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**UNIVERSITÉ
DE LORRAINE**



IAEM



THÈSE

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Positivité sur les variétés irréductibles holomorphes symplectiques

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ABSTRACT

In this thesis, we study some aspects of the positivity of divisors on irreducible holomorphic symplectic (IHS) manifolds. Fix a projective IHS manifold X of complex dimension $2n$.

Inspired by the work [BKS04] of Bauer, Küronya, and Szemberg, we show that the big cone of X has a locally finite decomposition into locally rational polyhedral subcones, called Boucksom-Zariski chambers. These subcones have a geometric meaning: on any of them, the volume function is expressed by a homogeneous polynomial of degree $2n$. Furthermore, in the interior of any Boucksom-Zariski chamber, the divisorial part of the augmented base locus of big divisors stays the same.

After analysing the big cone, we determine the structure of the pseudo-effective cone of X , generalising a result due to Kovács for K3 surfaces. In particular, we show that if the Picard number of X is at least 3, the pseudo-effective cone either is circular or does not contain circular parts and is equal to the closure of the cone generated by the prime exceptional divisor classes. From this result in convex geometry, we deduce some geometric properties of X and show the existence of rigid uniruled divisors on some singular symplectic varieties.

We study the behaviour of the asymptotic base loci of big divisors on X , and we provide a numerical characterisation for them. As a consequence of this numerical characterisation, we obtain a description for the dual of the cones $\text{Amp}_k(X)$, for any $1 \leq k \leq 2n$, where $\text{Amp}_k(X)$ is the convex cone of big divisor classes having an augmented base locus of dimension strictly smaller than k .

Using the divisorial Zariski decomposition, the Beauville-Bogomolov-Fujiki (BBF) form, and the decomposition of the big cone of X into Boucksom-Zariski chambers, we associate to any big divisor class α and a prime divisor E on X a polygon $\Delta_E(\alpha)$ whose geometry is related to the variation of the divisorial Zariski decomposition of α in the big cone. Its euclidean volume is expressed in terms of the BBF form and is independent of the choice of E . We show that these polygons fit in a convex cone $\Delta_E(X)$ as slices, globalising in this way the construction. To conclude, we show that under some hypothesis, the polygons $\Delta_E(\alpha)$ can be expressed as Minkowski sums of some polygons $\{\Delta_E(\beta_i)\}_{i \in I}$, for some big classes $\{\beta_i\}_{i \in I}$. Remarkably, these polygons behave like the Newton-Okounkov bodies of big divisors on smooth projective surfaces.

RÉSUMÉ

Dans cette thèse, nous étudions certains aspects de la positivité des diviseurs sur les variétés irréductibles holomorphes symplectiques (IHS). Fixons une variété IHS projective X de dimension complexe $2n$.

Inspirés par le travail [BKS04] de Bauer, Küronya et Szemberg, nous montrons que le cône big de X a une décomposition localement finie en sous-cônes localement rationnels polyédraux, qu'on appelle chambres de Boucksom-Zariski. Ces sous-cônes ont une signification géométrique : sur chacun d'eux, la fonction volume est exprimée par un polynôme homogène de degré $2n$. De plus, à l'intérieur de toute chambre de Boucksom-Zariski, la partie divisorielle du lieu base augmenté des diviseurs big reste la même.

Après avoir analysé le cône big, nous déterminons la structure du cône pseudo-effectif de X , généralisant ainsi un résultat de Kovács pour les surfaces K3. En particulier, nous montrons que si le nombre de Picard de X est au moins 3, le cône pseudo-effectif de X est soit circulaire, soit ne contient pas de parties circulaires et est égal à la clôture du cône engendré par les classes des diviseurs premiers exceptionnels. De ce résultat en géométrie convexe, nous déduisons quelques propriétés géométriques de X et nous montrons l'existence de diviseurs rigides unirégulés sur certaines variétés symplectiques singulières.

Nous étudions le comportement des lieux de base asymptotiques des diviseurs big sur X et nous en donnons une caractérisation numérique. En conséquence de cette caractérisation numérique, nous obtenons une description des duals des cônes $\text{Amp}_k(X)$, pour tout $1 \leq k \leq 2n$, où $\text{Amp}_k(X)$ est le cône convexe des classes des diviseurs big ayant un lieu base augmenté de dimension strictement plus petite que k .

En utilisant la décomposition divisorielle de Zariski, la forme de Beauville-Bogomolov-Fujiki (BBF) et la décomposition du cône big de X en chambres de Boucksom-Zariski, nous associons à toute classe de diviseurs big α et à un diviseur premier E sur X un polygone $\Delta_E(\alpha)$ dont la géométrie est liée à la variation de la décomposition divisorielle de α dans le cône big de X . Le volume euclidien est exprimé en termes de la forme BBF et est indépendant du choix de E . Nous montrons que ces polygones s'inscrivent dans un cône convexe $\Delta_E(X)$ sous forme de tranches, globalisant ainsi la construction. En conclusion, nous montrons que sous certaines hypothèses, les polygones $\Delta_E(\alpha)$ peuvent être écrits comme sommes de Minkowski de certains polygones $\{\Delta_E(\beta_i)\}_{i \in I}$, pour certaines classes big $\{\beta_i\}_{i \in I}$. Il est remarquable que ces polygones se comportent comme les corps de Newton-Okounkov des diviseurs big sur les surfaces projectives lisses.

PRÉSENTATION GÉNÉRALE EN FRANÇAIS

Le but de ce chapitre introductif est de décrire en français l'objectif de cette thèse et les résultats principaux qu'elle contient. Étant donnée une variété projective, il est intéressant et en même temps difficile de comprendre la positivité des fibrés en droites. Pour certaines classes de variétés projectives, la positivité des fibrés en droites est bien comprise et les surfaces projectives K3 en font partie. Les analogues en dimension supérieure des surfaces K3 sont les variétés irréductibles holomorphes symplectiques (aussi connues comme variétés hyper-Kählériennes compactes), pour lesquelles (à part les surfaces K3) la positivité des fibrés en droites n'est pas bien connue. Récemment, des progrès ont été faits par Chen Jiang, Ángel David Ríos Ortiz, Ulrike Riess et Mauro Varesco dans les travaux [Jia22], [Rie20], [Rie23], [Ort21], [Var22].

L'objectif principal de cette thèse est d'étudier certains aspects de la positivité des fibrés en droites sur les variétés irréductibles holomorphes symplectiques (IHS à partir de maintenant). Plus précisément, nous étudions certains aspects asymptotiques des diviseurs sur les variétés IHS projectives.

Le cône big d'une variété IHS

Le point de départ de la thèse est l'article [BKS04] de Bauer, Küronya et Szemberg sur les surfaces projectives lisses. Dans cet article, les auteurs obtiennent pour le cône big d'une surface projective lisse une décomposition localement finie en sous-cônes qui sont localement rationnels polyédraux. Ces sous-cônes sont appelés chambres de Zariski et ils correspondent aux régions du cône big où le support de la partie négative de la décomposition de Zariski est constant. En conséquence, la fonction de volume est polynomiale lorsqu'elle est restreinte à une chambre de Zariski quelconque. De plus, les auteurs observent que le lieu base augmenté des classes de diviseurs big est constant à l'intérieur de chaque chambre de Zariski. Les outils principaux utilisés par les auteurs pour obtenir ces résultats sont la décomposition de Zariski et la forme d'intersection sur une surface. Le deuxième groupe de cohomologie singulière (et aussi le groupe de Picard) d'une variété IHS est doté d'une forme quadratique non dégénérée à valeurs entières, connue sous le nom de forme de Beauville-Bogomolov-Fujiki (BBF), qui se comporte comme la forme d'intersection sur une surface lisse. De plus, grâce au travail pionnier de Boucksom, nous savons que sur toute variété IHS, il existe une décomposition pour chaque classe pseudo-effective analytique, connue sous le nom de décomposition divisorielle de Zariski, se comportant comme la décomposition de Zariski sur les surfaces. En conséquence, au moins en codimension 1, les variétés IHS se comportent de manière analogue aux surfaces lisses. Il est donc naturel de se demander si les résultats obtenus par Bauer, Küronya et Szemberg peuvent être généralisés au cas des variétés IHS. La réponse à cette question est affirmative et elle est contenue dans le théorème suivant.

Théorème (Theorem 0.1.2): Soit X une variété IHS projective de dimension complexe $2n$.

1. Il y a une décomposition localement finie du cône big $\text{Big}(X)$ en sous-cônes, que nous appellerons chambres de Boucksom-Zariski, qui sont localement rationnels polyédraux. Dans chaque chambre de Boucksom-Zariski le support de la partie négative de la décomposition divisorielle de Zariski est constant.
2. Pour chaque \mathbf{R} -diviseur big D , dont la décomposition divisorielle de Zariski est $D = P(D) + N(D)$, on a $\text{vol}(D) = (P(D))^{2n}$. En particulier, dans chaque chambre de Boucksom-Zariski, la fonction volume est exprimée par un polynôme homogène de degré $2n$, et elle est donc localement polynomiale.

La stratégie suivie pour démontrer le résultat ci-dessus est la même que celle adoptée par Bauer, Küronya et Szemberg dans leur article.

Grâce à [BKS04] nous savons que sur une surface projective lisse, les lieux base augmentés des classes big restent constants dans l'intérieur de chaque chambre de Zariski. D'autre part, dans le cas des variétés IHS, on peut facilement vérifier que dans l'intérieur des chambres de Boucksom-Zariski, les lieux base augmentés des classes big peuvent ne pas être constants. Mais c'était tout à fait prévisible, car la géométrie d'une variété de dimension plus grande de 2 est plus compliquée.

Le cône pseudo-effectif d'une variété IHS

Après avoir étudié le cône big d'une variété IHS projective, il était naturel d'étudier le cône pseudo-effectif d'une variété IHS projective, c'est-à-dire la clôture dans l'espace de Néron-Severi du cône big. Nous avons été motivés par un résultat de Kovács, selon lequel le cône pseudo-effectif (ou, de manière équivalente, le cône de Mori) d'une surface K3 lisse projective de nombre de Picard supérieur ou égal à 3, soit est circulaire et la surface ne contient pas de courbes rationnelles lisses, soit ne contient pas de parties circulaires et la surface contient des courbes rationnelles lisses. Nous savons que les variétés IHS se comportent comme les surfaces lisses en codimension 1. Il était donc naturel de se demander si un résultat analogue à celui de Kovács reste vrai pour les variétés IHS projectives de dimension supérieure et nous avons obtenu le résultat suivant.

Théorème (Theorem 0.2.2): Soit X une variété IHS projective dont le nombre de Picard est supérieur ou égal à 3. Alors, soit $\text{Neg}(X) = \emptyset$ et le cône pseudo-effectif $\overline{\text{Eff}(X)}$ de X est égal à $\overline{\text{Pos}(X)}$ est circulaire, ou

$$\overline{\text{Eff}(X)} = \overline{\sum_{E \in \text{Neg}(X)} \mathbf{R}^{\geq 0}[E]},$$

où $\text{Neg}(X)$ est l'ensemble des diviseurs premiers exceptionnels de X .

Rappelons que les diviseurs premiers exceptionnels sont les diviseurs premiers avec carré négatif par rapport à la forme de BBF. Si l'on veut adopter la stratégie de Kovács pour prouver le théorème ci-dessus, on se rend vite compte qu'il y a un problème : la classe α construite à partir d'un diviseur exceptionnel premier E , et apparaissant dans l'équation (3.3) du lemme 3.2.1 n'est pas effective, en général. En effet, alors que dans le cas des surfaces K3 l'effectivité de α découle directement du théorème de Riemann-Roch pour les surfaces, en dimension plus grande le théorème de Riemann-Roch pour les variétés IHS ne nous dit rien sur l'effectivité de α , principalement parce que nous n'avons pas de théorèmes d'annulation pour nous aider. Notre idée pour contourner ce problème est d'utiliser le groupe de monodromie de X . Plus précisément, dans notre approche initiale, nous avons utilisé la description explicite de $\text{Mon}_{\text{Hdg}}^2(X)$ pour les classes de déformation connues pour construire des classes effectives explicites (voir la Proposition 0.2.3). Une approche plus puissante en deux étapes, suggérée par un rapporteur anonyme, est d'utiliser le fait que (l'image de) $\text{Mon}_{\text{Hdg}}^2(X)$ dans $O^+(\text{Pic}(X))$ est d'indice fini pour prouver qu'elle agit sur la frontière du cône positif projectivisé avec orbites denses, et d'en déduire le théorème.

Avec le résultat ci-dessus, on déduit l'existence de diviseurs rigides unirrégulés sur certaines variétés symplectiques singulières.

Corollaire (Corollary 0.2.9): Soit Y une variété primitive symplectique projective singulière et \mathbf{Q} -factorielle, avec une résolution $f: X \rightarrow Y$, où X est une variété IHS projective dont le nombre de Picard est supérieur ou égal à 3. Alors Y contient un diviseur premier exceptionnel (et donc unirrégulé).

Lieux base asymptotiques

Le chapitre 4 découle de la nécessité de comprendre comment les lieux base asymptotiques des classes big sur une variété IHS projective varient lorsqu'ils sont perturbés dans l'espace de Néron-Severi et de trouver, si possible, une caractérisation numérique de ces lieux. Encore une fois, la décomposition divisorielle de Zariski et la forme de Beauville-Bogomolov-Fujiki jouent un rôle central et on a obtenu le résultat suivant.

Théorème (Theorem 0.3.1): Soit X une variété IHS projective et D un \mathbf{R} -diviseur big sur X .

1. Chaque composante irréductible de $\mathbf{B}_+(D), \mathbf{B}_-(D) = \mathbf{B}(D)$ est algébriquement coisotrope, et donc de dimension plus grande ou égale à $\dim(X)/2$.
2. Les composantes irréductibles divisorielles de $\mathbf{B}_+(D)$ sont les diviseurs premiers exceptionnels qui sont q_X -orthogonaux à $P(D)$. Pour toute composante irréductible

V de $\mathbf{B}_+(D)$, il existe des courbes rationnelles et contractibles couvrant V , et intersectant $P(D)$ négativement, ou trivialement, sur un modèle birationnel de X qui est IHS.

3. Les composantes irréductibles divisorielles de D sont les diviseur premiers exceptionnels contenus dans le support de $N(D)$. Pour chaque composante irréductible non divisorielle de $\mathbf{B}_-(D) = \mathbf{B}(D)$ il existe des courbes rationnelles et contractibles couvrant la composante, et intersectant $P(D)$ négativement sur un modèle birationnel de X qui est IHS.
4. $P(D)$ est instable si et seulement si $\mathbf{B}_+(D)$ contient une courbe $C \in \text{Cont}(X)$, telle que $C \notin \text{Ind}(f)$ pour une certaine application birationnelle $f: X \dashrightarrow X'$ de variétés IHS projectives, avec $f_*(P(D))$ nef, et $f_*(P(D)) \cdot f_*(C) = 0$. De plus, tous les nombres déstabilisants sont des nombres rationnels.

On dit qu'une courbe C sur une variété IHS projective X est contractible sur un modèle birationnel de X , s'il existe une autre variété IHS projective X' , une application birationnelle $f: X \dashrightarrow X'$, tel que $C \notin \text{Ind}(f)$ et (la classe de) $f_*(C)$ dans $N_1(X)_{\mathbf{R}}$ se trouve sur une face extrême de $\overline{\text{NE}}(X)$ qui est contractible, où $f_*(C)$ est la transformée stricte de C via f . De plus, on dit qu'un \mathbf{R} -diviseur D sur X est instable si $\mathbf{B}_-(D) \subsetneq \mathbf{B}_+(D)$. Dans l'énoncé ci-dessus, $\text{Cont}(X)$ dénote l'ensemble des courbes rationnelles qui sont contractibles sur un certain modèle birationnel de X .

Si X est une variété IHS projective de dimension $2n$, pour chaque $1 \leq k \leq 2n$ on définit $\text{Amp}_k(X)$ comme le cône des classes de diviseurs big α avec $\dim(\mathbf{B}_+(\alpha)) < k$. Nous avons obtenu pour le cône dual de $\text{Amp}_k(X)$ la description suivante.

Théorème (Theorem 0.3.4): Soit X une variété IHS projective de dimension $2n$. Alors,

$$\overline{\text{Amp}}_k(X) = \text{bMob}_k(X)^\vee,$$

pour chaque $1 < k < 2n$. De plus, si $k = 1, \dots, n$, on a $\overline{\text{Amp}}_k(X) = \text{Nef}(X) = \overline{\text{NE}}(X)^\vee$.

Le cône $\text{bMob}_k(X)$ ci-dessus est le cône des classes $\alpha \in N_1(X)_{\mathbf{R}}$ qui sur un modèle birationnel de X sont représentées par des courbes se déplaçant dans une famille couvrant une sous-variété de X de dimension au moins k . Le théorème ci-dessus répond, dans le cas particulier des variétés IHS projectives, à une question posée par Sam Payne.

Polygones de type Newton-Okounkov

La théorie des corps de Newton-Okounkov dans le cas des surfaces projectives lisses complexes est bien développée, grâce notamment aux travaux de Lazarsfeld-Mustață, Küronya-Lozovanu, Küronya-Lozovanu-Maclean ([LM09], [KL17a], [KLM12]). Etant donnée une surface algébrique

complexe lisse S et un drapeau général $F_\bullet = \{\text{pt}\} \subset C \subset S\}$, le corps de Newton-Okounkov d'un diviseur big sur S est déterminé par la façon dont la décomposition de Zariski varie lorsque nous perturbons le diviseur choisi le long de la direction négative induite par C et par la forme d'intersection sur la surface ([LM09, Theorem 6.4]). En particulier, si D est un diviseur big sur S , le corps de Newton-Okounkov de D par rapport au drapeau F_\bullet est donné par le polygone ([LM09, Theorem 6.4], [KLM12, Theorem B])

$$\Delta_{F_\bullet}(D) = \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq \mu_C(D), 0 \leq y \leq P(D_t) \cdot C\},$$

où $D_t := D - tC$, $\mu_C(D) := \sup_{t \in \mathbf{R}^{\geq 0}} (D - tC \text{ est big})$ et $P(D_t)$ est la partie positive de la décomposition de Zariski de D_t . Le volume euclidien de ce corps est égal à $\text{vol}(D)/2$, qui à son tour vaut $P(D)^2/2$ ([LM09, Theorem A]).

Dans le cas des variétés IHS projectives nous avons obtenu la décomposition du cône big en chambres de Boucksom-Zariski et cette décomposition explique comment la décomposition divisorielle de Zariski d'un diviseur big change dans le cône big quand on perturbe le diviseur choisi. En utilisant la décomposition du cône big en chambres de Boucksom-Zariski et la forme de BBF, nous pouvons associer à chaque diviseur big un polygone qui se comporte comme le corps de Newton-Okounkov d'un diviseur big sur une surface (pour un choix de drapeau général). Soit X une variété IHS projective, D un diviseur big sur X et E un diviseur premier sur X . Considérons le sous-ensemble de \mathbf{R}^2

$$\Delta_E^{\text{num}}(D) := \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq \mu_E(D), 0 \leq y \leq q_X(P(D_t), E)\},$$

où $D_t := D - tE$ et $\mu_E(D) := \sup\{t \in \mathbf{R}^{\geq 0} \mid D - tE \text{ est big}\}$.

Théorème (Theorem 5.2.5 and Proposition 5.2.7): Le sous-ensemble $\Delta_E^{\text{num}}(D)$ de \mathbf{R}^2 introduit ci-dessus est un polygone convexe. De plus, $\text{vol}(\Delta_E^{\text{num}}(D)) = q_X(P(D))/2$, où $\text{vol}(\Delta_E^{\text{num}}(D))$ est le volume euclidien du corps $\Delta_E^{\text{num}}(D)$. En particulier, nous avons

$$2^n c_X (\text{vol}(\Delta_E^{\text{num}}(D)))^n = c_X (q_X(P(D)))^n = \text{vol}(D) = (2n)! \cdot \text{vol}(\Delta(D)),$$

où c_X est la constante de Fujiki de X et $\Delta(D)$ est le corps de Newton-Okounkov classique de D (par rapport à n'importe quel drapeau complet).

Comme dans le cas des corps de Newton-Okounkov classiques, nous prouvons l'existence d'un cône convexe de dimension $\rho(X) + 2$ dont les "tranches" sont les polygones $\Delta_E^{\text{num}}(D)$ dans l'énoncé ci-dessus.

Théorème (Theorem 0.4.1): Pour chaque diviseur premier E sur une variété IHS projective X il existe un cône convexe $\Delta_E^{\text{num}}(X) \subset N^1(X)_{\mathbf{R}} \times \mathbf{R}^2$ avec la projection naturelle $p_1: \Delta_E^{\text{num}}(X) \rightarrow N^1(X)_{\mathbf{R}}$, tel que la fibre $p_1^{-1}(\zeta)$ est égale au polygone $\Delta_E^{\text{num}}(\zeta)$, pour tout ζ dans $\text{Big}(X)$.

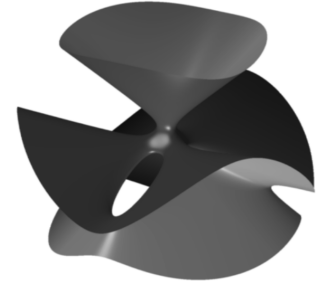
Pour conclure, en suivant [LS14], nous montrons que sous certaines hypothèses, les polygones $\Delta_E(\alpha)$ (avec α une classe big) peuvent être exprimés comme une somme de Minkowski de certains polygones $\{\Delta_E(\beta_i)\}_{i \in I}$, pour certaines classes mobiles $\{\beta_i\}_{i \in I}$. En particulier, nous avons le résultat suivant.

Théorème (Theorem 0.4.2): Soit X une variété IHS avec $\text{Eff}(X)$ rationnel polyédral et E un diviseur premier big. Alors, il existe un ensemble fini Ω de \mathbf{Q} -diviseurs mobiles tel que, pour chaque \mathbf{Q} -diviseur mobile D , il existe des nombres rationnels $\alpha_P(D)$ tels que $D = \sum_{P \in \Omega} \alpha_P(D)P$ et $\Delta_E(D) = \sum_{P \in \Omega} \alpha_P(D)\Delta_E(P)$.

La théorie des corps de Newton-Okounkov a suscité un grand intérêt ces dernières années. En effet, ces corps contiennent beaucoup d'informations asymptotiques sur les diviseurs, mais en général ils ne sont pas faciles à calculer. De plus, la dimension des corps de Newton-Okounkov classiques est égale à la dimension de la variété. Dans le cas des variétés IHS projectives nous avons les corps de Newton-Okounkov classiques et les polygones de type Newton-Okounkov dans le théorème ci-dessus. Ces polygones sont plus faciles à calculer et nous espérons qu'ils contiennent des informations fines sur la positivité des diviseurs comme dans le cas des corps de Newton-Okounkov classiques.

CONVENTIONS AND NOTATION

- If not otherwise stated, all varieties are defined over \mathbf{C} .
- In the second, third, fourth, and fifth chapters, X will always denote a projective irreducible holomorphic symplectic manifold of complex dimension $2n$.
- For us, a variety is always irreducible.



0. Introduction

In this introductory chapter, we motivate our work and carefully describe the content of the thesis. Given a projective variety, it is interesting and at the same time difficult to understand the positivity of line bundles. For certain classes of projective varieties, the positivity of line bundles is well understood and projective K3 surfaces are among them. The higher dimensional analogues of K3 surfaces are the irreducible holomorphic symplectic manifolds (also known as compact hyper-Kähler manifolds), for which (apart from K3 surfaces) the positivity of line bundles is not well understood, yet. Recently, some progress has been made by Chen Jiang, Ángel David Ríos Ortiz, Ulrike Riess, and Mauro Varesco in the works [Jia22], [Rie20], [Rie23], [Ort21], [Var22]. The main goal of this thesis is to study some aspects of the positivity of line bundles on projective irreducible holomorphic symplectic (IHS from now on) manifolds. More precisely, we study some asymptotic aspects of divisors on projective IHS manifolds.

An irreducible holomorphic symplectic (IHS) manifold is a simply connected compact Kähler manifold X such that $H^0(X, \Omega_X^2) \cong \mathbf{C}\sigma$, where σ is a holomorphic symplectic form. One of the reasons why these manifolds are much studied today is certainly that they form one of the three building blocks of compact Kähler manifolds with vanishing first (real) Chern class. At present, the known deformation classes of IHS manifolds are the following: the $K3^{[n]}$ and Kum_n deformation classes, for any $n \in \mathbf{N}$ (see [Bea83]), and the OG10, OG6 deformation classes, discovered by O'Grady (see [OGr99] and [OGr03] respectively). Thanks to the work [Bea83] of Beauville, there exists a quadratic form q_X on $H^2(X, \mathbf{C})$, behaving like the intersection form on a surface. The quadratic form q_X is non-degenerate and is known as the Beauville-Bogomolov-Fujiki (BBF for short) form. We refer the reader to [GHJ03; Huy99; Huy97; Huy03] for a thorough introduction to IHS manifolds.

The thesis is composed of four chapters, three of which can be found on the arXiv. The chapters are not independent of each other. There is a common thread between them all. We start by describing the second chapter, since the first chapter only contains all the needed background.

0.1 Introduction to the second chapter

In this chapter, we study the big cone, the algebraic pseudo-effective cone, and an asymptotic invariant of linear series of a projective IHS manifold.

The chapter is based on the first article [Den23] of the author, which was mostly inspired by the article [BKS04] of Bauer, Küronya and Szemberg. For this reason, we first recall what was done in [BKS04] in the case of surfaces, and we then move on to the case of projective IHS manifolds.

An effective \mathbf{R} -divisor D on a smooth projective surface S (over an algebraically closed field of arbitrary characteristic), can be written as $D = P(D) + N(D)$, where:

1. $P(D)$ is nef and is orthogonal to any irreducible component of $N(D)$ (with respect to the usual intersection form),
2. $N(D)$ is effective, so that $N(D) = \sum_i a_i N_i$, with $a_i \geq 0$ for every i , and the N_i are the irreducible components of $N(D)$,
3. The Gram matrix $(N_i \cdot N_j)_{i,j}$ of $N(D)$ is negative-definite, if $N(D) \neq 0$.

The divisor $P(D)$ is the *positive part* of D , while $N(D)$ is the *negative part* of D . The decomposition $D = P(D) + N(D)$ with the listed properties is unique and is known as the *Zariski decomposition* of D (for more details see for example [Bau09]).

Historical note. The Zariski decomposition was first obtained for effective \mathbf{Q} -divisors by Zariski in his fundamental paper [Zar62]. Then it was extended by Fujita in [Fuj79] to the case of pseudo-effective \mathbf{Q} -divisors. The existence and the uniqueness of the Zariski decomposition for effective \mathbf{R} -divisors can be easily deduced using the method of Bauer in [Bau09].

In [BKS04] the authors described the big cone of S in terms of variation of the Zariski decomposition in it.

Theorem 0.1.1 (Th. Bauer, A. Küronya, T. Szemberg): Let S be a smooth projective surface.

1. There is a locally finite decomposition of $\text{Big}(S)$ into locally rational polyhedral subcones, such that in each subcone the support of the negative part of the Zariski decomposition of the divisors is constant.
2. Let D be a big \mathbf{R} -divisor on S , with Zariski decomposition $D = P(D) + N(D)$. We have

$$\text{vol}(D) = (P(D))^2 = (D - N(D))^2.$$

Therefore on a chamber Σ_P on which the support of the negative part is constant, the volume is given by a homogeneous quadratic polynomial.

The subcones giving the decomposition in Theorem 0.1.1 are called *Zariski chambers* (for the definition and further details we refer the reader to [BKS04]).

For an arbitrary compact complex manifold, Boucksom in [Bou04] obtained a sort of "higher dimensional Zariski decomposition", called *divisorial Zariski decomposition* (see also the algebraic approach of Nakayama in [Nak04]). In particular, in the case of IHS manifolds, Boucksom characterised his decomposition in terms of the BBF form (for more details, see the subsection "*Divisorial Zariski decomposition*" in Chapter 1).

In the case of surfaces, the intersection form with its properties and Zariski decomposition played a fundamental role in obtaining Theorem 0.1.1. On the other hand, on an IHS manifold, we have the BBF form and the divisorial Zariski decomposition, so it seems natural to ask whether Theorem 0.1.1 can be generalised to the case of IHS manifolds. We were able to answer this question positively. In particular, we obtained the following theorem, which is the main result of the first chapter of this thesis.

Theorem 0.1.2: Let X be a projective irreducible holomorphic symplectic manifold of complex dimension $2n$.

1. There is a locally finite decomposition of the big cone $\text{Big}(X)$ of X into locally rational polyhedral subcones, called *Boucksom-Zariski chambers*, such that in each subcone the support of the negative part of the divisorial Zariski decomposition of the divisors is constant.
2. For any big \mathbf{R} -divisor D with divisorial Zariski decomposition $D = P(D) + N(D)$, we have $\text{vol}(D) = (P(D))^{2n}$. In particular, on each Boucksom-Zariski chamber, the volume function is expressed by a single homogeneous polynomial of degree $2n$, and so is locally polynomial.

Remark 0.1.3. In the statement of Theorem 0.1.2, item (1),

1. "locally finite" means that any point in $\text{Big}(X)$ has a neighbourhood meeting only a finite number of chambers;
2. "locally rational polyhedral subcones" means that any of these subcones is locally rational polyhedral at any big point of its closure in the Néron-Severi space, in the sense of Definition 1.1.5.

The strategy of our proof is based on that adopted by Bauer, Küronya and Szemberg in [BKS04].

Remark 0.1.4. Note that, in [BKS04], the authors also studied the behaviour of the augmented base loci (of divisors on a smooth projective surface). In this case, the problem led to another decomposition of the big cone into chambers, called *stability chambers*, on each of which the augmented base loci stay constant. It turned out that the stability chambers essentially agree with

the Zariski chambers (see Theorem 2.4.2). In general, in the case of a projective IHS manifold, the Boucksom-Zariski chambers do not agree with the stability chambers (see "Remark on the stability chambers" in Section 2.4). The stability chambers are studied more closely in Chapter 4.

The big cone of a smooth projective surface S can also be decomposed into *simple Weyl chambers*, which are defined as the connected components of

$$\text{Big}(S) \setminus \left(\bigcup_{C \in \text{Neg}(S)} C^\perp \right),$$

where $\text{Neg}(S)$ is the set of irreducible and reduced curves of negative square and, given a curve C , C^\perp is the hyperplane in the Néron-Severi space of classes orthogonal to the curve C , with respect to the usual intersection form. In several papers, the Zariski chambers and the simple Weyl chambers have been compared. First, Bauer and Funke in [BF12] studied the case of projective K3 surfaces. Then, Rams and Szemberg in [RS16], and, more recently, Hanumanthu and Ray in [HR20] studied the case of any smooth projective surface. We are mainly interested in the following result.

Theorem 0.1.5 ([RS16, Theorem 3]): Let S be a smooth projective surface. The following conditions are equivalent:

1. The interior of the Zariski chambers on S coincides with the simple Weyl chambers.
2. If two different reduced and irreducible negative curves C_1, C_2 on S satisfy $C_1 \cdot C_2 > 0$, then $C_1 \cdot C_2 \geq \sqrt{C_1^2 C_2^2}$.

The notion of a "simple Weyl chamber" can be naturally generalised to the case of a projective IHS manifold X , by replacing reduced and irreducible negative curves by prime exceptional divisors, which are the prime divisors on X with negative Beauville-Bogomolov-Fujiki square. In this way, as for surfaces, we obtain another decomposition of $\text{Big}(X)$ into chambers, which we also call *simple Weyl chambers*. As for surfaces, it is natural to compare the two decompositions of $\text{Big}(X)$ introduced so far. Our main result on this question is the following theorem, which is a natural generalisation of the above result by Rams and Szemberg.

Theorem 0.1.6: Let X be a projective IHS manifold. The following conditions are equivalent:

1. The interior of the Boucksom-Zariski chambers coincides with the simple Weyl chambers.
2. If two different prime exceptional divisors E_1, E_2 on X satisfy $q_X(E_1, E_2) > 0$, then $q_X(E_1, E_2) \geq \sqrt{q_X(E_1)q_X(E_2)}$.

Theorem 0.1.6 explains when the decomposition of $\text{Big}(X)$ into Boucksom-Zariski chambers

and the one into simple Weyl chambers coincide, and, to obtain it, we first generalise [HR20, Theorem 3.1, Theorem 3.2] (which are in turn a generalisation of the items in Theorem 2 of [BF12]) to our case (and these are interesting results in their own); this will be enough to deduce Theorem 0.1.6.

0.2 Introduction to the third chapter

This chapter is based on the second work [Den] of the author. We give a finer description of the pseudo-effective cone of a projective IHS manifold than that contained in the first chapter (more precisely in Corollary 2.1.6), and provide some geometric consequences.

The structure of the pseudo-effective cone of a projective IHS manifold X is similar to that of a smooth complex projective surface. Indeed, for a surface S we have

$$\overline{\text{Eff}}(S) = \overline{\text{Pos}}(S) + \sum_{C \in \text{Neg}(S)} \mathbf{R}^{\geq 0}[C]$$

(see [Kol96, Theorem 4.13]), where $\text{Neg}(S)$ is the set of irreducible and reduced curves of negative self-intersection in S , and $\text{Pos}(S)$ is the positive cone of S , i.e. the connected component of the set $\{\alpha \in N^1(S)_{\mathbf{R}} \mid \alpha^2 > 0\}$ containing the ample cone $\text{Amp}(S)$; while for a projective IHS manifold X we have

$$\overline{\text{Eff}}(X) = \overline{\text{Pos}}(X) + \sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E],$$

where $\text{Exc}(X)$ is the set of prime exceptional divisors in X , i.e. the prime divisors with negative BBF square (see Corollary 2.1.6), and $\text{Pos}(X)$ is the positive cone of X , i.e. the connected component of the set $\{\alpha \in N^1(X)_{\mathbf{R}} \mid q_X(\alpha) > 0\}$ containing the ample cone $\text{Amp}(X)$.

For a projective K3 surface (over an algebraically closed field of arbitrary characteristic), Kovács obtained a refinement for the structure of the pseudo-effective cone. In particular, he proved the following.

Theorem 0.2.1 ([Kov13], Theorem 1.1): Let S be a K3 surface of Picard number at least 3 over an algebraically closed field of arbitrary characteristic. Then either $\text{Exc}(S) = \emptyset$ and $\overline{\text{Eff}}(S) = \overline{\text{Pos}}(S)$ is circular, or

$$\overline{\text{Eff}}(S) = \overline{\sum_{C \in \text{Exc}(S)} \mathbf{R}^{\geq 0}[C]},$$

where $\text{Exc}(S)$ is the set of smooth rational curves contained in S .

Note that Kovács first proved the above result over \mathbf{C} ([Kov93, Theorem 1]). See also [Kov93, Theorem 2] for a complete description of the pseudo-effective cone in this case.

Replacing smooth rational curves by prime exceptional divisors, it is natural to ask whether the above result can be generalised to the case of an arbitrary projective IHS manifold, and we could answer affirmatively to this question. In particular, we obtained the following theorem, which is the main result of the second chapter of the present thesis.

Theorem 0.2.2: Let X be a projective IHS manifold of Picard number greater than or equal to 3. Then, either $\text{Exc}(X) = \emptyset$ and $\overline{\text{Eff}}(X) = \overline{\text{Pos}}(X)$ is circular, or

$$\overline{\text{Eff}}(X) = \overline{\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]},$$

where $\text{Exc}(X)$ is the set of prime exceptional divisors in X .

Note that the above result holds also in the singular setting (under some assumptions) and we refer the reader to Theorem 3.1.14 for a sketch of the proof in this case.

If one wants to adopt Kovács' strategy to prove Theorem 0.2.2, one quickly realises that there is a problem: the class α constructed starting from a prime exceptional divisor E and appearing in equation (3.3) of Lemma 3.2.1 is not effective, in general. Indeed, while in the case of K3 surfaces the effectivity of α directly follows from the Riemann-Roch Theorem, in the higher dimensional case the Riemann-Roch Theorem does not tell us anything about the effectivity of α , mainly because we do not have vanishing theorems helping us. Our idea to bypass this problem is to use the monodromy group $\text{Mon}_{\text{Hdg}}^2(X)$ of X . More precisely, in our original approach, we used the explicit description of $\text{Mon}_{\text{Hdg}}^2(X)$ for the known deformation classes. A more powerful approach in two steps, suggested by an anonymous referee, is to use the fact that (the image of) $\text{Mon}_{\text{Hdg}}^2(X)$ in $O^+(\text{Pic}(X))$ is of finite index to prove that it acts on the boundary of the projectivised positive cone with dense orbits and to deduce from that Theorem 0.2.2. That is what we do in Section 3.1.

Moreover, we use Kovács' approach to produce explicit effective integral divisors, either of BBF square 0, or with the same monodromy invariants as those of a given prime exceptional divisor, but this construction (apparently) works only for the known deformation classes of IHS manifolds.

Proposition 0.2.3: Let X be a projective IHS manifold belonging to one of the known deformation classes, carrying a prime exceptional divisor E , and of Picard number greater than or equal to 2. Then, for any primitive integral divisor D with class lying in $\text{Pos}(X)$, there exists an effective divisor of the form $xD - yE$ (with $x, y \in \mathbf{N}$) of BBF square 0, or with the same monodromy invariants as those of E .

Remark 0.2.4. The divisors in the above proposition are constructed as solutions of certain Pell equations. This does not come as a surprise. Indeed, Pell equations play an important role in describing cones of divisors of IHS manifolds (see for instance [BM13], [BC22]), and we hope that our explicit computations can be helpful in similar problems. It can happen that, in the situation of Proposition 0.2.3, both types of effective divisors exist. But, in general, this is not true (so, only one of the two types exists). However, we can construct explicit effective divisors with the same monodromy invariants as those of the prime exceptional divisor E in any case, if the Picard number of X is at least 3 (see Remark 3.2.2).

Theorem 0.2.2 describes the pseudo-effective cone of IHS manifolds of Picard number at least

3. In the case of Picard number 2, we have the following result of Oguiso.

Theorem 0.2.5 ([Ogu14, Theorem 1.3, item (2)]): Let X be a projective IHS manifold of Picard number 2. Then either the boundary rays of the movable cone $\overline{\text{Mov}}(X)$ are rational and $\text{Bir}(X)$ is a finite group or the boundary rays of $\overline{\text{Mov}}(X)$ are irrational and $\text{Bir}(X)$ is an infinite group.

From Theorem 0.2.5 one can easily deduce the following corollary providing a description for the pseudo-effective cone for projective IHS manifolds of Picard number 2 and for the proof of which we refer the reader to Chapter 3.

Corollary 0.2.6: Let X be a projective IHS manifold with $\rho(X) = 2$. Then either the boundary rays of $\overline{\text{Eff}}(X)$ are rational and $\text{Bir}(X)$ is a finite group or the boundary rays of $\overline{\text{Eff}}(X)$ are irrational and $\text{Bir}(X)$ is an infinite group.

We recall that for a projective IHS manifold X and a prime exceptional divisor E on it, we have a Hodge isometry ([Mar09, Corollary 3.6, item (2)])

$$R_E: H^2(X, \mathbf{Z}) \rightarrow H^2(X, \mathbf{Z}), \alpha \mapsto \alpha - \frac{2q_X([E], \alpha)}{q_X([E])} [E].$$

R_E is the reflection associated with E . We define W_{Exc} to be the subgroup of the group of isometries of $H^2(X, \mathbf{Z})$, which we denote by $O(H^2(X, \mathbf{Z}))$, generated by the reflections R_E , where E is any prime exceptional divisor on X .

Putting all together one obtains the following result, which is a higher dimensional analogue of a result due to Pjateckii-Šapiro and Šafarevič, and Sterk in the case of K3 surfaces ([Huy15, Corollary 4.7]).

Corollary 0.2.7: Let X be a projective IHS manifold belonging to one of the known deformation classes. Consider the following statements.

1. $\text{Eff}(X)$ is rational polyhedral.
2. $\text{Bir}(X)$ is finite.
3. $O(N^1(X))/W_{\text{Exc}}$ is finite.
4. X carries finitely many prime exceptional divisors.

Then, the statements (1), (2), (3) are equivalent and imply (4). Moreover, if X is of Picard number at least 3 and carries a prime exceptional divisor, all the above statements are equivalent.

We believe that the experts are aware of the equivalence between (1), (2) and (3) in Corollary

0.2.7. The result is interesting on its own, and we included it for the sake of completeness.

Now, we need to recall the definition of primitive symplectic variety.

Definition 0.2.8: A primitive symplectic variety is a normal compact Kähler variety Y with symplectic singularities such that $h^1(Y, \mathcal{O}_Y) = 0$ and $H^0(Y, \Omega_Y^{[2]})$ is generated by a holomorphic symplectic form σ .

We refer the reader to [BL22] for the general theory of primitive symplectic varieties.

Using the convex geometry of the pseudo-effective cone of projective IHS manifolds we deduce the existence of (rigid) uniruled divisors on certain primitive symplectic varieties.

Corollary 0.2.9: Let Y be a singular \mathbf{Q} -factorial projective primitive symplectic variety admitting a resolution $f: X \rightarrow Y$, with X a projective IHS manifold of Picard number $\rho(X) \geq 3$. Then Y carries a prime exceptional (hence uniruled) divisor.

The uniruledness of prime exceptional divisors on \mathbf{Q} -factorial primitive symplectic varieties has been proven in [LMP23, Theorem 1.2]. See also Corollary 3.1.15 for a more general version of Corollary 0.2.9.

0.3 Introduction to the fourth chapter

This chapter is based on the third work of the author [DO23], joint with Ángel David Ríos Ortiz.

Ample, nef, and semiample divisors play a central role in algebraic geometry. If Y is any normal, complex projective variety, to detect whether a Cartier divisor D is ample, nef or semiample one introduces the augmented base locus, the restricted base locus and the stable base locus of D , respectively, which are known as *asymptotic base loci*. In symbols, we denote these by $\mathbf{B}_+(D)$, $\mathbf{B}_-(D)$, $\mathbf{B}(D)$, respectively. An integral Cartier divisor D is ample (resp. nef, semiample) if and only if $\mathbf{B}_+(D) = \emptyset$ (resp. $\mathbf{B}_-(D) = \emptyset$, $\mathbf{B}(D) = \emptyset$).

It has been established by Takayama ([Tak08]) that, whenever Y is smooth and $K_Y \equiv 0$, given a big \mathbf{Q} -divisor D on Y , all the irreducible components of $\mathbf{B}_+(D)$, $\mathbf{B}_-(D)$, $\mathbf{B}(D)$ are uniruled. The above result was generalised to \mathbf{R} -divisors, and to adjoint divisors by Boucksom, Broustet, Pacienza ([BBP13, Theorem A]). Moreover, in [BBP13, Proof of Theorem A], one sees that for any irreducible component of the augmented (resp. restricted, stable) base locus of a big Cartier \mathbf{R} -divisor on Y , there exists a birational model Y' of Y on which the considered component can be contracted. Furthermore, [BBP13, Theorem A] also implies that $\mathbf{B}_+(D) = \mathbf{B}_-(D)$ in this case, for any big divisor D . Note that [BBP13, Theorem A] holds in more generality.

In this chapter, there are two major results. Before stating the first, we introduce some terminology.

We say that a big Cartier \mathbf{R} -divisor D on a normal complex projective variety Y is stable (resp. unstable) if $\mathbf{B}_+(D) = \mathbf{B}_-(D)$ (resp. $\mathbf{B}_-(D) \subsetneq \mathbf{B}_+(D)$). The stable divisor classes on Y form a dense

open subset of $\text{Big}(Y)$, the big cone of Y (i.e. the cone in $N^1(Y)_{\mathbf{R}}$ spanned by big integral divisor classes), by [ELMP06, Proposition 1.26]. Following [BKS04, Definition 2.8], we say that a real number λ is a destabilising number for a big, integral Cartier divisor D , with respect to an ample, integral Cartier divisor A , if $\mathbf{B}_+(D - \alpha A) \subsetneq \mathbf{B}_+(D - \beta A)$ for all rational numbers α, β with $\alpha < \lambda < \beta$.

In general, given a big Cartier divisor D on Y , the irreducible components of $\mathbf{B}_+(D), \mathbf{B}_-(D)$ cannot be described using the intersections of D with the curves on the variety (see Example 1.1.9). We recall that any pseudo-effective \mathbf{R} -divisor on a projective manifold admits a divisorial Zariski decomposition.

In the case of a smooth projective surface S , the augmented and restricted base loci can be characterised by using the divisorial Zariski decomposition. In particular, Bauer, Küronya, and Szemberg prove in [BKS04] that if D is any big divisor on S , $\mathbf{B}_+(D)$ is the union of the negative irreducible, reduced curves intersecting $P(D)$ trivially, while $\mathbf{B}_-(D)$ is the union of the curves supporting $N(D)$, where $P(D)$ (resp. $N(D)$) is the positive (resp. negative) part of the divisorial Zariski decomposition of D . Also, they show that all the destabilising numbers are rational numbers and that D is unstable if and only if there is a negative irreducible curve C on S such that $C \not\subset \text{Supp}(N(D))$, and $P(D) \cdot C = 0$. Here the point is that $P(D)$ encodes most of the positivity of D , and so it is possible to characterise $\mathbf{B}_+(D)$ using $P(D)$ and its intersections with the curves contained in S , while $\mathbf{B}_-(D)$ is determined by $N(D)$. In particular, on projective K3 surfaces, which are the projective irreducible holomorphic symplectic manifolds of dimension 2, the above tells us that all the asymptotic base loci of big divisors are a finite union of smooth rational curves.

We will say that a curve C on a projective IHS manifold X intersects negatively (resp. trivially) a divisor D and is contractible on some birational model of X , if there exist another projective IHS manifold X' , a birational map $f: X \dashrightarrow X'$, with $C \not\subset \text{Ind}(f)$ (the indeterminacy locus of f), such that $f_*(D) \cdot f_*(C) < 0$ (resp. $f_*(D) \cdot f_*(C) = 0$) and $f_*(C)$ lies on an extremal face F of $\overline{\text{NE}}(X')$ which is contractible, where $f_*(C)$ is the strict transform of C via f . We recall that a curve is extremal if its class in the Mori cone spans an extremal ray. We will denote the set of rational contractible curves by $\text{Cont}(X)$.

To conclude this prelude, we recall that the divisorial Zariski decomposition of pseudo-effective divisors on projective IHS manifolds is characterised with respect to the BBF form (see Subsection 1.3.2). Furthermore, as in the case of surfaces, the positivity of any pseudo-effective divisor on X is encoded by the positive part of this decomposition. Hence, if D is a big divisor on X , one may expect to find a characterisation of $\mathbf{B}_+(D), \mathbf{B}_-(D), \mathbf{B}(D)$ and of the stable classes, using the positive part and the negative part of the divisorial Zariski decomposition of D , together with the intersections of D with the rational curves contained in the variety.

Theorem 0.3.1: Let X be a projective irreducible holomorphic symplectic manifold and D a big \mathbf{R} -divisor on X .

1. All the irreducible components of $\mathbf{B}_+(D)$, $\mathbf{B}_-(D) = \mathbf{B}(D)$ are algebraically coisotropic, in particular, they have dimension at least $\dim(X)/2$.
2. The divisorial irreducible components of $\mathbf{B}_+(D)$ are the prime exceptional divisors which are q_X -orthogonal to $P(D)$. Any irreducible component V of $\mathbf{B}_+(D)$ is covered by rational, contractible curves, intersecting $P(D)$ negatively, or trivially, on some birational model of X which is an IHS manifold.
3. The divisorial irreducible components of $\mathbf{B}_-(D)$ are the prime exceptional divisors supporting $N(D)$. Any non-divisorial irreducible component of $\mathbf{B}_-(D) = \mathbf{B}(D)$ is covered by rational, contractible curves, intersecting $P(D)$ negatively on some birational model of X which is an IHS manifold.
4. $P(D)$ is unstable if and only if $\mathbf{B}_+(P(D))$ contains a curve $C \in \text{Cont}(X)$, such that $C \not\subset \text{Ind}(f)$ for some birational map $f: X \dashrightarrow X'$ of projective IHS manifolds, with $f_*(P(D))$ nef, and $f_*(P(D)) \cdot f_*(C) = 0$. Furthermore, all the destabilising numbers are rational numbers.

Let us state one particular consequence of item (1) of the theorem above, that should be of independent interest.

Corollary 0.3.2: Let $f: X \dashrightarrow X'$ be a birational map between projective IHS manifolds. Then (any irreducible component of) the indeterminacy locus of f is algebraically coisotropic.

Remark 0.3.3. Following the notation of Theorem 0.3.1, let D be any big divisor on X , V an irreducible component of $\mathbf{B}_+(D)$ (or $\mathbf{B}_-(D) = \mathbf{B}(D)$), and $\{C_t\}_t$ rational curves covering V , like those in the statement. Then, the birational model over which C_t can be contracted might depend on t . Furthermore, we provide an example showing that, in general, we cannot avoid considering the birational IHS models of the variety (cf. Subsection 4.5.2).

We now introduce some more terminology, to state the second main result of this paper.

Let Y be any smooth complex projective variety. The cone $\text{Amp}_k(X)$ (where k ranges from 1 to $\dim(Y)$) is the convex cone of divisor classes α on Y with $\dim(\mathbf{B}_+(\alpha)) < k$. It is known that $\overline{\text{NE}}(Y)^\vee$, the dual of the Kleiman-Mori cone $\overline{\text{NE}}(Y)$, is equal to the nef cone $\text{Nef}(Y) = \overline{\text{Amp}}(Y) = \overline{\text{Amp}}_1(Y)$ (see [ELMNP06, Theorem 1.1] for the last equality). This classical duality result is known as "Kleiman's criterion for amplitude" (cf. [Laz04, Theorem 1.4.29]), and it holds for any projective (non-necessarily smooth) variety. Later, Boucksom, Demailly, Păun, and Peternell proved that the dual of the pseudo-effective cone $\overline{\text{Eff}}(Y) = \overline{\text{Amp}}_d(Y)$ (where $d = \dim(Y)$) is equal to the cone of mobile curves $\overline{\text{Mob}}(Y) \subset \overline{\text{NE}}(Y)$, where $\text{Mob}(Y) \subset \text{NE}(Y)$ is the cone spanned by the

classes of curves moving in a family covering Y (cf. [BDPP13, Theorem 2.2]). Sam Payne proved that on a complete, \mathbf{Q} -factorial toric variety T , the closure of the cone $\text{Amp}_k(T)$ is dual to the cone $\text{bMob}_k(T)$ of curves which are birationally k -mobile, i.e. the closure of the cone spanned by classes which on some small \mathbf{Q} -factorial modification T' of T are represented by curves sweeping out the birational transform of k -dimensional subvarieties of T (cf. [Pay05, Theorem 2]). Furthermore, Payne asked whether such a result could have been generalised to other classes of varieties. Payne's result for toric varieties was generalised by Choi (cf. [Cho12, Theorem 1.1, Corollary 1.1]) to Mori dream spaces and \mathbf{Q} -factorial FT varieties (see [Cho12, Definition 4.4] for the definition of FT variety).

There are two obstructions for projective IHS manifolds to be Mori dream spaces: the group of birational self maps could be infinite, and, in general, it is not known whether a nef, integral divisor which is not big is semiample. The latter condition is always satisfied for the known deformation classes of IHS manifolds, hence a projective IHS manifold belonging to one of the known deformation classes is a Mori dream space if and only if its group of birational self maps is finite (see [Den, Corollary 3.11]), and Choi's result applies in these cases. Theorem 0.3.4 gives an affirmative answer to the question asked by Sam Payne (actually, as it is mentioned in Payne's article, it seems that the original question was raised by Robert Lazarsfeld and Olivier Debarre) in the case of arbitrary projective IHS manifolds.

We are now ready to state the second main result of this chapter.

Theorem 0.3.4: Let X be a projective IHS manifold of dimension $2n$. Then

$$\overline{\text{Amp}}_k(X) = \text{bMob}_k(X)^\vee, \quad (0.1)$$

for any $1 < k < 2n$, where $\dim(X) = 2n$. Furthermore, if $k = 1, \dots, n$, we have $\overline{\text{Amp}}_k(X) = \text{Nef}(X) = \overline{\text{NE}}(X)^\vee$.

For the formal definition of $\text{bMob}_k(X)$, we refer the reader to Section 4.3. We now explain the content of Theorem 0.3.4.

Let Y be any smooth complex projective variety. The cone $\text{Amp}_k(X)$ (where k ranges from 1 to $\dim(Y)$) is the convex cone of divisor classes α on Y with $\dim(\mathbf{B}_+(\alpha)) < k$. It is known that $\overline{\text{NE}}(Y)^\vee$, the dual of the Kleiman-Mori cone $\overline{\text{NE}}(Y)$, is equal to the nef cone $\text{Nef}(Y) = \overline{\text{Amp}}(Y) = \overline{\text{Amp}}_1(Y)$ (see [ELMNP06, Theorem 1.1] for the last equality). This classical duality result is known as "Kleiman's criterion for ampleness" ([Laz04, Theorem 1.4.29]), and it holds for any projective (non-necessarily smooth) variety. Later, Boucksom, Demailly, Păun, and Peternell proved that the dual of the pseudo-effective cone $\overline{\text{Eff}}(Y)$ is equal to the cone of mobile curves $\overline{\text{Mob}}(Y) \subset \overline{\text{NE}}(Y)$, where $\text{Mob}(Y) \subset \text{NE}(Y)$ is the cone spanned by the classes of the curves which move in a family covering Y ([BDPP13, Theorem 2.2]). Sam Payne proved that on a complete, \mathbf{Q} -factorial toric variety T , the closure of the cone $\text{Amp}_k(T)$ is dual to the cone $\text{bMob}_k(T)$ of curves which are birationally k -mobile, i.e. the closure of the cone spanned by classes which on some small \mathbf{Q} -

factorial modification T' of T are represented by curves sweeping out the birational transform of k -dimensional subvarieties of T ([Pay05, Theorem 2]). Furthermore, Payne asked whether such a result could be generalised to other classes of varieties. Payne's result for toric varieties was generalised by Choi ([Cho12, Theorem 1.1, Corollary 1.1]) to Mori dream spaces and \mathbf{Q} -factorial FT varieties (see [Cho12, Definition 4.4] for the definition of FT variety).

There are two obstructions for projective IHS manifolds to be Mori dream spaces: the group of birational self maps could be infinite, and, in general, it is not known whether a nef, integral divisor which is not big is semiample. The latter condition is always satisfied for the known deformation classes of IHS manifolds, hence a projective IHS manifold belonging to one of the known deformation classes is a Mori dream space if and only if its group of birational self maps is finite (see Corollary 3.1.13), and Choi's result applies in these cases. Theorem 0.3.4 gives an affirmative answer to the question asked by Sam Payne (actually, as it is mentioned in Payne's article, it seems that the original question was raised by Robert Lazarsfeld and Olivier Debarre) in the case of arbitrary projective IHS manifolds.

It is worth observing that if Y is any smooth, complex projective variety, a weaker description for the dual cones $\overline{\text{Amp}}_k(Y)^\vee$, and for the asymptotic base loci of pseudo-effective divisors has been provided by Brian Lehman ([Leh11]). In particular, items (1) and (2) of Theorem 0.3.1, and Theorem 0.3.4, are a strengthening of [Leh11, Theorem 1.5, Proposition 3.2, Corollary 3.3] (in the case of projective IHS manifolds).

0.4 Introduction to the fifth chapter

The theory of Newton-Okounkov bodies in the case of smooth complex projective surfaces is well developed, thanks in particular to the work of Lazarsfeld-Mustața, Küronya-Lozovanu, Küronya-Lozovanu-Maclean ([LM09], [KL17a], [KLM12]). Given a surface S and a general flag $F_\bullet = \{\text{pt} \subset C \subset S\}$, the Newton-Okounkov body of any big divisor on S is determined by the way the Zariski decomposition of D changes when we perturb D along the negative direction induced by C ([LM09, Theorem 6.4]). In particular, if D is a big divisor on S , the Newton-Okounkov body of D with respect to the flag F_\bullet is given by the polygon ([LM09, Theorem 6.4], [KLM12, Theorem B])

$$\Delta_{F_\bullet}(D) = \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq \mu_C(D), 0 \leq y \leq P(D_t) \cdot C\},$$

where $D_t := D - tC$ and $\mu_C(D) := \sup_{t \in \mathbf{R}^{\geq 0}} (D - tC \text{ is big})$. The Euclidean volume of this body is equal to $P(D)^2/2$ ([LM09, Theorem A]), where $P(D)$ is the positive part of the Zariski decomposition of D .

In the case of projective IHS manifolds, we obtained the decomposition of the big cone into Boucksom-Zariski chambers and this decomposition explains how the divisorial Zariski decomposition of a big divisor changes in the big cone when the divisor is perturbed. Using the decomposition of the big cone and the BBF pairing, we can associate to any big divisor a polygon behaving like the Newton-Okounkov body of a big divisor on a surface (computed with respect to a general flag). Let X be a projective IHS manifold, D a big divisor on X and E a prime divisor on

X. Consider the subset of \mathbf{R}^2

$$\Delta_E^{\text{num}}(D) := \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq \mu_E(D), 0 \leq y \leq q_X(P(D_t), E)\},$$

where $D_t := D - tE$ and $\mu_E(D) := \sup_{t \in \mathbf{R}^{\geq 0}} (D - tE \text{ is big})$. Below is the main result of the fifth chapter.

Theorem (Theorem 5.2.5 and Proposition 5.2.7): The subset $\Delta_E^{\text{num}}(D)$ of \mathbf{R}^2 introduced above is a convex polygon. Moreover, $\text{vol}(\Delta_E^{\text{num}}(D)) = q_X(P(D))/2$, where $\text{vol}(\Delta_E^{\text{num}}(D))$ is the Euclidean volume of the body $\Delta_E^{\text{num}}(D)$. In particular, we have

$$2^n c_X (\text{vol}(\Delta_E^{\text{num}}(D)))^n = c_X (q_X(P(D)))^n = \text{vol}(D) = (2n)! \cdot \text{vol}(\Delta(D)),$$

where c_X is the Fujiki constant of X and $\Delta(D)$ is the classical Newton Okounkov body of D with respect to any complete flag.

For the definition of c_X , we refer the reader to the first chapter of the thesis. As in the case of the classical Newton-Okounkov bodies, we show the existence of a convex cone of dimension $\rho(X) + 2$ whose "slices" are the polygons $\Delta_E^{\text{num}}(D)$ in the statement above.

Theorem 0.4.1: For each prime divisor E on a projective IHS manifold X there exists a convex cone $\Delta_E^{\text{num}}(X) \subset N^1(X)_{\mathbf{R}} \times \mathbf{R}^2$, together with the natural projection $p_1: \Delta_E^{\text{num}}(X) \rightarrow N^1(X)_{\mathbf{R}}$, such that the fibre $p_1^{-1}(\zeta)$ is equal to the polygon $\Delta_E^{\text{num}}(\zeta)$, for all ζ in $\text{Big}(X)$.

To conclude, following [LS14] we show that under some hypothesis, the polygons $\Delta_E(\alpha)$ (with α a big class) can be expressed as a Minkowski sum of some polygons $\{\Delta_E(\beta_i)\}_{i \in I}$, for some movable classes $\{\beta_i\}_{i \in I}$. In particular, we have the following.

Theorem 0.4.2: Let X be a projective IHS manifold with $\text{Eff}(X)$ rational polyhedral and E a big prime divisor. Then, there exists a finite set Ω of movable \mathbf{Q} -divisors such that, for any movable \mathbf{Q} -divisor D , there exist some rational numbers $\alpha_P(D)$ such that $D = \sum_{P \in \Omega} \alpha_P(D)P$ and $\Delta_E(D) = \sum_{P \in \Omega} \alpha_P(D)\Delta_E(P)$.

Newton-Okounkov bodies have attracted a great deal of interest in recent years. These objects contain a lot of asymptotic information about the divisors on projective varieties, but in general they are not easy to compute. In addition, the dimension of classical Newton-Okounkov bodies is equal to the dimension of the variety. In the case of projective IHS manifolds, we have the classical Newton-Okounkov bodies and the polygons of Newton-Okounkov type in the theorem above. These polygons are easier to compute and we hope that they contain fine information about the positivity of divisors as in the case of the classical Newton-Okounkov bodies.



1. Preparatory material

Throughout this chapter Y will be a normal projective variety of dimension d and X an IHS manifold of complex dimension $2n$.

1.1 Positivity notions

A prime divisor on Y will be a reduced and irreducible hypersurface. Recall that the Néron-Severi space of Y can be defined as $N^1(Y)_{\mathbf{R}} := N^1(Y) \otimes_{\mathbf{Z}} \mathbf{R}$, where $N^1(Y) := \text{Div}(Y)/\equiv$ is the Néron-Severi group of Y , $\text{Div}(Y)$ is the group of integral Cartier divisors of Y , and \equiv is the numerical equivalence relation. A Cartier divisor will be a formal linear combination (with coefficients in \mathbf{R}) of integral Cartier divisors. The class of a Cartier divisor D in $N^1(Y)_{\mathbf{R}}$ will be denoted by $[D]$.

Remark 1.1.1. For an IHS manifold X , the numerical and linear equivalence relations coincide.

Definition 1.1.2: An integral Cartier divisor D on Y is big if there exists a positive integer m such that $mD \equiv A + N$, where A is an ample integral Cartier divisor, and N is an effective integral Cartier divisor. When D is a Cartier \mathbf{Q} -divisor, we say that it is big if there exists a positive integer m such that mD is a big integral Cartier divisor. If D is any Cartier divisor, we say that it is big if $D = \sum_i a_i D_i$, where a_i is a positive real number and D_i is a big integral Cartier divisor, for any i .

The above definition tells us that, up to picking a sufficiently large multiple, a big integral Cartier divisor induces a rational map which is birational onto its image. Hence, in a certain sense, being big means being "generically ample". Note that bigness behaves well with respect to \equiv , namely, if $D \equiv D'$, D is big if and only if D' is.

An ample Cartier divisor is a Cartier divisor D of the form $D = \sum_i a_i D_i$, where any a_i is a positive real number, and any D_i is an ample integral Cartier divisor. A Cartier divisor D on Y is nef if $D \cdot C \geq 0$, for any irreducible and reduced curve C on Y . A Cartier divisor D is effective if $D = \sum_i a_i D_i$, with a_i is a non-negative real number, and D_i a prime Cartier divisor, for any i .

The ample cone $\text{Amp}(Y)$ is the convex cone in $N^1(Y)_{\mathbf{R}}$ of ample Cartier divisor classes. The nef cone $\text{Nef}(Y)$ is the convex cone in $N^1(Y)_{\mathbf{R}}$ of nef Cartier divisor classes. It is known that

$\text{Nef}(Y) = \overline{\text{Amp}(Y)}$, and that $\text{Amp}(Y)$ is the interior of $\text{Nef}(Y)$ (see Theorem 1.4.23 of [Laz04]). The big cone $\text{Big}(Y)$ of Y is the convex cone in $N^1(Y)_{\mathbf{R}}$ of big Cartier divisor classes. The pseudo-effective cone of Y is the closure of the big cone in $N^1(Y)_{\mathbf{R}}$. The effective cone $\text{Eff}(Y)$ of Y is the convex cone in $N^1(Y)_{\mathbf{R}}$ spanned by effective Cartier divisor classes. One can equivalently define the pseudo-effective cone of Y as the closure $\overline{\text{Eff}(Y)}$ of the effective cone in the Néron-Severi space. We have $\text{Amp}(Y) \subset \text{Big}(Y)$, and so $\text{Nef}(Y) \subset \overline{\text{Eff}(Y)}$.

Example 1.1.3. Consider $\pi: \text{Bl}_P \mathbf{P}^2 \rightarrow \mathbf{P}^2$, the blow up of \mathbf{P}^2 at a closed point P , and let l be a line of \mathbf{P}^2 not containing P . Then, π^*l is big and nef but not ample, because π^*l does not intersect the exceptional divisor. An example of a non-big divisor is given by a prime exceptional divisor E on X (see item (3) in Remark 2.1.1). For instance, a smooth rational curve on a smooth K3 surface is not big, as well as an elliptic curve on a K3 surface.

Definition 1.1.4:

1. A movable integral Cartier divisor on Y is an effective integral Cartier divisor D such that the linear series $|mD|$ has no divisorial components in its base locus, for any large enough and sufficiently divisible m .
2. We define the movable cone $\text{Mov}(Y)$ in $N^1(Y)_{\mathbf{R}}$ as the convex cone generated by the movable divisor classes.

We now recall two definitions concerned with the geometry of convex cones.

Definition 1.1.5: Let $K \subset \mathbf{R}^k$ be a closed convex cone with a non-empty interior.

- We say that K is locally (rational) polyhedral at $v \in \partial K$ if v has an open neighbourhood $U = U(v)$, such that $K \cap U$ is defined in U by a finite number of (rational) linear inequalities.
- Let U be an open subset of ∂K . We say that U is a circular part of K if there is no element in U at which K is locally polyhedral.

1.1.1 Asymptotic base loci

We recall the definition of the various asymptotic base loci and provide some of their basic properties. We refer the reader to [ELMNP06] for a comprehensive treatment of the argument. The stable base locus of an integral Cartier divisor D is defined as

$$\mathbf{B}(D) = \bigcap_{k \geq 1} \text{Bs}(|kD|),$$

where $\text{Bs}(|kD|)$ is the base locus of the linear series $|kD|$. The stable base locus of a Cartier \mathbf{Q} -divisor D on Y is defined as $\mathbf{B}(mD)$, where m is any positive integer such that mD integral.

Definition 1.1.6: Let D be a divisor on Y .

- The real *stable base locus* of D is

$$\mathbf{B}(D) := \bigcap \{ \text{Supp}(E) \mid E \text{ effective } \mathbf{R}\text{-divisor, } E \sim_{\mathbf{R}} D \}.$$

If D is a \mathbf{Q} -divisor, the real stable base locus of D coincides with the usual one ([BBP13, Proposition 1.1]).

- The *augmented base locus* of D is

$$\mathbf{B}_+(D) := \bigcap_{D=A+E} \text{Supp}(E),$$

where the intersection is taken over all the decompositions of the form $D = A + E$, where A and E are Cartier divisors such that A is ample and E is effective.

- The *restricted base locus* of D on Y is

$$\mathbf{B}_-(D) := \bigcup_A \mathbf{B}(D + A),$$

where the union is taken over all ample divisors A .

If D is a divisor on Y , we have the inclusions $\mathbf{B}_-(D) \subset \mathbf{B}(D) \subset \mathbf{B}_+(D)$ (see [ELMP06, Example 1.16], and [BBP13, Introduction]), which in general are strict, as it is shown by the following example.

Example 1.1.7. Consider the blow-up of the plane at 2 points, L be any line not passing through the two points, and L' its strict transform. Furthermore, let E_1, E_2 be the two exceptional divisors arising from the 2 points. Consider the divisor $D_i := L' + E_i$, for $i = 1, 2$. We observe that $\mathbf{B}_-(D_i) = \mathbf{B}(D_i) = E_i$, while $\mathbf{B}_+(D_i) = E_1 \cup E_2$ (it follows directly from [ELMP06, Examples 1.10, 1.11, 3.4]). For a strict (non-trivial) inclusion $\mathbf{B}_- \subsetneq \mathbf{B}$, with \mathbf{B}_- not algebraic, one can consider the beautiful example provided by Lesieutre (cf. [Les14]). For an example with $\emptyset \neq \mathbf{B}_- \subsetneq \mathbf{B} \subsetneq \mathbf{B}_+ \subsetneq Y$ see [TX23, Example 3.4, (iii)].

Remark 1.1.8. The augmented and restricted base loci on Y can be studied in the Néron-Severi space because they do not depend on the representative of the class we have chosen. In general, we cannot say the same for the (real) stable base locus. However, on IHS manifolds numerical and \mathbf{R} -linear equivalences coincide. Indeed, suppose that D, D' are numerically equivalent \mathbf{R} -divisors on X , then $D - D' = \sum_i a_i D_i$, where $a_i \in \mathbf{R}$ and D_i is a numerically trivial integral divisor, and hence also linearly equivalent to 0. In particular, D_i is a principal divisor, thus $D \sim_{\mathbf{R}} D'$. This implies that also the real stable base locus can be safely studied up to numerical equivalence in the case of projective IHS manifolds.

Note that if D is any Cartier divisor on Y , $\mathbf{B}_+(D) \subsetneq Y$ if and only if D is big, and $\mathbf{B}_+(D) = \emptyset$ if and only if D is ample. Moreover, $\mathbf{B}_-(D) \subsetneq Y$ if and only if D is pseudo-effective, and $\mathbf{B}_-(D) = \emptyset$ if and only if D is nef.

In general, it is not possible to describe the asymptotic base loci of a big divisor using the intersections of the divisor with curves, as the following example shows.

Example 1.1.9. Let Q be a point of \mathbf{P}^2 and $S' := \text{Bl}_Q(\mathbf{P}^2)$. Now, let S be the blow up of S' at a point P lying on the exceptional divisor E' of S' . If F is the exceptional divisor of S (i.e. the fiber over P), and E is the strict transform of E' on S , we have $F^2 = -1$, and $E^2 = -2$. Furthermore, the Gram matrix of $\{E, F\}$ is negative definite. Let D be a big and nef divisor that is orthogonal to both E, F , and consider the divisors $D_1 := D + E + F$, $D_2 := D + E + 3F$. Then $\mathbf{B}_-(D_2) = \mathbf{B}_-(D_1) = \text{Supp}(E) \cup \text{Supp}(F)$. But $D_1 \cdot E = -1 < 0$, $D_1 \cdot F = 0$, and $D_2 \cdot E = 1 > 0$.

We say that a big Cartier divisor D on Y is stable (resp. unstable) if $\mathbf{B}_+(D) = \mathbf{B}_-(D)$ (resp. $\mathbf{B}_-(D) \subsetneq \mathbf{B}_+(D)$). The stable divisor classes on Y form a dense open subset of $\text{Big}(Y)$, by [ELMP06, Proposition 1.26]. Following [BKS04, Definition 2.8], we say that a real number λ is a destabilising number for a big integral Cartier divisor D , with respect to an ample integral Cartier divisor A , if $\mathbf{B}_+(D - \alpha A) \subsetneq \mathbf{B}_+(D - \beta A)$ for all rational numbers α, β with $\alpha < \lambda < \beta$.

Definition 1.1.10: Let D be a stable big Cartier divisor on Y . The stability chamber of D is defined as

$$\text{SC}(D) := \{\alpha \in \text{Big}(Y) \mid \mathbf{B}_+(\alpha) = \mathbf{B}_+(D)\}.$$

1.1.2 Volumes

We introduce an asymptotic invariant for an integral Cartier divisor D on Y , which measures the asymptotic rate of growth of the spaces of global sections $H^0(Y, \mathcal{O}_Y(mD))$.

Definition 1.1.11: Let D be an integral Cartier divisor on Y . The volume of D is defined as

$$\text{vol}(D) := \limsup_{k \rightarrow \infty} \frac{h^0(Y, kD)}{k^d / d!}.$$

It is known that $\text{vol}(D) > 0$ if and only if D is big. One defines the volume of a Cartier \mathbf{Q} -divisor D by picking an integer k such that kD is integral, and by setting $\text{vol}(D) := \frac{1}{k^d} \text{vol}(kD)$. This definition does not depend on the integer k we have chosen (see Lemma 2.2.38 on [Laz04]). One can extend the notion of volume to every Cartier divisor. In particular, pick $D \in \text{Div}_{\mathbf{R}}(Y) = \text{Div}(Y) \otimes_{\mathbf{Z}} \mathbf{R}$ and let $\{D_k\}_k$ be a sequence of Cartier \mathbf{Q} -divisors converging to D in $N^1(Y)_{\mathbf{R}}$. We define

$$\text{vol}(D) := \lim_{k \rightarrow \infty} \text{vol}(D_k).$$

This number is independent of the choice of the sequence $\{D_k\}_k$ (see Theorem 2.2.44 of [Laz04]). Moreover, two numerically equivalent Cartier divisors have the same volume (see [Laz04, Propo-

sition 2.2.41]), hence the volume can be studied in $N^1(Y)_{\mathbf{R}}$, and by [Laz04, Theorem 2.2.44] it defines a continuous function $\text{vol}(-): N^1(Y)_{\mathbf{R}} \rightarrow \mathbf{R}$, called the *volume function*.

Now, let V be a subvariety of Y of dimension $m > 0$, D an integral Cartier divisor on Y . We have the natural restriction maps

$$r_k: H^0(Y, \mathcal{O}_Y(kD)) \rightarrow H^0(V, \mathcal{O}_V(kD)),$$

for any positive integer k . We define the restricted volume of D to V as

$$\text{vol}_{X|V}(D) := \limsup_{k \rightarrow \infty} \frac{\text{rank}(r_k)}{k^m}.$$

If D is any Cartier \mathbf{Q} -divisor, one picks k such that kD is integral, and defines $\text{vol}_{X|V}(D) := \frac{1}{k^m} \text{vol}_{X|V}(kD)$. To conclude, the following result generalises the listed properties of the classical volumes to the case of restricted volumes.

Theorem 1.1.12 ([ELMNP06, Theorem A]): Let $V \subset X$ be a subvariety of dimension $m > 0$ and let D be a Cartier \mathbf{Q} -divisor such that $V \not\subset \mathbf{B}_+(D)$. Then $\text{vol}_{X|V}(D) > 0$ and $\text{vol}_{X|V}(D)$ depends only on the numerical equivalence class of D . Furthermore, $\text{vol}_{X|V}(D)$ varies continuously as a function of the numerical equivalence class of D , and it extends uniquely to a continuous function

$$\text{vol}_{X|V}: \text{Big}^V(X)^+ \rightarrow \mathbf{R}^{\geq 0},$$

where $\text{Big}^V(X)^+$ denotes the set of all Cartier divisor classes ζ such that $V \not\subset \mathbf{B}_+(\zeta)$.

1.2 Lattices, hyperbolic geometry and Pell's equations

1.2.1 Some lattice theory

A lattice is a couple (L, b) , where L is a finitely generated, free abelian group, and b is a non-degenerate, integral valued, symmetric bilinear form $b: L \times L \rightarrow \mathbf{Z}$. If no confusion arises, we will simply write L . We say that L is even if b takes only even values. The signature $\text{sign}(b)$ of b is the signature of the natural extension $b_{\mathbf{R}}$ of b to $L \otimes_{\mathbf{Z}} \mathbf{R}$.

The divisibility $\text{div}_L(x)$ of an element $x \in L$ is defined as the non-negative generator of the ideal $b(x, L)$ in \mathbf{Z} . When no confusion arises, we will write $\text{div}(x)$ instead of $\text{div}_L(x)$.

As b is non-degenerate, one has an injective group homomorphism $L \hookrightarrow L^{\vee} := \text{Hom}_{\mathbf{Z}}(L, \mathbf{Z})$, by sending an element x of L to the element $b(x, -)$ of L^{\vee} . We will denote $b(x, -)$ by x when no confusion arises. Note that we have the following identification

$$L^{\vee} = \{x \in L \otimes_{\mathbf{Z}} \mathbf{Q} \mid b(x, y) \in \mathbf{Z} \text{ for all } y \in L\} \subset L \otimes_{\mathbf{Z}} \mathbf{Q}.$$

The discriminant group of L is the finite abelian group $A_L := L^{\vee} / L$. If A_L is trivial we say that L is unimodular. If f is an element of L^{\vee} we will denote its class in A_L by $[f]$. We say that an

element $x \in L$ is primitive if we cannot write $x = ax'$, where $a \in \mathbf{Z}^{>1}$, and $x' \in L$. In particular, x is non-zero.

An isometry of L is an automorphism of it (as an abelian group) preserving b . The group of isometries of L is denoted by $O(L)$. Suppose now that L has signature $\text{sign}(b) = (3, b_2 - 3)$, where $b_2 \in \mathbf{N}$, and $b_2 > 3$. Define the cone

$$C_L := \{v \in L \otimes_{\mathbf{Z}} \mathbf{R} \mid b(v) > 0\}.$$

In [Mar11, Lemma 4.1] it was proven that C_L has the homotopy type of S^2 , hence $H^2(C_L, \mathbf{Z}) \cong \mathbf{Z}$. Any isometry in $O(L)$ induces a homeomorphism of C_L , which in turn induces an automorphism of $H^2(C_L, \mathbf{Z}) \cong \mathbf{Z}$. This automorphism can act as 1 or -1 . We define $O^+(L)$ as the subgroup of the isometries of L acting trivially on $H^2(C_L, \mathbf{Z})$. The group $\widetilde{SO}^+(L)$ is the subgroup of the isometries of L of determinant 1, acting trivially on both A_L and $H^2(C_L, \mathbf{Z}) \cong \mathbf{Z}$.

Remark 1.2.1. Given a primitive element $x \in L$, we can consider the element $x/\text{div}(x)$ of L^\vee , which in turn gives the element $[x/\text{div}(x)]$ of A_L . Note that $\text{ord}([x/\text{div}(x)]) = \text{div}(x)$, hence $\text{div}(x)$ divides $|A_L|$. Indeed, suppose that the order is $k \leq \text{div}(x)$, then $kt = \text{div}(x)$ for some positive integer t . This implies that there exists an element $y \in L$ such that $x = ty$, and the primitivity of x forces t to be 1, and so $k = \text{div}(x)$. In particular, if M is the maximum among the orders of the elements of A_L , then $\text{div}(x) \leq M$.

The following is a result of Eichler, which will be very useful.

Lemma 1.2.2 (Eichler's criterion): Let L' be an even lattice and $L = U^2 \oplus L'$. Let $v, w \in L$ be two primitive elements such that the following hold:

- $b(v) = b(w)$.
- $[v/\text{div}(v)] = [w/\text{div}(w)]$ in A_L .

Then there exists an isometry $\iota \in \widetilde{SO}^+(L)$, such that $\iota(v) = w$.

This version of Eichler's criterion was taken from [MR20, Lemma 2.6].

Example 1.2.3. Given an IHS manifold X , one has the lattice $(H^2(X, \mathbf{Z}), q_X)$, where (with some abuse of notation) by q_X we mean the integral valued, symmetric bilinear form induced by the restriction of the BBF form to $H^2(X, \mathbf{Z})$. Note that, for any of the known IHS manifolds, the lattice $H^2(X, \mathbf{Z})$ is even and contains two copies of the hyperbolic lattice U so that we can use Eichler's criterion. The signature of q_X on $H^2(X, \mathbf{Z})$ is $(3, b_2(X) - 3)$, where $b_2(X)$ is the second Betti number of X . We will denote the discriminant group $A_{H^2(X, \mathbf{Z})}$ by A_X . If X is projective also $(N^1(X), q_X)$ is a lattice and in this case the signature of q_X is $(1, \rho(X) - 1)$, where $\rho(X)$ is the Picard number of X . We refer the reader to [GHJ03, Corollary 23.11] for a proof of these facts.

1.2.2 Some hyperbolic geometry

Consider a real vector space V of dimension $m + 1$ endowed with a non-degenerate quadratic form q of signature $(1, m)$. Thus, abstractly, (V, q) is isomorphic to \mathbf{R}^{m+1} with the quadratic form $x_0^2 - x_1^2 - \cdots - x_m^2$. The set

$$\{x \in V \mid q(x) > 0\}$$

has two connected components which are interchanged by $x \rightarrow -x$. We usually distinguish one of the two connected components, say $\mathcal{C} \subset V$, and call it the positive cone. Thus,

$$\{x \in V \mid q(x) > 0\} = \mathcal{C} \sqcup (-\mathcal{C}).$$

The subset $\mathcal{C}(1)$ of all $x \in \mathcal{C}$ with $q(x) = 1$ is isometric to the hyperbolic m -space

$$\mathbf{H}^m := \{x \in \mathbf{R}^{m+1} \mid x_0^2 - x_1^2 - \cdots - x_m^2 = 1, x_0 > 0\}.$$

By writing $\mathcal{C} \cong \mathcal{C}(1) \times \mathbf{R}_{>0} \cong \mathbf{H}^m \times \mathbf{R}_{>0}$, questions concerning the geometry of \mathcal{C} can be reduced to analogous ones for \mathbf{H}^m . We can equivalently view \mathbf{H}^m as the projectivisation of \mathcal{C} (in symbols $\mathbf{P}(\mathcal{C})$).

We write $O(V)$ for the orthogonal group $O(V, q)$, which is abstractly isomorphic to $O(1, m)$. By $O^+(V) \subset O(V)$ we denote the index two subgroup of transformations preserving the positive cone \mathcal{C} .

Now, let L be a lattice of signature $(1, m)$, and consider $V := L \otimes \mathbf{R}$, with the induced quadratic form. We can consider $O(L) \subset O(V)$, and $O^+(L) := O(L) \cap O^+(V)$.

Part of the above was taken from [Huy15, Chapter 8, Section 2].

Definition 1.2.4: Let Γ be a non-trivial discrete group of isometries of \mathbf{H}^m and choose a point $x \in \mathbf{H}^m$. The limit set $\Lambda(\Gamma) \subset \partial\mathbf{H}^m$ of Γ is the set of all the accumulation points of the orbit $\Gamma(x)$ in \mathbf{H}^m .

Definition 1.2.5: Let Γ be a non-trivial discrete group of isometries of \mathbf{H}^m . We say that Γ is non-elementary if $\Lambda(\Gamma)$ consists of at most two points.

Example 1.2.6. Let X be a projective IHS manifold of Picard number $m + 1$, and q_X the Beauville-Bogomolov-Fujiki (BBF) form of X . The above situation is represented by the couple $(N^1(X)_{\mathbf{R}}, q_X)$. It is known that $\text{Mon}_{\text{Hdg}}^2(X)$ acts on $\text{Pic}(X)$ via isometries, and preserves the positive cone $\text{Pos}(X)$ (which by abuse of notation we still denote by \mathcal{C}_X) of X . In particular, we have a natural map

$$\iota: \text{Mon}_{\text{Hdg}}^2(X) \rightarrow O^+(\text{Pic}(X)).$$

Let $\Gamma' \subset O^+(\text{Pic}(X))$ be the image of ι . Markman proved that Γ' is of finite index inside $O^+(\text{Pic}(X))$ ([Mar11, Lemma 6.23]). The projectivisation of $\text{Pos}(X)$ (i.e. $\mathbf{P}(\text{Pos}(X))$) is an m -dimensional

hyperbolic space, and we call it the *projectivised positive cone* (this terminology follows the one adopted by Amerik and Verbitsky in their works).

1.2.3 Pell's equations

A Pell equation is any diophantine equation of the form

$$x^2 - ky^2 = 1, \quad (1.1)$$

where k is a given positive integer. The fundamental solution of the equation (1.1) is the (positive) solution (x_1, y_1) minimising x . Once the fundamental solution is known, any other positive solution (and hence all the solutions) can be calculated recursively with the formulas

$$\begin{aligned} x_{j+1} &= x_1 x_j + k y_1 y_j \\ y_{j+1} &= x_1 y_j + y_1 x_j, \end{aligned}$$

and this is all we need from the theory of diophantine equations.

1.3 Generalities on IHS manifolds

For a general introduction to IHS manifolds, we refer the reader to [GHJ03]. Let X be an IHS manifold. Thanks to the work [Bea83] of Beauville, there exists a quadratic form on $H^2(X, \mathbf{C})$ generalising the intersection form on a surface. In particular, choosing the symplectic form σ of X in such a way that $\int_X (\sigma \bar{\sigma})^n = 1$, one can define

$$q_X(\alpha) := \frac{n}{2} \int_X (\sigma \bar{\sigma})^{n-1} \alpha^2 + (1-n) \left(\int_X \sigma^n \bar{\sigma}^{n-1} \alpha \right) \cdot \left(\int_X \sigma^{n-1} \bar{\sigma}^n \alpha \right),$$

for any $\alpha \in H^2(X, \mathbf{C})$. The quadratic form q_X is non-degenerate and is known as the Beauville-Bogomolov-Fujiki form (BBF form in what follows). Up to a rescaling, q_X takes integer values and is primitive on $H^2(X, \mathbf{Z})$ (for us q_X will always be integer-valued). We have the following important property for q_X .

Theorem 1.3.1 (Beauville-Bogomolov-Fujiki relation): There exists a positive constant $c_X \in \mathbf{R}^{>0}$, known as Fujiki constant, such that $q_X(\alpha)^n = c_X \int_X \alpha^{2n}$, for all $\alpha \in H^2(X, \mathbf{C})$. In particular, q_X can be renormalised such that q_X is a primitive integral quadratic form on $H^2(X, \mathbf{Z})$.

The BBF form shares many properties with the usual intersection form on a surface. Below are those that will be mostly used:

- $q_X(\alpha) > 0$ for any Kähler class α on X ([Huy99, p. 1.10]).
- If α is Kähler, $q_X(\alpha, N) > 0$ for any non-zero effective divisor N on X ([Huy99, p. 1.11]).

- $q_X(E, E') \geq 0$ for any couple of distinct prime divisors on X ([Bou04, Proposition 4.2, item (ii)]).

Note that q_X is invariant under deformation and its signature on $H^2(X, \mathbf{R})$ is $(3, b_2(X) - 3)$, while, when X is projective, its signature on $N^1(X)_{\mathbf{R}}$ is $(1, \rho(X) - 1)$, where $\rho(X)$ is the Picard number of X .

Definition 1.3.2: We say that a prime divisor E on X is exceptional if $q_X(E) < 0$.

Let $f: X \dashrightarrow X'$ be a bimeromorphic map, where X' is another IHS manifold. Recall that f restricts to an isomorphism $f: U \rightarrow U'$, where U, U' are open subsets such that $\text{codim}_X(X \setminus U), \text{codim}_{X'}(X' \setminus U') \geq 2$ (see for example [Huy99, paragraph 4.4]). Using the long exact sequence in cohomology with compact support and Poincaré duality, one has a chain of isomorphisms

$$H^2(X, \mathbf{R}) \cong H^2(U, \mathbf{R}) \cong H^2(U', \mathbf{R}) \cong H^2(X', \mathbf{R}). \quad (1.2)$$

The composition of the isomorphisms in (1.2), which we denote by f_* , is an isometry with respect to q_X and $q_{X'}$ and a morphism of Hodge structures (see for example [OGr97, Proposition I.6.2] for a proof). The inverse of f_* will be denoted by f^* (pull-back). Note that, when restricted to $\text{Pic}(X')$, f^* behaves like the pull-back of line bundles via morphisms in the usual sense because X and X' are isomorphic in codimension one.

1.3.1 Cones associated with IHS manifolds

We say that a divisor D on an IHS manifold X is q_X -nef if $q_X(D, E) \geq 0$, for any prime divisor E in X .

Definition 1.3.3: The q_X -nef cone of an IHS manifold X is the convex cone in $N^1(X)_{\mathbf{R}}$ of q_X -nef divisor classes. It will be denoted by $\text{Nef}_{q_X}(X)$.

The q_X -nef cone is closed since it is by definition the intersection of all the closed half-spaces of the form

$$D^{\geq 0} := \{\alpha \in N^1(X)_{\mathbf{R}} \mid q_X(\alpha, D) \geq 0\},$$

where D is a prime divisor on X . Moreover, it is non-empty, because it contains 0. Given a divisor D on X , with $D \neq 0$, we will frequently need to consider in $N^1(X)_{\mathbf{R}}$ the hyperplane of the classes that are q_X -orthogonal to D , i.e.

$$D^{\perp} := \{\alpha \in N^1(X)_{\mathbf{R}} \mid q_X(\alpha, D) = 0\}.$$

We recall that the positive cone $\mathcal{C}_X \subset H^{1,1}(X, \mathbf{R}) := H^{1,1}(X, \mathbf{C}) \cap H^2(X, \mathbf{R})$ is the connected component of the set $\{\alpha \in H^{1,1}(X, \mathbf{R}) \mid q_X(\alpha) > 0\}$ containing the Kähler cone \mathcal{K}_X . We will mostly

deal with the algebraic part of \mathcal{C}_X , which we denote by $\text{Pos}(X)$, and $\text{Pos}(X) := \{\alpha \in N^1(X)_{\mathbf{R}} \mid q_X(\alpha) > 0\}$. We still call $\text{Pos}(X)$ the positive cone of X .

Remark 1.3.4. The following holds

$$\text{Pos}(X) \subset \text{Big}(X). \quad (1.3)$$

Indeed, by [Huy99, Proposition 3.8] and [Dem92, Proposition 1.4], we have that

$$\text{Pos}(X) \subset \overline{\text{Eff}(X)}. \quad (1.4)$$

Taking the interiors (in $N^1(X)_{\mathbf{R}}$) of both members of (1.4) we obtain (1.3).

Definition 1.3.5: The fundamental exceptional chamber of the positive cone \mathcal{C}_X is defined as

$$\mathcal{FE}_X := \{\alpha \in \mathcal{C}_X \mid q_X(\alpha, [D]) > 0 \text{ for every prime exceptional divisor } D\}.$$

Lemma 1.3.6: Suppose that X is projective. We have $\text{Nef}_{q_X}(X) = \overline{\mathcal{FE}_X} \cap N^1(X)_{\mathbf{R}}$.

Proof:

We first show the inclusion \subset . Kleiman's Theorem for nef divisors inspired the following argument. The rational points in $\text{Nef}_{q_X}(X)$ are dense, hence we only need to prove that given an integral q_X -nef divisor D , we have $q_X(D) \geq 0$. Let H be an ample divisor on X . Then $q_X(D + tH) > 0$ for $t \in \mathbf{N}$ large enough. It follows that $D + tH \in \mathcal{C}_X$ for $t \gg 0$, because $[H]$ is effective and D is q_X -nef, and so $q_X(D + tH, H) > 0$. We claim that if $t_0 \in \mathbf{Q}$ satisfies $q_X(D + t_0H) > 0$, then $q_X(D + \frac{t_0}{2}H) > 0$. Indeed, up to clearing the denominators, we can assume that $t_0 \in \mathbf{N}$ in $D + t_0H$. It follows by Corollary 3.10 of [Huy99] that the class of $D + t_0H$ in $N^1(X)_{\mathbf{R}}$ is effective, and so

$$q_X(D, D + t_0H) \geq 0,$$

because D is q_X -nef. Then

$$q_X\left(D + \frac{t_0}{2}H\right) = [q_X(D) + q_X(D, t_0H)] + \frac{t_0^2}{4}q_X(H) = q_X(D, D + t_0H) + \frac{t_0^2}{4}q_X(H) > 0,$$

and the last inequality is true because $q_X(D, D + t_0H) \geq 0$, and $q_X(H) > 0$ (because $[H]$ is ample). Now, fixing $t \in \mathbf{N}$ large enough and considering the sequence of divisors

$$\left\{D + \frac{t}{2^m}H\right\}_m \rightarrow D,$$

using the continuity of q_X we are done. We now show \supset . It suffices to show that any class lying in the interior of $\overline{\mathcal{FE}_X} \cap N^1(X)_{\mathbf{R}}$ is q_X -nef. Let α be a class in $\text{int}\left(\overline{\mathcal{FE}_X} \cap N^1(X)_{\mathbf{R}}\right)$. Then, α is big, hence $\alpha = [A] + [N]$, with A ample and N effective. If there was a prime divisor E with $q_X(\alpha, E) < 0$,

E would be one of the irreducible components of N , and $q_X(E) < 0$, by [Bou04, Proposition 4.2] and because $q_X(A, E) > 0$. But this would contradict that $\alpha \in \overline{\mathcal{F}\mathcal{E}_X} \cap N^1(X)_{\mathbf{R}}$, hence α is q_X -nef. ■

Definition 1.3.7: The birational Kähler cone \mathcal{BK}_X of X is defined as

$$\mathcal{BK}_X = \bigcup_f f^* \mathcal{K}_{X'},$$

where f varies among all the bimeromorphic maps $f: X \dashrightarrow X'$, where X' is another IHS manifold.

By [Mar11, Proposition 5.6], the closures of $\mathcal{F}\mathcal{E}_X$ and \mathcal{BK}_X in \mathcal{C}_X coincide. Hence also their closures in $H^{1,1}(X, \mathbf{R})$ do coincide, and will be denoted by $\overline{\mathcal{F}\mathcal{E}_X}$ and $\overline{\mathcal{BK}_X}$.

Remark 1.3.8. Suppose that X is projective. We have the equalities

$$\overline{\text{Mov}(X)} = \overline{\mathcal{BK}_X} \cap N^1(X)_{\mathbf{R}} = \overline{\mathcal{F}\mathcal{E}_X} \cap N^1(X)_{\mathbf{R}}$$

(see [HT09, Theorem 7, Remark 9]), where the closure of $\text{Mov}(X)$ is taken in $N^1(X)_{\mathbf{R}}$. In particular, by Lemma 1.3.6, we obtain the equality $\overline{\text{Mov}(X)} = \text{Nef}_{q_X}(X)$.

The following result gives a remarkable property of $\text{Nef}_{q_X}(X)$, when X is projective, and will be useful later on.

Theorem 1.3.9 ([MZ13, Theorem 1.2]): If D is an effective and q_X -nef \mathbf{R} -divisor on X projective, there exist a projective IHS manifold X' and a birational map $f: X \dashrightarrow X'$ such that $f_*(D)$ is nef on X' .

Now we give a useful property of big and q_X -nef divisors.

Lemma 1.3.10: If D is any big and q_X -nef divisor on X , the BBF square of D is positive.

Proof:

Since D is big, we can write $D \equiv A + N$, with A ample and N effective. Then, $q_X(D) = q_X(D, A) + q_X(D, N)$. Now, $q_X(D, A) < 0$, because the class of D in $N^1(X)_{\mathbf{R}}$ is effective, and $q_X(D, N) \geq 0$, because D is q_X -nef. This concludes the proof. ■

Remark 1.3.11. If X is projective, by Remark 1.3.8, we obtain that the cones $\overline{\text{Eff}(X)}$ and $\overline{\text{Mov}(X)} = \text{Nef}_{q_X}(X)$ are mutually dual with respect to the BBF pairing.

We will need the following lemma, which is the key tool to produce circular parts in $\overline{\text{Eff}(X)}$, in the proof of Theorem 0.2.2.

Lemma 1.3.12 ([Kov93], Lemma 2.3): Let Q be a smooth compact quadric hypersurface in \mathbf{R}^k and $C \subset \mathbf{R}^k$ a compact convex set. Assume that $Q \not\subset C$, then there exists a non empty open subset U of Q such that $U \subset \partial(\text{Conv}(Q \cup C))$, where $\text{Conv}(Q \cup C)$ is the convex hull of $Q \cup C$.

Let $\mathcal{P}\text{eff}(X)$ be the analytic pseudo-effective cone of an IHS manifold X , as defined for example in [Bou04, Section 2.3]. If X is projective

$$\mathcal{P}\text{eff}(X) \cap N^1(X)_{\mathbf{R}} = \overline{\text{Eff}(X)} \quad (1.5)$$

([Dem92, Proposition 1.4]). The following result of Boucksom (which holds in a more general context) will be useful.

Theorem 1.3.13 ([Bou04, Theorem 3.19]): The boundary of the pseudo-effective cone $\mathcal{P}\text{eff}(X)$ of an IHS manifold X is locally polyhedral away from $\overline{\mathcal{BK}}_X$, with extremal rays generated by (the classes of) prime exceptional divisors.

The following conjecture holds for all known deformation classes of (projective) IHS manifolds (see [MO22], [MR20] for the O’Grady-type case, [Mat17] for the Kum_n -type and $\text{K3}^{[n]}$ -type case), but it is not known in general.

Conjecture 1 (SYZ). *Let X be a projective IHS manifold and D an integral, isotropic (with respect to the BBF form) divisor on X , such that $[D] \in \overline{\text{Mov}}(X)$. Then $\mathcal{O}_X(D)$ induces a rational Lagrangian fibration, i.e. there exists a birational map $f: X \dashrightarrow X'$, where X' is another projective IHS manifold, such that $f_*\mathcal{O}_X(D)$ induces a fibration (i.e. a surjective morphism with connected fibres) $X' \rightarrow B$ to a projective $\dim(X)/2$ -dimensional base B .*

Definition 1.3.14: Let X be a projective IHS manifold. We define the effective movable cone as $\overline{\text{Mov}}(X)^e := \overline{\text{Mov}}(X) \cap \text{Eff}(X)$ and $\overline{\text{Mov}}(X)^+$ as the convex hull of $\overline{\text{Mov}}(X) \cap N^1(X)_{\mathbf{Q}}$ in $N^1(X)_{\mathbf{R}}$.

Although the following lemma may be well-known to experts, we did not find it in the literature. As it is a useful result in this context, we decided to include it, with a proof.

Lemma 1.3.15: If X is a projective IHS manifold belonging to one of the known deformation classes, we have the equality $\overline{\text{Mov}}(X)^+ = \overline{\text{Mov}}(X)^e$.

Proof:

We start by observing that the interiors of the two cones coincide, hence we only have to check that everything works well at the level of the boundaries. We recall that the BBF form takes non-negative values on $\overline{\text{Mov}}(X)^+$ and $\overline{\text{Mov}}(X)^e$, by the proof of Lemma 1.3.6 and Remark 1.3.8. Let α be an integral divisor class belonging to $\partial\overline{\text{Mov}}(X)^+$. If $q_X(\alpha) > 0$ we are done, because

in that case α is a big class, by [Huy99, Proposition 3.8] and the fact that $q_X(\alpha, h) > 0$, for any ample divisor class h . If $q_X(\alpha) = 0$, the SYZ conjecture holds under our assumptions, so the class α is effective, hence the inclusion $\overline{\text{Mov}}(X)^+ \subset \overline{\text{Mov}}(X)^e$ is verified. Now, let α be an element of $\overline{\text{Mov}}(X)^e$ lying on its boundary. If α belongs to some wall $[E]^\perp$, for some prime exceptional divisor E , and $q_X(\alpha) > 0$, we are done, because $\overline{\text{Mov}}(X)$ is locally rational polyhedral away from $\partial\overline{\mathcal{C}}_X$ (see Corollary 2.2.8). It remains to check the case $0 \neq \alpha \in \partial\overline{\mathcal{C}}_X$ (and so $q_X(\alpha) = 0$). We can write $\alpha = \sum_{i=1}^k a_i [D_i]$, where the D_i are prime divisors and the a_i are positive integers. Let $W \subset N^1(X)_{\mathbf{R}}$ be the rational subspace spanned by the $[D_i]$. The class α q_X -intersects non-negatively every D_i , and $q_X(\alpha) = 0$, hence $q_X(\alpha, D_i) = 0$ for any i . By linear algebra, as the signature of q_X restricted to $N^1(X)_{\mathbf{R}}$ is $(1, \rho(X) - 1)$, it follows that a maximal, totally isotropic linear subspace of $N^1(X)_{\mathbf{R}}$ (with respect to q_X) must have dimension 1. Then

$$W' = \bigcap_{i=1}^k ([D_i]^\perp \cap W)$$

is a rational subspace of dimension 1. Indeed W' contains α , and there cannot be other elements linearly independent from α , because otherwise, we would have a totally isotropic linear subspace of $N^1(X)_{\mathbf{R}}$ of dimension greater than or equal to 2. We conclude that α is a multiple of a rational, movable class, and we are done.

Note that the above argument implies the following.

- $\overline{\text{Mov}}(X) = \overline{\text{Mov}}(X)^e$ for all the known projective IHS manifolds.
- The inclusion $\overline{\text{Mov}}(X)^+ \supset \overline{\text{Mov}}(X)^e$ holds for any projective IHS manifold.

■

The Kawamata-Morrison movable cone conjecture for projective IHS manifolds predicts the existence of a fundamental domain for the action of $\text{Bir}(X)$ on $\overline{\text{Mov}}(X)^e$, and the theorem below (proved by Markman) is a slightly weaker version.

Theorem 1.3.16 ([Mar11], Theorem 6.25): Let X be a projective IHS manifold. There exists a rational convex polyhedral cone Π in $\overline{\text{Mov}}(X)^+$, such that Π is a fundamental domain for the action of $\text{Bir}(X)$ on $\overline{\text{Mov}}(X)^+$.

Remark 1.3.17. If X is a projective IHS manifold belonging to one of the known deformation classes, combining Lemma 1.3.15 and Theorem 1.3.16, one deduces that Π is also a fundamental domain for the action of $\text{Bir}(X)$ on $\overline{\text{Mov}}(X)^e$.

1.3.2 Divisorial Zariski decompositions

On any compact complex manifold we have the divisorial Zariski decomposition, which was introduced by Boucksom in his fundamental paper [Bou04] (but see also Nakayama's construction, [Nak04]), and is a generalisation of the classical Zariski decomposition on surfaces. The

divisorial Zariski decomposition will play a central role in this thesis, and in the case of IHS manifolds Boucksom characterised it with respect to the BBF form. Its existence holds for arbitrary pseudo-effective analytic classes on compact complex manifolds, but as in this thesis we deal with projective IHS manifolds we recall what is a (actually the) divisorial Zariski decomposition only for pseudo-effective divisors on projective IHS manifolds.

Theorem 1.3.18 ([Bou04, Theorem 4.8]): Let D be a pseudo-effective \mathbf{R} -divisor on X projective. Then, D admits a divisorial Zariski decomposition, i.e. we can write $D = P(D) + N(D)$ in a unique way such that:

1. $P(D) \in \overline{\text{Mov}}(X)$.
2. $N(D)$ is an effective \mathbf{R} -divisor which is exceptional (when non-zero): the Gram matrix (for q_X) of its irreducible components is negative definite,
3. $P(D)$ is orthogonal (with respect to q_X) to any irreducible component of $N(D)$.

The divisor $P(D)$ ($N(D)$) is the *positive part* (resp. *negative part*) of D .

Remark 1.3.19. If the divisor D is effective, its positive part can be defined as the maximal q_X -nef subdivisor of D . This means that, if $D = \sum_i a_i D_i$, where the D_i 's are the irreducible components of D , the positive part of D is the maximal divisor $P(D) = \sum_i b_i D_i$, with respect to the relation " $P(D) \leq D$ if and only if $b_i \leq a_i$ for every i ", such that $q_X(P(D), E) \geq 0$, for any prime divisor E .

Remark 1.3.20. The divisorial Zariski decomposition is rational, namely, under the assumptions of Theorem 1.3.18, whenever D is a pseudo-effective \mathbf{Q} -divisor, $N(D)$ is a \mathbf{Q} -divisor.

Remark 1.3.21. We note that the divisorial Zariski decomposition behaves well with respect to the numerical equivalence relation. Indeed, let X be a projective IHS manifold and α an effective class in $N^1(X)_{\mathbf{R}}$. Suppose $\alpha = [E_1] = [E_2]$, where E_1, E_2 are effective \mathbf{R} -divisors. Write $E_i = P(E_i) + N(E_i)$ for the divisorial Zariski decomposition of E_i . Then $E_1 - E_2 = T$, where T is numerically trivial. Thus

$$P(E_1) + N(E_1) = (P(E_2) + T) + N(E_2),$$

and by the uniqueness of the divisorial Zariski decomposition, we must have $P(E_1) = P(E_2) + T$, because $P(E_2) + T$ is q_X -nef, q_X -orthogonal to $N(E_1)$ and $N(E_1)$ is exceptional by hypothesis. In particular, $[P(E_1)] = [P(E_2)]$ and $N(E_1) = N(E_2)$.

The positive part of the divisorial Zariski decomposition of a big divisor encodes most of the positivity of the divisor itself. Indeed, given any big \mathbf{Q} -divisor D on X , one has $H^0(X, mD) \cong H^0(X, mP(D))$, for m positive and sufficiently divisible ([Bou04, Theorem 5.5]), and also $\mathbf{B}_+(D) = \mathbf{B}_+(P(D))$, $\mathbf{B}_-(D) = \mathbf{B}_-(P(D))$ (see Lemma 4.1.2). Furthermore, D is big if and only if $P(D)$ is (see [Bou04, Proposition 3.8], or [KMPP19, Proposition 3.8]).

1.3.3 Monodromy operators

Let $\pi: \mathcal{X} \rightarrow S$ be a smooth and proper family of IHS manifolds over a connected analytic (possibly singular, non-reduced, reducible) base S , whose central fibre is a fixed IHS manifold X . Let $R^k\pi_*\mathbf{Z}$ be the k -th higher direct image of \mathbf{Z} . The space S is locally contractible, and the family π is topologically locally trivial (see for example [GHJ03, Theorem 14.5]). The sheaf $R^k\pi_*\mathbf{Z}$ is the sheafification of the presheaf

$$U \mapsto H^k(\pi^{-1}(U), \mathbf{Z}), \text{ for any open subset } U \subset S,$$

which is a constant presheaf, by the local contractibility of S and the local triviality of π . Then $R^k\pi_*\mathbf{Z}$ is a locally constant sheaf, i.e. a local system, for every $k \in \mathbf{N}$. Now, let $\gamma: [0, 1] \rightarrow S$ be a continuous path. Then $\gamma^{-1}(R^k\pi_*\mathbf{Z})$ is a constant sheaf for every k .

Definition 1.3.22:

1. Set $\pi^{-1}(\gamma(0)) = X$ and $\pi^{-1}(\gamma(1)) = X'$. The parallel transport operator $T_{\gamma, \pi}^k$ associated with the path γ and π is the isomorphism $T_{\gamma, \pi}^k: H^k(X, \mathbf{Z}) \rightarrow H^k(X', \mathbf{Z})$ between the stalks at 0 and 1 of the sheaf $R^k\pi_*\mathbf{Z}$, induced by the trivialisation of $\gamma^{-1}(R^k\pi_*\mathbf{Z})$. The isomorphism $T_{\gamma, \pi}^k$ is well defined on the fixed endpoints homotopy class of γ .
2. If γ is a loop, we obtain an automorphism $T_{\gamma, \pi}^k: H^k(X, \mathbf{Z}) \rightarrow H^k(X, \mathbf{Z})$. In this case $T_{\gamma, \pi}^k$ is called a monodromy operator.
3. The group of k -th monodromy operators on $H^k(X, \mathbf{Z})$ induced by π is

$$\text{Mon}^k(X)_\pi := \{T_{\gamma, \pi}^k \mid \gamma(0) = \gamma(1)\}$$

With the above definition, we can define the monodromy groups of an IHS manifold X .

Definition 1.3.23: The k -th monodromy group $\text{Mon}^k(X)$ of an IHS manifold X is defined as the subgroup of $\text{Aut}_{\mathbf{Z}}(H^k(X, \mathbf{Z}))$ generated by the subgroups of the type $\text{Mon}^k(X)_\pi$, where $\pi: \mathcal{X} \rightarrow S$ is a smooth and proper family of IHS manifolds over a connected analytic base. The elements of $\text{Mon}^k(X)$ are also called monodromy operators.

We are interested in the group $\text{Mon}^2(X)$, of which we will need the characterisation for some of the known IHS manifolds. Notice that $\text{Mon}^2(X) \subset O^+(H^2(X, \mathbf{Z}))$. We will mostly use three subgroups of $\text{Mon}^2(X)$. The first is $\text{Mon}_{\text{Bir}}^2(X)$, defined as the elements of $\text{Mon}^2(X)$ induced by the birational self maps of X (when X is projective). Indeed, any birational self-map induces a monodromy operator on $H^2(X, \mathbf{Z})$, by a result of Huybrechts ([Huy03, Corollary 2.7]). The second is $\text{Mon}_{\text{Hdg}}^2(X)$, namely the subgroup of the elements of $\text{Mon}^2(X)$ which are also Hodge isometries. The last one is W_{Exc} which was defined in the introduction.

Now, let $\text{Def}(X)$ be the Kuranishi deformation space of any IHS manifold X and $\mathcal{X} \rightarrow \text{Def}(X)$

the universal family. We recall that $\text{Def}(X)$ is smooth. Let 0 be the distinguished point of $\text{Def}(X)$ such that $\mathcal{X}_0 \cong X$. Set $\Lambda = H^2(X, \mathbf{Z})$. By the local Torelli Theorem, $\text{Def}(X)$ embeds holomorphically into the period domain

$$\Omega_\Lambda := \{p \in \mathbf{P}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C}) \mid q_X(p) = 0 \text{ and } q_X(p + \bar{p}) > 0\}$$

as an open (analytic) subset, via the local period map

$$\mathcal{P}: \text{Def}(X) \rightarrow \Omega_\Lambda, t \mapsto [H^{2,0}(\mathcal{X}_t)].$$

Now, let L be a non-trivial holomorphic line bundle on X . The Kuranishi deformation space of the couple (X, L) is defined as $\text{Def}(X, L) := \mathcal{P}^{-1}(c_1(L)^\perp)$, where $c_1(L)^\perp$ is a hyperplane in $\mathbf{P}(\Lambda \otimes_{\mathbf{Z}} \mathbf{C})$. The space $\text{Def}(X, L)$ is the locus in $\text{Def}(X)$ where $c_1(L)$ stays algebraic. Up to shrinking $\text{Def}(X)$ around 0 , we can assume that $\text{Def}(X)$ and $\text{Def}(X, L)$ are contractible. Let s be the flat section (with respect to the Gauss-Manin connection) of $R^2\pi_*\mathbf{Z}$ through $c_1(L)$ and $s_t \in H^{1,1}(\mathcal{X}_t, \mathbf{Z})$ its value at $t \in \text{Def}(X, L)$.

The following are the line bundles we will mostly be interested in.

Definition 1.3.24: A line bundle L on an IHS manifold X is stably exceptional if there exists a closed analytic subset $Z \subset \text{Def}(X, L)$, of positive codimension, such that the linear system $|L_t|$ consists of a prime exceptional divisor for every $t \in \text{Def}(X, L) \setminus Z$.

Prime exceptional divisors are stably exceptional, by [Mar10, Proposition 5.2]. By making use of parallel transport operators, we can define "stably exceptional classes".

Definition 1.3.25: Let X be a projective IHS manifold. A primitive, integral divisor class $\alpha \in N^1(X)$ (the Néron-Severi group of X) is stably exceptional if $q_X(\alpha, A) > 0$, for some ample divisor A , and there exist a projective IHS manifold X' and a parallel transport operator

$$f: H^2(X, \mathbf{Z}) \rightarrow H^2(X', \mathbf{Z}),$$

such that $kf(\alpha)$ is represented by a prime exceptional divisor on X' , for some integer k .

For example, a line bundle of self-intersection -2 , intersecting positively any ample line bundle on a (smooth) projective K3 surface is stably exceptional, and its class in the Néron-Severi space is stably exceptional. Note that any stably exceptional class (line bundle) is effective, by the semi-continuity Theorem.

Now, suppose that X is a projective IHS manifold of Kum_n -type, of dimension $2n$. Let $\mathcal{W} \subset O^+(H^2(X, \mathbf{Z}))$ be the subgroup acting on the discriminant group by ± 1 . Let $\chi: \mathcal{W} \rightarrow \{\pm 1\}$ be the character corresponding to the action of \mathcal{W} on the discriminant group. Let $\det: \mathcal{W} \rightarrow \{\pm 1\}$ be the determinant character. Let $\mathcal{N} \subset \mathcal{W}$ be the kernel of the product character $\det \cdot \chi$. The proposition below is due to Markman and was sent to Yoshioka via a letter. The proof we provide

is taken from the letter sent by Markman to Yoshioka and is due to Markman.

Proposition 1.3.26 (E. Markman): For $i = 1, 2$, let X_i be an irreducible holomorphic symplectic manifold deformation equivalent to a generalised Kummer variety of dimension $2n$ and $e_i \in H^2(X, \mathbf{Z})$ a primitive classe satisfying the two conditions $q_X(e_i) = 2(n+1)$, $(n+1) \mid \text{div}(e_i)$. There exists a parallel-transport operator $g: H^2(X_1, \mathbf{Z}) \rightarrow H^2(X_2, \mathbf{Z})$ satisfying $g(e_1) = e_2$, if and only if $\text{div}(e_1) = \text{div}(e_2)$, and $rs(e_1) = rs(e_2)$.

Proof:

The statement is analogous to part 2 of [Mar09, Proposition 1.8] and the proof is essentially the same. There are two differences.

1) The lattice $H^2(X, \mathbf{Z})$ for the Kum_n -type IHS manifolds is different from that of the $\text{K3}^{[n]}$ -type IHS manifolds. However, both lattices contain a primitive embedding of a unimodular lattice of co-rank 1, and this unimodular lattice contains, as a direct summand, the direct sum of three copies of the hyperbolic plane. This is all that is used in [Mar09], so the proof goes through verbatim to show that the invariants $\text{div}(e)$ and $rs(e)$ determine the \mathcal{W} -orbit of e .

2) The monodromy group of X coincides with \mathcal{N} (see [Mar23, Theorem 1.4] and [Mon16, Theorem 2.3]), and it is an index 2 subgroup of \mathcal{W} in the Kum_n -type case. It remains to show that each \mathcal{W} -orbit consists of only one \mathcal{N} -orbit. This is established for one \mathcal{W} -orbit in [MM11, Lemma 4.3]. The same argument applies for every \mathcal{W} -orbit of a primitive class e satisfying the conditions $q_X(e) = 2(n+1)$, $(n+1) \mid \text{div}(e)$. ■

In the proof of the above proposition, note that the monodromy invariant $rs(e)$ of the class e is defined as in the paragraph in [Mar09] preceding [Mar09, Proposition 1.7], except that we replace $n-1$ by $n+1$.

1.3.4 Wall divisors and rational curves

Recall that on an IHS manifold X we have $H_2(X, \mathbf{Z}) \cong H^2(X, \mathbf{Z})^* \subset H^2(X, \mathbf{Q})$. Hence, if γ is any curve class in $H_2(X, \mathbf{Z})$, there exists a unique \mathbf{Q} -divisor class $[D]$ on X such that $\alpha \cdot \gamma = q_X(\alpha, [D])$, for all \mathbf{R} -divisor classes α . The square of γ with respect to the BBF quadratic form is defined as $q_X(\gamma) := q_X(D) \in \mathbf{Q}$.

Definition 1.3.27: A wall divisor on X is an integral divisor D such that $q_X(D) < 0$ and $g(D)^\perp \cap \mathcal{BK}_X = \emptyset$, for any $g \in \text{Mon}_{\text{Hdg}}^2(X)$.

Since $q_X(D) < 0$ for any wall divisor D , the hyperplane $g(D)^\perp$ intersects \mathcal{C}_X . An important property of wall divisors is that they stay wall divisors under parallel transport ([Mon15, Theorem 1.3]). Furthermore, there is an important relation between them and (rational) extremal curves on X and its birational models.

Proposition 1.3.28 (Proposition 1.5, [Mon15]): Let h be an ample class on X . There is a bijective correspondence between primitive wall divisors intersecting positively h and negative extremal rays of the Mori cone of X (with respect to the BBF pairing).

Then, a wall divisor, up to the action of $\text{Mon}_{\text{Hdg}}^2(X)$, is dual to some (rational) extremal curve on some birational model of X .

Remark 1.3.29. The MBM classes defined in [AV15] are exactly the homology classes which are dual to wall divisors ([KLM15, Proposition 2.3, Remark 2.4])

1.3.5 A Hodge index-type Theorem for IHS manifolds

In what follows we provide a version of the *strong Hodge-index Theorem* for projective IHS manifolds, as we did not find it in the literature and it plays an important role in this thesis (in particular in the first chapter). We start with the following lemma of linear algebra.

Lemma 1.3.30: Let q be a quadratic form of signature $(1, \rho - 1)$ on a finite dimensional, real vector space V , with $\dim(V) = \rho$. Suppose that π is a 2-plane containing a positive direction. Then the signature of $q|_{\pi}$ is $(1, 1)$.

Proof:

Since π contains a positive direction, the signature of $q|_{\pi}$ is $(1, 1)$ or $(1, 0)$ (so in the second case $q|_{\pi}$ is degenerate). If $q|_{\pi}$ is degenerate then π contains no negative directions. But this is absurd because π intersects all the hyperplanes on which q_X is negative-definite (and we have one by hypothesis). Thus the signature is $(1, 1)$. ■

Proposition 1.3.31 (Hodge-type inequality): Let D, D' be two integral divisors on X projective, such that $q_X(D) > 0$. Then

1. $(q_X(D, D'))^2 \geq q_X(D)q_X(D')$,
2. Equality in the previous item holds if and only if D, D' are linearly dependent in $N^1(X)_{\mathbf{Q}}$, i.e. if and only if $rD \equiv D'$, for some $r \in \mathbf{Q}$, and so if and only if $aD \equiv bD'$ for some $a, b \in \mathbf{Z}$.

Proof:

If $[D'] = 0$ there is nothing to prove. So we can assume $[D'] \neq 0$.

1. Consider the subspace $\pi := \langle [D], [D'] \rangle \subset N^1(X)_{\mathbf{R}}$. The determinant of the matrix representing q_X restricted to π is $m := q_X(D)q_X(D') - q_X(D, D')^2$ and hence the signature of $(q_X)|_{\pi}$ is $(1, 1)$ if and only if $m < 0$, i.e. $(q_X(D, D'))^2 > q_X(D)q_X(D')$. The equality $m = 0$ holds true if and only if $\dim(\pi) = 1$, and we are done.

2. By the previous item we have $(q_X(D, D'))^2 = q_X(D)q_X(D')$ if and only if $[D]$ and $[D']$ are linearly dependent in $N^1(X)_{\mathbf{R}}$. But $[D], [D'] \in \text{Pic}(X)$, thus they must be linearly dependent in $N^1(X)_{\mathbf{Q}}$. It follows that $r[D] = [D']$ for some $0 \neq r \in \mathbf{Q}$, i.e. $a[D] = b[D']$ for some $a, b \in \mathbf{Z} \setminus \{0\}$. ■

Corollary 1.3.32: Let D, D' be two integral divisors on X , and assume $q_X(D) > 0$. Then, if $q_X(D, D') = 0$, we have $q_X(D') < 0$ or $[D'] = 0$.

Proof:

By item (1) of Proposition 1.3.31, we have

$$0 = (q_X([D], [D']))^2 \geq q_X(D)q_X(D').$$

Hence, since $q_X(D) > 0$, we have $q_X(D') \leq 0$. By contradiction, suppose that $q_X(D') = 0$, but $[D'] \neq 0$. As in this case

$$(q_X([D], [D']))^2 = 0 = q_X([D])q_X([D']),$$

by item (2) of Proposition 1.3.31, we have $b[D] = a[D']$, for some $a, b \in \mathbf{Z} \setminus \{0\}$. Thus

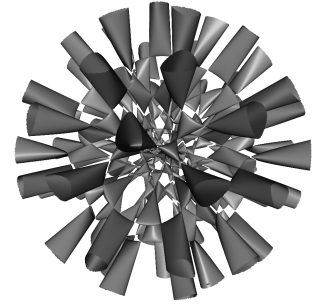
$$b^2 q_X([D]) = a^2 q_X([D']) = 0,$$

and so $b = 0$. Then $a[D'] = 0$, and since $N^1(X)$ is torsion free we must have $[D'] = 0$, which is clearly a contradiction. ■

Corollary 1.3.33 (Strong Hodge-index Theorem): Let D be an integral divisor on X , with $q_X(D) > 0$. Consider the decomposition $N^1(X)_{\mathbf{R}} = \mathbf{R}[D] \oplus D^{\perp}$. Then q_X is negative-definite on the hyperplane $[D]^{\perp}$.

Proof:

By Corollary 1.3.32 we have that $q_X(D') < 0$ for every non-zero rational class $[D'] \in D^{\perp}$. It remains to be proved that the same is true for each non-zero real class α q_X -orthogonal to D . Any basis \mathcal{B} for $H^{1,1}(X, \mathbf{Z})$ gives an integral basis \mathcal{B}' of D^{\perp} . The \mathbf{Q} -vector space $D_{\mathbf{Q}}^{\perp}$ spanned by \mathcal{B}' in D^{\perp} is the set rational points of D^{\perp} . But q_X is negative-definite on $D_{\mathbf{Q}}^{\perp}$, and \mathcal{B}' is also a basis for D^{\perp} , hence q_X is negative-definite on D^{\perp} , and this concludes the proof. ■



2. Boucksom-Zariski and Weyl chambers

2.1 Remarks on the pseudo-effective cone

In this section, we determine the structure of the *pseudo-effective* cone $\overline{\text{Eff}}(X)$.

Remark 2.1.1. We observe the following:

1. If D is an effective and big \mathbf{R} -divisor on X , then $P(D) > 0$. Indeed $D \equiv_{\text{num}} A + N$, for some A ample and N effective. Then $P(D) \geq A > 0$, as $P(D)$ is the maximal q_X -nef subdivisor of D (see Remark 1.3.19). It follows that an exceptional, effective \mathbf{R} -divisor cannot be big and that the class of a prime exceptional divisor D' belongs to the boundary of $\overline{\text{Eff}}(X)$.
2. If E is a prime exceptional divisor, then nE is fixed (i.e. $h^0(nE) = 1$) for every $n > 0$. Indeed, if nE were not fixed, there would be an effective divisor D linearly equivalent to mE and not containing E in its support, for some $0 < m \leq n$. Then, we would have $0 \leq m q_X(D, E) = m^2 q_X(E, E) < 0$, which is absurd.
3. Let D be a prime divisor with $q_X(D) = 0$, then $[D] \in \partial \overline{\text{Eff}}(X)$. Indeed, by [Bou04, Proposition 4.2, item (ii)], D is q_X -nef. If it were also big, then we would have $q_X(D) > 0$ by Lemma 1.3.10. Thus $[D]$ lies on the boundary of $\overline{\text{Eff}}(X)$.

Kleiman's criterion for amplitude on a projective variety inspired the proof of the following lemma.

Lemma 2.1.2: Let A be an ample divisor on X . Then the linear functional

$$q_X(-, A): N^1(X)_{\mathbf{R}} \rightarrow \mathbf{R}, \alpha \mapsto q_X(\alpha, A),$$

is positive on $\overline{\text{Eff}}(X) \setminus \{0\}$. In particular $\overline{\text{Eff}}(X)$ does not contain any line.

Proof:

As $[A]$ is Kähler, $q_X(-, A)$ is positive on $\text{Eff}(X) \setminus \{0\}$, hence it is nonnegative on $\overline{\text{Eff}}(X)$. Now, assume that $q_X(\alpha, A) = 0$ for some algebraic pseudo-effective class $\alpha \neq 0$. We note that there exists an

integral divisor D such that $q_X(\alpha, D) < 0^1$. But $D + nA$ is ample for $n \gg 1$. Hence we have

$$0 \leq q_X(\alpha, D + nA) = q_X(\alpha, D) < 0,$$

which is a contradiction. Now, let $l \subset \overline{\text{Eff}(X)}$ be a line. We can assume that l passes through the origin. Let $0 \neq \alpha \in l$ be an element. Then $-\alpha$ is also algebraic pseudo-effective and so $q_X(t\alpha + (1-t)(-\alpha), A) > 0$ for each $t \in [0, 1]$. This is clearly absurd, as 0 belongs to the segment joining α and $-\alpha$. ■

In fact, note that $\overline{\text{Eff}(Y)}$ does not contain lines for any projective variety Y . If the variety is smooth, the mentioned result is a particular case of [BFJ09, Proposition 1.3]. For a proof which works also in the non-smooth case, see Lemma 4.6 of [LM09].

Lemma 2.1.3: Let $K \subset \mathbf{R}^m$ be a closed convex cone not containing any line. If $\{x_m\}_m, \{y_m\}_m, \{z_m\}_m$ are three sequences in K such that $x_m = y_m + z_m$ for any m , and $\{x_m\}_m$ is bounded, then also $\{y_m\}_m$ and $\{z_m\}_m$ are bounded.

Proof:

Let $\|\cdot\|$ be any norm on \mathbf{R}^m . Assume by contradiction that $\{y_m\}_m$ is not bounded, then there exists a divergent subsequence $\{y_k\}_k$. Consider the sequences

$$\hat{x}_k = x_k / \|y_k\|, \hat{y}_k = y_k / \|y_k\|, \hat{z}_k = z_k / \|y_k\|.$$

Observe that $\|\hat{y}_k\| = 1$ for any k , and $\{\hat{x}_k\}_k \rightarrow 0$. Indeed $\{x_k\}_k$ is bounded, and $\{\|y_k\|\}_k \rightarrow \infty$. It follows that $\{\hat{y}_k\}_k$ has a limit point $y \in K$, and $\|y\| = 1$, thus $y \neq 0$. Denoting again by $\{\hat{y}_k\}_k$ a subsequence converging to y , we obtain $\{\hat{z}_k\}_k \rightarrow -y$, hence both y and $-y$ belong to K . This is absurd because K does not contain any line. ■

Corollary 2.1.4: Let E be a prime exceptional divisor on X . Then $[E]$ spans an extremal ray in $\overline{\text{Eff}(X)}$.

Proof:

We can decompose the algebraic pseudo-effective cone as

$$\overline{\text{Eff}(X)} = \mathbf{R}^{\geq 0}[E] + \overline{\text{Eff}(X) \cap E^{\geq 0}}. \quad (2.1)$$

Indeed, by Lemma 2.1.2 and Lemma 2.1.3, the cone $\mathbf{R}^{\geq 0}[E] + \overline{\text{Eff}(X) \cap E^{\geq 0}}$ is closed, and this implies the equality in (2.1), because clearly

$$\mathbf{R}^{\geq 0}[E] + \overline{\text{Eff}(X) \cap E^{\geq 0}} \subset \overline{\text{Eff}(X)}, \quad (2.2)$$

¹Indeed q_X is non-degenerate, and so we can find a class β satisfying $q_X(\alpha, \beta) \neq 0$. Without loss of generality, we can assume $q_X(\alpha, \beta) < 0$. If $\text{Pic}(X) \subset \alpha^{\geq 0}$, then $N^1(X)_{\mathbf{R}} = \alpha^{\geq 0}$, which is clearly absurd.

and also \supset holds in (2.2), as

$$\text{Eff}(X) \subset \mathbf{R}^{\geq 0}[E] + \text{Eff}(X) \cap E^{\geq 0} \subset \mathbf{R}^{\geq 0}[E] + \overline{\text{Eff}(X) \cap E^{\geq 0}}.$$

To prove extremality, assume now $[E] = \alpha + \alpha'$, with $\alpha = a[E] + \beta$, $\alpha' = a'[E] + \beta'$, and $\beta, \beta' \in \overline{\text{Eff}(X) \cap E^{\geq 0}}$. We have

$$(1 - a - a')[E] = \beta + \beta',$$

and so $a + a' \leq 1$, because otherwise, by Lemma 2.1.2, for any ample divisor A , we would have

$$0 > q_X((1 - a - a')E, A) = q_X(\beta + \beta', A) \geq 0,$$

thus $\beta + \beta' \in \mathbf{R}^{\geq 0}[E]$. On the other hand $\beta + \beta' \in \overline{\text{Eff}(X) \cap E^{\geq 0}}$, so we must have $\beta + \beta' = 0$, because

$$\left(\overline{\text{Eff}(X) \cap E^{\geq 0}}\right) \cap \mathbf{R}^{\geq 0}[E] = 0.$$

Indeed $\text{Eff}(X) \cap E^{\geq 0} \subset E^{\geq 0}$, and $E^{\geq 0}$ is closed. But

$$q_X(\beta, A) \geq 0, \text{ and } q_X(\beta', A) \geq 0,$$

and so Lemma 2.1.2 clearly implies $\beta = \beta' = 0$. ■

The next two results were inspired by [Kol96, Lemma 4.12, Theorem 4.13].

Lemma 2.1.5: If α is a pseudo-effective class generating an extremal ray in $\overline{\text{Eff}(X)}$, and $q_X(\alpha) < 0$, then there is a prime divisor E such that $\alpha \in \mathbf{R}^{\geq 0}[E]$.

Proof:

Let $\{\alpha_m\}_m$ be a sequence of effective classes converging to α . Set $\alpha_m = \sum_j a_{mj}[D_j]$, where the D_j are distinct prime divisors, and $a_{mj} > 0$. By the continuity of q_X , we have

$$q_X(\alpha) = \lim_{m \rightarrow \infty} q_X(\alpha, \alpha_m).$$

It follows that for m large enough $q_X(\alpha, \alpha_m) < 0$, and so there exists a prime divisor $E = D_{\bar{j}}$ such that $q_X(\alpha, E) < 0$. Thus each element of the sequence can be written as

$$\alpha_m = c_m E + \sum_{j \neq \bar{j}} a_{mj} D_j.$$

Then

$$0 > q_X(\alpha, E) = \lim_{m \rightarrow \infty} q_X(E, \alpha_m) = \lim_{m \rightarrow \infty} q_X \left(E, c_m E + \sum_{j \neq \bar{j}} a_{mj} D_j \right) \geq q_X(E) \lim_{m \rightarrow \infty} c_m.$$

It follows that $q_X(E) < 0$ and $\lim_{m \rightarrow \infty} c_m > 0$. Let $0 < \nu < \lim_{m \rightarrow \infty} c_m$. Then $D_m - \nu E$ is effective for $m \gg 1$, and $\alpha = \nu E + \lim_{m \rightarrow \infty} (D_m - \nu E)$. Since α spans an extremal ray, it follows that $[E] \in \mathbf{R}^{\geq 0} \alpha$.

■

Corollary 2.1.6: We have

$$\overline{\text{Eff}(X)} = \overline{\text{Pos}(X)} + \sum_E \mathbf{R}^{\geq 0}[E],$$

where the sum is over all the prime exceptional divisors.

Proof:

The inclusion \supset is obvious. It remains to show \subset , and it is sufficient to prove that every extremal ray of $\overline{\text{Eff}(X)}$ is contained in the right-hand side of the equality in the statement². Let $\mathbf{R}^{\geq 0}\alpha$ be an extremal ray. If $q_X(\alpha) \geq 0$, then $\alpha \in \overline{\text{Pos}(X)}$. If $q_X(\alpha) < 0$, by Lemma 2.1.5 there exists a prime divisor E such that $\mathbf{R}^{\geq 0}\alpha = \mathbf{R}^{\geq 0}[E]$, hence we are done. ■

From Corollary 2.1.6 it follows that:

1. Let $\alpha \in \overline{\text{Eff}(X)}$. If $q_X(\alpha) < 0$, then $\mathbf{R}^{\geq 0}\alpha$ is an extremal ray of $\overline{\text{Eff}(X)}$ if and only if $\mathbf{R}^{\geq 0}\alpha = \mathbf{R}^{\geq 0}[D]$, for some prime exceptional divisor D .
2. Let X be of Picard number $\rho(X) \geq 2$. If $\beta \in \overline{\text{Eff}(X)}$ spans an extremal ray of $\overline{\text{Eff}(X)}$ and $q_X(\beta) \geq 0$, then $q_X(\beta) = 0$. Note that $q_X(\beta) = 0$ does not necessarily imply that β spans an extremal ray (see Section 4 of this chapter for an example).

2.2 Boucksom-Zariski chambers

The goal of this section is to prove item (1) of Theorem 0.1.2. We start with the following useful result, which has already been proved by Boucksom in an analytic setting (see [Bou04, Proposition 3.8, item (iii)]). For the reader's convenience, we provide a proof in the framework of our algebraic approach to the divisorial Zariski decomposition.

Lemma 2.2.1: Let D be an effective and big \mathbf{R} -divisor on X , and $D = P(D) + N(D)$ its divisorial Zariski decomposition. Then $P(D)$ is a big and q_X -nef divisor.

Proof:

We only need to show that $[P(D)]$ is a big class. We know that $P(D) > 0$, by item (1) of Remark 2.1.1. By definition $D = \sum_i a_i D_i$, where the a_i are positive real numbers, and the D_i are integral and big. We can assume that each D_i is an effective and big \mathbf{Q} -divisor, so we can consider the divisorial Zariski decompositions $D_i = P(D_i) + N(D_i)$. By definition, for each i there exists an integer m_i such that $m_i D_i$ is integral and big. Then $m_i D_i = m_i P(D_i) + m_i N(D_i)$ is the divisorial Zariski decomposition of $m_i D_i$, moreover, by [KMPP19, Proposition 3.8], $m_i P(D_i)$ is a big \mathbf{Q} -divisor. It follows that also $P(D_i)$ is a big \mathbf{Q} -divisor. Thus we have $D = \sum_i (a_i P(D_i) + a_i N(D_i))$, and $P(D)$ is the unique maximal q_X -nef subdivisor of D (and it is also effective). This means that $P(D) \geq$

²Indeed $\overline{\text{Eff}(X)}$ does not contain lines, thus by Minkowski's Theorem it is spanned by its extremal rays.

$\sum_i a_i P(D_i)$, because $\sum_i a_i P(D_i)$ is a q_X -nef subdivisor of D . Thus we have $P(D) = \sum_i a_i P(D_i) + E$, where E is effective, and so we are done, because the sum of a big divisor and of an effective divisor is big. ■

Definition 2.2.2: Given a big \mathbf{R} -divisor D on X we define the q_X -null locus of D as

$$\text{Null}_{q_X}(D) := \{E \mid E \text{ is a prime divisor and } q_X(D, E) = 0\}.$$

Remark 2.2.3. We note that the q_X -null locus of a big \mathbf{R} -divisor D is finite. Indeed, let D' be a prime divisor lying in $\text{Null}_{q_X}(D)$. As D is big, we have $[D] = [A] + [N]$ in $N^1(X)_{\mathbf{R}}$, where A is ample and N is effective. As $q_X(A, D') > 0$, then D' is an irreducible component of N . It immediately follows that the cardinality of $\text{Null}_{q_X}(D)$ is bounded by that of the irreducible components of N , and so is finite.

Definition 2.2.4: Let D be an effective \mathbf{R} -divisor on X and $D = P(D) + N(D)$ its divisorial Zariski decomposition. We define

$$\text{Neg}_{q_X}(D) := \{E \mid E \text{ is an irreducible component of } N(D)\}.$$

Remark 2.2.5. We note that

1. given a big class α and a big \mathbf{R} -divisor D such that $\alpha = [D]$, we can define

$$\text{Null}_{q_X}(\alpha) := \text{Null}_{q_X}(D),$$

because q_X respects the numerical equivalence relation and so $\text{Null}_{q_X}(\alpha)$ does not depend on the representative of α we have chosen,

2. given an effective class α and $D \in \text{Div}_{\mathbf{R}}(X)$ an effective representative of α , we can define

$$\text{Neg}_{q_X}(\alpha) := \text{Neg}_{q_X}(D),$$

because from Remark 1.3.21 also $\text{Neg}_{q_X}(\alpha)$ does not depend on the effective representative of α we have chosen.

Remark 2.2.6. • Let D be a big \mathbf{R} -divisor on X , then $\text{Null}_{q_X}(D) \subset \text{Exc}(X)$, where $\text{Exc}(X)$ is the set of all prime exceptional divisors on X , i.e.

$$\text{Exc}(X) := \{E \mid E \text{ is a prime exceptional divisor on } X\}.$$

Indeed, let E be a prime divisor such that $q_X(D, E) = 0$. Since D is big, then $D \equiv A + N$, where A is an ample \mathbf{R} -divisor and N is an effective \mathbf{R} -divisor (see [Laz04, Proposition 2.2.22]). Thus we obtain

$$q_X(D, E) = q_X(A + N, E) = q_X(A, E) + q_X(N, E) = 0. \quad (2.3)$$

Now, the class of A in $N^1(X)_{\mathbf{R}} \subset H^{1,1}(X, \mathbf{R})$ is Kähler, thus $q_X(A, N) > 0$. The last equality in (2.3) implies $q_X(N, E) < 0$. By [Bou04, Proposition 4.2, item (ii)], E must be one of the irreducible components of N , and $q_X(E) < 0$.

- Let E, E' be two distinct prime exceptional divisors on X . Then $[E] \neq [E']$ in $N^1(X)_{\mathbf{R}}$. Indeed, if $q_X(E, E') \geq 0$ and $[E] = [E']$, then $0 > q_X(E) = q_X(E, E') \geq 0$ and this is clearly absurd.

Lemma 2.2.7: Let P be a big and q_X -nef \mathbf{R} -divisor on X . Then there is an open neighbourhood $U \subset \text{Big}(X)$ of $[P]$ in $N^1(X)_{\mathbf{R}}$ such that for all classes $[D] \in U$ we have

$$\text{Null}_{q_X}(D) \subset \text{Null}_{q_X}(P).$$

Proof:

Since the big cone is open, we can choose rational big classes $[D_1], \dots, [D_r]$ in $N^1(X)_{\mathbf{R}}$ such that P belongs to the interior of the closed cone $K := \sum_{i=1}^r \mathbf{R}^{\geq 0} [D_i]$. We can assume that D_1, \dots, D_r are effective divisors, because a big class is effective. If $D \in K$, we can have $q_X(D, D') < 0$ only for a finite number of prime divisors D' , by [Bou04, Proposition 4.2, item (ii)]. Indeed, if $q_X(D, D') < 0$, then D' must be an irreducible component of D . It follows that up to replacing each D_i with νD_i , for $\nu > 0$ rational and small enough, we can assume that

$$q_X(P + D_i, D') > 0 \tag{2.4}$$

for all prime divisors D' satisfying $q_X(P, D') > 0$. Now, consider an \mathbf{R} -divisor of the form $\sum_{i=1}^r \alpha_i (P + D_i)$, where $(\alpha_1, \dots, \alpha_r) \in (\mathbf{R}^{\geq 0})^r \setminus \{\vec{0}\}$. We note that if D' is a prime divisor such that

$$q_X \left(\sum_{i=1}^r \alpha_i (P + D_i), D' \right) = 0,$$

then $D' \in \text{Null}_{q_X}(P)$, using that P is q_X -nef and (2.4). Then it is clear that

$$\text{Null}_{q_X} \left(\sum_{i=1}^r \alpha_i (P + D_i) \right) \subset \text{Null}_{q_X}(P)$$

for each r -tuple $(\alpha_1, \dots, \alpha_r) \in (\mathbf{R}^{\geq 0})^r \setminus \{\vec{0}\}$. Therefore the open cone

$$U := \sum_{i=1}^r \mathbf{R}^{>0} [P + D_i]$$

is an open neighbourhood of P yielding the conclusion of the Lemma. ■

Corollary 2.2.8: For every class $\alpha \in \text{Big}(X) \cap \text{Nef}_{q_X}(X)$ there exists an open neighbourhood $U = U(\alpha)$ in $N^1(X)_{\mathbf{R}}$ and prime exceptional divisors $E_1, \dots, E_n \in \text{Exc}(X)$ such that

$$U \cap (\text{Big}(X) \cap \text{Nef}_{q_X}(X)) = U \cap (E_1^{\geq 0} \cap \dots \cap E_n^{\geq 0}).$$

Moreover, $\overline{\text{Nef}_{q_X}(X) \cap \text{Big}(X)} = \text{Nef}_{q_X}(X)$ is locally rational polyhedral at every big point.

Proof:

Let U be a neighbourhood of α as in Lemma 2.2.7. Notice that by construction \overline{U} is a rational polyhedral cone (i.e. spanned by a finite number of rational divisor classes). Every big class is effective, hence, by [Bou04, Proposition 4.2, item (ii)], we have the equality

$$\text{Big}(X) \cap \text{Nef}_{q_X}(X) = \text{Big}(X) \cap \left(\bigcap_{E \in \text{Exc}(X)} E^{\geq 0} \right),$$

and so

$$\begin{aligned} U \cap (\text{Big}(X) \cap \text{Nef}_{q_X}(X)) &= U \cap \text{Big}(X) \cap \left(\bigcap_{E \in \text{Exc}(X)} E^{\geq 0} \right) \\ &= U \cap \left(\bigcap_{E \in \text{Exc}(X)} E^{\geq 0} \right), \end{aligned} \tag{2.5}$$

because $U \subset \text{Big}(X)$ by construction (see Lemma 2.2.7). We note that α belongs to the intersection in (2.5). Moreover, if $E \in \text{Exc}(X)$, either $U \subset E^{\geq 0}$ or $U \cap E^{\perp} \neq 0$. Indeed, if $U \not\subset E^{\geq 0}$, we can find a class $[D'] \in U$ such that $q_X(D', D) < 0$ and since U is a convex cone, the segment joining D' and α is contained in U and must intersect E^{\perp} , hence $U \cap E^{\perp} \neq 0$. Now, if $U \subset E^{\geq 0}$ (for some $E \in \text{Exc}(X)$), we can omit $E^{\geq 0}$ from (2.5). Otherwise, we notice that in fact, we can have $U \cap E^{\perp} \neq 0$ only for finitely many prime exceptional divisors in $\text{Exc}(X)$. Indeed, if $[D'] \in U$, then $\text{Null}_{q_X}(D') \subset \text{Null}_{q_X}(\alpha)$ by construction, and $\text{Null}_{q_X}(\alpha)$ is finite by Remark 2.2.3. From the above argument, it follows that

$$U \cap (\text{Big}(X) \cap \text{Nef}_{q_X}(X)) = U \cap \left(\bigcap_{E \in \text{Null}_{q_X}(\alpha)} E^{\geq 0} \right),$$

hence U is the open neighbourhood of α we were looking for, and $\text{Nef}_{q_X}(X)$ is clearly locally rational polyhedral at α by definition. ■

The following easy remarks will be useful in the rest of the thesis.

Remark 2.2.9. Let D be an effective \mathbf{R} -divisor on X and set

$$\text{Neg}_{q_X}(D) = \{E_1, \dots, E_k\}.$$

Then the classes $[E_1], \dots, [E_k]$ are linearly independent in $N^1(X)_{\mathbf{R}}$. Indeed the Gram matrix $(q_X(E_i, E_j))_{i,j}$ is negative-definite, by Theorem 1.3.18, and so we cannot have $\sum_{i=1}^k \alpha_i [E_i] = 0$ for each k -tuple $(\alpha_1, \dots, \alpha_k) \neq \vec{0}$ of real numbers, as otherwise we would have $q_X(\sum_{i=1}^k \alpha_i [E_i]) = 0$.

Definition 2.2.10: Let P be a big and q_X -nef \mathbf{R} -divisor on X . The *Boucksom-Zariski chamber* of P is defined as

$$\Sigma_P := \left\{ D \in \text{Big}(X) \mid \text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P) \right\}.$$

We note that this definition makes sense, since by item (2) in Remark 2.2.5 $\text{Neg}_{q_X}(\alpha)$ does not depend on the representative of α we have chosen. Moreover, in general, these chambers are neither closed nor open.

Example 2.2.11. Let M be an \mathbf{R} -divisor whose class lies in $\text{int}(\text{Mov}(X))$. We observe that M is also big. Indeed

$$\overline{\text{Mov}(X)} = \text{Nef}_{q_X}(X) \subset \overline{\text{Eff}(X)}, \quad (2.6)$$

where the equality in (2.6) is justified by Remark 1.3.8. It follows that

$$\text{int}\left(\overline{\text{Mov}(X)}\right) = \text{int}(\text{Mov}(X)) \subset \text{Big}(X),$$

which implies that M is also big, hence it makes sense to consider the chamber associated with M . Then $\Sigma_M = \text{Big}(X) \cap \text{Nef}_{q_X}(X)$. Indeed, in Σ_M there are the big divisor classes α with $\text{Neg}_{q_X}(\alpha) = \text{Null}_{q_X}(M)$. But $\text{Null}_{q_X}(M) = \emptyset$, since $[M] \in \text{int}\left(\overline{\text{Mov}(X)}\right)$. Hence, Σ_M consists of the big divisor classes α with $N(\alpha) = 0$, that means $\Sigma_M = \text{Big}(X) \cap \text{Nef}_{q_X}(X)$.

Remark 2.2.12. If P is big and q_X -nef, the chamber Σ_P is a convex cone. To prove this, we have to show that if $\alpha, \beta \in \Sigma_P$, then also $\gamma := x\alpha + y\beta \in \Sigma_P$, for each couple (x, y) of positive numbers. Indeed, γ is certainly a big class. Moreover,

$$\text{Neg}_{q_X}(\alpha) = \text{Neg}_{q_X}(\beta) = \text{Neg}_{q_X}(P),$$

by definition of Σ_P . But the divisorial Zariski decomposition of γ is given by $P(\gamma) = P(\alpha) + P(\beta)$, and $N(\gamma) = N(\alpha) + N(\beta)$. It follows that also $\text{Neg}_{q_X}(\gamma) = \text{Neg}_{q_X}(P)$, and so we are done.

Recall that a face of a convex cone K is a subcone F such that if $v, v' \in K$, and $v + v' \in F$, then $v, v' \in F$. Let P be a big and q_X -nef \mathbf{R} -divisor and define

$$\text{Face}(P) := \text{Nef}_{q_X}(X) \cap \text{Null}_{q_X}(P)^\perp,$$

where

$$\text{Null}_{q_X}(P)^\perp := \bigcap_{D' \in \text{Null}_{q_X}(P)} (D')^\perp.$$

We observe that $\text{Face}(P)$ is a face of the q_X -nef cone. Indeed, suppose $\alpha + \beta \in \text{Face}(P)$, where $\alpha, \beta \in \text{Nef}_{q_X}(X)$. If $D' \in \text{Null}_{q_X}(P)$, then

$$0 = q_X(\alpha + \beta, D') = q_X(\alpha, D') + q_X(\beta, D').$$

Thus, since α and β are q_X -nef, we must have $q_X(\alpha, D') = q_X(\beta, D') = 0$, and so $\alpha, \beta \in \text{Face}(P)$.

Definition 2.2.13: A subset $S \subset \text{Exc}(X)$ is an *exceptional block* if $S = \{\emptyset\}$, or $S = \{E_1, \dots, E_k\}$ and the Gram matrix $(q_X(E_i, E_j))_{i,j}$ is negative-definite.

Lemma 2.2.14: Let S be an exceptional block. Then there exists a q_X -nef divisor P satisfying $\text{Null}_{q_X}(P) = S$.

Proof:

If $S = \{\emptyset\}$, any divisor in $\text{int}(\text{Mov}(X))$ will satisfy what is wanted (and we have such a divisor because X is projective). If $S = \{E_1, \dots, E_k\}$, pick a divisor $M \in \text{int}(\text{Mov}(X))$. We claim that P can be constructed explicitly as $P = M + \sum_{i=1}^k \lambda_i E_i$, where the λ_i are positive rational numbers. To prove the claim it is sufficient to check that:

1. the linear system \mathcal{S} of equations

$$q_X(M, E_j) + \sum_{i=1}^k q_X(E_j, E_i) \lambda_i = 0, \quad j = 1, \dots, k$$

admits a (unique) solution $(\lambda_1, \dots, \lambda_k) \in (\mathbf{Q}^{>0})^k$;

2. with such a solution $\text{Null}_{q_X}(P) = S$ holds.

Let us prove (1). For this purpose, we will need Lemma 4.1 of [BKS04]. Let G be the Gram matrix of the divisors in S . We know that G is negative-definite and so invertible. Then \mathcal{S} has a unique solution. Now, \mathcal{S} can be rewritten as

$$G^{-1} \cdot (q_X(M, E_1), \dots, q_X(M, E_k))^T = (-\lambda_1, \dots, -\lambda_k)^T.$$

The entries of G^{-1} are rational, and by Lemma 4.1 of [BKS04] are also non-positive. Moreover, each $q_X(M, E_j)$ is positive, because any element in $\text{int}(\text{Mov}(X))$ intersects positively (with respect to q_X) any prime exceptional divisor. From all the above it follows that the λ_i are positive and rational, and we are done. We now have to prove (2). Since $S \subset \text{Null}_{q_X}(P)$ by construction, we only have to verify that $\text{Null}_{q_X}(P) \subset S$. But this is immediate, because $P = M + \sum_{i=1}^k \lambda_i E_i$, and $q_X(P, D') = 0$ (where $D' \in \text{Null}_{q_X}(P)$) clearly implies $D' \in S$. ■

Lemma 2.2.15: Let P and P' be big and q_X -nef \mathbf{R} -divisors on X . The following hold:

1. $\Sigma_P = \Sigma_{P'}$ if and only if $\text{Face}(P) = \text{Face}(P')$.
2. $\Sigma_P \cap \Sigma_{P'} = \emptyset$, if $\text{Face}(P) \neq \text{Face}(P')$.
3. $\text{Big}(X)$ is the disjoint union of the Boucksom-Zariski chambers.

Proof:

1. We first show \Leftarrow . By definition of Σ_P and $\Sigma_{P'}$, it is sufficient to prove that $\text{Null}_{q_X}(P) = \text{Null}_{q_X}(P')$. If $\text{Face}(P) = \text{Face}(P')$, i.e. if

$$\text{Null}_{q_X}(P)^\perp \cap \text{Nef}_{q_X}(X) = \text{Null}_{q_X}(P')^\perp \cap \text{Nef}_{q_X}(X),$$

then $q_X(P, E) = 0$, for every prime divisor E belonging to $\text{Null}_{q_X}(P')$. Thus $\text{Null}_{q_X}(P') \subset \text{Null}_{q_X}(P)$, and by symmetry also the other inclusion holds. The implication \Rightarrow is straightforward, because since $\Sigma_P = \Sigma_{P'}$, then $\text{Null}_{q_X}(P) = \text{Null}_{q_X}(P')$, and so $\text{Face}(P) = \text{Face}(P')$.

2. The contrapositive of \Rightarrow in item (1) tells us that if $\text{Face}(P) \neq \text{Face}(P')$, then $\Sigma_P \neq \Sigma_{P'}$, and this clearly implies $\Sigma_P \cap \Sigma_{P'} = \emptyset$.

3. We clearly have the inclusion

$$\bigcup_{P \in \text{Big}(X) \cap \text{Nef}_{q_X}(X)} \Sigma_P \subset \text{Big}(X).$$

The inclusion \supset follows directly applying Lemma 2.2.14 to $\text{Neg}_{q_X}(D)$, where D is a big \mathbf{R} -divisor. ■

Corollary 2.2.16: There is a bijective correspondence between the Boucksom-Zariski chambers and the exceptional blocks.

Proof:

This is clear from the definition of Boucksom-Zariski chamber, from Lemma 2.2.14 and from the item (3) of Lemma 2.2.15. ■

Proposition 2.2.17: A big class α is on the boundary of some chamber Σ_P , with P a big and q_X -nef \mathbf{R} -divisor, if and only if $\text{Neg}_{q_X}(\alpha) \subsetneq \text{Null}_{q_X}(P(\alpha))$, where $P(\alpha)$ is the positive part of the divisorial Zariski decomposition of α .

Proof:

We first show \Rightarrow . Let $\alpha \in \partial \Sigma_P$ for some big and q_X -nef divisor P , and $N(\alpha) = \sum_{i=1}^k a_i N_i$. Let $\|\cdot\|$ be any norm on $N^1(X)_{\mathbf{R}}$. As α is on the boundary of Σ_P (of course this does not imply $\alpha \in \Sigma_P$), any small "ball" $B_\epsilon(\alpha) \subset \text{Big}(X)$ of centre α and radius ϵ will contain an element from another chamber. Thus, for every $\epsilon > 0$ small enough, we can find a class $\beta \in N^1(X)_{\mathbf{R}}$ of norm $\|\beta\| < \epsilon$, such that $\text{Neg}_{q_X}(\alpha + \beta) \neq \text{Neg}_{q_X}(\alpha)$. As q_X is non-degenerate on $N^1(X)_{\mathbf{R}}$ and on the subspace $\langle P(\alpha), N_1, \dots, N_K \rangle$, we can decompose the Néron-Severi space as

$$N^1(X)_{\mathbf{R}} = \langle P(\alpha), N_1, \dots, N_K \rangle \oplus \langle P(\alpha), N_1, \dots, N_K \rangle^\perp,$$

and so we can write $\beta = \beta' \oplus \beta''$, where

$$\beta' \in \langle P(\alpha), N_1, \dots, N_k \rangle \text{ and } \beta'' \in \langle P(\alpha), N_1, \dots, N_k \rangle^\perp{}^3.$$

If necessary, we can choose ϵ even smaller, in order to have

$$\text{Neg}_{q_X}(\alpha + \beta') = \text{Neg}_{q_X}(\alpha).$$

It follows that we can assume $\beta \in \langle P(\alpha), N_1, \dots, N_k \rangle^\perp$. Now, as by Lemma 2.2.1 $P(\alpha)$ is big and q_X -nef, we can take an open neighbourhood $U = U(P(\alpha))$ like the one in Lemma 2.2.7. As U is open, choosing ϵ even smaller (if necessary), we may assume that $P(\alpha) + \beta \in U$. Notice that, by construction, $\text{Neg}_{q_X}(\alpha + \beta) \neq \text{Neg}_{q_X}(\alpha)$. Thus $P(\alpha) + \beta$ cannot be q_X -nef, as otherwise the divisorial Zariski decomposition of $\alpha + \beta$ would be $(P(\alpha) + \beta) + N(\alpha)$, because $N(\alpha)$ is q_X -orthogonal to β , and so we would have $\text{Neg}_{q_X}(\alpha + \beta) = \text{Neg}_{q_X}(\alpha)$, which is absurd. It follows that we can find a prime divisor $E \in \text{Exc}(X) \setminus \{N_1, \dots, N_k\}$ such that $q_X(P(\alpha) + \beta, E) < 0$. But $q_X(P(\alpha), E) \geq 0$, because $P(\alpha)$ is q_X -nef. Hence there exists $t_0 \in [0, 1[$ satisfying

$$q_X(P(\alpha) + t_0\beta, E) = q_X(P(\alpha), E) + t_0q_X(\beta, E) = 0.$$

By the choice of U we know that $\text{Null}_{q_X}(P(\alpha) + t_0\beta) \subset \text{Null}_{q_X}(P(\alpha))$, and so $E \in \text{Null}_{q_X}(P(\alpha))$. Thus we obtained $\text{Neg}_{q_X}(\alpha) \subsetneq \text{Null}_{q_X}(P(\alpha))$.

The implication \Leftarrow is easier. Without loss of generality we can assume $\text{Null}_{q_X}(P(\alpha)) \setminus \text{Neg}_{q_X}(\alpha) = \{E\}$. Now, consider $\alpha + \epsilon[E]$, where $\epsilon > 0$. It is clear that the divisorial Zariski decomposition of $\alpha + \epsilon[E]$ is $P(\alpha) + (N(\alpha) + \epsilon[E])$, and so $\alpha + \epsilon[E]$ lies in a different chamber than the one to which α belongs (in particular, this chamber is exactly $\Sigma_{P(\alpha)}$), for every $\epsilon > 0$. But $\lim_{\epsilon \rightarrow 0} (\alpha + \epsilon[E]) = \alpha$, thus α is a limit point of a sequence contained in the chamber $\Sigma_{P(\alpha)}$, hence it must lie in the boundary of $\Sigma_{P(\alpha)}$. ■

Corollary 2.2.18: For every big and q_X -nef \mathbf{R} -divisor, the interior of a Boucksom-Zariski chamber Σ_P is

$$\text{int}(\Sigma_P) = \left\{ D \in \text{Big}(X) \mid \text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P) = \text{Null}_{q_X}(P(D)) \right\}.$$

Proof:

The inclusion \subset follows from the definition of Σ_P and Proposition 2.2.17. As for \supset , if D is such that $\text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P) = \text{Null}_{q_X}(P(D))$, clearly $D \in \Sigma_P$. Moreover $D \in \text{int}(\Sigma_P)$, again thanks to Proposition 2.2.17. ■

Remark 2.2.19. From Corollary 1.3.33 and Lemma 1.3.10 it follows that if P is a big and q_X -nef integral divisor, and if $\{E_1, \dots, E_k\} \subset \text{Null}_{q_X}(P)$, then each (non-zero) effective linear combination of the E_i is exceptional. In particular, if we choose nonnegative real numbers a_1, \dots, a_k , then

³Note that P_D, N_1, \dots, N_k are linearly independent in $N^1(X)_{\mathbf{R}}$.

$D' := P + (\sum_{i=1}^k a_i E_i)$ is exactly the divisorial Zariski decomposition of D' , i.e. $P(D') = P$ and $N(D') = \sum_{i=1}^k a_i E_i$.

Definition 2.2.20: Let P be a big and q_X -nef divisor. We define the *relative interior* of $\text{Face}(P)$ as the interior of $\text{Face}(P)$ in the topology of $\text{Null}_{q_X}(P)^\perp$. We will denote it by $\text{rel.int.Face}(P)$.

Example 2.2.21. Let M be an \mathbf{R} -divisor belonging to $\text{int}(\text{Mov}(X))$. Then $\text{rel.int.Face}(M) = \text{int}(\text{Nef}_{q_X}(X)) = \text{int}(\text{Mov}(X))$. Indeed $\text{Null}_{q_X}(M)^\perp = N^1(X)_{\mathbf{R}}$, and so $\text{rel.int.Face}(M)$ is the interior of the q_X -nef cone in the topology of $N^1(X)_{\mathbf{R}}$.

Remark 2.2.22. With the notation of the above definition, we note the following facts.

1. First of all we have $P \in \text{rel.int.Face}(P)$. Indeed, set $\{E_1, \dots, E_k\} = \text{Null}_{q_X}(P)$. Thanks to Corollary 2.2.8, we can find an open neighbourhood $U = U(P) \subset \text{Big}(X)$ such that

$$U \cap \text{Nef}_{q_X}(X) = U \cap (E_1^{\geq 0} \cap \dots \cap E_k^{\geq 0}).$$

It follows that

$$U \cap \text{Null}_{q_X}(P)^\perp \subset U \cap (E_1^{\geq 0} \cap \dots \cap E_k^{\geq 0}),$$

thus we have found an open neighbourhood of P (which is clearly non-empty as P belongs to it) in $\text{Null}_{q_X}(P)^\perp$ contained in $\text{Nef}_{q_X}(X)$, i.e. $P \in \text{rel.int.Face}(P)$.

2. It is important to observe that since $Q \in \text{rel.int.Face}(P)$, then we must have $\text{Null}_{q_X}(Q) = \text{Null}_{q_X}(P)$. Indeed $\text{Null}_{q_X}(P) \subset \text{Null}_{q_X}(Q)$ and also \supset must hold, as otherwise, for $E' \in \text{Null}_{q_X}(Q) \setminus \text{Null}_{q_X}(P)$, we would have $q_X(Q - \epsilon P, E') < 0$ for every $\epsilon > 0$, thus $Q - \epsilon P$ would not be q_X -nef and so Q could not lie in $\text{rel.int.Face}(P)$ (because otherwise $Q - \epsilon P$ would have been q_X -nef for every ϵ small enough).

Lemma 2.2.23: Let P be a big and q_X -nef \mathbf{R} -divisor on X and $Q \in \text{Face}(P)$. Then Q is big if $Q \in \text{rel.int.Face}(P)$.

Proof:

We can assume that P is integral by the proof of Lemma 2.2.15. Since $Q \in \text{rel.int.Face}(P)$, we can find $0 < \epsilon_0 \ll 1$ such that $Q - \epsilon_0 P$ is q_X -nef and so pseudo-effective. Indeed, by Remark 1.3.8 and Lemma 1.3.6, $\overline{\text{Mov}(X)} = \text{Nef}_{q_X}(X)$, and $\text{Mov}(X) \subset \text{Eff}(X)$ by definition, hence $\text{Nef}_{q_X}(X) = \overline{\text{Mov}(X)} \subset \overline{\text{Eff}(X)}$. Set $\{D_1, \dots, D_k\} = \text{Null}_{q_X}(P) = \text{Null}_{q_X}(Q)$. Now, P is big, so $P + \frac{1}{\epsilon_0}(a_1 D_1 + \dots + a_k D_k)$ is big for every nonnegative $a_1, \dots, a_k \in \mathbf{R}$. But

$$(Q - \epsilon_0 P) + (\epsilon_0 P + a_1 D_1 + \dots + a_k D_k) = Q + a_1 D_1 + \dots + a_k D_k =: Q_{a_1 \dots a_k}$$

is big, as it is the sum of a pseudo-effective divisor and a big divisor. Moreover, the positive part of the divisorial Zariski decomposition of $Q_{a_1 \dots a_k}$ is exactly Q and its negative part is exactly $\sum_{i=1}^k a_i D_i$, by Remark 2.2.19. Thus Q must be big by Lemma 2.2.1. \blacksquare

We now give a description of the big part of the closure of a given chamber in terms of q_X -null loci and q_X -negative loci.

Proposition 2.2.24: Let P be a big and q_X -nef \mathbf{R} -divisor. Then

$$\text{Big}(X) \cap \overline{\Sigma_P} = \left\{ D \in \text{Big}(X) \mid \text{Neg}_{q_X}(D) \subset \text{Null}_{q_X}(P) \subset \text{Null}_{q_X}(P(D)) \right\} \quad (2.7)$$

Proof:

Again, by Lemma 2.2.15, we can assume that P is integral. We first show \supset . Pick $D = P(D) + N(D)$ like in the right-hand side of (2.7), where as usual $P(D) + N(D)$ is the divisorial Zariski decomposition of D . As $\text{Null}_{q_X}(P) \subset \text{Null}_{q_X}(P(D))$, then $P(D) \in \text{Face}(P)$. Indeed we have $\text{Null}_{q_X}(P(D))^\perp \subset \text{Null}_{q_X}(P)^\perp$ and so $P(D) \in \text{Face}(P(D)) \subset \text{Face}(P)$. By Lemma 2.2.23 we can approximate $P(D)$ with a sequence $\{Q_n\}_n$ of big divisors contained in $\text{rel.int.Face}(P) \neq \emptyset$. Moreover, by Remark 2.2.22, we also have $\text{Null}_{q_X}(Q_n) = \text{Null}_{q_X}(P)$. On the other hand, if $\{D_1, \dots, D_j\} = \text{Null}_{q_X}(P) \setminus \text{Neg}_{q_X}(D)$, then, by Remark 2.2.19,

$$\left\{ N_n := N(D) + \frac{1}{n} \sum_{i=1}^j D_i \right\}_n$$

is a sequence of exceptional divisors converging to $N(D)$. If we let $D_n := Q_n + N_n$, we obtain a sequence $\{D_n\}_n$ converging to D and $Q_n = P(D_n)$ (resp. $N(D_n) = N_n$) is the positive (resp. negative) part of the divisorial Zariski decomposition of D_n . So, by construction,

$$\text{Neg}_{q_X}(D_n) = \text{Null}_{q_X}(P) = \text{Null}_{q_X}(Q_n) = \text{Null}_{q_X}(P(D_n)).$$

By Corollary 2.2.18, each D_n belongs to $\text{int}(\Sigma_P)$, and so $D \in \text{Big}(X) \cap \overline{\Sigma_P}$.

Now we prove \subset . Let $\{D_n\}_n$ be a sequence of divisors in $\text{int}(\Sigma_P)$ converging to D . If $D_n = P(D_n) + N(D_n)$ is the divisorial Zariski decomposition of each term in the sequence, then

$$\text{Neg}_{q_X}(D_n) = \text{Null}_{q_X}(P) = \text{Null}_{q_X}(P(D_n)),$$

which in turn implies

$$P(D_n) \in \text{Face}(P) \text{ and } N(D_n) \in \langle \text{Null}_{q_X}(P) \rangle.$$

Since $\text{Null}_{q_X}(P)^\perp$ and $\langle \text{Null}_{q_X}(P) \rangle$ are closed and orthogonal, it follows that $\{P(D_n)\}_n$ converges to a q_X -nef \mathbf{R} -divisor belonging to $\text{Null}_{q_X}(P)^\perp$, and $\{N(D_n)\}_n$ converges to an element in $\langle \text{Null}_{q_X}(P) \rangle$. Now, if we set $\text{Null}_{q_X}(P) = \{D_1, \dots, D_k\}$, we can write $N(D_n) = \sum_{i=1}^k a_i^{(n)} D_i$, and each $a_i^{(n)}$ is positive. As the D_j are linearly independent in $N^1(X)_{\mathbf{R}}$, it follows that $\{N(D_n)\}_n$ converges to an element of the form $\sum_{i=1}^k a_i D_i$, where some a_i can be 0. From all the above it is clear that

$$\{P(D_n)\}_n \rightarrow P(D) \in \text{Null}_{q_X}(P)^\perp \text{ and } \{N(D_n)\}_n \rightarrow N(D).$$

The first condition gives $\text{Null}_{q_X}(P) \subset \text{Null}_{q_X}(P(D))$, and the fact that $N(D) = \sum_{i=1}^k a_i D_i$ (where some a_i can be 0) implies that $\text{Neg}_{q_X}(D) \subset \text{Null}_{q_X}(P)$. \blacksquare

We are ready to describe geometrically the big part of the closure of any chamber. In what

follows $V^{\geq 0}(M)$ will denote the subcone of $N^1(X)_{\mathbf{R}}$ generated by a subset $M \subset N^1(X)_{\mathbf{R}}$ and $V^{> 0}(M)$ its interior.

Proposition 2.2.25: Let P be a big and q_X -nef \mathbf{R} -divisor. We have

$$\text{Big}(X) \cap \overline{\Sigma_P} = (\text{Big}(X) \cap \text{Face}(P)) + V^{\geq 0}(\text{Null}_{q_X}(P)).$$

Proof:

As usual, we can assume that P is integral. We first prove \subset . Let $D \in \text{Big}(X) \cap \overline{\Sigma_P}$. By Proposition 2.2.24 we know that this holds if and only if

$$\text{Neg}_{q_X}(D) \subset \text{Null}_{q_X}(P) \subset \text{Null}_{q_X}(P(D)).$$

Then $P(D) \in \text{Null}_{q_X}(P(D))^{\perp} \subset \text{Null}_{q_X}(P)^{\perp}$. We also know that $P(D)$ is big by Lemma 2.2.1 and q_X -nef by construction, thus $P(D) \in \text{Face}(P) \cap \text{Big}(X)$. Now, $\text{Neg}_{q_X}(D) \subset \text{Null}_{q_X}(P)$, thus $N(D) \in V^{\geq 0}(\text{Null}_{q_X}(P))$, hence $D \in (\text{Big}(X) \cap \text{Face}(P)) + V^{\geq 0}(\text{Null}_{q_X}(P))$.

Now we prove \supset . Suppose that

$$D \in (\text{Big}(X) \cap \text{Face}(P)) + V^{\geq 0}(\text{Null}_{q_X}(P)).$$

Write

$$D = Q + M,$$

where $Q \in \text{Big}(X) \cap \text{Face}(P)$, and $M \in V^{\geq 0}(\text{Null}_{q_X}(P))$. Clearly Q and M are q_X -orthogonal, by definition of $\text{Face}(P)$. Moreover, M is exceptional, by Corollary 1.3.32, and Q is q_X -nef. It follows that $Q = P(D)$ and $M = N(D)$, by the uniqueness of the divisorial Zariski decomposition. Of course we have $\text{Neg}_{q_X}(D) = \text{Neg}_{q_X}(M) \subset \text{Null}_{q_X}(P)$, and to conclude, as $Q = P(D) \in \text{Null}_{q_X}(P)^{\perp}$, we also have $\text{Null}_{q_X}(P) \subset \text{Null}_{q_X}(P(D))$, and so we are done by Proposition 2.2.24. \blacksquare

Corollary 2.2.26: Let P be a big and q_X -nef \mathbf{R} -divisor. Then $\overline{\Sigma_P}$ is locally rational polyhedral at every big point.

Proof:

We can assume that P is integral, because, according to the proof of Lemma 2.2.14, any Boucksom-Zariski chamber is associated with a big and q_X -nef integral divisor. Let $\alpha \in \text{Big}(X) \cap \overline{\Sigma_P}$ and write $\alpha = P(\alpha) + N(\alpha)$ for its divisorial Zariski decomposition. Moreover, set $\text{Null}_{q_X}(P) = \{D_1, \dots, D_k\}$. By Proposition 2.2.25 we have

$$\text{Big}(X) \cap \overline{\Sigma_P} = \text{Face}(P) \cap \text{Big}(X) + V^{\geq 0}(\text{Null}_{q_X}(P)).$$

The part $V^{\geq 0}(\text{Null}_{q_X}(P))$ is clearly rational. We know that $P(\alpha) \in \text{Face}(P) \cap \text{Big}(X)$, by Lemma 2.2.1. Let $U = U(P(\alpha))$ be a rational neighbourhood like the one in Lemma 2.2.7. Then, by

Corollary 2.2.8, we obtain

$$U \cap \text{Face}(P) \cap \text{Big}(X) = U \cap \text{Nef}_{q_X}(P) \cap \text{Null}_{q_X}(P)^\perp = U \cap (D_1^\perp \cap \cdots \cap D_k^\perp). \quad (2.8)$$

By virtue of (2.8), the neighbourhood $(U + V^{\geq 0}(\text{Null}_{q_X}(P))) \cap \overline{\Sigma_P}$ of α is cut out by finitely many rational closed half-spaces, hence we are done. ■

We now give a geometric description of the interior of any chamber.

Proposition 2.2.27: Let P be a big and q_X -nef \mathbf{R} -divisor. The interior of the chamber Σ_P is equal to

$$\text{rel.int.Face}(P) + V^{>0}(\text{Null}_{q_X}(P)).$$

Proof:

We can assume that P is integral, because, according to the proof of Lemma 2.2.14, any Boucksom-Zariski chamber is associated with a big and q_X -nef integral divisor. Let $D \in \text{int}(\Sigma_P)$. By Corollary 2.2.18 we have

$$\text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P) = \text{Null}_{q_X}(P(D)).$$

Then $\text{Face}(P) = \text{Face}(P(D))$, hence, by item (1) of Remark 2.2.22, $P(D) \in \text{rel.int.Face}(P)$. Moreover $N(D) \in V^{>0}(\text{Null}_{q_X}(P))$, because $\text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P)$, thus

$$D \in \text{rel.int.Face}(P) + V^{>0}(\text{Null}_{q_X}(P)).$$

Now, assume that $D \in \text{rel.int.Face}(P) + V^{>0}(\text{Null}_{q_X}(P))$. Write

$$D = Q + M,$$

where $Q \in \text{rel.int.Face}(P)$ and $M \in V^{>0}(\text{Null}_{q_X}(P))$. By Lemma 2.2.23 Q is big, hence D is big, as it is the sum of a big divisor and of an effective divisor. By Remark 2.2.19 M is exceptional. Moreover Q is q_X -nef, and $q_X(Q, M) = 0$. It follows from the uniqueness of the divisorial Zariski decomposition that $Q = P(D)$ and $M = N(D)$, i.e. $D = Q + M$ is the divisorial Zariski decomposition of D . Clearly we have $\text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P)$, and by item (2) of Remark 2.2.22 we also have $\text{Null}_{q_X}(P) = \text{Null}_{q_X}(P(D))$. Thus we have

$$\text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P) = \text{Null}_{q_X}(P(D)),$$

and by Corollary 2.2.18 we are done. ■

Our next goal is to prove that the decomposition of $\text{Big}(X)$ into Boucksom-Zariski chambers is locally finite, i.e. for each point in $\text{Big}(X)$ there exists a neighbourhood meeting only a finite number of Boucksom-Zariski chambers. The key to proving this result is the following.

Remark 2.2.28. Let D be a big \mathbf{R} -divisor and A an ample \mathbf{R} -divisor. Then

$$\text{Neg}_{q_X}(D + A) \subset \text{Neg}_{q_X}(D).$$

This follows directly from the fact that $P(D + A)$ is the maximal q_X -nef subdivisor of $D + A$, and $P(D + A) \geq P(D) + A$.

Proposition 2.2.29: The decomposition of the big cone of X into Boucksom-Zariski chambers is locally finite.

Proof:

Let $\text{Amp}(X)$ be the ample cone of X . It is sufficient to note that every big \mathbf{R} -divisor D has an open neighbourhood of the form $D' + \text{Amp}(X)$. Indeed $D \equiv A + N$, where A is ample and N is effective. We can write $A = \sum_i a_i A_i$, where the a_i are positive, and the A_i are integral and ample. If we choose $A' = \sum_{i=1}^k b_i A_i$, with $0 < b_i < a_i$ for every i , then $D' = (A - A') + N$ is still big, and $[D] \in [D'] + \text{Amp}(X) \subset N^1(X)_{\mathbf{R}}$. By Remark 2.2.28, every class α in this neighbourhood satisfy $\text{Neg}_{q_X}(\alpha) \subset \text{Neg}_{q_X}(D)$, and this clearly implies that $[D'] + \text{Amp}(X)$ meets only a finite number of Boucksom-Zariski chambers. ■

We are now ready to prove item (1) of Theorem 0.1.2.

Proof of Theorem 0.1.2, item (1):

By Proposition 2.2.15, $\text{Big}(X)$ is the disjoint union of the Boucksom-Zariski chambers, and by definition in each chamber the support of the negative part of the divisorial Zariski decomposition of the divisors is constant. By Corollary 2.2.8 and Corollary 2.2.26, every Boucksom-Zariski chamber is locally rational polyhedral, and by Proposition 2.2.29 the decomposition into Boucksom-Zariski chambers is locally finite. ■

To conclude this section, we show that the local finiteness of the given decomposition allows us to prove that the divisorial Zariski decomposition varies "continuously" in the big cone.

Proposition 2.2.30: Let $\{[D_n]\}_n$ be a sequence of big classes converging in $N^1(X)_{\mathbf{R}}$ to a big class $[D]$. If $D_n = P(D_n) + N(D_n)$ is the divisorial Zariski decomposition of D_n , and if $D = P(D) + N(D)$ is the divisorial Zariski decomposition of D , then $\{[P(D_n)]\}_n$ (respectively $\{[N(D_n)]\}_n$) converges to $[P(D)]$ (respectively $[N(D)]$).

Proof:

Assume first that $\{[D_n]\}_n \subset \Sigma_M$, for some M big and q_X -nef. We note that $\{[P(D_n)]\}_n \subset \text{Null}_{q_X}(M)^\perp$, hence $\{[P(D_n)]\}_n \rightarrow [P] \in \text{Null}_{q_X}(M)^\perp \cap \text{Nef}_{q_X}(X)$. Moreover $\{[N(D_n)]\}_n \subset \langle \text{Null}_{q_X}(M) \rangle$, hence $\{[N(D_n)]\}_n \rightarrow [N] \in \langle \text{Null}_{q_X}(M) \rangle$. Set $\text{Null}_{q_X}(M) = \{N_1, \dots, N_k\}$. For each n we can write $N(D_n) = \sum_{i=1}^k a_i^{(n)} N_i$, where $a_i^{(n)} > 0$ for each n , and for each $i = \{1, \dots, k\}$. Moreover the classes $[N_i]$ are linearly independent in $N^1(X)_{\mathbf{R}}$, because the matrix $(q_X(N_i, N_j))_{i,j}$ is negative-definite. Hence each $\{a_i^{(n)}\}_n$ converges to a number $a_i \geq 0$, thus we can write $N = \sum_{i=1}^k a_i N_i$. Putting everything together we obtained:

1. $q_X(P, N) = 0$,

2. $\sum_{i=1}^k a_i N_i = N$ is exceptional,
3. P is q_X -nef.

Then it is clear that $[P] = [P(D)]$ and $N = N(D)$. Assume now that $\{[D_n]\}_n$ is contained in more than one chamber. We know by Lemma 2.2.28 that $[D]$ has a neighbourhood of the form $[D'] + \text{Amp}(X)$, with $[D']$ a big class, and this neighbourhood meets only a finite number of Boucksom-Zariski chambers, by Proposition 2.2.29. Then one of these chambers must contain an infinite number of terms of $\{[D_n]\}_n$. Hence we are reduced to the case in which $\{[D_n]\}_n$ is contained in one chamber, and we are done. ■

2.3 Volume of divisors

The goal of this section is to prove item (2) of Theorem 0.1.2. In particular, we show that the volume function on $\text{Big}(X)$ is locally polynomial. Note that Bauer, Küronya and Szemberg in [BKS04, Subsection 3.3] provided a smooth projective threefold whose associated volume function is not locally polynomial.

The next two results have already been proved by Boucksom in his paper [Bou04] (see Proposition 3.20 and Proposition 4.12 respectively) in an analytic setting. For the reader's convenience, we provide an algebraic proof, which uses the continuity of the divisorial Zariski decomposition.

Lemma 2.3.1: Let D be a big \mathbf{R} -divisor on X , and $P(D) + N(D)$ its divisorial Zariski decomposition. Then $\text{vol}(D) = \text{vol}(P(D))$.

Proof:

Assume first that D is a \mathbf{Q} -divisor. By [KMPP19, Proposition 3.8] we know that the global sections of kD are given by the global sections of $kP(D)$, i.e.

$$H^0(X, kD) \cong H^0(X, kP(D)),$$

for any sufficiently divisible k . Moreover, the divisorial Zariski decomposition of kD is exactly $kP(D) + kN(D)$, for any k . Hence, for any sufficiently divisible k , we get

$$\text{vol}(D) := \frac{1}{k^{2n}} \text{vol}(kD) = \frac{1}{k^{2n}} \text{vol}(kP(D)) =: \text{vol}(P(D)).$$

Now, assume that D is an \mathbf{R} -divisor. Let $\{D_m\}_m$ be a sequence of big \mathbf{Q} -divisors converging to D . From the above we know that $\text{vol}(D_m) = \text{vol}(P(D_m))$ for any m . Moreover, by Proposition 2.2.30, we know that $\{P(D_m)\}_m$ converges to $P(D)$. By the continuity of the volume we obtain

$$\text{vol}(D) = \lim_{m \rightarrow +\infty} \text{vol}(P(D_m)) = \text{vol}(P(D)),$$

and we are done. ■

By Lemma 2.3.1 we know that $\text{vol}(D) = \text{vol}(P(D))$, moreover $P(D)$ is big and q_X -nef. For a big and nef \mathbf{R} -divisor we know that the volume is given by the top self-intersection (see [Laz04,

equation (2.9)), and a priori we cannot say the same for $P(D)$, as it is "only" q_X -nef, and of course we could have $\text{Nef}(X) \subsetneq \text{Nef}_{q_X}(X)$. The upshot is the following proposition.

Proposition 2.3.2: Let α be a big class, with divisorial Zariski decomposition $\alpha = P(\alpha) + N(\alpha)$. Then $\text{vol}(\alpha) = \text{vol}(P(\alpha)) = \int_X P(\alpha)^{2n} = c_X(q_X(P(\alpha)))^n$.

Proof:

It suffices to verify the assertion for α q_X -nef, by Proposition 2.3.1. First, assume that $\alpha \in \text{int}(\text{Mov}(X))$ is a rational class. Let $f: X \dashrightarrow X'$ be a birational map, such that $f_*(\alpha)$ is big and nef on the projective IHS manifold X' , we can do this by virtue of Theorem 1.3.9. From now on it is convenient to adopt the following convention: if D is any divisor, we will denote by D' its push-forward via f . Let D be a rational divisor representing α , and U (resp. U') the maximal open subset of X (resp. X') at which f (resp. f^{-1}) is regular. We know that $U \cong U'$ via f . Moreover $\text{codim}(X \setminus U) \geq 2$ and $\text{codim}(X' \setminus U') \geq 2$. Hence we have a canonical chain of isomorphisms

$$\text{Div}(X) \cong \text{Div}(U) \cong \text{Div}(U') \cong \text{Div}(X'),$$

which in turn induces a chain of isomorphisms

$$\text{Pic}(X) \cong \text{Pic}(U) \cong \text{Pic}(U') \cong \text{Pic}(X').$$

This implies that D' is big and nef because $H^0(X, kD) \cong H^0(X', kD')$ for every sufficiently divisible k and by choice of f . It follows that $\text{vol}(D) = \text{vol}(D')$. Now, consider the Fujiki relation for D

$$c_X(q_X(D))^n = \int_X [D]^{2n}.$$

As the Fujiki constant is a birational invariant, we have $c_X = c_{X'} =: c$. Using the Fujiki relation for D' on X' , we get the following chain of equalities

$$\int_X [D]^{2n} = c(q_X(D))^n = c(q_{X'}(D'))^n = \int_{X'} [D']^{2n} = \text{vol}(D'),$$

where the last equality in the equation above is justified by the fact that D' is big and nef. But $\text{vol}(D) = \text{vol}(D')$, thus $\int_X [D]^{2n} = \text{vol}(D)$, and in this case we are done. Now, assume that $\alpha = [D]$, where D is a q_X -nef \mathbf{R} -divisor. We can pick a sequence $\{D_k\}_{k \in \mathbf{N}}$ of big and q_X -nef rational divisors in $\text{int}(\text{Mov}(X))$ converging to D . By the above argument we have

$$\int_X [D_k]^{2n} = c_X(q_X(D_k))^n = \text{vol}(D_k),$$

for any k . By the continuity of the volume and of q_X , we get $c_X(q_X(D))^n = \text{vol}(D)$. Using the Fujiki relation for D we obtain

$$\int_X [D]^{2n} = c_X(q_X(D))^n = \text{vol}(D)$$

and we are done. ■

Now item (2) of Theorem 0.1.2 easily follows from the last proposition.

Proof of Theorem 0.1.2, item (2):

We already know from Proposition 2.3.1 and 2.3.2 that, for any big \mathbf{R} -divisor D with divisorial Zariski decomposition $D = P(D) + N(D)$, $\text{vol}(D) = (P(D))^{2n}$. Let Σ_P be a Boucksom-Zariski chamber, and set $\text{Null}_{q_X}(P) = \{D_1, \dots, D_k\}$. We know that the elements of $\text{Null}_{q_X}(P)$ are linearly independent in $N^1(X)_{\mathbf{R}}$. Hence we can choose a basis \mathcal{B} of $N^1(X)_{\mathbf{R}}$ of the form

$$\mathcal{B} = \{D_1, \dots, D_k, B_{k+1}, \dots, B_\rho\},$$

where ρ is the Picard number of X and $\{B_{k+1}, \dots, B_\rho\}$ is a basis for $\text{Null}_{q_X}(P)^\perp$. Let $D = P(D) + N(D) = \sum_i x_i B_i + N(D)$ be an element of Σ_P . We have

$$\text{vol}(D) = \text{vol}(P(D)) = (P(D))^{2n},$$

which is a homogeneous polynomial of degree $2n$ in the variables x_i , and we are done. ■

2.4 An example and a remark on the stability chambers

Let us start this section with the following example.

Boucksom-Zariski chambers on $\text{Hilb}^2(\mathbf{S})$. This example is based on the following result, which has been taken from the article [Rie20] of Ulrike Riess.

Proposition 2.4.1 ([Rie20, Lemma 3.2]): Let S be a K3 surface with $\text{Pic}(S) \cong \mathbf{Z} \cdot H_S$, for an ample line bundle with $(H_S)^2 = 2$. Consider the irreducible symplectic manifold $Y := \text{Hilb}^2(S)$, with the usual decomposition $\text{Pic}(Y) \cong \mathbf{Z} \cdot H \oplus \mathbf{Z} \cdot \delta$, where H is the line bundle associated to H_S . Then

1. $\mathcal{C}_Y \cap N^1(Y)_{\mathbf{R}} = \langle H + \delta, H - \delta \rangle$,
2. $\overline{\mathcal{BK}}_Y \cap N^1(Y)_{\mathbf{R}} = \langle H, H - \delta \rangle$,
3. $\text{Nef}(Y) = \langle H, 3H - 2\delta \rangle \subset N^1(Y)_{\mathbf{R}}$,
4. there is a unique other birational model Y' of Y which is an irreducible symplectic manifold. This satisfies

$$\text{Nef}(Y') = \langle 3H' - 2\delta', H' - \delta' \rangle \subset N^1(Y')_{\mathbf{R}},$$

where $H', \delta' \in \text{Pic}(Y')$ are the line bundles which correspond to H and δ via the birational transform.

With the notation of Proposition 2.4.1, we obtain the following picture, which has been drawn starting from Figure 1 in [Rie20].

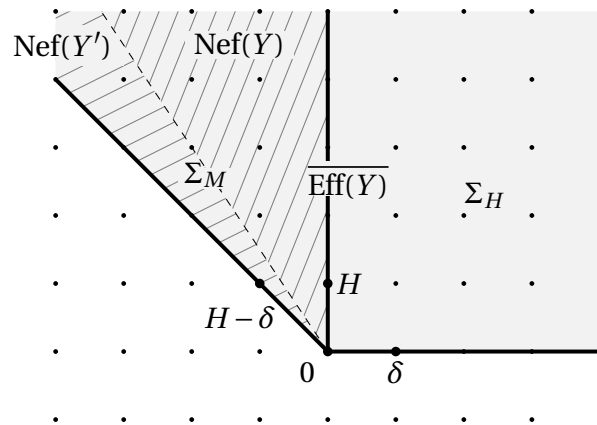


Figure 2.1: Boucksom-Zariski chambers on $\text{Hilb}^2(S)$

We only have one prime exceptional divisor, namely 2δ . It follows from Corollary 2.1.6 and item (1) of Proposition 2.4.1 that $\overline{\text{Eff}}(Y) = \langle H - \delta, \delta \rangle$. Clearly, we have two Boucksom-Zariski chambers. The first is Σ_M , where M is any element of $\text{int}(\text{Mov}(X))$, the second is the chamber associated (for example) to the big and q_Y -nef divisor H , namely Σ_H . It follows from Proposition 2.4.1 that $\overline{\Sigma}_M = \text{Nef}_{q_Y}(Y) = \langle H - \delta, H \rangle$, moreover $\overline{\Sigma}_H = \langle H, \delta \rangle$. We observe that the chamber Σ_M is neither closed nor open, while Σ_H is open.

Now we describe the volume function $\text{vol}(-)$ associated with Y . We start by observing that in this case, we have $c_Y = 3$ (see for example the table on page 78 in [Rap06]). By Proposition 2.3.2 and by the proof of item (2) of Theorem 0.1.2, we have the following description of the volume function.

1. **vol(-) on Σ_M :** let $[D] \in \Sigma_M$ and write $[D] = x[H] + z[H - \delta]$. Then

$$\text{vol}(D) = [D]^4 = 3 \cdot [q_Y(x[H] + z[H - \delta])]^2 = 12x^4 + 48x^2z^2 + 48x^3z.$$

2. **vol(-) on Σ_H :** let $[D] \in \Sigma_H$ and write $[D] = x[H] + z[\delta]$. Then

$$\text{vol}(D) = [P(D)]^4 = 3 \cdot [q_Y(x[H])]^2 = 3x^4(q_Y(H))^2 = 12x^4.$$

Outside the big cone $\text{vol}(-)$ is zero and this concludes its description.

We conclude this section with the following remark about the stability chambers (see Definition 1.1.10) on projective IHS manifolds.

Remark on the stability chambers. Let S be a smooth projective surface (over any algebraically closed field \mathbf{K}) and Y a normal projective variety. As we said in the introduction to this chapter, in [BKS04] the authors also studied how the augmented base loci of divisors vary in $N^1(S)_{\mathbf{R}}$, obtaining in this way the decomposition of $\text{Big}(S)$ into *stability chambers*.

The relevant result obtained in [BKS04] is the following.

Theorem 2.4.2: Let D be a stable big \mathbf{R} -divisor on S , and $\text{SC}(D)$ its stability chamber. Then

$$\text{int}(\text{SC}(D)) = \text{int}(\Sigma_{P(D)}),$$

where $\Sigma_{P(D)}$ is the Zariski chamber associated to the big and nef \mathbf{R} -divisor $P(D)$.

Theorem 2.4.2 tells us that on a smooth projective surface, the Zariski chambers and the stability chambers essentially coincide. Then, it is natural to ask the following question.

Question \blacklozenge . Let D be a stable big \mathbf{R} -divisor on a projective IHS manifold X , and $\text{SC}(D)$ its stability chamber. Is it true that

$$\text{int}(\text{SC}(D)) = \text{int}(\Sigma_{P(D)}),$$

where $\Sigma_{P(D)}$ is the Boucksom-Zariski chamber associated to the big and q_X -nef \mathbf{R} -divisor $P(D)$?

We immediately realised that this is not possible, and this is essentially due to the fact that the geometry of a higher dimensional projective IHS manifold is more intricate. We now explain why Question \blacklozenge cannot in general be answered positively.

Remark 2.4.3. Let A be any ample divisor on Y . One can show that $\text{SC}(A) = \text{Amp}(Y)$ (see for example [ELMP06, Proposition 1.5, Example 1.7]).

Let A be an ample divisor on a projective IHS manifold X . If Question \blacklozenge has an affirmative answer, then

$$\text{Amp}(X) = \text{SC}(A) = \text{int}(\Sigma_A).$$

But $\text{int}(\Sigma_A) = \text{int}(\text{Mov}(X))$, by Example 2.2.11. If X is the IHS manifold of Proposition 2.4.1, looking at Figure 2.1, we clearly have $\text{Amp}(X) \subsetneq \text{int}(\text{Mov}(X))$, thus Question \blacklozenge cannot be answered positively. This means that, in general, the augmented base loci are not constant in the interior of the Boucksom-Zariski chambers.

We will study asymptotic base loci of big divisors on IHS manifolds more carefully in the fourth chapter.

2.5 Simple Weyl chambers

In this section, we introduce the decomposition of the big cone of a projective IHS manifold into *simple Weyl chambers*. Similarly to what was done for surfaces, we compare this decomposition to that in Boucksom-Zariski chambers. Also, we prove Theorem 0.1.6.

Definition 2.5.1: The *simple Weyl chambers* on the projective IHS manifold X are defined as the connected components of the set

$$\text{Big}(X) \setminus \left(\bigcup_{D \in \text{Exc}(X)} D^\perp \right).$$

In this way, we obtain the decomposition of $\text{Big}(X)$ into simple Weyl chambers. The above definition does not give much information about the structure of the simple Weyl chambers, so it is natural to look for a concrete description of them. To begin with, consider an exceptional block S (see Definition 2.2.13) and define

$$W_S := \{ \alpha \in \text{Big}(X) \mid q_X(\alpha, D) < 0 \text{ for any } D \in S, \text{ and } q_X(\alpha, D) > 0 \text{ for any } D \in \text{Exc}(X) \setminus S \}.$$

We observe that W_S is a convex subset of the Néron-Severi space, and so is connected.

Lemma 2.5.2: The following assertions hold true.

1. The sets W_S are non-empty and open in $N^1(X)_{\mathbf{R}}$.
2. If S, S' are two exceptional blocks, then $W_S \cap W_{S'} = \emptyset$ if and only if $S' \neq S$.

Proof:

1. If $S = \{\emptyset\}$, then $W_S = \text{int}(\text{Mov}(X))$ and we are done. Assume then $S = \{D_1, \dots, D_k\}$. We first prove that W_S is not empty. Indeed, consider the divisor $D = M + \sum_{i=1}^k x_i D_i$ in the variables x_i , where M is any divisor lying in $\text{int}(\text{Mov}(X))$. We have to find values for the x_i such that the divisor D lies in W_S , i.e. $q_X(D, D_i) < 0$ for any $i = 1, \dots, k$, $q_X(D, D') > 0$ for any $D' \in \text{Exc}(X) \setminus S$, and $D \in \text{Big}(X)$. Let (a_1, \dots, a_k) be any k -tuple of negative real numbers, and consider the linear system \mathcal{S} defined by the equations

$$q_X(M, D_j) + \sum_i q_X(D_i, D_j) x_i = a_j, \quad j = 1, \dots, k.$$

It follows that

$$G^{-1} \cdot (q_X(M, D_1) - a_1, \dots, q_X(M, D_k) - a_k)^T = (-x_1, \dots, -x_k)^T,$$

where G is the Gram matrix of S , which is negative-definite and so invertible. Arguing as in Lemma 2.2.14 we obtain that the solution $(\lambda_1, \dots, \lambda_k)$ of \mathcal{S} is made of positive numbers. This implies that $D = M + \sum_{i=1}^k \lambda_i D_i$ is big and belongs to W_S .

To prove that W_S is open, as $N^1(X)_{\mathbf{R}}$ is finite-dimensional, it suffices to show that, for any $D \in W_S$, $D \pm \epsilon E$ belongs to W_S for ϵ small enough, where E any \mathbf{R} -divisor. As the big cone is open we can assume that $D \pm \epsilon E$ is big. Without loss of generality we can assume $q_X(E, D_i) \geq 0$ for any $i = 1, \dots, k$, so that $q_X(D - \epsilon E, D_i) < 0$. Also, choosing ϵ small enough

we have

$$q_X(D + \epsilon E, D_i) = q_X(D, D_i) + \epsilon q_X(E, D_i) < 0, \text{ for any } i = 1, \dots, k,$$

thus $q_X(D \pm \epsilon E, D_i) < 0$. It remains to check the intersection of $D \pm \epsilon E$ with any $D' \in \text{Exc}(X) \setminus S$. We observe that actually, we can have $q_X(D \pm \epsilon E, D') \leq 0$ only for a finite number of $D' \in \text{Exc}(X) \setminus S$. Indeed, as $D \pm \epsilon E$ is big, we have $D \pm \epsilon E \equiv A + N$, where A is ample and N is effective (both depending on the sign between D and ϵE). Hence

$$q_X(D \pm \epsilon E, D') = q_X(A + N, D') \leq 0$$

implies that D' is an irreducible component of N , and so we can have only a finite number of such D' . But then, up to taking ϵ even smaller, we clearly have

$$q_X(D \pm \epsilon E, D') > 0$$

for any $D' \in \text{Exc}(X) \setminus S$, and we are done.

2. The arrow \Rightarrow is obvious. Let us prove \Leftarrow . Without loss of generality, we can assume $S' \setminus S \neq \emptyset$. Let $D' \in S' \setminus S$. If $D \in W_S \cap W_{S'}$, then $q_X(D, D') < 0$ and $q_X(D, D') > 0$, which is clearly absurd. ■

The following proposition gives us the desired characterisation of the simple Weyl chambers.

Proposition 2.5.3: The simple Weyl chambers on X are exactly the sets $\{W_S\}_S$, where S varies among all the exceptional blocks.

Proof:

Let W be a simple Weyl chamber and pick any $D \in W$. We define

$$S_D := \{E \in \text{Exc}(X) \mid q_X(D, E) < 0\},$$

and observe that $S_D \subset \text{Neg}_{q_X}(D)$, and so S_D is an exceptional block. We now prove that $S := S_D$ is independent of the element of W we have chosen. Let $D' \neq D$ be another element of W . By contradiction, if $S_D \neq S_{D'}$, without loss of generality, we may pick an element $E \in S_D \setminus S_{D'}$, which means $q_X(D, E) < 0$ and $q_X(D', E) > 0$. Consider the linear functional

$$q_X(-, E): N^1(X)_{\mathbf{R}} \rightarrow \mathbf{R}, \text{ such that } q_X(-, E)(D'') := q_X(D'', E).$$

As W is connected and $q_X(-, E)$ is continuous, there exists $D'' \in W$ such that $q_X(-, E)(D'') = q_X(D'', E) = 0$, hence we reached a contradiction, because $W \subset \text{Big}(X) \setminus (\bigcup_{D \in \text{Exc}(X)} D^\perp)$. We conclude that $W \subset W_S$. Moreover, we must have $W = W_S$, because W is a connected component of $\text{Big}(X) \setminus (\bigcup_{D \in \text{Exc}(X)} D^\perp)$ and $W_S \subset \text{Big}(X) \setminus (\bigcup_{D \in \text{Exc}(X)} D^\perp)$ is also connected. To conclude, let S be any exceptional block. Since W_S is connected, then is contained in a simple Weyl chamber W .

But $W = W_{S'}$ for some exceptional block S' , and W_S is not empty by item (1) of Lemma 2.5.2. It follows by item (2) of Lemma 2.5.2 that $S = S'$, and so $W_S = W_{S'}$. ■

Corollary 2.5.4: There is a bijective correspondence between the exceptional blocks and the simple Weyl chambers on X .

Proof:

This is clear from Proposition 2.5.3. ■

Example 2.5.5. The easiest example of a simple Weyl chamber is that associated with \emptyset , i.e.

$$W_\emptyset = \text{int}(\text{Mov}(X)).$$

Indeed

$$\text{int}(\text{Mov}(X)) = \{\alpha \in N^1(X)_{\mathbf{R}} \cap \mathcal{C}_X \mid q_X(\alpha, E) > 0 \text{ for any } E \in \text{Exc}(X)\}.$$

We start with the following definition.

Definition 2.5.6: We say that the Boucksom-Zariski chambers on X are *numerically determined* if their interiors coincide with the simple Weyl chambers.

Since by Corollary 2.2.16 there is a bijective correspondence between the Boucksom-Zariski chambers and the exceptional blocks, from now on we will indicate the Boucksom-Zariski chamber associated with an exceptional block S with BZ_S . We start with the following relevant result, which tells us that even though the two decompositions can differ, already in dimension 2 (see [BF12, Examples]), the number of Boucksom-Zariski chambers is always the same as that of simple Weyl chambers (this happens also in the case of surfaces, see Theorem 2.5 in [HR20]).

Proposition 2.5.7: There is a bijective correspondence between the simple Weyl chambers and the Boucksom-Zariski chambers on X .

Proof:

This is obvious from Corollary 2.5.4 and Corollary 2.2.16. ■

The next result, which is a generalisation of Theorem 3.4 in [HR20], is a criterion for deciding whether a simple Weyl chamber W_S is contained in the Boucksom-Zariski chamber BZ_S .

Proposition 2.5.8: Let S be an exceptional block. Then $W_S \subset BZ_S$ is and only if the following condition holds: if $D \in \text{Exc}(X) \setminus S$ and $S \cup \{D\}$ is an exceptional block, then $q_X(D, D') = 0$ for any $D' \in S$.

Proof:

Note that we always have

$$W_\emptyset = \text{int}(\text{Mov}(X)) \subset \text{Big}(X) \cap \text{Nef}_{q_X}(X) = BZ_\emptyset,$$

so we can assume $S \neq \emptyset$. We first prove \Leftarrow . If we show that $\text{Neg}_{q_X}(D) = S$ for any $D \in W_S$, by definition of Boucksom-Zariski chamber we are done. Let $S = \{D_1, \dots, D_k\}$ and $D \in W_S$. As usual, let $D = P(D) + N(D)$ be the divisorial Zariski decomposition of D . As by hypothesis $q_X(D, D_i) < 0$, for any $i = 1, \dots, k$, and $P(D)$ is q_X -nef, it follows that any D_i is an irreducible component of $N(D)$. This proves $S \subset \text{Neg}_{q_X}(D)$. Now we have to prove $\text{Neg}_{q_X}(D) \subset S$. By contradiction, suppose

$$\text{Neg}_{q_X}(D) \setminus S = \{D'_1, \dots, D'_r\}.$$

We thus can write $N(D) = \sum_{i=1}^r a'_i D'_i + \sum_{j=1}^k a_j D_j$. By hypothesis, for any $i = 1, \dots, r$ and $j = 1, \dots, k$, we have $q_X(D'_i, D_j) = 0$. Since $D \in W_S$, by definition, we obtain

$$q_X(D, D'_j) = q_X(N(D), D'_j) = \sum_{i=1}^r a'_i q_X(D'_i, D'_j) =: b_j > 0, \text{ for any } j = 1, \dots, r. \quad (2.9)$$

If we denote by G the intersection matrix of $\{D'_1, \dots, D'_r\}$, the conditions contained in (2.9) can be resumed by the linear system

$$(a'_1, \dots, a'_r)G = (b_1, \dots, b_r),$$

which in turn gives

$$(a'_1, \dots, a'_r) = (b_1, \dots, b_r)G^{-1},$$

because G is invertible, as it is negative-definite. By Lemma 4.1 of [BKS04] the entries of G^{-1} are non-positive. It follows that $a'_i < 0$ for any $i = 1, \dots, r$, and this contradicts the effectivity of $N(D)$. Then $\text{Neg}_{q_X} \subset S$, and so $\text{Neg}_{q_X} = S$.

Now we prove the arrow \Rightarrow . By contradiction, assume that $W_S \subset BZ_S$, but $S = \{D_1, \dots, D_k\}$ does not satisfy the condition in the statement, so that there exists $E \in \text{Exc}(X) \setminus S$ such that $S \cup \{E\}$ is an exceptional block, and $q_X(E, D_i) > 0$ for some $D_i \in S$. We will show $W_S \not\subset BZ_S$ by constructing a divisor D belonging to $W_S \setminus BZ_S$. By Corollary 2.2.16 we can consider the Boucksom-Zariski chamber $BZ_{S \cup \{E\}}$ and we can pick an element $D' \in \text{int}(BZ_{S \cup \{E\}}) \neq \emptyset$. Note that, by Corollary 2.2.18, D' is such that $\text{Neg}_{q_X}(D') = \text{Null}_{q_X}(P(D')) = S \cup \{E\}$. Now, we will construct a big divisor D'' satisfying the following conditions:

- $\text{Supp}(N(D'')) = (\cup_{i=1}^k D_i) \cup E$.
- $q_X(D'', E) = q_X(N(D''), E) < 0$ and $q_X(D'', D_i) = q_X(N(D''), D_i) < 0$, for all $i = 1, \dots, k$.

Consider the divisor $xE + \sum_{i=1}^k x_i D_i$ in the variables x, x_1, \dots, x_k . Let (b, b_1, \dots, b_r) be any $(r+1)$ -

tuple of negative numbers and consider the linear system \mathcal{S} defined by the equations below.

$$\begin{aligned} q_X(E, E)x + \sum_{i=1}^k q_X(D_i, E)x_i &= b, \\ q_X(E, D_j)x + \sum_{i=1}^k q_X(D_i, D_j)x_i &= b_i, i = 1, \dots, k. \end{aligned}$$

As $S \cup \{E\}$ is an exceptional block, its Gram matrix G is negative-definite, and the entries of G^{-1} are non-positive by Lemma 4.1 of [BKS04]. It follows that the unique solution (c, c_1, \dots, c_k) of \mathcal{S} is made of positive numbers. Then the desired divisor D'' can be defined as $D'' := P(D') + cE + \sum_{i=1}^k c_i D_i$. Clearly $P(D'') = P(D')$, and $N(D'')$ is the remaining part in the definition of D'' . Now, consider the divisor

$$D := P(D'') + \frac{\min\{c, c_1, \dots, c_k\}}{2|q_X(E)|} E + \sum_{i=1}^k c_i D_i = P(D'') + N(D). \quad (2.10)$$

Note that D is big, and its divisorial Zariski decomposition is given by (2.10). Clearly $D \notin BZ_S$, because $\text{Neg}_{q_X}(D) \supsetneq S$. To conclude, we show that $D \in W_S$ by computing explicitly $q_X(D, E')$, where E' is any element of $\text{Exc}(X)$. Let $D_i \in S$ be any element of S . Then

$$q_X(D, D_i) = \frac{\min\{c, c_1, \dots, c_k\}}{2|q_X(E)|} q_X(E, D_i) + \sum_{j=1}^k c_j q_X(D_j, D_i) \leq q_X(N(D''), D_i) < 0.$$

Intersecting D with E we obtain

$$\begin{aligned} q_X(D, E) &= \frac{\min\{c, c_1, \dots, c_k\}}{2|q_X(E)|} q_X(E, E) + \sum_{j=1}^k c_j q_X(D_j, E) = \dots \\ &\dots = -\frac{1}{2} \min\{c, c_1, \dots, c_k\} + \sum_{j=1}^k c_j q_X(D_j, E) > 0, \end{aligned} \quad (2.11)$$

and the last inequality in (2.11) is true because, by assumption, there exists at least one $j \in \{1, \dots, k\}$ satisfying $q_X(D_j, E) > 0$. Intersecting D with any $E' \in \text{Exc}(X) \setminus (S \cup \{E\})$ we obtain

$$\begin{aligned} q_X(D, E') &= q_X(P(D''), E') + \frac{\min\{c, c_1, \dots, c_k\}}{2|q_X(E)|} q_X(E, E') + \dots \\ &\dots + \sum_{j=1}^k c_j q_X(D_j, E') \geq q_X(P(D''), E') > 0. \end{aligned} \quad (2.12)$$

The first inequality in (2.12), by [Bou04, Proposition 4.2, item (ii)], while the second because $P(D'') = P(D')$ is q_X -nef, and $\text{Null}_{q_X}(P(D')) = S \cup \{E\}$. With the last computation, we reached a contradiction, thus we are done. \blacksquare

Now we do the opposite, i.e. we exhibit a criterion explaining when the interior of a Boucksom-Zariski chamber BZ_S is contained in the simple Weyl chamber W_S .

Proposition 2.5.9: Let S be an exceptional block. Then $\text{int}(BZ_S) \subset W_S$ if and only if the prime exceptional divisors belonging to S are pairwise q_X -orthogonal.

Proof:

We first prove \Leftarrow . Set $S = \{D_1, \dots, D_k\}$. Let $D \in \text{int}(BZ_S)$, with divisorial Zariski decomposition $D = P(D) + N(D)$, where $N(D) = \sum_{i=1}^k a_i D_i$. By hypothesis we have $q_X(D_i, D_j) = 0$ for any $i \neq j$, thus

$$q_X(D, D_i) = q_X(N(D), D_i) = a_i q_X(D_i) < 0,$$

for any $D_i \in S$. Now, pick a divisor $E \in \text{Exc}(X) \setminus S$. Then $q_X(D, E) \geq 0$, by [Bou04, Proposition 4.2, item (ii)] and $P(D)$ is q_X -nef. If $q_X(D, E) = 0$, then $q_X(P(D), E) = 0$, and so $S = \text{Neg}_{q_X}(D) \neq \text{Null}_{q_X}(P(D))$. But this is absurd, because $D \in \text{int}(BZ_S)$ if and only if $\text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P(D))$, by Corollary 2.2.18. Hence $q_X(D, E) > 0$, and $\text{int}(BZ_S) \subset W_S$. We now show \Rightarrow . By contradiction, assume that $\text{int}(BZ_S) \subset W_S$, and there exist two elements of S which are not q_X -orthogonal, say D_1, D_2 . Let M be any divisor lying in $\text{int}(\text{Mov}(X))$. By Lemma 2.2.14 there exists a k -tuple of positive numbers $(\lambda_1, \dots, \lambda_k)$ such that $M + \sum_{i=1}^k \lambda_i D_i$ is q_X -nef with $\text{Null}_{q_X}(M + \sum_{i=1}^k \lambda_i D_i) = S$. Now, choose a k -tuple of positive numbers (μ_1, \dots, μ_k) such that $\mu_i > \lambda_i$ for any $i = 1, \dots, k$. Finally, pick a real number η satisfying $0 < \eta < \frac{\mu_2 - \lambda_2}{|q_X(D_1)|}$ and consider the divisor

$$D = M + (\lambda_1 + \eta)D_1 + \sum_{i=2}^k \mu_i D_i.$$

It follows by construction that the divisorial Zariski decomposition of D is given by $P(D) = M + \sum_{i=1}^k \lambda_i D_i$, and $N(D) = \eta D_1 + \sum_{i=2}^k (\mu_i - \lambda_i) D_i$. By construction $\text{Neg}_{q_X}(D) = \text{Null}_{q_X}(P(D))$, thus $D \in \text{int}(BZ_S)$, by Corollary 2.2.18. But

$$q_X(D, D_1) = \eta q_X(D_1) + \sum_{i=2}^k (\mu_i - \lambda_i) q_X(D_i, D_1) \geq \eta q_X(D_1) + (\mu_2 - \lambda_2) > 0, \quad (2.13)$$

and the last inequality in (2.13) is true by the choice of η . Thus $D \notin W_S$, which is a contradiction. \blacksquare

Before proving Theorem 0.1.6, we need the following chain of equivalences.

Lemma 2.5.10: The following assertions are equivalent:

1. If two prime exceptional divisors E_1, E_2 satisfy $q_X(E_1, E_2) > 0$, then $q_X(E_1, E_2) \geq \sqrt{q_X(E_1)q_X(E_2)}$.
2. If E_1, E_2 are prime exceptional divisors such that $\{E_1, E_2\}$ is an exceptional block, then $q_X(E_1, E_2) = 0$.
3. Let S be an exceptional block. If $E \in \text{Exc}(X) \setminus S$, and $S \cup \{E\}$ is an exceptional block, then $q_X(E, E') = 0$ for any $E' \in S$.

4. Let S be an exceptional block. Then the prime exceptional divisors of S are pairwise q_X -orthogonal.

Proof:

We first prove (1) \Rightarrow (2). Suppose that (2) does not hold true. Then there exist prime exceptional divisors E_1, E_2 such that $S = \{E_1, E_2\}$ is an exceptional block, and $q_X(E_1, E_2) \neq 0$. Then $q_X(E_1, E_2) > 0$, by [Bou04, Proposition 4.2, item (ii)]. Since the Gram matrix of S is negative-definite, we have

$$x^2 q_X(E_1) + y^2 q_X(E_2) + 2xy q_X(E_1, E_2) < 0, \quad (2.14)$$

for any couple $(x, y) \neq (0, 0)$. As $\sqrt{q_X(E_1)q_X(E_2)} \neq 0$, using the relation (2.14), we obtain

$$q_X(E_1, E_2) < \frac{[x\sqrt{-q_X(E_1)}]^2 + [y\sqrt{-q_X(E_2)}]^2}{2xy\sqrt{q_X(E_1)q_X(E_2)}} \cdot \sqrt{q_X(E_1)q_X(E_2)}. \quad (2.15)$$

If we choose $x = \sqrt{-q_X(E_1)}$ and $y = \sqrt{-q_X(E_2)}$, then (2.15) gives

$$q_X(E_1, E_2) < \sqrt{q_X(E_1)q_X(E_2)},$$

which contradicts item (1) of the statement.

Now we prove (2) \Rightarrow (1). We can assume that $\{E_1, E_2\}$ is not an exceptional block, as otherwise, its Gram matrix would be negative-definite, and so $q_X(E_1, E_2) = 0$ by hypothesis, and we are only interested in couples of exceptional divisors intersecting properly. By our assumption, there exists a couple (x, y) of real numbers of the same sign ⁴ satisfying

$$x^2 q_X(E_1) + y^2 q_X(E_2) + 2xy q_X(E_1, E_2) \geq 0. \quad (2.16)$$

As $\sqrt{q_X(E_1)q_X(E_2)} \neq 0$, the above inequality implies

$$\begin{aligned} q_X(E_1, E_2) &\geq \frac{[x\sqrt{-q_X(E_1)}]^2 + [y\sqrt{-q_X(E_2)}]^2}{2xy\sqrt{q_X(E_1)q_X(E_2)}} \cdot \sqrt{q_X(E_1)q_X(E_2)} \geq \dots \\ &\dots \geq \sqrt{q_X(E_1)q_X(E_2)}, \end{aligned} \quad (2.17)$$

and we are done. The implications (3) \Rightarrow (2) and (4) \Rightarrow (2) are trivial, so we prove (2) \Rightarrow (3). Let S be an exceptional block. If $S \cup \{E'\}$ is also exceptional for some $E' \in \text{Exc}(X) \setminus S$, and $E \in S$, then the block $\{E, E'\}$ is exceptional. By hypothesis we obtain $q_X(E, E') = 0$.

To conclude, we prove (2) \Rightarrow (4). Let S be an exceptional block. If E_1, E_2 belongs to S , also $\{E_1, E_2\}$ is an exceptional block, and by hypothesis we obtain $q_X(E_1, E_2) = 0$. \blacksquare

We are now ready to prove Theorem 0.1.6, which tells us when the Boucksom-Zariski chambers are numerically determined.

⁴Indeed, if such a couple were made of numbers with different signs, then the quantity (2.16) would be negative, which is not possible by our assumption.

Proof of Theorem 0.1.6:

We first prove (2) \Rightarrow (1). By Lemma 2.5.10, item (2) of Theorem 0.1.6 is equivalent to item (3) and (4) of Lemma 2.5.10. By Proposition 2.5.8 and 2.5.9 we obtain that $W_S \subset BZ_S$ and $\text{int}(BZ) \subset W_S$ for any exceptional block S , so that $W_S = \text{int}(BZ_S)$, and we are done. We now prove (1) \Rightarrow (2) in Theorem 0.1.6. As by hypothesis $W_S = \text{int}(BZ_S)$ for any exceptional block S , by Proposition 2.5.8 we obtain that item (3) of Lemma 2.5.10 holds true, and, by Lemma 2.5.10, this is equivalent to item (2) of Theorem 0.1.6. This concludes the proof. \blacksquare

The following criterion, explaining when a Boucksom-Zariski chamber intersects a given simple Weyl chamber, is a generalisation of Theorem 3.6 in [HR20].

Proposition 2.5.11: Let S, S' be two exceptional blocks. Then $W_{S'} \cap BZ_S \neq \emptyset$ if and only if $S' \subset S$ and each subset $T \subset S \setminus S'$ satisfies the following property: there exist two divisors, one in T and one in $S \setminus T$, intersecting properly (with respect to q_X).

Proof:

We first prove \Rightarrow . Let $D \in W_{S'} \cap BZ_S$ with divisorial Zariski decomposition $D = P(D) + N(D)$. Then for every $D' \in S'$ we have $q_X(D, D') < 0$. Moreover $q_X(N(D), D') < 0$, because $P(D)$ is q_X -nef, thus D' is an irreducible component of $N(D)$. But S coincides with the set of irreducible components of $N(D)$, hence $S' \subset S$. Now, suppose that there exists a subset $T \subset S \setminus S'$ not satisfying the condition in the statement. This means that $q_X(E, E') = 0$ for any $E \in T$ and $E' \in S \setminus T$ (by [Bou04, Proposition 4.2, item (ii)]). We can write

$$D = P(D) + N(D) = P(D) + \sum_{E_i \in T} a_i E_i + \sum_{E_j \in S \setminus T} b_j E_j.$$

Since $D \in W_{S'}$, and by our assumption, we have

$$q_X(D, E_j) = q_X\left(\sum_{E_i \in T} a_i E_i, E_j\right) > 0,$$

for any $E_j \in T \subset S \setminus S'$. The intersection matrix of T is negative-definite, hence, by Lemma 4.1 of [BKS04], the entries of T^{-1} are non-positive. It follows that the a_i 's appearing in $\sum_{E_i \in T} a_i E_i$ are negative, and this contradicts the effectivity of $N(D)$. Hence \Rightarrow is proved.

We now prove \Leftarrow . First, we show that there exists an element $D' \in W_{S'} \cap BZ_S$. We construct D' explicitly as

$$D' = M + \sum_{D'_i \in S'} \lambda_i D'_i + \sum_{D'_i \in S'} (\mu_i - \lambda_i) D'_i, \quad (2.18)$$

where M is any divisor lying in $\text{int}(\text{Mov}(X))$, and satisfying $P(D') = M + \sum_{D'_i \in S'} \lambda_i D'_i$, $N(D') = \sum_{D'_i \in S'} (\mu_i - \lambda_i) D'_i$. Arguing as in Lemma 2.2.14 we can find the $\lambda_i > 0$ in such a way that $\text{Null}_{q_X}(P(D')) = S'$, so that $P(D')$ is q_X -nef. Now, let $\{y_i < 0 \mid 1 \leq i \leq \text{card}(S')\}$ be a set of negative

real numbers, and consider the linear system of equations defined by

$$\sum_{D'_i \in S'} q_X(D'_i, D'_j) x_i = y_j, \text{ for any } D'_j \in S'.$$

Again, arguing as in Lemma 2.2.14, we obtain a unique solution for this linear system, made up of positive real numbers. Using the variables x_i to indicate the solution of the linear system above, we define $\mu_i := \lambda_i + x_i$. Defining D' as in (2.18), with the λ_i and the μ_i we introduced so far, clearly it holds $D' \in W_{S'} \cap BZ_{S'}$. Now, if $S = S'$ we are done, otherwise $S' \subsetneq S$, and we can consider the set

$$S'' := \{E'' \in S \setminus S' \mid q_X(E'', E') > 0 \text{ for some } E' \in S'\}.$$

We note that S'' is not empty by hypothesis. We describe how to construct a divisor D'' belonging to $W_{S'} \cap BZ_{S' \cup S''}$. We claim that D'' can be explicitly constructed of the form

$$D'' = M + \sum_{D''_i \in S'} \alpha_i D''_i + \sum_{D''_i \in S''} \beta_i D''_i + nN(D') + \sum_{D''_i \in S''} D''_i,$$

where

- n is any positive natural number, and the α_i, β_i are positive real numbers,
- $M \in \text{int}(\text{Mov}(X))$,
- $P(D'') = M + \sum_{D''_i \in S'} \alpha_i D''_i + \sum_{D''_i \in S''} \beta_i D''_i$,
- $N(D'') = nN(D') + \sum_{D''_i \in S''} D''_i$.

Indeed, arguing as in Lemma 2.2.14, we can clearly construct D'' having the listed properties. Now, if $E \in S'$, then

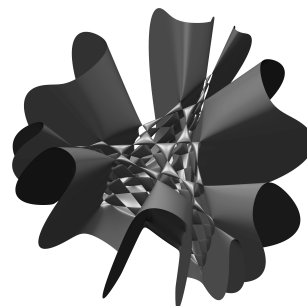
$$q_X(D'', E) = nq_X(N(D'), E) + q_X\left(\sum_{D''_i \in S''} D''_i, E\right).$$

As $D' \in W_{S'}$, we have $q_X(N(D'), E) < 0$, and choosing n large enough we will have $q_X(D'', E) < 0$ for any $E \in S'$. To conclude, let E be any prime exceptional divisor not belonging to S' . If $E \notin S''$, then $q_X(D'', E) > 0$, by construction of D'' . If $E \in S''$, by definition of S'' and by assumption, there exists an element E' in S' such that $q_X(E, E') > 0$. Now,

$$q_X(D'', E) = nq_X(N(D'), E) + q_X\left(\sum_{D''_i \in S''} D''_i, E\right),$$

all the terms of $q_X(N(D'), E)$ are non-negative and at least one of them is positive. Thus, increasing again n , if necessary, we obtain $q_X(D'', E) > 0$ for any $E \in \text{Exc}(X) \setminus S'$. We have thus constructed $D'' \in W_{S'} \cap BZ_{S' \cup S''}$. If $S = S' \cup S''$ we are done. Otherwise, we can repeat the above argument. Proceeding like this, since S is finite, we will end up with the desired divisor $D \in W_{S'} \cap BZ_S$. ■

Remark 2.5.12. By the proof of \Leftarrow in Proposition [2.5.11](#), we obtain that $W_S \cap BZ_S \neq \emptyset$, for any exceptional block S .



3. Pseudo-effective classes

3.1 The pseudo-effective cone of an IHS manifold

The aim of this section is to prove Theorem 0.2.2. We start with a remark about the difference between the definition of circular part used by Kovács and the one that we adopt.

Remark 3.1.1. Note that in the papers [Kov93], [Kov13] Kovács adopted another definition of "circular part" for a convex cone, in order to prove his Theorem 0.2.1. In particular, let K be as in Definition 1.1.5. We say that K is locally finitely generated at $v \in \partial K$ if there exists a finite set of points $S = \{v_1, \dots, v_k\} \subset K$ and a (not necessarily convex) subcone $C \subset K$ such that K is generated by S and C . For Kovács, an open subset $U \subset \partial K$ is a circular part of K if there is no point of U at which K is locally finitely generated. One can show that if K is locally polyhedral at a point, then K is locally finitely generated at that point. But the converse does not hold true, in general (see Example 3.1.2). However, in Theorem 0.2.2, a circular part of the pseudo-effective cone (if there is any) with respect to Definition 1.1.5 is circular with respect to Kovács' definition too because such a circular part would be contained in $\overline{\partial \text{Pos}(X)}$ (see "Proof of Theorem 1.2"). Hence, no problem arises in using our definition. Note that Definition 1.1.5 has also been adopted by Huybrechts in [Huy15, Chapter 8, Section 3], where he provided a slightly different proof of Theorem 0.2.1.

The following is an example of a locally finitely generated cone which is not locally polyhedral (at a point).

Example 3.1.2. Let C be the "ice cream" in Figure 3.1 and $C \subset \mathbf{R}^3 \subset \mathbf{R}^4$ an embedding of C in \mathbf{R}^4 , such that the vertex V of C does not coincide with the origin of \mathbf{R}^4 . Let K be the convex cone generated by C in \mathbf{R}^4 (via the embedding we have chosen). By our choice, the interior of this cone is not empty. Then K is locally finitely generated at V , but not locally polyhedral at V . Indeed, if S is the sphere appearing in Figure 3.1, the cone K is generated by V and the subcone K' of K generated by the sphere S embedded in \mathbf{R}^4 . The non-local polyhedrality of K at V is clear.

We now prove Corollary 0.2.6, which characterises the pseudo-effective cone of a projective IHS manifold of Picard number 2, using Oguiso's theorem [Ogu14, Theorem 1.3, item (2)].

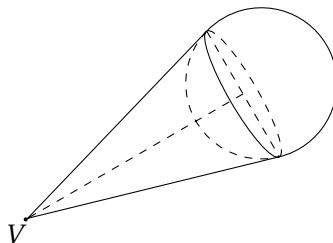


Figure 3.1: Hyperplane section of a locally finitely generated cone at a point.

Proof of Corollary 0.2.6:

We first observe that $\overline{\text{Eff}}(X)$ cannot have both one rational extremal ray and one irrational extremal ray. Indeed, if X does not contain any prime exceptional divisor then $\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X)$ (See Corollary 2.1.6) and we are done by Theorem 0.2.5. As by Theorem 1.3.13 and equality (1.5) the classes of the prime exceptional divisors span extremal rays in $\overline{\text{Eff}}(X)$, we only have to check the case " X contains only one prime exceptional divisor". In this $\overline{\text{Mov}}(X)$ is rational polyhedral by Theorem 0.2.5 and [Mar11, Theorem 6.17], and the extremal ray of $\overline{\text{Mov}}(X)$ spanned by an integral class of BBF square 0 is also extremal for $\overline{\text{Eff}}(X)$ and we are done. To conclude, it is sufficient to observe that $\overline{\text{Eff}}(X)$ is rational (resp. irrational) if and only if $\overline{\text{Mov}}(X)$ is rational (resp. irrational). Indeed, the perfect pairing induced by the BBF form gives the duality $\overline{\text{Eff}}(X) = \overline{\text{Mov}}(X)^*$, and we are done. ■

Remark 3.1.3. We observe that under the assumptions of Corollary 0.2.6, if X satisfies Conjecture 1 (e.g. if X belongs to one of the known deformation classes), whenever $\overline{\text{Eff}}(X)$ is rational we have $\overline{\text{Eff}}(X) = \text{Eff}(X)$, i.e. both extremal rays of the pseudo-effective cone are spanned by the classes of some integral, effective divisors. Also, note that both cases in Corollary 0.2.6 do occur (see for example [Rie20, Lemma 3.2] and [Ogu14, Proposition 5.3]).

Proof of Theorem 0.2.2:

Suppose that $\text{Exc}(X) = \emptyset$. In this case $\overline{\text{Eff}}(X) = \overline{\text{Pos}}(X)$, hence $\overline{\text{Eff}}(X)$ is circular.

Suppose that $\text{Exc}(X) \neq \emptyset$. We first show that $\overline{\text{Eff}}(X)$ does not contain any circular part. The orbit of E under the action of Γ' is made (up to a sign) of stably exceptional divisors (i.e. $\mathcal{O}_X(E)$ is a stably exceptional line bundle), and the walls $\{g(E)^\perp\}_{g \in \Gamma'}$ intersect the interior of the positive cone $\text{Pos}(X)$. Indeed the BBF square of E is negative. Let y be any class in $\text{Pos}(X)$. Consider the class

$$y' := y - \frac{q_X(E, y)}{q_X(E)}[E].$$

Clearly, $y' \in (E)^\perp$, and $q_X(y') = q_X(y', y) > 0$ (see Proposition 1.3.31), thus $y' \in E^\perp \cap \text{Pos}(X)$. Suppose by contradiction that $\overline{\text{Eff}}(X)$ contains a circular part \mathcal{C} . We can assume that $\mathcal{C} = \mathbf{R}^{>0}\mathcal{C}$. Then, \mathcal{C} is contained in $\partial\overline{\text{Mov}}(X) \cap \partial\overline{\text{Pos}}(X)$. Let x be a point lying in $E^\perp \cap \partial\overline{\text{Pos}}(X)$.

The projectivised positive cone $\mathbf{P}(\text{Pos}(X))/\Gamma'$ is a hyperbolic manifold, and since Γ' is of finite index in $O^+(\text{Pic}(X))$ and $\rho(X) \geq 3$, it is of finite volume ([AV16a, Section 3.2]). Then, by [Mar22, Proposition 5.1.9], as Γ' is non-elementary, $\Lambda(\Gamma') = \partial\mathbf{P}(\text{Pos}(X))$, and by [Sea15, Theorem 1.29] Γ' acts with dense orbits on $\partial\mathbf{P}(\text{Pos}(X))$. The rays $\{\mathbf{R}^{>0}(g(x))\}_{g \in \Gamma'}$ are contained in the positive cone $\text{Pos}(X)$. Then, the set of rays $\mathbf{R}^{>0}(g(x))$ (with $g \in \Gamma'$) contained in \mathcal{C} form a dense subset of the circular part \mathcal{C} , and any of them is orthogonal to the corresponding $g(E)$. This implies the existence of a class lying in $\text{Int}(\overline{\text{Mov}(X)})$, intersecting trivially some stably exceptional divisor, and this is a contradiction because any element in the interior of the movable cone intersects any stably exceptional divisor positively ([Mar11, Proposition 6.10]). Then, $\overline{\text{Eff}(X)}$ has no circular part. Now, using that $\overline{\text{Eff}(X)}$ does not contain any circular part, we show the equality $\overline{\text{Eff}(X)} = \overline{\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]}$. To do so, as $\overline{\text{Eff}(X)}$ does not contain lines (see for example Lemma 2.1.2), by Minkowski's Theorem on closed convex cones not containing lines, it suffices to show that $\overline{\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]}$ contains all the extremal rays of $\overline{\text{Eff}(X)}$. Suppose by contradiction that there exists an extremal ray R of $\overline{\text{Eff}(X)}$ not belonging to $\overline{\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]}$. Pick a non-zero element $v \in R$ (which must therefore satisfy $q_X(v) = 0$) and consider the hyperplane

$$H := \{x \in N^1(X)_{\mathbf{R}} \mid q_X(x, A) = q_X(v, A)\},$$

where A is any ample divisor. Clearly $Q := H \cap \overline{\partial\text{Pos}(X)}$ is a smooth compact quadric hypersurface in H . Indeed $q_X(v, A) > 0$ (see for example Lemma 2.1.2), hence H does not intersect $-\overline{\partial\text{Pos}(X)}$, so that $H \cap \overline{\partial\text{Pos}(X)}$ is non-degenerate. Moreover, up to a rescaling, any element $y \in \overline{\partial\text{Pos}(X)}$ is such that $q_X(y, A) = q_X(v, A)$, hence Q must be compact. Let N be the closure of the convex hull in H of all the elements $y \in H$ belonging to an extremal ray of the type $\mathbf{R}^{\geq 0}[E]$, where E is a prime exceptional divisor of X . Clearly, N is a compact convex subset of H . As $v \in Q \setminus N$, by Lemma 1.3.12, there exists an open subset U of Q contained in $\partial(\text{Conv}(Q \cup N))$, where $\text{Conv}(Q \cup N)$ is the convex hull of $Q \cup N$. Then $\mathbf{R}^{\geq 0}U$ is a circular part of $\overline{\text{Eff}(X)}$, and this contradicts the fact that $\overline{\text{Eff}(X)}$ does not contain circular parts. ■

Remark 3.1.4. For a somehow similar statement about the nef cone see [Mat18].

Corollary 3.1.5: Let X be a projective IHS manifold of Picard number at least 3, carrying a prime exceptional divisor. Then $\text{card}(\text{Exc}(X)) \geq \rho(X)$ and $\overline{\text{Eff}(X)}$ is rational polyhedral if and only if $\text{Exc}(X)$ is a finite set.

Proof:

By Theorem 0.2.2 we have, $\overline{\text{Eff}(X)} = \overline{\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]}$, and the latter equality is satisfied only if $\#\text{Exc}(X) \geq \rho(X)$, because the interior of $\overline{\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]}$ (which is non-empty) equals the interior of $\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]$. For the second part of the statement, if $\overline{\text{Eff}(X)}$ is rational polyhedral, X carries finitely many prime exceptional divisors, since any prime exceptional divisor spans an extremal ray of $\overline{\text{Eff}(X)}$ (by Corollary 2.1.4). Suppose now that $\text{Exc}(X)$ is a finite set. Since by Theorem 0.2.2 we have $\overline{\text{Eff}(X)} = \overline{\sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]}$ and any prime exceptional divisor spans an extremal ray, $\overline{\text{Eff}(X)}$ is rational polyhedral. ■

Example 3.1.6. Let S be a projective K3 surface of Picard number 2, carrying one (and only one) smooth rational curve. Then, by Corollary 3.1.5, $S^{[n]}$ carries at least one prime exceptional divisor coming neither from S , nor from the desingularisation of $S^{(n)}$, for any n .

Corollary 3.1.7: Let X be a projective IHS manifold of Picard number greater than 3, carrying a prime exceptional divisor. Then the nef cone $\text{Nef}(X)$ has no circular part.

Proof:

By Theorem 0.2.2, $\overline{\text{Mov}}(X)$ has no circular parts, and $\text{Nef}(X)$ is rational polyhedral away from the boundary of $\text{Pos}(X)$, thanks to a result of Kawamata ([Kaw88, Theorem 5.7]), hence also $\text{Nef}(X)$ has no circular parts. ■

In the proof of Corollary 0.2.7 we will need a theorem by Burnside and a result of Oguiso.

Theorem 3.1.8 ([Bur05], Main Theorem): Let G be a subgroup of $\text{GL}(n, \mathbb{C})$. Assume that there exists a positive integer d such that any element of G has order at most d . Then G is a finite group.

Lemma 3.1.9 ([Ogu14], Corollary 2.6): Let Y be a smooth projective variety with trivial canonical bundle, such that $h^1(Y, \mathcal{O}_Y) = 0$. Moreover, let $r: \text{Bir}(Y) \rightarrow \text{GL}(N^1(Y)_{\mathbb{R}})$ be the natural representation of $\text{Bir}(Y)$. Then the following hold.

1. $[\text{Bir}(Y) : \text{Aut}(Y)] = [r(\text{Bir}(Y)) : r(\text{Aut}(Y))]$.
2. If G is a subgroup of $\text{Bir}(Y)$, G is finite if and only if there is a positive integer d such that every element of $r(G)$ has order at most d .

Proof of Corollary 0.2.7:

We start by proving (1) \Leftrightarrow (2). If $\rho(X) = 2$ we are done, by Corollary 0.2.6. If $\rho(X) \geq 3$, consider the natural representation

$$f: \text{Bir}(X) \rightarrow \text{GL}(N^1(X)_{\mathbb{R}}).$$

We first prove (1) \Rightarrow (2). As $\text{Eff}(X)$ is rational polyhedral, X carries a finite number N of prime exceptional divisors, by Corollary 3.1.5. In particular, $\text{Eff}(X)$ does not contain circular parts, hence, by the second part of the proof of Theorem 0.2.2, the equality

$$\text{Eff}(X) = \sum_{E \in \text{Exc}(X)} \mathbf{R}^{\geq 0}[E]$$

holds in this case. If s is a birational self-map of X and $f(s)$ is the automorphism induced on $N^1(X)_{\mathbb{R}}$, we observe that $f(s)$ has order at most $N!$. Indeed, $f(s)$ acts on $\text{Exc}(X)$, hence an automorphism of $N^1(X)_{\mathbb{R}}$ is uniquely determined by $f(s)(\text{Exc}(X))$, because in $\text{Exc}(X)$ we can find a basis for the Néron-Severi space. Now we use Theorem 3.1.8 and Lemma 3.1.9 to conclude that

$\text{Bir}(X)$ is finite.

Now we prove $(2) \Rightarrow (1)$. As $\text{Bir}(X)$ is finite, $\overline{\text{Mov}(X)}^e$ is rational polyhedral, by Theorem 1.3.16 and Remark 1.3.17. Also, the equality $\overline{\text{Mov}(X)}^e = \text{Mov}(X) = \overline{\text{Mov}(X)}$ holds in this case. Hence, $\overline{\text{Mov}(X)}^* = \overline{\text{Eff}(X)} = \text{Eff}(X)$ is rational polyhedral, and we are done. We now prove $(2) \Leftrightarrow (3)$. Let $O^+(N^1(X))$ be the group of isometries of $N^1(X)$ preserving the positive cone $\text{Pos}(X)$. Clearly $O^+(N^1(X))$ has index 2 in $O(N^1(X))$. Moreover, by [Mar11, Lemma 6.23], the image of $\text{Mon}_{\text{Hdg}}^2(X)$ via the natural map $\rho: \text{Mon}_{\text{Hdg}}^2(X) \rightarrow O^+(N^1(X))$ has finite index in $O^+(N^1(X))$ and is isomorphic to a semi-direct product of W_{Exc} and $\text{Mon}_{\text{Bir}}^2(X)/\ker(\rho)$. Now, the kernel of ρ is finite by [Ogu14, Proposition 2.4], and looking at the chain of inclusions

$$W_{\text{Exc}} \subset \text{Im}(\rho) \subset O^+(N^1(X)) \subset O(N^1(X)),$$

the equivalence between (2) and (3) is easily checked.

The implication $(3) \Rightarrow (4)$ is obtained by $(3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (4)$. Suppose now that X carries a prime exceptional divisor and has the Picard number at least 3.

The implication $(4) \Rightarrow (1)$ is nothing but the "only if" part of Corollary 3.1.5, thus all the statements are equivalent in this case because we also have $(4) \Rightarrow (1) \Rightarrow (2) \Rightarrow (3)$. ■

Remark 3.1.10. Note that, given a projective IHS manifold X belonging to one of the known deformation classes, replacing $\text{Bir}(X)$ with $\text{Aut}(X)$ and the Kawamata-Morrison movable cone conjecture with the classical Kawamata-Morrison cone conjecture, stated for $\text{Nef}^e(X) = \text{Nef}(X) \cap \text{Eff}(X)$ (see [MY15, Theorem 1.3] and [AV16b, Section 5.2]), a similar argument to that given in the proof of $(1) \Leftrightarrow (2)$ in Corollary 0.2.7 can be used to prove that $\text{Nef}(X)$ is rational polyhedral if and only if $\text{Aut}(X)$ is finite.

We recall the following definitions.

Definition 3.1.11 ([HK00, Definition 1.8]): A small \mathbf{Q} -factorial modification of a normal projective variety Y is a birational map $g: Y \dashrightarrow Y'$, where Y' is again normal, projective, and \mathbf{Q} -factorial, and g is an isomorphism in codimension 1.

Definition 3.1.12: Let X be a normal and \mathbf{Q} -factorial projective variety. We say that X is a Mori dream space (MDS) if the following properties hold:

1. $\text{Pic}(X)$ is finitely generated (equivalently, $h^1(\mathcal{O}_X) = 0$);
2. $\text{Nef}(X)$ is generated by the classes of finitely many semiample divisors;
3. there is a finite collection of small \mathbf{Q} -factorial modifications $s_i: X \dashrightarrow X_i$, such that every X_i satisfies item (2) and

$$\text{Mov}(X) = \bigcup_i s_i^*(\text{Nef}(X_i)).$$

The corollary below explains when a projective IHS manifold is a MDS.

Corollary 3.1.13: Let X be a projective IHS manifold with $b_2(X) \neq 4$ and satisfying Conjecture 1. Then the following conditions are equivalent:

1. X is a MDS,
2. $\text{Bir}(X)$ is finite,
3. $0 < \rho(X) < 3$ and $\text{Eff}(X)$ is rational polyhedral, or $\rho(X) \geq 3$ and $\text{Exc}(X)$ is non-empty and finite.

Proof:

(1) \Leftrightarrow (2). Suppose that X is a MDS. Then $\text{Bir}(X)$ is finite, by definition of MDS. Suppose now that $\text{Bir}(X)$ is finite. By [AV16b, Theorem 3.17] the BBF square of the integral, primitive, and extremal classes of the Mori cones of the birational models of X is bounded, hence by [MY15, Corollary 1.5] the number of birational models of X is finite (up to isomorphism). This implies that (3) in Definition 3.1.12 is satisfied. The base point free Theorem, the hypothesis that X verifies Conjecture 1, and Remark 3.1.10 imply that also (2) in Definition 3.1.12 is satisfied. Thus X is a MDS.

(2) \Leftrightarrow (3). Follows directly from Corollary 0.2.7. ■

We conclude this section by proving Corollary 0.2.9. For some existence results of ample uniruled divisors on some primitive symplectic varieties see [BG23] and [LMP23].

Proof of Corollary 0.2.9:

As Y is \mathbf{Q} -factorial, any irreducible component of the exceptional locus of f is divisorial. By Corollary 3.1.5, $\text{card}(\text{Exc}(X)) \geq \rho(X)$. If A is any ample Cartier divisor on Y , any irreducible component of $\text{Exc}(f)$ is contained in $\mathbf{B}_+(f^*(A))$. In particular, by Corollary 1.3.32 and Proposition 4.1.2, the Gram matrix of the irreducible components of $\text{Exc}(f)$ is negative definite so that the classes in $N^1(X)_{\mathbf{R}}$ of the irreducible components of $\text{Exc}(f)$ are linearly independent. In particular, in $\text{Exc}(f)$ there are at most $\rho(X) - 1$ irreducible components. On the other hand, $\text{card}(\text{Exc}(X)) \geq \rho(X)$, hence there exists a prime exceptional divisor E on X which is not contracted by f . Then, $f_*(E)$ is a uniruled divisor on Y . ■

Adopting the same strategy to prove Theorem 0.2.2, we obtain the following version of Theorem 0.2.2 in the singular setting.

Theorem 3.1.14: Let X be a projective \mathbf{Q} -factorial primitive symplectic variety with terminal singularities of Picard number at least 3. Then, either $\overline{\text{Eff}(X)} = \overline{\text{Pos}(X)}$, or

$$\overline{\text{Eff}(X)} = \sum_E \overline{\mathbf{R}^{\geq 0}[E]},$$

where the sum runs over the prime exceptional divisors of X .

Proof:

All the theory about primitive symplectic varieties that is needed to prove the theorem is contained in the papers [BL22], [KMPP19], [LMP22], [LMP23], to which we refer the reader. Below we give an idea of the proof, which follows the lines of the proof of Theorem 0.2.2, highlighting the steps which to us seem more important. If there are no prime exceptional divisors, $\overline{\text{Eff}}(X) = \overline{\text{Pos}}(X)$ (here we used the singular version of Corollary 2.1.6), and we are done. Now, suppose that X contains a prime exceptional divisor E . Furthermore, suppose by contradiction that $\overline{\text{Eff}}(X)$ contains a circular part \mathcal{C} . One can show that $\overline{\text{Mov}}(X)$ is locally rational polyhedral away from the boundary of $\overline{\text{Pos}}(X)$ (more precisely, one only needs to adapt Corollary 2.2.8 to the singular setting). Hence, \mathcal{C} must be contained in $\partial\overline{\text{Mov}}(X) \cap \partial\overline{\text{Pos}}(X)$. Let x be a point lying in $E^\perp \cap \partial\overline{\text{Pos}}(X)$. Any prime exceptional divisor stays (up to a sign) stably exceptional under the action of the monodromy group $\text{Mon}_{\text{Hdg}}^{2,\text{lt}}(X)$ ([LMP22, Proposition 5.3, item (2)]). The image of $\text{Mon}_{\text{Hdg}}^{2,\text{lt}}(X)$ in $O^+(N^1(X))$ (which we denote by Γ') is of finite index by [LMP22, Lemma 6.3] and [BL22, Theorem 1.2, item (1)]. Moreover, the signature of the singular version of the BBF quadratic form on $N^1(X)$ is $(1, \rho(X) - 1)$. Then, $\mathbf{P}(\overline{\text{Pos}}(X))/\Gamma'$ is a hyperbolic manifold of finite volume (argue as in [AV16a, Section 3.2]). We obtain a contradiction arguing as in the smooth case and using the characterisation of $\text{int}(\overline{\text{Mov}}(X))$ provided in [LMP22, Proposition 5.9]. To show the equality $\overline{\text{Eff}}(X) = \sum_E \mathbf{R}^{\geq 0}[E]$, we argue as in the smooth case. ■

As a consequence of the above, we also obtain a more general statement for Corollary 0.2.9.

Corollary 3.1.15: Let X be a projective \mathbf{Q} -factorial primitive symplectic variety with non-terminal singularities of Picard number at least 2. Then X carries a prime exceptional (hence uniruled) divisor.

Proof:

Let $X' \rightarrow X$ be a \mathbf{Q} -factorial terminalisation of X . Then $\rho(X') \geq 3$, and by Theorem 3.1.14 X' carries at least $\rho(X') \geq 3$ prime exceptional divisors. To conclude the proof one argues as in the proof of Corollary 0.2.9. ■

3.2 Producing effective divisors

In this section, we show how to construct explicit effective integral divisors with some fixed monodromy invariants. We first need the following technical lemma, which is an adaptation of [Kov13, Lemma 3.1] to our case.

Lemma 3.2.1: Let X be a projective IHS manifold, D an integral divisor such that $[D] \in \overline{\text{Pos}}(X)$ and E an integral divisor such that $0 \neq [E] \in \partial\overline{\text{Eff}}(X)$. Let $t := \text{div}(E)$ be the divisibility of E . Consider the 2-plane $\pi := \langle [E], [D] \rangle \subset N^1(X)_{\mathbf{R}}$.

- (a) If $q_X(E) = 0$ there exists a pseudo-effective class $\alpha \in \pi \cap \overline{\text{Eff}(X)}$ represented by an integral divisor such that $q_X(\alpha) = 0$ and α and $[E]$ are on opposite sides of $\mathbf{R}^{\geq 0}[D]$.
- (b) If $q_X(E) < 0$ there exists an integral class α , such that $q_X(\alpha) = 0$ or $q_X(\alpha) = q_X(E)$ and α and $[E]$ are on opposite sides of $\mathbf{R}^{\geq 0}[D]$. Furthermore, only a positive multiple of the divisor defining α can be effective.

Proof:

Set $d = q_X(D)$, $bt = q_X(E, D)$ and $te = q_X(E)$. Note that $bt > 0$, for example by [MY15, Lemma 3.1].

(a) Let $\alpha = x[D] - y[E]$ be an element of π , for x, y real numbers. Then in this case $q_X(\alpha) = dx^2 - 2btxy$, and by choosing $x = 2bt$ and $y = d$, we obtain $\alpha = 2bt[D] - d[E]$ satisfying $q_X(\alpha) = 0$. Also, $q_X(\alpha, D) = btd > 0$, hence, by [MY15, Lemma 3.1], α belongs to $\overline{\text{Pos}(X)}$ and so is pseudo-effective. As $x[D] = \alpha + y[E]$, clearly α and $[E]$ are on opposite sides of $\mathbf{R}^{\geq 0}[D]$.

(b) Let $\alpha = tx[D] - y[E]$ be an element of π , for x, y real numbers. Then

$$q_X(\alpha) = t^2x^2d + y^2et - 2xyt^2b = te \left(\frac{tx^2d}{e} + y^2 - 2xyt \frac{b}{e} \right).$$

If we set $x' = y - \frac{xtb}{e}$, $y' = -\frac{x}{e}$ and $N = t^2b^2 - tde$, we can rewrite $q_X(\alpha)$ as

$$q_X(\alpha) = te \left[\left(y - \frac{xtb}{e} \right)^2 + tde \frac{x^2}{e^2} - t^2b^2 \frac{x^2}{e^2} \right] = te[(x')^2 - N(y')^2]. \quad (3.1)$$

If N is a square, then $q_X(\alpha) = te(x' - \sqrt{N}y')(x' + \sqrt{N}y')$. Choosing $x' = N$, $y' = \sqrt{N}$, i.e. $x = -e\sqrt{N}$, $y = N - \sqrt{N}tb$, we obtain the element $\alpha = -te\sqrt{N}[D] - (N - \sqrt{N}tb)[E]$ satisfying $q_X(\alpha) = 0$. We observe that $N - \sqrt{N}tb > 0$. If $b = 0$ or $b < 0$ this is trivial. If $b > 0$ and we assume $N - \sqrt{N}tb \leq 0$, we would have $N^2 - N(tb)^2 \leq 0$, which would imply $N - (tb)^2 \leq 0$ and this is a contradiction, because we have $N - (tb)^2 = -tde > 0$. Also, we note that the chosen α belongs to $\overline{\text{Pos}(X)}$. Indeed, it suffices to pick an element $\beta \in \overline{\text{Mov}(X)} \cap \text{Pos}(X) \cap [E]^\perp$ and to observe that $q_X(\beta, \alpha) = -\sqrt{N}teq_X([D], \beta) > 0$. For instance, let D' be an integral divisor whose class lies in $\text{int}(\overline{\text{Mov}(X)})$. One can choose $\beta := [D'] - \frac{q_X(D', E)}{q_X(E)}[E]$. Clearly $q_X(\beta, E) = 0$ and

$$q_X(\beta) = \frac{q_X(D')q_X(E) - q_X(D', E)^2}{q_X(E)} > 0$$

(for the latter inequality see for example Proposition 2.1.6). Clearly, also in this case α and $[E]$ are on opposite sides of $\mathbf{R}^{\geq 0}[D]$.

If N is not a square, the Pell equation

$$x'^2 - Ny'^2 = 1 \quad (3.2)$$

has infinitely many integer solutions (see [IR90], Pell's equation 17.5.2) and we may choose a

solution (x', y') of positive integers. Arguing as above we obtain a class

$$\alpha = -tey'[D] - (x' - tby')[E] \quad (3.3)$$

satisfying $q_X(\alpha) = te$. Also in this case we have $x' - tby' > 0$. Indeed, if $b = 0$ or $b < 0$ this is trivial. If $b > 0$ and we assume $x' - tby' \leq 0$, we would have $x'^2 - (tb)^2 y'^2 = 1 - tdey'^2 \leq 0$, which is clearly a contradiction. Also, as above, α and $[E]$ are on opposite sides of $\mathbf{R}^{\geq 0}[D]$. Now, suppose that a multiple of the divisor defining α (namely $D' := -tey'D - (x' - tby')E$) is effective. By contradiction, without loss of generality, we can assume that $-D'$ is effective. Then $-D' - tey'D = (x' - tby')E$ is big and this is clearly a contradiction because the latter belongs to the boundary of $\text{Big}(X)$. It follows that only a positive multiple of D' can be effective. ■

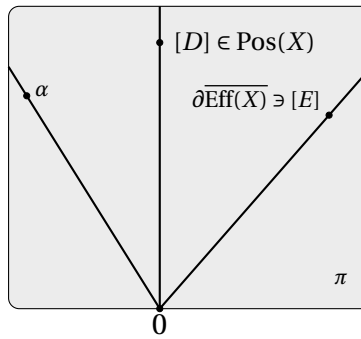


Figure 3.2: Picturing Lemma 3.2.1

Proof of Proposition 0.2.3:

With the notation of Lemma 3.2.1, suppose that $[E]$ is primitive in $H^2(X, \mathbf{Z})$. Then, we show that up to choosing a suitable solution for the Pell equation (3.2), the class α constructed in item (b) of Lemma 3.2.1 is primitive and effective. We adopt the notation of Lemma 3.2.1. It is natural to split the proof by distinguishing the deformation type.

- Suppose that X is of OG10-type. By [MO22, Proposition 3.1] we can only have $q_X(E) = -2$ and $\text{div}(E) = 1$ or $q_X(E) = -6$ and $\text{div}(E) = 3$. In the first case, the class α we get is of BBF square -2 and this implies that α is primitive. Furthermore, its divisibility must be one, because by Remark 1.2.1 $\text{div}(\alpha)$ divides $|A_X|$, and the discriminant group of X is $A_X \cong \mathbf{Z}/3\mathbf{Z}$ (the reader is referred to [Rap06] for the computation of the discriminant group of the 4 known deformation classes). It follows by [MO22, Proposition 5.4] and item (b) of Lemma 3.2.1 that the class α is stably exceptional, hence effective. In the second case, we have $q_X(\alpha) = -6$, and this again implies the primitivity of α . Also, $\alpha = 6y'[D] - (x' - 3by')[E]$, and for any $\gamma \in H^2(X, \mathbf{Z})$ we have $q_X(\alpha, \gamma) \equiv 0 \pmod{3}$. But $|A_X| = 3$, hence $\text{div}(\alpha) = 3$. Using again [MO22, Proposition 5.4] and item (b) of Lemma 3.2.1, we conclude that in this case, the class α is effective too.

- Suppose that X is of OG6-type. By [MR20, Lemma 6.4] it can only be $q_X(E) = -4$ and $\text{div}(E) = 2$ or $q_X(E) = -2$ and $\text{div}(E) = 2$. In both cases, the class α must be primitive. Furthermore, in the first case we have $\alpha = 4y'[D] - (x' - 2by')[E]$, so that $q_X(\alpha, \gamma) \equiv 0 \pmod{2}$, where γ is any

element of $H^2(X, \mathbf{Z})$. It follows that $\operatorname{div}(\alpha) \geq 2$, and as $A_X \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$, by Remark 1.2.1, we conclude that $\operatorname{div}(\alpha) = 2$. The same argument proves that also in the second case, the divisibility of α is 2. By [MR20, Proposition 5.4] and item (b) of Lemma 3.2.1 we conclude that the class α is effective. Notice that until now we have not needed the primitivity of $[D]$.

• Suppose that X is of $K3^{[n]}$ -type. By [Mar09, Theorem 1.12] we can have $\operatorname{div}(E) = 2(n-1)$ or $\operatorname{div}(E) = n-1$, and in both cases the BBF square of E is $-2(n-1)$.

First, suppose $\operatorname{div}(E) = 2(n-1)$. We observe that $\operatorname{div}(\alpha) = 2(n-1)$, because $\operatorname{div}(\alpha) \geq 2(n-1)$, and $q_X(\alpha) = 2(n-1)$. In this case, we have

$$\alpha = 2(n-1)y'[D] - [x' - 2(n-1)by'][E].$$

Now, we would like to find a solution (x', y') of the equation (3.2) satisfying $x' - 2(n-1)by' \equiv 1 \pmod{2(n-1)}$. We observe that $x' - 2(n-1)by' \equiv x' \pmod{2(n-1)}$, for any solution (x', y') , hence it suffices to find a solution such that $x' \equiv 1 \pmod{2(n-1)}$. Let (x_1, y_1) be the fundamental solution of the equation (3.2). The "second" solution of the equation is $x_2 = x_1^2 + Ny_1^2$, $y_2 = 2x_1y_1$, where in this case $N = 4(n-1)^2b^2 - 2(n-1)d$. Clearly $x_1^2 \equiv 1 \pmod{2(n-1)}$, hence $x_2 \equiv 1 \pmod{2(n-1)}$, and (x_2, y_2) is a solution we were looking for. We claim that, with respect to this choice of the solution of the Pell equation, the class α is primitive and $\operatorname{div}(\alpha) = 2(n-1)$. Indeed, $[D]$ and $[E]$ are linearly independent in $H^2(X, \mathbf{R})$. We observe that $\operatorname{g.c.d.}(2(n-1), x_2 - 2(n-1)by_2) = 1$, because $x_2 - 2(n-1)by_2 \equiv 1 \pmod{2(n-1)}$, and $\operatorname{g.c.d.}(x_2, y_2) = 1$, as they satisfy $x_2^2 - Ny_2^2 = 1$. Furthermore

$$2(n-1)b \cdot y_2 + [x_2 - 2(n-1)by_2] = x_2,$$

which implies that $\operatorname{g.c.d.}(y_2, x_2 - 2(n-1)by_2)$ divides x_2 . Then $\operatorname{g.c.d.}(y_2, x_2 - 2(n-1)by_2) = 1$, because $\operatorname{g.c.d.}(x_2, y_2) = 1$, and this implies the primitivity of α .

We now prove the effectivity of α . Recall that $q_X(\alpha) = te[(x')^2 - N(y')^2]$ (see the equality (3.1)). To do so, first observe that $\left[\frac{\alpha}{2(n-1)} \right] = \left[-\frac{E}{2(n-1)} \right]$ in $A_X \cong \mathbf{Z}/2(n-1)\mathbf{Z}$, hence, using Lemma 1.2.2 and changing the sign of $-[E]$ with the reflection $R_E \in \operatorname{Mon}^2(X)$, we obtain an isometry $\iota \in O^+(H^2(X, \mathbf{Z}))$, sending $[E]$ to α , acting as -1 on A_X . Indeed, R_E acts as -1 on A_X , and the isometry given by Lemma 1.2.2 acts trivially on A_X . By [Mar10, Lemma 4.2], the isometry ι belongs to $\operatorname{Mon}^2(X)$, and by [Mar09, Proposition 9.16, item (3)], the monodromy orbit of α is determined by $\operatorname{div}(\alpha)$ and $rs(\alpha)$, hence $rs(\alpha) = rs([E])$ (see [Mar09] for the definition of the latter invariant). Hence α is an effective class, by item (2) of Lemma 3.2.1 and Markman's characterisation of stably exceptional classes on $K3^{[n]}$ -type IHS manifolds ([Mar11, Theorem 9.17]).

Now, suppose that $\operatorname{div}(E) = n-1$. We would like to find a solution (x', y') of the equation (3.2) satisfying $x' - (n-1)by' \equiv 1 \pmod{2(n-1)}$. Also in this case the "second" solution (x_2, y_2) of the equation (3.2) yields the conclusion. Indeed $N = (n-1)^2b^2 - 2(n-1)d$ in this case, and we have

$$x_2 = x_1^2 + Ny_1^2 \equiv 1 + 2Ny_1^2 \equiv 1 \pmod{2(n-1)}.$$

But y_2 is even, hence $x_2 - (n-1)by_2 \equiv 1 \pmod{2(n-1)}$. Now, we observe that with respect to the solution (x_2, y_2) of (3.2), we have $\operatorname{div}(\alpha) = n-1$. Indeed, $\operatorname{div}(\alpha) \geq n-1$, and we can have either

$\operatorname{div}(\alpha) = n - 1$, or $\operatorname{div}(\alpha) = 2(n - 1)$, because $q_X(\alpha) = 2(n - 1)$. If N is even, we have that x' is odd and y' is even, for any solution (x', y') . If N is odd, then we can have two possibilities: x_1 is odd and y_1 is even or x_1 is even and y_1 is odd. In any case, we will have that x_2 is odd and y_2 is even. By definition of divisibility, there exists an element $\gamma \in H^2(X, \mathbf{Z})$ such that $q_X(E, \gamma) = n - 1$, thus $q_X(\alpha, \gamma) \equiv -x_2(n - 1) \pmod{2(n - 1)}$. As x_2 is odd, $q_X(\alpha, \gamma)$ cannot be divided by $2(n - 1)$, hence $\operatorname{div}(\alpha) = n - 1$. Arguing as in the case of divisibility $2(n - 1)$, we see that α is primitive. To show that α is effective in this case, just note that $[\alpha/(n - 1)] = [-E/(n - 1)]$ in A_X , and, arguing as in the case of divisibility $2(n - 1)$, we obtain an isometry $\iota \in O^+(H^2(X, \mathbf{Z}))$, sending $[E]$ to α , and acting as -1 on A_X . This isometry is a monodromy operator, by [Mar10, Lemma 4.2]. By [Mar09, Proposition 9.16, item (3)], the monodromy orbit of α is determined by $\operatorname{div}(\alpha)$ and $rs(\alpha)$, hence $rs(\alpha) = rs([E])$. Again, by item (b) of Lemma 3.2.1 and [Mar09, Theorem 1.12], we conclude that the class α is effective.

- Suppose now that X is of Kum_n -type. By [Yos16, Proposition 5.4] we can have $\operatorname{div}(E) = 2(n + 1)$ or $\operatorname{div}(E) = n + 1$, and in both cases the BBF square of E is $2(n + 1)$. Recall that $A_X \cong \mathbf{Z}/2(n + 1)\mathbf{Z}$ in this case. The proof goes exactly as in the $K3^{[n]}$ -type case. Also in this case the "second" solution (x_2, y_2) of the equation (3.2) satisfies the congruence $x_2 - \operatorname{div}(E)by_2 \equiv 1 \pmod{2(n + 1)}$, hence $\alpha = 2(n + 1)[D] - [x_2 - \operatorname{div}(E)by_2][E]$ and E are such that $\left[\frac{\alpha}{\operatorname{div}(E)} \right] = \left[-\frac{E}{\operatorname{div}(E)} \right]$ in A_X . Arguing as in the $K3^{[n]}$ -type case, we conclude that $\operatorname{div}(\alpha) = \operatorname{div}(E)$ (if $\operatorname{div}(E) = 2(n + 1)$, this is true for any solution of (3.2), if $\operatorname{div}(E) = n + 1$, this is true if we choose the solution (x_2, y_2) of (3.2)), and α is primitive (with respect to the solution (x_2, y_2)). Again, arguing exactly as in the $K3^{[n]}$ -type case, we obtain an isometry $\iota \in O^+(H^2(X, \mathbf{Z}))$ of determinant -1 , acting as -1 on A_X , and sending $[E]$ to α . By Markman's and Mongardi's characterisation of $\operatorname{Mon}^2(X)$ ([Mar23, Theorem 1.4] and [Mon16, Theorem 2.3]), we conclude that the isometry ι lies in $\operatorname{Mon}^2(X)$. By Proposition 1.3.26, the monodromy orbit of α is determined by $\operatorname{div}(\alpha)$ and $rs(\alpha)$, hence $rs(\alpha) = rs([E])$. Then the class α is effective, by item (b) of Lemma 3.2.1 and Yoshioka's characterisation of the stably exceptional classes on Kum_n -type IHS manifolds ([Yos16, Proposition 5.4]). ■

We now show how the computations of this section allow us to prove Theorem 0.2.2 for the known deformation classes of IHS manifolds, without the use of hyperbolic geometry. The proof follows Kovács' and Huybrechts' strategies (see [Huy15, Section 8] for the proof of Kovács' result by Huybrechts).

Alternative proof of Theorem 0.2.2:

We show that $\overline{\operatorname{Eff}(X)}$ does not contain any circular part, the rest of the proof is the same as above. Let E be a prime exceptional divisor and D any integral divisor with class lying in $\operatorname{Pos}(X)$. Without loss of generality, we can assume that the classes $[E]$ and $[D]$ are primitive in $H^2(X, \mathbf{Z})$. Set $q_X(E) = te$, where $\operatorname{div}(E) = t$. Assume by contradiction that $\overline{\operatorname{Eff}(X)}$ has a circular part \mathcal{C} . By Theorem 1.3.13 and equality (1.5), \mathcal{C} must be contained in $\partial\overline{\operatorname{Mov}(X)} \cap \partial\overline{\operatorname{Eff}(X)}$. Also, $\overline{\operatorname{Mov}(X)}$ is locally polyhedral away from the boundary of $\operatorname{Big}(X)$ (see Corollary 2.2.8), so that we can assume $\mathcal{C} = \mathbf{R}^{\geq 0}\mathcal{C} \subset \partial\overline{\operatorname{Pos}(X)} \cap \partial\overline{\operatorname{Eff}(X)}$. Then there exists a neighbourhood U of \mathcal{C} in $\overline{\operatorname{Eff}(X)}$ such that $q_X(\beta) \geq 0$ for any $\beta \in U$. By Lemma 3.2.1 and the proof of Proposition 0.2.3, for any

primitive integral divisor D whose class lies in $\text{Pos}(X)$, on the opposite side of D (with respect to E) there exists a class α , represented by an integral divisor, being either pseudo-effective and of BBF square 0 or effective and of BBF square te . As $[D]$ approaches \mathcal{C} , we can only have $q_X(\alpha) = 0$, with α pseudo-effective. This means that \mathcal{C} contains an integral class α of BBF square 0. Now, consider the class $\alpha_\epsilon = (1 - \epsilon)\alpha + \epsilon[E]$, where $\epsilon > 0$ is rational. Clearly, for ϵ small enough α_ϵ is effective. Also we note that $q_X(\alpha, E) > 0$, as otherwise we would have $q_X(\alpha_\epsilon) = -2\epsilon^2 q_X(\alpha, E) + 2\epsilon q_X(\alpha, E) - 2\epsilon^2 < 0$, which would contradict $\alpha \in \overline{\partial\text{Eff}(X)} \cap \overline{\partial\text{Pos}(X)}$. As by assumption $\rho(X) \geq 3$, we can find an integral class α' of negative BBF square, lying in $(\mathbf{R}\beta \oplus \mathbf{R}[E])^\perp$. We now define

$$\alpha_k := -2k^2 q_X(\alpha') (q_X(\alpha, E))^3 \alpha - 2k (q_X(\alpha, E))^2 \alpha' + [E]. \quad (3.4)$$

An easy computation shows that $q_X(\alpha_k) = q_X(E) = te$ and that for $k \gg 0$ $q_X(\alpha_k, [A]) > 0$, where A is any ample divisor. We want to prove that for k large enough the classes α_k are effective. For this purpose, we need to distinguish the different deformation types. Notice that the classes α_k are primitive independently of the deformation type we have chosen. Indeed, $[E], \alpha$ and α' are linearly independent in $H^2(X, \mathbf{R})$, and the class $[E]$ is primitive, whence any α_k is primitive.

- Suppose that X is of OG10-type. As remarked in the proof of Proposition 0.2.3, by [MO22, Proposition 6.4], we have $q_X(E) = -2$ or $q_X(E) = -6$ and $\text{div}(E) = 3$. In the first case $q_X(\alpha_k) = -2$. By [MO22, Proposition 5.4], α_k is stably exceptional, hence effective, and we are done. In the second case, we have

$$\alpha_k = -54k^2 g^3 q_X(\alpha') \alpha - 18k g^2 \alpha' + [E],$$

where $q_X(\alpha, E) = 3g$, hence $\text{div}(\alpha_k) = 3$. If k is large enough, by [MO22, Proposition 5.4], α_k is a stably exceptional class. Hence, for k large enough, the classes $\{\alpha_k\}_k$ are effective.

- Suppose that X is of OG6-type. As said in the proof of Proposition 0.2.3 we have $q_X(E) = -2$ and $\text{div}(E) = -2$ or $q_X(E) = -4$ and $\text{div}(E) = 2$. Then, arguing as in the proof of Proposition 0.2.3, we conclude that in both cases the classes α_k have divisibility 2 and using [MR20, Proposition 5.4] we conclude that α_k is stably exceptional, hence effective for k large enough.

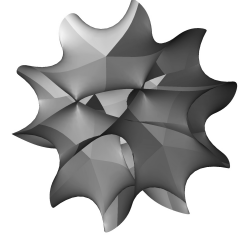
- Suppose that X is of $K3^{[n]}$ -type. We can have $\text{div}(E) = -2(n-1)$ or $\text{div}(E) = n-1$, and in any case $q_X(E) = -2(n-1)$, which implies $q_X(\alpha_k) = 2(n-1)$. As the coefficient of $[E]$ in α is 1, we conclude that $\text{div}(\alpha) = \text{div}(E)$. Further, arguing as in the proof of Proposition 0.2.3, we see that $[\alpha_k / \text{div}(\alpha_k)] = [E / \text{div}(E)]$ in A_X . Hence we obtain an isometry $\iota \in \widetilde{SO}^+(H^2(X, \mathbf{Z}))$ (so that ι belongs to $\text{Mon}^2(X)$) sending $[E]$ to α_k . By [Mar09, Proposition 9.16, item (3)], the monodromy orbit of α_k is determined by $\text{div}(\alpha_k)$ and $rs(\alpha_k)$, hence $rs(\alpha_k) = rs([E])$. As for k large enough we have $q_X(\alpha_k, [A]) > 0$, by [Mar11, Theorem 9.17] we conclude that the α_k are effective for k large enough.

- Suppose that X is of Kum_n -type. We can have $\text{div}(E) = -2(n+1)$ or $\text{div}(E) = n+1$, and in any case $q_X(E) = -2(n+1)$ ([Yos16, Proposition 5.4]), which implies $q_X(\alpha_k) = -2(n+1)$ (recall that the BBF square of α_k equals that of E). As for the $K3^{[n]}$ -type case, the classes α_k have divisibility $\text{div}(\alpha_k) = \text{div}(E)$. Again, $[\alpha_k / \text{div}(\alpha_k)] = [E / \text{div}(E)]$ in A_X , hence, by Lemma 1.2.2, we get an isometry $\iota \in \widetilde{SO}^+(H^2(X, \mathbf{Z}))$ sending $[E]$ to α_k . The isometry ι has determinant 1 and

acts trivially on A_X , hence, by Markman's and Mongardi's characterisation of $\text{Mon}^2(X)$, it lies in $\text{Mon}^2(X)$. By Proposition 1.3.26, the monodromy orbit of α_k is determined by $\text{div}(\alpha_k)$, $rs(\alpha_k)$, hence $rs(\alpha_k) = rs([E])$. We conclude that α_k is effective for k large enough, by [Yos16, Proposition 5.4].

Dividing both members of (3.4) by $2k^2$ we see that the sequence of rays $\{\mathbf{R}^{>0}\alpha_k\}_k$ converges to $\mathbf{R}^{>0}\alpha$ and this contradicts the circularity of $\overline{\text{Eff}(X)}$ at γ (in particular the fact that locally around γ in $\overline{\text{Eff}(X)}$ the BBF form is nonnegative). ■

Remark 3.2.2. We observe that, if in Proposition 0.2.3 we are in the first situation, using the classes α_k defined in equation (3.4), we can construct explicit effective divisors with the same monodromy invariants to those of a given prime exceptional divisor E .



4. Asymptotic base loci and cones of k -ample divisors

4.1 Characterisation of asymptotic base loci

This section is devoted to the study of the asymptotic base loci of big divisors on projective IHS manifolds.

Proposition 4.1.1: Let D be any big \mathbf{R} -divisor on X . Then $\mathbf{B}_+(D) = \mathbf{B}_+(P(D)) \cup \text{Supp}(N(D))$, and $\mathbf{B}_-(D) = \mathbf{B}_-(P(D)) \cup \text{Supp}(N(D))$.

Proof:

Write $D = A + N$, where A and N are ample and effective respectively. Then $P(D) = A + N'$, where $N' \leq N$ is effective (see Remark 1.3.19), and so $\text{Supp}(N') \subset \text{Supp}(N)$. This implies that $\mathbf{B}_+(P(D)) \subset \mathbf{B}_+(D)$. Moreover $\text{Supp}(N(D)) \subset \mathbf{B}_+(D)$, hence $\mathbf{B}_+(P(D)) \cup \text{Supp}(N(D)) \subset \mathbf{B}_+(D)$. On the other hand, writing $P(D) = A' + N'$, with A' and N' ample and effective respectively, we obtain that $D = A' + N' + N(D)$, hence $\mathbf{B}_+(D) \subset \mathbf{B}_+(P(D)) \cup \text{Supp}(N(D))$, and we are done. By definition,

$$\mathbf{B}(D) = \bigcap_{E \equiv D} \text{Supp}(E),$$

and we know by [BBP13, Theorem A] that, for any big divisor D , $\mathbf{B}_-(D) = \mathbf{B}(D)$. If we write $D = P(D) + N(D)$, the above equality becomes

$$\mathbf{B}_-(D) = \bigcap_{E \equiv D} \text{Supp}(P(E) + N(E)),$$

and for any effective divisor E which is numerically equivalent to D , we have $\text{Supp}(N(E)) = \text{Supp}(N(D))$ (see Remark 1.3.21), so that

$$\mathbf{B}_-(D) = \left(\bigcap_{E \equiv D} \text{Supp}(P(E)) \right) \cup \text{Supp}(N(D)).$$

If E' is an effective divisor numerically equivalent to $P(D)$, $E' + N(D) \equiv P(D) + N(D)$. On the other hand, if E is an effective divisor numerically equivalent to D , $P(E)$ is numerically equivalent to

$P(D)$ (see Remark 1.3.21). This implies that

$$\mathbf{B}_-(D) = \left(\bigcap_{E \equiv P(D)} \text{Supp}(E) \right) \cup \text{Supp}(N(D)).$$

But $\bigcap_{E \equiv P(D)} \text{Supp}(E) = \mathbf{B}(P(D)) = \mathbf{B}_-(P(D))$, and this concludes the proof. ■

In particular, to study the asymptotic base loci of big divisors on projective IHS manifolds, we can restrict ourselves to big, movable divisors.

Proposition 4.1.2: Let D be a big \mathbf{R} -divisor. Then

$$\mathbf{B}_+(D) = \mathbf{B}_+(P(D))$$

and the divisorial part $\mathbf{B}_+(D)_{\text{div}}$ of $\mathbf{B}_+(D)$ is $\text{Null}_{q_X}(P(D))$.

Proof:

By Proposition 4.1.1, as $\text{Supp}(N(D)) \subset \text{Null}_{q_X}(P(D))$, we only need to check that

$$\mathbf{B}_+(P(D))_{\text{div}} = \text{Null}_{q_X}(P(D)).$$

Suppose $E \notin \mathbf{B}_+(P(D))$, then $q_X(P(D), E) > 0$. Indeed,

$$\mathbf{B}_+(D) := \bigcap_{\substack{D=A+N \\ A \text{ ample} \\ N \text{ effective}}} \text{Supp}(N),$$

hence there exists a decomposition $D = A + N$ such that $E \notin \text{Supp}(N)$. As $q_X(A, E) > 0$ ([Huy99, p. 1.11]) and q_X is an intersection product we are done. This implies that if E is any prime divisor satisfying $q_X(P(D), E) = 0$, then $E \in \mathbf{B}_+(P(D))$, so that $\mathbf{B}_+(P(D)) \supset \text{Null}_{q_X}(P(D))$. Now, let E be any prime divisor lying in $\mathbf{B}_+(P(D))$. By [MZ13, Theorem 1.2] there exists a birational model X' of X and a birational map $f: X \dashrightarrow X'$ such that $f_*(P(D))$ is big and nef. Furthermore, arguing as in [BBP13, Proof of Theorem A], we obtain that the strict transform E' of E via f is an irreducible component of $\mathbf{B}_+(f_*P(D))$. But then, by [ELMNP06, Theorem 5.2, item (b)] and [ELMNP06, Example 5.5], we have $(f_*P(D))^{2n-1} \cdot E' = 0$, and we have ([GHJ03, Exercise 23.2])

$$q_X(f_*P(D), E') \int (f_*P(D))^{2n} = q_X(f_*P(D)) \int (f_*P(D))^{2n-1} \cdot E' = 0,$$

which forces $q_X(f_*P(D), E') = 0$, because $f_*(P(D))$ is big and nef, and hence

$$\int (f_*P(D))^{2n} > 0.$$

But then also $q_X(P(D), E) = 0$, because f_* is an isometry (with respect to the BBF forms), hence we conclude that $E \in \text{Null}_{q_X}(P(D))$. ■

Remark 4.1.3. We observe that if D lies in $\text{int}(\text{Mov}(X))$, then $\mathbf{B}_+(D)$ does not contain divisorial

irreducible components. Indeed, by assumption $q_X(D, E) > 0$ for any prime exceptional divisor E , hence $\mathbf{B}_+(D)$ does not have any divisorial irreducible component, by the above proposition.

Definition 4.1.4: Let D be any big \mathbf{R} -divisor, and fix a log MMP for the pair $(X, \epsilon P(D))$, with $0 < \epsilon \ll 1$. We define $\text{MBM}(D)$ to be the set of rational curves in $\text{Cont}(X)$ which are flipped while running the chosen log MMP for $(X, \epsilon P(D))$, or contracted at the end of the chosen log MMP for $(X, \epsilon P(D))$.

Remark 4.1.5. Let us explain Definition 4.1.4. Given any big \mathbf{R} -divisor D on X , a curve $C \in \text{MBM}(D)$ satisfies one of the two following possibilities:

- There exists a birational map $f: X \dashrightarrow X'$, arising from the chosen log MMP for the pair $(X, \epsilon P(D))$, with $C \not\subset \text{Ind}(f)$, such that $C' = f_*(C)$ (the strict transform of C via f) is flipped at the next step of the chosen log MMP.

- The curve C "survives" until the end of the chosen log MMP. Let $f: X \dashrightarrow X'$ be the model on which $f_*(P(D)) = P(D')$ (where $D' = f_*(D)$) is nef (i.e. where the chosen log MMP terminates). In this case $f_*(C) = C'$ is contracted by the morphism induced by a big and nef integral divisor class lying in the relative interior of the minimal (with respect to the dimension) extremal face of the nef cone of X' containing the class of $P(D')$. Clearly, in this case $P(D')$ is not ample.

A priori, the elements of $\text{MBM}(D)$ could depend on the log MMP we have chosen and any time we write $\text{MBM}(D)$, this has to be thought of with respect to a fixed log MMP for $(X, \epsilon P(D))$.

Remark 4.1.6. Let D be any big Cartier \mathbf{R} -divisor on any normal complex projective variety Y . We observe that if C is an irreducible curve with $C \not\subset \mathbf{B}_+(D)$ then $D \cdot C > 0$. Indeed, by definition of $\mathbf{B}_+(D)$, we can write $P = A + E$, where A is Cartier and ample, E is Cartier and effective, and $C \not\subset \text{Supp}(E)$. It follows that if $D \cdot C \leq 0$ for some irreducible curve C , $C \subset \mathbf{B}_+(D)$. Note that if C is a curve lying in $\mathbf{B}_+(D)$, it is not true that $D \cdot C \leq 0$, in general, also when $[D] \in \text{int}(\overline{\text{Mov}(X)})$ (see Subsection 4.5.2 for a counterexample).

The following proposition is a consequence of [MZ13, Theorem 1.2], [BBP13, Proof of Theorem A], and [Kaw91, Theorem 1]. It gives a recipe to compute the augmented base locus of any big divisor on a projective IHS manifold. The main limitation of this method is that to compute the augmented base locus of a big divisor one must know how to run a log MMP for the positive part of the divisor.

Proposition 4.1.7: Let D be any big \mathbf{R} -divisor on a projective IHS manifold X , then

$$\mathbf{B}_+(D) = \bigcup_{C \in \text{MBM}(P(D))} \text{Supp}(C).$$

Proof:

Since $\mathbf{B}_+(D) = \mathbf{B}_+(P(D))$, we can assume that D is movable (i.e. its class lies in $\text{Mov}(X)$).

By running the log MMP associated with $\text{MBM}(D)$, for the pair $(X, \varepsilon D)$, for sufficiently small $\varepsilon \in \mathbf{Q}$, we obtain a finite sequence of log flips (cf. [MZ13, Theorem 1.2])

$$\begin{array}{ccccccc}
 X_0 = X & \xrightarrow{\varphi_0} & X_1 & \xrightarrow{\varphi_1} & X_2 & \xrightarrow{\dots} & X_{n-1} & \xrightarrow{\varphi_{n-1}} & X_n \\
 \swarrow \pi_0^- & & \swarrow \pi_1^- & & \swarrow \pi_{n-1}^- & & \swarrow \pi_{n-1}^- & & \swarrow \pi_{n-1}^- \\
 & & Z_0 & & Z_1 & & & & Z_{n-1} \\
 & & \nwarrow \pi_0^+ & & \nwarrow \pi_1^+ & & \nwarrow \pi_{n-1}^+ & & \nwarrow \pi_{n-1}^+
 \end{array} \tag{4.1}$$

such that, if we set $D_0 = D$, $D_i := \varphi_{i-1,*} D_{i-1}$, and

$$\varphi := \varphi_{n-1} \circ \dots \circ \varphi_0,$$

for any $i = 1, \dots, n$, $\varphi_*(D)$ is big and nef on X_n . Without loss of generality, we may assume that D_n is not ample. So, let $f: X_n \rightarrow Y$ be the birational morphism contracting the extremal face $D_n^\perp \cap \overline{\text{NE}}(X)$ of the Mori cone $\overline{\text{NE}}(X)$ ([KM98, Theorem 3.7]). Now, choose a point x_0 in $\mathbf{B}_+(D)$, and define recursively $x_i := \varphi_{i-1}(x_{i-1})$ (whenever we can do it), for any $i = 1, \dots, n-1$. Suppose that $x_0 \in \text{Exc}(\pi_0^-)$. Then, by [Kaw91, Theorem 1] there exists a rational curve C passing through x_0 that is contracted by π_0 . Clearly $D \cdot C < 0$, hence $C \subset \text{MBM}(D)$.

If $x_0 \notin \text{Exc}(\pi_0^-)$, let V_0 be an irreducible component of $\mathbf{B}_+(D)$ containing it. Arguing as in [BBP13, Proof of Theorem A], we see that the strict transform V_1 of V_0 via φ_0 is an irreducible component of $\mathbf{B}_+(D_1)$. If $x_1 \in \text{Exc}(\pi_1^-)$, arguing as above, we find a rational curve C_1 on X_1 containing x_1 that is contracted by π_1^- . Moreover, we have $D_1 \cdot C_1 < 0$. The strict transform C of this curve via φ_0^{-1} is a rational curve passing through x_0 which belongs to $\mathbf{B}_+(D)$. It is important to point out that it could hold $D \cdot C \geq 0$, even though $D_1 \cdot C_1 < 0$. If $x_1 \notin \text{Exc}(\pi_1^-)$, the strict transform V_2 of V_1 via φ_1 is an irreducible component of $\mathbf{B}_+(D_2)$ and we can continue like this for finitely many times. If it never happens that $x_i \in \text{Exc}(\pi_i^-)$, for any $i = 1, \dots, n-1$, then $x_n \in \text{Exc}(f) = \mathbf{B}_+(D_n)$, and D_n is nef. The face $D_n^\perp \cap \overline{\text{NE}}(X_n)$ which is $(D_n - \varepsilon A)$ -negative (and $D_n - \varepsilon A$ is big for $\varepsilon \ll 0$), is contracted by f . Now, we choose $0 < \varepsilon \ll 1$ such that $D - \varepsilon A$ is a big \mathbf{Q} -divisor. Using the Kleiman's criterion for f -ampleness (cf. [KM98, Theorem 1.44]), we see that $-(D_n - \varepsilon A)$ is f -ample. Up to a rescaling, we can assume that $(X, D_n - \varepsilon A)$ is klt. Then, using again [Kaw91, Theorem 1], we conclude that through x_n passes a rational curve C_n which is contracted by f , and $D_n \cdot C_n = 0$. We conclude that the strict transform C of C_n via φ^{-1} is a rational curve passing through x_0 , contained in $\mathbf{B}_+(D)$. This shows that $\mathbf{B}_+(D) \subset \text{MBM}(D)$.

On the other hand, let C be a curve lying in $\text{MBM}(D)$. Suppose that at a certain point the strict transform C_i of C via φ_{i-1} is contracted by π_i^- . Then $C_i \subset \mathbf{B}_+(D_i)$. Let V_i be the irreducible component of $\mathbf{B}_+(D_i)$ containing C_i . Arguing as in [BBP13, Proof of Theorem A], we conclude that the strict transform V of V_i via $(\varphi_{i-1} \circ \dots \circ \varphi_1 \circ \varphi)^{-1}$ is an irreducible component of $\mathbf{B}_+(D)$, and hence $C \subset \mathbf{B}_+(D)$. Otherwise, $D_n \cdot C_n = 0$, because D_n is nef and C_n can be contracted. But then $C_n \subset \mathbf{B}_+(D_n)$, by Remark 4.1.6, and arguing once again as in [BBP13, Theorem A], we obtain that $C \subset \mathbf{B}_+(D)$ and $\text{MBM}(D) \subset \mathbf{B}_+(D)$, and we are done. ■

Remark 4.1.8. Let D be any big \mathbf{R} -divisor on X . It is important to point out that if an irreducible component of $\mathbf{B}_+(D)$, or $\mathbf{B}_-(D)$ can be contracted on X , it is covered by a family of rational curves

which are contracted by the morphism contracting the component, by [Kaw91, Theorem 1].

Remark 4.1.9. It is worth observing that, for an integral divisor D , Proposition 4.1.7 is consistent with [BCL14, Theorem A]. Indeed, adopting the same notation of Proposition 4.1.7, every φ_i is a birational map which is an isomorphism away from its indeterminacy locus (see for example [Huy99, 4.4-(i)]). Moreover $\text{Ind}(\varphi_i) = \text{Exc}(\pi_i^-)$, for any $i = 1, \dots, n-1$. Since $\text{codim}(\text{Ind}(\varphi_i)) \geq 2$ we have that

$$H^0(X, D_0) \cong H^0(X, D_i) \quad (4.2)$$

for any D_i . Moreover, as D_n is big and nef, mD_n is globally generated for any integer $m \gg 0$, by the base point free theorem.

Let ψ be the morphism induced by $|mD_n|$, for m large enough. Then, the map $\psi \circ \varphi$ is induced by mD , for any $m \gg 0$. Let $f_{|mD|}$ be the map to some projective space induced by $|mD|$, for any $m \gg 0$. By [BCL14, Theorem 1], $\mathbf{B}_+(D)$ coincides with the locus where $f_{|mD|}$ is not an isomorphism. But $f_{|mD|} = \psi \circ \varphi$, because of 4.2, and in Proposition 4.1.7 we proved that $\mathbf{B}_+(D) = \text{MBM}(D)$, which is exactly the locus where $\psi \circ \varphi$ (and so $f_{|mD|}$) is not an isomorphism.

Corollary 4.1.10: Let D be any big \mathbf{R} -divisor on X . Then, the divisorial part of $\mathbf{B}_-(D)$ consists of the prime exceptional divisors supporting the negative part of D , while the higher codimension irreducible components of $\mathbf{B}_-(D)$ are union of curves in $\text{MBM}(P(D))$ which at some step of the log MMP associated with $\text{MBM}(P(D))$ get flipped.

Proof:

The divisorial irreducible components of $\mathbf{B}_-(D)$ are the prime exceptional divisors supporting $N(D)$ by [KL17a, Theorem 4.1], for example. Now, by Proposition 4.1.1, we can assume that D is movable (i.e. $N(D) = 0$). Let $f: X \dashrightarrow X'$ be the birational map coming from the log MMP associated with $\text{MBM}(D)$. If D is rational, we are done, by the proof of Proposition 4.1.7. If D is irrational, we can find rational, big and nef divisors D'_1, \dots, D'_k on X' , such that $f_*(D) = \sum_i a_i D'_i$ (with $a_i > 0$ for any i), because $\text{Nef}(X')$ is locally rational polyhedral in the big cone. If we set $D_i = f^*(D'_i)$, we have $D = \sum_i a_i D_i$, so that $\mathbf{B}_-(D_i) = \text{Ind}(f)$, for any i . Clearly, $\text{Ind}(f) \subset \mathbf{B}_-(D)$. On the other hand $\mathbf{B}_-(D) \subset \cup_i \mathbf{B}_-(D_i) = \text{Ind}(f)$, and we are done. ■

Proposition 4.1.11: Let D be any big \mathbf{R} -divisor on X , and suppose $\rho(X) = 2$. Then, either the irreducible components of $\mathbf{B}_+(D)$ are all divisorial, or they are all non-divisorial. Moreover, if $\mathbf{B}_+(D)$ is divisorial, it is irreducible.

Proof:

As $\mathbf{B}_+(D) = \mathbf{B}_+(P(D))$, we can assume that D is movable. Write D as $D = A - B$, where A and B are ample divisors on X . We can consider for any $\lambda \in [0, 1]$ the divisor $D_\lambda = A - \lambda B$. If $0 \leq \lambda_1 \leq \lambda_2 \leq 1$, $\mathbf{B}_+(D_{\lambda_1}) \subset \mathbf{B}_+(D_{\lambda_2})$. Suppose E is a prime divisor lying in $\mathbf{B}_+(D)$. Then, there exists $\lambda_1 \in [0, 1]$ such that $q_X(P(D_{\lambda_1}), E) = 0$. Suppose that for some $\lambda_2 \geq \lambda_1$ another component V joins. Up to going on a birational model of X , we can assume that V is covered by a family of rational curves

$\{C_t\}_t$, such that $P(D_{\lambda_2}) \cdot C_t = 0$, for any t . But we also have $q_X(P(D_{\lambda_2}), E) = 0$. This implies that the divisor dual to any of the curves C_t is a positive multiple of E because otherwise, C_t would intersect negatively any ample divisor. Thus $E \cdot C_t < 0$, and hence $C_t \subset E$. We conclude that $V \subset E$, thus $V = E$, because V was assumed to be an irreducible component of $\mathbf{B}_+(D)$, and hence $\mathbf{B}_+(D)$ is irreducible. On the other hand, if $\mathbf{B}_+(D)$ has a non-divisorial irreducible component, all the irreducible components must be non-divisorial, because otherwise, repeating a similar argument to the one above, all the irreducible components would be contained in a divisorial irreducible component of $\mathbf{B}_+(D)$, and this would give a contradiction. ■

4.1.1 The asymptotic base loci are algebraically coisotropic

For the next results, we need to introduce the notion of *algebraically coisotropic subvarieties*, introduced by Voisin in [Voi16]. Let X be an IHS manifold of dimension $2n$ with symplectic form $\sigma \in H^{2,0}(X)$. Let $P \subset X$ be a subvariety of codimension d and P_{reg} its regular locus. Then, the restriction of σ to any $p \in P_{\text{reg}}$ factors as

$$\sigma|_{P,p} : T_{P,p} \longrightarrow T_{X,p} \xrightarrow{\cong} \Omega_{X,p} \longrightarrow \Omega_{P,p} \quad (4.3)$$

where the first and last maps are given by the inclusion and restriction respectively. We say that P is *coisotropic* if $\sigma|_{P,p}$ has rank $2n - 2d$ for every $p \in P_{\text{reg}}$; if the codimension of P is n , i.e. $\sigma|_{P_{\text{reg}}} = 0$, we say that P is *Lagrangian*.

Definition 4.1.12 ([Voi16, Definition 0.5]): A subvariety $P \subset X$ of codimension d is *algebraically coisotropic* if it is coisotropic and admits a rational map $\phi : P \dashrightarrow B$ onto a variety of dimension $2n - 2d$ such that $\sigma|_P = \phi^* \sigma_B$ for some $\sigma_B \in H^{2,0}(B)$.

This last notion has received considerable attention in recent years, due to its connections with the Chow group of projective IHS manifolds. Examples of algebraically coisotropic subvarieties for some of the known IHS manifolds were constructed in [Voi16] (see also [KLM15; Lin20]), and we will provide more. Recall that a symplectic resolution $\pi : X \rightarrow Z$ is a birational morphism from an IHS manifold X onto a normal variety Z .

Proposition 4.1.13 ([BL21, Proposition 4.12]): Every irreducible component P of the exceptional locus of a symplectic resolution $\pi : X \rightarrow Z$ is algebraically coisotropic and the coisotropic fibration of P is given by the restriction $\pi|_P : P \rightarrow B := \pi(P)$. Moreover, the general fibre of $\pi|_P$ is rationally connected.

From the above proposition, we deduce the following.

Corollary 4.1.14: Let D be a big \mathbf{R} -divisor on X . For any irreducible component P of $\mathbf{B}_+(D)$, $\mathbf{B}(D)$ or $\mathbf{B}_-(D)$ there exists a birational map $\phi : X \dashrightarrow X'$ such that $\phi_*(P)$ is an algebraically coisotropic variety. In particular, P has dimension at least $\dim(X)/2$.

Proof:

By [ELMP06, Lemma 1.14] and [BBP13, Proposition 2.8] it is enough to prove the Corollary for any irreducible component of $\mathbf{B}_+(D)$. But this follows from the fact that any irreducible component of $\mathbf{B}_+(D)$ is contractible on some birational model of X , and from Proposition 4.1.13. ■

As a refinement of the so far conjectural Bloch-Beilinson filtration in the Chow group of an IHS manifold, Voisin introduced the following filtration.

Definition 4.1.15 ([Voi16, Definition 0.2]): Let X be a projective IHS manifold. We define $S_i X \subset X$ to be the set of points in X whose orbit under rational equivalence has dimension $\geq i$. The filtration S_\bullet is then defined by letting $S_i \text{CH}_0(X)$ be the subgroup of $\text{CH}_0(X)$ generated by classes of points $x \in S_i X$.

Following Huybrechts [Huy14], we say that a subvariety $Z \subset X$ such that all points of Z are rationally equivalent in X is a *constant cycle subvariety*. Proposition 4.1.13 implies that the fibres of the coisotropic fibration of P are rationally connected, in particular, they are constant cycle subvarieties. The following clarifies the connection between constant cycle subvarieties and algebraically coisotropic varieties.

Theorem 4.1.16 ([Voi16, Theorem 1.3]): Let Z be a codimension i subvariety of a projective IHS manifold X . Assume that any point of Z has an orbit of dimension $\geq i$ under rational equivalence in X (that is $Z \subset S_i X$). Then Z is algebraically coisotropic and the fibres of the isotropic fibration are i -dimensional orbits of X for rational equivalence.

Proposition 4.1.17: Let D be a big \mathbf{R} -divisor on X . For any irreducible component P of $\mathbf{B}_+(D)$, $\mathbf{B}(D)$ or $\mathbf{B}_-(D)$ of codimension i we have that $P \subset S_i X$. In particular, P is algebraically coisotropic.

Proof:

As in Corollary 4.1.14 it is enough to prove the statement for an irreducible component $P \subset \mathbf{B}_+(D)$ of codimension i . There exists a birational map $\phi : X \dashrightarrow X'$ that induces a birational map $\phi|_P : P \dashrightarrow P'$. The proof of Proposition 4.1.13 implies that $P' \subset S_i X'$. Therefore we need to prove that this is the case for P . Notice that the contraction map $\pi|_{P'} : P' \rightarrow B'$ is the *MRC fibration* of P' (see [Kol96, Section IV.5], or [GHS03], or [Voi, Section 3] for properties of the MRC fibration). Therefore, by restricting to a possible smaller open subset we can assume that the rational map $\phi : P \dashrightarrow B'$ induced by composition is the MRC fibration of P as well. In particular, since ϕ is almost holomorphic, we can assume that there exists an open subset $U \subset P$ such that $\phi|_U : U \rightarrow B'$

is flat and the fibres are rationally connected subvarieties of P . Fix an ample class on P , then the fibres of the map $\phi|_U : U \rightarrow B'$ have the same Hilbert polynomial with respect to the fixed ample class. The relative Hilbert scheme (or the Kontsevich space of stable maps) is proper [Kol96], hence for every point $p \in P$ there exists a limit of the rationally connected varieties $\phi^{-1}(b)$ containing p . Since the limits of rationally connected varieties are rationally chain connected, we obtain that the orbit under rational equivalence in P is of dimension $\geq i$, therefore $P \subset S_i X$. We conclude that the variety is algebraically coisotropic by Theorem 4.1.1. ■

We conclude this section by proving the first three items of Theorem 0.3.1.

Proof of Theorem 0.3.1, item (1):

The irreducible components of $\mathbf{B}_+(D), \mathbf{B}_-(D) = \mathbf{B}(D)$ are algebraically coisotropic by Proposition 4.1.17. The bound from below for the dimension of any such irreducible component follows from the definition of algebraically coisotropic subvariety. The statement about the divisorial irreducible components of $\mathbf{B}_+(D)$ (resp. $\mathbf{B}_-(D)$) is proven in Proposition 4.1.2 (resp. Corollary 4.1.10). The rest directly follows from Proposition 4.1.7 and Corollary 4.1.10. This concludes the proof of the first three items of Theorem 0.3.1. ■

4.2 Stable classes and destabilising numbers

In this section, we describe the parts of the big cone of a projective IHS manifold where the augmented base loci stay constant.

Remark 4.2.1. By [BBP13, Theorem A], a big divisor D on X is unstable if and only if $\mathbf{B}(D) \subsetneq \mathbf{B}_+(D)$.

Proposition 4.2.2: A big, movable divisor D on X is unstable if and only if there exist a birational hyper-Kähler model X' of X , a birational map $f : X \dashrightarrow X'$, a rational curve $C \in \text{Cont}(X)$, with $C \not\subset \text{Ind}(f)$, such that $f_*(D)$ is nef on X' , and $f_*(D) \cdot f_*(C) = 0$.

Proof:

If the divisor D is unstable, then $\mathbf{B}(D) \subsetneq \mathbf{B}_+(D)$. Let p be a point lying in $\mathbf{B}_+(D)$ but not in $\mathbf{B}(D)$, and consider a log MMP

$$\begin{array}{c}
 \xrightarrow{\quad f \quad} \\
 X = X_1 \dashrightarrow^{\varphi_1} X_2 \dashrightarrow^{\varphi_2} \cdots \dashrightarrow^{\varphi_{k-1}} X_k
 \end{array} \tag{4.4}$$

for the pair $(X, \epsilon D)$. We know that $\mathbf{B}(D) = \text{Ind}(f)$, hence p does not belong to the indeterminacy locus of f . By Proposition 4.1.7 and Corollary 4.1.10, there exists a rational curve C passing through p , belonging to $\text{MBM}(D)$, which survives until the end of the log MMP we have chosen. In particular, we must have $f_*(D) \cdot f_*(C) = 0$, because, since C lies in $\text{MBM}(D)$, it must be contracted by the morphism induced by a certain line bundle, whose class lies in the relative interior of

the minimal extremal face of the nef cone of X_k containing $f_*(D)$. Now, suppose $f: X \dashrightarrow X'$ is a birational map of IHS manifolds and $C \in \text{Cont}(X)$ a rational curve with $C \not\subset \text{Ind}(f)$, such that $f_*(D)$ is nef on X' , and $f_*(D) \cdot f_*(C) = 0$, we want to show that $\mathbf{B}(D) \subsetneq \mathbf{B}_+(D)$, i.e. that D is unstable. The birational map f is a composition of log flips. For simplicity, let us keep the notation of diagram (4.4). Note that $\varphi_{i-1,*}(C)$ is not contained in the indeterminacy locus of φ_i , for any $i = 1, \dots, k-1$, and we set $\varphi_{0,*}(C) = C$. Again, the curve $\varphi_{k-1,*}(C)$ is contracted by the morphism induced by a certain line bundle, whose class lies in the relative interior of the minimal extremal face of the nef cone of X_k containing $f_*(D)$, because $f_*(D) \cdot f_*(C) = 0$. This implies that $C \in \text{MBM}(D)$ (here $\text{MBM}(D)$ is taken with respect to the log MMP in diagram (4.4)), and so, by Proposition 4.1.7, we have that $C \subset \mathbf{B}_+(D)$ and $C \not\subset \mathbf{B}(D) = \text{Ind}(f)$, hence D is unstable. ■

Proposition 4.2.3: Let λ be a destabilising number for a big, integral divisor D on X , with respect to an ample, integral divisor A . If $D - \lambda A$ is big, λ is rational.

Proof:

We know that $D - \lambda A$ is unstable if and only if $\mathbf{B}(D - \lambda A) \subsetneq \mathbf{B}_+(D - \lambda A)$. In particular, up to going on a birational model of X , if $D - \lambda A$ is unstable, there exists a rational curve C , such that $P(D - \lambda A) \cdot C = 0$, by Proposition 4.2.2. Then, λ is forced to be a rational number, and this concludes the proof. ■

We point out that, in general, the destabilising numbers are not rational numbers. An example where these numbers are irrational can be found in [BKS04], as well as the proof that the destabilising numbers on smooth projective surfaces are rational.

Remark 4.2.4. Note that, in the above proposition, the assumption on $D - \lambda A$ to be big cannot be dropped. Indeed, if λ is a destabilising number for $D - \lambda A$, the divisor $D - \lambda A$ is pseudo-effective but not big, and it lies on an irrational ray of $\overline{\text{Eff}}(X)$, λ must be irrational (because otherwise, $D - \lambda A$ would be rational).

We now put all the above together to prove the item (4) of Theorem 0.3.1.

Proof of Theorem 0.3.1, item (2):

The part concerning unstable divisors is proven in Proposition 4.2.2, while the part concerning the destabilising numbers is proven in Proposition 4.2.3. ■

The rest of this section is devoted to study how the asymptotic base loci of a big divisor vary when perturbing the divisor.

Definition 4.2.5: Let D be a stable big \mathbf{R} -divisor on X . The stability chamber of D is defined as

$$\text{SC}(D) := \{D' \in \text{Big}(X) \mid \mathbf{B}_+(D') = \mathbf{B}_+(D)\}.$$

Notice that by [ELMP06, Proposition 1.26] and [ELMP06, Proposition 1.24, item (iii)], it follows

that any stability chamber in $\text{Big}(X)$ has a non-empty interior, thus, being stable is an open condition.

We notice that, in general, the stability chambers are not convex subcones of the big cone. An example of this pathology is provided in [LMR20, Example 3.1] for Mori dream spaces. We provide an example of this pathology on a projective IHS manifold of $K3^{[2]}$ -type which is not a Mori dream space (see Subsection 4.5.2). This cannot happen on surfaces (hence in particular on $K3$ surfaces), because in that case, the augmented base loci stay constant in the interior of the Zariski chambers (see [BKS04, Section 1] for the definition of Zariski chamber, and [BKS04, Theorem 2.2] for the cited result), which are convex subcones of the big cone.

Definition 4.2.6: We say that a big divisor D on X is stable in codimension 1 if $\mathbf{B}_+(D)_{\text{div}} = \mathbf{B}_-(D)_{\text{div}}$.

Below is a corollary to Proposition 4.1.2.

Corollary 4.2.7: Let D be a big divisor on X . Then, D is stable in codimension 1 if and only if $\text{Null}_{q_X}(P(D)) = \text{Neg}_{q_X}(D)$, if and only if (the class of) D does not lie on the boundary of any Boucksom-Zariski chamber.

Proof:

This directly follows from Proposition 4.1.2 and Proposition 2.2.17. ■

Corollary 4.2.8: Let D be any big divisor on X . Then, D is unstable if and only if $\text{Neg}_{q_X}(D) \subsetneq \text{Null}_{q_X}(P(D))$, or $P(D)$ is unstable in codimension greater than or equal to 2 (i.e. $\mathbf{B}_+(P(D))$ has an irreducible component V of codimension greater than or equal to 2, with $V \not\subset \mathbf{B}_-(P(D))$).

Proof:

This directly follows from Corollary 4.2.7 and from the fact that $\mathbf{B}_+(D) = \mathbf{B}_+(P(D))$. ■

Lemma 4.2.9: Suppose that any big class $\gamma \in \overline{\text{Mov}}(X)$ on a projective IHS manifold X satisfies $\gamma \cdot C \leq 0$, for any curve of $\text{Cont}(X)$ contained in $\mathbf{B}_+(\gamma)$. Then, $\text{SC}(D)$ is a convex subcone of $\text{Big}(X)$ for any big divisor D .

Proof:

Let α, β be two classes lying in $\text{SC}(D)$. We want to prove that $\alpha + \beta \in \text{SC}(D)$. One easily checks that $\mathbf{B}_+(\alpha' + \beta') \subset \mathbf{B}_+(\alpha') \cup \mathbf{B}_+(\beta')$ ([ELMP06, Example 1.9]), for any $\alpha', \beta' \in N^1(X)_{\mathbf{R}}$, and so $\mathbf{B}_+(\alpha + \beta) \subset \mathbf{B}_+(\alpha)$. Suppose by contradiction that $p \in \mathbf{B}_+(\alpha)$ but $p \notin \mathbf{B}_+(\alpha + \beta)$. Let $C \in \text{Cont}(X)$ be a rational curve contained in $\mathbf{B}_+(\alpha) = \mathbf{B}_+(\beta)$ passing through p . Then $C \notin \mathbf{B}_+(\alpha + \beta)$, and so $(P(\alpha + \beta)) \cdot C =$

$(P(\alpha) + P(\beta)) \cdot C > 0$, whereas $P(\alpha + \beta) \cdot C \leq 0$, by assumption, and this is a contradiction. We conclude that $\mathbf{B}_+(\alpha + \beta) = \mathbf{B}_+(\alpha) = \mathbf{B}_+(\beta)$. \blacksquare

The above lemma tells us that, on IHS manifolds, to have a non-convex stability chamber, there must exist some curve in $\text{Cont}(X)$, and some movable divisor class γ , such that $C \in \mathbf{B}_+(\gamma)$, and $\gamma \cdot C > 0$.

4.3 Duality for cones of k -ample divisors

In this section, we prove Theorem 0.3.4.

Definition 4.3.1: Let $\overline{\text{Amp}}_k(X)$ be the closure of the convex cone $\text{Amp}_k(X)$ spanned by divisor classes γ with $\dim(\mathbf{B}_+(\gamma)) \leq k - 1$. We say that a class lying in $\overline{\text{Amp}}_k(X)$ is k -ample.

Remark 4.3.2. Note that the terminology introduced in the above definition is not standard. Indeed, another notion of " k -ampleness" is introduced by Totaro in [Tot13] and has been intensively studied.

Let $\phi : X \dashrightarrow X'$ be a birational map to a projective IHS manifold X' . Since $N^1(X)_{\mathbf{R}}$ and $N^1(X')_{\mathbf{R}}$ are isomorphic under ϕ_* , their dual spaces $N_1(X)_{\mathbf{R}}$ and $N_1(X')_{\mathbf{R}}$ are also isomorphic. Under this isomorphism, any class $\alpha \in N_1(X')_{\mathbf{R}}$ can be pulled-back to a class in $N_1(X)_{\mathbf{R}}$. We define

$$\overline{\text{Mob}}_k(X, X') \subset N_1(X)_{\mathbf{R}} \quad (4.5)$$

to be the image of the convex cone generated by numerical classes of irreducible curves C in X' moving in a family that sweeps out the birational image of a subvariety of X of dimension at least k , via the isomorphism $N_1(X)_{\mathbf{R}} \cong N_1(X')_{\mathbf{R}}$. Now, consider the cone

$$\sum_{X \dashrightarrow X'} \overline{\text{Mob}}_k(X, X'),$$

where the sum is taken over all birational maps $X \dashrightarrow X'$ such that X' is a projective IHS manifold.

Definition 4.3.3: We define the cone of birationally k -mobile curves of X as

$$\text{bMob}_k(X) := \sum_{X \dashrightarrow X'} \overline{\text{Mob}}_k(X, X').$$

Example 4.3.4. We have $\overline{\text{Amp}}_{2n-1}(X) = \overline{\text{Mov}}(X)$ and $\text{bMob}_{2n-1}(X)^{\vee} \subset \overline{\text{Mov}}(X)$. Indeed, the equality follows from the fact that $\text{Amp}_{2n-1}(X) = \text{Int}(\text{Mov}(X))$. For the inclusion $\text{bMob}_{2n-1}(X)^{\vee} \subset \overline{\text{Mov}}(X)$, if E is a prime exceptional divisor that can be contracted via $\pi : X \rightarrow Y$ (with Y normal and projective), the class of a general fibre of π is given by $-2 \frac{q_X(E, -)}{q_X(E)}$ ([Mar09, Corollary 3.6, part 1]). If E is not contractible, it can be contracted on a birational model $f : X \dashrightarrow X'$ (see [Dru11, Proposition 1.4], or [BBP13, Proof of Theorem A]), and, on X' , either the homology class $-2 \frac{q_X(E', -)}{q_X(E')}$, or the homology class $-\frac{q_X(E', -)}{q_X(E')}$, is represented by a rational curve ([Mar09, Corollary

3.6, part 1)) moving in a family sweeping out the strict transform E' of E in X' via f . Now, $\overline{\text{Mov}}(X)$ consists of the pseudo-effective classes q_X -intersecting non-negatively any prime exceptional divisor. Indeed, any class in $\overline{\text{Mov}}(X)$ is q_X -nef, i.e. q_X -intersect non-negatively any prime divisor. On the other hand, suppose that α is a pseudo-effective class q_X -intersecting non-negatively any prime exceptional divisor. If $\alpha \notin \overline{\text{Mov}}(X)$, we have $\alpha = P(\alpha) + N(\alpha)$, with $N(\alpha) \neq 0$. Then, since $P(\alpha) \in \overline{\text{Mov}}(X)$ and q_X is an intersection product (i.e. $q_X(E, E') \geq 0$ for any two distinct prime divisors E, E' on X), there must be some prime exceptional divisor supporting $N(\alpha)$, and q_X -intersecting negatively α . But this contradicts the assumption on α . Then, $\alpha = P(\alpha) \in \overline{\text{Mov}}(X)$, and we obtain the desired inclusion.

We now prove Theorem 0.3.4.

Proof of Theorem 0.3.4:

The inclusion \subset is clear. Hence, we are left to show \supset . We first observe that

$$\text{bMob}_k(X)^\vee \subset \overline{\text{Eff}}(X).$$

Indeed, $\overline{\text{Mob}}(X) \subset \text{bMob}_k(X)$, thus, passing to the duals, we obtain $\text{bMob}_k(X)^\vee \subset \overline{\text{Eff}}(X)$ ([BDPP13, Theorem 2.2]). Then, the interior of $\text{bMob}_k(X)^\vee$ is contained in $\text{Big}(X)$. It suffices then to show that

$$\text{int}(\text{bMob}_k(X)^\vee) \subset \text{Amp}_k(X).$$

Suppose that D is a big divisor with class lying in $\text{int}(\text{bMob}_k(X)^\vee)$, but not in $\text{Amp}_k(X)$. Then, there exists an irreducible component V of $\mathbf{B}_+(D)$ with $\dim(V) \geq k$. By Example 4.3.4 $D \in \text{Int}(\overline{\text{Mov}}(X))$, then, by [BBP13, Proof of Theorem A], there exists a class α and a birational map $\phi : X \dashrightarrow X'$, with X' a projective IHS manifold, inducing a birational map $\phi|_V : V \dashrightarrow V'$ (where V' is the strict transform of V via ϕ), and such that $\phi_*(\alpha)$ is the class of a curve which moves in a family sweeping out V' , and with $\phi_*(D) \cdot \phi_*(\alpha) \leq 0$. But then $D \cdot \alpha \leq 0$, and $\alpha \in \text{bMob}_k(X)$. This is a contradiction, because D intersects positively any class lying in $\text{bMob}_k(X)$, as $D \in \text{int}(\text{bMob}_k(X)^\vee)$. The second part of the statement directly follows from Corollary 4.1.14. ■

4.4 A decomposition of $\text{Eff}(X)$ into chambers of Mori-type

The birational geometry of a Mori dream space is encoded by its effective cone, and, more in particular, by the decomposition of its effective cone into Mori chambers. On IHS manifolds, the minimal model program works nicely ([MZ13, Theorem 4.1], [LP16, Theorem 1.2]), and, as already remarked, projective IHS manifolds are close to being Mori dream spaces. For this reason, also on projective IHS manifolds, one may suspect the existence of a decomposition of the effective cone into chambers of "Mori-type", resuming the birational geometry of the considered variety. In this section, we show that we indeed have such a decomposition for $\text{Eff}(X)$, which we still call decomposition into *Mori chambers*.

Fix Y to be a normal complex \mathbf{Q} -factorial projective variety.

Definition 4.4.1: Let $f: Y \dashrightarrow Z$ be a dominant rational map, with Z normal and projective. We say that f is a rational contraction if there exists a resolution of f

$$\begin{array}{ccc} Y' & & \\ \downarrow \mu & \searrow f' & \\ Y & \dashrightarrow f & Z, \end{array}$$

where Y' is smooth and projective, μ is birational, and for every μ -exceptional effective divisor E' on Y' , we have $f'_*(\mathcal{O}_{Y'}(E')) = \mathcal{O}_Z$.

Remark 4.4.2. Let Y' be a projective \mathbf{Q} -factorial variety. If $s: Y' \dashrightarrow Y$ is a small \mathbf{Q} -factorial modification and $c: Y \dashrightarrow Z$ is a rational contraction, with Z normal and projective, also $c \circ s$ is a rational contraction.

If in the above definition, f is a morphism, f is a contraction in the usual sense.

Now, let D be an effective divisor on X . Then, either $D \in \overline{\text{Mov}}(X)$, so that $D \in f^{-1}(\text{Nef}(X'))$, for some projective irreducible holomorphic symplectic X' , and some birational map $f: X \dashrightarrow X'$, or D has some divisorial stable base locus. In the second case, let $D = P + N$ be the divisorial Zariski decomposition of D , and $f: X \dashrightarrow X'$ be any birational map making P nef on X' (projective IHS). We consider the face of $\overline{\text{Mov}}(X)$

$$F = \text{Neg}_{q_X}(D)^\perp \cap \overline{\text{Mov}}(X)$$

and

$$\text{Neg}_{q_X}(D)^\perp := \{\alpha \in N^1(X)_{\mathbf{R}} \mid q_X(\alpha, E) = 0 \text{ for any prime divisor } E \text{ supporting } N\}.$$

Let $c: X' \rightarrow Y'$ be the birational morphism contracting the face $F'^\vee \subset \overline{\text{NE}}(X')$ (where $P' = f_*(P)$, and $F' = f_*(F)$) to the normal \mathbf{Q} -factorial symplectic variety Y' (we can do this because the Gram matrix of the irreducible components of $N' = f_*(N)$ is negative definite, if $N' \neq 0$). In particular, we have $c^{-1}(\text{Nef}(Y')) = F'$, and so $(c \circ f)^{-1}(\text{Nef}(Y')) = F$. Then, $D \in F \cap \text{Eff}(X) + V^{\geq 0}(\text{Neg}_{q_X}(D))$, where $V^{\geq 0}(\text{Neg}_{q_X}(D))$ is the conic convex hull of $\text{Neg}_{q_X}(D)$. Vice versa, let $f: X \dashrightarrow Y'$ be a birational contraction to a normal, \mathbf{Q} -factorial projective variety. Then, f factors through a birational map $g: X \dashrightarrow X'$, with X' a projective IHS manifold, and a surjective birational morphism $c: X' \rightarrow Y'$ with connected fibres. As Y' is \mathbf{Q} -factorial, the irreducible components of the exceptional locus $\text{Exc}(c)$ are divisorial. Let $S = \{E'_1, \dots, E'_k\}$ be the set of irreducible components of $\text{Exc}(c)$. The Gram matrix of S must be negative definite (if S is not empty), and $c^{-1}(\text{Nef}(Y')) = S^\perp \cap \text{Nef}(X')$. Set $F = (c \circ f)^*(\text{Nef}(Y'))$. Then, any divisor of the form $D = A + B$, where $A \in F \cap \text{Eff}(X)$, and $B = \sum_i a_i f^*(E'_i)$, $a_i \geq 0$ for every i , is effective. Furthermore, A (resp. B) is the positive (resp. negative) part of the divisorial Zariski decomposition of D . We conclude that the class of any effective divisor D on X lies in the subcone of $\text{Eff}(X)$

$$F \cap \text{Eff}(X) + V^{\geq 0}(\text{Neg}_{q_X}(D)), \quad (4.6)$$

where $F = \text{Neg}_{q_X}(D)^\perp \cap \overline{\text{Mov}(X)}$, and induces a birational contraction to a normal \mathbf{Q} -factorial projective variety. Vice versa, any birational contraction to a normal \mathbf{Q} -factorial projective variety induces a subcone of $\text{Eff}(X)$ of the form 4.6. We call the subcones of $\text{Eff}(X)$ of the form 4.6 *Mori chambers* of X . Note that two different Mori chambers can intersect only on the boundary, i.e. the interiors of two different Mori chambers are disjoint. We resume all the above with the following proposition.

Proposition 4.4.3: Let X be a projective IHS manifold. Then

$$\text{Eff}(X) = \cup_i \mathcal{C}_i,$$

where

$$\mathcal{C}_i = g_i^*(\text{Nef}(Y_i)) \cap \text{Eff}(X) + \sum_j \mathbf{R}^{\geq 0} E_i^j,$$

$g_i: X \dashrightarrow Y_i$ is a birational contraction, with Y_i projective, normal and \mathbf{Q} -factorial, and the E_i^j 's are the prime exceptional divisors contracted by g_i , for any i .

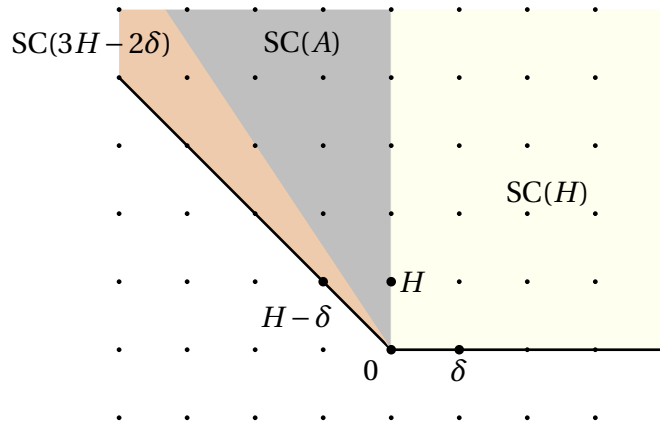
The decomposition of $\text{Eff}(X)$ into Mori chambers induces a decomposition of $\text{Big}(X)$ into convex subcones, which we still call decomposition of $\text{Big}(X)$ into *Mori chambers*. In the interior of any Mori chamber of $\text{Big}(X)$, the stable base locus of the divisor classes is constant. However, a locus of the big cone where the stable base locus is constant could be bigger than the interior of a Mori chamber (see Remark 4.5.7 for an example). It is worth observing that the decomposition of $\text{Big}(X)$ into Boucksom-Zariski chambers is related to that in Mori chambers. Indeed, the interior of any Boucksom-Zariski chamber different from $\overline{\text{Mov}(X)} \cap \text{Big}(X)$ is equal to the interior of a unique Mori chamber (and vice versa). The point is that the Boucksom-Zariski chambers should be thought of as "stability chambers in codimension 1". In particular, in the interior of each Boucksom-Zariski chamber, the divisorial augmented base locus is constant, and a big divisor D lies on the boundary of some Boucksom-Zariski chamber if and only if D is *unstable in codimension one*, i.e. if and only if $\mathbf{B}_-(D)_{\text{div}} \subsetneq \mathbf{B}_+(D)_{\text{div}}$, and this follows directly from Proposition 2.2.17, and Proposition 4.1.2. This is coherent with the fact that the big classes in $\text{int}(\text{Mov}(X))$ have an empty divisorial augmented base locus, though the big classes in $\text{Mov}(X)$ could have some non-divisorial augmented base locus.

4.5 Examples

In this section, we illustrate our results with several examples.

4.5.1 Hilbert schemes of points on K3 surfaces

Let S_d be a K3 surface such that $\text{Pic}(S_d) \cong \mathbf{Z} \cdot H_{S_d}$, with $H_{S_d}^2 = 2d$. Consider the projective IHS manifold $\text{Hilb}^2(S_d)$ with the usual decomposition $\text{Pic}(\text{Hilb}^2(S_d)) \cong \mathbf{Z} \cdot H \oplus \mathbf{Z} \cdot \delta$, where H is the line bundle associated to H_S and δ is half the divisor corresponding to non-reduced subschemes

Figure 4.1: Stability chambers on $\text{Hilb}^2(S_1)$

or the exceptional divisor of the Hilbert-Chow morphism $\text{Hilb}^2(S_d) \rightarrow \text{Sym}^2(S_d)$. The effective cone of $\text{Hilb}^2(S_d)$ and its decomposition into birational chambers is completely determined by the solution of some Pell equations due to the work of Bayer-Macri, see [BM13, Lemma 13.1, Proposition 13.3]. For polarised K3 surfaces S_d of low degrees one can give a precise description of the stability chambers of $\text{Big}(\text{Hilb}^2(S_d))$.

Example 4.5.1. Assume that $d = 1$. Then S_1 is a double covering of \mathbf{P}^2 ramified along a sextic curve. Let $\pi : S \rightarrow \mathbf{P}^2$ be such a covering. We have:

1. $\text{Eff}(\text{Hilb}^2(S_1)) = \langle H - \delta, \delta \rangle$,
2. $\text{Mov}(\text{Hilb}^2(S_1)) = \langle H - \delta, H \rangle$,
3. $\text{Nef}(\text{Hilb}^2(S_1)) = \langle 3H - 2\delta, H \rangle$.

We see that there are three stability chambers: $\text{SC}(A)$, which is associated with any ample divisor A , the chamber $\text{SC}(H)$, and $\text{SC}(3H - 2\delta)$. We can describe all of them geometrically:

1. In $\text{SC}(A)$ the augmented base locus is empty.
2. In $\text{SC}(H)$ the augmented base locus is given by the exceptional divisor $E \equiv 2\delta$. The divisor H is big and nef, and the morphism induced by $|H|$ is nothing but the Hilbert-Chow morphism $\text{Hilb}^2(S_d) \rightarrow \text{Sym}^2(S_d)$ (whose exceptional locus is given by E).
3. In $\text{SC}(3H - 2\delta)$ the augmented base locus is given by the plane \mathbf{P}^2 , obtained using the fibers of the double covering π . The other birational model of $\text{Hilb}^2(S_1)$ is obtained by taking the *Mukai flop* along the plane \mathbf{P}^2 , i.e. the blow-up of \mathbf{P}^2 in $\text{Hilb}^2(S_1)$, followed by the contraction of the exceptional divisor in the other direction.

The decomposition of the big cone into stability chambers is given in the picture 4.1 below.

Example 4.5.2. With the same notation as above, let $d = 2$. Then $S_2 \subset \mathbf{P}^3$ is a quartic surface, and we get the following decomposition of the effective cone:

1. $\text{Eff}(\text{Hilb}^2(S_2)) = \langle 2H - 3\delta, \delta \rangle$,
2. $\text{Mov}(\text{Hilb}^2(S_2)) = \text{Nef}(\text{Hilb}^2(S_1)) = \langle 3H - 4\delta, H \rangle$.

Notice that in this case, we have an involution $\iota : \text{Hilb}^2(S_2) \rightarrow \text{Hilb}^2(S_2)$ given by the residual intersection with the line spanned by two different points. Let $E = 2\delta$ be the exceptional divisor. Then, the involution ι does not fix E , because a tangent line which is not bi-tangent maps a non-reduced double point to two points. Moreover, one can check that the class of $\iota(E)$ in the Néron-Severi space is $\iota^*(E) = 2H - 3\delta$, and this is represented by a prime exceptional divisor different from E . All the asymptotic base loci of big divisors are divisorial in this case, and there are three stability chambers: $\text{SC}(A)$ where A is any ample divisor on $\text{Hilb}^2(S_2)$, the chamber $\text{SC}(H)$, and $\text{SC}(3H - 4\delta)$, see Figure 4.2. The first two chambers are as in Example 4.5.1. In the third, the augmented base locus is given by $i(E)$, and we have $\text{SC}(2H - 3\delta) = \langle 2H - 3\delta, 3H - 4\delta \rangle \cap \text{Big}(\text{Hilb}^2(S_2))$.

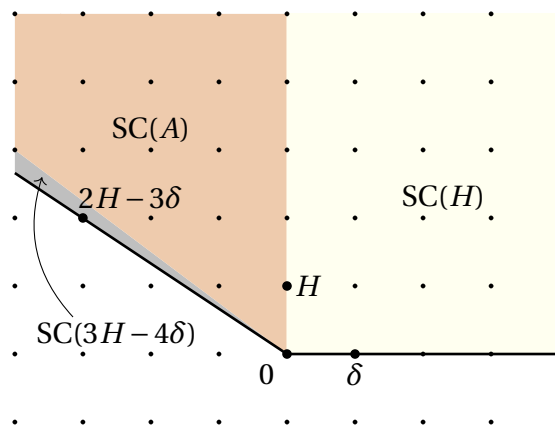


Figure 4.2: Stability chambers on $\text{Hilb}^2(S_2)$

Example 4.5.3. Let $d = 3$, then $S_3 \subset \mathbf{P}^4$ is the complete intersection of a quadric $Q \subset \mathbf{P}^4$ and a cubic $T \subset \mathbf{P}^4$. The decomposition of the effective cone is:

1. $\text{Eff}(\text{Hilb}^2(S_3)) = \langle H - 2\delta, \delta \rangle$,
2. $\text{Mov}(\text{Hilb}^2(S_3)) = \text{Nef}(\text{Hilb}^2(S_3)) = \langle 2H - 3\delta, H \rangle$.

This case also corresponds to divisorial stability chambers. As before we have three stability chambers: $\text{SC}(A)$ where A is an ample divisor on $\text{Hilb}^2(S_3)$, the chamber $\text{SC}(H)$ and $\text{SC}(3H - 4\delta)$. In the third stability chamber, the augmented base locus is given by an irreducible divisor D . One can describe D geometrically (see [GO20, Lemma 4.5]) as follows: the quadric Q is a hyperplane section of $\text{Gr}(2, 4) \subset \mathbf{P}^5$ under the Plücker embedding, then $D \cong \mathbf{P}(\mathcal{U}^\vee|_S)$, where \mathcal{U} is the tautological rank 2 bundle on $\text{Gr}(2, 4)$. A computation of its Chern classes yields that D is not isomorphic to E .

4.5.2 The example of Hassett and Tschinkel

We now want to describe the stability chambers of the Fano variety of lines of certain smooth cubic fourfolds containing a smooth cubic scroll. We will see that in this case not all the stability chambers are convex subcones of the big cone. In [HT10] the authors gave a complete description of the movable cone of such a variety. The main result of Hassett and Tschinkel we are interested in is the following.

Theorem 4.5.4 ([HT10, Theorem 7.4]): Suppose that Y is a smooth cubic fourfold containing a smooth cubic scroll T with

$$H^4(Y, \mathbf{Z}) \cap H^{2,2}(Y, \mathbf{C}) = \mathbf{Z}h^2 + \mathbf{Z}T$$

and let $F = F_0$ denote the variety of lines on Y . Then we have an infinite sequence of Mukai flops

$$\cdots F_2^\vee \dashrightarrow F_1^\vee \dashrightarrow F_0 \dashrightarrow F_1 \dashrightarrow F_2 \cdots$$

with isomorphisms between every other flop in this sequence

$$\cdots F_2^\vee \cong F_0 \cong F_2 \cdots \quad \text{and} \quad \cdots F_1^\vee \cong F_1 \cdots \quad (4.7)$$

The positive cone of F can be expressed as the union of the nef cones of the models $\cdots, F_1^\vee, F_0, F_1, \cdots$:

$$\cdots, \text{cone}(\alpha_2^\vee, \alpha_1^\vee), \text{cone}(\alpha_1^\vee, \alpha_1), \text{cone}(\alpha_1, \alpha_2), \cdots \quad (4.8)$$

We refer the reader to [HT10] for all the geometric details of this example (which is quite intricate). We explain below the terminology of Theorem 4.5.4.

The birational geometry of F is governed by the two planes P, P^\vee and the surface S contained in F ([HT10, Proposition 4.1]). Let $f_1 = f_P: F \dashrightarrow F_1$ be the Mukai flop of F along P , and $f_2 = f_{P \cup S}: F \dashrightarrow F_2$ the birational map obtained by flopping the plane $P \subset F$, first, and after $S_1 \subset F_1$, where S_1 is the strict transform of the surface S via f_1 (we can perform this Mukai flop because S_1 is a plane in F_1). One has $F_2 \cong F$ ([HT10, Theorem 6.2]), so that f_2 gives a birational self map of order 2, i.e. an involution. Analogously we define $f_1^\vee: F \dashrightarrow F_1^\vee$ to be the Mukai flop of F along P^\vee , and $f_2^\vee: F \dashrightarrow F_2^\vee$ the birational map obtained by flopping the plane $P^\vee \subset F$, first, and after $S_1^\vee \subset F_1^\vee$, where S_1^\vee is the strict transform of the surface S via f_1^\vee (we can perform this flop too because S_1^\vee is a plane in F_1^\vee). Also, f_1^\vee gives a birational involution. The composition $f_2 \circ f_2^\vee$ is an element of $\text{Bir}(F)$ of infinite order. The group $\text{Bir}(F)$ is generated by f_2^\vee and f_2 , and any log MMP for pairs of the type (F, D) , with $[D] \in \text{Mov}(F)$ starts by flopping first either P or P^\vee , because the two extremal rays of the Mori cone of F are generated by the classes of a line in P and P^\vee . To conclude, we define the birational map $f_3: F \dashrightarrow F_3$ as the composition of f_2 with the Mukai flop of F_2 along the strict transform P_2^\vee via f_2 of P^\vee (which is a plane), and the birational map

$f_3^\vee : F \dashrightarrow F_3^\vee$ as the composition of f_2^\vee with the Mukai flop of F_2^\vee along the strict transform P_2 via f_2^\vee of P (which also, in this case, is a plane).

Now, let τ be $\alpha([T])$, where $\alpha : H^4(Y, \mathbf{Z}) \rightarrow H^2(F, \mathbf{Z})$ is the Abel-Jacobi map, $[T]$ is the class of the cubic scroll T contained in X , and $g = \alpha(h^2)$, where h^2 is the square of the hyperplane class of the cubic fourfold. The pseudo-effective cone of F is $\overline{\text{Eff}}(F) = \text{cone}(g - (3 - \sqrt{6})\tau, (3 + \sqrt{6})\tau - g)$. Note that we do not have prime exceptional divisors on F , hence

$$\text{Big}(F) = \mathcal{C}_F \cap N^1(F)_{\mathbf{R}} = \text{Mov}(F) = \text{Eff}(F). \quad (4.9)$$

One has $q_F(g) = q_F(g, \tau) = 6$, and $q_F(\tau) = 2$. Consider the following table of classes in $N^1(F)_{\mathbf{R}}$.

$$\begin{array}{l|l|l} \alpha_3^\vee = 39\tau - 7g & \rho_3^\vee = 8\tau - 3/2 \cdot g & f_{2,*}(\rho_3^\vee) = \text{dual of the class of a line in } P_2^\vee \\ \alpha_2^\vee = 9\tau - g & \rho_2^\vee = 2\tau - g/2 & f_{1,*}(\rho_2^\vee) = \text{dual of the class of a line in } S_1^\vee \\ \alpha_1^\vee = g + 3\tau & \rho_1^\vee = \tau - g/2 & \rho_1^\vee = \text{dual of the class of a line in } P^\vee \\ \alpha_1 = 7g - 3\tau & \rho_1 = 3/2 \cdot g - \tau & \rho_1 = \text{dual of the class of line in } P \\ \alpha_2 = 17g - 9\tau & \rho_2 = 7/2 \cdot g - 2\tau & f_{1,*}(\rho_2) = \text{dual of the class of a line in } S_1 \\ \alpha_3 = 71g - 39\tau & \rho_3 = 29/2 \cdot g - 8\tau & f_{2,*}(\rho_3) = \text{dual of the class of a line in } P_2 \end{array}$$

With the above notation, we obtain the following decomposition of the big cone into stability chambers.

Proposition 4.5.5: We have for $\text{Big}(F)$ six stability chambers. The first is $\text{Amp}(F)$, the ample cone of F . In the second, SC_1 , the augmented base locus consists of the plane P . In the third, SC_1^\vee , the augmented base locus is given by the plane P^\vee . In the fourth, SC_2 , the augmented base locus is given by $P \cup S$, and in the fifth, SC_2^\vee , the augmented base locus is given by the union $P^\vee \cup S$. The sixth stability chamber, which is not a convex cone, is denoted by SC_3 , and there the augmented base locus is given by the union $P \cup P^\vee \cup S$.

Proof:

The ample cone $\text{Amp}(F)$ is the stability chamber where the augmented base locus is empty, which we denote by SC_0 . A log MMP for any divisor lying in $f_P^*(\text{Amp}(F_1))$ is given by f_P . Then, the augmented base locus of any class lying in $f_P^*(\text{Amp}(F_1))$ is given by the plane P . Note that the rays α_1 and α_2 are part of the unstable locus (i.e. the regions of the big cone where $\mathbf{B}_- \subsetneq \mathbf{B}_+$). Then, $\text{SC}_1 = f_P^*(\text{Amp}(F_1)) \cup \alpha_1$ gives a second stability chamber. Similarly, the augmented base locus of any class lying in $f_P^*(\text{Amp}(F_1^\vee))$ is given by P^\vee , and so $\text{SC}_1^\vee = f_P^*(\text{Amp}(F_1^\vee)) \cup \alpha_1^\vee$ gives the third stability chamber. The augmented base locus of any class lying in $f_{P \cup S}^*(\text{Amp}(F_2))$ is given by $P \cup S$, because $f_{P \cup S}$ gives a log MMP for any such class. It follows that $\text{SC}_2 = f_{P \cup S}^*(\text{Amp}(F_2)) \cup \alpha_2$ is a stability chamber. Analogously, we see that the augmented base locus of any class lying in $f_{P^\vee \cup S}^*(\text{Amp}(F_2^\vee))$ is given by $P^\vee \cup S$, and hence $\text{SC}_2^\vee = f_{P^\vee \cup S}^*(\text{Amp}(F_2^\vee)) \cup \alpha_2^\vee$ is another stability chamber. To conclude, we see that the augmented base locus of any class lying in $f_{P \cup S \cup P^\vee}^*(\text{Amp}(F_3))$ is given by $P \cup S \cup P^\vee$, because $f_{P \cup S \cup P^\vee}$ gives a log MMP for these classes. But

the same is true for any element lying in $f_{P^\vee \cup S \cup P}^*(\text{Amp}(F_3^\vee))$. After this step, the augmented base locus stabilises "on both sides" of the big cone (i.e. on the left of the ray α_3^\vee , and on the right of α_3), and it is always equal to $P^\vee \cup S \cup P$. The sixth chamber, denoted by SC_3 , is given by $\text{Big}(F)$ minus the interior of the convex cone spanned by α_3 and α_3^\vee . In particular, we observe that it is not a convex subcone of $\text{Big}(F)$. This is consistent with Lemma 4.2.9. Indeed, for example, if γ is any class lying in SC_3 (resp. SC_3^\vee), and if l^\vee (resp. l) is any line lying in P^\vee (resp. P), we see that $\gamma \cdot l^\vee > 0$ (resp. $\gamma \cdot l > 0$), but $l^\vee \subset \mathbf{B}_+(\gamma) = \mathbf{B}_-(\gamma)$ (resp. $l \subset \mathbf{B}_+(\gamma) = \mathbf{B}_-(\gamma)$). In particular, to characterise numerically the asymptotic base loci, we cannot avoid considering the birational models of the variety. To conclude, we observe that unstable locus of $\text{Big}(F)$ is given by the union of rays $\alpha_1 \cup \alpha_1^\vee \cup \alpha_2 \cup \alpha_2^\vee \cup \alpha_3 \cup \alpha_3^\vee$. ■

Remark 4.5.6. Let F be as in Theorem 4.5.4. According to Theorem 0.3.4, we provide an example of a class in $N_1(F)_\mathbf{R}$ which on F_2 is represented by a rational curve moving in a family sweeping out a subvariety of F_2 , which is the strict transform of a 2-dimensional subvariety of F . Following the notation of the table above, consider the class $\rho_3 \in N_1(F)_\mathbf{R}$, and $f_{2,*}(\rho_3) \in N_1(F_2)_\mathbf{R}$. Then, the class dual to $f_{2,*}(\rho_3)$ is represented by a line moving in a family sweeping out the lagrangian plane $f_{2,*}(P^\vee)$ (the strict transform of P^\vee via f_2). The extremal rays of the Mori cone of F are generated by $q_F(\rho_1, -)$, and $q_F(\rho_1^\vee, -)$, and an easy calculation shows $q_F(\rho_3, -) = \frac{5}{2}q_F(\rho_1^\vee, -) + \frac{21}{2}q_F(\rho_1, -)$.

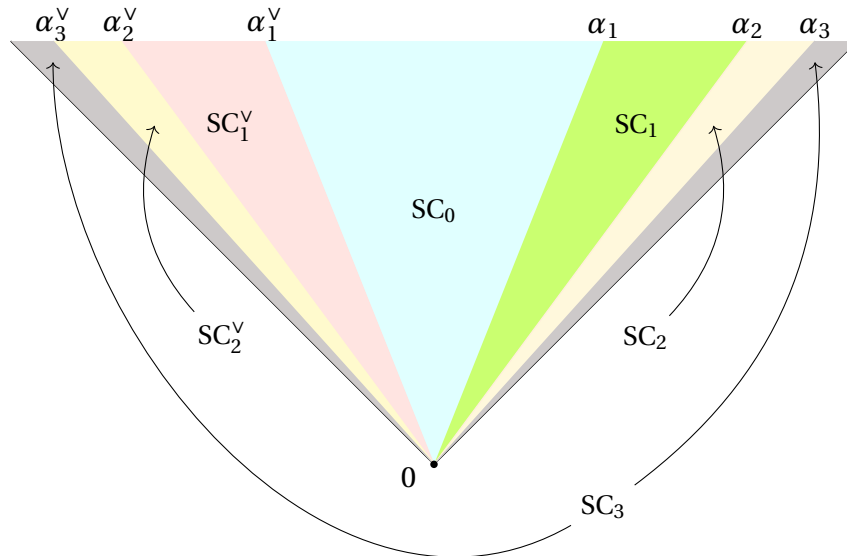


Figure 4.3: Representation of the stability chambers on F

Remark 4.5.7. Notice that in this example the decomposition of $\text{Eff}(F) = \text{Big}(F)$ into Mori chambers is strictly finer than the one into stability chambers. Indeed, while any Mori chamber is equal to $f^{-1}(\text{Nef}(F'))$ for some birational model F' of F so that we have infinitely many ones, the stability chambers are six.

We would like to conclude this section with a result which is interesting on its own.

If X is a projective IHS manifold carrying a birational self-map of infinite order, $\text{Bir}(X)$ is an infinite group. This happens for instance in the above example. Indeed, as it is proven in [HT10], with the same notation of above, the map $f_{P \cup S}$ gives an involution $\iota \in \text{Bir}(X)$, and $f_{P^\vee \cup S}$ induces a second involution $\iota^\vee \in \text{Bir}(X)$. An element of infinite order is then $\iota \circ \iota^\vee$ (but also $\iota^\vee \circ \iota$). We show below that also the converse is true, using Theorem 3.1.8, and a result of Cattaneo-Fu.

Proposition 4.5.8: Suppose that $\text{Bir}(X)$ is infinite. Then $\text{Bir}(X)$ contains an element of infinite order.

Proof:

By [CF19, Theorem 7.1] the number of conjugacy classes of finite subgroups of $\text{Bir}(X)$ is finite. Then, there exists an integer N such that any element of finite order in $\text{Bir}(X)$ has at most order N . Suppose by contradiction that $\text{Bir}(X)$ does not contain elements of infinite order. Then the image of the natural representation

$$\rho: \text{Bir}(X) \rightarrow \text{GL}(N^1(X)_{\mathbb{C}})$$

is a group of finite order, by Theorem 3.1.8. But then $\text{Bir}(X)$ is finite by [Ogu14, Corollary 2.6], and this is a contradiction. ■

4.5.3 A non-equidimensional augmented base locus

Let S be a projective K3 surface carrying a smooth rational curve $C \cong \mathbf{P}^1$. Consider the n -plane $\mathbf{P}^n \cong C^{[n]} \subset S^{[n]}$. Let $\text{cont}: S \rightarrow S'$ be the morphism contracting the curve C . We have an induced morphism $\text{cont}^{(n)}: S^{(n)} \rightarrow S'^{(n)}$, contracting $C^{(n)}$. The n -plane $C^{[n]}$ is not contracted by the Hilbert-Chow morphism $\pi: S^{[n]} \rightarrow S^{(n)}$, hence $C^{[n]} \not\subset E$, where E is the exceptional divisor coming from π . Let D be the wall divisor related to $C^{[n]}$. Then, if A is any ample, integral, Cartier divisor on $S'^{(n)}$, we have $D' := (\pi \circ \text{cont}^{(n)})^*(A) \in D^\perp \cap E^\perp$, and $E, C^{[n]}$ are two irreducible components of $\mathbf{B}_+(D')$ of different dimension.

4.6 Final remarks and some questions

In this section, we ask two questions naturally arising from our work.

4.6.1 Is the restricted base locus Zariski-closed?

In general, given a pseudo-effective divisor D on a smooth projective variety Y , the restricted base locus $\mathbf{B}_-(D)$ is not Zariski closed. The first examples of non-Zariski closed restricted base loci were provided by John Leusieutre (cf. [Les14]), and we do not know other examples of this pathology.

In the case of a projective IHS manifold X , if $\text{Bir}(X)$ is finite, $\mathbf{B}_-(D)$ is Zariski closed for any pseudo-effective divisor D . In any case, if D is big on X , as $\mathbf{B}_-(D) = \mathbf{B}(D)$, all big divisors have Zariski-closed restricted base locus. Hence, the only classes that might have a non-Zariski closed restricted base locus on IHS manifolds, are the pseudo-effective but not big ones. In the example of Hassett and Tshinkel we observe that all pseudo-effective classes have a Zariski closed restricted base locus. Indeed, in that case, the only potential rays with a non-Zariski closed restricted base locus are the ones spanned by $\alpha = g - (3 - \sqrt{6})\tau$ and $\beta = (3 + \sqrt{6})\tau - g$. Using that $\mathbf{B}_-(D) = \cup_{m \in \mathbf{N}} \mathbf{B}_+(D + A_m)$, where $\{A_m\}_{m \in \mathbf{N}}$ is a sequence of ample divisors with $\|A_m\| \rightarrow 0$, and that $\mathbf{B}_+(\alpha + A_m)$ and $\mathbf{B}_+(\beta + A_m)$ stabilise for $m \gg 0$, we see that $\mathbf{B}_-(\beta) = \mathbf{B}_-(\alpha) = P \cup P^\vee \cup S$, so that all pseudo-effective divisors on F have a Zariski-closed restricted base locus. Then, it is natural to ask the following question, which will be investigated by the authors in future works.

Question 1. *Let X be a projective IHS manifold and D any pseudo-effective divisor on X . Is $\mathbf{B}_-(D)$ Zariski closed?*

4.6.2 Is the Voisin filtration preserved under birational maps?

Our results shed light on Conjecture [Voi16, Conjecture 0.4] and also motivate the following question about the birational invariance of the Voisin filtration.

Question 2. *Let $f : X \dashrightarrow X'$ be a birational map between projective IHS manifolds. Is it true that the induced isomorphism*

$$\text{CH}_0(X) \cong \text{CH}_0(X') \tag{4.10}$$

preserves the filtration S_\bullet ?

In the case of moduli space of Bridgeland stable sheaves on a K3 surface, the above question was answered affirmatively very recently in (cf. [LZ22, Corollary 3.3]).



5. Polygons of Newton-Okounkov type

The starting point of this chapter is the following formula.

$$q_X(\alpha, \beta) \int \alpha^{2n} = q_X(\alpha) \int \alpha^{2n-1} \beta, \text{ for any } \alpha, \beta \in H^2(X, \mathbf{C}) \quad (5.1)$$

The formula can be found in [GHJ03, Exercise 23.2], but be careful, there is a typo in there. As usual, given any big divisor D on X , we will denote by $D = P(D) + N(D)$ the divisorial Zariski decomposition of D .

5.1 Restricted volumes

In this section, we provide a first application of the formula (5.1). In particular, we compute the restricted volume of any big \mathbf{R} -divisor on a projective IHS manifold to a chosen prime divisor.

Lemma 5.1.1: Let E be a prime divisor on a normal projective variety Y , and $f: Y \dashrightarrow Y'$ a small \mathbf{Q} -factorial modification of Y (Definition 3.1.11). Suppose that, given a Cartier divisor D on Y , the Weil divisor $D' = f_*(D)$ is Cartier. Furthermore, let $E' = f_*(E)$ be the strict transform of E via f . Then, the complex vector spaces $\text{Im}(H^0(Y, \mathcal{O}_Y(mD)) \rightarrow H^0(E, \mathcal{O}_E(mD)))$, $\text{Im}(H^0(Y', \mathcal{O}_{Y'}(mD')) \rightarrow H^0(E', \mathcal{O}_{E'}(mD')))$ have the same dimension, for any m , so that $\text{vol}_{X|E}(D) = \text{vol}_{X'|E'}(D')$.

Proof:

A section $s \in H^0(Y, \mathcal{O}_Y(mD))$ vanishes on E if and only if $s' = f_*(s) \in H^0(Y', \mathcal{O}_{Y'}(mD'))$ vanishes on E' . As $H^0(Y, \mathcal{O}_Y(mD)) \cong H^0(Y', \mathcal{O}_{Y'}(mD'))$ (because f is a small \mathbf{Q} -factorial modification) we are done. ■

Proposition 5.1.2: Let E be a prime divisor on a projective IHS manifold X and D any big divisor with $E \notin \mathbf{B}_+(D)$. Then

$$\text{vol}_{X|E}(D) = P(D)^{2n-1} \cdot E = \frac{q_X(P(D), E) \cdot \text{vol}(D)}{q_X(P(D))}.$$

Proof:

Let $f: X \dashrightarrow X'$ be a birational model on which $f_*(P(D)) = P(D')$ is nef (where $D' = f_*(D)$) and set $E' = f_*(E)$. As $\text{Big}^E(X)^+ \subset \text{Big}(X)$ is open, we can find a sequence $\{\gamma_n\}_n$ of rational big divisor classes converging to $[D]$, which in turn gives (via push-forward) a sequence of rational big divisor classes $\{\gamma'_k\}_k$ in $\text{Big}^{E'}(X')^+$, converging to $[D']$. By [Mat13, Proposition 3.1] we have $\text{vol}_{X|E}(\gamma_k) = \text{vol}_{X|E}(P(\gamma_k))$. Moreover, by Lemma 5.1.1, we have $\text{vol}_{X|E}(\gamma_k) = \text{vol}_{X'|E'}(\gamma'_k)$, and hence $\text{vol}_{X|E}(P(\gamma)) = \text{vol}_{X'|E'}(P(\gamma'))$. By the continuity of the divisorial Zariski decomposition (see Proposition 2.2.30) we obtain $\{P(\gamma_k)\}_k \rightarrow P(\gamma)$, and $\{P(\gamma'_k)\}_k \rightarrow P(\gamma')$. By the continuity of $\text{vol}_{X|E}(-)$ (resp. $\text{vol}_{X'|E'}(-)$) on $\text{Big}^E(X)^+$ (resp. $\text{Big}^{E'}(X')^+$) we conclude that $\text{vol}_{X|E}(D) = \text{vol}_{X'|E'}(D')$. Now, $\text{vol}_{X'|E'}(D') = \text{vol}_{X'|E'}(P(D')) = P(D')^{2n-1} \cdot E'$, and using the formula (5.1) we have

$$q_{X'}(P(D'), E') \int (P(D'))^{2n} = q_{X'}(P(D')) \cdot (P(D'))^{2n-1} \cdot E',$$

in particular, using the Fujiki relation and the equality $\text{vol}_{X'|E'}(P(D')) = P(D')^{2n-1} \cdot E'$, we have

$$c_{X'} \cdot (q_{X'}(P(D')))^n \cdot q_{X'}(P(D'), E') = q_{X'}(P(D')) \cdot \text{vol}_{X'|E'}(P(D')).$$

On X we have

$$c_X \cdot (q_X(P(D)))^n \cdot q_X(P(D), E) = q_X(P(D)) \cdot (P(D))^{2n-1} \cdot E,$$

and it is known that f induces an isometry between $\text{Pic}(X)$ and $\text{Pic}(X')$ (with respect to the BBF forms) so that $q_{X'}(P(D'), E') = q_X(P(D), E)$, and $q_X(P(D)) = q_X(P(D'))$. Moreover, the Fujiki constant is a birational invariant, thus $c_X = c_{X'}$. Putting everything together we obtain

$$\begin{aligned} \text{vol}_{X'|E'}(P(D')) &= c_{X'}(q_{X'}(P(D')))^{n-1} q_{X'}(P(D'), E') \\ &= c_X(q_X(P(D)))^{n-1} q_X(P(D), E) = P(D)^{2n-1} \cdot E. \end{aligned} \tag{5.2}$$

But $\text{vol}_{X'|E'}(P(D')) = \text{vol}_{X|E}(P(D))$, hence $\text{vol}_{X|E}(P(D)) = P(D)^{2n-1} \cdot E$. ■

Let Y be a normal complex projective variety. It was proven in Theorem 0.1.2 that the volume function on a projective IHS manifold is locally polynomial. Using that for a prime Cartier divisor E the function $\text{vol}_{Y|E}(-)$ (the restricted volume function to E) is the derivative of $\text{vol}_Y(-)$ along the direction in $N^1(Y)_{\mathbf{R}}$ induced by E (see [LM09, Corollary C], or [BFJ09, Corollary C]), one deduces that if the volume function is locally polynomial, also $\text{vol}_{Y|E}(-)$ is. As a consequence of the above proposition, we provide below a proof of the local polynomiality of $\text{vol}_{X|E}(-)$, without using the mentioned differentiability result.

Corollary 5.1.3: For a prime divisor E on X , let $\text{Big}^E(X) \subset \text{Big}(X)$ be the open subcone of the big cone of classes not containing E in their augmented base locus. Then, the function $\text{vol}_{X|E}(-): \text{Big}^E(X) \rightarrow \mathbf{R}^{\geq 0}$ is locally polynomial. Furthermore, $\text{vol}_{X|E}(-)$ takes rational values on big, rational divisor classes.

Proof:

Choose a Boucksom-Zariski chamber Σ_S , associated with an exceptional block $S = \{E_1, \dots, E_k\}$ not containing E (see Corollary 2.2.16), and let $\mathcal{B} = \{P_1, \dots, P_j\}$ be a basis for S^\perp . Then, using equation (5.2), we obtain

$$(\text{vol}_{X|E}(-))|_{\Sigma_S} = c_X \left(q_X \left(\sum_{i=1}^j x_i P_i \right) \right)^{n-1} \cdot q_X \left(\sum_{i=1}^j x_i P_i, E \right),$$

which is a homogeneous polynomial of degree $2n - 1$. Then $\text{vol}_{X|E}(-)$ is locally polynomial, because $\text{Big}^E(X)^+$ is covered by the Boucksom-Zariski chambers of the type Σ_S , with S such that $E \notin S$. The rationality of $\text{vol}_{X|E}(-)$ on rational classes follows by the rationality of the Fujiki constant, and of q_X on $N^1(X)_{\mathbf{Q}}$. ■

5.2 Polygons of Newton-Okounkov type

In this section, to any big divisor D on a projective IHS manifold X , we associate a 2-dimensional convex body. We compute the euclidean volume of these bodies, and we deduce that they are in fact (possibly irrational) polygons.

Definition 5.2.1: Let D be any big divisor on X , and E a prime divisor. We define the threshold

$$\mu_E(D) := \sup\{t > 0 \mid D_t \text{ is big}\},$$

where $D_t := D - tE$.

Lemma 5.2.2: Let E be a prime divisor on X , and D a big divisor. Then, $E \notin \mathbf{B}_-(D)$ implies $E \notin \mathbf{B}_+(D_t)$, for any $t \in]0, \mu_E(D)[$.

Proof:

Recall that the divisorial part of $\mathbf{B}_-(D)$ equals the union of the irreducible components of $N(D)$ (see for example [KL17a, Theorem 4.1]), we can therefore assume $q_X(E) < 0$, otherwise, there is nothing to prove. By [KL17a, Lemma 4.3], $E \notin \mathbf{B}_-(D_t)$, for any $t \geq 0$. We have only to verify that E does not belong to $\text{Null}_{q_X}(P(D_t))$. Suppose by contradiction it does. We have $P(D) + N(D) = P(D_t) + N(D_t) + tE$, and we can find a big and q_X -nef integral divisor M satisfying $\text{Null}_{q_X}(P(D_t)) = \text{Null}_{q_X}(M)$ (see Lemma 2.2.14). Then, the Hodge-index Theorem implies that the Gram matrix of $N(D_t) + tE$ is negative definite, which implies $N(D) = N(D_t) + tE$, and this is absurd because $E \notin \mathbf{B}_-(D)$ by assumption. ■

Construction. Let D be any big \mathbf{R} -divisor on X , and E a prime divisor not lying in $\mathbf{B}_-(D)$. Then $q_X(P(D), E) > 0$, for any $t \in]0, \mu_E(D)[$, by Lemma 5.2.2 and Proposition 4.1.2. We define (inspired

by [KL17b, Exercise 2.2.21]) the following 2-dimensional numerical body

$$\Delta_E^{\text{num}}(D) := \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq \mu_E(D), 0 \leq y \leq q_X(P(D_t), E)\},$$

where $P(D_t)$ is the positive part of the divisorial Zariski decomposition of D_t .

Remark 5.2.3. In Construction 5.2, the body $\Delta_E^{\text{num}}(D)$ is called *numerical* to be consistent with [KL17b, Definition 2.2.20], in the case of surfaces.

Remark 5.2.4. Note that the boundary of $\Delta_E^{\text{num}}(D)$ is defined by a piecewise linear function. Indeed, by Theorem 0.1.2, $\text{Big}(X)$ admits a locally finite decomposition into locally rational polyhedral subcones, called Boucksom-Zariski chambers. Also, in each Boucksom-Zariski chamber the divisorial Zariski decomposition varies linearly, hence $\partial\Delta_E^{\text{num}}(D)$ is piecewise linear.

We now state and prove the main result of this chapter.

Theorem 5.2.5: Let D be any big \mathbf{R} -divisor D on a projective IHS manifold X and E a prime divisor not lying in $\mathbf{B}_-(D)$. Then the body $\Delta_E^{\text{num}}(D)$ is convex and its euclidean volume equals $q_X(P(D))/2$. This gives in particular the relations

$$2^n c_X(\text{vol}(\Delta_E^{\text{num}}(D)))^n = c_X(q_X(P(D)))^n = \text{vol}(D) = (2n)! \cdot \text{vol}(\Delta(D)),$$

where c_X is the Fujiki constant of X , and $\Delta(D)$ is the Newton-Okounkov body of D with respect to any complete flag.

Proof:

We first show that $\Delta_E^{\text{num}}(D)$ is convex. Let $\text{Big}^E(X)^+$ be the convex open subcone of $\text{Big}(X)$ consisting of the big classes α such that $E \notin \mathbf{B}_+(\alpha)$. We claim that the function $q_X(-, E)$ is concave on $\text{Big}^E(X)^+$. If we prove the claim we are done, because by Lemma 5.2.2, the class of any element of the form $D - tE$, with $t \in]0, \mu_E(D)[$ lies in $\text{Big}^E(X)^+$. Let D_1, D_2 two big divisors whose classes lie in $\text{Big}^E(X)^+$. It suffices to show that

$$q_X(P(D_1 + D_2), E) \geq q_X(P(D_1), E) + q_X(P(D_2), E).$$

We observe that $P(D_1 + D_2) \geq P(D_1) + P(D_2)$, because $P(D_1 + D_2)$ is the maximal q_X -nef subdivisor of $D_1 + D_2$. Furthermore, as $D_1 + D_2 = P(D_1) + P(D_2) + N(D_1) + N(D_2)$, $P(D_1 + D_2)$ is obtained by adding to $P(D_1) + P(D_2)$ something coming from $N(D_1) + N(D_2)$. But $E \notin \text{Supp}(N(D_1) + N(D_2))$, hence $q_X(P(D_1 + D_2), E) > q_X(P(D_1), E) + q_X(P(D_2), E)$, and we are done. Now we compute the volume of $\Delta_E^{\text{num}}(D)$. Using relation (5.1), we have

$$q_X(P(D_t), E) \int P(D_t)^{2n} = q_X(P(D_t)) \int P(D_t)^{2n-1} E, \text{ for any } t. \quad (5.3)$$

By definition of $\Delta_E^{\text{num}}(D)$,

$$\text{vol}(\Delta_E^{\text{num}}(D)) = \int_0^{\mu_E(D)} q_X(P(D_t), E) dt.$$

By Proposition 5.1.2 $\int P(D_t)^{2n-1} E = \text{vol}_{X|E}(P(D_t))$. From equation (5.3) we obtain

$$\text{vol}(\Delta_E^{\text{num}}(D)) = \int_0^{\mu_E(D)} q_X(P(D_t), E) dt = \int_0^{\mu_E(D)} q_X(P(D_t)) \cdot \frac{\text{vol}_{X|E}(P(D_t))}{\text{vol}(P(D_t))} dt. \quad (5.4)$$

By [LM09, Corollary 4.25, item (iii)], in the interval $]0, \mu_E(D)[$ the following holds

$$\frac{d}{dt}(\text{vol}_X(P(D_t))) = -2n \cdot \text{vol}_{X|E}(P(D_t)). \quad (5.5)$$

Also, using the Fujiki relation, we have

$$q_X(P(D_t)) = \frac{(\int P(D_t)^{2n})^{1/n}}{(c_X)^{1/n}} = \frac{\text{vol}(P(D_t))^{1/n}}{(c_X)^{1/n}}.$$

Then the equality (5.4) becomes

$$\text{vol}(\Delta_E^{\text{num}}(D)) = -\frac{1}{2n \cdot (c_X)^{1/n}} \int_0^{\mu_E(D)} -2n \cdot (\text{vol}(P(D_t)))^{\frac{1}{n}-1} \text{vol}_{X|E}(P(D_t)) dt,$$

and hence, by equality (5.5),

$$\text{vol}(\Delta_E^{\text{num}}(D)) = -\frac{1}{2n \cdot (c_X)^{1/n}} \left[\frac{(\text{vol}(P(D_t)))^{1/n}}{1/n} \right]_0^{\mu_E(D)} = \frac{1}{2 \cdot (c_X)^{1/n}} \text{vol}(P(D))^{1/n},$$

where the last equality in the above formula comes from the fact that $\text{vol}(P(D_{\mu_E(D)})) = 0$, because $P(D_{\mu_E(D)})$ is not big, by definition of the threshold $\mu_E(D)$. Using once again the Fujiki relation we obtain $2 \cdot \text{vol}(\Delta_E^{\text{num}}(D)) = q_X(P(D))$, as wanted. ■

Remark 5.2.6. Suppose that a prime divisor E belongs to $\mathbf{B}_-(D)$, for some big divisor D on X . Let ν be the coefficient of E in $N(D)$. Then we have

$$2 \cdot \text{vol}(\Delta_E^{\text{num}}(D - \nu E)) = q_X(P(D)),$$

because $P(D) = P(D - \nu E)$. Hence, up to "subtracting" νE from D , we can define the body $\Delta_E^{\text{num}}(D)$, also if $E \in \mathbf{B}_-(D)$. In particular, if E is any prime divisor, we define $\Delta_E^{\text{num}}(D) := \Delta_E^{\text{num}}(D - \nu E)$.

With the following, it turns out that the body $\Delta_E^{\text{num}}(D)$ is, in fact, a polygon.

Proposition 5.2.7: Let D be a big rational divisor on X , E be a prime divisor on X and ν the coefficient of E in $N(D)$. Then $\Delta_E^{\text{num}}(D - \nu E)$ is a polygon.

Proof:

The proof goes exactly as in [KLM12, Proposition 2.1]. By Remark 5.2.4 we only have to check that along the segment $[0, \mu_E(D)]$ we pass through a finite number of Boucksom-Zariski chambers. Without loss of generality, we can assume $\nu = 0$. Set $D' := D - \mu_E(D)E$ and $D'_s := D' + sE$, for

$s \in [0, \mu_E(D)]$. We claim that the function $s \mapsto N(D'_s)$ is non-increasing along $[0, \mu_E(D)]$, i.e. if $0 \leq s' < s \leq \mu_E(D)$, $N(D'_{s'}) - N(D'_s)$ is effective. If we prove the claim we are done. We have

$$P(D'_{s'}) = D'_{s'} - N(D'_{s'}) = (D'_s - (s - s')E) - N(D'_{s'}).$$

As $P(D'_{s'})$ is q_X -nef and $P(D'_s)$ is the maximal q_X -nef subdivisor of $D'_{s'}$,

$$P(D'_s) - P(D'_{s'}) = (s - s')E + N(D'_{s'}) - N(D'_s)$$

is effective. Then we are left to show that E is not contained in the support of $N(D'_s)$, for any $s \in [0, \mu_E(D)]$. Indeed, if $E \in \text{Supp}(N(D'_s))$ for some $s \in [0, \mu_E(D)]$, for any $\lambda > 0$ the divisorial Zariski decomposition of $D'_{s+\lambda}$ would be $D'_{s+\lambda} = P(D'_s) + N(D'_s) + \lambda E$, with $P(D'_s) = P(D'_{s+\lambda})$ and $N(D'_{s+\lambda}) = N(D'_s) + \lambda E$. This would imply $E \in \text{Supp}(N(D'_{\mu_E(D)})) = \text{Supp}(N(D))$, and this is absurd. ■

Remark 5.2.8. Let D be a big rational divisor on X , E be a prime divisor on X and ν the coefficient of E in $N(D)$. Then all the vertices of the polygon $\Delta_E^{\text{num}}(D - \nu E)$ contained in the set $\{[\nu, \mu_E(D)] \times \mathbf{R}\}$ have rational coordinates. Indeed, as already said in Remark 5.2.4, $\text{Big}(X)$ admits a decomposition into locally rational polyhedral subcones, hence all the break-points with the first coordinate lying in $[\nu, \mu_E(D)[$ are rational.

Proposition 5.2.9: Let D be a rational divisor on X , and E a prime divisor not lying in $\mathbf{B}_-(D)$. Then $\mu_E(D)$ is algebraic.

Proof:

Without loss of generality, we can assume that $\Delta_E^{\text{num}}(D)$ is a trapezium. We know by Theorem 0.2.2 that the volume of $\Delta_E^{\text{num}}(D)$ equals $q_X(P(D))/2$, which is a rational number. The last vertex of $\Delta_E^{\text{num}}(D)$ has coordinates $(\mu_E(D), q_X(P(D) - \mu_E(D)E, E))$. Then, $\mu_E(D)$ is a solution of a quadratic equation with rational coefficients, hence it is algebraic. ■

We now globalise Construction 5.2 by proving Theorem 0.4.1.

Proof of Theorem 0.4.1:

We define

$$\Delta_E^{\text{num}}(X) := \{(\zeta, t, y) \in \text{Eff}(X) \times \mathbb{R}^2 \mid \nu_E(\zeta) \leq t \leq \mu_E(\zeta), 0 \leq y \leq q_X(P(\zeta_t), E)\},$$

where $\nu_E(\zeta)$ is the coefficient of E in $N(\zeta)$. If $(\zeta, t, y) \in \Delta_E^{\text{num}}(X)$, clearly $\lambda \cdot (\zeta, t, y) \in \Delta_E^{\text{num}}(X)$, for any positive real number λ , and so $\Delta_E^{\text{num}}(X)$ is a cone. We have to check that $\Delta_E^{\text{num}}(X)$ is convex, namely, if $(\zeta', t', y'), (\zeta'', t'', y'')$ belong to $\Delta_E^{\text{num}}(X)$, we have to show that also $(\zeta + \zeta', t + t', y + y')$ belongs to $\Delta_E^{\text{num}}(X)$. We start by observing that, if $\zeta - \mu_E(\zeta)E$ and $\zeta' - \mu_E(\zeta')E$ lie on the same extremal face of $\overline{\text{Eff}(X)}$, the threshold $\mu_E(\zeta + \zeta')$ is equal to $\mu_E(\zeta) + \mu_E(\zeta')$. If $\zeta - \mu_E(\zeta)E$ and $\zeta' - \mu_E(\zeta')E$ lie on different extremal faces, the divisor $\zeta + \zeta' - (\mu_E(\zeta) + \mu_E(\zeta'))E$ is big. It follows that $\mu_E(\zeta + \zeta') \geq \mu_E(\zeta) + \mu_E(\zeta')$. In particular, we obtain $\nu_E(\zeta + \zeta') \leq t + t' \leq \mu_E(\zeta + \zeta')$. Now, we have to

prove that $q_X(P((\zeta + \zeta')_{t+t'}), E) \geq y + y'$ and that the equality $q_X(P((\zeta + \zeta')_{t+t'}), E) = 0$ could occur only if $q_X(P(\zeta_t), E) = q_X(P(\zeta'_{t'}), E) = 0$. By definition of $\Delta_E^{\text{num}}(D)$, we have

$$E \notin \mathbf{B}_-(\zeta_t) \cup \mathbf{B}_-(\zeta'_{t'}) = \text{Supp}(N(\zeta_t)) \cup \text{Supp}(N(\zeta'_{t'})).$$

Furthermore $(\zeta + \zeta')_{t+t'} = \zeta_t + \zeta'_{t'}$, so that

$$P((\zeta + \zeta')_{t+t'}) = P(\zeta_t + \zeta'_{t'}) = P(\zeta_t) + P(\zeta'_{t'}) + N,$$

where N is an effective divisor not containing E in its support. Then $q_X(P(\zeta_t + \zeta'_{t'}), E) \geq 0$, because $q_X(P(\zeta_t), E) \geq 0$, $q_X(P(\zeta'_{t'}), E) \geq 0$, since $P(\zeta_t)$ and $P(\zeta')$ are q_X -nef, and $q_X(N, E) \geq 0$, by [Bou04, Proposition 4.2, item (ii)], as $E \notin \text{Supp}(N)$. Clearly, $q_X(P(\zeta_t + \zeta'_{t'}), E) = 0$ could occur only if E belongs to $\mathbf{B}_+(\zeta_t)$ and $\mathbf{B}_+(\zeta'_{t'})$. We conclude that $q_X(P(\zeta_t + \zeta'_{t'}), E) \geq y + y' \geq 0$, thus $(\zeta + \zeta', t + t', y + y') \in \Delta_E^{\text{num}}(\zeta + \zeta')$, and we are done. ■

Corollary 5.2.10: Let ζ, ζ' be two big divisor classes, and E a prime divisor on X . We have $\Delta_E^{\text{num}}(\zeta) + \Delta_E^{\text{num}}(\zeta') \subset \Delta_E^{\text{num}}(\zeta + \zeta')$, and hence we obtain the inequality

$$q_X(P(\zeta + \zeta'))^{1/2} \geq q_X(P(\zeta))^{1/2} + q_X(P(\zeta'))^{1/2}. \quad (5.6)$$

Using the Fujiki relation, we recover the log-concavity relation for the volume function, i.e.

$$\text{vol}_X(\zeta + \zeta')^{\frac{1}{2n}} \geq \text{vol}_X(\zeta)^{\frac{1}{2n}} + \text{vol}_X(\zeta')^{\frac{1}{2n}}. \quad (5.7)$$

Proof:

The inequality 5.6 is obtained by using the Brunn-Minkowski Theorem, while the inequality 5.7 is obtained from 5.6 using the Fujiki relation. ■

Remark 5.2.11. Notice that the inequality (5.7) was obtained by Lazarsfeld for any projective variety ([Laz04, Theorem 11.4.9]). His proof makes use of Fujita's approximation Theorem ([Laz04, Theorem 11.4.4]). Hence, the inequality (5.6) can be directly obtained using the Fujiki relation and Lazarsfeld's result. Our proof (of both inequalities) for IHS manifolds does not make use of Fujita's approximation Theorem.

Following [SS16], we now show that, under certain assumptions, the cone $\Delta_E^{\text{num}}(X)$ is rational polyhedral, providing a set of rational generators.

Proposition 5.2.12: Let E be a non-exceptional prime divisor. Suppose that $\text{Eff}(X)$ is rational polyhedral, as well as the closure of any Boucksom-Zariski chamber of $\text{Big}(X)$. Let $\{D_i\}_i$ be a set of rational generators of the extremal rays of the closure of the Boucksom-Zariski chambers. Then, $\Delta_E^{\text{num}}(X)$ is rational polyhedral, generated by the sets $T := \{([D_i], 0, q_X(P(D_i), E))\}_i$, $T' := \{([D_i], 0, 0)\}_i$, and the element $([E], 1, 0)$.

Proof. We first show that the cone generated by T , T' and $([E], 1, 0)$ is contained in $\Delta_E^{\text{num}}(X)$. It suffices to show that $([E], 1, 0) \in \Delta_E^{\text{num}}(X)$ and

$$([D_i], 0, q_X(P(D_i), E)), ([D_i], 0, 0) \in \Delta_E^{\text{num}}(X),$$

for any i . We observe that $\Delta_E(E)$ contains the point $(1, 0)$, because $\mu_E(E) = 1$, hence $([E], 1, 0) \in \Delta_E^{\text{num}}(X)$. The elements of T and T' belong to $\Delta_E^{\text{num}}(X)$ by definition of $\Delta_E^{\text{num}}(D_i)$ and because $E \notin \mathbf{B}_-(D_i)$, as E is not exceptional. We now show that $\Delta_E^{\text{num}}(X)$ is contained in the cone generated by T, T' and $(E, 1, 0)$. Let $([D], a, b)$ be an element of $\Delta_E^{\text{num}}(X)$. Set $D_a = D - aE$ and let $D_a = P(D_a) + N(D_a)$ be the divisorial Zariski decomposition of D_a . There exists a big and q_X -nef divisor P such that D_a belongs to the closure $\overline{\Sigma_P}$ of the Boucksom-Zariski chamber Σ_P associated with P . Let $\{D_{t_j}\}_j$ be the generators of $\overline{\Sigma_P}$. Then, $D_a = \sum_j a_j D_{t_j}$, with $a_j \geq 0$, for any j . In particular, $P(D_a) = \sum_j a_j P(D_{t_j})$ and

$$0 \leq b \leq q_X(P(D_a), E) = \sum_j a_j q_X(P(D_j), E).$$

Then, there exists $c \in [0, 1]$ such that $b = c \sum_j a_j q_X(P(D_j), E)$. Some easy calculations show that

$$([D_a], 0, b) = c \sum_j a_j ([D_j], 0, q_X(P(D_j), E)) + (1 - c) \sum_j a_j ([D_j], 0, 0).$$

If we add $(a[E], a, 0)$ to both members of the equality above, we obtain

$$([D], a, b) = c \sum_j a_j ([D_j], 0, q_X(P(D_j), E)) + (1 - c) \sum_j a_j ([D_j], 0, 0) + a([E], 1, 0),$$

which concludes the proof. ■

5.3 Minkowski decompositions

Following what was done in [LS14] in the case of surfaces, in this section, we investigate under which conditions the polygons $\Delta_E^{\text{num}}(D)$ can be decomposed into simpler ones of the same type. More precisely, let D be a big rational divisor and E a prime divisor on X . We ask the following question.

Question: Is there a set Ω of rational big divisors such that the bodies $\Delta_E^{\text{num}}(D)$ decompose as (rational) Minkowski sum of the bodies $\Delta_E(-)$ associated to divisors in Ω ? If yes, can we choose Ω to be finite?

We recall that the Minkowski sum of two non-empty subsets A and B of a real vector space V is defined as

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Theorem 0.4.2 gives an affirmative answer to the above question, under the assumption that $\text{Eff}(X)$ is rational polyhedral (see Corollaries 0.2.6, 0.2.7).

Definition 5.3.1: With the notation of Theorem 0.4.2, the set Ω is called a Minkowski basis.

From now on we assume that $\text{Eff}(X)$ is rational polyhedral.

To any Boucksom-Zariski chamber of $\text{Big}(X)$ we will associate an element of Ω . Let $\Sigma = \Sigma_P$ be a Boucksom-Zariski chamber associated with an integral divisor $P \in \text{Mov}(X) \cap \text{Big}(X)$. Then, the negative part of the divisorial Zariski decomposition of any element in Σ is supported on a set $S = \{E_1, \dots, E_k\}$ of prime exceptional divisors. Let E be a big prime divisor, and consider the vector subspace $W_{S \cup E}$ of $N^1(X)_{\mathbf{R}}$ generated by S and E . Then, $(W_{S \cup E}) \cap S^\perp$ is a one dimensional vector subspace of $N^1(X)_{\mathbf{R}}$ (because $S \cup E$ spans a $(k+1)$ -dimensional vector subspace of $N^1(X)_{\mathbf{R}}$, and S^\perp is a $(\rho(X) - k)$ -dimensional vector subspace of $N^1(X)_{\mathbf{R}}$), spanned by some rational element of the form $D = xE + \sum_{i=1}^k x_i E_i$. We claim that x and the x_i 's are all of the same sign, so that either D or $-D$ lies in $\text{Mov}(X)$. Let G be the Gram matrix of S . Indeed, by [BKS04, Lemma 4.1] the entries of G^{-1} are non-positive. Then, for a fixed $x \neq 0$, the solution of the linear system

$$G \cdot (x_1, \dots, x_k)^T = -x (q_X(E, E_1), \dots, q_X(E, E_k))^T$$

has the same sign of that of x , because $q_X(E, E_j) \geq 0$, for any $j = 1, \dots, k$. Choosing x rational and positive, we obtain a rational generator D_Σ of $(W_{S \cup E}) \cap S^\perp$ which lies in $\text{Mov}(X)$. Clearly, up to clearing the denominators of $D_\Sigma = xE + \sum_{i=1}^k x_i E_i$, we can assume that D_Σ is integral. We observe that D_Σ is big, because E is, and $\sum_{i=1}^k x_i E_i$ is effective.

We can now describe a Minkowski basis Ω for X . For any Boucksom-Zariski chamber Σ we add to Ω the integral divisor D_Σ constructed above, and for any extremal ray R of $\text{Mov}(X)$ of BBF square 0, we add to Ω an integral generator of R .

We observe that since we are assuming that $\text{Eff}(X)$ is rational polyhedral (and so also $\text{Mov}(X)$ is), Ω is finite. Furthermore, $\mu_E(D_\Sigma) = x$, and, for any $0 < t < \mu_E(D)$, D_t belongs to Σ , by construction.

We now prove Theorem 0.4.2.

Proof of Theorem 0.4.2:

Let D be a rational movable divisor on X . If D is not big, $q_X(D) = 0$, and so $\lambda D = D'$ for some $D' \in \Omega$, $\lambda \in \mathbf{N}$ ([MY15, Lemma 3.1]). Then $\Delta_E(D) = \frac{1}{\lambda} \Delta_E(D')$. Otherwise, let Σ be the Boucksom-Zariski chamber associated with P , and D_Σ the element of Ω attached to P . Define the movable threshold

$$\tau = \tau(D) := \sup \{t > 0 \mid D - tD_\Sigma \text{ is movable}\}.$$

Clearly τ is a rational number, and $D' := D - \tau D_\Sigma$ belongs to $\text{Face}(D) := \text{Null}_{q_X}(D)^\perp \cap \text{Mov}(X)$. If $D' = 0$, we are done. Otherwise, we claim that

$$\Delta_E^{\text{num}}(D) = \Delta_E^{\text{num}}(D') + \tau \Delta_E^{\text{num}}(D_\Sigma). \quad (5.8)$$

To show equality 5.8 we first observe that if $0 \leq t \leq \mu_E(\tau D_\Sigma)$, the divisor $D - tE = D' + (\tau D_\Sigma - tE)$ is effective, because D' is effective, and $\tau D_\Sigma - tE = (\tau \cdot x - t)E + \sum_i x_i E_i$ is effective, since $\mu_E(\tau D_\Sigma) = \tau \cdot x$. Also, note that $\mu_E(D) \geq \mu_E(\tau D_\Sigma)$. Then, for any $0 \leq t \leq \mu_E(\tau D_\Sigma)$ we have

$$P(D - tE) = P(D' + \tau D_\Sigma - tE) = P(D') + P(\tau D_\Sigma - tE).$$

Now, let t be such that $\mu_E(\tau D_\Sigma) \leq t \leq \mu_E(D)$. If $\mu_E(\tau D_\Sigma) = \mu_E(D)$, D' is not big, and we are done. Otherwise D' is big, and, arguing as in Proposition 5.2.7, we see that $D' - (t - \mu_E(\tau D_\Sigma))E$ is big for any $t \in [\mu_E(\tau D_\Sigma), \mu_E(D)[$, and $N(D - tE) \geq \sum_i x_i E_i$. In particular,

$$P(D - tE) = P(D' - (t - \mu_E(\tau D_\Sigma))E),$$

for any t in $[\mu_E(\tau D_\Sigma), \mu_E(D)]$. Resuming, we obtained

$$P(D_t) = \begin{cases} P(D') + P(\tau D_\Sigma - tE), & \text{if } 0 \leq t \leq \mu_E(\tau D_\Sigma), \\ P(D' - (t - \mu_E(\tau D_\Sigma))E), & \text{if } \mu_E(\tau D_\Sigma) \leq t \leq \mu_E(D), \end{cases}$$

and so

$$P(D_t) = \begin{cases} P(D') + (\tau \cdot x - t)E, & \text{if } 0 \leq t \leq \tau \cdot x, \\ P(D' - (t - \mu_E(\tau D_\Sigma))E), & \text{if } \tau \cdot x \leq t \leq \mu_E(D). \end{cases}$$

Then

$$q_X(P(D_t), E) = \begin{cases} q_X(P(D'), E) + q_X(P(\tau D_\Sigma - tE), E), & \text{if } 0 \leq t \leq \mu_E(\tau D_\Sigma), \\ q_X(P(D' - (t - \mu_E(\tau D_\Sigma))E), E), & \text{if } \mu_E(\tau D_\Sigma) \leq t \leq \mu_E(D), \end{cases}$$

that means $\Delta_E^{\text{num}}(D) = \Delta_E^{\text{num}}(D') + \tau \Delta_E^{\text{num}}(D_\Sigma)$. Now, as we are assuming that D' is big, there exists a prime exceptional divisor not lying in $\text{Null}_{q_X}(D')$. Indeed, if $\text{Null}_{q_X}(D')$ coincides with the set of prime exceptional divisors of X , we would have $q_X(D') = 0$, and this would contradict the bigness of D' , by Lemma 1.3.10. Then, the set of prime exceptional divisors of X not lying in $\text{Null}_{q_X}(D')$ is finite because we are assuming that $\text{Eff}(X)$ is rational polyhedral. We claim that $\text{Null}_{q_X}(D')$ strictly contains $\text{Null}_{q_X}(P(D))$. Indeed, if not, there would exist $0 < \epsilon \ll 1$ such that $D - (\tau + \epsilon)D_\Sigma$ lies in $\text{Mov}(X)$, and this would contradict the maximality of τ . Then, we can consider the Boucksom-Zariski chamber associated with D' , which is then different to that associated with D , by Lemma 2.2.15. Then, we can repeat the above procedure to decompose $\Delta_E^{\text{num}}(D')$, as we did for $\Delta_E^{\text{num}}(D)$. This can be done finitely many times because at any step the dimension of $\text{Face}(D') = \text{Null}_{q_X}(D')^\perp \cap \text{Mov}(X)$ decreases of at least 1, and for at most $\rho(X)$ times because the dimension of $\text{Face}(D')$ is at most $\rho(X)$. The proof is then concluded. \blacksquare

If we allow a Minkowski basis to be infinite and we suppose that $\overline{\text{Eff}(X)} = \text{Eff}(X)$ (which does not imply that $\text{Eff}(X)$ is polyhedral), Theorem 0.4.2 holds in more generality. In particular, we have the following corollary.

Corollary 5.3.2: Let E be any big prime divisor on X and suppose that $\overline{\text{Eff}(X)} = \text{Eff}(X)$. Then, there exists a set $\Omega = \Omega(E)$ of \mathbf{Q} -divisors whose classes lie in $\overline{\text{Mov}(X)}$, such that, if D is any big \mathbf{Q} -divisor, there exist some rational numbers $\{\alpha_P(D)\}_{P \in \Omega}$ such that $P(D) = \sum_{P \in \Omega} \alpha_P(D)P$ and, up to a translation, $\Delta_E^{\text{num}}(D) = \sum_{P \in \Omega} \alpha_P(D)\Delta_E^{\text{num}}(P)$. In particular, for any big \mathbf{Q} -divisor D , the polygon $\Delta_E^{\text{num}}(D)$ is rational.

Proof. The proof is the same as that of Theorem 0.4.2. ■

Remark 5.3.3. Theorem 0.4.2 and Corollary 5.3.2 hold also for any big \mathbf{R} -divisor D on X . The only difference is that the coefficients $\alpha_P(D)$ appearing in the Minkowski decomposition might not be rational.

5.4 Examples

We start by computing some polygons of Newton-Okounkov type on Hilbert schemes of points on K3 surfaces.

Example 5.4.1. Let X be $\text{Hilb}^2(S)$, where S is a projective K3 surface with $\text{Pic}(S) = \mathbf{Z} \cdot H_S$ and $H_S^2 = 2$. Let E' be a general member of the linear system $|H - \delta|$, and E the exceptional divisor of the Hilbert-Chow morphism $S^{[2]} \rightarrow S^{(2)}$. We compute the polygons $\Delta_E^{\text{num}}(H)$, $\Delta_{E'}^{\text{num}}(3H - E)$.

We have $\mu_E(H) = 1/2$, because $P(H_t) = H_t = H - tE$ for any $t \in [0, 1/2]$, and $H - \frac{1}{2}E = H - \delta$ is not big, but pseudo-effective. Then,

$$\Delta_E^{\text{num}}(H) = \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq 1/2, 0 \leq y \leq 8t\}$$

is a triangle. The volume of $\Delta_E^{\text{num}}(H)$ is given by $q_X(H)/2 = 1$.

Now, we consider $\Delta_{E'}^{\text{num}}(3H - E)$. In this case $\mu_{E'}(3H - E) = 3$, because $3H - E - 3(H - \frac{1}{2}E) = \frac{3}{2}E$, which is effective, but not big. We have

$$P(3H - E - tE') = \begin{cases} 3H - E - tE', & \text{if } 0 \leq t \leq 2, \\ (3 - t)H, & \text{if } 2 \leq t \leq 3. \end{cases}$$

It follows that

$$\Delta_{E'}^{\text{num}}(3H - E) = \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq 2, 0 \leq y \leq 2 \text{ or } 2 \leq t \leq 3, 0 \leq y \leq 6 - 2t\},$$

which is a trapezium. In this case, the euclidean volume of $\Delta_{E'}^{\text{num}}(3H - E)$ is $q_X(3H - E)/2 = 5$.

We now provide a set of generators G for $\Delta_E^{\text{num}}(X)$. We have 2 Boucksom-Zariski chambers, Σ_H and Σ_M , where M is any movable divisor. Furthermore, $\overline{\Sigma_M} = \langle H - \delta, H \rangle$, $\overline{\Sigma_H} = \langle H, \delta \rangle$. In particular, $\text{Eff}(X)$ is rational polyhedral, as well as the Boucksom-Zariski chambers. Then, we can use Proposition 5.2.12 and pick

$$G = \{([2H - E], 0, 0), ([H], 0, 0), ([E], 0, 0), ([2H - E], 0, 8), ([E], 1, 0)\}.$$

We now give an example of a K3 surface S such that the classical Newton-Okounkov body $\Delta_{F_\bullet}(D)$ of any big \mathbf{Q} -divisor D with respect to a general admissible flag $F_\bullet = \{\{pt\} \subset C \subset S\}$ is a rational polygon. In this case $\Delta_{F_\bullet}(D) = \Delta_C^{\text{num}}(D)$, for any big \mathbf{R} -divisor D .

Example 5.4.2. Let S be a K3 surface such that $\text{Pic}(S)$ is isomorphic to \mathbf{Z}^3 , with intersection form given by the matrix

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

One can show (cf. [Tot12]) that S has a unique elliptic fibration $S \rightarrow \mathbf{P}^1$, induced by some nef line bundle L , and infinitely many smooth rational curves, each of which is a section of $S \rightarrow \mathbf{P}^1$. The ray spanned by L in $\text{Nef}(S)$ is extremal, and is the only extremal ray of square 0 of the nef cone. Let $C \subset S$ be a big curve. Applying Corollary 5.3.2, we find an *infinite* set of nef \mathbf{Q} -divisors Ω , such that any nef \mathbf{Q} -divisor is a linear combination (with coefficients in \mathbf{Q}) of elements in Ω , and any polygon of Newton-Okounkov type $\Delta_C^{\text{num}}(D)$ is, up to translation, the Minkowski sum (with coefficients in \mathbf{Q}) of some of the polygons $\{\Delta_C^{\text{num}}(P)\}_{P \in \Omega}$. If we choose $\{pt\}$ to be a general smooth point of C and set

$$F_\bullet := \{\{pt\} \subset C \subset S\},$$

we clearly have $\Delta_C^{\text{num}}(D) = \Delta_{F_\bullet}(D)$, for any big \mathbf{R} -divisor D . In particular, since any of the $\Delta_C^{\text{num}}(D)$ is rational whenever D is rational, also any of the $\Delta_{F_\bullet}(D)$ is.

The following example shows that the bodies $\Delta_C^{\text{num}}(D)$ on a projective K3 surface S are in some sense preserved when passing to the Hilbert scheme of points $S^{[n]}$.

Example 5.4.3. Let S be a projective K3 surface and $F_\bullet = \{\{pt\} \subset C \subset S\}$ a general admissible flag. Let D be any big \mathbf{R} -divisor on S . By the generality of F_\bullet and [LM09, Theorem 6.4], the classical Newton-Okounkov body $\Delta_{F_\bullet}(D)$ is

$$\Delta_{F_\bullet}(D) = \Delta_C^{\text{num}}(D) = \{(t, y) \in \mathbf{R}^2 \mid 0 \leq t \leq \mu_C(D), 0 \leq y \leq P(D_t) \cdot C\}.$$

Now, let $X = S^{[n]}$ be the Hilbert scheme of n points on S . Any irreducible curve C' on S is canonically associated with a prime divisor on X . This correspondence induces a homomorphism $\iota: \text{Div}_{\mathbf{R}}(S) \rightarrow \text{Div}_{\mathbf{R}}(X)$, which in turn induces the usual injective group homomorphism $\iota': \text{Pic}(S) \hookrightarrow \text{Pic}(X)$, which embeds $\text{Pic}(S)$ in $\text{Pic}(X)$ as a sublattice. In particular, $C'^2 = q_X(\iota'([C']))$, and to any exceptional block on S (i.e. the Gram matrix of the elements of S , with respect to q_X , is negative definite) corresponds an exceptional block on the Hilbert scheme X . Furthermore, ι preserves the divisorial Zariski decomposition of effective divisors, i.e. if D' is an effective divisor on S , with (divisorial) Zariski decomposition $P(D') + N(D')$, $\iota(P(D')) + \iota(N(D'))$ is the divisorial Zariski decomposition of $\iota(D')$. Indeed, it suffices to check that the class of $\iota(P(D'))$ belongs to $\overline{\text{Mov}}(X)$. Write $P(D') = \sum_i a_i E_i$, with $a_i > 0$ and E_i prime for any i . Then, $\iota(P(D'))$ is effective, hence we are left to check that $\iota(P(D'))$ q_X -intersect non-negatively any prime exceptional divisor on X . But this follows from the fact that $P(D')$ is nef on S and [Bou04, Proposition 4.2, item (ii)]. It

follows that a divisor D' on S is big if and only if $\iota(D')$ is because ι preserves the divisorial Zariski decomposition, and by [Huy99, Proposition 3.8]. Then, $\mu_C(D) = \mu_{\iota(C)}(\iota(D))$, and since ι' respects the BBF forms of S and X , we obtain the equality $\Delta_F(D) = \Delta_{E_C}^{\text{num}}(\iota(D))$. In particular, the natural injective linear map $N^1(S)_{\mathbf{R}} \times \mathbf{R}^2 \rightarrow N^1(X)_{\mathbf{R}} \times \mathbf{R}^2$ maps the cone $\Delta_F(S) = \Delta_C^{\text{num}}(S)$ into $\Delta_{\iota(C)}^{\text{num}}(X)$.

The next example shows that the polygons $\Delta_E^{\text{num}}(D)$ might be irrational.

Example 5.4.4. Consider the Fano variety of lines $F = F(Y)$ of a smooth cubic fourfold Y containing a smooth cubic scroll T with

$$H^4(Y, \mathbf{Z}) \cap H^{2,2}(Y, \mathbf{C}) = \mathbf{Z}h^2 + \mathbf{Z}T.$$

By the work of Beauville and Donagi, F is a projective IHS manifold of dimension 4. We have $\text{Big}(F) = \text{Eff}(F) = \text{Mov}(F) = \mathcal{C}_F$ (cf. [HT10] and [Den, Theorem 1.2]), and $\overline{\text{Eff}(F)}$ is irrational. Then, for any big integral divisor D on X , there exists a prime divisor E such that the polygon $\Delta_E^{\text{num}}(D)$ is an irrational triangle.

BIBLIOGRAPHY

- [AV15] Ekaterina Amerik and Misha Verbitsky. “Rational Curves on Hyperkähler Manifolds”. In: *Int. Math. Res. Not. IMRN* 2015.23 (2015), pp. 13009–13045 (cit. on p. 32).
- [AV16a] Ekaterina Amerik and Misha Verbitsky. “Hyperbolic geometry of the ample cone of a hyperkähler manifold”. In: *Res. Math. Sci.* 3 (July 2016) (cit. on pp. 69, 73).
- [AV16b] Ekaterina Amerik and Misha Verbitsky. “Collections of Orbits of Hyperplane Type in Homogeneous Spaces, Homogeneous Dynamics, and Hyperkähler Geometry”. In: *Int. Math. Res. Not. IMRN* 2020 (2016), pp. 25–38 (cit. on pp. 71, 72).
- [Bau09] Thomas Bauer. “A simple proof for the existence of Zariski decompositions on surfaces”. In: *J. Algebraic Geom.* 18 (2009), pp. 789–793 (cit. on p. 2).
- [BBP13] Sébastien Boucksom, Amaël Broustet, and Gianluca Pacienza. “Uniruledness of stable base loci of adjoint linear systems via Mori theory”. In: *Math. Z.* 275.1-2 (2013), pp. 499–507 (cit. on pp. 8, 17, 81–84, 87, 88, 91, 92).
- [BC22] Pietro Beri and Alberto Cattaneo. “On birational transformations of Hilbert schemes of points on K3 surfaces”. In: *Math. Z.* 301 (June 2022), pp. 1–18 (cit. on p. 6).
- [BCL14] Sébastien Boucksom, Salvatore Cacciola, and Angelo Felice Lopez. “Augmented base loci and restricted volumes on normal varieties”. In: *Math. Z.* 278.3-4 (2014), pp. 979–985 (cit. on p. 85).
- [BDPP13] Sébastien Boucksom, Jean-Pierre Demailly, Mihai Păun, and Thomas Peternell. “The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension”. In: *J. Algebraic Geom.* 22.2 (2013), pp. 201–248 (cit. on pp. 11, 92).
- [Bea83] Arnaud Beauville. “Variétés Kählériennes dont la première classe de Chern est nulle”. In: *J. Differential Geom.* 18.4 (1983), pp. 755–782 (cit. on pp. 1, 22).
- [BF12] Thomas Bauer and Michael Funke. “Weyl and Zariski chambers on K3 surfaces”. In: *Forum Math.* 28 (2012), pp. 1159–1166 (cit. on pp. 4, 5, 58).
- [BFJ09] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. “Differentiability of volumes of divisors and a problem of teissier”. In: *J. Algebraic Geom.* 18 (2009), pp. 279–308 (cit. on pp. 36, 104).
- [BG23] Valeria Bertini and Annalisa Grossi. “Rational curves on primitive symplectic varieties of OG_6^S -type”. In: *Math. Z.* 304 (June 2023) (cit. on p. 72).
- [BKS04] Thomas Bauer, Alex Küronya, and Tomasz Szemberg. “Zariski chambers, volumes, and stable base loci”. In: *J. Reine Angew. Math.* 576 (2004), pp. 209–233 (cit. on pp. iii, v, vii, viii, 2, 3, 9, 18, 43, 51, 54, 59, 60, 63, 89, 90, 111).

- [BL21] Benjamin Bakker and Christian Lehn. “A global Torelli theorem for singular symplectic varieties”. In: *J. Eur. Math. Soc. (JEMS)* 23.3 (2021), pp. 949–994 (cit. on p. 86).
- [BL22] Benjamin Bakker and Christian Lehn. “The global moduli theory of symplectic varieties”. In: *J. Reine Angew. Math.* 2022.790 (2022), pp. 223–265 (cit. on pp. 8, 73).
- [BM13] Arend Bayer and Emanuele Macrì. “MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations”. In: *Invent. Math.* 198 (Jan. 2013), pp. 505–590 (cit. on pp. 6, 95).
- [Bou04] Sébastien Boucksom. “Divisorial Zariski decompositions on compact complex manifolds”. In: *Ann. Sci. Ec. Norm. Supér. Ser. 4*, 37.1 (2004), pp. 45–76 (cit. on pp. 3, 23, 25–28, 35, 38, 40, 41, 51, 60–63, 109, 114).
- [Bur05] William Burnside. “On criteria for the finiteness of the order of a group of linear substitutions”. In: *Proc. Lond. Math. Soc. (3)* s2-3 (1 1905), pp. 435–440 (cit. on p. 70).
- [CF19] Andrea Cattaneo and Lie Fu. “Finiteness of Klein actions and real structures on compact hyperkähler manifolds”. In: *Math. Ann.* 375.3-4 (2019), pp. 1783–1822 (cit. on p. 100).
- [Cho12] Sung Rak Choi. “Duality of the cones of divisors and curves”. In: *Math. Res. Lett.* 19 (2012), pp. 403–416 (cit. on pp. 11, 12).
- [Dem92] Jean-Pierre Demailly. “Regularization of closed positive currents and Intersection theory”. In: *J. Algebraic Geom.* 1 (1992), pp. 361–409 (cit. on pp. 24, 26).
- [Den] Francesco Antonio Denisi. *Pseudo-effective classes on projective irreducible holomorphic symplectic manifolds*. Submitted. URL: <https://arxiv.org/pdf/2205.15148.pdf> (cit. on pp. 5, 11, 115).
- [Den23] Francesco Antonio Denisi. “Boucksom–Zariski and Weyl Chambers on Irreducible Holomorphic Symplectic Manifolds”. In: *Int. Math. Res. Not. IMRN* 2023 (20 2023), pp. 17156–17204 (cit. on p. 2).
- [DO23] Francesco Antonio Denisi and Ángel David Ríos Ortiz. *Asymptotic base loci on hyper-Kähler manifolds*. <https://arxiv.org/abs/2304.01773>. 2023 (cit. on p. 8).
- [Dru11] Stéphane Druel. “Quelques remarques sur la décomposition de Zariski divisorielle sur les variétés dont la première classe de Chern est nulle”. In: *Math. Z.* 267.1-2 (2011), pp. 413–423 (cit. on p. 91).
- [ELMNP06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustață, Michael Nakamaye, and Mihnea Popa. “Restricted volumes and base loci of linear series”. In: *Amer. J. Math.* 131 (2006), pp. 607–651 (cit. on pp. 10, 11, 16, 19, 82).

- [ELMP06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustață, and Mihnea Popa. “Asymptotic invariants of stable base loci”. In: *Ann. Inst. Fourier (Grenoble)* 56.6 (2006), pp. 1701–1734 (cit. on pp. 9, 17, 18, 55, 87, 89, 90).
- [Fuj79] Takao Fujita. “On Zariski problem”. In: *Proc. Japan Acad. Ser. A Math. Sci.* 55 (1979), pp. 106–110 (cit. on p. 2).
- [GHJ03] Mark Gross, Daniel Huybrechts, and Dominic Joyce. *Calabi-Yau Manifolds and Related Geometries. Lectures at a Summer School in Nordfjordeid, Norway, June 2001*. Springer-Verlag, 2003 (cit. on pp. 1, 20, 22, 29, 82, 103).
- [GHS03] Tom Graber, Joe Harris, and Jason Starr. “Families of rationally connected varieties”. In: *J. Amer. Math. Soc.* 16.1 (2003), pp. 57–67 (cit. on p. 87).
- [GO20] Frank Gounelas and John Christian Ottem. “Remarks on the positivity of the cotangent bundle of a K3 surface”. In: *Épijournal Géom. Algébrique* 4 (2020), Art. 8, 16 (cit. on p. 96).
- [HK00] Yi Hu and Sean Keel. “Mori dream spaces and GIT.” In: *Michigan Mathematical Journal* 48.1 (2000), pp. 331–348 (cit. on p. 71).
- [HR20] Krishna Hanumanthu and Nabanita Ray. “Weyl and Zariski chambers on projective surfaces”. In: *Forum Math.* 32 (2020), pp. 1027–1037 (cit. on pp. 4, 5, 58, 63).
- [HT09] Brendan Hassett and Yuri Tschinkel. “Moving and ample cones of holomorphic symplectic fourfolds”. In: *Geom. Funct. Anal.* 19 (2009), pp. 1065–1080 (cit. on p. 25).
- [HT10] Brendan Hassett and Yuri Tschinkel. “Flops on holomorphic symplectic fourfolds and determinantal cubic hypersurfaces”. In: *J. Inst. Math. Jussieu* 9.1 (2010), pp. 125–153 (cit. on pp. 97, 100, 115).
- [Huy03] Daniel Huybrechts. “The Kähler cone of a compact hyperkähler manifold”. In: *Math. Ann.* 326.3 (2003), pp. 499–513 (cit. on pp. 1, 29).
- [Huy14] Daniel Huybrechts. “Curves and cycles on K3 surfaces”. In: *Algebr. Geom.* 1.1 (2014). With an appendix by C. Voisin, pp. 69–106 (cit. on p. 87).
- [Huy15] Daniel Huybrechts. *Lectures on K3 surfaces*. Available online: <https://www.math.uni-bonn.de/people/huybrech/K3Global.pdf>. Cambridge University Press, 2015 (cit. on pp. 7, 21, 67, 77).
- [Huy97] Daniel Huybrechts. “Birational symplectic manifolds and their deformations”. In: *J. Differential Geom.* 45.3 (1997), pp. 488–513 (cit. on p. 1).
- [Huy99] Daniel Huybrechts. “Compact Hyperkähler Manifolds: Basic Results”. In: *Invent. Math.* 135 (1999), pp. 63–113 (cit. on pp. 1, 22–24, 27, 82, 85, 115).
- [IR90] Kenneth Ireland and Michael Rosen. *A classical introduction to modern number theory*. Springer-Verlag, 1990 (cit. on p. 74).

- [Jia22] Chen Jiang. “Positivity of Riemann-Roch polynomials and Todd classes of hyperkähler manifolds”. In: *J. Algebraic Geom.* 32 (2022), pp. 239–269 (cit. on pp. [vii](#), [1](#)).
- [Kaw88] Yujiro Kawamata. “Crepant Blowing-Up of 3-Dimensional Canonical Singularities and Its Application to Degenerations of Surfaces”. In: *Ann. of Math.* 2nd ser. 127.1 (1988), pp. 93–163 (cit. on p. [70](#)).
- [Kaw91] Yujiro Kawamata. “On the length of a extremal rational curve”. In: *Invent. Math.* 105.3 (1991), pp. 609–612 (cit. on pp. [83–85](#)).
- [KL17a] Alex Küronya and Victor Lozovanu. “Positivity of Line Bundles and Newton-Okounkov Bodies”. In: *Doc. Math.* 22 (2017), pp. 1285–1302 (cit. on pp. [x](#), [12](#), [85](#), [105](#)).
- [KL17b] Alex Küronya and Victor Lozovanu. “Geometric aspects of Newton-Okounkov bodies”. In: *Banach Center Publications* 116 (2017) (cit. on p. [106](#)).
- [KLM12] Alex Küronya, Victor Lozovanu, and Catriona Maclean. “Convex bodies appearing as Okounkov bodies of divisors”. In: *Adv. Math.* 229.5 (2012), pp. 2622–2639 (cit. on pp. [x](#), [xi](#), [12](#), [107](#)).
- [KLM15] Andreas Knutsen, Margherita Lelli-Chiesa, and Giovanni Mongardi. “Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds”. In: *Trans. Amer. Math. Soc.* 371 (2015), pp. 1403–1438 (cit. on pp. [32](#), [86](#)).
- [KM98] Janos Kollár and Shigefumi Mori. *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Mathematics. Cambridge University Press, 1998 (cit. on p. [84](#)).
- [KMPP19] Michał Kapustka, Giovanni Mongardi, Gianluca Pacienza, and Piotr Pokora. On the Boucksom-Zariski decomposition for irreducible symplectic varieties and bounded negativity. <https://arxiv.org/abs/1911.03367>. 2019 (cit. on pp. [28](#), [38](#), [51](#), [73](#)).
- [Kol96] János Kollár. *Rational curves on algebraic varieties*. Vol. 32. Springer-Verlag, Berlin, 1996 (cit. on pp. [5](#), [37](#), [87](#), [88](#)).
- [Kov13] Sándor Kovács. “The Cone of Curves of K3 Surfaces Revisited”. In: *Birational Geometry, Rational Curves, and Arithmetic*. New York, NY: Springer New York, 2013, pp. 163–169 (cit. on pp. [5](#), [67](#), [73](#)).
- [Kov93] Sándor Kovács. “The cone of curves of a K3 surface”. In: *Math. Ann.* 300.4 (1993), pp. 681–691 (cit. on pp. [5](#), [26](#), [67](#)).
- [Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry I. Classical Setting: Line Bundles and Linear Series*. Springer Nature, 2004 (cit. on pp. [10](#), [11](#), [16](#), [18](#), [19](#), [39](#), [51](#), [109](#)).
- [Leh11] Brian Lehmann. *The movable cone via intersections*. 2011. URL: <https://arxiv.org/pdf/1111.3928.pdf> (cit. on p. [12](#)).

- [Les14] John Lesieutre. “The diminished base locus is not always closed”. In: *Compos. Math.* 150.10 (2014), pp. 1729–1741 (cit. on pp. 17, 100).
- [Lin20] Hsueh-Yung Lin. “Lagrangian constant cycle subvarieties in Lagrangian fibrations”. In: *Int. Math. Res. Not. IMRN* 1 (2020), pp. 14–24 (cit. on p. 86).
- [LM09] Robert Lazarsfeld and Mircea Mustață. “Convex bodies associated to linear series”. In: *Ann. Sci. Éc. Norm. Super.* 4th ser. 42.5 (2009), pp. 783–835 (cit. on pp. x, xi, 12, 36, 104, 107, 114).
- [LMP22] Christian Lehn, Giovanni Mongardi, and Gianluca Pacienza. *The Morrison–Kawamata cone conjecture for singular symplectic varieties*. <https://arxiv.org/abs/2207.14754>. 2022 (cit. on p. 73).
- [LMP23] Christian Lehn, Giovanni Mongardi, and Gianluca Pacienza. “Deformations of rational curves on primitive symplectic varieties and applications”. In: *Algebr. Geom.* 10.2 (2023), pp. 199–227 (cit. on pp. 8, 72, 73).
- [LMR20] Antonio Laface, Alex Massarenti, and Rick Rischter. “On Mori chamber and stable base locus decompositions”. In: *Trans. Amer. Math. Soc.* 373.3 (2020), pp. 1667–1700 (cit. on p. 90).
- [LP16] Christian Lehn and Gianluca Pacienza. “Deformations of singular symplectic varieties and termination of the log minimal model program”. In: *Algebr. Geom.* 3.4 (2016) (cit. on p. 92).
- [ŁS14] Patrycja Łuszcz-Świdecka and David Schmitz. “Minkowski decomposition of Okounkov bodies on surfaces”. In: *J. Algebra* 414 (2014), pp. 159–174 (cit. on pp. xii, 13, 110).
- [LZ22] Zhiyuan Li and Ruxuan Zhang. “Beauville–Voisin Filtrations on Zero-Cycles of Moduli Space of Stable Sheaves on K3 Surfaces”. In: *Int. Math. Res. Not. IMRN* (2022). In press. DOI: [10.1093/imrn/rnac161](https://doi.org/10.1093/imrn/rnac161) (cit. on p. 101).
- [Mar09] Eyal Markman. “Prime exceptional divisors on holomorphic symplectic varieties and monodromy-reflections”. In: *Kyoto J. Math.* 53 (Dec. 2009), pp. 345–403 (cit. on pp. 7, 31, 76–78, 91).
- [Mar10] Eyal Markman. “Integral constraints on the monodromy group of the hyperkähler resolution of a symmetric product of a K3 surface”. In: *Internat. J. of Math.* 2 (2010), pp. 169–223 (cit. on pp. 30, 76, 77).
- [Mar11] Eyal Markman. “A survey of Torelli and monodromy results for holomorphic-symplectic varieties, Proc. of the conference ‘Complex and Differential Geometry’”. In: *Springer Proc. Math.* 8 (2011), pp. 189–220 (cit. on pp. 20, 21, 25, 27, 68, 69, 71, 76, 78).
- [Mar22] Bruno Martelli. *An Introduction to Geometric Topology*. 2nd edition. Available at http://people.dm.unipi.it/martelli/Geometric_topology.pdf. 2022 (cit. on p. 69).

- [Mar23] Eyal Markman. “The monodromy of generalized Kummer varieties and algebraic cycles on their intermediate Jacobians”. In: *J. Eur. Math. Soc. (JEMS)* 25.1 (2023), pp. 231–321 (cit. on pp. 31, 77).
- [Mat13] Shin-ichi Matsumura. “Restricted volumes and divisorial Zariski decompositions”. In: *Amer. J. Math.* 135.3 (2013), pp. 637–662 (cit. on p. 104).
- [Mat17] Daisuke Matsushita. “On isotropic divisors on irreducible symplectic manifolds”. In: *Higher Dimensional Algebraic Geometry: In honour of Professor Yujiro Kawamata’s sixtieth birthday* 74 (2017), pp. 291–312 (cit. on p. 26).
- [Mat18] Daisuke Matsushita. *On subgroups of an automorphism group of an irreducible symplectic manifold*. <https://arxiv.org/abs/1808.10070>. 2018 (cit. on p. 69).
- [MM11] Eyal Markman and Sukhendu Mehrotra. “Hilbert schemes of K3 surfaces are dense in moduli”. In: *Math. Nachr.* 290.5-6 (Dec. 2011), pp. 876–884 (cit. on p. 31).
- [MO22] Giovanni Mongardi and Claudio Onorati. “Birational geometry of irreducible holomorphic symplectic tenfolds of O’Grady type”. In: *Math. Z.* 300 (2022), pp. 3497–3526 (cit. on pp. 26, 75, 78).
- [Mon15] Giovanni Mongardi. “A note on the Kähler and Mori cones of hyperkähler manifolds”. In: *Asian J. Math.* 19 (2015), pp. 583–592 (cit. on pp. 31, 32).
- [Mon16] Giovanni Mongardi. “On the monodromy of irreducible symplectic manifolds”. In: *Algebr. Geom.* 3 (3 2016), pp. 385–391 (cit. on pp. 31, 77).
- [MR20] Giovanni Mongardi and Antonio Rapagnetta. “Monodromy and birational geometry of O’Grady’s sixfolds”. In: *J. Math. Pures Appl. (9)* 146 (2020), pp. 31–68 (cit. on pp. 20, 26, 75, 76, 78).
- [MY15] Eyal Markman and Kōta Yoshioka. “A Proof of the Kawamata–Morrison Cone Conjecture for Holomorphic Symplectic Varieties of $K3^{[n]}$ or Generalized Kummer Deformation Type”. In: *Int. Math. Res. Not. IMRN* 2015 (24 2015), pp. 13563–13574 (cit. on pp. 71, 72, 74, 111).
- [MZ13] Daisuke Matsushita and De-Qi Zhang. “Zariski F -decomposition and Lagrangian fibration on hyperkähler manifolds”. In: *Math. Res. Lett.* 20.5 (2013), pp. 951–959 (cit. on pp. 25, 82–84, 92).
- [Nak04] Noboru Nakayama. *Zariski-decomposition and Abundance, MJS Memoirs*. Vol. 14. Mathematical Society of Japan, 2004 (cit. on pp. 3, 27).
- [OGr03] Kieran Gregory O’Grady. “A new six dimensional irreducible symplectic variety”. In: *J. Algebraic Geom.* 12 (2003), pp. 435–505 (cit. on p. 1).
- [OGr97] Kieran Gregory O’Grady. “The weight-two Hodge structure of moduli spaces of sheaves on a K3 surface”. In: *J. Algebraic Geom.* 6 (1997), pp. 599–644 (cit. on p. 23).
- [OGr99] Kieran Gregory O’Grady. “Desingularized moduli spaces of sheaves on a K3”. In: *J. Reine Angew. Math.* 512 (1999), pp. 49–117 (cit. on p. 1).

- [Ogu14] Keiji Oguiso. “Automorphism groups of Calabi-Yau manifolds of Picard number 2”. In: *J. Algebraic Geom.* 23 (2014), pp. 775–795 (cit. on pp. 7, 67, 68, 70, 71, 100).
- [Ort21] Ángel David Ríos Ortiz. *Higher Gaussian Maps on K3 surfaces*. <https://arxiv.org/abs/2112.09101>. 2021 (cit. on pp. vii, 1).
- [Pay05] Sam Payne. “Stable base loci, movable curves, and small modifications, for toric varieties”. In: *Math. Z.* 253 (2005), pp. 421–431 (cit. on pp. 11, 12).
- [Rap06] Antonio Rapagnetta. “On the Beauville form of the known irreducible symplectic varieties”. In: *Math. Ann.* 340 (2006), pp. 77–95 (cit. on pp. 54, 75).
- [Rie20] Ulrike Rieß. “On the non-divisorial base locus of big and nef line bundles of $K3^{[2]}$ -type varieties”. In: *Proc. Roy. Soc. Edinburgh Sect. A* 151.1 (2020), pp. 52–78 (cit. on pp. vii, 1, 53, 54, 68).
- [Rie23] Ulrike Rieß. “Base divisors of big and nef line bundles on irreducible symplectic varieties”. In: *Ann. Inst. Fourier (Grenoble)* 73.2 (2023), pp. 609–633. URL: <https://aif.centre-mersenne.org/articles/10.5802/aif.3530/> (cit. on pp. vii, 1).
- [RS16] Slawomir Rams and Tomasz Szemberg. “When are Zariski chambers numerically determined?” In: *Forum Math.* 24 (2016), pp. 609–625 (cit. on p. 4).
- [Sea15] José Seade. *Limit sets of discrete groups*. Notes from a school at the ICTP, Trieste. <https://indico.ictp.it/event/a14294/session/34/contribution/142/material/0/0.pdf>. 2015 (cit. on p. 69).
- [SS16] David Schmitz and Henrik Seppänen. “On the polyhedrality of global Okounkov bodies”. In: *Adv. Geom.* 16.1 (2016), pp. 83–91 (cit. on p. 109).
- [Tak08] Shigeharu Takayama. “On the uniruledness of stable base loci”. In: *J. Differential Geom.* 78.3 (2008), pp. 521–541 (cit. on p. 8).
- [Tot12] Burt Totaro. “Algebraic surfaces and hyperbolic geometry”. In: *Current Developments in Algebraic Geometry* (2012). MSRI Publications 59, Cambridge. URL: https://www.math.ucla.edu/~totaro/papers/public_html/algebraic.pdf (cit. on p. 114).
- [Tot13] Burt Totaro. “Line bundles with partially vanishing cohomology”. In: *J. Eur. Math. Soc. (JEMS)* 15.3 (2013), pp. 731–754 (cit. on p. 91).
- [TX23] Nikolaos Tsakanikas and Zhixin Xie. *Comparison and uniruledness of asymptotic base loci*. 2023. URL: <https://arxiv.org/abs/2309.01031> (cit. on p. 17).
- [Var22] Mauro Varesco. *Towards generic base point freeness for generalized Kummer varieties in low dimension*. <https://arxiv.org/abs/2211.11485>. 2022 (cit. on pp. vii, 1).

- [Voi] Claire Voisin. “Fibrations in algebraic geometry and applications”. In: *Hyperbolicity properties of algebraic varieties*. Vol. 56. Soc. Math. France, Paris, pp. 319–353 (cit. on p. [87](#)).
- [Voi16] Claire Voisin. “Remarks and questions on coisotropic subvarieties and 0-cycles of hyper-Kähler varieties”. In: *K3 surfaces and their moduli*. Vol. 315. Progr. Math. 2016, pp. 365–399 (cit. on pp. [86](#), [87](#), [101](#)).
- [Yos16] Kōta Yoshioka. “Bridgeland’s Stability and the Positive Cone of the Moduli Spaces of Stable Objects on an Abelian Surface”. In: *Adv. Stud. Pure Math.* 69 (2016), pp. 473–537 (cit. on pp. [77–79](#)).
- [Zar62] Oscar Zariski. “The Theorem of Riemann-Roch for High Multiples of an Effective Divisor on an Algebraic Surface”. In: *Ann. of Math.* 76 (1962), pp. 106–110 (cit. on p. [2](#)).