be analytic and bounded in the domain \mathcal{O}_λ^\pm .

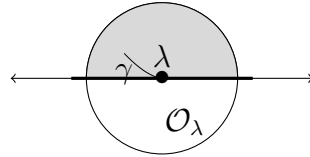


Figure 2.2: **Outgoing spectral singularity located inside the essential spectrum.** The figure shows an example of a curve $\gamma : (0, 1] \rightarrow \mathcal{O}_\lambda^+$, where $\mathcal{O}_\lambda^+ = \mathcal{O}_\lambda \cap \mathbb{C}^+$ and \mathcal{O}_λ is a complex neighborhood of λ . The thick line represents the essential spectrum of H .

Likewise, in the case of a regular spectral point λ located at a spectral endpoint of H , the limit (2.53) does not depend on the continuous curve $\gamma : (0, 1] \rightarrow \mathbb{C} \setminus [\lambda, \infty)$ (or

$\gamma : (0, 1] \rightarrow \mathbb{C} \setminus (-\infty, \lambda]$, depending on whether (2.46) or (2.47) holds). This is the reason why the incoming and outgoing spectral singularities coincide. See Figure 2.3.

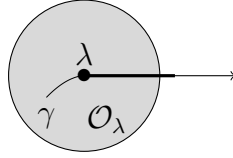


Figure 2.3: **Spectral singularity located at an endpoint.** The figure shows an example of a curve $\gamma : (0, 1] \rightarrow \mathcal{O}_\lambda \setminus [\lambda, \infty)$, where \mathcal{O}_λ is a complex neighborhood of λ . The thick line represents the essential spectrum of H .

We conclude this section with the following consequence of the previous propositions.

Corollary 2.4.10.

1. Let λ be in the interior of $\sigma_{\text{ess}}(H)$. Suppose that there exists $r > 0$ such that the maps (2.48) extend by continuity to $\mathring{D}(\lambda, r) \cap \bar{\mathbb{C}}^\pm$. If λ is an outgoing/incoming regular spectral point of H , then there exists a complex neighborhood \mathcal{O}_λ of λ such that H has no eigenvalue in $\bar{\mathcal{O}}_\lambda^\pm := \mathcal{O}_\lambda \cap \bar{\mathbb{C}}^\pm$,

$$\sigma_{\text{p}}(H) \cap \bar{\mathcal{O}}_\lambda^\pm = \emptyset.$$

2. Let $\lambda \in \sigma_{\text{ess}}(H)$ be a spectral endpoint of H such that (2.46) holds. Suppose that there exists $r > 0$ such that the map (2.49) extends by continuity to $\{\lambda + \nu, |\nu| < r, 0 \leq \arg(\nu) \leq 2\pi\}$. If λ is a regular spectral point of H , then there exists a complex neighborhood \mathcal{O}_λ of λ such that H has no eigenvalue in \mathcal{O}_λ ,

$$\sigma_{\text{p}}(H) \cap \mathcal{O}_\lambda = \emptyset.$$

The same holds if (2.47) holds instead of (2.46), assuming that there exists $r > 0$ such that the map (2.50) extends by continuity to $\{\lambda + \nu, |\nu| < r, -\pi \leq \arg(\nu) \leq \pi\}$.

Proof. It suffices to combine Propositions 2.4.7 and 2.4.9. □

2.4.6 Spectral singularities for dissipative operators

In this section, we focus on the particular case where the operator H is dissipative. Recall that $H = H_0 + V$ with $V \in \mathcal{B}(\mathcal{H})$. We write

$$V = V_1 - iV_2, \quad V_1 = CW_1C, \quad V_2 = CW_2C,$$

where $W_1 := \text{Re}(W)$ and $W_2 := -\text{Im}(W)$. Here the real and imaginary parts of a bounded operator $A \in \mathcal{B}(H)$ are defined as usual by $\text{Re}(A) := \frac{1}{2}(A + A^*)$, $\text{Im}(A) := \frac{1}{2i}(A - A^*)$. We suppose that

$$W_2 \geq 0,$$

so that

$$H = H_0 + V_1 - iV_2 =: H_{V_1} - iV_2$$

is indeed dissipative. Our purpose is to prove that if λ is an outgoing regular spectral point of the self-adjoint part $H_{V_1} = H_0 + V_1$, then λ is also an outgoing regular spectral point for H . In other words, adding the ‘dissipative part’ $-iV_2$ cannot create outgoing spectral singularities.

We begin with recalling the following easy lemma (see [16, Lemma 6.1] or [35, Lemma 3.1]). We include a proof for the convenience of the reader.

Lemma 2.4.11. *Let $\lambda \in \mathbb{R}$ be a real eigenvalue of H . Then λ is an eigenvalue of H_{V_1} and*

$$\text{Ker}(H - \lambda) \subset \mathcal{H}_p(H_{V_1}) \cap \text{Ker}(V_2). \quad (2.54)$$

Proof. Let $u \in \text{Ker}(H - \lambda)$, $u \neq 0$. Then

$$\lambda \|u\|_{\mathcal{H}}^2 = \langle u, Hu \rangle_{\mathcal{H}} = \langle u, H_{V_1}u \rangle_{\mathcal{H}} - i \|W_2^{\frac{1}{2}}Cu\|_{\mathcal{H}}^2.$$

Since $\lambda \in \mathbb{R}$, identifying the real and imaginary parts, we obtain that $u \in \text{Ker}(W_2^{\frac{1}{2}}C) \subset \text{Ker}(V_2)$ and therefore $Hu = H_{V_1}u = \lambda u$. This establishes (2.54). \square

Note that Definition 2.2.2 of a regular spectral point of H applies to H_{V_1} as well. In other words, $\lambda \in \sigma_{\text{ess}}(H_{V_1})$ is an outgoing/incoming regular spectral point of H_{V_1} if

$$C\mathcal{R}_{V_1}(\lambda \pm i0^+)CW_1 := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_{V_1}(\lambda \pm i\varepsilon)CW_1$$

exists in $\mathcal{B}(\mathcal{H})$. Otherwise, λ is an outgoing/incoming spectral singularity of H_{V_1} . Moreover, Proposition 2.4.1 applied to H_{V_1} shows that λ is an outgoing/incoming regular spectral point of H_{V_1} if and only if $\text{Id} + C\mathcal{R}_0(\lambda \pm i0^+)CV_1$ is invertible in $\mathcal{B}(\mathcal{H})$, which is also equivalent to $\text{Id} + \mathcal{R}_0(\lambda \pm i0^+)V_1'$ being invertible in $\mathcal{B}(\mathcal{H}'_C)$.

The next proposition is the main result of this subsection.

Proposition 2.4.12. *Suppose that Hypothesis 2.5 holds. Let $\lambda \in \sigma_{\text{ess}}(H)$ and suppose that the limits (2.39) exist in the norm topology of $\mathcal{B}(\mathcal{H})$. If λ is an outgoing regular spectral point of H_{V_1} then λ is an outgoing regular spectral point of H .*

Proof. Suppose that λ is an outgoing regular spectral point of H_{V_1} . Suppose also, by contradiction, that λ is an outgoing spectral singularity of H . By Theorem 2.3.1, there exists $\Psi \in \mathcal{H}_C^+$ such that

$$\Psi = -\mathcal{R}_0(\lambda + i0^+)V'\Psi \quad \text{and} \quad (H' - \lambda)\Psi = 0. \quad (2.55)$$

Recall from Section 2.2.3 that $V' \in \mathcal{B}(\mathcal{H}'_C, \mathcal{H}_C)$ is given by $V' = CWC'$. Since $W = W_1 - iW_2$, this yields

$$\langle V'\Psi, \Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C} = \langle W_1C'\Psi, C'\Psi \rangle_{\mathcal{H}} + i \langle W_2C'\Psi, C'\Psi \rangle_{\mathcal{H}},$$

and hence, since $W_2 \geq 0$,

$$\operatorname{Im} (\langle V'\Psi, \Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C}) = \langle W_2 C'\Psi, C'\Psi \rangle_{\mathcal{H}} \geq 0. \quad (2.56)$$

Now, using (2.55), we have

$$\begin{aligned} \operatorname{Im} (\langle V'\Psi, \Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C}) &= -\operatorname{Im} \langle V'\Psi, \mathcal{R}_0(\lambda + i0^+)V'\Psi \rangle_{\mathcal{H}_C, \mathcal{H}'_C} \\ &= \frac{-1}{2i} (\langle WC'\Psi, C\mathcal{R}_0(\lambda + i0^+)V'\Psi \rangle_{\mathcal{H}} - \langle C\mathcal{R}_0(\lambda + i0^+)V'\Psi, WC'\Psi \rangle_{\mathcal{H}}) \\ &= \frac{-1}{2i} (\langle WC'\Psi, C\mathcal{R}_0(\lambda + i0^+)V'\Psi \rangle_{\mathcal{H}} - \langle WC'\Psi, C\mathcal{R}_0(\lambda - i0^+)V'\Psi \rangle_{\mathcal{H}}) \\ &= \lim_{\varepsilon \rightarrow 0^+} -\varepsilon \langle WC'\Psi, C\mathcal{R}_0(\lambda + i\varepsilon)\mathcal{R}_0(\lambda - i\varepsilon)V'\Psi \rangle_{\mathcal{H}} \\ &= \lim_{\varepsilon \rightarrow 0^+} -\varepsilon \langle \mathcal{R}_0(\lambda - i\varepsilon)V'\Psi, \mathcal{R}_0(\lambda - i\varepsilon)V'\Psi \rangle_{\mathcal{H}} \leq 0. \end{aligned} \quad (2.57)$$

Equations (2.56) and (2.57) imply that $W_2 C'\Psi = 0$. Inserting this into (2.55), we obtain that

$$\Psi = -\mathcal{R}_0(\lambda + i0^+)V'_1\Psi \quad \text{and} \quad (H'_0 + V'_1 - \lambda)\Psi = 0.$$

which is a contradiction since λ is a regular spectral point of H_{V_1} . \square

Remark 2.4.13. *The previous proof actually shows that, under the conditions of Proposition 2.4.12,*

$$\lambda \text{ is an outgoing spectral singularity of } H \Rightarrow W_2 C'\Psi = 0. \quad (2.58)$$

Since C' is injective, this in turn yields

$$W_2 \text{ is injective} \Rightarrow H \text{ has no outgoing spectral singularities.}$$

In the context of Schrödinger operators, one can also combine (2.58) with the unique continuation principle (see e.g. [76, Theorem XIII.63]) to obtain Proposition 2.3.7.

Proof of Proposition 2.3.7. Let $\lambda > 0$. Assuming that (2.30) holds with $\sigma > 1$, suppose by contradiction that λ is an outgoing spectral singularity of H . Then, by Proposition 2.3.5, there exists $\Psi \in L^2_{-\sigma/2}$, $\Psi \neq 0$, such that

$$(-\Delta + V(x) - \lambda)\Psi = 0.$$

This implies that $\Psi \in H^2_{\text{loc}}(\mathbb{R}^3)$ and that $|\Delta\Psi(x)| \leq (\|V\|_{L^\infty} + \lambda)|\Psi(x)|$ for a.e. $x \in \mathbb{R}^3$. Now, since there exists a non-trivial open set U such that $\operatorname{Im}(V) < 0$ on U , (2.58) and the fact that $C(x) = \langle x \rangle^{-\sigma/2}$ imply that $\Psi(x) = 0$ on U . By [76, Theorem XIII.63], we conclude that $\Psi = 0$. This is a contradiction.

The same argument holds for $\lambda = 0$ under the condition that (2.30) holds with $\sigma > 2$. \square

2.5 Spectral resolution formula and spectral subspaces

In this section, we begin with constructing a functional calculus for H . In Section 2.5.1, we consider the simplest case of intervals without spectral singularities, next, in Section 2.5.2, we construct a regularized functional calculus in intervals possibly containing spectral singularities. The latter is subsequently used in Section 2.5.3 to establish the spectral resolution formula stated in Proposition 2.3.10. Finally we prove Theorems 2.3.2 and 2.3.3 in Sections 2.5.4 and 2.5.5, respectively.

2.5.1 Functional calculus in intervals not containing spectral singularities

Consider first a closed interval $I \subset \mathbb{R}$ that does not contain any spectral singularity of H . We will furthermore assume that a limiting absorption principle holds for H in I , in the sense that there exists $\varepsilon_0 > 0$ such that

$$\sup_{\operatorname{Re}(z) \in I, \pm \operatorname{Im}(z) \in (0, \varepsilon_0)} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (2.59)$$

Note that, in the same way as for H_0 , Fatou's Theorem and (2.59) yield that the limits $C\mathcal{R}_H(\lambda \pm i0^+)CW$ exist for almost every $\lambda \in I$, in the norm topology of $\mathcal{B}(\mathcal{H})$, and that the map $I \ni \lambda \mapsto C\mathcal{R}_H(\lambda \pm i0^+)CW \in \mathcal{B}(\mathcal{H})$ is bounded. The main purpose of this subsection is then to define a spectral projection for H in I by mimicking Stone's formula, setting

$$\mathbf{1}_I(H) := \operatorname{w}\text{-}\lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (2.60)$$

The next proposition shows that, under our assumptions, $\mathbf{1}_I(H)$ is a well-defined non-orthogonal projection. The proof is similar to that given in [35] for dissipative operators. For the convenience of the reader, a sketch of the proof of Proposition 2.5.1 focusing on the differences with [35] is reported in Appendix 2.6.3.

Proposition 2.5.1. *Suppose that Hypothesis 2.1 holds. Let $I \subset \mathbb{R}$ be a closed interval and suppose that there exists $\varepsilon_0 > 0$ such that (2.59) holds. Then the weak limit (2.60) exists in $\mathcal{B}(\mathcal{H})$ and we have*

$$\mathbf{1}_{I_1}(H)\mathbf{1}_{I_2}(H) = \mathbf{1}_{I_1 \cap I_2}(H), \quad (2.61)$$

for any closed intervals $I_1, I_2 \subset I$ without spectral singularity, with the convention that $\mathbf{1}_\emptyset(H) = 0$. In particular, $\mathbf{1}_I(H)$ is a projection. Its adjoint is given by

$$\mathbf{1}_I(H)^* = \mathbf{1}_I(H^*) = \operatorname{w}\text{-}\lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (\mathcal{R}_{H^*}(\lambda + i\varepsilon) - \mathcal{R}_{H^*}(\lambda - i\varepsilon)) d\lambda. \quad (2.62)$$

Proof. See Appendix 2.6.3. □

We note the following representation formula which follows from our proof:

$$\begin{aligned} \mathbf{1}_I(H) &= \mathbf{1}_I(H_0) - \frac{1}{2i\pi} \int_I \mathcal{R}_0(\lambda \mp i0^+) CW C \mathcal{R}_0(\lambda \pm i0^+) d\lambda \\ &\quad + \frac{1}{2i\pi} \int_I \mathcal{R}_0(\lambda \pm i0^+) CW C \mathcal{R}_H(\lambda \pm i0^+) CW C \mathcal{R}_0(\lambda \pm i0^+) d\lambda, \end{aligned} \quad (2.63)$$

in the sense of quadratic forms on $\mathcal{H} \times \mathcal{H}$. We recall that, for all $u \in \mathcal{H}$, $\lambda \mapsto C \mathcal{R}_0(\lambda \pm i0^+)u$ denotes the limit of $\lambda \mapsto C \mathcal{R}_0(\lambda \pm i\varepsilon)u$ in $L^2(\mathbb{R}; \mathcal{H})$ as $\varepsilon \rightarrow 0^+$, while, for a.e. λ , $C \mathcal{R}_H(\lambda \pm i0^+) CW$ is the limit of $C \mathcal{R}_H(\lambda \pm i\varepsilon) CW$ as $\varepsilon \rightarrow 0^+$, in the norm topology of $\mathcal{B}(\mathcal{H})$.

Under the same assumptions and using similar arguments, we also have the following functional calculus. We denote by $C_b(I)$ the set of bounded continuous functions on I .

Proposition 2.5.2. *Under the conditions of Proposition 2.5.1, the map*

$$C_b(I) \ni f \mapsto f(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I f(\lambda) (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda \in \mathcal{B}(\mathcal{H}) \quad (2.64)$$

is a Banach algebra morphism. Moreover, for all $t \in \mathbb{R}$,

$$e^{itH} \mathbf{1}_I(H) = \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I e^{it\lambda} (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda \quad (2.65)$$

and for all $z_0 \in \rho(H)$,

$$\mathcal{R}_H(z_0) \mathbf{1}_I(H) = \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (\lambda - z_0)^{-1} (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (2.66)$$

Proof. See Appendix 2.6.3. □

We mention that this functional calculus uniquely extends to a Borel functional calculus, i.e. an algebra morphism $L^\infty(I) \ni f \rightarrow f(H)$. See e.g. [44, Theorem 2.4].

2.5.2 ‘Regularized’ functional calculus

Our next concern is to regularize the definition (2.60) in the case where the spectral interval I contains spectral singularities. More generally, we will now consider a closed interval $I \subset \mathbb{R}$ and a bounded holomorphic function

$$h : U \rightarrow \mathbb{C}, \quad \{z \in \mathbb{C}, \text{Re}(z) \in I, |\text{Im}(z)| \leq \varepsilon_0\} \subset U,$$

with $\varepsilon_0 > 0$ and U open, such that

$$\sup_{\text{Re}(z) \in I, \pm \text{Im}(z) \in (0, \varepsilon_0)} \|h(z)\| \|C \mathcal{R}_H(z) CW\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (2.67)$$

We will also assume that

$$\lambda \mapsto \sup_{0 < \varepsilon < \varepsilon_0} |h'(\lambda \pm i\varepsilon)| \in L^2(I), \quad (2.68)$$

where h' stands for the derivative of h . The ‘regularized spectral projection’ for H in I is then defined by

$$(h\mathbf{1}_I)(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (h(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))d\lambda. \quad (2.69)$$

In this context, Proposition 2.5.1 should be modified as follows.

Proposition 2.5.3. *Suppose that Hypothesis 2.1 holds. Let $I \subset \mathbb{R}$ be a closed interval. Suppose that there exist $\varepsilon_0 > 0$ and a bounded holomorphic function h defined on a complex neighborhood of $\{z \in \mathbb{C}, \text{Re}(z) \in I, |\text{Im}(z)| \leq \varepsilon_0\}$ such that (2.67) and (2.68) hold. Then the weak limit (2.69) exists in $\mathcal{B}(\mathcal{H})$ and its adjoint is given by*

$$(h\mathbf{1}_I)(H)^* = (\bar{h}\mathbf{1}_I)(H^*) = \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (\overline{h(\lambda + i\varepsilon)}\mathcal{R}_{H^*}(\lambda + i\varepsilon) - \overline{h(\lambda - i\varepsilon)}\mathcal{R}_{H^*}(\lambda - i\varepsilon))d\lambda.$$

Proof. See Appendix 2.6.3. □

It is not difficult to verify that, if H is self-adjoint, then $(h\mathbf{1}_I)(H) = h(H)\mathbf{1}_I(H)$ (see the proof of Proposition 2.5.3). However, under the assumptions of Proposition 2.5.3, this formula does *not* make sense since $\mathbf{1}_I(H)$ is ill-defined in general.

Similarly as in (2.63), our proof will show that

$$\begin{aligned} (h\mathbf{1}_I)(H) &= (h\mathbf{1}_I)(H_0) - \frac{1}{2i\pi} \int_I h(\lambda)\mathcal{R}_0(\lambda \mp i0^+)CWCR_0(\lambda \pm i0^+)d\lambda \\ &\quad + \frac{1}{2i\pi} \int_I h(\lambda)\mathcal{R}_0(\lambda \pm i0^+)CWCR_H(\lambda \pm i0^+)CWCR_0(\lambda \pm i0^+)d\lambda, \end{aligned} \quad (2.70)$$

in the sense of quadratic forms on $\mathcal{H} \times \mathcal{H}$.

One can also define a ‘regularized functional calculus’ on the set of functions

$$\mathcal{C}_{\text{b,reg}}(I) := \{f : I \rightarrow \mathbb{C}, \exists g \in \mathcal{C}_b(I), f = hg\}.$$

Proposition 2.5.4. *Under the conditions of Proposition 2.5.3, the map*

$$\mathcal{C}_{\text{b,reg}}(I) \ni f \mapsto f(H) \in \mathcal{B}(\mathcal{H}),$$

where

$$f(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I g(\lambda)(h(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))d\lambda$$

is an algebra morphism and there exists $c > 0$ such that

$$\|f(H)\|_{\mathcal{B}(\mathcal{H})} \leq c\|g\|_{L^\infty}, \quad (2.71)$$

for all $f \in C_{b,\text{reg}}(I)$, with $f = hg$. Moreover, for all $t \in \mathbb{R}$,

$$e^{itH}(h\mathbf{1}_I)(H) = \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I e^{it\lambda} (h(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)) d\lambda \quad (2.72)$$

and for all $z_0 \in \rho(H)$,

$$\mathcal{R}_H(z_0)(h\mathbf{1}_I)(H) = \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (\lambda - z_0)^{-1} (h(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (2.73)$$

Proof. See Appendix 2.6.3. □

Again, this functional calculus uniquely extends to a Borel functional calculus. For other definitions of functional calculi for general operators on Banach spaces under an assumption of polynomial growth of the resolvent near the real axis, we refer to [18, 44].

2.5.3 Spectral resolution formula

We now turn to the proof of the resolution formula stated in Proposition 2.3.10. It relies in particular on the following resolvent bounds.

Lemma 2.5.5. *Suppose that Hypothesis 2.1 holds. Let $I \subset \mathbb{R}$ be a closed interval and suppose that there exists $\varepsilon_0 > 0$ such that (2.59) holds.*

(i) *There exists $c > 0$ such that, for a.e. $\lambda \in I$, for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\|\mathcal{R}_H(\lambda \pm i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \leq c\varepsilon^{-1}.$$

(ii) *There exists $c > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, for all $u \in \mathcal{H}$,*

$$\int_I \|\mathcal{R}_H(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda \leq c\varepsilon^{-1} \|u\|_{\mathcal{H}}^2.$$

Proof. See Appendix 2.6.3. □

To prove Proposition 2.3.10, we will rely on the following construction. Assume that Hypotheses 2.2 and 2.3 hold. Let $\sigma_{\text{disc,real}}(H) = \{e_1, \dots, e_p\}$ be the set of real, discrete eigenvalues of H , with $e_1 < \dots < e_p$. Note that $\sigma_{\text{disc,real}}(H)$ is finite by Hypothesis 2.2. Let $\delta > 0$ be the distance between $\sigma_{\text{disc,real}}(H)$ and $\sigma_{\text{ess}}(H)$. For $\varepsilon > 0$ small enough, we consider the complex open set U_ε such that $\sigma(H) \subset U_\varepsilon$ and the boundary of U_ε is given by (i) small circles surrounding each discrete eigenvalue of H and no other point of $\sigma(H)$, (ii) rectangles whose opposite sides are given by the complex segments $[e_\ell + \delta/2 + i\varepsilon, e_{\ell+1} - \delta/2 + i\varepsilon]$ and $[e_\ell + \delta/2 - i\varepsilon, e_{\ell+1} - \delta/2 - i\varepsilon]$ and (iii) the curve given by the complex segments $[e_p + \delta/2 \pm i\varepsilon, \varepsilon^{-3} \pm i\varepsilon]$, $[e_p + \delta/2 - i\varepsilon, e_p + \delta/2 + i\varepsilon]$ and the (long) circle arc centered at the origin and joining the complex points $\varepsilon^{-3} + i\varepsilon$ and $\varepsilon^{-3} - i\varepsilon$. The

circles and rectangles defined by (i), (ii) are oriented counterclockwise, while the curve defined by (iii) is oriented clockwise. See Figure 2.4. We denote by $\Gamma_{(i)}$ the union of curves defined by (i) and by $\Gamma_{\varepsilon, \sharp}$ the unions of curves defined by \sharp , where \sharp stands for (ii) or (iii). If $\sigma_{\text{disc,real}}(H)$ is empty, then $\Gamma_{\varepsilon, (ii)}$ is absent and we replace $e_p - \delta > 2$ by -1 in the definition of $\Gamma_{\varepsilon, (iii)}$ (fixing arbitrarily -1 as a real number such that $-1 < \inf \sigma_{\text{ess}}(H)$).

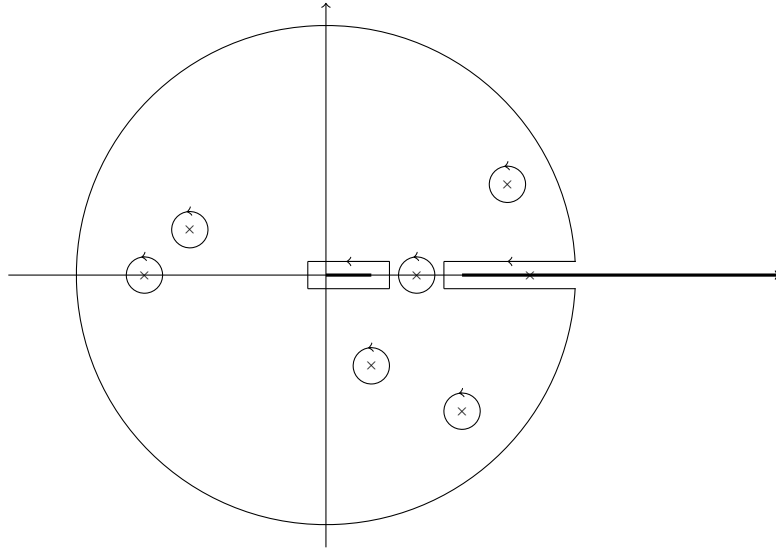


Figure 2.4: **The contour** Γ_ε . The crosses and thick lines represent the eigenvalues and essential spectrum of H , respectively.

Note that the proof of Proposition 2.3.10 is the only place where we use the assumption that H_0 is semi-bounded from below. Without much efforts, we can replace this assumption by the condition that H_0 has a spectral gap.

We are now ready to prove Proposition 2.3.10. Recall that the function r has been defined in (2.36), for some $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$.

Proof of Proposition 2.3.10. In order to be able to apply Proposition 2.5.4 for a suitable function h , it is convenient to add a decaying term to the regularizing function r . Hence we set

$$\tilde{r}(z) := (z - z_0)^{-1} r(z) = (z - z_0)^{-(\nu_1 + \dots + \nu_n + \nu_\infty + 1)} \prod_{j=1}^n (z - \lambda_j)^{\nu_j}, \quad (2.74)$$

so that \tilde{r} satisfies (2.68) in any interval $I \subset \mathbb{R}$. Let $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. We note that $\sigma(H) \cup \{\infty\} \subset U_\varepsilon \cup \{\infty\}$ and that $U_\varepsilon \cup \{\infty\}$ is an open set in $\bar{\mathbb{C}}$. The function \tilde{r} is extended to $\bar{\mathbb{C}}$ by setting $\tilde{r}(\infty) = 0$. By the Riesz-Dunford functional calculus (see e.g. [24, Section VIII.9]), using in particular that \tilde{r} is analytic in a neighborhood of

$\sigma(H) \cup \{\infty\}$ in $\bar{\mathbb{C}}$, we have that

$$\tilde{r}(H) = -\frac{1}{2i\pi} \int_{\Gamma_\varepsilon} \tilde{r}(z) \mathcal{R}_H(z) dz. \quad (2.75)$$

We consider successively the contributions to this integral from $\Gamma_{(i)}$, $\Gamma_{\varepsilon,(ii)}$ and $\Gamma_{\varepsilon,(iii)}$. The contribution from $\Gamma_{(i)}$ gives, by definition, the Riesz projection onto the discrete spectral subspace of H ,

$$-\frac{1}{2i\pi} \int_{\Gamma_{(i)}} \tilde{r}(z) \mathcal{R}_H(z) dz = \tilde{r}(H) \Pi_{\text{disc}}(H). \quad (2.76)$$

The contribution from $\Gamma_{\varepsilon,(ii)}$ gives, for each rectangle, four terms. The integrals over the vertical segments are of order $\mathcal{O}(\varepsilon)$. This easily follows from the fact that $z \mapsto |\tilde{r}(z)| \|\mathcal{R}_H(z)\|_{\mathcal{B}(\mathcal{H})}$ is uniformly bounded on these segments, whose lengths are equal to 2ε . The sum of the integrals over the horizontal segments can be rewritten as

$$\frac{1}{2i\pi} \int_{e_\ell + \delta/2}^{e_{\ell+1} - \delta/2} (\tilde{r}(\lambda + i\varepsilon) \mathcal{R}_H(\lambda + i\varepsilon) - \tilde{r}(\lambda - i\varepsilon) \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda.$$

Applying Proposition 2.5.3, we deduce that

$$\begin{aligned} & \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_{e_\ell + \delta/2}^{e_{\ell+1} - \delta/2} (\tilde{r}(\lambda + i\varepsilon) \mathcal{R}_H(\lambda + i\varepsilon) - \tilde{r}(\lambda - i\varepsilon) \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda \\ &= \int_{e_\ell + \delta/2}^{e_{\ell+1} - \delta/2} \tilde{r}(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) d\lambda. \end{aligned}$$

Therefore, the contribution to (2.75) from $\Gamma_{\varepsilon,(ii)}$ gives

$$\text{w-lim}_{\varepsilon \rightarrow 0^+} -\frac{1}{2i\pi} \int_{\Gamma_{\varepsilon,(ii)}} \tilde{r}(z) \mathcal{R}_H(z) dz = \frac{1}{2i\pi} \sum_{\ell=1}^{p-1} \int_{e_\ell + \delta/2}^{e_{\ell+1} - \delta/2} \tilde{r}(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) d\lambda. \quad (2.77)$$

It remains to consider the contribution from $\Gamma_{\varepsilon,(iii)}$. As before, the integral over the small vertical segment is of order $\mathcal{O}(\varepsilon)$. The sum of the integrals over the horizontal segments can be rewritten as

$$\frac{1}{2i\pi} \int_{e_p + \delta/2}^{\varepsilon^{-3}} (\tilde{r}(\lambda + i\varepsilon) \mathcal{R}_H(\lambda + i\varepsilon) - \tilde{r}(\lambda - i\varepsilon) \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda.$$

First, we note that

$$\int_{\varepsilon^{-3}}^{\infty} (\tilde{r}(\lambda + i\varepsilon) \mathcal{R}_H(\lambda + i\varepsilon) - \tilde{r}(\lambda - i\varepsilon) \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda = \mathcal{O}(\varepsilon).$$

Indeed, we have $|\tilde{r}(\lambda \pm i\varepsilon)| \leq c|\lambda|^{-1}$ for λ large enough and hence, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \int_{\varepsilon^{-3}}^{\infty} |\langle u, \tilde{r}(\lambda \pm i\varepsilon)\mathcal{R}_H(\lambda \pm i\varepsilon)v \rangle_{\mathcal{H}}| d\lambda &\leq c\varepsilon^{\frac{3}{2}}\|u\|_{\mathcal{H}} \left(\int_{\varepsilon^{-3}}^{\infty} \|\mathcal{R}_H(\lambda \pm i\varepsilon)v\|_{\mathcal{H}}^2 d\lambda \right)^{\frac{1}{2}} \\ &\leq c\varepsilon\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}, \end{aligned}$$

the second inequality being a consequence of Lemma 2.5.5(ii). Hence

$$\begin{aligned} &\frac{1}{2i\pi} \int_{e_p+\delta/2}^{\varepsilon^{-3}} \langle u, (\tilde{r}(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - \tilde{r}(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))v \rangle_{\mathcal{H}} d\lambda \\ &= \frac{1}{2i\pi} \int_{e_p+\delta/2}^{\infty} \langle u, (\tilde{r}(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - \tilde{r}(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))v \rangle_{\mathcal{H}} d\lambda + \mathcal{O}(\varepsilon)\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}}. \end{aligned}$$

Combining this with Proposition 2.5.4, we obtain that

$$\begin{aligned} &\text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2i\pi} \int_{e_p+\delta/2}^{\varepsilon^{-3}} (\tilde{r}(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - \tilde{r}(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)) d\lambda \\ &= \frac{1}{2i\pi} \int_{e_p+\delta/2}^{\infty} \tilde{r}(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) d\lambda. \end{aligned}$$

The integral over the circle arc in $\Gamma_{\varepsilon,(\text{iii})}$ can be estimated as follows. Denote by $\mathcal{C}_{\varepsilon}$ this circle arc. For $z \in \mathcal{C}_{\varepsilon}$, $|\text{Im}(z)| \leq \varepsilon_0$, Lemma 2.5.5(i) implies that $\|\mathcal{R}_H(z)\|_{\mathcal{B}(\mathcal{H})} \leq c\varepsilon^{-1}$. Using in addition that $|\tilde{r}(z)| \leq C|z|^{-1} = \mathcal{O}(\varepsilon^3)$ for $z \in \mathcal{C}_{\varepsilon}$, we obtain

$$\left| \int_{\mathcal{C}_{\varepsilon} \cap \{|\text{Im}(z)| \leq \varepsilon_0\}} \tilde{r}(z)\mathcal{R}_H(z) dz \right| = \mathcal{O}(\varepsilon).$$

For $z \in \mathcal{C}_{\varepsilon}$, $|\text{Im}(z)| \geq \varepsilon_0$, it suffices to use that $\|\mathcal{R}_H(z)\|_{\mathcal{B}(\mathcal{H})} \leq c$ together with $|\tilde{r}(z)| \leq C|z|^{-1} = \mathcal{O}(\varepsilon^3)$ to conclude that

$$\left| \int_{\mathcal{C}_{\varepsilon} \cap \{|\text{Im}(z)| \geq \varepsilon_0\}} \tilde{r}(z)\mathcal{R}_H(z) dz \right| = \mathcal{O}(\varepsilon^3).$$

Putting together the previous estimates, we have shown that

$$\text{w-lim}_{\varepsilon \rightarrow 0^+} -\frac{1}{2i\pi} \int_{\Gamma_{\varepsilon,(\text{iii})}} \tilde{r}(z)\mathcal{R}_H(z) dz = \frac{1}{2i\pi} \int_{e_p+\delta/2}^{\infty} \tilde{r}(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) d\lambda. \quad (2.78)$$

Equations (2.75), (2.76), (2.77) and (2.78), together with the fact that $\mathcal{R}_H(\lambda + i0^+) = \mathcal{R}_H(\lambda - i0^+) = \mathcal{R}_H(\lambda)$ if $\lambda \in \rho(H)$, yield

$$\tilde{r}(H) = \tilde{r}(H)\Pi_{\text{disc}}(H) + \frac{1}{2i\pi} \int_{\sigma_{\text{ess}}(H)} \tilde{r}(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) d\lambda. \quad (2.79)$$

It remains to show that one can replace \tilde{r} by r in the previous equation. For the first two terms, we have that $\tilde{r}(H) = \mathcal{R}_H(z_0)r(H)$ by definition, while (2.73) in Proposition 2.5.4 implies

$$\begin{aligned} & \int_{\sigma_{\text{ess}}(H)} \tilde{r}(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) d\lambda \\ &= \mathcal{R}_H(z_0) \int_{\sigma_{\text{ess}}(H)} r(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) d\lambda. \end{aligned}$$

Hence, applying $(H - z_0)$ to both sides of (2.79), we obtain (2.37). This concludes the proof. \square

2.5.4 Proof of Theorem 2.3.2

Now we prove Theorem 2.3.2. We recall that $\mathcal{H}_p^\pm(H)$ is the vector space spanned by all generalized eigenstates corresponding to eigenvalues $\lambda \in \mathbb{C}$ such that $\mp \text{Im}\lambda > 0$ and that $\mathcal{H}_{\text{ads}}^\pm$ is the closure of $\{u \in \mathcal{H}, \lim_{t \rightarrow \infty} \|e^{\pm itH}u\|_{\mathcal{H}} = 0\}$. We begin with proving the following easy inclusion, $\mathcal{H}_p^\pm(H) \subset \mathcal{H}_{\text{ads}}^\pm(H)$, which holds under the assumption that H has finitely many eigenvalues with finite algebraic multiplicities.

Proposition 2.5.6. *Suppose that Hypothesis 2.2 holds. Then*

$$\mathcal{H}_p^\pm(H) \subset \mathcal{H}_{\text{ads}}^\pm(H).$$

Proof. We prove that $\mathcal{H}_p^+(H) \subset \mathcal{H}_{\text{ads}}^+(H)$. Let $\lambda \in \sigma_{\text{disc}}(H)$ with $\text{Im}(\lambda) < 0$ and let $u \in \text{Ran}(\Pi_\lambda(H))$. For all $t > 0$, we estimate

$$\|e^{-itH}u\|_{\mathcal{H}} \leq \frac{1}{2\pi} \int_{\gamma} e^{t\text{Im}(z)} \|\mathcal{R}_H(z)u\|_{\mathcal{H}} dz,$$

where γ is the circle defined as in (2.5). Since $\gamma \subset \mathbb{C}^-$, we have

$$\|e^{-itH}u\|_{\mathcal{H}} \leq \frac{e^{-t\delta}}{2\pi} \int_{\gamma} \|\mathcal{R}_H(z)u\|_{\mathcal{H}} dz,$$

for some $\delta > 0$. Hence $u \in \mathcal{H}_{\text{ads}}^+(H)$. The proof of $\mathcal{H}_p^-(H) \subset \mathcal{H}_{\text{ads}}^-(H)$ is analogous. \square

To prove the converse inclusion, we will use the following easy lemma. Recall that

$$\Pi_{\text{disc}}^\pm(H) = \sum_{\lambda \in \sigma_{\text{disc}}(H), \mp \text{Im}(\lambda) > 0} \Pi_\lambda(H), \quad \Pi_{\text{disc}}^0(H) = \sum_{\lambda \in \sigma_{\text{disc}}(H), \text{Im}(\lambda) = 0} \Pi_\lambda(H).$$

Lemma 2.5.7. *Suppose that Hypothesis 2.2 holds. Then*

$$\mathcal{H}_{\text{ads}}^\pm(H) \subset \text{Ker}(\Pi_{\text{disc}}^\mp(H) + \Pi_{\text{disc}}^0(H)).$$

Proof. Let $u \in \mathcal{H}_{\text{ads}}^+(H)$. We have

$$(\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H))e^{-itH}u = e^{-itH}(\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H))u. \quad (2.80)$$

The restriction of H to $\text{Ran}(\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H)) = \mathcal{H}_{\text{disc}}^-(H) \oplus \mathcal{H}_{\text{disc}}^0(H)$ is a linear mapping from a finite dimensional space to itself, whose eigenvalues have non-negative imaginary parts. Hence, by Lyapunov's Theorem,

$$\lim_{t \rightarrow \infty} \|e^{-itH}(\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H))u\|_{\mathcal{H}} \neq 0 \quad \text{unless} \quad (\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H))u = 0.$$

Since the left-hand-side of (2.80) tends to 0 as $t \rightarrow \infty$ (for $u \in \mathcal{H}_{\text{ads}}^+(H)$), we conclude that indeed $(\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H))u = 0$. Hence we have proven that $\mathcal{H}_{\text{ads}}^+(H) \subset \text{Ker}(\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H))$. The proof of $\mathcal{H}_{\text{ads}}^-(H) \subset \text{Ker}(\Pi_{\text{disc}}^+(H) + \Pi_{\text{disc}}^0(H))$ is similar. \square

Now we are ready to prove Theorem 2.3.2. We will use the regularizing function r defined in (2.36), for some $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$.

Proof of Theorem 2.3.2. In view of Proposition 2.5.6, it remains to show that $\mathcal{H}_{\text{ads}}^{\pm}(H) \subset \mathcal{H}_{\text{p}}^{\pm}(H)$. We prove that $\mathcal{H}_{\text{ads}}^+(H) \subset \mathcal{H}_{\text{p}}^+(H)$, the inclusion $\mathcal{H}_{\text{ads}}^-(H) \subset \mathcal{H}_{\text{p}}^-(H)$ can be proven in the same way.

Let $u \in \mathcal{H}_{\text{ads}}^+(H)$. By Proposition 2.3.10, we can write

$$r(H)u = \Pi_{\text{disc}}(H)r(H)u + r_{\text{ess}}(H)u, \quad (2.81)$$

where

$$r_{\text{ess}}(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\sigma_{\text{ess}}(H)} r(\lambda)(\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon))d\lambda$$

is a bounded operator.

Clearly, since $r(H)$ is bounded and commutes with e^{-itH} , we have that $r(H)u \in \mathcal{H}_{\text{ads}}^+(H)$. Lemma 2.5.7 then implies that $r(H)u \in \text{Ker}(\Pi_{\text{disc}}^-(H) + \Pi_{\text{disc}}^0(H))$. Inserting this into (2.81) gives

$$r(H)u = \Pi_{\text{disc}}^+(H)r(H)u + r_{\text{ess}}(H)u. \quad (2.82)$$

Now we show that $u \in \text{Ker}(r_{\text{ess}}(H))$. We have $\text{Ker}(r_{\text{ess}}(H)) = \text{Ran}(r_{\text{ess}}(H)^*)^{\perp}$ where, by Proposition 2.5.3,

$$r_{\text{ess}}(H)^* = \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_{\sigma_{\text{ess}}(H)} \overline{r(\lambda)}(\mathcal{R}_{H^*}(\lambda + i\varepsilon) - \mathcal{R}_{H^*}(\lambda - i\varepsilon))d\lambda.$$

For all $v = r_{\text{ess}}(H)^*w \in \text{Ran}(r_{\text{ess}}(H)^*)$, we can write

$$|\langle v, u \rangle_{\mathcal{H}}| = |\langle e^{-itH^*}v, e^{-itH}u \rangle_{\mathcal{H}}| \leq \|e^{-itH^*}v\|_{\mathcal{H}} \|e^{-itH}u\|_{\mathcal{H}}. \quad (2.83)$$

By (2.71) and (2.72) in Proposition 2.5.4, we have

$$\|e^{-itH^*}v\|_{\mathcal{H}} = \|e^{-itH^*}r_{\text{ess}}(H)^*w\|_{\mathcal{H}} \leq c\|w\|_{\mathcal{H}}.$$

Inserting this into (2.83), letting $t \rightarrow \infty$ and using that $u \in \mathcal{H}_{\text{ads}}^+(H)$, we obtain that $\langle v, u \rangle = 0$. Hence $u \in \text{Ker}(r_{\text{ess}}(H))$ and therefore (2.82) reduces to

$$r(H)u = \Pi_{\text{disc}}^+(H)r(H)u. \quad (2.84)$$

We have proven that $r(H)u$ belongs to $\mathcal{H}_{\text{disc}}^+(H) = \mathcal{H}_p^+(H)$. Now the Riesz-Dunford functional calculus shows that the restriction of $r(H)$ to $\mathcal{H}_p^+(H)$ is bounded invertible. Hence we deduce that $u \in \mathcal{H}_p^+(H)$. This concludes the proof. \square

2.5.5 Proof of Theorem 2.3.3

Recall that $\mathcal{H}_p(H)$ is the closure of the vector space spanned by all generalized eigenvectors of H , and that the absolutely continuous spectral subspace of H has been defined in Section 2.2.2. We want to prove that $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_p(H^*)^\perp$. We begin with the following proposition which only requires that $\mathcal{H}_p(H)$ be finite dimensional.

Proposition 2.5.8. *Suppose that Hypothesis 2.2 holds. Then*

$$\mathcal{H}_{\text{ac}}(H) \subset \text{Ran}(\text{Id} - \Pi_p(H)) = \mathcal{H}_p(H^*)^\perp.$$

Proof. Let $u \in \mathcal{M}(H)$. We decompose

$$u = \Pi_p(H)u + (\text{Id} - \Pi_p(H))u.$$

Suppose by contradiction that $\Pi_p(H)u \neq 0$. Then there exists $v \in \mathcal{H}_p(H^*)$ such that $\langle u, v \rangle_{\mathcal{H}} = 1$. Indeed, if $\langle u, v \rangle_{\mathcal{H}} = 0$ for all $v \in \mathcal{H}_p(H^*)$, then $u \in \mathcal{H}_p(H^*)^\perp = \text{Ker}(\Pi_p(H))$.

Now, if $v \in \mathcal{H}_p(H^*)$ satisfies $\langle u, v \rangle = 1$, the map $t \mapsto \langle e^{-itH}u, v \rangle_{\mathcal{H}}$ cannot belong to $L^2(\mathbb{R}, \mathbb{C})$ since

$$\langle e^{-itH}u, v \rangle_{\mathcal{H}} = \langle u, e^{itH^*}v \rangle_{\mathcal{H}}$$

and the restriction of iH^* to $\mathcal{H}_p(H^*)$ is a linear mapping on a finite dimensional vector space.

This proves that $\Pi_p(H)u = 0$ and hence that $\mathcal{M}(H) \subset \text{Ran}(\text{Id} - \Pi_p(H))$. Finally taking the closure of $\mathcal{M}(H)$, we get the proposition. \square

To prove the converse inclusion, we will use the following easy lemma. Recall that r has been defined in (2.36), for some $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 2.5.9. *Suppose that Hypotheses 2.2 and 2.3 hold. Then*

$$\text{Ran}(r(H)(\text{Id} - \Pi_p(H))) \text{ is dense in } \mathcal{H}_p(H^*)^\perp.$$

Proof. Since H commutes with $\Pi_p(H)$, $r(H)$ preserves $\mathcal{H}_p(H^*)^\perp = \text{Ran}(\text{Id} - \Pi_p(H))$. The adjoint of $r(H)(\text{Id} - \Pi_p(H))$ is given by

$$[r(H)(\text{Id} - \Pi_p(H))]^* = \left[\prod_{j=1}^n (H^* - \lambda_j)^{\nu_j} \right] \mathcal{R}_{H^*}(\bar{z}_0)^{(\nu_1 + \dots + \nu_n + \nu_\infty)} (\text{Id} - \Pi_p(H^*)).$$

Since the restriction of H^* to $\text{Ran}(\text{Id} - \Pi_p(H^*))$ has no eigenvalues, the right-hand-side of the previous equation restricted to $\text{Ran}(\text{Id} - \Pi_p(H^*))$ is an injective operator, which concludes the proof of the lemma. \square

Now we prove Theorem 2.3.3.

Proof of Theorem 2.3.3. By Proposition 2.5.8, we know that $\mathcal{H}_{\text{ac}}(H) \subset \mathcal{H}_p(H^*)^\perp$. To prove that $\mathcal{H}_p(H^*)^\perp \subset \mathcal{H}_{\text{ac}}(H)$, since $\mathcal{H}_{\text{ac}}(H)$ is closed, it suffices by Lemma 2.5.9 to show that

$$\text{Ran}(r(H)(\text{Id} - \Pi_p(H))) \subset \mathcal{H}_{\text{ac}}(H).$$

Let $u = r(H)(\text{Id} - \Pi_p(H))v \in \text{Ran}(r(H)(\text{Id} - \Pi_p(H)))$. Let $(v_n)_{n \in \mathbb{N}} = (Cw_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{H}_C = \text{Ran}(C)$ such that $v_n \rightarrow v$ as $n \rightarrow \infty$ (recall that \mathcal{H}_C is dense in \mathcal{H}). We claim that

$$u_n := r(H)(\text{Id} - \Pi_p(H))v_n \in \mathcal{H}_{\text{ac}}(H). \quad (2.85)$$

To prove (2.85), applying Proposition 2.3.10 and the fact that $(\text{Id} - \Pi_p(H))\Pi_{\text{disc}}(H) = 0$, we first observe that, for all $\varphi \in \mathcal{H}$,

$$\begin{aligned} \langle u_n, \varphi \rangle &= \langle r(H)(\text{Id} - \Pi_p(H))v_n, \varphi \rangle \\ &= -\frac{1}{2i\pi} \int_{\Lambda} r(\lambda) \langle (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+))v_n, \psi \rangle d\lambda, \end{aligned}$$

where, to simplify notations, we set $\Lambda := \sigma_{\text{ess}}(H)$ and $\psi := (\text{Id} - \Pi_p(H^*))\varphi$. Next we apply (2.72) in Proposition 2.5.4 and Plancherel's Theorem to obtain

$$\begin{aligned} &\int_{\mathbb{R}} |\langle e^{-itH}u_n, \varphi \rangle|_{\mathcal{H}}^2 dt \\ &= \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_H(\lambda - i0^+) - \mathcal{R}_H(\lambda + i0^+))v_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda. \end{aligned} \quad (2.86)$$

Using the resolvent identity

$$\mathcal{R}_H(z) = \mathcal{R}_0(z) - \mathcal{R}_0(z)V\mathcal{R}_0(z) + \mathcal{R}_0(z)V\mathcal{R}_H(z)V\mathcal{R}_0(z),$$

for all $z \in \rho(H)$, we decompose

$$\begin{aligned} &\int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_H(\lambda - i\varepsilon) - \mathcal{R}_H(\lambda + i\varepsilon))v_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda \\ &\leq \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda - i\varepsilon) - \mathcal{R}_0(\lambda + i\varepsilon))v_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda \\ &\quad + \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda - i\varepsilon)V\mathcal{R}_0(\lambda - i\varepsilon))v_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda \\ &\quad + \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda + i\varepsilon)V\mathcal{R}_0(\lambda + i\varepsilon))v_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda \\ &\quad + \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda - i\varepsilon)V\mathcal{R}_H(\lambda - i\varepsilon)V\mathcal{R}_0(\lambda - i\varepsilon))v_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda \\ &\quad + \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda + i\varepsilon)V\mathcal{R}_H(\lambda + i\varepsilon)V\mathcal{R}_0(\lambda + i\varepsilon))v_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda. \end{aligned} \quad (2.87)$$

We claim that each term of the right-hand-side of the previous equation is bounded by $c_n \|\varphi\|_{\mathcal{H}}^2$, for some positive constant c_n depending on n . We estimate each term separately.

For the first term in the right-hand-side of (2.87), it suffices to use that H_0 is a self-adjoint operator with purely absolutely continuous spectrum (by Hypothesis 2.1), which yields

$$\int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda - i0^+) - \mathcal{R}_0(\lambda + i0^+)) v_n, \psi \rangle|^2 d\lambda \leq c_n \|r\|_{L^\infty}^2 \|\psi\|_{\mathcal{H}}^2. \quad (2.88)$$

Next, remembering that $v_n = Cw_n$ and $V = CWC$, the second and third terms in the right-hand-side of (2.87) are estimated as

$$\begin{aligned} & \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda \pm i\varepsilon)CWC\mathcal{R}_0(\lambda \pm i\varepsilon)) Cw_n, \psi \rangle|_{\mathcal{H}}^2 d\lambda \\ & \leq \|W\|_{\mathcal{B}(\mathcal{H})}^2 \|r\|_{L^\infty}^2 \sup_{\lambda \in \Lambda} \left(\|C\mathcal{R}_0(\lambda \pm i\varepsilon)C\|_{\mathcal{B}(\mathcal{H})}^2 \right) \|w_n\|_{\mathcal{H}}^2 \int_{\Lambda} \|C\mathcal{R}_0(\lambda \mp i\varepsilon)\psi\|_{\mathcal{H}}^2 d\lambda \\ & \leq c_n \|\psi\|_{\mathcal{H}}^2, \end{aligned} \quad (2.89)$$

where we used Hypothesis 2.1 (and (2.24)) in the second inequality.

Finally, to estimate the fourth and fifth terms in the right-hand-side of (2.87), we write similarly

$$\begin{aligned} & \int_{\Lambda} |r(\lambda) \langle (\mathcal{R}_0(\lambda \pm i\varepsilon)CWC\mathcal{R}_H(\lambda \pm i\varepsilon)CWC\mathcal{R}_0(\lambda \pm i\varepsilon)) Cw_n, \psi \rangle_{\mathcal{H}}|^2 d\lambda \\ & \leq \|W\|_{\mathcal{B}(\mathcal{H})}^2 \sup_{\lambda \in \Lambda} \left(|r(\lambda)| \|\mathcal{C}\mathcal{R}_H(\lambda \pm i\varepsilon)C\|_{\mathcal{B}(\mathcal{H})}^2 \right) \sup_{\lambda \in \Lambda} \left(\|C\mathcal{R}_0(\lambda \pm i\varepsilon)C\|_{\mathcal{B}(\mathcal{H})}^2 \right) \|w_n\|_{\mathcal{H}}^2 \\ & \quad \times \int_{\Lambda} \|C\mathcal{R}_0(\lambda \mp i\varepsilon)\psi\|_{\mathcal{H}}^2 d\lambda \\ & \leq c_n \|\psi\|_{\mathcal{H}}^2, \end{aligned} \quad (2.90)$$

where we used Hypotheses 2.1 and 2.3 in the second inequality.

Inserting (2.87)–(2.90) into (2.86) and using that $\|\psi\|_{\mathcal{H}} \leq \|\varphi\|_{\mathcal{H}}$, we deduce that

$$\int_{\mathbb{R}} |\langle e^{-itH} u_n, \varphi \rangle|_{\mathcal{H}}^2 dt \leq C_n \|\varphi\|_{\mathcal{H}}^2.$$

Therefore, $u_n \in \mathcal{H}_{\text{ac}}(H)$ for all $n \in \mathbb{N}$. Since $u_n \rightarrow u$ in \mathcal{H} as $n \rightarrow \infty$, and since $\mathcal{H}_{\text{ac}}(H)$ is closed, this implies that $u \in \mathcal{H}_{\text{ac}}(H)$ and hence the proof of the theorem is complete. \square

Our last proposition shows that in the case where H is dissipative, Hypothesis 2.4 can be dropped in the statement of Theorem 2.3.3. Using the notations from Section 2.4.6, we know that the only possible generalized eigenvectors corresponding to a real eigenvalue of H are eigenvectors in the usual sense, and that they are also eigenvectors of H_{V_1} (and of H^*). Especially if $\lambda \in \mathbb{R}$ is an eigenvalue of H , then $\text{Ker}((H - \lambda)^2) = \text{Ker}(H - \lambda)$ (see [35, lemma 3.3]) and we have

$$u \in \text{Ker}(H - \lambda) \Rightarrow u \in \text{Ker}(H_{V_1} - \lambda) \cap \text{Ker}(V_2) \Rightarrow u \in \text{Ker}(H^* - \lambda),$$

see also Lemma 2.4.11. Choosing an orthogonal basis $\{e_1, \dots, e_n\}$ in $\text{Ker}(H - \lambda)$, the spectral projection corresponding to λ can then be defined in the usual way, setting

$$\Pi_\lambda(H)u := \sum_{j=1}^n \langle e_j, u \rangle e_j, \quad u \in \mathcal{H}.$$

One readily checks that $\Pi_\lambda(H)^* = \Pi_\lambda(H^*)$. The spectral projection $\Pi_p(H)$ onto the point spectral subspace of H can then be defined as in Section 2.2.2. Modifying the previous proof in a straightforward way, we deduce the following.

Proposition 2.5.10. *Suppose that Hypotheses 2.1-2.3 hold and that H is dissipative, $\text{Im}(H) \leq 0$. Then*

$$\mathcal{H}_{\text{ac}}(H) = \text{Ran}(\text{Id} - \Pi_p(H)) = \mathcal{H}_p(H^*)^\perp.$$

2.6 Appendix

2.6.1 Proof of Proposition 2.2.1

In this appendix, we give a proof of properties of spectral projections associated to embedded eigenvalues.

Proof of Proposition 2.2.1. We are in the case where λ is an embedded eigenvalue of H . Let $(\varphi_k)_{1 \leq k \leq m_\lambda}$ a basis of $\text{Ker}((H - \lambda)^{m_\lambda})$ permitting to define $\Pi_\lambda(H)$ (see (2.13)). Then the properties are proved in the following way

- $\Pi_\lambda(H)^2 = \Pi_\lambda(H)$: Let $u \in \mathcal{H}$, then

$$\begin{aligned} \Pi_\lambda(H)\Pi_\lambda(H)u &= \sum_{k=1}^n \langle J\varphi_k, \Pi_\lambda(H)u \rangle_{\mathcal{H}} \varphi_k = \sum_{k,l=1}^n \langle J\varphi_l, \varphi_k \rangle_{\mathcal{H}} \langle J\varphi_k, u \rangle_{\mathcal{H}} \varphi_l \\ &= \sum_{k=1}^n \langle J\varphi_k, u \rangle_{\mathcal{H}} \varphi_k = \Pi_\lambda(H)u. \end{aligned}$$

- $\forall u \in \mathcal{D}(H_0)$, $\Pi_\lambda(H)Hu = H\Pi_\lambda(H)u$: Let $u \in \mathcal{D}(H_0)$, then

$$\begin{aligned}
\Pi_\lambda(H)Hu &= \sum_{k=1}^n \langle J\varphi_k, Hu \rangle_{\mathcal{H}} \varphi_k = \sum_k^n \langle JH\varphi_k, u \rangle_{\mathcal{H}} \varphi_k \\
&= \sum_{k,l=1}^n \langle J\varphi_l, u \rangle_{\mathcal{H}} \langle J\varphi_l, H\varphi_k \rangle_{\mathcal{H}} \varphi_k \\
&= \sum_{k,l=1}^n \langle J\varphi_l, u \rangle_{\mathcal{H}} \langle J\varphi_k, H\varphi_l \rangle_{\mathcal{H}} \varphi_k \\
&= \sum_{l=1}^n \langle J\varphi_l, u \rangle_{\mathcal{H}} H\varphi_l \\
&= H\Pi_\lambda(H)u.
\end{aligned}$$

- $(\Pi_\lambda(H))^* = \Pi_\lambda(H^*)$: Let $u, v \in \mathcal{H}$, then

$$\begin{aligned}
\langle \Pi_\lambda(H)u, v \rangle_{\mathcal{H}} &= \sum_{k=1}^n \overline{\langle J\varphi_k, u \rangle_{\mathcal{H}}} \langle \varphi_k, v \rangle_{\mathcal{H}} \\
&= \left\langle u, \sum_{k=1}^n \langle \varphi_k, v \rangle_{\mathcal{H}} J\varphi_k \right\rangle_{\mathcal{H}} = \langle u, \Pi_\lambda(H^*)v \rangle_{\mathcal{H}}.
\end{aligned}$$

□

2.6.2 Appendix to Section 2.4

In this appendix, we provide the proofs of Propositions 2.4.1 and 2.4.7. They consist in a suitable adaptation of the corresponding proofs in [34] where the particular case of dissipative operators have been considered.

Proof of Proposition 2.4.1. We prove the result in the case of an outgoing regular spectral point, the proof in the case of an incoming regular spectral point is identical.

First we prove that (i) \Rightarrow (ii). Suppose that λ is an outgoing regular spectral point of H . There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, $\mathcal{R}_H(\lambda + i\varepsilon)$ exists in $\mathcal{B}(\mathcal{H})$. The resolvent identity gives

$$(\text{Id} - C\mathcal{R}_H(\lambda + i\varepsilon)CW)(\text{Id} + C\mathcal{R}_0(\lambda + i\varepsilon)CW) = \text{Id}. \quad (2.91)$$

Using that the limits in (2.39) exist, we obtain from (2.91) that

$$(\text{Id} - C\mathcal{R}_H(\lambda + i0^+)CW)(\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW) = \text{Id}.$$

Thus $\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW$ is injective, and hence invertible in $\mathcal{B}(\mathcal{H})$ by the Fredholm alternative.

Next we prove that (ii) \Rightarrow (i). Suppose that $\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW$ is invertible in $\mathcal{B}(\mathcal{H})$. Suppose by contradiction that λ is an accumulation point of eigenvalues of H located in $\lambda + i(0, \infty)$. Then there exists a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive real numbers such that $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ and, for all $n \in \mathbb{N}$, vectors $u_n \in \mathcal{H}$, $\|u_n\|_{\mathcal{H}} = 1$, such that

$$(H - (\lambda + i\varepsilon_n))u_n = 0.$$

Applying $C\mathcal{R}_0(\lambda + i\varepsilon_n)$ to this equations yields

$$C\mathcal{R}_0(\lambda + i\varepsilon_n)(H - (\lambda + i\varepsilon_n))u_n = (\text{Id} + C\mathcal{R}_0(\lambda + i\varepsilon_n)CW)Cu_n = 0.$$

Since $\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW$ is invertible in $\mathcal{B}(\mathcal{H})$, for n large enough, $\text{Id} + C\mathcal{R}_0(\lambda + i\varepsilon_n)CW$ is also invertible. Therefore $C\mathcal{R}_0(\lambda + i\varepsilon_n)(H - (\lambda + i\varepsilon_n))$ is injective, which is a contradiction since $u_n \neq 0$.

It remains to show that $C\mathcal{R}_H(\lambda + i\varepsilon)CW$ converges in $\mathcal{B}(\mathcal{H})$ as $\varepsilon \rightarrow 0^+$. Since for $\varepsilon > 0$ small enough $\text{Id} + C\mathcal{R}_0(\lambda + i\varepsilon)CW$ is invertible in $\mathcal{B}(\mathcal{H})$, (2.91) gives

$$C\mathcal{R}_H(\lambda + i\varepsilon)CW = (\text{Id} + C\mathcal{R}_0(\lambda + i\varepsilon)CW)^{-1} - \text{Id},$$

This proves that

$$C\mathcal{R}_H(\lambda + i0^+)CW = (\text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW)^{-1} - \text{Id}$$

exists in $\mathcal{B}(\mathcal{H})$. □

Before proving Proposition 2.4.7, we recall the proof of Lemma 2.4.2 which was used several times in the main text.

Proof of Lemma 2.4.2. Consider for instance the operator $\mathcal{R}_0(\lambda + i\varepsilon)C$. Let $\varepsilon > 0$, $u \in \mathcal{H}$. We have

$$\begin{aligned} \|\mathcal{R}_0(\lambda + i\varepsilon)Cu\|_{\mathcal{H}}^2 &= \langle \mathcal{R}_0(\lambda + i\varepsilon)Cu, \mathcal{R}_0(\lambda + i\varepsilon)Cu \rangle_{\mathcal{H}} \\ &= \frac{1}{2i\varepsilon} \langle Cu, [\mathcal{R}_0(\lambda + i\varepsilon) - \mathcal{R}_0(\lambda - i\varepsilon)]Cu \rangle_{\mathcal{H}} \\ &= \frac{1}{\varepsilon} \text{Im}(\langle u, C\mathcal{R}_0(\lambda + i\varepsilon)Cu \rangle_{\mathcal{H}}). \end{aligned}$$

Since the limits (2.39) exist, there exists $c_0 > 0$ such that

$$\sup_{\varepsilon > 0} \text{Im}(\langle u, C\mathcal{R}_0(\lambda + i\varepsilon)Cu \rangle_{\mathcal{H}}) \leq c_0^2 \|u\|_{\mathcal{H}}^2.$$

Hence

$$\|\mathcal{R}_0(\lambda + i\varepsilon)Cu\|_{\mathcal{H}}^2 \leq \frac{1}{\varepsilon} c_0^2 \|u\|_{\mathcal{H}}^2.$$

This proves the lemma for $\mathcal{R}_0(\lambda + i\varepsilon)C$. The proof for $\mathcal{R}_0(\lambda - i\varepsilon)C$ is identical. □

Proof of Proposition 2.4.7. Again, we prove the result in the case of an outgoing regular spectral point, the proof in the case of an incoming regular spectral point being identical.

(ii) \Rightarrow (i) is obvious. We prove that (i) \Rightarrow (ii). Suppose that λ is an outgoing regular spectral point. By Proposition 2.4.1,

$$A(\lambda) := \text{Id} + C\mathcal{R}_0(\lambda + i0^+)CW$$

is invertible in $\mathcal{B}(\mathcal{H})$. Since the maps in (2.48) extend by continuity to $\mathring{D}(\lambda, r) \cap \bar{\mathcal{C}}^\pm$, there exists a compact interval $K_\lambda \subset \mathbb{R}$ whose interior contains λ such that, for all $\mu \in K_\lambda$, $A(\mu)$ is invertible. By Proposition 2.4.1, this implies that each $\mu \in K_\lambda$ is not an accumulation point of eigenvalues located in $\mu + i(0, \infty)$ and that, for all $\mu \in K_\lambda$, $C\mathcal{R}_H(\mu + i0^+)CW$ exists in $\mathcal{B}(\mathcal{H})$. Finally, the fact that the limit

$$C\mathcal{R}_H(\mu + i0^+)CW = \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\mu + i\varepsilon)CW = (\text{Id} + C\mathcal{R}_0(\mu + i0^+)CW)^{-1} - \text{Id}$$

is uniform in $\mu \in K_\lambda$ follows from the continuity of the map $z \mapsto (\text{Id} + C\mathcal{R}_0(z)CW)^{-1}$ on $\mathring{D}(\lambda, r) \cap \bar{\mathcal{C}}^+$. \square

2.6.3 Appendix to Section 2.5

Proof of Propositions 2.5.1 and 2.5.2. To prove the existence of the weak limit in (2.60), we use twice the resolvent formula, which gives for $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$ small enough,

$$\begin{aligned} \mathcal{R}_H(\lambda \pm i\varepsilon) &= \mathcal{R}_0(\lambda \pm i\varepsilon) - \mathcal{R}_0(\lambda \pm i\varepsilon)V\mathcal{R}_0(\lambda \pm i\varepsilon) \\ &\quad + \mathcal{R}_0(\lambda \pm i\varepsilon)V\mathcal{R}_H(\lambda \pm i\varepsilon)V\mathcal{R}_0(\lambda \pm i\varepsilon). \end{aligned} \quad (2.92)$$

Stone's formula for the self-adjoint operator H_0 shows that

$$\text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (\mathcal{R}_0(\lambda + i\varepsilon) - \mathcal{R}_0(\lambda - i\varepsilon)) d\lambda = \mathbf{1}_I(H_0) \quad (2.93)$$

in $\mathcal{B}(\mathcal{H})$. Since for all $u \in \mathcal{H}$, $C\mathcal{R}_0(\lambda \pm i\varepsilon)u$ converge in $L^2(I; \mathcal{H})$ as $\varepsilon \rightarrow 0^+$ by Hypothesis 2.1, we deduce that the weak limits

$$\text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I \mathcal{R}_0(\lambda \pm i\varepsilon)V\mathcal{R}_0(\lambda \pm i\varepsilon)d\lambda = \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I \mathcal{R}_0(\lambda \pm i\varepsilon)CW C\mathcal{R}_0(\lambda \pm i\varepsilon)d\lambda \quad (2.94)$$

exist in $\mathcal{B}(\mathcal{H})$. For the last term from (2.92), we write

$$\begin{aligned} &\text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I \mathcal{R}_0(\lambda \pm i\varepsilon)CW C\mathcal{R}_H(\lambda \pm i\varepsilon)CW C\mathcal{R}_0(\lambda \pm i\varepsilon)d\lambda \\ &= \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I \mathcal{R}_0(\lambda \pm i0^+)CW C\mathcal{R}_H(\lambda \pm i\varepsilon)CW C\mathcal{R}_0(\lambda \pm i0^+)d\lambda, \end{aligned} \quad (2.95)$$

where we used that, for all $u \in \mathcal{H}$, $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i\varepsilon)u$ converge in $L^2(I; \mathcal{H})$ as $\varepsilon \rightarrow 0^+$, together with the fact that $C\mathcal{R}_H(\lambda \pm i\varepsilon)CW$ is uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$ by (2.59).

Since $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i0^+)u$ belongs to $L^2(I, \mathcal{H})$ for all $u \in \mathcal{H}$, combining the fact that $C\mathcal{R}_H(\lambda \pm i\varepsilon)CW$ converges to $C\mathcal{R}_H(\lambda \pm i0^+)CW$ in $\mathcal{B}(\mathcal{H})$ for a.e. $\lambda \in I$ and again that $C\mathcal{R}_H(\lambda \pm i\varepsilon)CW$ is uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$, we obtain

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \int_I \langle u, \mathcal{R}_0(\lambda \pm i\varepsilon)CW C\mathcal{R}_H(\lambda \pm i\varepsilon)CW C\mathcal{R}_0(\lambda \pm i\varepsilon)v \rangle d\lambda \\ &= \int_I \langle C\mathcal{R}_0(\lambda \mp i0^+)u, WC\mathcal{R}_H(\lambda \pm i0^+)CW C\mathcal{R}_0(\lambda \pm i0^+)v \rangle d\lambda, \end{aligned} \quad (2.96)$$

by Lebesgue's dominated convergence theorem.

Equations (2.92)–(2.96) prove that the weak limit in (2.60) exists. Moreover, for all $u, v \in \mathcal{H}$, we have that

$$\begin{aligned} \langle u, \mathbb{1}_I(H)v \rangle &= \langle u, \mathbb{1}_I(H_0)v \rangle - \frac{1}{2i\pi} \int_I \langle C\mathcal{R}_0(\lambda \mp i0^+)u, WC\mathcal{R}_0(\lambda \pm i0^+)v \rangle d\lambda \\ &\quad + \frac{1}{2i\pi} \int_I \langle C\mathcal{R}_0(\lambda \mp i0^+)u, WC\mathcal{R}_H(\lambda \pm i0^+)CW C\mathcal{R}_0(\lambda \pm i0^+)v \rangle d\lambda. \end{aligned}$$

Equation (2.62) is easily proven, while (2.61), (2.65) and (2.66) follow as in [35]. Then the Banach algebra morphism property (2.64) follows using arguments below and the projection property. \square

Proof of Propositions 2.5.3 and 2.5.4. The proof has the same structure as that of Propositions 2.5.1 and 2.5.2, with the following modifications. First, (2.93) is replaced by the following argument. We write

$$\begin{aligned} & \int_I (h(\lambda + i\varepsilon)\mathcal{R}_0(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_0(\lambda - i\varepsilon)) d\lambda \\ &= \int_I h(\lambda)(\mathcal{R}_0(\lambda + i\varepsilon) - \mathcal{R}_0(\lambda - i\varepsilon)) d\lambda \\ &\quad + \int_I ([h(\lambda + i\varepsilon) - h(\lambda)]\mathcal{R}_0(\lambda + i\varepsilon) - [h(\lambda - i\varepsilon) - h(\lambda)]\mathcal{R}_0(\lambda - i\varepsilon)) d\lambda. \end{aligned}$$

For the first term, since H_0 is self-adjoint, we have

$$\text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I h(\lambda)(\mathcal{R}_0(\lambda + i\varepsilon) - \mathcal{R}_0(\lambda - i\varepsilon)) d\lambda = h(H_0). \quad (2.97)$$

For the second term, we use the mean-value Theorem together with the Cauchy-Schwarz inequality, writing for all $u, v \in \mathcal{H}$,

$$\begin{aligned} & \int_I |\langle u, [h(\lambda \pm i\varepsilon) - h(\lambda)]\mathcal{R}_0(\lambda \pm i\varepsilon)v \rangle_{\mathcal{H}}| d\lambda \\ & \leq \varepsilon \|u\|_{\mathcal{H}} \left(\int_I \left(\sup_{0 < \varepsilon < \varepsilon_0} |h'(\lambda \pm i\varepsilon)| \right)^2 d\lambda \right)^{\frac{1}{2}} \left(\int_I \|\mathcal{R}_0(\lambda \pm i\varepsilon)v\|_{\mathcal{H}}^2 d\lambda \right)^{\frac{1}{2}}. \end{aligned} \quad (2.98)$$

The first integral is bounded by (2.68). The second integral can be rewritten as

$$\begin{aligned} \int_I \|\mathcal{R}_0(\lambda \pm i\varepsilon)v\|_{\mathcal{H}}^2 d\lambda &= \int_I \langle v, \mathcal{R}_0(\lambda \mp i\varepsilon)\mathcal{R}_0(\lambda \pm i\varepsilon)v \rangle_{\mathcal{H}} d\lambda \\ &= \frac{1}{2i\varepsilon} \int_I \langle v, (\mathcal{R}_0(\lambda - i\varepsilon) - \mathcal{R}_0(\lambda + i\varepsilon))v \rangle_{\mathcal{H}} d\lambda, \end{aligned} \quad (2.99)$$

from which we deduce that

$$\left(\int_I \|\mathcal{R}_0(\lambda \pm i\varepsilon)v\|_{\mathcal{H}}^2 d\lambda \right)^{\frac{1}{2}} \leq c\varepsilon^{-\frac{1}{2}}\|v\|_{\mathcal{H}}. \quad (2.100)$$

Together with (2.97), (2.98) and (2.99), this implies that

$$\text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (h(\lambda + i\varepsilon)\mathcal{R}_0(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_0(\lambda - i\varepsilon))d\lambda = h(H_0).$$

The rest of the proof follows in the same way as in the proof of Propositions 2.5.1 and 2.5.2 (see in particular (2.94) and (2.95)), using that $h(\lambda \pm i\varepsilon)$ and $h(\lambda \pm i\varepsilon)C\mathcal{R}_H(\lambda \pm i\varepsilon)CW$ are uniformly bounded in $\varepsilon \in (0, \varepsilon_0)$. \square

In the proof of the spectral resolution formula stated in Proposition 2.3.10, we used Lemma 2.5.5 which we now prove. The arguments are similar to those used in the previous proofs.

Proof of Lemma 2.5.5. To prove (i), it suffices to use the resolvent equation (2.92) together with the fact that $V = CWC$ and the estimates $\|\mathcal{R}_0(\lambda \pm i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \leq \varepsilon^{-1}$, $\|C\mathcal{R}_0(\lambda \pm i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \leq c\varepsilon^{-1/2}$ (see Lemma 2.4.2) and $\|C\mathcal{R}(H \pm i\varepsilon)CW\|_{\mathcal{B}(\mathcal{H})} \leq c$ by (2.59).

To prove (ii), we use again (2.92), writing

$$\begin{aligned} \int_I \|\mathcal{R}_H(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda &\leq 2 \int_I \|\mathcal{R}_0(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda \\ &\quad + 2 \int_I \|\mathcal{R}_0(\lambda \pm i\varepsilon)CWC\mathcal{R}_0(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda \\ &\quad + 2 \int_I \|\mathcal{R}_0(\lambda \pm i\varepsilon)CWC\mathcal{R}_H(\lambda \pm i\varepsilon)CWC\mathcal{R}_0(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda. \end{aligned}$$

By (2.100), the first term is bounded $c\varepsilon^{-1}$. The same holds for the second and third terms, using again that $\|C\mathcal{R}_0(\lambda \pm i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \leq c\varepsilon^{-1/2}$ and $\|C\mathcal{R}(H \pm i\varepsilon)CW\|_{\mathcal{B}(\mathcal{H})} \leq c$. \square

Chapter 3

Scattering theory for some non-self-adjoint operators

We consider a non-self-adjoint H given as the perturbation of a self-adjoint operator H_0 . We suppose that H is of the form $H = H_0 + CWC$ where C is a bounded metric operator, relatively compact with respect to H_0 , and W is bounded. We suppose that $C(H_0 - z)^{-1}C$ is uniformly bounded in $z \in \mathbb{C} \setminus \mathbb{R}$. We define the regularized wave operators associated to H and H_0 by $W_{\pm}(H, H_0) := \text{s-lim}_{t \rightarrow \infty} e^{\pm itH} r_{\mp}(H) \Pi_p(H^*)^{\perp} e^{\mp itH_0}$ where $\Pi_p(H^*)$ is the projection onto the direct sum of all the generalized eigenspace associated to eigenvalue of H^* and r_{\mp} is a rational function that regularizes the ‘incoming/outgoing spectral singularities’ of H . We prove the existence and study the properties of the regularized wave operators. In particular we show that they are asymptotically complete if H does not have any spectral singularity.

3.1 Introduction

We are interested in this paper in the scattering theory for non-self-adjoint operators. Non-self-adjoint ‘Hamiltonians’ in Quantum Mechanics are particularly relevant in various context, see e.g. [4, 62] and references therein for a detailed exposition.

In particular, non-conservative phenomena may be described by effective or phenomenological non-self-adjoint operators. A famous model involving non-self-adjoint operators is the *nuclear optical model* describing the interaction of a neutron (or a proton) and a nucleus in Nuclear Physics. It was introduced by Feshbach, Porter and Weisskopf in [37] as an empirical model taking into account the formation of a compound nucleus. In this model, if the neutron is initially (at time $t = 0$) in the normalized state φ_0 , then it evolves into the unnormalized state $\varphi_t = e^{-itH} \varphi_0$ at time t , where φ_t is the solution to the Schrödinger equation

$$i\partial_t \varphi_t = H\varphi_t. \tag{3.1}$$

Here $H = -\Delta + V(x)$ is a dissipative operator acting on $L^2(\mathbb{R}^3, \mathbb{C})$, with $\text{Im}(V(x)) \leq 0$. Part of the energy of the neutron may be transferred to the nucleus, eventually leading to

the capture of the neutron by the nucleus. The transfer of energy is mathematically illustrated by the dissipative nature of the ‘Hamiltonian’ H . In particular, given a normalized state φ_0 , the probability that the neutron escapes from the nucleus is given by

$$\rho_{\text{scatt}}(\varphi_0) = 1 - \lim_{t \rightarrow \infty} \|e^{-itH}\varphi_0\|_{L^2}^2.$$

and $\rho_{\text{abs}}(\varphi_0)$, the probability of absorption of the neutron by the nucleus, is given by

$$\rho_{\text{abs}}(\varphi_0) = 1 - \rho_{\text{scat}}(\varphi_0).$$

If the neutron is initially in a state whose probability of scattering is not zero, then it is expected that there exists a scattering state $\varphi \in \mathcal{H}$ such that $\|\varphi\|_{L^2(\mathbb{R}^3, \mathbb{C})} = \rho_{\text{scatt}}(\varphi_0)$ and

$$\lim_{t \rightarrow \infty} \|e^{-itH}\varphi_0 - e^{-itH_0}\varphi\|_{L^2(\mathbb{R}^3, \mathbb{C})} = 0.$$

This motivated the development of a scattering theory for non-self-adjoint operators. The nuclear optical model leads to predictions that correspond to experimental scattering data to a high precision. Theoretical justifications of the model have been given in [38–40] (see also [4, 41, 50]). We mention the works [32, 34, 35] for an abstract dissipative scattering theory and [72, 85] for dissipative Schrödinger operators. Moreover, scattering theory for non-self-adjoint operators on Hilbert spaces has been considered in other contexts, see for example [54] for the construction of local wave operators assuming that a limiting absorption principle holds, [87] for one-dimensional Schrödinger operators with a complex potential on the half-line, [88] for an abstract framework on scattering theory for non-self-adjoint operators under an assumption of Kato’s smoothness of the perturbation, and [55] in the case where the perturbation is not too large in Kato’s sense.

We consider here an abstract class of operators of the form $H = H_0 + V$ acting on a Hilbert space \mathcal{H} , where H_0 is self-adjoint with absolutely continuous spectrum and V is a bounded operator relatively compact with respect to H_0 . In particular, the essential spectrum of H and that of H_0 coincide. We suppose that V decomposes into the form $V = CWC$, with W bounded and C a bounded operator such that

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|C(H_0 - z)^{-1}C\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (3.2)$$

Such factorizations go back to the seminal work of Kato [55].

We aim at constructing and studying the wave operators associated to H and H_0 . As H is a perturbation of a self-adjoint operator by a bounded operator, $-iH$ is the generator of a strongly continuous group of evolution. We suppose in addition that H (and hence H^*) has a finite number of eigenvalues with finite algebraic multiplicities. We can then define the *regularized wave operators* associated to H and H_0 by

$$W_{\pm}(H, H_0) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \Pi_{\text{p}}(H^*)^{\perp} r_{\mp}(H) e^{-itH_0}. \quad (3.3)$$

Here $\Pi_{\text{p}}(H^*)^{\perp}$ is the projection onto the orthogonal of the point spectral subspace of H^* , and $r_{\mp}(H)$ are rational functions of H regularizing the incoming/outgoing spectral

singularities of H . See below for precise definitions. In our context, spectral singularities are defined as points of the essential spectrum of H where H does not satisfy a suitable limiting absorption principle (see Definition 3.3.1 below). For Schrödinger operators with a bounded, compactly supported potential, spectral singularities correspond to real resonances (see e.g. [25]).

We are not aware of such a definition of regularized wave operators previously in the literature, even in context of Schrödinger operators. In this paper, we prove that, under suitable assumptions, the regularized wave operators $W_{\pm}(H, H_0)$ exist, are injective, and such that their ranges are dense in $\text{Ran}(\Pi_p(H^*))^{\perp}$. (We will recall from chapter 2 that $\text{Ran}(\Pi_p(H^*))^{\perp}$ coincide with the ‘absolutely continuous spectral subspace’ of H , see Subsection 3.3.4 for the definition of $\mathcal{H}_{\text{ac}}(H)$). These properties therefore extend the corresponding well-known properties that hold for wave operators in unitary scattering theory, which supports our definition of $W_{\pm}(H, H_0)$.

Finally we show that if H has no spectral singularity, then wave operators are *asymptotically complete*, in the sense that they define bijections from \mathcal{H} to $\text{Ran}(\Pi_p(H^*))^{\perp}$. A consequence of Asymptotic Completeness is that the solutions of the Schrödinger equation (3.1) with initial conditions in $\text{Ran}(\Pi_p(H^*))^{\perp}$ are uniformly bounded in time.

Notation. In the following, if \mathcal{H}_1 and \mathcal{H}_2 are two Hilbert spaces, $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ stands for the set of continuous linear operators from \mathcal{H}_1 to \mathcal{H}_2 . If $\mathcal{H}_1 = \mathcal{H}_2$, we simplify the notation by setting $\mathcal{B}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1)$. We let $\mathcal{R}_B(z) := (B - z)^{-1}$ the resolvent of an operator B . For H_0 , we denote $\mathcal{R}_0(z)$ its resolvent. Moreover $D(\lambda, \varepsilon)$ is the open disk centered at λ with radius ε and $\mathbb{C}^{\pm} = \{z \in \mathbb{C}, \pm \text{Im}(z) > 0\}$. Finally, the set of all integers between 1 and n is denoted $\llbracket 1, n \rrbracket$ and A^{cl} is the closure of the set A .

3.2 Abstract setting

3.2.1 The model

Consider $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ a complex separable Hilbert space and an operator on \mathcal{H} of the form

$$H := H_0 + V, \quad (3.4)$$

where H_0 is self-adjoint and semi-bounded from below and $V \in \mathcal{B}(\mathcal{H})$ is a bounded operator. In particular, H is a closed operator with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$ and its adjoint is given by

$$H^* = H_0 + V^*, \quad \mathcal{D}(H^*) = \mathcal{D}(H_0).$$

Without loss of generality, we suppose that $H_0 \geq 0$.

As H is a perturbation of the self-adjoint operator H_0 by the bounded operator V , $-iH$ generates a strongly continuous one-parameter group $\{e^{-itH}\}_{t \in \mathbb{R}}$ which satisfies

$$\|e^{-itH}\|_{\mathcal{B}(\mathcal{H})} \leq e^{|t|\|V\|_{\mathcal{B}(\mathcal{H})}}, \quad t \in \mathbb{R},$$

(see e.g [19] or [28]).

We assume that there exists an operator $C \in \mathcal{B}(H)$, relatively compact with respect to H_0 , such that V is of the form

$$V = CWC, \quad (3.5)$$

with $W \in \mathcal{B}(\mathcal{H})$. Finally, we require that C be a *metric operator*, which means that C is a positive and injective (see e.g. [3]).

3.2.2 Spectral subspaces, spectral projection

We let $\sigma(H)$ be the spectrum of H and $\rho(H) = \mathbb{C} \setminus \sigma(H)$ its resolvent set. We define the point spectrum of H as the set of all eigenvalues of H ,

$$\sigma_p(H) := \{\lambda \in \mathbb{C}, \text{Ker}(H - \lambda) \neq \{0\}\}.$$

For each eigenvalue λ , we define its algebraic multiplicity $m_\lambda(H)$ as the dimension of the generalized eigenspace associated to λ

$$m_\lambda(H) := \dim \left(\bigcup_{k=1}^{\infty} \text{Ker}((H - \lambda)^k) \right).$$

If λ is an isolated eigenvalue of H , we denote by

$$\Pi_\lambda(H) := \frac{1}{2\pi i} \int_\gamma (z \text{Id} - H)^{-1} dz, \quad (3.6)$$

the usual Riesz projection, where γ is a circle oriented counterclockwise and centered at λ , of sufficiently small radius (so that λ is the only point of the spectrum of H contained in the interior of γ) is finite dimensional. The discrete spectrum of H , $\sigma_{\text{disc}}(H)$, is the set of all isolated eigenvalues λ such that the range of the associated Riesz projection is finite dimensional.

As V is a relatively compact perturbation of H_0 , the essential spectrum $\sigma_{\text{ess}}(H) := \sigma(H) \setminus \sigma_{\text{disc}}(H)$ and the essential spectrum of H_0 coincide. Moreover, the discrete spectrum $\sigma_{\text{disc}}(H)$ is at most countable and can only accumulate at points of $\sigma_{\text{ess}}(H)$. See Figure 3.1. We define in addition the set of eigenvalues embedded in the essential spectrum of H :

$$\sigma_{\text{emb}}(H) := \sigma_p(H) \cap \sigma_{\text{ess}}(H).$$

3.2.2.1 Eigenspaces corresponding to isolated eigenvalues

For $\lambda \in \sigma_{\text{disc}}(H)$, since the restriction of H to $\text{Ran}(\Pi_\lambda(H))$ may have a nontrivial Jordan form, $\text{Ran}(\Pi_\lambda(H))$ is in general spanned by generalized eigenvectors of H associated to λ , i.e., by vectors $u \in \mathcal{D}(H^k)$ such that $(H - \lambda)^k u = 0$ for some $1 \leq k \leq m_\lambda(H)$. Moreover the dimension of the range of $\Pi_\lambda(H)$ satisfies $\dim \text{Ran}(\Pi_\lambda(H)) = m_\lambda(H)$. We set

$$\mathcal{H}_{\text{disc}}(H) := \text{Span} \{u \in \text{Ran}(\Pi_\lambda(H)), \lambda \in \sigma_{\text{disc}}(H)\}^{\text{cl}},$$

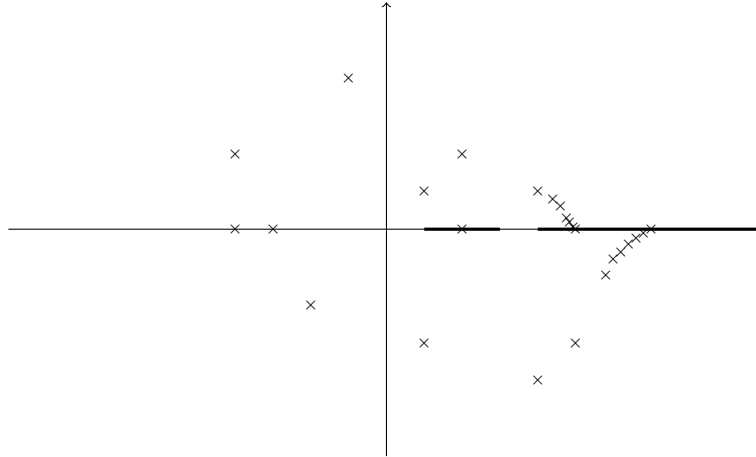


Figure 3.1: **Spectrum of H .** The essential spectrum of H , contained in $[0, \infty)$, is represented by thick lines and coincides with that of H_0 . The eigenvalues of H are represented by crosses. The discrete spectrum of H consists of isolated eigenvalues of finite algebraic multiplicities which may accumulate at any point of the essential spectrum. The point spectrum of H may also contain eigenvalues embedded in the essential spectrum.

where A^{cl} stands for the closure of a subset $A \subset \mathcal{H}$. We will sometimes assume that the discrete spectrum of H is finite. The spectral projection $\Pi_{\text{disc}}(H)$ onto $\mathcal{H}_{\text{disc}}(H)$ is then defined by

$$\Pi_{\text{disc}}(H) := \sum_{\lambda \in \sigma_{\text{disc}}(H)} \Pi_{\lambda}(H). \quad (3.7)$$

3.2.2.2 Eigenspaces corresponding to embedded eigenvalues

If λ is an eigenvalue of H embedded in its essential spectrum then the Riesz projection corresponding to λ is ill-defined. Under some additional conditions, however, one can define the spectral projection $\Pi_{\lambda}(H)$ as follows.

In the following (see Hypothesis 3.4 below), we will suppose the existence of a *antiunitary* operator $J \in \mathcal{B}(\mathcal{H})$ verifying

$$J\mathcal{D}(H_0) \subset \mathcal{D}(H_0) \quad \text{and} \quad \forall u \in \mathcal{D}(H_0), \quad JHu = H^*Ju. \quad (3.8)$$

In particular, J establishes a one-to-one correspondence between $\text{Ker}((H - \lambda)^k)$ and $\text{Ker}((H^* - \bar{\lambda})^k)$ for all $k \in \mathbb{N}$ and hence

$$m_{\lambda}(H) = m_{\bar{\lambda}}(H^*).$$

To shorten notation, let $m_{\lambda} = m_{\lambda}(H) = m_{\bar{\lambda}}(H^*)$. In order to define projections onto the generalized eigenspace associated to embedded eigenvalues, we will suppose that for each embedded eigenvalue $\lambda \in \sigma_{\text{ess}}(H)$, m_{λ} is finite and the symmetric bilinear form

$$\text{Ker}((H - \lambda)^{m_{\lambda}}) \times \text{Ker}((H - \lambda)^{m_{\lambda}}) \ni (u, v) \mapsto \langle Ju, v \rangle_{\mathcal{H}} \quad \text{is non-degenerate.} \quad (3.9)$$

This implies that there exists a basis $(\varphi_k)_{1 \leq k \leq m_\lambda}$ of $\text{Ker}((H - \lambda)^{m_\lambda})$ such that

$$\langle J\varphi_i, \varphi_j \rangle_{\mathcal{H}} = \delta_{ij}, \quad 1 \leq i, j \leq m_\lambda.$$

Thus we can define the spectral projection $\Pi_\lambda(H)$ onto the generalized eigenspace corresponding to λ as

$$\Pi_\lambda(H)u = \sum_{k=1}^{m_\lambda} \langle J\varphi_k, u \rangle_{\mathcal{H}} \varphi_k, \quad u \in \mathcal{H}. \quad (3.10)$$

It is not difficult to observe that $\Pi_\lambda(H)$ is a projection commuting with H , such that $\Pi_\lambda(H) \in \mathcal{B}(\mathcal{H})$ and $\Pi_\lambda(H)^* = \Pi_\lambda(H^*)$. For more detail about this projection, see subsection 2.2.2.2.

3.2.2.3 Projection onto the point spectrum

In the following we will suppose that H has a finite number of eigenvalues with finite algebraic multiplicities, see Hypothesis 3.2. Thus the sum of all the projections associated to generalized eigenspaces of H , defined by

$$\Pi_p(H) := \sum_{\lambda \in \sigma_p(H)} \Pi_\lambda(H)$$

is well-defined. Next we define the point spectral subspace of H as the range of $\Pi_p(H)$,

$$\mathcal{H}_p(H) := \text{Ran}(\Pi_p(H)).$$

Finally, we observe that

$$\mathcal{H}_p(H) = \sum_{\lambda \in \sigma_p(H)} \text{Ker}(H - \lambda)^{m_\lambda}.$$

3.3 Assumptions and main results

3.3.1 Hypotheses

In this section we detail our main abstract assumptions. In Section 3.3.3 we will show that they are satisfied in the case of complex Schrödinger operators, under suitable assumptions on the potential.

In our first hypothesis, we require that, at any point of its essential spectrum, H_0 satisfies a *limiting absorption principle* with weight C .

Hypothesis 3.1 (Limiting absorption principle for H_0). *We have*

$$\sup_{z \in \mathbb{C}^\pm} \|C\mathcal{R}_0(z)C\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (3.11)$$

Note that (3.11) implies (see e.g. [13, Proposition 4.1]) that the spectrum of H_0 is purely absolutely continuous, i.e. that $\sigma_{\text{pp}}(H_0) = \emptyset$, $\sigma_{\text{ac}}(H_0) = \sigma(H_0)$, $\sigma_{\text{sc}}(H_0) = \emptyset$, where $\sigma_{\text{pp}}(H_0)$, $\sigma_{\text{ac}}(H_0)$, $\sigma_{\text{sc}}(H_0)$ stand for the usual pure point, absolutely continuous and singular continuous spectra of the self-adjoint operator H_0 .

Moreover, by Fatou's Theorem, (3.11) implies that the limits $C\mathcal{R}_0(\lambda \pm i0^+)C$ exist for almost every $\lambda \in \sigma_{\text{ess}}(H)$, in the norm topology of $\mathcal{B}(\mathcal{H})$, and that the map $\mathbb{R} \ni \lambda \mapsto C\mathcal{R}_0(\lambda \pm i0^+)C \in \mathcal{B}(\mathcal{H})$ is bounded (observe that $C\mathcal{R}_0(\lambda \pm i0^+)C = C\mathcal{R}_0(\lambda)C$ if $\lambda \in \mathbb{R} \setminus \sigma_{\text{ess}}(H)$).

Note also that Hypothesis 3.1 implies (see [55] or [76, Theorem XIII.25 and its corollary]) that C is *relatively smooth* with respect to H_0 in the sense of Kato, i.e. that there exists a constant c_0 such that

$$\forall u \in \mathcal{H}, \quad \int_{\mathbb{R}} \|Ce^{-itH_0}u\|_{\mathcal{H}}^2 dt \leq c_0^2 \|u\|_{\mathcal{H}}^2. \quad (3.12)$$

Recall that (3.12) is equivalent to

$$\forall u \in \mathcal{H}, \quad \int_{\mathbb{R}} \left(\|C\mathcal{R}_0(\lambda - i0^+)u\|_{\mathcal{H}}^2 + \|C\mathcal{R}_0(\lambda + i0^+)u\|_{\mathcal{H}}^2 \right) d\lambda \leq 2\pi c_0^2 \|u\|_{\mathcal{H}}^2, \quad (3.13)$$

where $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i0^+)u$ denotes the limit of $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i\varepsilon)u$ in $L^2(\mathbb{R}; \mathcal{H})$ as $\varepsilon \rightarrow 0^+$.

Next we assume that the point spectral subspace of H is finite.

Hypothesis 3.2 (Eigenvalues of H). *H has only a finite number of eigenvalues with finite algebraic multiplicities.*

Hypothesis 3.2 prevents the essential spectrum of H from having an accumulation point of eigenvalues. It does not exclude, however, the presence of eigenvalues embedded in the essential spectrum of H .

Our next hypothesis concerns the spectral singularities of H . First we recall the definition of a regular spectral point.

Definition 3.3.1 (Regular spectral point and spectral singularity).

1. Let $\lambda \in \sigma_{\text{ess}}(H)$.

(i) We say that λ is an outgoing/incoming regular spectral point of H if λ is not an accumulation point of eigenvalues located in $\lambda \pm i(0, \infty)$ and if the limit

$$C\mathcal{R}_H(\lambda \pm i0^+)CW := \lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\lambda \pm i\varepsilon)CW \quad (3.14)$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$. If λ is not an outgoing/incoming regular spectral point, we say that λ is an outgoing/incoming spectral singularity of H .

(ii) We say that λ is a regular spectral point of H if λ is both an outgoing and an incoming regular spectral point of H . If λ is not a regular spectral point, we say that λ is a spectral singularity of H .

2. We say that infinity is an outgoing/incoming regular spectral point of H if there exists $m > 0$ such that for all $\lambda > m$, λ is an outgoing/incoming regular spectral point and such that the map

$$[m, \infty) \ni \lambda \mapsto C\mathcal{R}_H(\lambda \pm i0^+)CW \in \mathcal{B}(\mathcal{H})$$

is bounded. If infinity is not outgoing/incoming regular spectral point of H , we say that H has an outgoing/incoming spectral singularity at infinity.

Next, we introduce the notion of order of a spectral singularity.

Definition 3.3.2 (Order of a spectral singularity).

1. Let $\lambda \in \sigma_{\text{ess}}(H)$ be an outgoing/incoming spectral singularity of H . We say that λ is a spectral singularity of finite order if there exist an integer n and $\varepsilon > 0$ such that

$$\sup_{z \in D(\lambda, \varepsilon) \cap \mathbb{C}^\pm} |\lambda - z|^n \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (3.15)$$

Otherwise we say that λ is an outgoing/incoming spectral singularity of infinite order. If λ is an outgoing/incoming spectral singularity of finite order we define its order as the smallest integer satisfying (3.15).

2. If H has an outgoing/incoming spectral singularity at infinity, we say that infinity is an outgoing/incoming spectral singularity of finite order if there exists an integer n , $\varepsilon_0 > 0$, $m > 0$ and $z_0 \in \rho(H) \setminus \mathbb{R}$, such that

$$\sup_{\substack{\text{Re}(z) > m \\ \pm \text{Im}(z) < \varepsilon_0}} |z - z_0|^{-n} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (3.16)$$

Otherwise we say that infinity is an outgoing/incoming spectral singularity of infinite order. If infinity is an outgoing/incoming spectral singularity of finite order we define its order as the smallest integer satisfying (3.16).

Remark that λ is a spectral singularity of finite order means that, in a neighborhood of λ , the map $z \mapsto C\mathcal{R}_H(z)CW$ blows up at most polynomially as z approaches λ . Moreover if the map $z \mapsto C\mathcal{R}_H(z)CW$ has a meromorphic continuation across $\sigma_{\text{ess}}(H)$, then the spectral singularities of H correspond to poles of the meromorphic continuation of the weighted resolvent of H , and the order of the spectral singularity correspond to the order of the pole. This is in particular the case for Schrödinger operators (see 3.3.3).

In the following, we assume that H has a finite number of spectral singularities and that each spectral singularity has a finite order.

Hypothesis 3.3 (Spectral singularities for H). H only has a finite number of outgoing/incoming spectral singularities in $\sigma_{\text{ess}}(H) \cup \{\infty\}$ and each spectral singularity has a finite order. Moreover, for all closed interval $I \subset \sigma_{\text{ess}}(H)$ not containing any spectral singularity, there exists $\varepsilon_0 > 0$ such that

$$\sup_{\substack{\text{Re}(z) \in I \\ \pm \text{Im}(z) \in (0, \varepsilon_0)}} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

This hypothesis has the following consequence. Let $\lambda_1, \dots, \lambda_n \in \sigma_{\text{ess}}(H)$ be the spectral singularities of H belonging to $\sigma_{\text{ess}}(H)$, of order $\nu_1, \dots, \nu_n < \infty$, respectively, and let ν_∞ be the order of ∞ in the case where ∞ is a spectral singularity (otherwise, we use the convention that $\nu_\infty = 0$). Then there exists $\varepsilon_0 > 0$ such that

$$\sup_{\substack{\text{Re}(z) \in \sigma_{\text{ess}}(H) \\ \pm \text{Im}(z) \in (0, \varepsilon_0)}} \frac{1}{|z - z_0|^{\nu_\infty}} \left(\prod_{j=1}^n \frac{|z - \lambda_j|^{\nu_j}}{|z - z_0|^{\nu_j}} \right) \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty, \quad (3.17)$$

where z_0 is an arbitrary complex number such that $z_0 \in \rho(H)$, $z_0 \in \mathbb{C} \setminus \mathbb{R}$. Note that the factors $|z - \lambda_j|^{\nu_j}$ ‘regularize’ the singularities of $z \mapsto C\mathcal{R}_H(z)CW$ as z approaches λ_j . Dividing them by $|z - z_0|^{\nu_j}$ produces bounded terms. The factor $|z - z_0|^{-\nu_\infty}$ regularizes a possible singularity at ∞ .

Observe that since $\lambda_1, \dots, \lambda_n$ are the only spectral singularities of H , for all $\lambda \in \sigma_{\text{ess}}(H) \setminus \{\lambda_1, \dots, \lambda_n\}$, the limits $C\mathcal{R}_H(\lambda \pm i0^+)CW$ exist in the norm topology of $\mathcal{B}(\mathcal{H})$. The condition (3.17) then implies that the maps

$$\sigma_{\text{ess}}(H) \setminus \{\lambda_1, \dots, \lambda_n\} \ni \lambda \mapsto \frac{1}{|\lambda - z_0|^{\nu_\infty}} \left(\prod_{j=1}^n \frac{|\lambda - \lambda_j|^{\nu_j}}{|\lambda - z_0|^{\nu_j}} \right) C\mathcal{R}_H(\lambda \pm i0^+)CW \in \mathcal{B}(\mathcal{H}) \quad (3.18)$$

are bounded. Since embedded eigenvalues are outgoing and incoming spectral singularities (see section 2.4.4), Hypothesis 3.3 also concerns possible embedded eigenvalues. In the following we will denote by r_j the function that regularizes the spectral singularity λ_j , i.e.

$$r_j(z) := \frac{(z - \lambda_j)^{\nu_j}}{(z - z_0)^{\nu_j}}. \quad (3.19)$$

In the same way, r_∞ stands for the function that regularizes the spectral singularity at ∞ , i.e.

$$r_\infty(z) := (z - z_0)^{-\nu_\infty}. \quad (3.20)$$

Moreover we let r_+ be the product of all the functions r_j where λ_j is an outgoing spectral singularity, and r_- be the product of all the functions r_j where λ_j is an incoming spectral singularity.

We also mention the following ‘local version’ of (3.17) which will be useful in the sequel. Obviously, since H has a finite number n of spectral singularities, it also has a finite number of outgoing and incoming spectral singularities. Let J_1, \dots, J_n be compact intervals and J_∞ be an interval such that i) for each $j \in \llbracket 1, n \rrbracket$, the interior J_j^{int} of J_j contains the outgoing/incoming spectral singularity λ_j and no other outgoing/incoming spectral singularity, ii) $J_1^{\text{int}}, \dots, J_n^{\text{int}}, J_\infty^{\text{int}}$ are disjoint, and iii) $\sigma_{\text{ess}}(H) = \bigcup_{j=1}^n J_j \cup J_\infty$. As, for $j \in \llbracket 1, n \rrbracket \cup \{\infty\}$, r_k is invertible in J_j for all $k \in \llbracket 1, n \rrbracket \cup \{\infty\} \setminus \{j\}$, we deduce from (3.17) the following local limiting absorption principle:

$$\sup_{\substack{\text{Re}(z) \in J_j \\ \pm \text{Im}(z) \in (0, \varepsilon_0)}} |r_j(z)| \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

Finally, in order to define the spectral projections onto the generalized eigenspaces associated to embedded eigenvalues, we require the existence of a conjugate operator J satisfying, in particular, $JH = H^*J$.

Hypothesis 3.4 (Conjugate operator). *There exists an antiunitary operator $J : \mathcal{H} \rightarrow \mathcal{H}$ such that*

$$(i) \quad J\mathcal{D}(H_0) \subset \mathcal{D}(H_0) \text{ and } \forall u \in \mathcal{D}(H_0), \quad JH_0u = H_0Ju.$$

$$(ii) \quad JC = CJ \text{ and } JW = W^*J.$$

Moreover, for all embedded eigenvalues $\lambda \in \sigma_{\text{ess}}(H)$, the symmetric bilinear form

$$\text{Ker}((H - \lambda)^{m_\lambda}) \times \text{Ker}((H - \lambda)^{m_\lambda}) \ni (u, v) \mapsto \langle Ju, v \rangle \quad \text{is non-degenerate.} \quad (3.21)$$

This condition ensures that the spectral projections (3.10) onto the generalized eigenspaces associated to embedded eigenvalues are well-defined. See the discussion in section 2.2.2.2 for more details.

3.3.2 Main results

In this section we state our main results. Recall from (3.3) that the wave operators associated to H and H_0 are defined by

$$W_{\pm}(H, H_0) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \Pi_p(H^*)^{\perp} r_{\mp}(H) e^{-itH_0}$$

We recall that $\Pi_p(H^*)$ is the sum of all the spectral projections onto the generalized eigenspaces associated to the eigenvalues of H^* (either isolated or embedded into the essential spectrum), and that $\Pi_p(H^*)$ is well-defined thanks to Hypotheses 3.2 and 3.4. The projection $\Pi_p(H^*)^{\perp} = \text{Id} - \Pi_p(H^*)$ is needed to project out the generalized eigenspace associated to H , and the ‘regularizing operator’ $r_{\mp}(H)$ is needed to regularize the incoming/outgoing spectral singularities of H . See section 3.4.2.1 for more details about the justification of this definition. In the same way as for (3.3), this ‘non-stationary’ definition of wave operators taking into account possible spectral singularities seem to be new. See however e.g. [55] for general definitions of non necessarily unitary wave operators, under assumptions ensuring the absence of spectral singularities.

Our first main result is the existence of the wave operators $W_{\pm}(H, H_0)$ as well as properties about their ranges and kernels.

Theorem 3.3.3. *Suppose that Hypotheses 3.1-3.4 hold. Then the wave operators $W_{\pm}(H, H_0)$ exist. Moreover they are injective and their ranges are dense in $\mathcal{H}_p(H^*)^{\perp}$.*

Once the existence of the wave operators $W_{\pm}(H, H_0)$ is proven, it is not difficult to show that they satisfy the usual intertwining properties (see Proposition 3.4.7). One of the ingredients to the properties of these operators is the studying of

$$W_{\pm}(H_0, H) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_0} r_{\pm}(H) \Pi_p(H)^{\perp} e^{-itH}$$

in section ???. Because $\{e^{-itH_0}\}_{t \in \mathbb{R}}$ is unitary, the existence of $W_{\pm}(H_0, H)$ implies that for all $u \in \text{Ran } W_{\pm}(H_0, H)$ there exists $v \in \mathcal{H}$ such that

$$\lim_{t \rightarrow \pm\infty} \|e^{-itH} \Pi_{\mathfrak{p}}(H^*)^{\perp} r_{\pm}(H)v - e^{-itH_0}u\| = 0. \quad (3.22)$$

And (3.22) leads to $u = W_{\pm}(H_0, H)v$ and

$$e^{-itH_0}u = \Pi_{\mathfrak{p}}(H^*)^{\perp} r_{\pm}(H)e^{-itH}v + o(1), \quad t \rightarrow \pm\infty.$$

Next we want to study the invertibility of the wave operator $W_{\pm}(H, H_0)$. We say that $W_{\pm}(H, H_0)$ is asymptotically complete if its range is closed. In particular, if the wave operator $W_{\pm}(H, H_0)$ is asymptotically complete, then it is invertible in $\mathcal{B}(\mathcal{H}, \mathcal{H}_{\mathfrak{p}}(H^*)^{\perp})$. The next theorem give sufficient conditions for asymptotic completeness.

Theorem 3.3.4. *Suppose that Hypotheses 3.1-3.4 hold and that H does not have any spectral singularity. Then the wave operators $W_{\pm}(H, H_0)$ are asymptotically complete.*

A similar result is proved in [35] for dissipative operators. In particular, the previous theorem implies that, if H does not have any spectral singularity, then H and H_0 are similar in the sense that

$$\forall u \in \mathcal{H}_{\mathfrak{p}}(H^*)^{\perp} \cap \mathcal{D}(H_0), \quad H_0u = W_{\pm}(H, H_0)^{-1}HW_{\pm}(H, H_0)u.$$

Another consequence of Theorem 3.3.4 is the following result.

Theorem 3.3.5. *Suppose that Hypotheses 3.1-3.4 hold. Suppose that H does not have any spectral singularities. Then there exist $m_1 > 0$, and $m_2 > 0$ such that*

$$\forall t \in \mathbb{R}, \quad \forall u \in \mathcal{H}_{\mathfrak{p}}(H^*)^{\perp}, \quad m_1 \|u\|_{\mathcal{H}} \leq \|e^{-itH}u\|_{\mathcal{H}} \leq m_2 \|u\|_{\mathcal{H}}.$$

The second inequality shows that solutions of the Schrödinger equation

$$\begin{cases} i\partial_t u_t = H u_t \\ u_0 \in \mathcal{H}_{\mathfrak{p}}(H^*)^{\perp} \end{cases}$$

cannot blow up as $t \rightarrow \pm\infty$ and that the norm of u_t is controlled by the norm of the initial state u_0 . Related results have been established in [46] for non-self-adjoint Schrödinger operators and in [35] for abstract dissipative operators.

Finally, using the definition of spectral projection associated to H in an interval $I \subset \sigma_{\text{ess}}(H)$ without spectral singularities (see (3.26) below), we can define the local wave operators associated to H and H_0 on I by

$$W_{\pm}(H, H_0, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \mathbf{1}_I(H) e^{-itH_0}.$$

The next theorem then shows that $W_{\pm}(H, H_0, I)$ are asymptotically complete in the sense that the range of $W_{\pm}(H, H_0, I)$ is equal to the range of $\mathbf{1}_I(H)$.

Theorem 3.3.6. *Suppose that Hypotheses 3.1-3.3 hold. Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval without any spectral singularity. Then $W_{\pm}(H, H_0, I)$ is invertible in $\mathcal{B}(\text{Ran}(\mathbf{1}_I(H_0)), \text{Ran}(\mathbf{1}_I(H)))$.*

Similar results have been obtained in [35] in abstract dissipative scattering theory, and in [54] for a class of non-self-adjoint Schrödinger operators.

3.3.3 Applications to Schrödinger operators

In this section we will show that our results apply to non-self-adjoint Schrödinger operators. We consider

$$\mathcal{H} = L^2(\mathbb{R}^d, \mathbb{C}), \quad d \geq 3 \text{ odd}, \quad H_0 = -\Delta, \quad \mathcal{D}(H_0) = H^2(\mathbb{R}^d, \mathbb{C}),$$

where $H^2(\mathbb{R}^d, \mathbb{C})$ is the usual Sobolev's space. It is well-known that $\sigma_{\text{ess}}(H) = [0, \infty)$ and that

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \|\langle x \rangle^{-\delta} (-\Delta - z)^{-1} \langle x \rangle^{-\delta}\|_{\mathcal{B}(\mathcal{H})} < \infty, \quad \delta > 1,$$

where $\langle x \rangle = (1 + x^2)^{1/2}$. Hence choosing $C(x) = \langle x \rangle^{-\delta}$ with $\delta > 1$, we see that Hypothesis 3.1 is satisfied.

We suppose that

$$V \in L_c^\infty(\mathbb{R}^d, \mathbb{C}) = \{u \in L^\infty(\mathbb{R}^d) \text{ compactly supported}\}. \quad (3.23)$$

It then follows from [43] that H has only a finite number of discrete eigenvalues with finite algebraic multiplicities. In particular, Hypothesis 3.2 is satisfied.

To verify that Hypothesis 3.3 holds, we rely on the theory of resonances. Resonances are defined as poles of the meromorphic extension of the weighted resolvent, see e.g [25]. More precisely, assuming (3.23), the map

$$\{z \in \mathbb{C}, \text{Im}(z) > 0\} \ni z \mapsto (H - z^2)^{-1} : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$$

is meromorphic and extends to a meromorphic map

$$\mathbb{C} \ni z \mapsto (H - z^2)^{-1} : L_c^2(\mathbb{R}^d) \rightarrow L_{\text{loc}}^2(\mathbb{R}^d),$$

where where $L_c^2(\mathbb{R}^d) := \{u \in L^2(\mathbb{R}^d), u \text{ is compactly supported}\}$ and $L_{\text{loc}}^2(\mathbb{R}^d) := \{u : \mathbb{R}^d \rightarrow \mathbb{C}, u \in L^2(K) \text{ for all compact set } K \subset \mathbb{R}^d\}$. Poles of the extension of the resolvent of H are called resonances. To a real resonance $\pm\lambda_0$ with $\lambda_0 \geq 0$ corresponds an outgoing or/and an incoming spectral singularity λ_0^2 in the sense of Definition 3.3.1. The order of the pole of the resonance coincides with the order of the spectral singularity. Hence, if for example λ_0 is an outgoing spectral singularity, there exists \mathcal{V}_{λ_0} a closed real neighborhood of λ_0 whose interior contains λ_0 and no other resonance, such that

$$\sup_{\substack{\text{Re}(z) \in \mathcal{V}_\lambda \\ \text{Im}(z) < \varepsilon_0}} |\lambda_0 - z|^{\nu_0} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty$$

where ν_0 is the order of the pole of λ_0^2 and $\varepsilon_0 > 0$ is small enough. Furthermore, under the assumption (3.23), H only has finitely many spectral singularities $\{\lambda_1, \dots, \lambda_n\}$ of finite orders, and no spectral singularity at infinity. This shows that Hypothesis 3.3 is satisfied. We refer to [25] for more details about the theory of resonances and [35] for more details about the relation between resonances and spectral singularities.

Applying Theorems 3.3.3, 3.3.4 and 3.3.5, we obtain the following proposition.

Proposition 3.3.7. *Suppose that $V \in L_c^\infty(\mathbb{R}^d, \mathbb{C})$ and that for each eigenvalue embedded in the essential spectrum of H , the symmetric bilinear form*

$$\text{Ker}((H - \lambda)^{m_\lambda}) \ni (u, v) \mapsto \int_{\mathbb{R}^d} u(x)v(x)dx \quad (3.24)$$

is non-degenerate. Then the wave operators $W_\pm(H, H_0)$ associated to $H = -\Delta + V$ and $H_0 = -\Delta$ exist, are injective and their ranges are dense in $\mathcal{H}_p(H)^\perp$.

Moreover if H does not have any real resonance, then $W_\pm(H, H_0)$ are asymptotically complete and there exist $m_1 > 0$ and $m_2 > 0$ such that, for all $u \in \mathcal{H}_p(H)^\perp$,

$$\forall t \in \mathbb{R}, \quad m_1 \|u\|_{\mathcal{H}} \leq \|e^{itH}u\|_{\mathcal{H}} \leq m_2 \|u\|_{\mathcal{H}}.$$

Note that if H has no embedded eigenvalue, then the condition (3.24) is not necessary. We mention that the condition (3.24) has been considered in [89] and [1], to study the long-time behavior of solutions to the Schrödinger equation with a complex potential.

Finally we apply Theorem 3.3.6.

Proposition 3.3.8. *Suppose that $V \in L_c^\infty(\mathbb{R}^d, \mathbb{C})$. Let $I \subset [0, \infty)$ be a closed interval without any resonance. Then the local wave operators $W_\pm(H, H_0, I)$ exist and are asymptotically complete.*

3.3.4 Organization of the paper and main ingredients

In this section we describe some of the main tools we will use in this paper.

To prove the existence of the wave operators $W_\pm(H, H_0)$, it is useful to characterize the spectral subspace $\mathcal{H}_p(H^*)^\perp$. We define the absolutely continuous spectral subspace of H as the closure of $\mathcal{M}(H)$ in \mathcal{H} , where

$$\mathcal{M}(H) := \left\{ u \in \mathcal{H}, \forall v \in \mathcal{H}, \int_{\mathbb{R}} |\langle e^{itH}u, v \rangle_{\mathcal{H}}|^2 dt \leq c_u \|v\|_{\mathcal{H}}^2 \right\}. \quad (3.25)$$

It follows from chapter 2 that, under our assumptions, $\mathcal{H}_{ac}(H)$ coincide with $\mathcal{H}_p(H^*)^\perp$. We will see that there exists $c > 0$ such that, for all $u \in \mathcal{H}_{ac}(H)$,

$$\int_0^\infty \|Cr_\pm(H)e^{\mp itH}u\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2.$$

The existence of the wave operators will then follow from this property, the equality $\mathcal{H}_{ac}(H) = \mathcal{H}_p(H^*)^\perp$ and Cook's method (see [76] or [94]).

The properties concerning the kernel and the range of the regularized wave operators mainly follow from the direct sum decomposition,

$$\mathcal{H}_{ac}(H) \oplus \mathcal{H}_p(H) = \mathcal{H},$$

given in theorem 2.3.3. This is a generalization of the well-known spectral decomposition for self-adjoint operators without singular continuous spectrum. Note that this direct sum is J -orthogonal.

If $I \subset \sigma_{\text{ess}}(H)$ is a closed interval without any spectral singularity, we can define the spectral projection associated to H in I by setting

$$\mathbb{1}_I(H) := \frac{1}{2\pi i} \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (3.26)$$

It follows from section 2.5 that, under our assumptions, (3.26) exists and is a projection. Moreover (3.26) induces a bounded functional calculus in the sense that

$$\forall t \in \mathbb{R}, \quad e^{itH} \mathbb{1}_I(H) := \frac{1}{2\pi i} \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I e^{it\lambda} (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (3.27)$$

In particular, $\{e^{itH}\}_{t \in \mathbb{R}}$ is uniformly bounded on the range of $\mathbb{1}_I(H)$. Such spectral projections for non self-adjoint operators have been considered by several authors, see [35] for dissipative operators, [47] and [48] for general non-self-adjoint operators, [54] for wave operators using a stationary approach and [69] and [71] for differential operators.

If $I \subset \sigma_{\text{ess}}(H)$ is a closed interval whose interior contains spectral singularities, then we define the ‘regularized spectral projection’ associated to H in I by setting

$$(h\mathbb{1}_I)(H) := \frac{1}{2\pi i} \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I (h(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (3.28)$$

where h is a function that regularizes the singularity of the resolvent at the spectral singularity. More precisely, assuming in particular Hypothesis 3.3, h is chosen as

$$h(z) := \prod_{j \in \mathcal{S}_I} r_j(z), \quad \mathcal{S}_I := \{j \in \llbracket 1, n \rrbracket \cup \{\infty\}, \lambda_j \in I \text{ is a spectral singularity}\}, \quad (3.29)$$

where r_j are defined in (3.19)–(3.20). Similarly as above, under our assumptions, (3.28) exists and defines a Borel functional calculus in the sense that for all $t \in \mathbb{R}$,

$$e^{itH} (h\mathbb{1}_I)(H) := \frac{1}{2\pi i} \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I e^{it\lambda} (h(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - h(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)) d\lambda. \quad (3.30)$$

See section 2.5.2. In particular, $\{e^{itH}\}_{t \in \mathbb{R}}$ is uniformly bounded on the range of $(h\mathbb{1}_I)(H)$.

Finally, we will use the following identity (see proposition 2.3.10) which generalizes the well-known resolution of identity for self-adjoint operators to our context:

$$\begin{aligned} r(H) &= r(H)\Pi_{\text{disc}}(H) \\ &+ \frac{1}{2\pi i} \text{w-lim}_{\varepsilon \rightarrow \infty} \int_{\sigma_{\text{ess}}(H)} (r(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - r(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)) d\lambda, \end{aligned} \quad (3.31)$$

where r is the regularizing function of Hypothesis 3.3.

3.4 The Wave operators

3.4.1 Preliminary results

We begin with preliminary results that will allow us to prove the existence and study the wave operators in the sequel.

Proposition 3.4.1. *Suppose that Hypothesis 3.1 holds. Let $A \in \mathcal{B}(\mathcal{H})$ be such that there exists $c > 0$ satisfying*

$$\forall u \in \mathcal{H}, \quad \int_0^\infty \|C e^{\pm itH^*} A^* u\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2.$$

Then the strong limits

$$\text{s-lim}_{t \rightarrow \pm\infty} A e^{itH} e^{-itH_0} \quad \text{and} \quad \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_0} e^{-itH^*} A^*$$

exist.

We can state a second version of the last proposition.

Proposition 3.4.2. *Suppose that Hypothesis 3.1 holds. Let $A \in \mathcal{B}(\mathcal{H})$ be such that there exists $c > 0$ satisfying*

$$\forall u \in \mathcal{H}, \quad \int_0^\infty \|C e^{\pm itH} A u\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2.$$

Then the strong limits

$$\text{s-lim}_{t \rightarrow \pm\infty} e^{itH_0} e^{-itH} A \quad \text{and} \quad \text{s-lim}_{t \rightarrow \pm\infty} A e^{itH^*} e^{-itH_0}$$

exist.

The proofs of Propositions 3.4.1 and 3.4.2 are standard and based on Cook's method (see e.g. [76] or [94]). For the convenience of the reader, we give a proof of Proposition 3.4.1 in Appendix 3.6.

3.4.2 The regularized wave operators $W_{\pm}(H, H_0)$ and $W_{\pm}(H^*, H_0)$

We recall the following definition. Note that $\Pi_{\text{ac}}(H)$ is the projection onto the absolutely continuous spectral subspace of H defined in section 3.3.4.

Definition 3.4.3. *The regularized wave operators associated to H and H_0 (respectively to H^* and H_0) are defined by*

$$\begin{aligned} W_{\pm}(H, H_0) &:= \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \Pi_{\text{ac}}(H) r_{\mp}(H) e^{-itH_0}, \\ W_{\pm}(H^*, H_0) &:= \text{s-lim}_{t \rightarrow \pm\infty} e^{itH^*} \Pi_{\text{ac}}(H) \bar{r}_{\mp}(H^*) e^{-itH_0}. \end{aligned}$$

In this section we prove the existence and study the properties of these wave operators. However, since Definition 3.4.3 is not standard, we begin with a few remarks justifying it.

3.4.2.1 Remarks on Definition 3.4.3 of the regularized wave operators

To fix the ideas, we consider here the wave operator $W_-(H, H_0)$. In the usual setting, when H and H_0 are self-adjoint operators acting on the same Hilbert space and when H_0 has purely absolutely continuous spectrum, the common definition of the wave operators $W_-(H, H_0)$ is

$$W_-(H, H_0) := \text{s-lim}_{t \rightarrow -\infty} e^{itH} e^{-itH_0}. \quad (3.32)$$

Assuming for simplicity that H has finitely many eigenvalues with finite multiplicities and no singular continuous spectrum, we can write, for all $t \geq 0$ and $u \in \mathcal{H}$,

$$e^{itH} e^{-itH_0} u = e^{itH} \Pi_{\text{ac}}(H) e^{-itH_0} u + e^{itH} \Pi_{\text{p}}(H) e^{-itH_0} u,$$

and since $\Pi_{\text{p}}(H)$ is compact and H_0 has purely absolutely continuous spectrum, it is well-known that

$$\Pi_{\text{p}}(H) e^{-itH_0} u \xrightarrow[t \rightarrow -\infty]{} 0 \quad (3.33)$$

(see e.g [94, Lemma 1.4.1]). Hence proving the existence of (3.32) is equivalent to proving the existence of

$$\text{s-lim}_{t \rightarrow -\infty} e^{itH} \Pi_{\text{ac}}(H) e^{-itH_0}.$$

In our context, however, the group $\{e^{itH}\}$ is not uniformly bounded ($\|e^{itH} \Pi_{\text{p}}(H)\|_{\mathcal{B}(\mathcal{H})}$ may even blow up exponentially as $t \rightarrow -\infty$). Therefore (3.33) fails in general.

Still, a natural definition in our context might be

$$W_-(H, H_0) := \text{s-lim}_{t \rightarrow -\infty} e^{itH} \Pi_{\text{ac}}(H) e^{-itH_0}. \quad (3.34)$$

The next proposition however shows that if H has an outgoing spectral singularity, then C is not locally relatively smooth with respect to H , and hence the usual Cook criterion do not apply to prove the existence of (3.34). The purpose of adding a regularizing operator in the definition of $W_-(H, H_0)$ (see Definition 3.4.3) is precisely to overcome this issue.

Proposition 3.4.4. *Suppose that the assumptions 3.1-3.4 hold. Suppose there exists a compact interval of $J \subset \sigma_{\text{ess}}(H)$ without eigenvalue of H and $u \in \mathcal{H}$ such that*

$$\lim_{\varepsilon \rightarrow 0^+} \int_J \|C\mathcal{R}_H(\lambda - i\varepsilon)CWu\|_{\mathcal{H}}^2 d\lambda = \infty.$$

Then, C is not H -smooth in the interval J .

Proof. Let $v = CWu$. Parseval's theorem gives

$$\int_0^\infty \|C e^{itH} \Pi_{\text{ac}}(H) v\|_{\mathcal{H}}^2 dt = \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{\sigma_{\text{ess}}(H)} \|C\mathcal{R}_H(\lambda - i\varepsilon) \Pi_{\text{ac}}(H) v\|_{\mathcal{H}}^2 d\lambda.$$

Fixing $\varepsilon > 0$ small enough, we have

$$\begin{aligned} \int_{\sigma_{\text{ess}}(H)} \|C\mathcal{R}_H(\lambda - i\varepsilon)\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda &\geq \int_J \|C\mathcal{R}_H(\lambda - i\varepsilon)\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ &\geq \frac{1}{2} \int_J \|C\mathcal{R}_H(\lambda - i\varepsilon)v\|_{\mathcal{H}}^2 d\lambda - \frac{1}{2} \int_J \|C\mathcal{R}_H(\lambda - i\varepsilon)\Pi_{\text{p}}(H)v\|_{\mathcal{H}}^2 d\lambda. \end{aligned}$$

As J does not contain any eigenvalue of H ,

$$\lim_{\varepsilon \rightarrow 0^+} \int_J \|C\mathcal{R}_H(\lambda - i\varepsilon)\Pi_{\text{p}}(H)v\|_{\mathcal{H}}^2 d\lambda$$

exists in \mathcal{H} . Moreover, by assumption

$$\lim_{\varepsilon \rightarrow 0^+} \int_J \|C\mathcal{R}_H(\lambda - i\varepsilon)v\|_{\mathcal{H}}^2 d\lambda = \int_J \|C\mathcal{R}_H(\lambda - i\varepsilon)CWu\|_{\mathcal{H}}^2 d\lambda = \infty.$$

This prove that C is not H -smooth. □

Based on the previous proposition, we believe that, in general, the limit (3.34) does not exist. Another argument supporting our Definition 3.4.3 come from the properties proven later in this section, namely that $W_{\pm}(H, H_0)$ exist, are injective and with ranges dense in $\mathcal{H}_{\text{ac}}(H)$.

To conclude this subsection, we underline that we could also have defined the wave operators by using the regularizing operator $r(H)$ instead of $r_{\pm}(H)$ in Definition 3.4.3. The advantage would have been to be able to rely on the functional calculus of section 2.5 to prove the existence of the wave operators. See the discussion about local wave operators in Section 3.4.4 for related arguments proving the existence of wave operators using the functional calculus. The ‘inconvenience’ of using $r(H)$ instead of $r_{\pm}(H)$ in Definition 3.4.3 is that ‘a part of’ $r(H)$ is superfluous for the strong limit to exist, which may have the effect of somehow projecting out scattering states from the range of the wave operators.

We mention that such definitions of wave operators with an ‘identification’ operator appearing between the two evolution groups has been considered by Kato in [57], in a general setting where the evolution groups act on different Hilbert spaces. In the unitary case, various properties of the wave operators depending on the assumptions made on the identification operator are given in [94].

3.4.2.2 Existence of $W_{\pm}(H, H_0)$ and $W_{\pm}(H^*, H_0)$

Now we turn to the proof of the existence of the regularized wave operators $W_{\pm}(H, H_0)$ and $W_{\pm}(H^*, H_0)$. It suffices to apply Propositions 3.4.1 and 3.4.2 with $A = r_{\mp}(H)\Pi_{\text{ac}}(H)$. The following lemma shows that the conditions of Propositions 3.4.1 and 3.4.2 are indeed satisfied.

Lemma 3.4.5. *Suppose that Hypotheses 3.1-3.4 hold. Then there exists $c_0 > 0$ such that, for all $u \in \mathcal{H}$,*

$$\int_0^\infty \|Cr_{\mp}(H)e^{\pm itH}\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 dt \leq c_0 \|u\|_{\mathcal{H}}^2, \quad (3.35)$$

and

$$\int_0^\infty \|C\bar{r}_{\mp}(H^*)e^{\pm itH^*}\Pi_{\text{ac}}(H^*)u\|_{\mathcal{H}}^2 dt \leq c_0 \|u\|_{\mathcal{H}}^2. \quad (3.36)$$

Lemma 3.4.5 proves that $Cr_-(H)$ is locally smooth with respect to H in the sense of Kato in $[0, \infty)$, and that $Cr_+(H)$ is locally smooth with respect to H in the sense of Kato in $(-\infty, 0]$. Note also that if H does not have any spectral singularity, then $r_{\pm}(H) = \text{Id}$ and this lemma shows that C is globally smooth with respect to H .

The proof of Lemma 3.4.5 is based on the following observation. If λ_j is a spectral singularity of order 1, then, for all $z \in \rho(H) \setminus \{z_0\}$, if $j \neq \infty$,

$$\begin{aligned} r_j(H) - r_j(z) &= (H - \lambda_j)\mathcal{R}_H(z_0) - (z - \lambda_j)(z - z_0)^{-1} \\ &= (H - z)\mathcal{R}_H(z_0)(\lambda_j - z_0)(z - z_0)^{-1}. \end{aligned}$$

Likewise, if $j = \infty$,

$$\begin{aligned} r_\infty(H) - r_\infty(z) &= \mathcal{R}_H(z_0) - (z - z_0)^{-1} \\ &= (H - z)\mathcal{R}_H(z_0)(z - z_0)^{-1}. \end{aligned}$$

Next, if λ_j is a spectral singularity of order μ_j , then, if $j \neq \infty$,

$$\begin{aligned} r_j(H) - r_j(z) &= ((H - \lambda_j)\mathcal{R}_H(z_0))^{\nu_j} - ((\lambda_j - z)(\lambda_j - z_0)^{-1})^{\nu_j} \\ &= \left((H - z)\mathcal{R}_H(z_0) \left(\frac{\lambda_j - z_0}{z - z_0} \right) \right)^{\nu_j} \left(\sum_{k=0}^{\nu_j-1} ((H - \lambda_j)\mathcal{R}_H(z_0))^k \left(\frac{z - \lambda_j}{z - z_0} \right)^{\nu_j-1-k} \right), \end{aligned} \quad (3.37)$$

while if $j = \infty$,

$$r_\infty(H) - r_\infty(z) = ((H - z)\mathcal{R}_H(z_0)(z - z_0)^{-1})^{\nu_\infty} \left(\sum_{k=0}^{\nu_\infty-1} \mathcal{R}_H(z_0)^k (z - z_0)^{k+1-\nu_\infty} \right).$$

Now we are ready to prove Lemma 3.4.5.

Proof of Lemma 3.4.5. Let $u \in \mathcal{H}$. We will prove that there exists $c > 0$ such that

$$\int_0^\infty \|Cr_+(H)e^{-itH}\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2. \quad (3.38)$$

The other inequalities can be obtained in the same way. Parseval's identity gives

$$\int_0^\infty \|Cr_+(H)e^{-itH}\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 dt = \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}} \|Cr_+(H)\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 d\lambda$$

Let n_+ denote the number of incoming spectral singularities. We partition $\sigma_{\text{ess}}(H)$ into a finite union of $n_+ + 2$ interval J_j 's such that i) J_0 is an interval of the form $(-\infty, a]$ with $a < 0$, $|a| \geq \|V\|_{\mathcal{B}(\mathcal{H})} + 1$, ii) for all $j \in \llbracket 1, n_+ \rrbracket$, J_j is a compact interval whose interior contains the incoming spectral singularity λ_j and no other incoming spectral singularities and iii) J_{n_++1} is an interval of the form $[b, \infty)$, with b large enough, containing a possible singularity at infinity. We then obtain that, for all $\varepsilon > 0$ small enough,

$$\int_{\mathbb{R}} \|Cr_+(H)\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 d\lambda = \sum_{j=0}^{n_++1} \int_{J_j} \|Cr_+(H)\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 d\lambda.$$

Therefore, to prove (3.38), it suffices to prove that for all $j \in \llbracket 0, n_+ + 1 \rrbracket$, there exists $c_j > 0$ such that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{J_j} \|Cr_+(H)\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 d\lambda \leq c_j \|u\|_{\mathcal{H}}^2. \quad (3.39)$$

First, we estimate the integral over J_0 . As H_0 is self-adjoint and $J_0 \subset (-\infty, 0)$, we remark that for all $\lambda \in J_0$, for all $\varepsilon > 0$ small enough,

$$\|\mathcal{R}_0(\lambda + i\varepsilon)\| \leq \frac{1}{\text{dist}(\lambda + i\varepsilon, \sigma(H_0))} \leq \frac{1}{|\lambda + i\varepsilon|}.$$

Next, using the resolvent identity, we have

$$\begin{aligned} \|\mathcal{R}_H(\lambda + i\varepsilon)\|_{\mathcal{H}} &\leq \|\mathcal{R}_0(\lambda + i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} + \|\mathcal{R}_0(\lambda + i\varepsilon)V\mathcal{R}_H(\lambda + i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \\ &\leq \frac{1}{|\lambda + i\varepsilon|} + \|V\|_{\mathcal{B}(\mathcal{H})} \frac{1}{|\lambda + i\varepsilon|} \|\mathcal{R}_H(\lambda + i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \\ &\leq \frac{1}{|\lambda + i\varepsilon|} + \|V\|_{\mathcal{B}(\mathcal{H})} \frac{1}{|a| - |\varepsilon|} \|\mathcal{R}_H(\lambda + i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \end{aligned}$$

As $\|V\|_{\mathcal{B}(\mathcal{H})} (|a| - |\varepsilon|)^{-1} < 1$ we have that for all $\lambda \in J_0$, for all $\varepsilon > 0$ small enough,

$$\|\mathcal{R}_H(\lambda + i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \leq \frac{b}{|\lambda + i\varepsilon|},$$

for some positive constant b . Therefore we deduce that there exists $c_0 > 0$ such that

$$\int_{J_0} \|C\mathcal{R}_H(\lambda + i\varepsilon)r(H)\Pi_{\text{ac}}(H)u\|^2 d\lambda \leq c_0 \int_{J_0} \frac{1}{|\lambda + i\varepsilon|^2} d\lambda \|u\|_{\mathcal{H}}^2.$$

As J_0 is a closed interval not containing 0, Lebesgue's dominated theorem shows that

$$\lim_{\varepsilon \rightarrow 0^+} \int_{J_0} \frac{1}{|\lambda + i\varepsilon|^2} d\lambda$$

exists. This proves (3.39) for $j = 0$.

Now let $j \in \llbracket 1, n_+ + 1 \rrbracket$. Writing

$$\tilde{r}_j(H) := \prod_{\substack{k=1 \\ k \neq j}}^{n_++1} r_k(H) \in \mathcal{B}(\mathcal{H}),$$

and setting $v = \tilde{r}_j(H)u$, we have

$$\begin{aligned} \int_{J_j} \|Cr_+(H)\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)u\|_{\mathcal{H}}^2 d\lambda &= \int_{J_j} \|Cr_j(H)\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ &\leq 2 \int_{J_j} \|C(r_j(H) - r_j(\lambda + i\varepsilon))\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \end{aligned} \quad (3.40)$$

$$+ 2 \int_{J_j} |r_j(\lambda + i\varepsilon)|^2 \|C\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda. \quad (3.41)$$

To estimate (3.40), we use (3.37): for all $j \in \llbracket 1, n_+ \rrbracket$, we have

$$\begin{aligned} &\int_{J_j} \|C(r_j(H) - r_j(\lambda + i\varepsilon))\mathcal{R}_H(\lambda + i\varepsilon)\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ &= \int_{J_j} \left\| C \left(\mathcal{R}_H(z_0) \left(\frac{\lambda_j - z_0}{\lambda + i\varepsilon - z_0} \right) \right)^{\nu_j} A(\lambda + i\varepsilon)(H - (\lambda + i\varepsilon))^{\nu_j - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}}^2 d\lambda, \end{aligned}$$

where

$$A(\lambda + i\varepsilon) := \left(\sum_{k=0}^{\nu_j - 1} ((H - \lambda_j)\mathcal{R}_H(z_0))^k \left(\frac{\lambda + i\varepsilon - \lambda_j}{\lambda + i\varepsilon - z_0} \right)^{\nu_j - 1 - k} \right).$$

Clearly, for all $\lambda \in J_j$,

$$\lim_{\varepsilon \rightarrow 0^+} C \left(\mathcal{R}_H(z_0) \left(\frac{\lambda_j - z_0}{\lambda + i\varepsilon - z_0} \right) \right)^{\nu_j} A(\lambda + i\varepsilon)(H - (\lambda + i\varepsilon))^{\nu_j - 1} \Pi_{\text{ac}}(H)v,$$

exists in \mathcal{H} . We claim that there exists $\varepsilon_0 > 0$ such that

$$\sup_{\lambda \in J_j, \varepsilon < \varepsilon_0} \left\| C \left(\mathcal{R}_H(z_0) \left(\frac{\lambda_j - z_0}{\lambda + i\varepsilon - z_0} \right) \right)^{\nu_j} A(\lambda + i\varepsilon)(H - (\lambda + i\varepsilon))^{\nu_j - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}} < \infty. \quad (3.42)$$

Indeed, it's clear that there exists $\varepsilon_1 > 0$ such that

$$\sup_{\lambda \in J_j, 0 < \varepsilon < \varepsilon_1} \|A(\lambda + i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

A direct computation shows that $\lambda \mapsto \mathcal{R}_H(z_0)(H - \lambda + i\varepsilon)$ is uniformly bounded in $\mathcal{B}(\mathcal{H})$ on J_j with a bound independant of ε , thus there exist $b_j > 0$ and $\varepsilon_2 > 0$ such that

$$\sup_{\lambda \in J_j, 0 < \varepsilon < \varepsilon_2} \left| \frac{\lambda_j - z_0}{\lambda + i\varepsilon - z_0} \right|^{\nu_j} \|C\mathcal{R}_H(z_0)^{\nu_j}(H - (\lambda + i\varepsilon))^{\nu_j - 1} \Pi_{\text{ac}}(H)v\|_{\mathcal{H}} < b_j \|v\|_{\mathcal{H}}.$$

So we get (3.42). Finally, using dominated convergence theorem, the map

$$J_j \ni \lambda \mapsto C \left(\mathcal{R}_H(z_0) \left(\frac{\lambda_j - z_0}{\lambda - z_0} \right) \right)^{\nu_j} A(\lambda)(H - \lambda)^{\nu_j - 1} \Pi_{\text{ac}}(H)v$$

is in $L^2(J_j, \mathcal{H})$ and there exists $a_j > 0$ such that

$$\begin{aligned} & \int_{J_j} \left\| C \left(\mathcal{R}_H(z_0) \left(\frac{\lambda_j - z_0}{\lambda - z_0} \right) \right)^{\nu_j} A(\lambda)(H - \lambda)^{\nu_j - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}}^2 d\lambda \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{J_j} \left\| C \left(\mathcal{R}_H(z_0) \left(\frac{\lambda_j - z_0}{\lambda + i\varepsilon - z_0} \right) \right)^{\nu_j} A(\lambda + i\varepsilon)(H - (\lambda + i\varepsilon))^{\nu_j - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}}^2 d\lambda \\ &\leq a_j \|u\|_{\mathcal{H}}^2. \end{aligned} \quad (3.43)$$

where in the last inequality we used that $v = \tilde{r}_j(H)u$ and that $r_j(H)$ is bounded. For the integral over J_{n+1} we have

$$\begin{aligned} & \int_{J_{n+1}} \|C(r_\infty(H) - r_\infty(\lambda + i\varepsilon)) \mathcal{R}_H(\lambda + i\varepsilon) \Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ &= \int_{J_{n+1}} \left\| CA(\lambda + i\varepsilon) (\mathcal{R}_H(z_0) (\lambda + i\varepsilon - z_0)^{-1})^{\nu_\infty} (H - (\lambda + i\varepsilon))^{\nu_\infty - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}}^2 d\lambda, \end{aligned}$$

where

$$A(\lambda + i\varepsilon) := \left(\sum_{k=0}^{\nu_\infty - 1} \mathcal{R}_H(z_0)^k (\lambda + i\varepsilon - z_0)^{k+1-\nu_\infty} \right).$$

It is clear that for all $j \in J_{n+1}$,

$$\lim_{\varepsilon \rightarrow 0^+} CA(\lambda + i\varepsilon) (\mathcal{R}_H(z_0) (\lambda + i\varepsilon - z_0)^{-1})^{\nu_\infty} (H - (\lambda + i\varepsilon))^{\nu_\infty - 1} \Pi_{\text{ac}}(H)v$$

exists in \mathcal{H} . Moreover, there exists $\varepsilon_1 > 0$ such that for all $\lambda \in J_{n+1}$, for all $\varepsilon > 0$ smaller than ε_1 ,

$$\|CA(\lambda + i\varepsilon)\|_{\mathcal{B}(\mathcal{H})} \leq \left(\nu_\infty \sup_{k \in \llbracket 0, \nu_\infty - 1 \rrbracket} \left(|\lambda + i\varepsilon_1 - z_0|^{k+1-\nu_\infty} \|C\mathcal{R}_H(z_0)\|_{\mathcal{B}(\mathcal{H})}^k \right) \right) < \infty,$$

and a direct computation shows that

$$\mathcal{R}_H(z_0)(H - (\lambda + i\varepsilon)) = \text{Id} + \mathcal{R}_H(z_0)(z_0 - (\lambda + i\varepsilon)).$$

So for all ε smaller than ε_1 ,

$$\begin{aligned} & \left\| (\mathcal{R}_H(z_0) (\lambda + i\varepsilon - z_0)^{-1})^{\nu_\infty} (H - (\lambda + i\varepsilon))^{\nu_\infty - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}} \\ &\leq \frac{\|\mathcal{R}_H(z_0)\|_{\mathcal{B}(\mathcal{H})}}{|\lambda + i\varepsilon_1 - z_0|^{\nu_\infty}} \left(1 + \|\mathcal{R}_H(z_0)\|_{\mathcal{B}(\mathcal{H})} |z_0 - (\lambda + i\varepsilon_1)| \right)^{\nu_\infty - 1} \|\Pi_{\text{ac}}(H)v\|_{\mathcal{H}} \\ &= \mathcal{O}(\lambda^{-1}), \quad (\lambda \rightarrow \infty). \end{aligned}$$

Thus the map

$$J_{n_++1} \ni \lambda \mapsto \frac{\|\mathcal{R}_H(z_0)\|_{\mathcal{B}(\mathcal{H})}}{|(\lambda + i\varepsilon_1 - z_0)|^{\nu_\infty}} \left(1 + \|\mathcal{R}_H(z_0)\|_{\mathcal{B}(\mathcal{H})} |z_0 - (\lambda + i\varepsilon_1)|\right)^{\nu_\infty - 1} \|\Pi_{\text{ac}}(H)v\|_{\mathcal{H}},$$

is in $L^2(J_{n_++1}, \mathcal{H})$. Finally with Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon \rightarrow 0^+} \int_{J_{n_++1}} \|C(r_\infty(H) - r_\infty(\lambda + i\varepsilon)) \mathcal{R}_H(\lambda + i\varepsilon) \Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda$$

exists and there exists $a_\infty > 0$ such that

$$\begin{aligned} & \int_{J_{n_++1}} \left\| CA(\lambda) (\mathcal{R}_H(z_0) (\lambda - z_0)^{-1})^{\nu_\infty} (H - \lambda)^{\nu_\infty - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}}^2 d\lambda \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{J_{n_++1}} \left\| CA(\lambda + i\varepsilon) (\mathcal{R}_H(z_0) (\lambda + i\varepsilon - z_0)^{-1})^{\nu_\infty} (H - (\lambda + i\varepsilon))^{\nu_\infty - 1} \Pi_{\text{ac}}(H)v \right\|_{\mathcal{H}}^2 d\lambda \\ &\leq a_\infty \|u\|_{\mathcal{H}}^2. \end{aligned} \quad (3.44)$$

It remains to estimate (3.41). We use again the resolvent identity. For all $j \in \llbracket 1, n_+ + 1 \rrbracket$, and for all $\varepsilon > 0$ small enough, we have

$$\begin{aligned} & \int_{J_j} |r_j(\lambda + i\varepsilon)|^2 \|C\mathcal{R}_H(\lambda + i\varepsilon) \Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ &= \int_{J_j} |r_j(\lambda + i\varepsilon)|^2 \|C\mathcal{R}_0(\lambda + i\varepsilon) \Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ & \quad + \int_{J_j} |r_j(\lambda + i\varepsilon)|^2 \|C\mathcal{R}_H(\lambda + i\varepsilon) CW C\mathcal{R}_0(\lambda + i\varepsilon) \Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda. \end{aligned}$$

As for all $j \in \llbracket 1, n_+ + 1 \rrbracket$,

$$\sup_{\substack{\lambda \in J_j \\ 0 < \varepsilon < \varepsilon_0}} |r_j(\lambda + i\varepsilon)| \|C\mathcal{R}_H(\lambda + i\varepsilon) CW\|_{\mathcal{B}(\mathcal{H})} < \infty$$

by Hypothesis 3.3, and since C is relatively smooth with respect to H_0 , there exists $b_j > 0$ such that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \int_{J_j} |r_j(\lambda + i\varepsilon)|^2 \|C\mathcal{R}_0(\lambda + i\varepsilon) \Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ & \quad + \lim_{\varepsilon \rightarrow 0^+} \int_{J_j} |r_j(\lambda + i\varepsilon)|^2 \|C\mathcal{R}_H(\lambda + i\varepsilon) CW C\mathcal{R}_0(\lambda + i\varepsilon) \Pi_{\text{ac}}(H)v\|_{\mathcal{H}}^2 d\lambda \\ & \leq b_j \|u\|_{\mathcal{H}}^2. \end{aligned} \quad (3.45)$$

Combining (3.43), (3.44) and (3.45), we obtain (3.38), which concludes the proof of the lemma. \square

Proposition 3.4.6. *Suppose that Hypotheses 3.1-3.4 hold. Then the wave operators $W_\pm(H, H_0)$ and $W_\pm(H^*, H_0)$ exist.*

Proof. It suffices to remark that $\Pi_{\text{ac}}(H)r_\pm(H)$ commute with H and then apply Lemma 3.4.5 together with Propositions 3.4.1 and 3.4.2. \square

3.4.2.3 Properties of the wave operators

We derive in this section the main properties of the regularized wave operators. First we have that they satisfy the usual intertwining properties.

Proposition 3.4.7. *Suppose that Hypotheses 3.1-3.4 hold. Then for all $t \in \mathbb{R}$, for all $u \in \mathcal{H}$,*

$$e^{itH}W_{\pm}(H, H_0)u = W_{\pm}(H, H_0)e^{itH_0}u \quad \text{and} \quad e^{itH^*}W_{\pm}(H^*, H_0)u = W_{\pm}(H^*, H_0)e^{itH_0}u. \quad (3.46)$$

In particular, $W_{\pm}(H, H_0)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$, $W_{\pm}(H^, H_0)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ and for all $u \in \mathcal{D}(H_0)$,*

$$HW_{\pm}(H, H_0)u = W_{\pm}(H, H_0)H_0u \quad \text{and} \quad H^*W_{\pm}(H^*, H_0)u = W_{\pm}(H^*, H_0)H_0u. \quad (3.47)$$

The proof is standard. We will recall it in Appendix 3.6. The last property shows that, on the range of $W_{\pm}(H, H_0)$, the evolution group generated by H acts like the evolution group generated by H_0 .

In the following proposition we show that the regularized wave operators are injective and that their ranges are dense in the absolutely continuous spectral subspace.

Proposition 3.4.8. *Suppose Hypotheses 3.1-3.4 hold. Then $W_{\pm}(H, H_0)$ and $W_{\pm}(H^*, H_0)$ are injective and we have*

$$\text{Ran}(W_{\pm}(H, H_0))^{\text{cl}} = \mathcal{H}_{\text{ac}}(H) \quad \text{and} \quad \text{Ran}(W_{\pm}(H^*, H_0))^{\text{cl}} = \mathcal{H}_{\text{ac}}(H^*).$$

Proof. We show that $W_+(H, H_0)$ is injective. The proof for the other wave operators is similar. To show that $W_+(H, H_0)$ is injective, we compute its kernel. Let $u \in \text{Ker}(W_+(H, H_0))$ then

$$\lim_{t \rightarrow \infty} \|e^{itH}\Pi_{\text{ac}}(H)r_-(H)e^{-itH_0}u\|_{\mathcal{H}} = 0.$$

Since $r(H) = r_-(H)\tilde{r}(H)$ for some bounded operator $\tilde{r}(H)$, we also have that

$$\lim_{t \rightarrow \infty} \|e^{itH}\Pi_{\text{ac}}(H)r(H)e^{-itH_0}u\|_{\mathcal{H}} = 0.$$

Next we claim that for all $t \in \mathbb{R}$, for all $v \in \mathcal{H}$,

$$\|e^{itH}\Pi_{\text{ac}}(H)r(H)v\|_{\mathcal{H}} \geq \|\Pi_{\text{ac}}(H)r(H)^2v\|_{\mathcal{H}}. \quad (3.48)$$

Indeed, it follows from the functional calculus 3.31 (see also 2.5.2) that, for all $t \in \mathbb{R}$,

$$\|e^{itH}\Pi_{\text{ac}}(H)r(H)v\|_{\mathcal{H}} \leq \|\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}.$$

Hence, for all $t \in \mathbb{R}$,

$$\|\Pi_{\text{ac}}(H)r(H)v\|_{\mathcal{H}} = \|e^{-itH}\Pi_{\text{ac}}(H)r(H)e^{itH}v\|_{\mathcal{H}} \leq \|e^{itH}\Pi_{\text{ac}}(H)v\|_{\mathcal{H}}. \quad (3.49)$$

Applying the last inequality to $r(H)u$, we obtain (3.48).

Next, for all $t \in \mathbb{R}$, applying (3.48) to $v = e^{-itH_0}u$, we have

$$\|e^{itH}\Pi_{\text{ac}}(H)r(H)e^{-itH_0}u\|_{\mathcal{H}} \geq \|\Pi_{\text{ac}}(H)r(H)^2e^{-itH_0}u\|_{\mathcal{H}}.$$

Then we write

$$r(H)^2 = (r(H) - r(H_0))r(H) + r(H_0)(r(H) - r(H_0)) + r(H_0)^2$$

Proceeding by double induction over the number of spectral singularities and the order of each spectral singularity, it is not difficult to see that $r(H) - r(H_0)$ is a compact operator since C is relatively compact with respect to H_0 . Hence $r(H)^2 - r(H_0)^2$ is also compact. Using the triangular inequality, we obtain that

$$\begin{aligned} \|e^{itH}\Pi_{\text{ac}}(H)r(H)e^{-itH_0}u\|_{\mathcal{H}} &\geq \|\Pi_{\text{ac}}(H)r(H_0)^2e^{-itH_0}u\|_{\mathcal{H}} \\ &- \|(\Pi_{\text{ac}}(H)(r(H)^2 - r(H_0)^2))e^{-itH_0}u\|_{\mathcal{H}}. \end{aligned}$$

As $r(H)^2 - r(H_0)^2$ is compact, the second term in the right hand side of the last inequality vanishes when t goes to ∞ . Moreover, since u is in the kernel of the wave operators, we have

$$0 = \lim_{t \rightarrow \infty} \|e^{itH}\Pi_{\text{ac}}(H)e^{-itH_0}r(H_0)^2u\|_{\mathcal{H}},$$

and since $\Pi_{\text{p}}(H)$ is a compact operator,

$$0 = \lim_{t \rightarrow \infty} \|\Pi_{\text{p}}(H)e^{-itH_0}r(H_0)^2u\|_{\mathcal{H}}.$$

Using $\Pi_{\text{p}}(H) + \Pi_{\text{ac}}(H) = \text{Id}$, we then deduce from the previous estimates that

$$0 = \lim_{t \rightarrow \infty} \|e^{-itH_0}r(H_0)^2u\|_{\mathcal{H}}.$$

As the evolution group associated to H_0 is unitary and $r(H_0)^2$ is injective (since H_0 has no eigenvalue), we conclude that $u = 0$, which proves that the wave operators are injective.

To show that $\text{Ran}(W_+(H, H_0))^{\text{cl}} = \mathcal{H}_{\text{ac}}(H)$, it suffices to use that $\text{Ker}((W_+(H, H_0))^*) = \mathcal{H}_{\text{p}}(H^*)$ by Propositions 3.4.12 and 3.4.13 proven below. \square

Clearly, the last proposition shows that the regularized wave operators $W_{\pm}(H, H_0)$ are invertible from \mathcal{H} to their range. In Section 3.5, we will give conditions in order to have that $W_{\pm}(H, H_0)$ are invertible in $\mathcal{B}(\mathcal{H}, \mathcal{H}_{\text{ac}}(H))$.

3.4.3 The regularized wave operators $W_{\pm}(H_0, H)$ and $W_{\pm}(H_0, H^*)$

As shown by the proof of Proposition 3.4.8, an important ingredient to study the regularized wave operators $W_{\pm}(H_0, H)$ is to study their adjoint, in order to use the duality property between the kernel of an operator and the orthogonal of its range. Formally, the adjoint of $W_{\pm}(H_0, H)$ are given by $W_{\pm}(H^*, H)$ where $W_{\pm}(H^*, H)$ are defined as follows.

Definition 3.4.9. *The wave operators associated to H_0 and H (respectively H_0 and H^*) are defined by*

$$\begin{aligned} W_{\pm}(H_0, H) &:= \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_0} \Pi_{\text{ac}}(H) r_{\pm}(H) e^{-itH}, \\ W_{\pm}(H_0, H^*) &:= \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_0} \Pi_{\text{ac}}(H^*) \bar{r}_{\pm}(H^*) e^{-itH^*}. \end{aligned}$$

In the remainder of this section we state the existence and study the properties of these wave operators.

3.4.3.1 Existence of $W_{\pm}(H_0, H)$ and $W_{\pm}(H_0, H^*)$

The existence of $W_{\pm}(H_0, H)$ and $W_{\pm}(H_0, H^*)$ follows in the same way as in the previous section.

Proposition 3.4.10. *Suppose that Hypotheses 3.1-3.4 hold. Then the wave operators $W_{\pm}(H_0, H)$ and $W_{\pm}(H_0, H^*)$ exist.*

Proof. It suffices to apply Propositions 3.4.1 and 3.4.2 together with Lemma 3.4.5. \square

We mention the following interesting consequence of Proposition 3.4.10. Together with the Banach-Steinhaus Theorem, the last proposition shows that

$$\sup_{t \geq 0} \|e^{-itH} r_+(H) \Pi_{\text{ac}}(H)\|_{\mathcal{B}(\mathcal{H})} < \infty \quad \text{and} \quad \sup_{t \geq 0} \|e^{itH} r_-(H) \Pi_{\text{ac}}(H)\|_{\mathcal{B}(\mathcal{H})} < \infty, \quad (3.50)$$

and

$$\sup_{t \geq 0} \|e^{-itH^*} \bar{r}_+(H^*) \Pi_{\text{ac}}(H^*)\|_{\mathcal{B}(\mathcal{H})} < \infty \quad \text{and} \quad \sup_{t \geq 0} \|e^{itH^*} \bar{r}_-(H^*) \Pi_{\text{ac}}(H^*)\|_{\mathcal{B}(\mathcal{H})} < \infty. \quad (3.51)$$

In other words, when restricted to $\text{Ran}(r_{\mp}(H) \Pi_{\text{ac}}(H))$, the semigroups generated by $\pm iH$ are uniformly bounded (roughly speaking $\Pi_{\text{ac}}(H)$ projects out the generalized eigenstates of H , while $r_{\mp}(H)$ ‘projects out’ the outgoing/incoming states corresponding to spectral singularities). This should be compared to the result that can be obtained using the functional calculus of 2.5.2, which only gives that the groups generated by $\pm iH$ are uniformly bounded when restricted to the range of $r(H) \Pi_{\text{ac}}(H)$.

3.4.3.2 Properties of $W_{\pm}(H_0, H)$ and $W_{\pm}(H_0, H^*)$

The following intertwining properties follow in the same way as in the proof of Proposition 3.4.7 (see Appendix 3.6).

Proposition 3.4.11. *Suppose that Hypotheses 3.1-3.4 hold. Then for all $t \in \mathbb{R}$, for all $u \in \mathcal{H}$,*

$$e^{itH_0} W_{\pm}(H_0, H) = W_{\pm}(H_0, H) e^{itH} \quad \text{and} \quad e^{itH_0} W_{\pm}(H_0, H^*) = W_{\pm}(H_0, H^*) e^{itH} u.$$

In particular, $W_{\pm}(H_0, H)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$, $W_{\pm}(H_0, H^*)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ and for all $u \in \mathcal{D}(H_0)$,

$$H_0 W_{\pm}(H_0, H)u = W_{\pm}(H_0, H)H_0 u \quad \text{and} \quad H_0 W_{\pm}(H_0, H^*)u = W_{\pm}(H_0, H^*)H^* u.$$

The next proposition relates the wave operators $W_{\pm}(H, H_0)$ to the adjoints of the wave operators $W_{\mp}(H_0, H^*)$.

Proposition 3.4.12. *Suppose that Hypotheses 3.1-3.4 hold. Then*

$$(W_{\pm}(H, H_0))^* = W_{\pm}(H_0, H^*) \quad \text{and} \quad (W_{\pm}(H^*, H_0))^* = W_{\pm}(H_0, H)$$

Proof. We prove that $(W_{\pm}(H, H_0))^* = W_{\pm}(H_0, H^*)$ the second equality is obtained in the same way. For $u, v \in \mathcal{H}$, we have

$$\begin{aligned} \langle W_{\pm}(H, H_0)u, v \rangle_{\mathcal{H}} &= \lim_{t \rightarrow \pm\infty} \langle e^{itH} \Pi_{\text{ac}}(H) r_{\mp}(H) e^{-itH_0} u, v \rangle_{\mathcal{H}} \\ &= \lim_{t \rightarrow \pm\infty} \langle u, e^{itH_0} \bar{r}_{\pm}(H^*) \Pi_{\text{ac}}(H^*) e^{-itH^*} v \rangle_{\mathcal{H}} \\ &= \langle u, W_{\pm}(H_0, H^*)v \rangle_{\mathcal{H}}. \end{aligned}$$

This proves the result. □

It should be noted that Proposition 3.4.12 and Hypothesis 3.4 yield

$$\|W_{\pm}(H, H_0)\|_{\mathcal{B}(\mathcal{H})} = \|W_{\pm}(H_0, H^*)\|_{\mathcal{B}(\mathcal{H})} = \|W_{\mp}(H^*, H_0)\|_{\mathcal{B}(\mathcal{H})} = \|W_{\mp}(H_0, H)\|_{\mathcal{B}(\mathcal{H})}.$$

In the next proposition we characterize the ranges and kernels of $W_{\pm}(H_0, H)$ and $W_{\pm}(H_0, H^*)$. Its proof relies on the characterization of the set of asymptotically disappearing states of H , defined by

$$\mathcal{H}_{\text{ads}}^{\pm}(H) := \left\{ u \in \mathcal{H}, \lim_{t \rightarrow \pm\infty} \|e^{-itH} u\|_{\mathcal{H}} = 0 \right\}^{\text{cl}}. \quad (3.52)$$

It follows from theorem 2.3.2 that, under our assumptions, $\mathcal{H}_{\text{ads}}(H)$ coincide with the direct sum of the generalized eigenspaces associated to eigenvalues with negative/positive imaginary part, that is

$$\mathcal{H}_{\text{ads}}^{\pm}(H) = \mathcal{H}_{\text{p}}^{\pm}(H) := \{u \in \ker(H - \lambda)^{m_{\lambda}}, \lambda \in \sigma_{\text{p}}(H), \mp \text{Im}(\lambda) < 0\}, \quad (3.53)$$

where we recall that m_{λ} is the algebraic multiplicity of the eigenvalue λ .

Proposition 3.4.13. *Suppose that Hypotheses 3.1-3.4 hold. Then*

$$\text{Ker}(W_{\pm}(H_0, H)) = \mathcal{H}_{\text{ac}}(H^*)^{\perp} = \mathcal{H}_{\text{p}}(H), \quad \text{Ker}(W_{\pm}(H_0, H^*)) = \mathcal{H}_{\text{ac}}(H)^{\perp} = \mathcal{H}_{\text{p}}(H^*),$$

and

$$\text{Ran}(W_{\pm}(H_0, H))^{\text{cl}} = \mathcal{H} = \text{Ran}(W_{\pm}(H_0, H^*))^{\text{cl}}.$$

Proof. It follows from theorem 2.3.3 that $\mathcal{H}_{\text{ac}}(H^*)^\perp = \mathcal{H}_{\text{p}}(H)$ and likewise for $\mathcal{H}_{\text{ac}}(H)^\perp$. We prove that $\text{Ker}(W_+(H_0, H)) = \mathcal{H}_{\text{p}}(H)$, the equalities $\text{Ker}(W_-(H_0, H)) = \mathcal{H}_{\text{p}}(H)$ and $\text{Ker}(W_\pm(H_0, H^*)) = \mathcal{H}_{\text{p}}(H^*)$ are proven in the same way. It follows from the definition of $W_+(H_0, H)$ that $\mathcal{H}_{\text{p}}(H) \subset \text{Ker}(W_+(H_0, H))$. Hence we only need to prove the reverse inclusion.

Let $u \in \text{Ker}(W_+(H_0, H))$. Using the unitarity of the group associated to H_0 , we have that

$$\lim_{t \rightarrow \infty} \|e^{-itH} r_+(H) \Pi_{\text{ac}}(H) u\|_{\mathcal{H}} = \lim_{t \rightarrow \infty} \|e^{itH_0} r_+(H) \Pi_{\text{ac}}(H) e^{-itH} u\|_{\mathcal{H}} = 0.$$

Thus $r_+(H) \Pi_{\text{ac}}(H) u$ belongs to $\mathcal{H}_{\text{ads}}^+(H)$. By the remark above, this implies that $r_+(H) \Pi_{\text{ac}}(H) u \in \mathcal{H}_{\text{p}}^+(H) \subset \mathcal{H}_{\text{p}}(H)$. Together with the fact that $r_-(H) \Pi_{\text{ac}}(H) u \in \mathcal{H}_{\text{ac}}(H)$, this yields

$$r_-(H) \Pi_{\text{ac}}(H) u = 0. \quad (3.54)$$

Finally as $r_-(H)$ is injective on $\mathcal{H}_{\text{ac}}(H)$ (because the restriction of H to $\mathcal{H}_{\text{ac}}(H)$ has no point spectrum), we obtain that $\Pi_{\text{ac}}(H) u = 0$. This proves that $u \in \mathcal{H}_{\text{p}}(H)$.

The fact that $\text{Ran}(W_\pm(H_0, H))^{\text{cl}} = \mathcal{H}$ follows from the facts that $W_\pm(H^*, H_0)$ are injective by Proposition 3.4.8 and that $(W_\pm(H_0, H))^* = W_\pm(H^*, H_0)$ by Proposition 3.4.12. \square

3.4.4 The local regularized wave operators

It can be useful to define local wave operators on an interval I of the essential spectrum. In this section we define the local regularized wave operators and we give their properties.

We recall that given a closed interval $I \subset \sigma_{\text{ess}}(H_0)$, h stands for the function defined in (3.29) which regularizes the spectral singularities of H in I . It allows one to consider a regularized spectral projection $(h\mathbb{1}_I)(H)$ as in (3.28) which in turn induces a functional calculus (3.30).

Definition 3.4.14. *Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval. We define the local regularized wave operators on the interval I by*

$$W_\pm(H, H_0, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} (h\mathbb{1}_I)(H) e^{-itH_0}, \quad W_\pm(H^*, H_0, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH^*} (\bar{h}\mathbb{1}_I)(H^*) e^{-itH_0}.$$

Note that in the particular case where H does not have any spectral singularity in I , then the local wave operators associated to H and H_0 reduce to

$$W_\pm(H, H_0, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \mathbb{1}_I(H) e^{-itH_0} \quad W_\pm(H^*, H_0, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH^*} \mathbb{1}_I(H^*) e^{-itH_0}.$$

Since $\mathbb{1}_I(H)$ is a projection, then, in this case, $\text{Ran}(W_\pm(H, H_0, I)) \subset \text{Ran}(\mathbb{1}_I(H))$.

3.4.4.1 Existence of the local regularized wave operator

As in Section 3.4.2 for the regularized wave operators $W_\pm(H, H_0)$, the proof of the existence of the local regularized wave operators $W_\pm(H, H_0, I)$, is based on Proposition 3.4.1. In order to apply this proposition, we need to prove that $C(h\mathbb{1}_I)(H)$ is relatively smooth

with respect to H_0 , which is the purpose on the following lemma. Note that the proof, based on the functional calculus (3.30), follows a different route from that of Lemma 3.4.5. The inconvenient of functional calculus is that we need to regularized both outgoing and incoming spectral singularities.

Lemma 3.4.15. *Suppose that Hypotheses 3.1-3.4 hold. Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval. Then $C(h\mathbf{1}_I)(H)$ is relatively smooth with respect to H in the sense of Kato and $C(h\mathbf{1}_{H^*})(I)$ is relatively smooth with respect to H^* .*

Proof. It suffices to show that there exists $c > 0$ such that, for all $u \in \mathcal{H}$,

$$\int_0^\infty \|Ce^{itH}(h\mathbf{1}_I)(H)\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2, \quad (3.55)$$

the proof of

$$\int_0^\infty \|Ce^{-itH}(h\mathbf{1}_I)(H)u\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2$$

being similar. Let $u, v \in \mathcal{H}$. Using the integral representation (3.28) of $(h\mathbf{1}_I)(H)$, for all $t \in \mathbb{R}$, we have

$$\begin{aligned} \langle v, Ce^{itH}(h\mathbf{1}_I)(H)u \rangle_{\mathcal{H}} &= \langle v, Ce^{itH_0}h(H_0)\mathbf{1}_I(H_0)u \rangle_{\mathcal{H}} \\ &- \frac{1}{2\pi i} \int_I e^{it\lambda} h(\lambda) \langle C\mathcal{R}_0(\lambda \mp i0^+)Cv, W C\mathcal{R}_0(\lambda \pm i0^+)u \rangle_{\mathcal{H}} d\lambda \\ &+ \frac{1}{2\pi i} \int_I e^{it\lambda} h(\lambda) \langle C\mathcal{R}_{H^*}(\lambda \mp i0^+)C W C\mathcal{R}_0(\lambda \mp i0^+)Cv, W C\mathcal{R}_0(\lambda \pm i0^+)u \rangle_{\mathcal{H}} d\lambda. \end{aligned} \quad (3.56)$$

Let $g_1^\pm : \mathbb{R} \rightarrow \mathbb{C}$ and $g_2^\pm : \mathbb{R} \rightarrow \mathbb{C}$ be the functions defined by

$$\begin{aligned} g_1^\pm(\lambda) &= h(\lambda) \langle W C\mathcal{R}_0(\lambda \mp i0^+)Cv, C\mathcal{R}_0(\lambda \pm i0^+)u \rangle_{\mathcal{H}} \mathbf{1}_I(\lambda) \\ &= h(\lambda) \langle v, C\mathcal{R}_0(\lambda \pm i0^+)C W C\mathcal{R}_0(\lambda \pm i0^+)u \rangle_{\mathcal{H}} \mathbf{1}_I(\lambda), \end{aligned}$$

and

$$\begin{aligned} g_2(\lambda) &= h(\lambda) \langle W C\mathcal{R}_{H^*}(\lambda \mp i0^+)C W C\mathcal{R}_0(\lambda \mp i0^+)Cv, C\mathcal{R}_0(\lambda \mp i0^+)u \rangle_{\mathcal{H}} \mathbf{1}_I(\lambda) \\ &= h(\lambda) \langle v, C\mathcal{R}_0(\lambda \pm i0^+)C W C\mathcal{R}_H(\lambda \pm i0^+)C W C\mathcal{R}_0(\lambda \mp i0^+)u \rangle_{\mathcal{H}} \mathbf{1}_I(\lambda). \end{aligned}$$

We claim that g_1^\pm and g_2^\pm are in $L^1(\mathbb{R}, \mathbb{C}) \cap L^2(\mathbb{R}, \mathbb{C})$. Indeed for all $\lambda \in I$, by Hypotheses 3.1 and 3.3 and the Cauchy-Schwarz inequality, we have

$$|g_1^\pm(\lambda)|^2 \leq \|h\|_\infty^2 \sup_{z \in \mathbb{C}^\pm} \|C\mathcal{R}_0(z)C\|_{\mathcal{B}(\mathcal{H})}^2 \|W\|_{\mathcal{B}(\mathcal{H})} \|C\mathcal{R}_0(\lambda \pm i0^+)u\|_{\mathcal{H}}^2 \|v\|_{\mathcal{H}}^2, \quad (3.57)$$

and

$$\begin{aligned} |g_2^\pm(\lambda)|^2 &\leq \left(\sup_{\substack{\text{Re}(z) \in I \\ |\text{Im}(z)| \leq \varepsilon_0}} (|h(z)| \|C\mathcal{R}_H(z)C W\|_{\mathcal{B}(\mathcal{H})})^2 \right) \left(\sup_{z \in \mathbb{C}^\pm} \|C\mathcal{R}_0(z)C\|_{\mathcal{B}(\mathcal{H})}^2 \right) \\ &\times \|W\|_{\mathcal{B}(\mathcal{H})}^2 \|C\mathcal{R}_0(\lambda \pm i0^+)u\|_{\mathcal{H}}^2 \|v\|_{\mathcal{H}}^2. \end{aligned} \quad (3.58)$$

Since C is relatively smooth with respect to H_0 , the function $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i0^+)u$ is square integrable on I . Moreover, by the same argument and the Cauchy-Schwarz inequality, there exists $c_1 > 0$ such that

$$\begin{aligned} \int_{\mathbb{R}} |g_1^\pm(\lambda)| \, d\lambda &\leq \left(\int_I |h(z)| \|W C \mathcal{R}_0(\lambda \mp i0^+) C v\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} \left(\int_I \|C \mathcal{R}_0(\lambda \pm i0^+) u\|_{\mathcal{H}}^2 \right)^{\frac{1}{2}} \\ &\leq c_1 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}. \end{aligned}$$

In the same way, we have

$$\int_{\mathbb{R}} |g_2(\lambda)| \, d\lambda \leq c_2 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}.$$

Using (3.56), we deduce that there exists $c_3 > 0$ such that

$$\begin{aligned} \int_0^\infty |\langle v, C e^{itH}(h\mathbf{1}_I)(H)u \rangle_{\mathcal{H}}|^2 \, dt &\leq c_3 \left(\int_{\mathbb{R}} |\langle v, C e^{itH_0} \mathbf{1}_I(H_0)u \rangle_{\mathcal{H}}|^2 \, dt \right. \\ &\quad \left. + \int_{\mathbb{R}} |\check{g}_1^\pm(t)|^2 \, dt + \int_{\mathbb{R}} |\check{g}_2^\pm(t)|^2 \, dt \right), \end{aligned}$$

where for $i \in \{1, 2\}$, \check{g}_i^\pm is the inverse Fourier transform of g_i^\pm , i.e.

$$\check{g}_i^\pm(t) := \int_{\mathbb{R}} e^{it\lambda} g_i^\pm(\lambda) \, d\lambda$$

As g_1^\pm and g_2^\pm are in $L^2(\mathbb{R}, \mathbb{C})$, Plancherel's equality implies that there exists $c_4 > 0$ such that

$$\begin{aligned} \int_0^\infty |\langle v, C e^{itH}(h\mathbf{1}_I)(H)u \rangle_{\mathcal{H}}|^2 \, dt &\leq c_4 \left(\int_{\mathbb{R}} |\langle v, C e^{itH_0} h(H_0) \mathbf{1}_I(H_0)u \rangle_{\mathcal{H}}|^2 \, dt \right. \\ &\quad \left. + \int_{\mathbb{R}} |g_1^\pm(\lambda)|^2 \, d\lambda + \int_{\mathbb{R}} |g_2^\pm(\lambda)|^2 \, d\lambda \right). \end{aligned}$$

Inserting (3.57)–(3.58) into the last inequality and using that C is relatively smooth with respect to H_0 , we obtain

$$\int_0^\infty |\langle v, C e^{itH}(h\mathbf{1}_I)(H)u \rangle_{\mathcal{H}}|^2 \, dt \leq c_5 \|u\|_{\mathcal{H}}^2 \|v\|_{\mathcal{H}}^2.$$

Using $\|C e^{itH}(h\mathbf{1}_I)(H)u\|_{\mathcal{H}} = \lim_{n \rightarrow \infty} \langle v_n, C e^{itH}(h\mathbf{1}_I)(H)u \rangle_{\mathcal{H}}$ for some normalized sequence $(v_n)_{n \in \mathbb{N}}$ together with Fatou's lemma, this leads to (3.55). \square

Using this lemma we obtain the existence of the local regularized wave operators.

Proposition 3.4.16. *Suppose that Hypotheses 3.1–3.4 hold. Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval. Then $W_{\pm}(H, H_0, I)$ and $W_{\pm}(H^*, H_0, I)$ exist.*

Proof. It suffices to apply Lemma 3.4.15 together with Propositions 3.4.1 and 3.4.2. \square

Similarly as before, the adjoints of $W_{\mp}(H^*, H_0, I)$ and $W_{\mp}(H, H_0, I)$ are formally given by

$$W_{\pm}(H_0, H, I) := \text{s-lim}_{t \rightarrow \infty} e^{itH_0}(h\mathbf{1}_I)(H)e^{-itH} \quad W_{\pm}(H_0, H^*, I) := \text{s-lim}_{t \rightarrow \infty} e^{itH_0}(\bar{h}\mathbf{1}_I)(H^*)e^{-itH^*}.$$

We then have the following existence result.

Proposition 3.4.17. *Suppose that Hypotheses 3.1-3.4 hold. Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval. Then the wave operators $W_{\pm}(H_0, H, I)$ and $W_{\pm}(H_0, H^*, I)$ exist and satisfy*

$$W_{\pm}(H_0, H, I)^* = W_{\mp}(H^*, H_0, I), \quad W_{\pm}(H_0, H^*, I)^* = W_{\mp}(H, H_0, I).$$

Proof. Again, it suffices to apply Lemma 3.4.15 together with Propositions 3.4.1 and 3.4.2 to obtain the existence. The adjoint properties follow from a direct computation. \square

3.4.4.2 Properties of the local regularized wave operators

In this section we reformulate the properties of regularized wave operators stated in Section 3.4.2 for the local regularized wave operators. First we have the following intertwining properties whose proofs, left to the reader, rely on the same arguments as those used in the proof of Proposition 3.4.7.

Proposition 3.4.18. *Let $I \subset \sigma_{\text{ess}}(H_0)$ be a closed interval. Suppose that Hypotheses 3.1-3.4 hold. Then for all $t \in \mathbb{R}$, for all $u \in \mathcal{H}$,*

$$e^{itH}W_{\pm}(H, H_0, I)u = W_{\pm}(H, H_0, I)e^{itH_0}u,$$

and

$$e^{itH^*}W_{\pm}(H^*, H_0, I)u = W_{\pm}(H^*, H_0, I)e^{itH_0}u.$$

In particular, $W_{\pm}(H, H_0, I)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$, $W_{\pm}(H^*, H_0, I)\mathcal{D}(H_0) \subset \mathcal{D}(H_0)$ and for all $u \in \mathcal{D}(H_0)$,

$$HW_{\pm}(H, H_0, I)u = W_{\pm}(H, H_0, I)H_0u \quad \text{and} \quad H^*W_{\pm}(H^*, H_0, I)u = W_{\pm}(H^*, H_0, I)H_0u.$$

Moreover,

$$\begin{aligned} (r\mathbf{1}_I)(H)W_{\pm}(H, H_0, I)u &= W_{\pm}(H, H_0, I)r(H_0)\mathbf{1}_I(H_0)u, \\ (r\mathbf{1}_I)(H^*)W_{\pm}(H^*, H_0, I)u &= W_{\pm}(H^*, H_0, I)r(H_0)\mathbf{1}_I(H_0)u. \end{aligned} \tag{3.59}$$

Note that (3.59) relies on the existence of the spectral projection and the intertwining property for the resolvent of H and H_0 , where the intertwining property for the resolvent of H and H_0 is proved by using the Laplace's transform of the resolvent.

The next two propositions characterize the kernels and ranges of $W_{\pm}(H, H_0, I)$ and $W_{\pm}(H_0, H, I)$. The proof of Proposition 3.4.19 relies on the same idea as that of Proposition 3.4.8, up to some technicalities due to the use of the regularized functional calculus (3.30). Recall that h stands for the function defined in (3.29) which regularizes the spectral singularities of H in I .

Proposition 3.4.19. *Suppose that Hypotheses 3.1-3.4 hold. Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval. Then*

$$\text{Ker}(W_{\pm}(H, H_0, I)) = \text{Ker}(\mathbf{1}_I(H_0)) \quad \text{and} \quad \text{Ker}(W_{\pm}(H^*, H_0, I)) = \text{Ker}(\mathbf{1}_I(H_0)).$$

Moreover,

$$\text{Ran}(W_{\pm}(H, H_0, I))^{\text{cl}} = \text{Ran}((r\mathbf{1}_I)(H))^{\text{cl}},$$

and

$$\text{Ran}(W_{\pm}(H, H_0^*, I))^{\text{cl}} = \text{Ran}((r\mathbf{1}_I)(H^*))^{\text{cl}}.$$

Proof. We show that $\text{Ker}(W_+(H, H_0, I)) = \text{Ker}(\mathbf{1}_I(H_0))$, the proof the other equalities characterizing the kernels are similar. Let $u \in \text{Ker}(W_+(H, H_0, I))$, then

$$\lim_{t \rightarrow \infty} \|e^{itH}(h\mathbf{1}_I)(H)e^{-itH_0}u\|_{\mathcal{H}} = 0.$$

Composing by $(r\mathbf{1}_I)(H)$ and using the intertwining properties of Proposition 3.4.18, we obtain

$$\lim_{t \rightarrow \infty} \|e^{itH}(h\mathbf{1}_I)(H)e^{-itH_0}\mathbf{1}_I(H_0)r(H_0)u\|_{\mathcal{H}} = 0.$$

Next since $r = h\tilde{r}$ for some rational function \tilde{r} , we also have

$$\lim_{t \rightarrow \infty} \|e^{itH}(r\mathbf{1}_I)(H)e^{-itH_0}\mathbf{1}_I(H_0)r(H_0)u\|_{\mathcal{H}} = 0.$$

We claim that for all $t \in \mathbb{R}$, for all $v \in \mathcal{H}$,

$$\|e^{itH}(h\mathbf{1}_I)(H)e^{-itH_0}v\|_{\mathcal{H}} \geq \|(r\mathbf{1}_I)(H)^2e^{-itH_0}v\|_{\mathcal{H}}. \quad (3.60)$$

Indeed, for all $t \in \mathbb{R}$, by the functional calculus (3.30) that $\{e^{itH}\}_{t \in \mathbb{R}}$ is uniformly bounded on the range of $h(H)$. Thus we have

$$\|e^{-itH}(r\mathbf{1}_I)(H)v\|_{\mathcal{H}} \leq \|v\|_{\mathcal{H}}.$$

Thus for all $t \in \mathbb{R}$,

$$\|(r\mathbf{1}_I)(H)v\|_{\mathcal{H}} = \|e^{-itH}(r\mathbf{1}_I)(H)e^{itH}v\|_{\mathcal{H}} \leq \|e^{itH}v\|_{\mathcal{H}}.$$

Applying the last inequality to $(r\mathbf{1}_I)(H)e^{itH_0}v$, we obtain (3.60). Now (3.60) yields

$$\lim_{t \rightarrow \infty} \|(r\mathbf{1}_I)(H)^2e^{-itH_0}\mathbf{1}_I(H_0)r(H_0)u\|_{\mathcal{H}} = 0.$$

and hence

$$\lim_{t \rightarrow \infty} \|(r\mathbf{1}_I)(H)^3e^{-itH_0}\mathbf{1}_I(H_0)r(H_0)u\|_{\mathcal{H}} = 0. \quad (3.61)$$

Now let $I^c := (\sigma_{\text{ess}}(H) \setminus I)^{\text{cl}}$. We claim that

$$\lim_{t \rightarrow \infty} \|(r\mathbf{1}_{I^c})(H)^3e^{-itH_0}\mathbf{1}_I(H_0)r(H_0)u\|_{\mathcal{H}} = 0. \quad (3.62)$$

Indeed, it follows again from the functional calculus (3.30) that there exists $c > 0$ such that, for all $t \in \mathbb{R}$,

$$\|(r\mathbf{1}_{I^c})(H)^3 e^{-itH_0} \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}} \leq c \|(r\mathbf{1}_{I^c})(H) e^{-itH} (r\mathbf{1}_{I^c})(H) e^{-itH_0} \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}}.$$

Letting t go to ∞ yields

$$\limsup_{t \rightarrow \infty} \|(r\mathbf{1}_{I^c})(H)^3 e^{-itH_0} \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}} \leq c \|(r\mathbf{1}_{I^c})(H) W_+(H, H_0, I^c) \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}}.$$

Finally using the intertwining properties of Proposition 3.4.18 together with the facts that $\mathbf{1}_{I \cap I^c}(H_0) = 0$ since H_0 has purely absolutely continuous spectrum, we obtain (3.62). Proceeding in the same way, we have that

$$\limsup_{t \rightarrow \infty} \|(r\mathbf{1}_I)(H) (r\mathbf{1}_{I^c})(H)^2 e^{-itH_0} \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}} = 0.$$

and

$$\limsup_{t \rightarrow \infty} \|(r\mathbf{1}_{I^c})(H) (r\mathbf{1}_I)(H)^2 e^{-itH_0} \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}} = 0.$$

Summing the last two previous limits with (3.61) and (3.62) gives

$$\limsup_{t \rightarrow \infty} \|(r\mathbf{1}_{\text{ess}})(H)^3 e^{-itH_0} r(H_0) \mathbf{1}_I(H_0) u\|_{\mathcal{H}} = 0. \quad (3.63)$$

Now $r(H)^3 \Pi_p(H)$ is compact under our assumptions and therefore

$$\lim_{t \rightarrow \infty} \|r(H)^3 \Pi_p(H) e^{-itH_0} \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}} = 0. \quad (3.64)$$

Summing (3.63) and (3.64), it follows from the spectral resolution formula (3.31) and the triangular inequality that

$$\limsup_{t \rightarrow \infty} \|r(H)^3 e^{-itH_0} r(H_0) \mathbf{1}_I(H_0) u\|_{\mathcal{H}} = 0.$$

Writing

$$r(H)^3 = (r(H)^2(r(H) - r(H_0)) + r(H)(r(H) - r(H_0))r(H_0) + (r(H) - r(H_0))r(H_0)^2 + r(H_0)^3),$$

and using that $r(H) - r(H_0)$ is compact, we obtain

$$\lim_{t \rightarrow \infty} \|(r(H)^3 - r(H_0)^3) e^{-itH_0} \mathbf{1}_I(H_0) r(H_0) u\|_{\mathcal{H}} = 0.$$

and hence

$$\lim_{t \rightarrow \infty} \|e^{-itH_0} \mathbf{1}_I(H_0) r^4(H_0) u\|_{\mathcal{H}} = 0.$$

Since $r(H_0)^4$ is injective (because H_0 has no eigenvalue) and $\{e^{-itH_0}\}_{t \in \mathbb{R}}$ is unitary, we conclude that $u \in \text{Ker}(\mathbf{1}_I(H_0))$.

Conversely, let $u \in \text{Ker}(\mathbb{1}_I(H_0))$. Then

$$W_+(H, H_0, I)(r\mathbb{1}_I)(H_0)u = 0$$

and using the intertwining properties of Proposition 3.4.18, we also have

$$(r\mathbb{1}_I)(H)W_+(H, H_0, I)u = 0. \quad (3.65)$$

Moreover, using that

$$(r\mathbb{1}_{I^c})(H)(r\mathbb{1}_I)(H) = 0,$$

as follows from the functional calculus (3.30), where $I^c := (\sigma_{\text{ess}}(H) \setminus I)^{\text{cl}}$, we obtain

$$(r\mathbb{1}_{I^c})(H)W_+(H, H_0, I)u = 0. \quad (3.66)$$

Now, we claim that

$$r(H)\Pi_p(H)W_+(H, H_0, I)u = 0. \quad (3.67)$$

Indeed, it suffices to prove that $W_+(H, H_0, I)u \in \mathcal{H}_{\text{ac}}(H)$, where we recall that $\mathcal{H}_{\text{ac}}(H)$ is defined as the closure of $\mathcal{M}(H)$, see (3.25). Since H_0 has purely absolutely continuous spectrum, there exists a sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{M}(H_0)$ such that $(u_n)_{n \in \mathbb{N}}$ converges to u when $n \rightarrow \infty$. Using the intertwining properties of Proposition 3.4.18, we deduce that, for all $n \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbb{R}} |\langle e^{itH}W_+(H, H_0, I)u_n, v \rangle_{\mathcal{H}}|^2 dt &= \int_{\mathbb{R}} |\langle e^{itH_0}u_n, W_+(H, H_0, I)^*v \rangle_{\mathcal{H}}|^2 dt \\ &\leq c_{u_n} \|W_+(H, H_0, I)\|_{\mathcal{B}(\mathcal{H})} \|v\|_{\mathcal{H}}^2. \end{aligned}$$

Thus $(W_+(H, H_0, I)u_n)_{n \in \mathbb{N}} \subset \mathcal{M}(H)$, which implies (3.67). Finally, using the spectral resolution formula (3.31) together with (3.65), (3.66) and (3.67) we obtain

$$r(H)W_+(H, H_0, I)u = 0.$$

Since the restriction $r(H)$ to $\mathcal{H}_{\text{ac}}(H)$ is injective, it follows that $u \in \text{Ker}(W_+(H, H_0, I))$.

The characterizations of the ranges follow from Propositions 3.4.17 and 3.4.20. \square

Now we turn to the study of the kernels and ranges of $W_{\pm}(H_0, H, I)$.

Proposition 3.4.20. *Suppose that Hypotheses 3.1-3.4 hold. Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval. Then*

$$\text{Ker}((h\mathbb{1}_I)(H)) = \text{Ker}(W_{\pm}(H_0, H, I)) \quad \text{and} \quad \text{Ker}((\bar{h}\mathbb{1}_I)(H^*)) = \text{Ker}((W_{\pm}(H_0, H^*, I)).$$

Moreover we have

$$\text{Ran}(W_{\pm}(H_0, H, I))^{\text{cl}} = \text{Ran}(\mathbb{1}_I(H_0)) \quad \text{and} \quad \text{Ran}(W_{\pm}(H_0, H^*, I))^{\text{cl}} = \text{Ran}(\mathbb{1}_I(H_0)).$$

Proof. We show that $\text{Ker}(W_+(H_0, H, I)) = \text{Ker}((h\mathbf{1}_I)(H))$, the proof the other equalities characterizing the kernels are similar. Let $u \in \text{Ker}(W_+(H_0, H, I))$. Then

$$\lim_{t \rightarrow \infty} \|e^{itH_0} e^{-itH} (h\mathbf{1}_I)u\|_{\mathcal{H}} = 0$$

and therefore, by unitarity of e^{itH_0} ,

$$\lim_{t \rightarrow \infty} \|e^{-itH} (h\mathbf{1}_I)u\|_{\mathcal{H}} = 0.$$

Hence $(h\mathbf{1}_I)u \in \mathcal{H}_{\text{ads}}(H)^+$ (see (3.52)) which in turn yields $(h\mathbf{1}_I)u \in \mathcal{H}_p^+(H)$ by (3.53). Thus we have

$$(h\mathbf{1}_I)(H)u = \Pi_p^+(H)(h\mathbf{1}_I)(H)u.$$

Using (3.28) it is not difficult to verify that, for all $v \in \mathcal{H}$,

$$\langle \Pi_p^+(H)(h\mathbf{1}_I)(H)u, v \rangle_{\mathcal{H}} = 0.$$

Therefore $\Pi_p^+(H)(h\mathbf{1}_I)(H)u = 0$ and hence $u \in \text{Ker}((h\mathbf{1}_I)(H))$.

Conversely, if $u \in \text{Ker}((h\mathbf{1}_I)(H))$, then by definition of $W_+(H_0, H, I)$, we have $u \in \text{Ker}(W_+(H_0, H, I))$.

The characterizations of the ranges follow from Propositions 3.4.17 and 3.4.19. \square

3.5 Asymptotic completeness

In this section, we study a notion of asymptotic completeness for the wave operator $W_{\pm}(H, H_0)$.

Definition 3.5.1. *We say that the wave operator $W_{\pm}(H, H_0)$ or $W_{\pm}(H^*, H_0)$ is asymptotically complete if its range is closed.*

This definition is motivated by Proposition 3.4.8. Indeed, we already know that, under our assumptions, the wave operators $W_{\pm}(H, H_0)$ are injective and that their ranges are dense in $\mathcal{H}_{\text{ac}}(H)$. Hence if the wave operators are asymptotically complete, they are bijective. In the following we give conditions ensuring that asymptotic completeness holds. Note that $W_-(H, H_0)$ may be asymptotically complete while $W_+(H, H_0)$ is not.

3.5.1 Asymptotic completeness and spectral singularities

Our first result shows that if H does not have any spectral singularity, then the wave operators are asymptotically complete. A similar result has been proved in [35] for dissipative operators.

We recall that if H has no outgoing/incoming spectral singularities, then the wave operators $W_{\pm}(H, H_0)$ and $W_{\pm}(H_0, H)$ reduce to

$$W_{\pm}(H, H_0) = \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \Pi_{\text{ac}}(H) e^{-itH_0}, \quad W_{\pm}(H_0, H) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH_0} \Pi_{\text{ac}}(H) e^{-itH}.$$

Theorem 3.5.2. *Suppose that Hypotheses 3.1-3.4 hold. Suppose that H does not have any spectral singularity. Then $W_{\pm}(H, H_0)$ and $W_{\pm}(H^*, H_0)$ are asymptotically complete.*

Proof. We prove that $W_+(H, H_0)$ is asymptotically complete. By proposition 3.4.8, we already know that $\text{Ran}(W_+(H, H_0)) \subset \mathcal{H}_{\text{ac}}(H)$. Hence, to prove that the range of $W_+(H, H_0)$ is closed, it suffices to show that $\mathcal{H}_{\text{ac}}(H) \subset \text{Ran}(W_+(H, H_0))$. Let $u \in \mathcal{H}_{\text{ac}}(H)$. Then

$$u = \Pi_{\text{ac}}(H)\Pi_{\text{ac}}(H)u = \Pi_{\text{ac}}(H)e^{itH}e^{-itH_0}e^{itH_0}e^{-itH}\Pi_{\text{ac}}(H)u.$$

As e^{itH} is uniformly bounded in $t \in \mathbb{R}$ by (3.50), letting $t \rightarrow \infty$, we easily obtain

$$u = W_+(H, H_0)W_+(H_0, H^*)u.$$

Thus $u \in \text{Ran}(W_+(H, H_0))$, which concludes the proof. \square

Under the conditions of Theorem 3.5.2, we see in particular that $W_+(H, H_0)$ is a left inverse of $W_+(H_0, H)$ and, as $W_+(H, H_0)$ is invertible in $\mathcal{B}(\mathcal{H}, \mathcal{H}_{\text{ac}}(H))$, we have

$$W_+(H, H_0)^{-1} = W_+(H_0, H).$$

Note also that the intertwining properties show that

$$Hu = W_+(H, H_0)H_0W_+(H_0, H)u, \quad \forall u \in \mathcal{D}(H_0) \cap \mathcal{H}_{\text{ac}}(H),$$

and

$$e^{itH}u = W_+(H, H_0)e^{itH_0}W_+(H, H_0)u, \quad \forall u \in \mathcal{H}_{\text{ac}}(H), \quad \forall t \in \mathbb{R}.$$

In other words, the restriction of H to $\mathcal{H}_{\text{ac}}(H)$ is similar to H_0 .

The next result is a reformulation of Theorem 3.3.5.

Theorem 3.5.3. *Suppose that Hypotheses 3.1-3.4 hold. Suppose that H does not have spectral singularities. Then there exist $m_1 > 0$, and $m_2 > 0$ such that*

$$m_1 \|u\|_{\mathcal{H}} \leq \|e^{itH}u\|_{\mathcal{H}} \leq m_2 \|u\|_{\mathcal{H}}, \quad \forall u \in \mathcal{H}_{\text{ac}}(H), \quad \forall t \in \mathbb{R}.$$

Proof. Let $u \in \mathcal{H}_{\text{ac}}(H)$. Using the functional calculus (3.27), we deduce that there exists $m_2 > 0$ such that for all $t \in \mathbb{R}$, for all $v \in \mathcal{H}$,

$$\begin{aligned} |\langle v, e^{itH}u \rangle_{\mathcal{H}}| &= \left| \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_{\sigma_{\text{ess}}(H)} \langle v, e^{it\lambda} (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) u \rangle_{\mathcal{H}} d\lambda \right| \\ &\leq m_2 \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}. \end{aligned}$$

Thus

$$\|e^{itH}u\|_{\mathcal{H}} \leq m_2 \|u\|_{\mathcal{H}}.$$

For the other inequality, we use that for all $t \in \mathbb{R}$,

$$\|e^{itH}u\|_{\mathcal{H}} = \|W_+(H, H_0)e^{itH_0}W_+(H_0, H)u\|_{\mathcal{H}}.$$

By the remarks above, $W_+(H, H_0)$ and $W_+(H_0, H)$ are invertible in $\mathcal{B}(\mathcal{H}, \mathcal{H}_{\text{ac}}(H))$ and $\mathcal{B}(\mathcal{H}_{\text{ac}}(H), \mathcal{H})$, respectively, and the evolution group generated by H_0 is unitary. Hence there exists $m_1 > 0$ such that

$$\|e^{itH}u\|_{\mathcal{H}} \geq m_1 \|u\|_{\mathcal{H}}$$

This concludes the proof. \square

3.5.2 Completeness of the local wave operators

We recall that if I is a closed interval contained in $\sigma_{\text{ess}}(H)$ such that H does not have any spectral singularity in I , the local wave operators associated to H and H_0 reduce to

$$W_{\pm}(H, H_0, I) := \text{s-lim}_{t \rightarrow \pm\infty} e^{itH} \mathbf{1}_I(H) e^{-itH_0}$$

The next theorem shows that in this case $W_{\pm}(H, H_0, I)$ are asymptotically complete in a natural sense.

Theorem 3.5.4. *Suppose that Hypotheses 3.1-3.4 hold. Let $I \subset \sigma_{\text{ess}}(H)$ be a closed interval not containing any spectral singularity of H . Then $W_{\pm}(H, H_0, I)$ and $W_{\pm}(H^*, H_0, I)$ are asymptotically complete in the sense that*

$$\text{Ran}(W_+(H, H_0, I)) = \text{Ran}(\mathbf{1}_I(H)), \quad \text{Ran}(W_+(H^*, H_0, I)) = \text{Ran}(\mathbf{1}_I(H^*)).$$

Proof. We prove the equality for $W_+(H, H_0, I)$. The proof of the equalities for $W_-(H, H_0, I)$ and $W_{\pm}(H^*, H_0, I)$ are similar. If I does not have any spectral singularity, then with the intertwining property,

$$W_+(H, H_0, I) = \mathbf{1}_I(H)W_+(H, H_0, I) = W_+(H, H_0, I)\mathbf{1}_I(H_0). \quad (3.68)$$

So by proposition 3.4.19 and with (3.68) the restriction of $W_+(H, H_0, I)$ to $\text{Ran}(\mathbf{1}_I(H_0))$ is injective and $\text{Ran}(W_+(H, H_0, I)) \subset \text{Ran}(\mathbf{1}_I(H))$. We will show the reverse inclusion. Let $u \in \text{Ran}(\mathbf{1}_I(H))$, then

$$u = \mathbf{1}_I(H)u = e^{itH} \mathbf{1}_I(H) e^{-itH_0} e^{itH_0} e^{-itH} \mathbf{1}_I(H).$$

As $t \mapsto e^{itH} \mathbf{1}_I(H) e^{-itH_0}$ is uniformly bounded in $\mathcal{B}(\mathcal{H})$, then

$$\begin{aligned} u &= e^{itH} \mathbf{1}_I(H) e^{-itH_0} W_+(H_0, H, I)u + o(1), \quad (t \rightarrow \infty) \\ &= W_+(H, H_0, I)W_+(H_0, H, I)u \end{aligned}$$

This prove the reverse inclusion. \square

The previous result shows that if H does not have any spectral singularity in I , then the inverses of $W_{\pm}(H, H_0, I)$ are $W_{\pm}(H_0, H, I)$. It should be noted that in order to construct the local wave operators on an interval I not containing any spectral singularity, it suffices to have the existence of the spectral projection in I . Next it follows from Theorem 3.3.6 that they are invertible, and hence in particular injective with closed range. In the general case where I contains spectral singularities, on the other hand, we relied on the whole spectral structure of H in order to prove that $W_{\pm}(H, H_0, I)$ are injective (see Proposition 3.4.19).

3.5.3 Remarks on the notion of asymptotic completeness

Theorem 3.5.2 shows that if H does not have any spectral singularities then the wave operators $W_{\pm}(H, H_0)$ are asymptotically complete. We conjecture that if H has a spectral singularity, then the regularized wave operators are *not* asymptotically complete. We give in this section partial results supporting this conjecture.

First we consider regularized wave operators with an ‘over-regularizing’ operator in the following sense. For the sake of simplicity, suppose that H has only one incoming spectral singularity λ_0 of finite order ν_0 . Consider the regularized wave operator

$$\widetilde{W}_+(H, H_0) := \text{s-lim}_{t \rightarrow \infty} e^{itH} \Pi_{\text{ac}}(H) (H - \lambda_0)^{\nu_0+1} \mathcal{R}_H(z_0)^{\nu_0+1} e^{-itH_0}.$$

This definition should be compared to the Definition 3.4.3 where the regularizing operator is $(H - \lambda_0)^{\nu_0} \mathcal{R}_H(z_0)^{\nu_0}$ instead of $(H - \lambda_0)^{\nu_0+1} \mathcal{R}_H(z_0)^{\nu_0+1}$. In particular we have that

$$\widetilde{W}_+(H, H_0) = (H - \lambda_0) \mathcal{R}_H(z_0) W_+(H, H_0),$$

where $W_+(H, H_0)$ is the regularized wave operator associated to H and H_0 from Definition 3.4.3. Under our assumptions, it follows from Proposition 3.13 that $\widetilde{W}_+(H, H_0)$ exists, is injective and its range is dense in $\mathcal{H}_{\text{ac}}(H)$. The next proposition shows that $\widetilde{W}(H, H_0)$ is not surjective.

Proposition 3.5.5. *Suppose that Hypotheses 3.1-3.4 hold. Then $\widetilde{W}(H, H_0)$ is not asymptotically complete.*

Proof. Suppose that $\widetilde{W}(H, H_0)$ is asymptotically complete. Then its range is closed and hence there exists $c > 0$ such that, for all $u \in \mathcal{H}$,

$$\left\| \widetilde{W}(H, H_0) u \right\|_{\mathcal{H}} \geq c \|u\|_{\mathcal{H}}.$$

It then follows from (3.50) that there exists $c_1 > 0$ such that

$$c_1 \lim_{t \rightarrow \infty} \left\| (H - \lambda_0) \mathcal{R}_H(z_0) e^{-itH_0} u \right\|_{\mathcal{H}} \geq c \|u\|_{\mathcal{H}}.$$

As $(H - \lambda_0) \mathcal{R}_H(z_0) - (H_0 - \lambda_0) \mathcal{R}_{H_0}(z_0)$ is compact, we deduce that

$$c_1 \lim_{t \rightarrow \infty} \left\| (H_0 - \lambda_0) \mathcal{R}_{H_0}(z_0) e^{-itH_0} u \right\|_{\mathcal{H}} \geq c \|u\|_{\mathcal{H}}.$$

Finally, using that $\{e^{-itH_0}\}_{t \in \mathbb{R}}$ is unitary and that $\mathcal{R}_{H_0}(z_0)$ is bounded, this implies that there exists $c_2 > 0$ such that

$$c_2 \left\| (H_0 - \lambda_0) u \right\|_{\mathcal{H}} \geq c \|u\|_{\mathcal{H}}.$$

This is a contradiction because this would imply that $H_0 - \lambda_0$ is bijective (since it is self-adjoint and injective with closed range), which is impossible since λ_0 is in the essential spectrum of H_0 . \square

Our last result shows that if H has an incoming spectral singularity and if $-iH$ is the generator of a strongly continuous one parameter semigroup which is uniformly bounded for positive times, then $W_+(H, H_0)$ is not asymptotically complete. It's especially the case if $\text{Im}(V) \leq 0$ because in this case $-iH$ becomes dissipative and thus generates a contraction semigroup.

Proposition 3.5.6. *Suppose that Hypotheses 3.1-3.4 hold. If $\{e^{itH}\}_{\pm t \geq 0}$ is uniformly bounded and H has an incoming/outgoing spectral singularity then $W_{\pm}(H, H_0)$ are not asymptotically complete.*

Proof. We prove the proposition for $W_+(H, H_0)$. The proof for $W_-(H, H_0)$ is very similar. Suppose that $W_+(H, H_0)$ is asymptotically complete. Then there exists $c > 0$ such that for all $u \in \mathcal{H}$,

$$\|W_+(H, H_0)u\|_{\mathcal{H}} \geq c \|u\|_{\mathcal{H}}.$$

Using that $\{e^{itH}\}_{t \geq 0}$ is uniformly bounded and arguing as in the proof of Proposition 3.5.5, we have that there exists $c_1 > 0$ such that, for all $u \in \mathcal{H}$,

$$\|r_-(H_0)u\|_{\mathcal{H}} \geq c_1 \|u\|_{\mathcal{H}}.$$

Since $r_-(H_0)$ is self-adjoint and injective, this would imply that it is bijective. However $r_-(H_0)$ is not bijective by definition, because spectral singularities of H are points of the essential spectrum of H_0 . \square

3.6 Proofs of Propositions 3.4.1 and 3.4.7

In this appendix, for completeness, we recall the proofs of some standard results about wave operators.

Proof of Proposition 3.4.1. We suppose that there exist $c > 0$ such that

$$\int_0^{\infty} \|Ce^{-itH^*} A^* u\|_{\mathcal{H}}^2 dt \leq c \|u\|_{\mathcal{H}}^2. \quad (3.69)$$

and we prove that the strong limit $s\text{-}\lim_{t \rightarrow \infty} Ae^{itH} e^{-itH_0}$ exist. The existence of the other strong limits are similar. We will show that for all $u \in \mathcal{H}$,

$$\lim_{t \rightarrow \infty} Ae^{itH} e^{-itH_0} u$$

exists in \mathcal{H} . We claim that for all $t > 0$, for all $u \in \mathcal{H}$, we have

$$Ae^{itH} e^{-itH_0} u = Au + i \int_0^t Ae^{isH} C W C e^{-isH_0} u ds. \quad (3.70)$$

Indeed, let $u \in \mathcal{D}(H_0)$ and $t \in \mathbb{R}$, then as $\{e^{itH}\}_{t \in \mathbb{R}}$ preserves $\mathcal{D}(H_0)$, then

$$\begin{aligned} Ae^{itH}e^{-itH_0}u &= Au - \int_0^t \partial_s (Ae^{isH}e^{-isH_0}u) \, ds \\ &= Au - \int_0^t Ae^{isH}(iH - iH_0)e^{-isH_0}u \, ds \\ &= Au - i \int_0^t Ae^{isH}Ve^{-isH_0}u \, ds. \end{aligned}$$

As the equality

$$Ae^{itH}e^{-itH_0}u = Au - i \int_0^t Ae^{isH}Ve^{-isH_0}u \, ds$$

makes sense if $u \in \mathcal{H}$, using that $\mathcal{D}(H_0)$ is dense in \mathcal{H} together with Lebesgue's dominated convergence theorem, we deduce that (3.70) holds.

To prove that the limit in the left hand side of (3.70) exists when $t \rightarrow \infty$, we show that the integral in the right hand side of (3.70) converges when $t \rightarrow \infty$. To do this, we show that the integral satisfies a Cauchy criterion. As C is relatively smooth with respect to H_0 , it follows from (3.4.5) that there exists $c_0 > 0$ such that, for all $x, y \in \mathbb{R}$,

$$\begin{aligned} &\left\| \int_x^y Ae^{isH}CWCe^{-isH_0}u \, ds \right\|_{\mathcal{H}} \\ &= \sup_{\substack{v \in \mathcal{H} \\ \|v\|_{\mathcal{H}}=1}} \left| \int_x^y \langle Ce^{-isH^*}A^*v, WCe^{-isH_0}u \rangle_{\mathcal{H}} \, ds \right| \\ &\leq \sup_{\substack{v \in \mathcal{H} \\ \|v\|_{\mathcal{H}}=1}} \int_x^y |\langle Ce^{-isH^*}Av, WCe^{-isH_0}u \rangle_{\mathcal{H}}| \, ds \\ &\leq \sup_{\substack{v \in \mathcal{H} \\ \|v\|_{\mathcal{H}}=1}} \left(\int_x^y \|Ce^{-isH^*}Av\|_{\mathcal{H}}^2 \, ds \right)^{1/2} \left(\int_x^y \|WCe^{-isH_0}u\|_{\mathcal{H}}^2 \, ds \right)^{1/2} \\ &\leq c_0 \|u\|_{\mathcal{H}}, \end{aligned}$$

where we used (3.69) and (3.12) in the last inequality. Since \mathcal{H} is complete the map $s \mapsto Ae^{isH}Ve^{-isH_0}u$ belongs to $L^1([0, \infty), \mathcal{H})$ and hence the limit of (3.70) exists when $t \rightarrow \infty$. \square

Proof of Proposition 3.4.7. We show the proposition for $W_+(H, H_0)$, the proof is similar for the other wave operators. Let $u \in \mathcal{H}$ and $t \in \mathbb{R}$. Then

$$\begin{aligned} e^{itH}W_+(H, H_0)u &= e^{itH}e^{isH}\Pi_{\text{ac}}(H)r_-(H)e^{-isH_0}u + o(1), \quad (s \rightarrow \infty) \\ &= e^{i(t+s)H}\Pi_{\text{ac}}(H)r_-(H)e^{-i(t+s)H_0}e^{itH_0}u + o(1), \quad (s \rightarrow \infty) \\ &= e^{irH}\Pi_{\text{ac}}(H)r_-(H)e^{-irH_0}e^{itH_0}u + o(1), \quad (r \rightarrow \infty) \\ &= W_+(H, H_0)e^{itH_0}u. \end{aligned}$$

This shows (3.46). To prove that $W_+(H, H_0)$ preserves $\mathcal{D}(H_0)$, it suffices to use that

$$\mathcal{D}(H_0) = \left\{ u \in \mathcal{H}, \quad \lim_{t \rightarrow 0^+} \frac{1}{-it} (e^{-itH} - \text{Id}) u \text{ exists in } \mathcal{H} \right\}.$$

together with 3.46. To prove (3.47), we use that

$$Hu = \lim_{t \rightarrow 0} \frac{1}{-it} (e^{-itH} - \text{Id}) u$$

and (3.46) again. □

Chapitre 4

Solutions de l'équation de Schrödinger associée à une perturbation non bornée d'un opérateur non-auto-adjoint

Dans ce chapitre nous considérons une perturbation non bornée et relativement compacte d'un opérateur auto-adjoint agissant sur un espace de Hilbert. Le caractère non-borné de la perturbation ne permet pas d'assurer, en général, que l'opérateur Hamiltonien engendre un groupe d'évolution fortement continu à un paramètre. Le premier objectif est ainsi de dégager des hypothèses sur le modèle pour obtenir l'existence d'un tel groupe. Nous montrons ensuite comment les résultats obtenus dans les chapitres précédents peuvent être généralisés au contexte présent.

4.1 Motivations et présentation du modèle

4.1.1 Présentation du modèle

Considérons un opérateur auto-adjoint H_0 de domaine $\mathcal{D}(H_0)$ agissant sur un espace de Hilbert \mathcal{H} , un opérateur C auto-adjoint de domaine $\mathcal{D}(C)$ dense dans \mathcal{H} et $W \in \mathcal{B}(\mathcal{H})$ tel que $\text{Ran}(W) \subset \mathcal{D}(C)$. Nous définissons l'opérateur V de domaine $\mathcal{D}(C)$ par

$$\forall u \in \mathcal{D}(C), \quad Vu = CWCu$$

et nous supposons que V est relativement compact par rapport à H_0 . En particulier $\mathcal{D}(C) \subset \mathcal{D}(H_0)$ et pour tout $z \in \rho(H_0)$ l'opérateur $V\mathcal{R}_0(z)$ est un opérateur compact. Enfin nous définissons l'opérateur total H par

$$H := H_0 + V, \quad \mathcal{D}(H) := \mathcal{D}(H_0).$$

Cette hypothèse sur V a pour conséquence que H est un opérateur fermé et que le spectre essentiel de H et celui de H_0 coïncident (voir [76, Corrolaire XIII.4.2]). Ce qui diffère des

chapters précédents est que l'on ne fait pas l'hypothèse que V est borné. Dans la littérature, on trouve des résultats assurant l'existence de groupe d'évolution pour des perturbations bornées d'opérateurs auto-adjoint et pour des perturbations non-bornées satisfaisant une condition de régularité pour la topologie d'opérateur par rapport au groupe associé à l'opérateur non perturbé (voir [15, Chapitre 3]). Sous les hypothèses que nous faisons ici, à notre connaissance il n'y a pas de résultat général assurant l'existence d'un groupe d'évolution associé à $-iH$. En particulier, l'existence de solutions non triviales pour l'équation de Schrödinger (4.1)

$$\begin{cases} \partial_t u(t) = Hu(t) \\ u_0 \in \mathcal{D}(H_0), \end{cases} \quad (4.1)$$

n'est pas garantie. Nous souhaitons dégager des hypothèses sur ce modèle, dans la continuité des travaux des chapitres 2 et 3 pour construire un groupe d'évolution associé à $-iH$. Un exemple satisfaisant les conditions décrites dans ce chapitre, mais pas celles décrites dans les chapitres précédents, serait un opérateur de Schrödinger sur $L^2(\mathbb{R}^3, \mathbb{C})$ avec un potentiel V de la forme $V(x) = \frac{e^{-|x|}}{|x|}$, où $|\cdot|$ est la norme euclidienne sur \mathbb{R}^3 . La singularité en 0 de ce potentiel fait que celui-ci n'est pas dans $L^\infty(\mathbb{R}^3, \mathbb{C})$ et donc que V n'est pas un opérateur borné sur $L^2(\mathbb{R}^3, \mathbb{C})$. Notons qu'ici en plus du fait que le potentiel n'est pas un opérateur borné, nous affaiblissons les hypothèses sur C dans le sens où celui-ci n'est ni supposé positif ni injectif. L'hypothèse associée à la positivité de C est uniquement utilisée dans le chapitre 2 pour construire le triplet de Gelfand et montrer le théorème 2.3.1 et l'hypothèse concernant l'injectivité de C est utilisée dans le même chapitre pour prouver le théorème 2.3.3 et, encore une fois, pour construire le triplet de Gelfand. Nous ne nous attacherons pas à pas montrer de résultat similaire ici, nous n'avons donc pas besoin de telles hypothèses sur C .

Dans la suite nous renvoyons aux notations et aux explications du chapitre 2 pour les notions de spectre essentiel.

4.1.2 Hypothèses

Les hypothèses que nous présentons ici vont être très similaires à celles des chapitres 2 et 3, cependant nous devons tenir compte du fait que C n'est pas borné.

Hypothèse 4.1. H_0 a un spectre purement absolument continu.

Comme C n'est pas supposé injectif, l'hypothèse 4.2 ci dessous ne suffira pas à assurer que le spectre de H_0 est purement absolument continu comme c'est le cas au chapitre 2 (voir par exemple la remarque sous l'hypothèse 2.1). L'hypothèse 4.1 est donc nécessaire ici pour rester dans la continuité des travaux précédents.

Hypothèse 4.2. C est relativement lisse au sens de Kato par rapport à H_0 , c'est-à-dire qu'il existe $c_0 > 0$ telle que

$$\forall u \in \mathcal{H}_{ac}(H_0), \quad \int_{\mathbb{R}} \left(\|C\mathcal{R}_0(\lambda - i0^+)u\|_{\mathcal{H}}^2 + \|C\mathcal{R}_0(\lambda + i0^+)u\|_{\mathcal{H}}^2 \right) d\lambda \leq 2\pi c_0^2 \|u\|_{\mathcal{H}}^2. \quad (4.2)$$

Ici $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i0^+)u$ est la limite de $\lambda \mapsto C\mathcal{R}_0(\lambda \pm i\varepsilon)u$ dans $L^2(\mathbb{R}; \mathcal{H})$ quand $\varepsilon \rightarrow 0^+$ (voir [55]). Pour rester dans la continuité des travaux précédents, nous aurions pu supposer que la résolvante de H_0 avec les poids C se prolonge en un opérateur borné et satisfait un principe d'absorption limite (voir par exemple 2.1). Ce n'est cependant ici pas nécessaire pour la suite de ce chapitre.

Hypothèse 4.3. *Pour tout $z \in \rho(H)$ l'opérateur $C\mathcal{R}_H(z)C$ défini sur $\mathcal{D}(C)$ se prolonge en un opérateur borné dans $\mathcal{B}(\mathcal{H})$.*

Notons que comme $D(H) = D(H_0)$, on a $\mathcal{D}(H) \subset \mathcal{D}(C)$ car C est relativement compact par rapport à H_0 . De plus cette hypothèse est très similaire à l'hypothèse 4.2.

L'hypothèse 4.3 permet de définir la notion de point spectral régulier pour H . Cette notion est semblable à celle que nous avons donnée à la section 2.2.2, nous la rappelons ici.

Definition 4.1.1 (Point spectral régulier et singularité spectrale). *Soit $\lambda \in \sigma_{\text{ess}}(H)$.*

- i) *On dit que λ est un point spectral régulier sortant/entrant si λ n'est pas un point d'accumulation de valeurs propres localisées en $\lambda \pm i]0, +\infty[$ et si la limite*

$$\lim_{\varepsilon \rightarrow 0^+} C\mathcal{R}_H(\lambda \pm i\varepsilon)CW$$

existe dans $\mathcal{B}(\mathcal{H})$. Si λ n'est pas un point spectral régulier sortant/entrant de H , on dit que λ est une singularité spectrale sortante/entrante de H .

- ii) *On dit que λ est un point spectral régulier de H si λ est un point spectral régulier entrant et sortant de H . Sinon on dit que λ est une singularité spectrale de H .*

- iii) *On dit que l'infini est un point spectral régulier sortant/entrant de H s'il existe $m > 0$ tel que pour tout $\lambda > m$, λ est un point spectral régulier sortant/entrant et si l'application*

$$[m, +\infty[\ni \lambda \mapsto C\mathcal{R}_H(\lambda \pm i0^+)CW \in \mathcal{B}(\mathcal{H})$$

est bornée. Si l'infini n'est pas un point spectral régulier sortant/entrant de H on dit que H a une singularité spectrale sortante/entrante à l'infini.

Soulignons qu'à part la construction du triplet de Gelfand faite à la section 2.2.3.1, les propriétés démontrées à la section 2.4 sont toujours valables si l'on suppose que les résolvantes à poids de H et de H_0 se prolongent en des opérateurs bornés.

Nous rappelons aussi la définition de l'ordre d'une singularité spectrale.

Definition 4.1.2 (Ordre d'une singularité spectrale). *i) Soit $\lambda \in \sigma_{\text{ess}}(H)$ une singularité spectrale sortante/entrante. On dit que λ est une singularité spectrale d'ordre fini s'il existe un entier n et $\varepsilon > 0$ tels que*

$$\sup_{z \in D(\lambda, \varepsilon) \cap \mathbb{C}^\pm} |\lambda - z|^n \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < +\infty \quad (4.3)$$

Si on dit que λ est une singularité spectrale d'ordre infini. Si λ est une singularité spectrale sortante/entrante d'ordre fini, on définit son ordre comme le plus petit entier satisfaisant (4.3).

- ii) Si H possède une singularité spectrale sortante/entrante à l'infini, on dit que l'infini est une singularité spectrale sortante/entrante d'ordre fini s'il existe un entier n , $\varepsilon_0 > 0$, $m > 0$ et $z_0 \in \rho(H) \setminus \mathbb{R}$ tels que

$$\sup_{\substack{\operatorname{Re}(z) > m \\ \pm \operatorname{Im}(z) < \varepsilon_0}} |z - z_0|^{-n} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty \quad (4.4)$$

Si on dit que l'infini est une singularité spectrale sortante/entrante d'ordre infini. Si l'infini est une singularité spectrale sortante/entrante d'ordre fini, on définit son ordre comme le plus petit entier satisfaisant (4.4).

Cela nous amène à l'hypothèse sur les singularités spectrales de H .

Hypothèse 4.4. H a un nombre fini de singularités spectrales et chaque singularité spectrale est d'ordre fini. De plus, pour tout intervalle fermé $I \subset \sigma_{\text{ess}}(H_0)$, ne contenant pas de singularités spectrales, il existe $\varepsilon_0 > 0$ tel que

$$\sup_{\substack{\operatorname{Re}(z) \in I \\ \pm \operatorname{Im}(z) \in]0, \varepsilon_0[}} \|C\mathcal{R}_H(z)CW\|_{\mathcal{B}(\mathcal{H})} < \infty.$$

Pour toute singularité spectrale λ_j , nous noterons r_j la fonction définie sur $\mathbb{C} \setminus \{z_0\}$ par

$$r_j(z) = \frac{(z - \lambda_j)^{\nu_j}}{(z - z_0)^{\nu_j}},$$

régularisant la singularité spectrale λ_j , où ν_j l'ordre de la singularité spectrale et $z_0 \in \rho(H)$. Nous noterons r_∞ la fonction régularisant la singularité spectrale à l'infini définie sur $\mathbb{C} \setminus \{z_0\}$ par

$$r_\infty(z) = \frac{1}{(z - z_0)^{\nu_\infty}},$$

où ν_∞ est l'ordre de la singularité spectrale à l'infini. Si I est un intervalle de $\sigma_{\text{ess}}(H)$, en notant alors \mathcal{S}_I les indices de l'ensemble des singularités spectrales de H dans l'intervalle I classés par ordre croissant, on définit sur $\mathbb{C} \setminus \{z_0\}$ la fonction r régularisant toutes les singularités spectrales de H par

$$r_I(z) = \prod_{j \in \mathcal{S}_I} r_j(z).$$

De plus, si $I = \sigma_{\text{ess}}(H)$, on note simplement r cette fonction.

Rappelons que le théorème de Fatou combiné avec l'hypothèse 4.4 assure en plus que

$$\lim_{\varepsilon \rightarrow 0^+} r(\lambda \pm i\varepsilon)C\mathcal{R}_H(\lambda \pm i\varepsilon)CW$$

existe dans $\mathcal{B}(\mathcal{H})$ pour presque tout $\lambda \in \sigma_{\text{ess}}(H)$. Pour plus de détails concernant cette hypothèse, nous renvoyons aux remarques sous les hypothèses 2.3 et 3.3.

4.2 Opérateurs spectraux

Dans cette section nous allons expliquer pourquoi les projections spectrales et les projections spectrales régularisées ainsi que leur calcul fonctionnel associé (introduit à la section 2.5) ont toujours du sens ici et ce qui diffère dans les preuves de leur existence.

4.2.1 Existence des opérateurs spectraux

Soit I un intervalle fermé de $\sigma_{\text{ess}}(H)$. On définit les projections spectrales « régularisées » associées à l'opérateur H sur I par

$$(r_I \mathbf{1}_I)(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (r_I(\lambda + i\varepsilon) \mathcal{R}_H(\lambda + i\varepsilon) - r_I(\lambda - i\varepsilon) \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda, \quad (4.5)$$

où r_I est la fonction régularisant les singularités spectrales sur l'intervalle I (voir la remarque sous l'hypothèse 4.4). Si I ne contient pas de singularité spectrale on aura $r_I = 1$ et on définira

$$\mathbf{1}_I(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda, \quad (4.6)$$

la projection spectrale associée à H sur l'intervalle I . En particulier les propriétés vérifiées pour $(r_I \mathbf{1}_I)(H)$ sont également vérifiées pour $\mathbf{1}_I(H)$. La proposition suivante établit l'existence de $(r_I \mathbf{1}_I)(H)$ sous nos hypothèses.

Proposition 4.2.1. *Supposons que les hypothèses 4.1-4.4 sont satisfaites. Soit I un intervalle fermé de $\sigma_{\text{ess}}(H)$, alors $(r_I \mathbf{1}_I)(H)$ existe et appartient à $\mathcal{B}(\mathcal{H})$. De plus si I ne contient pas de singularité spectrale alors $\mathbf{1}_I(H)$ est une projection.*

Avec les hypothèses 4.2 et 4.3, C est relativement lisse au sens de Kato par rapport à H_0 et la résolvante à poids de H se prolonge en un opérateur borné, la preuve de la proposition 4.2.1 s'obtient en répétant la preuve de la proposition 2.5.3. De plus, au sens des formes quadratiques sur $\mathcal{H} \times \mathcal{H}$, l'opérateur $(r_I \mathbf{1}_I)(H)$ s'écrit

$$\begin{aligned} (r_I \mathbf{1}_I)(H) &= r(H_0) \mathbf{1}_I(H_0) - \frac{1}{2\pi i} \int_I r_I(\lambda) \mathcal{R}_0(\lambda \pm i0^+) C W C \mathcal{R}_0(\lambda \pm i0^+) d\lambda \\ &\quad + \frac{1}{2\pi i} \int_I r_I(\lambda) \mathcal{R}_0(\lambda \pm i0^+) C W C \mathcal{R}_H(\lambda \pm i0^+) C W C \mathcal{R}_0(\lambda \pm i0^+) d\lambda, \end{aligned}$$

où $\lambda \mapsto C \mathcal{R}_0(\lambda \pm i0^+)$ est la limite dans $L^2(I, \mathcal{H})$ de $\lambda \mapsto C \mathcal{R}_0(\lambda \pm i\varepsilon)$ quand $\varepsilon \rightarrow 0^+$ et $\lambda \mapsto r_I(\lambda) C \mathcal{R}_H(\lambda \pm i0^+) C W$ est la limite presque partout de $\lambda \mapsto r_I(\lambda \pm i\varepsilon) C \mathcal{R}_H(\lambda \pm i\varepsilon) C W$ dans $\mathcal{B}(\mathcal{H})$.

4.2.2 Calcul fonctionnel régularisé

Dans cette section nous allons définir un calcul fonctionnel continu pour H en utilisant les projections spectrales que nous avons introduites à la section précédente.

Soit I un intervalle fermé de $\sigma_{\text{ess}}(H)$, notons $\mathcal{C}_b(I)$ l'ensemble des fonctions continues bornées sur I . Il est bien connu que muni de la norme $\|\cdot\|_\infty$, où

$$\forall f \in \mathcal{C}_b(I), \quad \|f\|_\infty = \sup_{x \in I} |f|,$$

$\mathcal{C}_b(I)$ est un espace vectoriel normé complet.

Nous définissons l'espace vectoriel des fonctions régularisé sur I par

$$\mathcal{C}_{b,\text{reg}}(I) := \{f : I \rightarrow \mathbb{C}, \exists g \in \mathcal{C}_b(I), f = r_I g\},$$

que nous munissons de la norme

$$\begin{aligned} \|\cdot\|_{\text{reg}} : \mathcal{C}_{b,\text{reg}}(I) &\rightarrow \mathbb{R}_+ \\ f &\mapsto \|g\|_\infty. \end{aligned}$$

On alors peut vérifier que $\mathcal{C}_{b,\text{reg}}(I)$ est complet pour la norme $\|\cdot\|_{\text{reg}}$. De plus, on définit l'espace vectoriel des polynômes régularisés par

$$\mathcal{P}_{\text{reg}}(I) := \{r_I P \mid P \in \mathbb{R}[x]\},$$

où $\mathbb{R}[x]$ est l'ensemble des polynômes à coefficients dans \mathbb{R} à une indéterminée et si I est borné, alors $\mathcal{P}_{\text{reg}}(I)$ est dense dans $\mathcal{C}_{b,\text{reg}}(I)$ pour la topologie de $\|\cdot\|_{\text{reg}}$.

La proposition suivante établit alors un calcul fonctionnel régularisé pour H .

Proposition 4.2.2. *Supposons que les hypothèses 4.1-4.4 sont satisfaites. Soit I un intervalle fermé de $\sigma_{\text{ess}}(H)$, alors l'application*

$$\begin{aligned} \mathcal{T} : \mathcal{C}_{b,\text{reg}}(I) &\longrightarrow \mathcal{B}(\mathcal{H}) \\ f &\mapsto (f\mathbf{1}_I)(H), \end{aligned}$$

où

$$(f\mathbf{1}_I)(H) := \text{w-lim}_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I g(\lambda) (r_I(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)) d\lambda$$

est bien définie et est un morphisme continu d'espaces de Banach. De plus $(f\mathbf{1}_I)(H)$ s'écrit au sens des formes quadratiques par

$$\begin{aligned} (f\mathbf{1}_I)(H) &= f(H_0)\mathbf{1}_I(H_0) - \frac{1}{2\pi i} \int_I g(\lambda)r_I(\lambda)\mathcal{R}_0(\lambda \pm i0^+)CWCR_0(\lambda \pm i0^+)d\lambda \\ &+ \frac{1}{2\pi i} \int_I g(\lambda)r_I(\lambda)\mathcal{R}_0(\lambda \pm i0^+)CWCR_H(\lambda \pm i0^+)CWCR_0(\lambda \pm i0^+)d\lambda. \end{aligned} \quad (4.7)$$

Avant de démontrer la proposition 4.2.2 nous allons montrer un lemme qui nous sera utile dans la suite.

Lemme 4.2.3. *Supposons que les hypothèses 4.1-4.4 soient satisfaites. Soit $I \subset \sigma_{\text{ess}}(H)$ un intervalle fermé, alors pour tout $u \in \mathcal{H}$,*

$$\int_I \|r_I(\lambda \pm i\varepsilon)\mathcal{R}_H(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda = \mathcal{O}\left(\frac{1}{\varepsilon}\right) \|u\|_{\mathcal{H}}^2, \quad \varepsilon \rightarrow 0^+. \quad (4.8)$$

Démonstration. Soit $u \in \mathcal{H}$, soit $\varepsilon > 0$ assez petit, alors

$$\begin{aligned} & \int_I \|r_I(\lambda \pm i\varepsilon)\mathcal{R}_H(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda \\ &= \int_I \langle r_I(\lambda \pm i\varepsilon)u, r_I(\lambda \pm i\varepsilon)\mathcal{R}_{H^*}(\lambda \mp i\varepsilon)\mathcal{R}_H(\lambda \pm i\varepsilon)u \rangle_{\mathcal{H}} d\lambda. \end{aligned} \quad (4.9)$$

Or d'après l'identité de la résolvante,

$$\begin{aligned} \mathcal{R}_{H^*}(\lambda \pm i\varepsilon) - \mathcal{R}_H(\lambda \mp i\varepsilon) &= 2i\mathcal{R}_{H^*}(\lambda \mp i\varepsilon)C \operatorname{Im}(W)C\mathcal{R}_H(\lambda \pm i\varepsilon) \\ &\mp 2i\varepsilon\mathcal{R}_{H^*}(\lambda \mp i\varepsilon)\mathcal{R}_H(\lambda \pm i\varepsilon). \end{aligned}$$

Ainsi (4.9) se réécrit,

$$\begin{aligned} & \int_I \|r_I(\lambda \pm i\varepsilon)\mathcal{R}_H(\lambda \pm i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda \\ &= \pm \frac{1}{2i\varepsilon} \int_I \langle r_I(\lambda \pm i\varepsilon)u, r_I(\lambda \pm i\varepsilon)(\mathcal{R}_{H^*}(\lambda \pm i\varepsilon) - \mathcal{R}_H(\lambda \mp i\varepsilon))u \rangle_{\mathcal{H}} d\lambda \end{aligned} \quad (4.10)$$

$$\mp \frac{1}{\varepsilon} \int_I \langle r_I(\lambda \pm i\varepsilon)C\mathcal{R}_H(\lambda \pm i\varepsilon)u, r_I(\lambda \pm i\varepsilon) \operatorname{Im}(W)C\mathcal{R}_H(\lambda \pm i\varepsilon)u \rangle_{\mathcal{H}} d\lambda. \quad (4.11)$$

On utilise encore l'identité de la résolvante :

$$\begin{aligned} \forall z \in \rho(H), \quad \mathcal{R}_H(z) &= \mathcal{R}_0(z) - \mathcal{R}_H(z)V\mathcal{R}_0(z) \\ &= \mathcal{R}_0(z) - \mathcal{R}_0(z)V\mathcal{R}_0(z) + \mathcal{R}_0(z)V\mathcal{R}_H(z)V\mathcal{R}_0(z). \end{aligned}$$

En appliquant alors la première égalité à (4.11) et la deuxième à (4.10) et en utilisant les mêmes techniques que pour prouver que les projections spectrales régularisées existent (voir les preuves des propositions 2.5.1 et 2.5.3 en appendice 2.6.3), ce qui est possible grâce aux hypothèses 4.2 et 4.4, on obtient que les intégrales dans (4.10) et (4.11) sont uniformément bornées par une constante indépendante de ε multiplié par $\|u\|_{\mathcal{H}}^2$, ce qui montre (4.8). \square

Démonstration de la proposition 4.2.2. Commençons par nous placer dans le cas où I est un intervalle compact. Montrons que pour tout $n \in \mathbb{N}^*$, pour tout $u, v \in \mathcal{D}(H_0)$,

$$\langle \mathcal{T}(x \mapsto x^n r_I(x))u, v \rangle = \langle H^n(r_I \mathbb{1}_I)(H)u, v \rangle_{\mathcal{H}}. \quad (4.12)$$

Pour $n = 1$ on a

$$\begin{aligned}
& \langle H(r_I \mathbf{1}_I)(H)u, v \rangle_{\mathcal{H}} \\
&= \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I \langle H(r_I(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))u, v \rangle_{\mathcal{H}} d\lambda \\
&= \lim_{\varepsilon \rightarrow 0^+} \frac{-\varepsilon}{2\pi} \int_I \langle (r_I(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) + r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))u, v \rangle_{\mathcal{H}} d\lambda \\
&+ \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi i} \int_I \langle \lambda(r_I(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))u, v \rangle_{\mathcal{H}} d\lambda \\
&= \lim_{\varepsilon \rightarrow 0^+} \frac{-\varepsilon}{2\pi} \int_I \langle (r_I(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) + r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))u, v \rangle_{\mathcal{H}} d\lambda \\
&+ \langle \mathcal{T}(x \mapsto xr_I(x))u, v \rangle_{\mathcal{H}}. \tag{4.13}
\end{aligned}$$

Il suffit donc de montrer que le premier terme dans le membre de droite de (4.13) est nul pour obtenir (4.12). Pour tout $\varepsilon > 0$ assez petit, on a

$$\begin{aligned}
& \frac{\varepsilon}{2\pi} \int_I \langle (r_I(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) + r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))u, v \rangle_{\mathcal{H}} d\lambda \\
&= \frac{\varepsilon}{2\pi} \int_I \langle (r_I(\lambda + i\varepsilon)\mathcal{R}_H(\lambda + i\varepsilon) - r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon))u, v \rangle_{\mathcal{H}} d\lambda \\
&+ \lim_{\varepsilon \rightarrow 0^+} \frac{\varepsilon}{\pi} \int_I \langle r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)u, v \rangle_{\mathcal{H}} d\lambda. \tag{4.14}
\end{aligned}$$

Comme la projection spectrale régularisée existe sur l'intervalle I , le premier terme du membre de droite de (4.14) tend vers 0 quand $\varepsilon \rightarrow 0$. Par ailleurs comme I est borné, d'après l'inégalité de Cauchy-Schwarz, on a

$$\int_I \|r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)u\|_{\mathcal{H}} d\lambda \leq |I| \left(\int_I \|r_I(\lambda - i\varepsilon)\mathcal{R}_H(\lambda - i\varepsilon)u\|_{\mathcal{H}}^2 d\lambda \right)^{\frac{1}{2}},$$

et avec le lemme 4.2.3, on a que le deuxième terme du membre de (4.14) tend vers 0 quand $\varepsilon \rightarrow 0^+$. Finalement, en procédant ensuite par récurrence sur n on obtient (4.12). De plus, pour tout $u \in \mathcal{D}(H_0)$, on a

$$\mathcal{T}(x^n r_I)u = H^n(r_I \mathbf{1}_I)(H)u.$$

Par ailleurs comme les monômes régularisés forment une base de l'espace vectoriel des polynômes régularisés, on obtient que pour tout $P \in \mathcal{P}_{\text{reg}}(I)$, pour tout $u \in \mathcal{D}(H_0)$,

$$\mathcal{T}(P)u = P(H)(r_I \mathbf{1}_I)(H)u.$$

En particulier l'équation (4.7) est satisfaite pour tout $P \in \mathcal{P}_{\text{reg}}(I)$. Finalement comme l'ensemble des polynômes régularisés sur I est dense dans l'ensemble des fonctions régularisées continues bornées sur I , alors en utilisant par exemple [76, Théorème I.7] \mathcal{T}

se prolonge de façon unique à $\mathcal{C}_{b,\text{reg}}(I)$, et en particulier, l'équation (4.7) se prolonge à $\mathcal{C}_{b,\text{reg}}(I)$. Ceci prouve que \mathcal{T} existe et est bien définie si I est borné. De plus, il est clair que dans ce cas \mathcal{T} satisfait la propriété de morphisme. Par ailleurs, en utilisant (4.7), comme pour tout $f \in \mathcal{C}_{b,\text{reg}}$, pour tout $u, v \in \mathcal{H}$, on a

$$\begin{aligned} \langle \mathcal{T}(f)u, v \rangle_{\mathcal{H}} &= \langle f(H_0)\mathbf{1}_I(H_0)u, v \rangle_{\mathcal{H}} \\ &- \frac{1}{2\pi i} \int_I g(\lambda)r_I(\lambda) \langle C\mathcal{R}_0(\lambda \mp i0^+)u, W C\mathcal{R}_0(\lambda \pm i0^+)v \rangle_{\mathcal{H}} d\lambda \\ &+ \frac{1}{2\pi i} \int_I g(\lambda) \langle C\mathcal{R}_0(\lambda \mp i0^+)u, W r_I(\lambda)C\mathcal{R}_H(\lambda \pm i0^+)C W C\mathcal{R}_0(\lambda \pm i0^+)v \rangle_{\mathcal{H}} d\lambda, \end{aligned}$$

et qu'avec l'hypothèse 4.2, C est relativement lisse au sens de Kato par rapport à H_0 , et qu'avec l'hypothèse 4.4, $\lambda \mapsto r_I(\lambda)C\mathcal{R}_H(\lambda \pm i0^+)C W$ est uniformément bornée dans $\mathcal{B}(\mathcal{H})$ sur I et que g est bornée sur I , alors

$$|\langle \mathcal{T}(f)u, v \rangle_{\mathcal{H}}| \leq \|g\|_{\infty} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \leq \|f\|_{\text{reg}} \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \quad (4.15)$$

ce qui montre que \mathcal{T} est continue et satisfait l'équation fonctionnelle

$$\mathcal{T}(f) = (gr_I\mathbf{1}_I)(H)$$

dans $\mathcal{B}(\mathcal{H})$.

Maintenant, si I n'est pas borné, alors en approchant I par une suite d'intervalles bornés $(I_n)_{n \in \mathbb{N}}$ et en utilisant que (4.15) ne dépend pas de I , on peut alors faire tendre n vers $+\infty$ dans l'équation (4.7) et grâce au théorème de convergence dominée, on obtient que \mathcal{T} existe, est bien définie pour tout $I \subset \sigma_{\text{ess}}(H)$ et que c'est un morphisme d'espaces de Banach. \square

Si I ne contient pas de singularité spectrale, alors le calcul fonctionnel se réécrit pour les fonctions appartenant à $\mathcal{C}_b(I)$ et \mathcal{T} est défini par

$$\forall f \in \mathcal{C}_b(I), \quad \mathcal{T}(f) = \frac{1}{2\pi i} \text{w-lim}_{\varepsilon \rightarrow 0^+} \int_I f(\lambda) (\mathcal{R}_H(\lambda + i\varepsilon) - \mathcal{R}_H(\lambda - i\varepsilon)) d\lambda.$$

De plus le fait que $\mathbf{1}_I(H)$ soit une projection dans ce cas fait de \mathcal{T} un morphisme d'algèbres de Banach.

4.2.3 Construction du groupe d'évolution

Nous avons à présent tous les outils nécessaires pour construire le groupe d'évolution associé à $-iH$. Soit $I \subset \sigma_{\text{ess}}(H)$. Pour tout $t \in \mathbb{R}$, la fonction $I \ni x \mapsto g_t(x) := e^{-itx}r_I(x)$ appartient à $\mathcal{C}_{b,\text{reg}}(I)$. On peut donc appliquer le calcul fonctionnel régularisé de la proposition 4.2.2 à la suite de fonctions $(g_t)_{t \in \mathbb{R}}$. Ceci permet de construire une solution faible à l'équation de Schrödinger (4.1).

Proposition 4.2.4. *Supposons que les hypothèses 4.1-4.4 sont satisfaites. Soit I un intervalle fermé de $\sigma_{\text{ess}}(H)$. Alors pour tout $u_0 \in \mathcal{D}(H_0)$, l'application*

$$\mathbb{R} \ni t \mapsto \mathcal{T}(I \ni x \mapsto e^{-itx} r_I(x)) u_0$$

est solution au sens des formes quadratiques de l'équation de Schrödinger.

Démonstration. Supposons pour commencer que I est borné. Pour tout $t \in \mathbb{R}$, posons $g_t : I \ni x \mapsto e^{-itx} r_I(x)$, alors comme g_t appartient à $\mathcal{C}_{\text{b,reg}}(I)$, d'après la proposition 4.2.2, $\mathcal{T}(g_t)$ est bien défini et est un opérateur borné. Soit $u \in \mathcal{D}(H)$, montrons que $\mathbb{R} \ni t \mapsto \mathcal{T}(g_t)u \in \mathcal{H}$ est solution de l'équation de Schrödinger avec condition initiale u . Soit $v \in \mathcal{H}$, alors d'après la proposition 4.2.2, on a

$$\begin{aligned} \langle v, \mathcal{T}(g_t)u \rangle_{\mathcal{H}} &= \langle v, e^{-itH_0} r_I(H_0) \mathbb{1}_I(H_0) u \rangle_{\mathcal{H}} \\ &- \frac{1}{2\pi i} \int_I e^{-it\lambda} r_I(\lambda) \langle C\mathcal{R}_0(\lambda \mp i0^+) u, W C\mathcal{R}_0(\lambda \pm i0^+) v \rangle_{\mathcal{H}} d\lambda \\ &+ \frac{1}{2\pi i} \int_I e^{-it\lambda} \langle C\mathcal{R}_0(\lambda \mp i0^+) v, W r_I(\lambda) C\mathcal{R}_H(\lambda \pm i0^+) C W C\mathcal{R}_0(\lambda \pm i0^+) u \rangle_{\mathcal{H}} d\lambda. \end{aligned}$$

En utilisant que $t \mapsto e^{-itH_0}$ est un groupe d'évolution et le théorème de dérivation sous le signe intégrale dans les deux derniers termes du membre de droite de l'équation précédente, (les dominations proviennent du fait que C est relativement lisse par rapport à H_0 d'après l'hypothèse 4.2, que $\lambda \mapsto r_I(\lambda) C\mathcal{R}_H(\lambda \pm i0^+) C W$ sont bornées sur I d'après l'hypothèse 4.4 et que I est borné) alors on a,

$$\begin{aligned} \partial_t (\langle v, \mathcal{T}(g_t)u \rangle_{\mathcal{H}}) &= \langle v, -iH_0 e^{-itH_0} r_I(H_0) \mathbb{1}_I(H_0) u \rangle_{\mathcal{H}} \\ &- \frac{1}{2\pi i} \int_I (-i\lambda) e^{-it\lambda} r_I(\lambda) \langle C\mathcal{R}_0(\lambda \mp i0^+) u, W C\mathcal{R}_0(\lambda \pm i0^+) v \rangle_{\mathcal{H}} d\lambda \\ &+ \frac{1}{2\pi i} \int_I (-i\lambda) e^{-it\lambda} \langle C\mathcal{R}_0(\lambda \mp i0^+) v, W r_I(\lambda) C\mathcal{R}_H(\lambda \pm i0^+) C W C\mathcal{R}_0(\lambda \pm i0^+) u \rangle_{\mathcal{H}} d\lambda. \end{aligned}$$

Puis en utilisant la définition de la projection spectrale régularisée, on obtient

$$\begin{aligned} \partial_t (\langle v, \mathcal{T}(g_t)u \rangle_{\mathcal{H}}) &= \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0^+} \int_I \langle v, -i\lambda e^{-it\lambda} (r_I(\lambda + i\varepsilon) \mathcal{R}_H(\lambda + i\varepsilon) - r_I(\lambda - i\varepsilon) \mathcal{R}_H(\lambda - i\varepsilon)) u \rangle_{\mathcal{H}} d\lambda \\ &= \langle v, -iH\mathcal{T}(g_t)u \rangle_{\mathcal{H}}, \end{aligned} \tag{4.16}$$

où (4.16) a été obtenu de la même façon que (4.13), ce qui montre le résultat.

Maintenant si I n'est pas un intervalle borné, alors en l'approchant par une suite d'intervalles compacts $(I_n)_{n \in \mathbb{N}}$, on peut montrer que

$$\frac{1}{2\pi i} \int_{I_n} \langle v, -iH e^{-itH} r_I(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) u \rangle_{\mathcal{H}} d\lambda$$

converge uniformément, par rapport à t vers

$$\frac{1}{2\pi i} \int_I \langle v, -iHe^{-itH} r_I(\lambda) (\mathcal{R}_H(\lambda + i0^+) - \mathcal{R}_H(\lambda - i0^+)) u \rangle_{\mathcal{H}} d\lambda,$$

ce qui permet d'obtenir le résultat. \square

La proposition suivante montre que si $I \subset \sigma_{\text{ess}}(H)$ est un intervalle ne contenant pas de singularité spectrale alors la solution de l'équation de Schrödinger (4.1) donnée grâce au calcul fonctionnel est en fait naturellement associée à un groupe d'évolution.

Proposition 4.2.5. *Supposons que les hypothèses 4.1-4.4 sont satisfaites. Soit I un intervalle fermé de $\sigma_{\text{ess}}(H)$ ne contenant pas de singularité spectrale, alors pour tout $u_0 \in \mathcal{D}(H_0)$, l'application*

$$\mathbb{R} \ni t \mapsto \mathcal{T}(I \ni x \mapsto e^{-itx})u_0$$

est solution au sens des formes quadratiques de l'équation de Schrödinger (4.1). De plus cette application satisfait la propriété de groupe.

L'existence de la solution provient de la proposition 4.2.4. La propriété de groupe est liée au fait que dans le cas où I est un intervalle sans singularité spectrale, \mathcal{T} est un morphisme d'algèbres.

4.3 Perspectives

Dans ce chapitre nous avons montré comment construire une solution de l'équation de Schrödinger pour une condition $u \in \mathcal{D}(H_0)$. Cependant, si on décompose le spectre de H comme réunion disjointe du spectre discret et du spectre essentiel et que l'on démontre une formule de décomposition spectrale similaire à celle de la proposition 2.3.10, ce qui est caché ici est qu'en construisant une solution de (4.1) pour une condition initiale $u \in \mathcal{D}(H_0)$, on a projeté u sur le sous-espace spectral associé à l'intervalle $I \subset \sigma_{\text{ess}}(H)$ que l'on considère. Une perspective envisagée est de construire une solution de (4.1) sans projeter sur un sous-espace spectral. Pour cela, l'idée serait de munir $\text{Ran}(r(H))$ d'une topologie satisfaisante afin de voir les opérateurs spectraux comme des opérateurs de $\mathcal{B}(\text{Ran}(r(H)), \mathcal{H})$, puis de construire un calcul fonctionnel similaire à celui de la proposition 4.2.2. Enfin, obtenir une formule de décomposition spectrale similaire à celle de la proposition 2.3.10 permettrait de construire une solution de (4.1) pour toute condition initiale $u \in \text{Ran}(r(H)) \cap \mathcal{D}(H)$. Il faut souligner d'ailleurs que $\text{Ran}(r(H))$ est dense dans $\mathcal{H}_p(H^*)^\perp$. De plus, une telle construction devrait permettre de caractériser également les états qui disparaissent à l'infini. Enfin grâce au calcul fonctionnel construit ci-dessus, on peut penser qu'il est raisonnable (en faisant les hypothèses nécessaires) de définir un sous-espace spectral absolument continu comme au chapitre 2 et que celui-ci satisfera la caractérisation d'orthogonalité avec $\mathcal{H}_p(H^*)$.

Bibliographie

- [1] M. Aafarani, *Large time behavior of solutions to Schrödinger equation with complex-valued potential*, J. Math. Pures Appl., 150, (2021), 64–111.
- [2] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 2, (1975), 151–218.
- [3] J-P. Antoine and C. Trapani, *Partial inner product spaces, metric operators and generalized hermicity*, J.Phys. A : Math Theor., 46, (2013), 21p.
- [4] F. Bagarello, J.-P. Gazeau, H.F Szafraniec, M. Znojil, *Non-Selfadjoint Operators in Quantum Physics. Mathematical Aspects*, Hoboken, NJ : John Wiley & Sons, Inc, Hoboken, NJ, 2015.
- [5] C.M. Bender, S. Boettcher. *Real spectra in non-Hermitian Hamiltonians having PT symmetry*, Phys. Rev. Lett., 80, (1998), 5243–5246.
- [6] J. Behrndt, A. F. M. ter Elst, F. Gesztesy, *The Generalized Birman-Schwinger Principle*, arXiv :2005.01195, (2020).
- [7] N. Bohr. *Neutron capture and nuclear constitution*, Nature, 137, (1936), 344–348.
- [8] N. Boussaid, A. Comech, *Limiting absorption principle and virtual levels of operators in Banach spaces*, Ann. Math. Qué., 46, No. 1, (2022), 161–180.
- [9] D. Borisov and D. Krejcirik, *PT-symmetric waveguides*, Integral Equations Operator Theory, 62, (2008), no. 4, 489–515.
- [10] D. Borisov and D. Krejcirik, *The effective Hamiltonian for thin layers with non-Hermitian Robin-type boundary conditions*, Asympt. Anal., 76, (2012), 49–59.
- [11] V.Bruneau, E.M. Ouhabaz *Lieb-Thirring estimates for non-self-adjoint Schrödinger operators*, Journal of Mathematical Physics 49, 093504 (2008).
- [12] L. Cossetti, L. Fanelli and D. Krejcirik, *Absence of eigenvalues of Dirac and Pauli Hamiltonians via the method of multipliers*, Comm. Math. Phys., 379, (2020), 633–691.

- [13] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger operators, with application to quantum mechanics and global geometry*, Springer Study edition, Texts and Monographs in Physics, 1987.
- [14] E. B. Davies. *Two-channel Hamiltonians and the optical model of nuclear scattering*, Ann. Inst. H. Poincaré Sect. A (N.S.), 29, (1978), 395–413.
- [15] E. B. Davies. *One-Parameter Semigroups*, Academic Press Inc. (London) LTD, (1980).
- [16] E. B. Davies. *Nonunitary scattering and capture. I. Hilbert space theory*, Comm. Math. Phys. (3) 71, (1980), 277–288.
- [17] E. B. Davies. *Nonunitary scattering and capture. II. Quantum dynamical semigroup theory*. Ann. Inst. H. Poincaré Sect. A (N.S.), 32(4) :361–375, 1980.
- [18] E.B. Davies, *The functional calculus*, J. Lond. Math. Soc., 52, (1995), 166–176.
- [19] E. B. Davies. *Linear operators and their spectra*, volume 106 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2007.
- [20] J. Dieudonné. *Quasi-Hermitian operators*, Proceedings of the International Symposium on Linear Spaces, (1961), 115–123.
- [21] W.H. Dickhoff and R.J. Charity. *Recent developments for the optical model of nuclei*, Prog. Part. Nucl. Phys., 105, (2019), 252–299.
- [22] P. Dorey, C. Dunning, and R. Tateo, *Spectral equivalences, Bethe ansatz equations, and reality properties in PT-symmetric quantum mechanics*, J. Phys. A, 34, (2001), 5679–5704.
- [23] N. Dunford. *A survey of the theory of spectral operators*, Bull. Amer. Math. Soc., 64, (1958), 217–274.
- [24] N. Dunford and J. T. Schwartz. *Linear operators. Part III : Spectral operators*, Interscience Publishers [John Wiley & Sons, Inc.], New York-London-Sydney, 1971.
- [25] S. Dyatlov and M. Zworski. *Mathematical theory of scattering resonances*, AMS studies in Mathematics 200, 2019.
- [26] R. El-Ganainy, K.G. Makris, M. Khajavikhan, Z.H. Musslimani, S. Rotter and D. N. Christodoulides, *Non-Hermitian physics and PT symmetry*, Nature Physics, 14, (2018), 11–19.
- [27] D. E. Edmunds and W. D. Evans. *Spectral theory and differential operators*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1987.

- [28] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.
- [29] V. Enss, *Asymptotic Completeness for Quantum Mechanical Potentials Scattering*, Commun. math. Phys. 61, 285-291 (1978).
- [30] V. Enss, *Asymptotic Observables on Scattering States*, Commun. Math. Phys. 89, 245-268 (1963).
- [31] M. Falconi, J. Faupin, J. Fröhlich and B. Schubnel, *Scattering Theory for Lindblad Master Equations*, Comm. Math. Phys., (3) 350, (2017), 1185–1218.
- [32] J. Faupin, *Generic nature of asymptotic completeness in dissipative scattering theory*, Rev. Math. Phys., 33, No. 1, (2021), 23 p.
- [33] L. Fanelli, D. Krejcirik and L. Vega, *Spectral stability of Schrodinger operators with subordinated complex potentials*, J. Spectr. Theory, 8, (2018), 575–604.
- [34] J. Faupin and F. Nicoleau, *Scattering matrices for dissipative quantum systems*, J. Funct. Anal., 9, (2019), 3062–3097.
- [35] J. Faupin and J. Fröhlich, *Asymptotic completeness in dissipative scattering theory*, Adv. Math., 340, (2018), 300–362.
- [36] J. Faupin and N. Frantz *Spectral decomposition of some non self-adjoint operators* preprint on ArXiv 2203.12406 (2022)
- [37] H. Feshbach, C. Porter and V. Weisskopf. *Model for nuclear reactions with neutrons*, Phys. Rev., 96, (1954), 448–464.
- [38] H. Feshbach, *Unified theory of nuclear reactions*, Ann. Phys. (NY), (1958), 357–390.
- [39] H. Feshbach, *The optical model and its justification*, Ann. Rev. Nucl. Sci., 8, (1958).
- [40] H. Feshbach, *A unified theory of nuclear reactions II*. Ann. Phys. (NY), (1962), 287–313.
- [41] H. Feshbach, *Theoretical Nuclear Physics, Nuclear Reactions*, Wiley, New York, 1992.
- [42] R. L. Frank, *Eigenvalue bounds for Schrödinger operators with complex potentials. III*, Trans. Amer. Math. Soc., 370, (2018), 219–240.
- [43] R. L. Frank, A. Laptev, and O. Safronov, *On the number of eigenvalues of Schrödinger operators with complex potentials*, J. Lond. Math. Soc., (2) 94, (2016), 377–390.
- [44] V. Georgescu, C. Gérard and D. Häfner, *Boundary values of resolvents of selfadjoint operators in Krein spaces*, J. Funct. Anal., 265, No. 12, (2013), 3245–3304.

- [45] F. Gesztesy, Y. Latushkin, M. Mitrea and M. Zinchenko, *Non-self-adjoint operators, infinite determinants and some applications*, Russ. J. Math. Phys., 12, (2005), 43–471.
- [46] M. Goldberg, *A dispersive bound for three-dimensional Schrödinger operators with zero energy eigenvalues*, Comm. Partial Differential Equations, 35, (2010), 1610–1634.
- [47] C. Goldstein, *Perturbation of non-selfadjoint operators. I*, Arch. Rational Mech. Anal., 37, (1970), 268–296.
- [48] C. Goldstein, *Perturbation of non-selfadjoint operators. II*, Arch. Rational Mech. Anal., 42, (1971), 380–402.
- [49] M. Hansmann and D. Krejcirik, *The abstract Birman-Schwinger principle and spectral stability*, J. Anal. Math., 148, (2022), 361–398.
- [50] P. E. Hodgson, *The nuclear optical model*, Rep. Prog. Phys., 34, (1971), 765–819.
- [51] J. S. Howland, *Puiseux series for resonances at an embedded eigenvalue*, Pacific J. Math., 55, (1974), 157–176.
- [52] G. E. Huige, *Perturbation theory of some spectral operators*, Comm. Pure Appl. Math., 24, (1971), 741–757.
- [53] A. Jensen and T. Kato, *Spectral properties of Schrödinger operators and time-decay of the wave functions*, Duke Math. J., 46, (1979), no. 3, 583–611.
- [54] T. Kako and K. Yajima, *Spectral and Scattering Theory for a Class of Non-Selfadjoint Operator* Sci. Papers College Gen. Ed. Univ. Tokyo, 26, (1976).
- [55] T. Kato, *Wave operators and similarity for some non-selfadjoint operators*, Math. Ann., 162, (1965/1966), 258–279.
- [56] T. Kato, *Perturbation theory for linear operators* Die Grundlehren der mathematischen Wissenschaften, Band 132, Springer-Verlag New York, Inc., New York, 1966.
- [57] T. Kato, *Scattering theory with two Hilbert spaces*, J. Funct. Anal. 1 (1967) 342–369
- [58] T. Kato and T. Kuroda, *The Abstract Theory of Scattering*, Rocky Mountain, Journal of Mathematics, Volume 1, Number 1, Winter 1971
- [59] T. Kato, *On the Cook-Kuroda Criterion in Scattering Theory*, Commun. Maths. Phys. 67, 85–90 (1979).
- [60] A.V. Kiselev and S. Naboko, *Nonsself-adjoint operators with almost Hermitian spectrum : Cayley identity and some questions of spectral structure*, Ark. Mat., 47, (2009), 91–125.

- [61] D. Krejcirik, *Calculation of the metric in the Hilbert space of a PT -symmetric model via the spectral theorem*, J. Phys. A : Math. Theor., 41, (2008), 244012.
- [62] D. Krejcirik, *Mathematical aspects of quantum mechanics with non-self-adjoint operators*, Habilitation Thesis, Universitas Carolina Pragensis, 2017.
- [63] D. Krejcirik, H. Bila and M. Znojil, *Closed formula for the metric in the Hilbert space of a PT -symmetric model*, J. Phys. A, 39, (2006), 10143–10153.
- [64] D. Krejcirik, P. Siegl, M. Tater and J. Viola, *Pseudospectra in non-Hermitian quantum mechanics*, J. Math. Phys., 56, (2015), 103513.
- [65] R. Konno and S. T. Kuroda, *On the finiteness of perturbed eigenvalues*, J. Fac. Sci., Univ. Tokyo, Sec. I, 13, (1966), 55–63.
- [66] S. T. Kuroda *Scattering theory for differential operators, I, operator theory*, J. Math. Sco. Japan, Vol. 25, No. 1, 1973.
- [67] S. Longhi, *Bloch Oscillations in Complex Crystals with PT Symmetry*, Phys. Rev. Lett., 103, (2009), 123601.
- [68] P. A. Martin, *Scattering theory with dissipative interactions and time delay*, Nuovo Cimento B, 30, (1975), 217–238.
- [69] K. Mochizuki, *On the large perturbation by a class of non-selfadjoint operators*, J. Math. Soc. Japan, 19, (1967), 123–158.
- [70] K. Mochizuki, *Eigenfunction expansions associated with the Schrödinger operator with a complex potential and the scattering theory*, Publ. Res. Inst. Math. Sci. Ser. A, 4, (1968/69), 419–466.
- [71] K. Mochizuki. *Eigenfunction expansions associated with the Schrödinger operator with a complex potential and the scattering inverse problem*, Proceedings of the Japan Academy, Proc. Japan Acad. 43(7), (1967), 638-643.
- [72] K. Mochzuki. *Scattering theory for wave equations with dissipative terms*, J. Operator Theory, 14(1) :57-66, 1985.
- [73] A. Mostafazadeh, *Pseudo-Hermiticity versus PT symmetry : II. A complete characterization of non-Hermitian Hamiltonians with a real spectrum*, J. Math. Phys., 43, (2002), 2814–2816.
- [74] S.N. Naboko, *Absolutely continuous spectrum of a nondissipative operator, and a functional model. I*, Zap. Nauchn. Sem. Leningrad. Otdel Mat. Inst. Steklov., 65, (1976), 90–02 (Russian). English transl. : J. Soviet Math., 16, (1981), 1109–1117.
- [75] V. Petkov, *Scattering theory for hyperbolic operators*, volume 21 of Studies in Mathematics and its Applications, North-Holland Publishing Co., Amsterdam, 1989.

- [76] M. Reed and B. Simon. *Methods of modern mathematical physics. I–IV*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975–1980.
- [77] A. Regensburger, Ch. Bersch, M.-A. Miri, G. Onishchukov, D. N. Christodoulides, and U. Peschel, *Parity-time synthetic photonic lattices*, Nature, 488, (2012), 167–171.
- [78] J. Royer, *Limiting absorption principle for the dissipative Helmholtz equation*, Communication in Partial Differential Equations 25, (2010), N°8, P.1458-1489.
- [79] J. Royer, *Mourre’s commutators method for a dissipative form perturbation*, Journal of Operator Theory 76 (2016), n°2, p351-385.
- [80] W. Rudin, *Function theory in the Unit Ball of \mathbb{C}^n* , New York, NY : Springer New York, 1980.
- [81] Ch. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, *Observation of parity-time symmetry in optics*, Nature Phys., 6, (2010), 192–195.
- [82] J. Schwartz, *Some non-selfadjoint operators*, Comm. Pure Appl. Math., 13, (1960), 609–639.
- [83] F. G. Scholtz, H. B. Geyer, and F. J. W. Hahne. *Quasi-Hermitian operators in quantum mechanics and the variational principle*, Annals of Physics, 213, (1992), 74–101.
- [84] K. C. Shin, *On the reality of the eigenvalues for a class of PT-symmetric oscillators*, Commun. Math. Phys., 229, (2002), 543–564.
- [85] B. Simon. *Phase space analysis of simple scattering systems : extensions of some work of Enss*, Duke Math. J., 46, (1979), 119–168.
- [86] S. A. Stepin. *Scattering and spectral singularities for some dissipative operators of mathematical physics*, J. Nonlinear Math. Phys., 11(suppl.), (2004), 194–203.
- [87] S.A. Stepin. *Complex Potentials : Bound States, Quantum Dynamics and Wave Operators*, Semigroups of Operators - Theory and Application (Bedlewo Poland October 2013), Springer-Verag (2015), 287-297.
- [88] S.A. Stepin, *Complete Wave operators in non-self-adjoint Kato model of smooth perturbation theory*, Russian Journal of Mathematical Physics **26**, 94-108 (2019).
- [89] X.P. Wang, *Gevrey estimates of the resolvent and sub-exponential time-decay for the heat and Schrödinger semigroups*, J. Math. Pures Appl., 135, (2020), 284–338.
- [90] X. P. Wang, *Number of eigenvalues for dissipative Schrödinger operators under perturbation*, J. Math. Pures Appl., 96, (2011) 409–422.

- [91] X. P. Wang. *Time-decay of semigroups generated by dissipative Schrödinger operators* J. Differential Equations, 253, (2012), 3523–3542.
- [92] X. P. Wang and L. Zhu, *On the wave operator for dissipative potentials with small imaginary part*, Asymptot. Anal., 86, (2014), 49–57.
- [93] Z. Wen, C.M. Bender, *PT-symmetric potentials having continuous spectra*, J. Phys. A, 53, (2020), 375302.
- [94] D. R. Yafaev. *Mathematical scattering theory*, volume 105 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1992.
- [95] D. R. Yafaev. *Mathematical scattering theory, Analytic theory*, Mathematical Surveys and Monographs 158. American Mathematical Society, Providence, RI, 2010.

Abstract

In this thesis, we consider non-self-adjoint operators in Hilbert spaces of the form $H = H_0 + CWC$, where H_0 is self-adjoint, W is bounded and C is bounded and relatively compact with respect to H_0 . We suppose that C is a metric operator and that $C(H_0 - z)^{-1}C$ is uniformly bounded in $z \in \mathbb{C} \setminus \mathbb{R}$. We define the spectral singularities of H as the points of the essential spectrum $\lambda \in \sigma_{\text{ess}}(H)$ such that $C(H - \lambda \pm i\varepsilon)^{-1}CW$ does not have a limit as $\varepsilon \rightarrow 0^+$.

First we study the spectral structure of this model. We prove that the spectral singularities of H are in one-to-one correspondence with the eigenvalues, associated to resonant states, of an extension of H to a larger Hilbert space. Next, we show that the asymptotically disappearing states for H , i.e. the set of vectors φ such that $e^{\pm itH}\varphi \rightarrow 0$ as $t \rightarrow \infty$, coincide with the generalized eigenstates of H corresponding to eigenvalues $\lambda \in \mathbb{C}$, $\mp \text{Im}(\lambda) > 0$. Finally, we define the absolutely continuous spectral subspace of H and show that it satisfies $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H^*)^\perp$, where $\mathcal{H}_{\text{p}}(H^*)$ stands for the point spectral subspace of H^* . We thus obtain a direct sum decomposition of the Hilbert spaces in terms of spectral subspaces of H .

Next we study the asymptotic behavior of $e^{\pm itH}\varphi$ as $t \rightarrow \infty$ through scattering theory. We define the regularized wave operators associated to H and H_0 by $W_\pm(H, H_0) := \text{s-lim}_{t \rightarrow \infty} e^{\pm itH} r_\mp(H) \Pi_{\text{p}}(H^*)^\perp e^{\mp itH_0}$ where $\Pi_{\text{p}}(H^*)$ is the projection onto $\mathcal{H}_{\text{p}}(H^*)$ and r_\mp is a rational function that regularizes the ‘incoming/outgoing spectral singularities’ of H . We prove the existence and study the properties of the regularized wave operators. In particular we show that they are asymptotically complete if H does not have any spectral singularity.

Finally we construct solutions to the abstract Schrödinger equation associated to H in the case where the non-self-adjoint perturbation CWC is unbounded.

Our results apply to Schrödinger operators with complex potentials.

Résumé

Dans cette thèse, nous considérons un opérateur non auto-adjoint sur un espace de Hilbert de la forme $H = H_0 + CWC$, où H_0 est auto-adjoint, W est un opérateur borné et C est un opérateur relativement compact par rapport à H_0 . De plus on suppose que C est un opérateur positif et injectif tel que $C\mathcal{R}_0(z)C$ est uniformément borné pour $z \in \mathbb{C} \setminus \mathbb{R}$. Nous définissons les singularités spectrales de H comme l'ensemble des $\lambda \in \sigma_{\text{ess}}(H_0)$ du spectre de H tel que $C\mathcal{R}_H(\lambda \pm i0^+)CW$ n'a pas de limite quand $\varepsilon \rightarrow 0^+$.

D'abord nous étudions la structure spectrale de ce modèle. Nous prouvons que les singularités spectrales de H sont en bijection avec les valeurs propres, associées à des états résonants, d'une extension de H à un espace de Hilbert plus gros. Ensuite nous montrons que les états qui disparaissent à l'infini pour H , c'est à dire l'ensemble des vecteurs φ tels que $e^{\pm itH}\varphi \rightarrow 0$ quand $t \rightarrow \infty$, coïncident avec l'ensemble des combinaisons linéaires de vecteurs propres généralisés de H associés à des valeurs propres $\lambda \in \mathbb{C}$, $\mp \text{Im}(\lambda) > 0$. Finalement, nous définissons le sous-espace spectral absolument continu de H et montrons qu'il satisfait l'égalité $\mathcal{H}_{\text{ac}}(H) = \mathcal{H}_{\text{p}}(H^*)^\perp$, où $\mathcal{H}_{\text{p}}(H^*)$ est le sous-espace spectral ponctuel de H^* . Ainsi nous obtenons une décomposition en somme directe de l'espace de Hilbert en terme de sous-espace spectraux de H .

Ensuite nous étudions le comportement asymptotique des états $e^{\pm itH}\varphi$ du système au temps t à l'aide de la théorie de la diffusion. Nous définissons les opérateurs d'onde associés à H et H_0 par $W_\pm(H, H_0) := \text{s-lim}_{t \rightarrow \infty} e^{\pm itH} r_\mp(H) \Pi_{\text{p}}(H^*)^\perp e^{\mp itH_0}$, où $\Pi_{\text{p}}(H^*)$ est la projection sur $\mathcal{H}_{\text{p}}(H^*)$ et r_\mp est une fonction rationnelle régularisant les singularités spectrales entrantes/sortantes de H . Nous prouvons l'existence de ces opérateurs d'onde régularisés et étudions leurs propriétés. En particulier, nous montrons qu'ils sont asymptotiquement complets si H n'a pas de singularités spectrales.

Enfin, nous construisons des solutions de l'équation de Schrödinger abstraite associée à H dans le cas où la perturbation CWC n'est plus bornée.

Nos résultats s'appliquent à des opérateurs de Schrödinger avec des potentiels à valeurs complexes.