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## Multi-type bi-sexual branching processes

Nicolás Zalduendo Vidal

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# Processus de branchement bi-sexués multi-types

## THÈSE

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pour l'obtention du

**Doctorat de l'Université de Lorraine**

(Mention Mathématiques Appliquées)

par

M. Nicolás Zalduendo Vidal

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# Notation

Through this manuscript  $\mathbb{R}$  represent the set of all real numbers,  $\mathbb{N} = \{0, 1, 2, \dots\}$  and  $\mathbb{N}^* = \{1, 2, \dots\}$ . We have also that:

- $\mathbb{R}_+ = [0, +\infty)$ .
- $\mathbb{R}_+^p$ , for  $p \in \mathbb{N} \setminus \{0\}$ , is the set of all  $p$ - dimensional vectors with entries on  $\mathbb{R}_+$ .
- For  $i \in \{1, \dots, p\}$ ,  $e_i$  stands for the  $i$ -th canonical vector of  $\mathbb{R}^p$ .
- For  $p \in \mathbb{N}$ ,  $\mathbf{1}_p \in \mathbb{R}^p$  with  $\mathbf{1}_p = (1, 1, \dots, 1)$ .
- $\mathbb{N}^p$ , for  $p \in \mathbb{N} \setminus \{0\}$ , is the set of all  $p$ - dimensional vectors with entries on  $\mathbb{N}$ .
- $\mathbb{R}_+^{d \times d}$ , for  $d \in \mathbb{N} \setminus \{0\}$  is the set of all  $d \times d$  matrices with entries on  $\mathbb{R}_+$ .
- $\mathbb{N}^{d \times d}$ , for  $d \in \mathbb{N} \setminus \{0\}$  is the set of all  $d \times d$  matrices with entries on  $\mathbb{N}$ .
- For  $x \in \mathbb{R}^p$ ,  $|\cdot|$  stands for the 1-norm. Namely,  $|x| = |x_1| + \dots + |x_p|$ .
- $S = \{x \in \mathbb{R}_+^p, |x| = 1\}$  is the unitary ball for the 1-norm.
- $S^* = \{x \in \mathbb{R}_+^p, |x| = 1 \text{ and } x_i > 0 \text{ for all } i \in \{1, \dots, p\}\}$  is the elements of  $S$  with strictly positive components.

We also consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  where all our random variable are defined. The associated expectation of  $\mathbb{P}$  is  $\mathbb{E}$ . In addition,

- $L^1 := L^1(\Omega, \mathcal{F}, \mathbb{P})$  is the space of all the  $\mathbb{P}$ -integrable random variables.
- $\stackrel{(d)}{=}$  stands for equality in distribution.



# Introduction Générale

*This introduction is a direct translation of the english version that can be found in page 13.*

Cette thèse est centrée sur l'étude du processus de branchement bi-sexué multi-type. Par bi-sexué, nous entendons que ce processus modélise l'évolution en temps discret d'une population où des femelles et des mâles doivent se rassembler et former des couples dans le but de se reproduire. Par multi-type, nous entendons qu'il existe plusieurs types de femelles et de mâles dans la population. Ce type (ou cette caractéristique) joue un rôle dans la manière dont ils se reproduisent. Par conséquent, le processus que nous cherchons à étudier est un processus de Markov multidimensionnel. Dans cette introduction, nous donnons un aperçu très général de l'état de l'art autour de ce modèle. Cette introduction n'a, cependant, pas pour but d'être exhaustive. Dans la première section, nous partons du processus de population le plus simple (le processus de Galton-Watson en temps discret) et allons jusqu'à la construction de notre modèle. Ensuite, dans la deuxième partie de cette introduction, nous présentons l'organisation de ce manuscrit avec une brève description des contributions que l'on peut trouver dans chaque chapitre. Les contributions scientifiques et les perspectives de cette thèse sont présentées à la fin de cette introduction.

## Quelques rappels sur les processus de branchement

### Le modèle classique

Les processus de branchement figurent sans aucun doute parmi les modèles qui ont été étudiés de manière très approfondie dans le domaine des probabilités au cours des dernières décennies. Initialement formulés il y a plus de cent ans par Galton et Watson, ces modèles (couramment connus dans la littérature sous le nom de processus de Galton-Watson) ont évolué depuis leur idée originelle très simple vers des processus plus complexes.

Le modèle de base est très simple et intuitif : imaginons que nous avons une collection d'objets. Après une unité de temps, soit ils se divisent en créant de nouveaux objets identiques, soit ils meurent et cessent d'exister (nous pouvons penser aux bactéries subissant le processus de mitose, ou aux particules qui peuvent se diviser elles-mêmes). Les nouveaux objets peuvent se diviser à nouveau, produisant de nouveaux objets, et ce processus continue indéfiniment, ou jusqu'à ce qu'il n'y ait plus d'objet. Nous supposons que toutes ces divisions se produisent indépendamment les unes des autres. Si nous appelons  $Z_n$  le nombre d'objets présents au temps  $n \in \mathbb{N}$ , alors la collection  $(Z_n)_{n \in \mathbb{N}}$  est appelée le processus de Galton-Watson.

La Figure 1 est une illustration de la réalisation des deux premières générations d'un processus de branchement de Galton-Watson, initialisé avec  $Z_0 = 3$ . En rouge, nous pouvons observer la descendance d'un objet.

Si nous considérons maintenant que tous les objets ne sont pas identiques, mais qu'ils ont une caractéristique (ou un type) associée, nous parlons d'un processus de branchement de Galton-

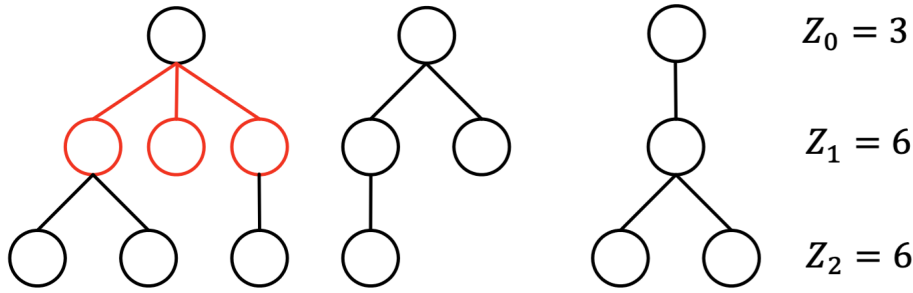


Figure 1: Réalisation d'un processus de Galton-Watson avec  $Z_0 = 3$

Galton-Watson multi-type. Pour mieux comprendre, imaginons que nous avons  $d \in \mathbb{N}$  différents types d'objets, et que chacun d'entre eux peut produire de nouveaux descendants indépendamment les uns des autres. Supposons également que la loi de probabilité de la descendance d'un objet est entièrement déterminée par son type. Alors, le processus de branchement de Galton-Watson multi-type  $(Z_n)_{n \in \mathbb{N}}$  est le processus de Markov défini sur  $\mathbb{N}^d$  tel que, pour  $n \in \mathbb{N}$ , le vecteur  $Z_n$  représente le nombre d'objets de chaque type présents au temps  $n$ .

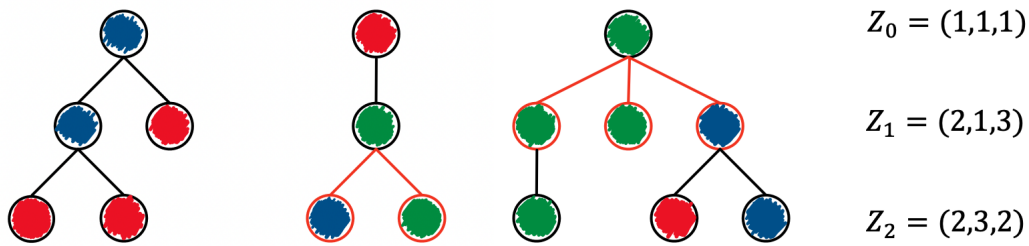


Figure 2: Réalisation d'un processus de Galton-Watson multi-type avec  $Z_0 = (1, 1, 1)$

La Figure 2 est une illustration d'une réalisation des deux premières générations d'un processus de branchement de Galton-Watson multi-type. Dans cet exemple,  $d = 3$ , chaque type est représenté par une couleur différente (bleu, rouge ou vert) et  $Z_0 = (1, 1, 1)$ . Nous soulignons que, en raison de la manière dont notre processus est construit, deux particules du même type donnent une descendance suivant la même distribution de probabilité. Dans la figure, nous avons mis en évidence, par des branches rouges, la descendance de deux particules vertes, qui sont échantillonnées selon la même loi.

Nous tenons à souligner que pour ces deux modèles, la reproduction se produit de manière *asexuée*. Par conséquent, cela n'inclut pas le cas des populations plus complexes où il est nécessaire que deux individus se réunissent pour se reproduire (dans ce cas, nous parlons de reproduction *bi-sexuée*).

### Le modèle bi-sexué

La première modélisation d'une population avec des interactions sexuelles en utilisant des processus de branchement remonte à presque soixante ans, et a été introduit par Daley [Dal68]. On peut considérer ce processus comme un processus de branchement avec une étape intermédiaire d'appariement entre chaque génération. Pour donner une idée générale, imaginons une population

composée de femelles et de mâles. Ils ne peuvent pas se reproduire par eux-mêmes, ils ont donc besoin d'interagir et de former des unités d'appariement (également appelées couples) pour se reproduire. Chaque unité d'appariement peut se reproduire indépendamment des autres et donner naissance à de nouveaux individus de l'un des deux sexes. De nouveau, ces nouveaux individus doivent former de nouvelles unités d'appariement pour se reproduire, car ils ne peuvent pas le faire seuls. Ce processus se répète indéfiniment en produisant de nouveaux individus.

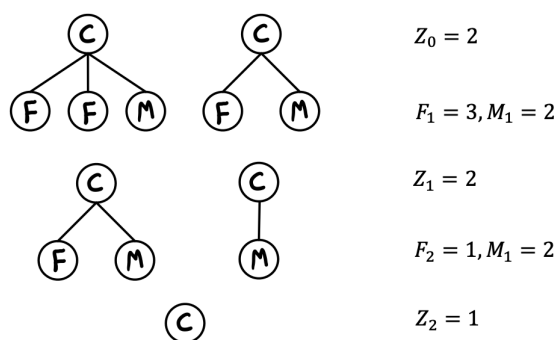


Figure 3: Réalisation d'un processus de branchement de Galton-Watson bi-sexué avec  $Z_0 = 2$  et fonction d'appariement  $\xi(x, y) = \min\{x, y\}$ .

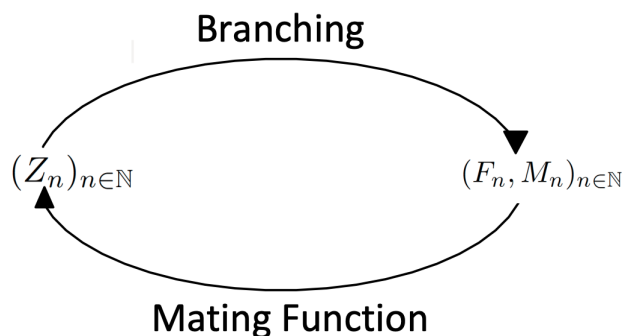


Figure 4: Dynamique en deux étapes pour le processus de branchement de Galton-Watson bi-sexué.

L'appariement a lieu grâce à une fonction déterministe  $\xi : \mathbb{N}^2 \rightarrow \mathbb{N}$  appelée la fonction d'appariement. L'idée est que si à un certain moment dans la population il y a  $x$  femelles et  $y$  mâles, alors ils forment  $\xi(x, y)$  unités d'appariement. La Figure 3 illustre un exemple de l'un de ces processus où nous utilisons la fonction d'appariement  $\xi(x, y) = \min\{x, y\}$ . Nous observons que nous commençons avec une population initiale composée de deux unités d'appariement (le **C** dans la condition initiale signifie "Couple"). Elles engendrent une descendance, formant au total 3 femelles et 2 mâles. Ces descendants forment ensuite  $\min\{3, 2\} = 2$  unités d'appariement, et ainsi de suite.

L'idée que le lecteur doit garder à l'esprit est que ce processus doit être considéré comme un processus en deux étapes. Nous entendons par là que, si à un moment donné nous avons  $Z_n$  unités d'appariement, pour obtenir la génération suivante  $Z_{n+1}$ , nous passons d'abord par une première étape de branchement (création de la descendance) et une deuxième étape d'appariement (formation des nouveaux couples) (voir Figure 4). Nous soulignons que, comme l'étape d'appariement ne donne que le nombre de couples créés, nous perdons la trace de la généalogie du processus (en d'autres termes, nous n'enregistrons pas *comment* les couples sont formés, mais nous savons seulement *combien* il y en a).

Les principales questions qui se posent pour ce type de modèles sont les suivantes :

1. Extinction : Supposons la condition très raisonnable que  $\xi(0, 0) = 0$ . Autrement dit, lorsqu'il n'y a pas d'individu, aucun couple n'est formé. Alors,  $\{0\}$  est un état absorbant. Si à un certain moment  $n \in \mathbb{N}$ , nous avons  $Z_n = 0$ , alors nous disons que le processus s'est éteint. Une question naturelle est de savoir si nous pouvons trouver des conditions sous lesquelles cela se produit avec probabilité égale à un. De plus, on souhaite que ces conditions soient à la fois nécessaires et suffisantes. En d'autres termes, si ces conditions

ne sont pas remplies, on souhaite que la probabilité que le processus ne touche jamais  $\{0\}$  soit positive.

2. Convergence à long terme : Si l'on sait que le processus survit indéfiniment, une deuxième question est de savoir si l'on peut décrire le comportement de ce processus à long terme, c'est-à-dire lorsque  $n \rightarrow +\infty$ . Si le processus diverge, alors une question classique est de trouver la vitesse de divergence en fonction des paramètres du processus.

Ces deux questions ont déjà été résolues pour le processus de Galton-Watson classique présenté précédemment dans cette introduction (à la fois dans les cas uni-type et multi-type). La principale difficulté pour aborder ces deux questions dans le modèle bi-sexué réside dans la présence de la fonction d'appariement. En effet, du fait de l'ajout de cette deuxième étape, les principales techniques utilisées pour aborder ces questions dans le cas classique ne s'appliquent pas à ce modèle.

Néanmoins, des réponses à ces questions ont été données sous des hypothèses très simples, en supposant en plus que la fonction d'appariement  $\xi$  est une fonction *superadditive*. Le résultat d'extinction peut être trouvé dans [DHT86], tandis que pour le comportement à long terme, le lecteur peut consulter [GM96, GM97b]. Les détails de ces résultats peuvent également être trouvés dans le **Chapitre 1**, où nous donnons un état de l'art plus complet pour ces processus. Pour d'autres références, le lecteur peut également consulter les excellentes revues d'Alsmeyer [Als02], Hull [Hul03] et Molina [Mol10].

## Structure du manuscrit et contributions

Cette thèse est composée de six chapitres et d'une annexe. Dans le **Chapitre 1**, nous présentons un état de l'art plus détaillé pour les processus de branchement que nous avons exposés au début de cette introduction. Nous commençons par quelques résultats généraux sur le processus de Galton-Watson classique, puis nous répertorions les nombreux résultats existants concernant son homologue bi-sexué.

### Un modèle multidimensionnel général

Le **Chapitre 2** est entièrement dédié à la construction du processus de branchement bi-sexué multi-type. Ce processus est décrit par la chaîne de Markov  $(Z_n)_{n \in \mathbb{N}}$ , définie sur  $\mathbb{N}^p$ , où  $p$  est un entier positif fixé. La construction peut être résumée comme suit :

1. Si à l'instant  $n \in \mathbb{N}$ , le processus est dans la configuration  $(Z_{n,1}, \dots, Z_{n,p})$  d'unités d'appariement (ce qui signifie qu'il y a  $Z_{n,i}$  unités d'appariement de type  $i$ ), alors chacune d'entre elles se reproduira et donnera naissance à des femelles et des mâles de différents types. Nous notons qu'une unité d'appariement peut donner naissance à une descendance de plus d'un type. De plus, nous supposons qu'il y a  $q_f \in \mathbb{N}$  (resp.  $q_m \in \mathbb{N}$ ) différents types de femelles (resp. de mâles).
2. Le nombre total de femelles (resp. de mâles) produits sera noté  $F_{n+1} \in \mathbb{N}^{q_f}$  (resp.  $M_{n+1} \in \mathbb{N}^{q_m}$ ). Ils formeront la génération  $(n+1)$  d'unités d'appariement à l'aide d'une fonction déterministe  $\xi : \mathbb{N}^{q_f} \times \mathbb{N}^{q_m} \rightarrow \mathbb{N}^p$ , appelée fonction d'appariement. Par conséquent, nous avons  $Z_{n+1} = \xi(F_{n+1}, M_{n+1})$ .

Ainsi, nous avons la même structure en deux étapes que pour le processus bi-sexué uni-type ( $p = 1$ ) présenté en début d'introduction. Nous donnons maintenant une définition plus formelle de ces deux étapes : étant donné  $Z_0 \in \mathbb{N}^p$ , pour tout  $n \in \mathbb{N}$ , nous avons

- **L'étape de branchement** : Pour  $j \in \{1, \dots, q_f\}$  et  $k \in \{1, \dots, q_m\}$ , nous définissons le nombre de femelles de type  $j$  et le nombre de mâles de type  $k$  présents à la génération  $(n + 1)$  comme suit :

$$F_{n+1,j} = \sum_{i=1}^p \sum_{\ell=1}^{Z_{n,i}} X_{i,j}^{(n,\ell)} \quad \text{et} \quad M_{n+1,k} = \sum_{i=1}^p \sum_{\ell=1}^{Z_{n,i}} Y_{i,k}^{(n,\ell)}, \quad (1)$$

où pour tout  $i \in \{1, \dots, p\}$ ,  $(X_{i,\cdot}^{(n,\ell)}, Y_{i,\cdot}^{(n,\ell)})_{n,\ell \in \mathbb{N}}$  est une famille de vecteurs aléatoires i.i.d. dont les hypothèses sont présentées dans la Section 2.1 et sont omises dans cette introduction.

- **L'étape d'appariement** : Les femelles et les mâles forment la génération  $(n + 1)$  d'unités d'appariement de la manière suivante :

$$Z_{n+1} = \xi((F_{n+1,1}, \dots, F_{n+1,q_f}), (M_{n+1,1}, \dots, M_{n+1,q_m})). \quad (2)$$

On suppose également que  $\xi(0, 0) = 0$ , ce qui fait de  $(Z_n)_{n \in \mathbb{N}}$  une chaîne de Markov avec un état absorbant en  $\{0\}$ .

Dans tous les chapitres de 2 à 5,  $\xi : \mathbb{N}^{q_f} \times \mathbb{N}^{q_m} \rightarrow \mathbb{N}^p$  est considérée comme une fonction supraditive, c'est-à-dire

$$\xi(x_1 + x_2, y_1 + y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2), \quad \text{pour tous } x_1, x_2 \in \mathbb{N}^{q_f}, y_1, y_2 \in \mathbb{N}^{q_m}.$$

Cette hypothèse garantit en particulier que la fonction  $\mathfrak{M} : \mathbb{R}_+^p \rightarrow (\mathbb{R}_+ \cup \{+\infty\})^p$  définie par

$$\mathfrak{M}(z) = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_1 \mid Z_0 = \lfloor kz \rfloor)}{k}$$

est bien définie pour tout  $z \in \mathbb{R}_+^p$  (bien qu'elle puisse prendre la valeur  $+\infty$ ). Pour  $z \in \mathbb{R}_+^p$ ,  $\mathfrak{M}(z)$  est appelé le *taux de croissance asymptotique* du processus dans la direction  $z$ .

En choisissant correctement les paramètres du processus, on peut retrouver le modèle bi-sexué uni-type et les modèles asexués (uni-type et multi-type). Si l'on calcule la fonction  $\mathfrak{M}$  pour ces trois cas particuliers, on constate que  $\mathfrak{M}$  est une fonction linéaire. En particulier, dans le cas asexué,  $\mathfrak{M}$  est une matrice dont la plus grande valeur propre décrit le comportement du processus à long terme. Comme nous le verrons plus tard dans le Chapitre 3, cette fonction est concave dans le cas bi-sexué général. Néanmoins, sous des hypothèses supplémentaires, on peut prouver (voir Chapitre 2, Section 2.3.1) que lorsque  $\mathfrak{M}$  prend des valeurs finies, il existe un unique  $\lambda^* > 0$ , ainsi qu'un unique vecteur  $z^* \in \mathbb{R}_+^p$  avec des composantes strictement positives et  $|z^*| = 1$ , qui sont des éléments propres de  $\mathfrak{M}$ . Autrement dit, tels que

$$\mathfrak{M}(z^*) = \lambda^* z^*.$$

Comme nous le verrons, ces éléments propres joueront le même rôle que dans le cas multi-type asexué.

Dans ce qui suit, nous énonçons nos principaux résultats en faisant référence aux chapitres où ils peuvent être trouvés.

## Lois des Grands Nombres

L'intuition derrière le fait d'avoir une fonction d'appariement superadditive est que nous devons concentrer notre étude du processus dans les cas où la population est grande. Cela est étudié dans le **Chapitre 3**, où nous développons deux lois des grands nombres (LGN).

**LGN pour une population initiale large** (Voir Théorème 22, Page 60). Ce résultat lie le comportement du processus à un temps fixé  $n \in \mathbb{N}$  avec les itérations de la fonction  $\mathfrak{M}$  lorsque la population initiale est grande et croît dans une direction fixée. Il se présente comme suit : si nous notons  $(Z_n^m)_{n \in \mathbb{N}}$  un processus de branchement bi-sexué multi-type avec une condition initiale  $Z_0 = z_m$ , où  $(z_m)_{m \in \mathbb{N}}$  est une suite aléatoire telle que  $z_m \sim_{m \rightarrow +\infty} mz_\infty$ , pour  $z_\infty \in \mathbb{R}_+^p$  un vecteur déterministe, alors, pour tout  $n \in \mathbb{N}$

$$Z_n^m \sim_{m \rightarrow +\infty} \mathfrak{M}^n(mz_\infty), \text{ presque sûrement.}$$

Sous des hypothèses supplémentaires, une convergence  $L^1$  est également démontrée. Ce résultat est déjà apparu dans le cas uni-type dans [DHT86], mais dans le cas particulier où  $z_m = mz_\infty$ ,  $n = 1$  et seulement avec une convergence presque sûre. Par conséquent, nous étendons ce résultat non seulement à un cadre multi-type, mais aussi avec une condition initiale aléatoire et pour toutes les valeurs de  $n \in \mathbb{N}$ . De plus, la convergence  $L^1$  est également un résultat nouveau dans le cas uni-type. Ces convergences jouent un rôle important dans la démonstration des résultats suivants.

**LGN trajectorielle** (Voir Théorème 30, Page 72). Le but de ce résultat est d'étudier le comportement de la trajectoire complète du processus, lorsqu'il est initialisé avec une condition initiale suffisamment grande. Ce résultat est présenté dans le cas où  $\mathfrak{M}$  est fini et où sa valeur propre est  $\lambda^* > 1$ . On a que, pour tous  $\varepsilon, \eta \in (0, 1)$ , il existe  $r > 0$  tel que, si  $|Z_0| > r$ , alors

1. Avec une probabilité d'au moins  $1 - \eta$ , nous avons  $Z_{n_0} \neq 0$  et, pour tous les  $n \geq n_0$ ,

$$(1 - \varepsilon)\mathfrak{M}(Z_n) \leq Z_{n+1} \leq (1 + \varepsilon)\mathfrak{M}(Z_n), \quad (3)$$

où  $n_0 \in \mathbb{N}$  est une valeur déterministe dépendant uniquement de  $\mathfrak{M}$  (et non des paramètres  $\varepsilon, \eta, r$ ), et les inégalités sont prises composante par composante.

2. En utilisant cela, nous pouvons prouver que dans l'éventualité où le processus n'est pas éteint, il croît dans la direction donnée par  $z^*$ , le vecteur propre de  $\mathfrak{M}$ , c'est-à-dire,

$$\frac{Z_n}{|Z_n|} \xrightarrow{n \rightarrow +\infty} z^*. \quad (4)$$

Le premier résultat est un nouveau résultat, même dans le cas uni-type. Pour le deuxième point, nous retrouvons un résultat classique déjà existant dans le processus de branchement de Galton-Watson asexué, mais maintenant pour le cadre bi-sexué multi-type. Ces résultats sont ensuite utilisés pour étudier le comportement en temps long du processus.

## Comportement à long temps

Dans le **Chapitre 4**, nous étudions le comportement du processus quand le temps  $n$  tend vers l'infini. Nous établissons des conditions nécessaires et suffisantes pour l'extinction presque sûre et étudions la convergence du processus dans le cas de survie.

**Conditions d'extinction** (Voir Théorème 37, Page 88). Nous prouvons que dans le cas où  $\mathfrak{M}$  est fini, le processus s'éteint presque sûrement si et seulement si  $\lambda^* \leq 1$ . Si  $\mathfrak{M}$  n'est pas



fini, alors le processus peut survivre avec une probabilité positive. Ce résultat a été prouvé pour le processus bi-sexué uni-type dans [DHT86], mais seulement dans le cas où  $\mathfrak{M}$  est fini. Nous complétons le résultat en fournissant une preuve pour le cas où  $\mathfrak{M}$  prend une valeur infinie.

**Propriétés de convergence** (Voir Théorème 39, Page 94). Nous prouvons que pour tout  $z \in \mathbb{N}^p$ , il existe une variable aléatoire  $\mathcal{C}$  telle que

$$\frac{Z_n}{(\lambda^*)^n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}(\cdot | Z_0=z)\text{-p.s.}} \mathcal{C}z^*. \quad (5)$$

De plus, nous prouvons que pour les valeurs de  $z \in \mathbb{N}^p$  telles que  $\mathcal{C}$  est non dégénérée en 0, alors, à un événement  $\mathbb{P}(\cdot | Z_0 = z)$ -négligeable près,

$$\{\mathcal{C} = 0\} = \{\exists n \in \mathbb{N}, Z_n = 0\}. \quad (6)$$

Enfin, nous donnons une série de conditions sous lesquelles la convergence  $L^1$  a lieu et  $\mathcal{C}$  est non dégénérée.

Le résultat (5) est déjà connu dans la littérature pour les modèles asexués et le modèle bi-sexué uni-type. D'autre part, (6) est un résultat nouveau, même pour le processus de branchement bi-sexué uni-type.

### Comportement quasi-limite dans le cas sous-critique

Dans le **Chapitre 5**, nous nous intéressons à l'existence de distributions quasi-stationnaires dans le cas où  $\lambda^* < 1$ . Nous prouvons que, en fonction des hypothèses en place pour le processus, nous pouvons avoir un nombre infini, un nombre fini ou une unique distribution quasi-stationnaire. Plus précisément,

1. Sous l'hypothèse que la fonction d'appariement est sous-affine, il existe un nombre infini de distributions quasi-stationnaires (voir Théorème 49, Page 112). Pour prouver ce résultat, nous prouvons d'abord un résultat général pour les processus de Feller, puis nous l'appliquons au cas particulier du processus de branchement bi-sexué.
2. Sous les hypothèses précédentes, mais en supposant également une régularité supplémentaire pour la fonction  $\mathfrak{M}$  et une condition de moment sur la distribution de la descendance, nous prouvons que le processus admet une famille finiment engendrée de distributions quasi-stationnaires (voir Théorème 50, Page 114).
3. Sous les hypothèses précédentes, mais avec l'hypothèse supplémentaire que le processus est irréductible et aperiodique, nous prouvons qu'il existe une unique distribution quasi-limite pour les distributions initiales à support compact (voir Théorème 53, Page 118).

Ces résultats sont connus depuis des décennies pour le processus de Galton-Watson asexué classique, mais ils sont nouveaux pour le modèle bi-sexué en général, y compris le processus uni-type.

### Croissance illimitée des modèles de population bi-sexuée

Dans le **Chapitre 6**, nous présentons deux résultats principaux concernant les conditions suffisantes pour que la probabilité

$$\mathbb{P}(\mathcal{X}_n \xrightarrow[n \rightarrow +\infty]{} +\infty \mid \mathcal{X}_0 = x)$$

soit strictement positive pour certains  $x \in \mathbb{R}_+^q$ . Lorsque cela arrive, nous disons que  $(\mathcal{X}_n)_{n \in \mathbb{N}}$  a une croissance illimitée avec une probabilité positive. Plus précisément, ces résultats donnent des conditions pour lesquelles le processus  $(g(\mathcal{X}_n))_{n \in \mathbb{N}}$  diverge vers l'infini, où  $g : \mathbb{R}_+^q \rightarrow \mathbb{R}_+$  est une fonction appropriée. Ces résultats généralisent un résultat déjà existant sur la croissance illimitée établi dans [GMM05], pour les modèles qui croissent dans toutes les directions (ce qui n'est pas le cas, par exemple, pour certains processus de branchement bi-sexués). Ces résultats sont ensuite appliqués à différents modèles de populations à deux sexes. Voici quelques-uns des exemples que nous abordons.

1. Nous considérons un modèle de processus de branchement bi-sexué multi-type où la fonction d'appariement est une fonction aléatoire. Dans ce cas, nous considérons une fonction qui est presque sûrement croissante. Nous donnons des conditions pour une croissance illimitée et nous accordons une attention particulière au cas où la fonction d'appariement aléatoire est superadditive avec une probabilité positive. Ce modèle est présenté dans l'Exemple 13, à la page 129.
2. Considérons un processus de branchement bi-sexué multi-type surcritique, c'est-à-dire qu'il survit avec une probabilité positive. Si la fonction d'appariement de ce processus est une certaine fonction  $\xi$ , nous construisons un nouveau processus, mais avec une fonction d'appariement  $\xi + \varphi$ , où  $\varphi$  est une perturbation aléatoire. Nous donnons des conditions sur  $\varphi$  pour que le nouveau processus reste surcritique. Ce modèle est présenté dans l'Exemple 14, à la page 130.
3. Un processus de naissance et de mort à deux sexes en temps continu est un processus de naissance et de mort à 2 types (femelles et mâles), mais où le taux de naissance est une fonction qui dépend des deux types. Nous considérons deux exemples particuliers de ce processus, inspirés par la version en temps discret des processus de branchement bi-sexués, et nous donnons des conditions nécessaires et suffisantes pour leur extinction presque sûre. Le premier des deux résultats peut être obtenu comme un cas particulier d'un résultat plus général dans [KK73], tandis que le second est un résultat complètement nouveau. Ces modèles se trouvent dans la Section 6.4, à la page 138.

## Contributions Scientifiques

Au cours de cette thèse, les articles scientifiques suivants ont été produits.

- Les chapitres 2, 3 et 4 sont basés sur l'article :  
C. Fritsch, D. Villemonais et N. Zaldueño  
*The multi-type bisexual Galton-Watson branching process* (accepté dans Annales de l'Institut Henri Poincaré).  
ArXiv e-prints : [2206.09622](https://arxiv.org/abs/2206.09622).
- Le chapitre 5 est basé sur l'article :  
C. Fritsch, D. Villemonais et N. Zaldueño  
*Quasi-limiting behaviour of the subcritical multi-type bisexual Galton-Watson branching process*.  
Pre-print (à soumettre) : [Lien](#).

Le contenu du chapitre 6 fera partie d'un article, actuellement en cours de production.

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De plus, l'article suivant a été produit au cours de cette thèse, mais son contenu n'est pas inclus dans ce manuscrit, car il est lié à un sujet complètement différent.

- Y. Yalanda et N. Zaldueño  
*Restricted maximum of non-intersecting Brownian bridges* (soumis).  
ArXiv e-prints : [2302.11432](https://arxiv.org/abs/2302.11432).

## Perspectives

La principale contribution de ce travail réside dans le développement d'un modèle général pour un processus de branchement de Galton-Watson bi-sexué multi-type. Au cours de la majeure partie de ce travail, la fonction d'appariement du processus est considérée comme une fonction superadditive, et pour ce type de modèles, deux lois des grands nombres sont développées dans le chapitre 3 pour les cas où la population est grande. Pour compléter ce tableau, des théorèmes de limite centrale doivent être développés pour comprendre pleinement le comportement de la population lorsqu'elle est grande. De plus, il serait intéressant d'étendre ces résultats à des modèles non-superadditifs. La superadditivité de la fonction d'appariement est un ingrédient clé dans les preuves développées dans ce manuscrit, bien qu'elle implique que la population se comporte toujours de manière coopérative. Des modèles plus complexes dans lesquels l'appariement permet un comportement compétitif entre des individus de différents types doivent être pris en compte.

Dans le chapitre 5, l'existence d'une infinité de distributions quasi-stationnaires est prouvée pour le cas où le processus est sous-critique. Cependant, la question de savoir si des distributions quasi-stationnaires existent dans le cas critique est encore ouverte. Pour un processus de branchement bi-sexué critique, le comportement critique est défini en fonction du comportement du processus lorsque la population tend vers l'infini. Ainsi, un processus peut être critique tout en manifestant un comportement sous-critique pour une population finie. Une première approche consisterait à trouver un moyen de mesurer la "vitesse" à laquelle ce processus se rapproche d'un comportement critique lorsque la population devient grande.

L'un des modèles manquants dans la théorie des processus de branchement bi-sexués est un modèle en temps continu. Bien que deux exemples soient traités dans ce manuscrit pour le cas de processus de naissance et de mort au chapitre 6, une analyse plus complète pour un modèle général en temps continu doit être fournie. L'une des principales difficultés lorsqu'on traite des processus bi-sexués en temps continu est le fait qu'ils doivent toujours être écrits comme des processus de plusieurs types : soit nous suivons le nombre de femelles et de mâles (un processus à deux types), soit le nombre d'unités d'appariement, de femelles seules et de mâles seuls (un processus à trois types). Ainsi, nos développements pour le processus de branchement bi-sexué multi-type discret, et en particulier l'existence d'éléments propres pour ce modèle, peuvent s'avérer utiles lors de l'analyse de son homologue continu.

En définitive, il existe plusieurs directions de recherche futures basées sur les fondements posés dans cette thèse, notamment des études approfondies sur le comportement des processus de branchement bi-sexués multi-types, et l'application de ces modèles à des scénarios du monde réel en biologie des populations, en épidémiologie et en écologie.



# General Introduction

This thesis is centered in the study of the multi-type bisexual branching processes. With bisexual, we mean that this process models the evolution in discrete time of a population where females and males need to come together and form couples with the purpose of accomplishing reproduction. With multi-type, we mean that in the population there are several *types* of females and males. This type (or characteristic) plays a role in the way they mate and reproduce. Hence, the process we aim to study is a multidimensional Markov process. This introduction will not delve into exhaustive details but will provide a general background on the state of the art around this model. In order to do so, we start from the simplest population process (the discrete time Galton-Watson process) and build up our model from there. We do this in the first section. Then, in the second part of this introduction, we present the organization of this manuscript with a brief description of the contributions that can be found in each chapter. The scientific contributions and the perspectives are presented at the end.

## Some background on branching processes

### The classical model

Branching processes are, without a doubt, among the models that have been the more extensively studied within the field of probability in the last decades. Originally formulated more than one hundred years ago by Galton and Watson (for whom these models are also commonly known in the literature as *Galton-Watson* processes), these models have evolved from their original and very simple idea to more complex processes.

The basic model is very simple and intuitive: imagine you have a collection of objects. After one unit of time, they either divide and create new identical objects or they die and cease to exist (we can think of bacteria experiencing the process of mitosis, or particles that can divide themselves). The new objects can divide again, producing new objects, and this process goes indefinitely, or until they are no more objects. We assume that all the divisions happen independently of each other. If we call  $Z_n$  the number of objects present at time  $n \in \mathbb{N}$ , then the collection  $(Z_n)_{n \in \mathbb{N}}$  is called the Galton-Watson process.

Figure 5 is an illustration of the realization of the first two generations of a Galton-Watson process, initialized with  $Z_0 = 3$ . In red, the offspring of one object.

If we now consider that not all the objects are identical, but that they have a characteristic (or type) associated, we speak of a multi-type Galton-Watson branching process. To set some ideas imagine that there are  $d \in \mathbb{N}$  different types of objects, and that all of them can produce new descendents independently of each other. We assume also that the probability distribution for the offspring of an object is entirely determined by its type. Then, the multi-type Galton-Watson process  $(Z_n)_{n \in \mathbb{N}}$  is the Markov process defined on  $\mathbb{N}^d$  such that, for  $n \in \mathbb{N}$ , the vector  $Z_n$  represents the number of objects of each type present at time  $n$ .

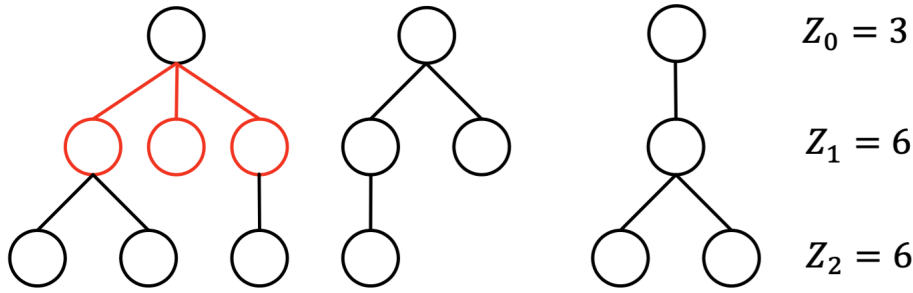


Figure 5: A realization of a Galton-Watson process with  $Z_0 = 3$

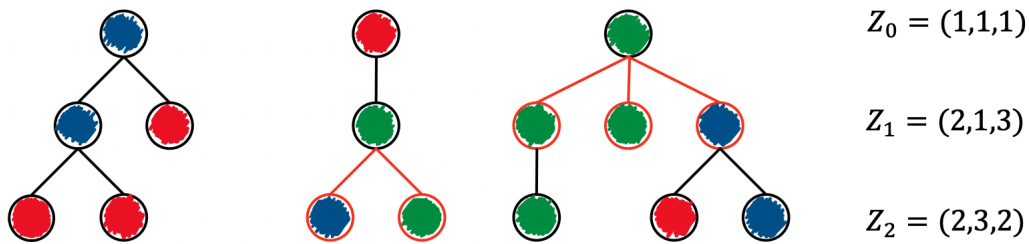


Figure 6: A realization of a multi-type Galton-Watson process with  $Z_0 = (1, 1, 1)$

In Figure 6 we have an illustration of a realization of the first two generations of a multi-type Galton-Watson process. In this case,  $d = 3$ , each type represented with a different color (blue, red or green) and  $Z_0 = (1, 1, 1)$ . We remark that, due to the way our process is constructed, two particles of the same type give offspring following the same probability distribution. In the figure, we have highlighted the offspring of two green particles, which are sampled following the same law.

We emphasize that for these two models the reproduction happens *asexually*. Hence, it does not include the case of more complex populations where it is necessary that two individuals come together to accomplish the reproduction (we say in this case that the reproduction is *bisexual*).

### The bisexual model

The first attempt to model a population with sexual interactions using branching processes came almost sixty years ago, and is a model introduced by Daley [Dal68] in 1968. It can be thought as a branching process with an extra mating step. To give the general idea consider a population formed by females and males. They can not reproduce by themselves, so they need to interact and form mating units to achieve reproduction (also called couples). Each mating unit can reproduce independently of each other and beget new individuals that are of one of two sexes. Again, these new individuals need to form new mating units to reproduce, since they can not do it alone. This process repeats producing new individuals, and we can continue indefinitely.

The mating occurs with the help of a deterministic function  $\xi : \mathbb{N}^2 \rightarrow \mathbb{N}$  called the mating function. The idea is that if at a certain time in the population there are  $x$  females and  $y$  males, then they mate forming  $\xi(x, y)$  mating units. In Figure 7, an illustration of an example of one of these processes where we use the mating function  $\xi(x, y) = \min\{x, y\}$ . We observe that we start with an initial population of two mating units (the  $\mathbf{C}$  in the initial condition stands for

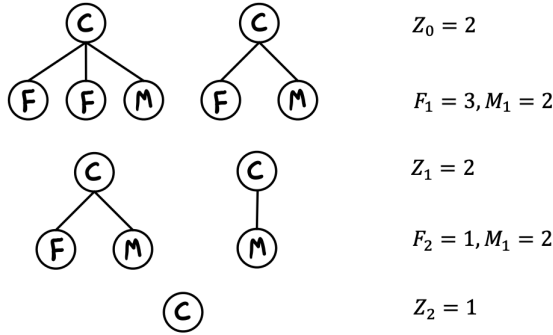


Figure 7: A realization of a bisexual Galton-Watson branching process with  $Z_0 = 2$  and mating function  $\xi(x, y) = \min\{x, y\}$ .

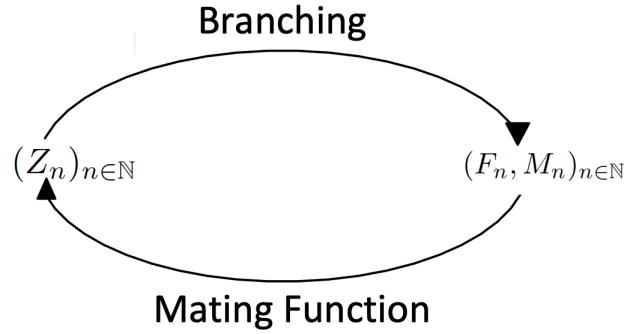


Figure 8: Two-steps dynamic for the bisexual Galton-Watson branching process.

“Couple”). They beget offspring forming in total 3 females and 2 males. Then, they mate forming  $\min\{3, 2\} = 2$  mating units, and we can continue with this process.

The idea that the reader must keep in mind is that this process needs to be considered as a two step process. We mean this in the sense that, if at a giving time we have  $Z_n$  mating units, to obtain the next generation  $Z_{n+1}$  we go through a first step of branching (creation of offspring) and a second step of mating (formation of the new couples) (see Figure 8). We remark that, since the mating step only gives the number of couples that are created, we lose track of the genealogy of the process (in other words, we do not keep record on *how* the couples are formed, but we only know *how many* there are).

The main questions that will concern us for this kind of models are:

1. Extinction: Assume the very reasonable condition that  $\xi(0, 0) = 0$ . That is, when there are no individuals, no couples are formed. Then,  $\{0\}$  is an absorbing state. If at some time  $n \in \mathbb{N}$  we have  $Z_n = 0$ , then we say that the process is extinct. A natural question is whether we can find conditions under which this occurs with probability one. Moreover, one desires that these conditions are both necessary and sufficient. In other words, if these conditions do not hold, one wants that the probability that the process never touches  $\{0\}$  is positive.
2. Long time convergence: If one knows that the process survives indefinitely, then a second question is whether one can describe what is the behaviour of this process in the long time limit  $n \rightarrow +\infty$ . If the process diverges, then a classical question is to find the speed of divergence in terms of the parameters of the process.

These two questions have already been solved for the classical Galton-Watson process presented early in this introduction (both in the single and multi-type cases). The main difficulty to address these two questions in the bisexual model is the presence of the mating function. In fact, the addition of this second step makes that the main techniques used to attack these questions in the classical case, do not apply for this model. Nevertheless, answers to these questions have been given under very simple assumptions and assuming in addition that the mating function  $\xi$  is any *superadditive* function. The extinction result can be found in [DHT86], whilst for the long time behaviour the reader can consult [GM96, GM97b]. The details of these results can

also be found in **Chapter 1**, where we give a more complete state of the art for these processes. For other references, the reader can also consult the excellent surveys of Alsmeyer [Als02], Hull [Hul03] and Molina [Mol10].

## Objectives

If we now consider that the population is formed by individuals with different characteristics (and then, they form different types of mating units), then we need to consider a multidimensional version of the bisexual model. In the literature, this issue has been considered for limited cases with very particular mating functions (see e.g. [Hul98, GHMM06, GMM09, AGM11]) mainly to model the evolution of genes linked to the  $Y$  chromosome (see also Chapter 1, Section 1.3). So far, no general mathematical description for a multi-type bisexual model has yet been established. The objective of this thesis is to fill this gap.

### Specific Objectives of this Thesis

- To construct a multi-type bisexual branching process with the help of a multidimensional mating function and establish the basic assumptions under which we will work.
- To develop results on the asymptotic behaviour of the process for large population, such as laws of large numbers.
- To study the existence of necessary and sufficient conditions for almost sure extinction.
- To study the existence of quasi-stationary distributions in the cases where the extinction of the process is certain.
- On the event where the process survives indefinitely, to study the convergence properties related to the long time behaviour of the process. In particular, study if one can estimate the rates at which the process grows in terms of the parameters of the process.
- To establish results on the long time behaviour for more general models that consider random mating or a continuous time setting.

## Structure of the manuscript and contributions

This thesis is composed of six chapters and one appendix. In **Chapter 1** we present a more detailed state of the art for the branching processes we presented at the beginning of this introduction. We start with some general results on the classical Galton-Watson process to then list the several results existing around its bisexual counterpart.

### A general multidimensional model

**Chapter 2** is dedicated entirely to the construction of the multi-type bisexual branching process. This process is the Markov Chain defined on  $\mathbb{N}^p$ , namely  $(Z_n)_{n \in \mathbb{N}}$ , where  $p$  is a fixed positive integer. The construction can be summarized as follows:

1. If at a given time  $n \in \mathbb{N}$ , there is a configuration  $(Z_{n,1}, \dots, Z_{n,p})$  of mating units (meaning that there are  $Z_{n,i}$  mating units of type  $i$ ), then each one of them will reproduce and give birth to females and males of different types. We remark that a mating unit may produce



offspring of more than one type. Moreover, we assume that there are  $q_f \in \mathbb{N}$  (resp.  $q_m \in \mathbb{N}$ ) different types of females (resp. of males).

2. The total number of females (resp. of males) produced will be denote  $F_{n+1} \in \mathbb{N}^{q_f}$  (resp.  $M_{n+1} \in \mathbb{N}^{q_m}$ ). They will form the  $(n+1)$ -th generation of mating units with the help of a deterministic function  $\xi : \mathbb{N}^{q_f} \times \mathbb{N}^{q_m} \rightarrow \mathbb{N}^p$ , called the mating function. Hence, we have  $Z_{n+1} = \xi(F_{n+1}, M_{n+1})$ .

Thus, we have the same two steps structure for the one step distribution of the process. We now give a more formal definition of these two steps: given  $Z_0 \in \mathbb{N}^p$ , for all  $n \in \mathbb{N}$  we have

- **The branching step:** For  $j \in \{1, \dots, q_f\}$  and  $k \in \{1, \dots, q_m\}$ , we set the of number females of type  $j$  and the number of males of type  $k$  present in the  $(n+1)$ -th generation as

$$F_{n+1,j} = \sum_{i=1}^p \sum_{\ell=1}^{Z_{n,i}} X_{i,j}^{(n,\ell)} \text{ and } M_{n+1,k} = \sum_{i=1}^p \sum_{\ell=1}^{Z_{n,i}} Y_{i,k}^{(n,\ell)}, \quad (7)$$

where for all  $i \in \{1, \dots, p\}$ ,  $(X_{i,\cdot}^{(n,\ell)}, Y_{i,\cdot}^{(n,\ell)})_{n,\ell \in \mathbb{N}}$  is a family of i.i.d. random vectors whose assumptions are presented in Section 2.1 and are omitted in this introduction.

- **The mating step:** The females and males mate forming the  $(n+1)$ -th generation of mating units

$$Z_{n+1} = \xi((F_{n+1,1}, \dots, F_{n+1,q_f}), (M_{n+1,1}, \dots, M_{n+1,q_m})). \quad (8)$$

It is also assumed that  $\xi(0,0) = 0$  making  $(Z_n)_{n \in \mathbb{N}}$  a Markov chain with an absorbing state at  $\{0\}$ .

In all chapters from 2 to 5,  $\xi : \mathbb{N}^{q_f} \times \mathbb{N}^{q_m} \rightarrow \mathbb{N}^p$  is considered to be a superadditive function, i.e.

$$\xi(x_1 + x_2, y_1 + y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2), \text{ for all } x_1, x_2 \in \mathbb{N}^{q_f}, y_1, y_2 \in \mathbb{N}^{q_m}.$$

This assumption in particular guarantees that the function  $\mathfrak{M} : \mathbb{R}_+^p \rightarrow (\mathbb{R}_+ \cup \{+\infty\})^p$  given by

$$\mathfrak{M}(z) = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_1 \mid Z_0 = \lfloor kz \rfloor)}{k}$$

is well defined for all  $z \in \mathbb{R}_+^p$  (although it can take the value  $+\infty$ ). For  $z \in \mathbb{R}_+^p$ ,  $\mathfrak{M}(z)$  is called the *asymptotic growth rate* of the process in the direction  $z$ .

By a correct choice of the parameters of the process, one can recover the single-type bisexual model and both (single-type and multi-type) asexual models. If we compute the function  $\mathfrak{M}$  for these three particular cases, it turns out that  $\mathfrak{M}$  is a linear function. In particular, in the asexual case  $\mathfrak{M}$  is a matrix whose largest eigenvalue describes the behaviour of the process in the long time. As we will see later in Chapter 3, this function is concave in the general case. Nevertheless, under mild assumptions one can prove (see Chapter 2, Section 2.3.1) that when  $\mathfrak{M}$  takes finite values, then there exists a unique  $\lambda^* > 0$ , and a unique  $z^* \in \mathbb{R}_+^p$  with strictly positive components and  $|z^*| = 1$  that are eigenelements of  $\mathfrak{M}$ . That is, such that

$$\mathfrak{M}(z^*) = \lambda^* z^*.$$

As we will see, these eigenelements will play the same role they play in the asexual multi-type case.

In what follows, we state our main results making reference to the chapters where they can be found.

## Laws of Large Numbers

The intuition behind the fact of having a superadditive mating function is that we need to focus our study of the process in the cases where the population is large. This is studied in **Chapter 3**, where we develop two laws of large numbers.

**LLN for large initial population** (See Theorem 22, Page 60). This result links the behaviour of the process at a fixed time  $n \in \mathbb{N}$  with the iterations of the function  $\mathfrak{M}$  when the initial population is large and grows in a fixed direction. It goes as follows: if we denote by  $(Z_n^m)_{n \in \mathbb{N}}$  a multi-type bisexual branching process with initial condition  $Z_0 = z_m$ , where  $(z_m)_{m \in \mathbb{N}}$  is a random sequence such that  $z_m \sim_{m \rightarrow +\infty} m z_\infty$ , for  $z_\infty \in \mathbb{R}_+^p$  a deterministic vector, then, for all  $n \in \mathbb{N}$

$$Z_n^m \sim_{m \rightarrow +\infty} \mathfrak{M}^n(m z_\infty), \text{ almost surely.}$$

Under extra assumptions, an  $L^1$  convergence is also proved. This result already appeared in the single-type case in [DHT86], but in the particular case where  $z_m = m z_\infty$ ,  $n = 1$  and only with almost sure convergence. Hence, we extend this result not only to a multi-type setting, but also with a random initial condition and for all  $n \in \mathbb{N}$ . Additionally, the  $L^1$  convergence is also an original result. This convergence plays an important role in the proof of the results to follow.

**LLN for the trajectory** (See Theorem 30, Page 72). The goal of this result is to study the behaviour of the complete trajectory of the process, when initialized with a large initial condition. This result is presented for the case when  $\mathfrak{M}$  is finite and its eigenvalue holds  $\lambda^* > 1$ . It states that, for all  $\varepsilon, \eta \in (0, 1)$ , there exists  $r > 0$  such that, if  $|Z_0| > r$ , then

1. With probability at least  $1 - \eta$ , we have  $Z_{n_0} \neq 0$  and, for all  $n \geq n_0$ ,

$$(1 - \varepsilon)\mathfrak{M}(Z_n) \leq Z_{n+1} \leq (1 + \varepsilon)\mathfrak{M}(Z_n), \quad (9)$$

where  $n_0 \in \mathbb{N}$  is a deterministic value depending only on  $\mathfrak{M}$  (and not on the parameters  $\varepsilon, \eta, r$ ), and the inequalities are componentwise.

2. Using this, we can prove that on the event that the process is not extinct, then it grows in the direction given by  $z^*$ , the eigenvector of  $\mathfrak{M}$ , i.e.,

$$\frac{Z_n}{|Z_n|} \xrightarrow{n \rightarrow +\infty} z^*. \quad (10)$$

The first result is a new result even in the single-type bisexual case. For the second point, we recover a classic result already existing in the asexual Galton-Watson branching process, but now for the multi-type bisexual setting. These results are then used to study the long time behaviour of the process.

## Long time behaviour

In **Chapter 4** we study the behaviour of the process in the limit when the time goes to infinity. We exhibit necessary and sufficient conditions for the almost sure extinction and study the convergence of the process in the survival event.

**Extinction conditions** (See Theorem 37, Page 88). We prove that in the case where  $\mathfrak{M}$  is finite, the process is almost sure extinct if and only if  $\lambda^* \leq 1$ . If  $\mathfrak{M}$  is not finite, then the process can survive with positive probability. This result was proved for the single-type bisexual process in [DHT86], although only for the case where  $\mathfrak{M}$  is finite. We complete the picture providing a proof for the case where  $\mathfrak{M}$  takes an infinite value.

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**Convergence properties** (See Theorem 39, Page 94). We prove that for all  $z \in \mathbb{N}^p$  there exists a random variable  $\mathcal{C}$  such that

$$\frac{Z_n}{(\lambda^*)^n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}(\cdot | Z_0 = z) - \text{a.s.}} \mathcal{C}z^*. \quad (11)$$

In addition, we prove that for the values of  $z \in \mathbb{N}^p$  such that  $\mathcal{C}$  is non-degenerate at 0, then up to a  $\mathbb{P}(\cdot | Z_0 = z)$ -negligible event,

$$\{\mathcal{C} = 0\} = \{\exists n \in \mathbb{N}, \quad Z_n = 0\}. \quad (12)$$

Finally, we give a series of conditions under which the convergence holds in  $L^1$  and  $\mathcal{C}$  is non-degenerate.

The result in (11) is already known in the literature for asexual and single-type bisexual models. On the other hand, (12) is a new result even for the single-type bisexual branching process.

### Quasi-limiting behaviour in the subcritical case

In **Chapter 5** we are interested in the existence of quasi-stationary distributions for the case where  $\lambda^* < 1$ . We prove that, depending on the assumptions that are in place for the process we may have an infinite number of, a finite number of or a unique quasi-stationary distribution. More precisely,

1. Under the assumption that the mating function is sub-affine, there are infinitely many quasi-stationary distributions (see Theorem 49, Page 112). To prove this result we prove first a general result for Feller processes and then we apply it for the particular case of the bisexual branching process.
2. Under the previous assumptions, but assuming also extra regularity for the function  $\mathfrak{M}$  and a moment condition on the offspring distribution, we prove that the process admits a finitely generated family of quasi-stationary distributions (see Theorem 50, Page 114).
3. Under the previous assumptions, but with the extra hypothesis that the process is irreducible and aperiodic, we prove that there is a unique quasi-limiting distribution for compactly supported initial distributions (see Theorem 53, Page 118).

These results are known for decades for the classical asexual Galton-Watson process, but are new for the bisexual model in general, including the single-type process.

### Unlimited growth of two-sex population models

Given a Markov Chain  $(\mathcal{X}_n)_{n \in \mathbb{N}}$  defined on  $\mathbb{R}_+^q$ , in **Chapter 6** we give two main results concerning sufficient conditions such that the probability

$$\mathbb{P}(\mathcal{X}_n \xrightarrow[n \rightarrow +\infty]{} +\infty \mid \mathcal{X}_0 = x)$$

is strictly positive for some  $x \in \mathbb{R}_+^q$ . When this happens, we say that  $(\mathcal{X}_n)_{n \in \mathbb{N}}$  has unlimited growth with positive probability. More precisely, these results give conditions such that the process  $(g(\mathcal{X}_n))_{n \in \mathbb{N}}$  diverges to infinity, where  $g : \mathbb{R}_+^q \rightarrow \mathbb{R}_+$  is a suitable function. These results generalize an already existing result on unlimited growth established in [GMM05], for models that grow in every direction (which is not the case, for instance, for certain bisexual branching processes). These results are then applied for different models of two-sex populations. Here we mention some of the examples that we deal with.

1. We consider a model of multi-type bisexual Galton-Watson branching process where the mating function is a random function. In this case, we consider a function that is almost surely increasing. We give conditions for unlimited growth and give special attention to the case where the random mating function is superadditive with positive probability. This model is in Example 13, in page 129.
2. Consider a supercritical multi-type bisexual Galton-Watson branching process, i.e. that survives with a positive probability. If the mating function of this process is a certain function  $\xi$ , we build a new process but with mating function  $\xi + \varphi$ , where  $\varphi$  is random perturbation. We give conditions over  $\varphi$  such that the new process stays supercritical. This model is presented in Example 14, in page 130.
3. A continuous time two-sex birth and death process is a 2-type (females and males) birth and death process, but where the rate of birth is a function that depends on both types. We consider two particular examples of this process, inspired by discrete time version of bisexual branching processes, and give necessary and sufficient conditions for their certain extinction. The first of the two results can be obtained as a particular case of a more general result in [KK73], whilst the second is a completely new result. This models are in Section 6.4, in page 138.

## Scientific Contributions

During this PhD, the following scientific articles were produced.

- Chapters 2, 3 and 4 are based on the article:  
C. Fritsch, D. Villemonais and N. Zaldueño  
*The multi-type bisexual Galton-Watson branching process* (accepted in Annales de l'Institut Henri Poincaré).  
ArXiv e-prints: [2206.09622](https://arxiv.org/abs/2206.09622).
- Chapter 5 is based on the article:  
C. Fritsch, D. Villemonais and N. Zaldueño  
*Quasi-limiting behaviour of the subcritical multi-type bisexual Galton-Watson branching process*.  
Pre-print (to be submitted): [Link](#).

The contents of Chapter 6 will be part of an article, currently in production.

In addition, the following article was completed during this PhD, but its content is not included in this manuscript, since it is related to a completely different subject.

- Y. Yalanda and N. Zaldueño  
*Restricted maximum of non-intersecting Brownian bridges* (submitted)  
ArXiv e-prints: [2302.11432](https://arxiv.org/abs/2302.11432).

## Perspectives

The main contribution of this work is the development of a general model for a multi-type bisexual Galton-Watson branching process. Through most of this work, the mating function of the process is considered to be a superadditive function, and for this type of models two laws of large numbers

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are developed in Chapter 3 for the cases when the population is large. To complete the picture, central limit theorems need to be developed to fully understand the behaviour of the population when it is large. Moreover, it would be interesting to extend these results to non-superadditive models. The superadditivity of the mating function is a key ingredient in the proofs developed in this manuscript, although it implies that the population behaves always in a cooperative way. More complex models where the mating allows a competitive behaviour between individuals of different types need to be taken into account.

In Chapter 5, the existence of infinitely many quasi-stationary distributions is proved for the case where the process is subcritical. However whether quasi-stationary distributions exist in the critical case, is still an open question. For a critical bisexual branching process, the critical behaviour is defined based on the behaviour of the process as the population tends to infinity. Thus, a process may be critical whilst evidencing a subcritical behaviour for finite population. A first approach would be to find a way to measure the “speed” at which this process approaches a critical behaviour when the population becomes larger.

One of the models that is missing in the theory of bisexual branching processes is a model in continuous time. Although two examples are treated in this manuscript for the case of birth and death processes in Chapter 6, a more complete analysis for a general model in continuous time needs to be provided. One of the main difficulties while dealing with continuous time bisexual processes is the fact that they always need to be written as multi-type processes: either we keep track of the number of females and males (a two types process) or the number of mating units, single females and single males (a three types process). Hence, our developments in the multi-type bisexual discrete process, and in particular the existence of eigenelements for this model, may come in handy when analysing its continuous counterpart.

Overall, there are several directions for future research based on the foundations laid in this thesis, including further investigations into the behavior of multi-type bisexual branching processes, and the application of these models to real-world scenarios in population biology, epidemiology, and ecology.



# Chapter 1

## The Theory of Bisexual Branching Processes

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This chapter is intended to be a survey around the existing results in the theory of bisexual branching processes. This summary of results is by no means exhaustive, but it allows us to have an idea on the current state of the theory and on the different techniques employed in the demonstrations. We focus mainly on discrete time processes, although some models in continuous time are discussed near the end of this chapter. Since this kind of processes appears as an extension of the classical Galton-Watson process (from now on, referred to as the *asexual* Galton-Watson process), we first introduce a summary on the different results for this process both in one and in higher dimension (Section 1.1). We then introduce the bisexual process in one dimension (Section 1.2), beginning with its first results and some particular examples, all the way to the different variations of this model present in the literature. We finish with a list of existing results in higher dimension (Section 1.3) and some results in continuous time (Section 1.4).

### 1.1 The Galton-Watson branching process

The Galton-Watson process was introduced by Henry Watson [Wat73] to answer a question related to the extinction of surnames in the families of the British aristocracy which was formulated by sir Francis Galton that same year [Gal73]. This process can be described as follows: imagine objects that can generate additional objects of the same kind, independently of each other, by

an asexual reproduction or division (particles, bacteria, etc). Assume we have an initial set of these objects at time zero (or the 0–th generation). After one unit of time all of these objects may (or not) have generated a random number of descendents. The set of all the offspring is considered to be the first generation. This process is repeated obtaining a second generation, and we can keep going infinitely many times or until there are no objects anymore. If we assume that all the objects are indistinguishable, then we refer to this process as the *single-type* (asexual) Galton-Watson process, which corresponds to a Markov Chains with values on  $\mathbb{N}$ . If there is a finite number of different objects, then we speak of the *multi-type* (asexual) Galton-Watson process, and in this case it is a Markov Chain defined in a state space with higher dimension. We first focus on the study of the former, to finish this section with the latter. For both processes, we follow mainly the books of T. Harris [Har63] and K. Athreya and P. Ney [AV01].

### 1.1.1 Single-type asexual model

We consider the model recently discussed, where all objects are identical in terms of the law of their offspring distribution. The process previously described can be formalized as follows. Consider a probability distribution  $(p_k)_{k \in \mathbb{N}}$  on  $\mathbb{N}$  for the number of particles that a single particle can produce as offspring. Then, we define the process  $(Z_n)_{n \in \mathbb{N}}$  as a Markov Chain on  $\mathbb{N}$  given by,

$$Z_{n+1} = \sum_{i=1}^{Z_n} L_i^{(n)}, \text{ for all } n \in \mathbb{N}, \quad (1.1)$$

where  $Z_0 = z_0 \in \mathbb{N}$  is given, and  $(L_i^{(n)})_{i, n \in \mathbb{N}}$  is a family of i.i.d. random variables with common probability distribution  $(p_k)_{k \in \mathbb{N}}$ .

To avoid trivialities we always assume that  $p_0 + p_1 < 1$  (i.e. there is a positive probability that a single particle produces two or more descendents). Note that if it is not the case, then the population is bounded by  $Z_0$  at all times. In addition, in the discussion to follow, and to simplify our notation, we consider always that  $L$  is a random variable with probability distribution given by  $(p_k)_{k \in \mathbb{N}}$ . We assume that  $L \in L^1$ , i.e. that the number of particles produced by a single parent is an integrable random variable, and set  $m = \mathbb{E}(L)$ .

One of the most important and useful properties of this process is the branching property. It can be expressed in the following manner: given  $z \in \mathbb{N}$ , denote  $(Z_n(z))_{n \in \mathbb{N}}$  a Galton-Watson process issued from  $Z_0 = z$ . Then, for all  $k \in \mathbb{N}$  and all  $n \in \mathbb{N}$ ,

$$Z_n(k) \stackrel{(d)}{=} \sum_{\ell=1}^k Z_n^{(\ell)}(1), \quad (1.2)$$

where  $Z_n^{(1)}(1), \dots, Z_n^{(k)}(1)$  are  $k$  independent copies of a Galton-Watson process issued from  $Z_0^{(\ell)}(1) = 1$ , for all  $\ell \in \{1, \dots, k\}$ , and  $\stackrel{(d)}{=}$  entails for equality in distribution. This property plays a fundamental role in the study of conditions under which the process gets absorbed at 0 in finite time (also known as the extinction of the process). In fact, this property allows us to:

1. Restraint the study of extinction conditions to the particular case where  $Z_0 = 1$ . In fact, if  $q$  denotes the probability that the process eventually hits  $\{0\}$  starting from one particle in the zero-th generation, then thanks to (1.2), with probability  $q^j$  the process will hit  $\{0\}$  if  $Z_0 = j$ . This entails that the process will be almost surely extinct in finite time starting from  $Z_0 = 1$ , if and only if it will be almost surely extinct in finite time for any initial condition.



2. Find an explicit expression for the probability of extinction, in terms of the *probability generating functions* of the process.

### Extinction Conditions

As previously stated, to obtain extinction condition on the process for general initial condition, it is sufficient to study the probability of extinction starting from one individual, defined by

$$q = \mathbb{P}(\exists n \in \mathbb{N}, Z_n = 0 \mid Z_0 = 1). \quad (1.3)$$

The following result gives a necessary and sufficient condition for the almost sure extinction with respect to the mean number of offspring.

**Theorem 1.** *Assume that  $L \in L^1$  and let set  $m = \mathbb{E}(L)$  then*

$$q = 1 \iff m \leq 1.$$

In other words, the process almost surely goes to extinction if and only if the expected number of offspring is less than or equal to one.

The proof of this result is based on the study of the probability generating function (from now on, PGF) of the variable  $L$ , defined by the function  $f_L : [0, 1] \rightarrow \mathbb{R}$  given by  $f_L(s) = \mathbb{E}(s^L)$ . In fact, if  $L \in L^1$ , we can show (for more details, we refer to Chapter 1 of [AV01]) that  $q$  is the smallest fixed point of  $f_L$  in  $[0, 1]$ :

$$q = \inf\{s \in [0, 1] : f_L(s) = s\}.$$

Due to the previous result, a process with  $m < 1$  is called a *subcritical* process, with  $m = 1$  a *critical* process, whilst when  $m > 1$  is called a *supercritical* one.

### Long time behaviour

Given a supercritical ( $m > 1$ ) Galton-Watson process  $(Z_n)_{n \in \mathbb{N}}$ , we are interested on the behaviour in the long time limit in the event of survival (after proper rescaling). We start by noticing that if we define

$$W_n = \frac{Z_n}{m^n},$$

then  $(W_n)_{n \in \mathbb{N}}$  forms a non-negative martingale. In fact, from the Markov property,

$$\mathbb{E}(W_{n+1} \mid Z_n) = \frac{1}{m^{n+1}} \mathbb{E}(Z_{n+1} \mid Z_n) = \frac{1}{m^{n+1}} m Z_n = W_n.$$

Hence, by the Martingale Convergence Theorem, there exists a non-negative random variable  $W$  such that

$$W_n = \frac{Z_n}{m^n} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} W.$$

Of course, this reasoning does not depend on the fact that  $m > 1$ , but due to Theorem 1 we know that if it is not the case, then  $W = 0$ . Moreover, we have the following clear inclusion

$$\{\exists n \in \mathbb{N} : Z_n = 0\} \subseteq \{W = 0\}. \quad (1.4)$$

A natural question then is if whether we can impose extra conditions on the process (particularly on the offspring distribution) to obtain the almost sure equality in (1.4). The following Theorem (known as the  $L \log L$ -type criteria) presents the answer.

**Theorem 2.** [Kesten - Stigum Theorem, see [KS66]] The following statements are equivalent.

- (a)  $\{W = 0\} = \{\exists n \in \mathbb{N} : Z_n = 0\}$ .
- (b)  $\mathbb{P}(W = 0) = q$ , with  $q$  defined in (1.3).
- (c)  $\mathbb{E}(W) = 1$ .
- (d)  $\mathbb{E}(L \log L) < +\infty$ .

It can also be proved that, in the case where one (then, all) of the previous statements are true, then the convergence of  $(W_n)_{n \in \mathbb{N}}$  towards  $W$  holds in  $L^1$ .

### 1.1.2 Multi-type asexual model

The multi-type version of the previous model is its natural extension to a higher (but in this case, finite) dimension. Consider that now the objects that reproduce asexually (bacteria, particles) are of different types. The number of possible types is finite and equal to  $d \in \mathbb{N}^*$ , a positive integer that will be fixed throughout this section.

We define the multi-type process as the Markov Chain taking values on  $\mathbb{N}^d$  given by

$$Z_{n+1} = \sum_{i=1}^d \sum_{k=1}^{Z_{n,i}} L_{i,j}^{(k,n)}, \quad (1.5)$$

where  $Z_0 = z_0 \in \mathbb{N}^d$  is given, and for every  $i \in \{1, \dots, d\}$ , the family  $((L_{i,1}^{(k,n)}, \dots, L_{i,d}^{(k,n)}))_{k,n \in \mathbb{N}}$  stands for an i.i.d. family of random vectors with a common probability distribution  $(p_i(v))_{v \in \mathbb{N}^d}$ , taking values on  $\mathbb{N}^d$  and describing the offspring of the next generation. The idea is that for any  $i \in \{1, \dots, d\}$  and  $v = (v_1, \dots, v_d) \in \mathbb{N}^d$ ,  $p_i(v)$  stands for the probability that a single particle of type  $i$  has as offspring exactly  $v_1$  particles of type 1,  $v_2$  particles of type 2, and so on. In the discussion to follow, we denote the vector  $L_i = (L_{i,1}, \dots, L_{i,d})$  as a random vector which has this probability distribution. We assume that for any  $i_1 \neq i_2$ ,  $L_{i_1}$  is independent of  $L_{i_2}$ , which translates into the fact that every particle reproduces independently of other particles. We also assume that all random variables  $L_{i,j}$ , for  $i, j \in \{1, \dots, d\}$ , are integrable.

To avoid trivialities we assume that for all  $i \in \{1, \dots, d\}$ ,

$$\sum_{\substack{v \in \mathbb{N}^d \\ |v|=1}} p_i(v) < 1.$$

In other words, we are not considering the case where every particle has exactly one descendent. In the literature, such a process is called a *singular* process.

We remark that if we consider  $d = 1$ , we recover the single-type process discussed in the previous section.

Before moving to the question of conditions for certain extinction, we want to remark that, as in the case of the single-type asexual model, here we also have the branching property. As in the single-type case, this property is very useful for two main reasons: first, to find conditions for almost sure extinction we only have to work with a process initialized with only one particle; and second, we can find an explicit equation that, when solved, gives the probability of extinction. As in the case with  $d = 1$ , this equation corresponds to a fixed point problem in terms of the PGFs.

In the multi-type case, the branching property can be stated as follows. For any  $z \in \mathbb{N}^d$  let denote  $(Z_n(z))_{n \in \mathbb{N}}$  a multi-type Galton-Watson process for which  $Z_0 = z$ . Then, for all  $z \in \mathbb{N}^d$  and all  $j \in \{1, \dots, d\}$

$$Z_{n,j} = \sum_{i=1}^d \sum_{k=1}^{z_i} \hat{Z}_{n,j}^{(k,i)}(e_i),$$

where  $(\hat{Z}^{(k,i)}(e_i))_{k \in \mathbb{N}}$  are i.i.d. multi-type Galton Watson process issued from  $e_i$ , the  $i$ -th canonical vector on  $\mathbb{N}^d$ .

### The condition for almost sure extinction

For  $i \in \{1, \dots, d\}$ , denote by  $\mathbb{P}_i$  the probability measure  $\mathbb{P}(\cdot \mid Z_0 = e_i)$  and by  $\mathbb{E}_i$  its corresponding expectation. The objective for the following definitions is to focus on the case where  $Z_0 \in \{e_1, \dots, e_d\}$ .

We define the matrix  $M \in \mathbb{R}_+^{d \times d}$  given by

$$M_{i,j} = \mathbb{E}_i(Z_{1,j}), \text{ for all } i, j \in \{1, \dots, d\}. \quad (1.6)$$

In other words,  $M_{i,j}$  is the expected number of particles of type  $j$  produced by a single particle of type  $i$ . Note that in the case where  $d = 1$ , then  $M$  is just a real number and it is equal to  $m$ , the expected number of offspring.  $M$  is called the *expectation matrix* of the process  $(Z_n)_{n \in \mathbb{N}}$ .

The main tool to study the extinction conditions for the process is the Perron-Frobenius Theorem, which guarantees the existence of a largest positive eigenvalue for the expectation matrix. Under the assumption that the matrix  $M$  is positively regular (that is, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $M^n$  has strictly positive entries), then this theorem guarantees the existence of a positive eigenvalue  $\rho^*$  that is greater in absolute value than any other eigenvalue;  $\rho^*$  corresponds to a right eigenvector  $u^* \in \mathbb{R}^d$  and a left eigenvector  $v^* \in \mathbb{R}^d$ , both with all positive entries, which are the only non-negative eigenvectors. This eigenvalue is also known as Perron-Frobenius root in the literature. In the special case where  $d = 1$  we have  $\rho^* = m$ . As we see in the next theorem,  $\rho^*$  plays a similar role as the one that  $m$  played in the single-type case.

We define for all  $i \in \{1, \dots, d\}$ , the probability of extinction issued from one individual of type  $i$  as

$$q_i = \mathbb{P}(\exists n \in \mathbb{N}, |Z_n| = 0 \mid Z_0 = e_i). \quad (1.7)$$

**Theorem 3.** *Assume that the expectation matrix  $M$  defined by (1.6) is positively regular and set  $\rho^*$  its Perron-Frobenius root. Then,*

$$q_i = 1, \text{ for all } i \in \{1, \dots, d\} \iff \rho^* \leq 1.$$

Analogously to the single-type case, a process with  $\rho^* < 1$  is called *subcritical*, if  $\rho^* = 1$  then it is called *critical*, whilst if  $\rho^* > 1$  it is called *supercritical*.

### The probability of extinction

Regarding the probability of extinction, the branching property allows us to get an explicit equation using the PGFs of the process, similarly to what it is done in the single-type case. To get into the details, define for  $i \in \{1, \dots, d\}$  the PGF of  $L_i$ , given by  $f_i : [0, 1]^d \rightarrow [0, 1]$ , where for  $(s_1, \dots, s_d) \in [0, 1]^d$ ,

$$f_i(s_1, \dots, s_d) = \mathbb{E} \left( s_1^{L_{i,1}} \cdot \dots \cdot s_d^{L_{i,d}} \right) = \sum_{v \in \mathbb{N}^d} p_i(v) s_1^{v_1} \cdot \dots \cdot s_d^{v_d}.$$

Set  $F : [0, 1]^d \rightarrow [0, 1]^d$  the function defined by

$$(s_1, \dots, s_d) \in [0, 1]^d \mapsto F(s_1, \dots, s_d) = (f_1(s_1, \dots, s_d), \dots, f_d(s_1, \dots, s_d)).$$

Then, if  $\rho^* > 1$ , we have that  $q = (q_1, \dots, q_d) \in [0, 1]^d$  defined by (1.7) is the only solution to the fixed point equation

$$F(s_1, \dots, s_d) = (s_1, \dots, s_d), \text{ with } (s_1, \dots, s_d) \in [0, 1]^d.$$

### Convergence properties

We finish the study of this process with some results on the long time behaviour for the supercritical case, similar to the ones presented in the single-type case.

**Theorem 4.** [Kesten-Stigum Theorem (see [KS66])] Assume  $\rho^* > 1$ . There exist a random vector  $W$  of dimension  $d$  and a non-negative real random variable  $C$  such that

1. We have the following convergence for the renormalized process

$$\frac{1}{(\rho^*)^n} Z_n \xrightarrow[n \rightarrow +\infty]{a.s.} W.$$

2. With probability one, the random vector has the same direction that the left eigenvector corresponding to  $\rho^*$ ,  $v^*$ , namely

$$W = Cv^*.$$

3. One has either

$$\mathbb{E}(C \mid Z_0 = e_i) = u_i^*, \text{ for all } 1 \leq i \leq d \tag{1.8}$$

with  $u^*$  the right eigenvector corresponding to  $\rho^*$ , or

$$\mathbb{P}(C = 0 \mid Z_0 = e_i) = 1, \text{ for all } 1 \leq i \leq d.$$

4. Moreover, (1.8) holds if and only if

$$\mathbb{E}(L_{i,j} \log L_{i,j}) < +\infty \text{ for all } 1 \leq i, j \leq d.$$

## 1.2 The single-type bisexual branching process

The classical Galton-Watson process presented in the previous section does not take into account the distinction between sexes of individuals. Although this issue can be overcome by a multi-type process with 2 types, such a process neglects to model a population for which the interaction between sexes is necessary for reproduction. In fact, Galton-Watson processes in general assume that the reproduction can happen asexually, or that the non-reproductive part (usually the males) plays no role in the dynamics of the process. For instance, as we saw in Theorem 1, a population with just one individual in the asexual Galton-Watson process can survive with positive probability, which is not the case in populations that need at least two individuals to beget offspring. The processes that take the formation of couples (this action is also called *mating*) into account for the reproduction are denominated *bisexual* Galton-Watson branching processes. The first of these models was introduced in 1968 by Daley in [Dal68], and can be described as follows: consider a population that can be divided into two sexes (females and males)

which evolves in discrete time. Each unit of time  $n \in \mathbb{N}$  can be thought as a generation unit, and for simplicity, the individuals alive at time  $n \in \mathbb{N}$  will be called the  $n$ -th generation. Nevertheless, discrete time can also represent smaller units of time, for example, different repetitions of the estrous cycle in female mammals (also known as the “on heat” period), when the mating occurs and they reproduce (see for instance [Bro89]). The reproduction can only be carried by *mating units*, which are units formed by one female and one male of the same generation who come together for the purpose of procreation. The process studied by Daley corresponds to the process counting the number of mating units in each generation. Starting from  $Z_0$  mating units in the initial (or zero-th) generation, the process is formed by the continuing iteration of the following steps.

1. Consider that at a given generation  $n \in \mathbb{N}$  there are  $Z_n \in \mathbb{N}$  mating units. Each one of these units will reproduce independently and will give birth to a certain (random) number of individuals, each one of them either a female or male. The distribution of the number of individuals is identical for all the mating units and it is specified through a probability distribution  $(p_{j,k})_{j,k \in \mathbb{N}}$  defined on  $\mathbb{N}^2$ . In other words,  $p_{j,k} \in [0, 1]$  represents the probability that a single mating unit has exactly  $j$  females and  $k$  males as offspring.
2. Denote by  $X_{n,\ell}$  and  $Y_{n,\ell}$  the number of females and males (respectively) produced by the  $\ell$ -th mating unit (labelled in an arbitrary way) present in the  $n$ -th generation. We have that  $(X_{n,\ell}, Y_{n,\ell})_{n,k \in \mathbb{N}}$  forms a family of independent and identically distributed random vectors with common distribution given by  $(p_{j,k})_{j,k \in \mathbb{N}}$ . Denote the total number of females (respectively males) produced by the  $n$ -th generation of mating units as  $F_{n+1}$  (respectively  $M_{n+1}$ ). They form the  $(n+1)$ -th generation of individuals. In other words,

$$F_{n+1} = \sum_{\ell=1}^{Z_n} X_{n,\ell} \text{ and } M_{n+1} = \sum_{\ell=1}^{Z_n} Y_{n,\ell}. \quad (1.9)$$

We remark that we obtain a similar structure to the one defining the asexual Galton-Watson process (see equation (1.1)), as the production of offspring is independent for all mating units.

3. The females and males in the  $(n+1)$ -th generation form a certain number of mating units in this generation determined by a deterministic function  $\xi : \mathbb{N}^2 \rightarrow \mathbb{N}$ , namely

$$Z_{n+1} = \xi(F_{n+1}, M_{n+1}). \quad (1.10)$$

The function  $\xi$  is known in the literature as the *mating function*.

The process  $(Z_n)_{n \in \mathbb{N}}$  formed by the constant iteration of steps (1.9) and (1.10) is known as the single-type bisexual Galton-Watson branching process. It is assumed that this process is repeated indefinitely. It is also assumed that the empty sum is equal to 0, and so  $(Z_n)_{n \in \mathbb{N}}$  forms a discrete time homogeneous Markov Chain. The parameters of the process are  $Z_0$ ,  $\xi$  and  $(p_{j,k})_{j,k \in \mathbb{N}}$  and they must be specified in any realization of the process.

The organization of the rest of this section is as follows. We start with some early (and sometimes partial) results present in the literature in Section 1.2.1. In this section we will focus mainly in Daley’s results and some results that followed concerning almost sure extinction. In Section 1.2.2 we state a more general result concerning the certain extinction of processes where the mating function is considered to be superadditive. Later, a result around the probability

of extinction is stated in Section 1.2.3 and conditions for the convergence of the process are discussed in Section 1.2.4. Finally, some additional results for variations of this process are enlisted in Section 1.2.5

### 1.2.1 Some results of the literature on specific examples

In this section we will give the first results present in the literature, mainly concerning the question of almost sure extinction for some specific examples.

We start by remarking two particular properties that the offspring distribution may hold. These two properties are sometimes assumed throughout the literature since they may simplify the computations. In other cases, they are useful to build counterexamples. We refer to them as **Model A** and **Model B**, and they correspond to the following extra assumptions.

- (a) **Model A:** Assume that each mating unit produces a random number of individuals according to a probability distribution  $(\hat{p}_j)_{j \in \mathbb{N}}$ . Each individual in the offspring is female with probability  $\alpha \in [0, 1]$  and male with probability  $1 - \alpha$ , independently of the other offspring. Consequently, we recover  $(p_{j,k})_{j,k \in \mathbb{N}}$  via the relation

$$p_{j,k} = \binom{j+k}{j} \alpha^j (1-\alpha)^k \hat{p}_{j+k}, \text{ for all } j, k \in \mathbb{N}. \quad (1.11)$$

To avoid trivialities, Daley also assumed that  $\alpha \in (0, 1)$  and that  $\hat{p}_0 + \hat{p}_1 < 1$ .

- (b) **Model B:** Assume that each mating unit produces a certain number of females and a certain number of males independently of each other. In other words, if  $(p_j^X)_{j \in \mathbb{N}}$  (respectively  $(p_k^Y)_{k \in \mathbb{N}}$ ) is the probability distribution of the number of females (respectively, of males) produced by a single mating unit, then

$$p_{j,k} = p_j^X p_k^Y, \text{ for all } j, k \in \mathbb{N}. \quad (1.12)$$

Conditions (a) and (b) are equivalent when the distribution for the total number of individuals produced by a mating unit is distributed according to a Poisson random variable.

In general, we will not assume that any of these extra assumptions are in place unless we clearly state it in advance.

### Daley's mating functions

Daley imposed the following restrictions on the mating function  $\xi$ :

- When there are no individuals in a given generation, then no mating units can be produced, i.e.  $\xi(0, 0) = 0$ . Similarly, if at a given time there are only individuals of one of the two sexes, there can be no mating units, meaning  $\xi(x, 0) = \xi(0, y) = 0$  for all  $x, y \in \mathbb{N}$ . This fact also implies that  $Z$  is a Markov Chain with absorption at  $\{0\}$ .
- If the number of individuals of one of the sexes increases, then the number of mating units they produce cannot decrease. In other words,  $\xi$  must be a non-decreasing function on both arguments.

Whilst recapitulating Daley's results in [Als02], Alsmeyer also added a third assumption on the mating function, that hold for all the examples treated by Daley.

- If the number of both females and males within a generation increases to infinity, then so must do the number of mating units they form. This is formally expressed as

$$\lim_{(x,y) \rightarrow +\infty} \xi(x,y) = +\infty.$$

Alsmeyer called a mating function that holds all three assumptions a *common sense mating function*.

**Example 1.** Daley studied two examples of mating functions:

1. [**Completely Promiscuous Mating**] This model (also known as *cows and bulls* model) considers that males have infinitely great reproductive power and so the appearance of one male in the population is enough to ensure that every female forms a mating unit with such male. This is done by considering the mating function

$$\xi(x,y) = \begin{cases} x & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$$

2. [**Perfect Fidelity with Promiscuous Mating**] Here we consider that females practice perfect fidelity, in the sense that each female can form at most one mating unit in every generation. On the other hand, males mate promiscuously and can form up to  $d$  mating units, where  $d$  is a fixed positive integer, if there are enough females available. This can be expressed as

$$\xi(x,y) = \min\{x, dy\}.$$

One example where this mating function can make sense is in the model of a population where at every unit of time  $n \in \mathbb{N}$ , a female can breed offspring from one male at a time, but males can have multiple partners within this period. In the special case when  $d = 1$ , we refer to this model as *perfect fidelity mating*.

One can check that both mating functions are *common sense mating functions* in the way described by Alsmeyer.

The first natural question that arises is whether we can find conditions under which there is almost sure extinction. The first difficulty whilst addressing this question is the lack of the branching property. In fact, it is not hard to convince oneself that, in general, a statement equivalent to that of equation (1.2) for the case of asexual branching processes does not hold in the bisexual case. As an example, consider the case of completely promiscuous mating and consider that **Model B** is in place (see (1.12)).

Denote for  $n \in \mathbb{N}$  the function  $F_n(k) = \mathbb{P}_n(Z_1 = k)$ . Given the assumption in **Model B**, we have for all  $k \in \mathbb{N}$  that

$$\begin{aligned} F_1(k) &= \mathbb{P}_1(Z_1 = k) \\ &= \mathbb{P}(X_{1,1} = k, Y_{1,1} > 0) \\ &= \mathbb{P}(X_{1,1} = k)(1 - \mathbb{P}(Y_{1,1} = 0)) \\ &= p_k^X(1 - p_0^Y), \end{aligned}$$

and that

$$\begin{aligned}
 F_2(k) &= \mathbb{P}_2(Z_1 = k) \\
 &= \mathbb{P}(X_{1,1} + X_{1,2} = k, Y_{1,1} + Y_{1,2} > 0) \\
 &= \mathbb{P}(X_{1,1} + X_{1,2} = k)(1 - \mathbb{P}(Y_{1,1} + Y_{1,2} = 0)) \\
 &= \sum_{\ell=1}^k p_{k-\ell}^X p_{\ell}^X (1 - (p_0^Y)^2).
 \end{aligned}$$

On the other hand, if the branching property were in place, then

$$F_2(k) = \sum_{\ell=0}^k F_1(k - \ell)F_1(\ell).$$

Hence, combining these two facts we have that

$$(1 - p_0^Y)^2 = 1 - (p_0^Y)^2.$$

Thus, we have either  $p_0^Y = 0$ , in which  $(Z_n)_{n \in \mathbb{N}}$  has the same distribution as an asexual Galton-Watson process with offspring distribution  $(p_k^X)_{k \in \mathbb{N}}$ ; or  $p_0^Y = 1$ , in which case  $\mathbb{P}(Z_n = 0 \text{ for all } n) = 1$ . We conclude that the branching property does not hold in a non-trivial case. Hence, one can not restraint the study of extinction conditions to the case  $\{Z_0 = 1\}$ . Moreover, one can easily construct a process where  $\mathbb{P}_1(Z_n \rightarrow 0) = 1$  but  $\mathbb{P}_k(Z_n \rightarrow 0) < 1$  for  $k \in \mathbb{N}$  large enough.

Daley in his work conjectured that if one defines

$$Q = \lim_{n \rightarrow +\infty} \mathbb{P}(Z_n = 0 \mid Z_0 > 0),$$

then  $Q = 1$  if and only if  $\mathbb{E}(Z_1 \mid Z_0 = j) \leq j$  for all large value of  $j \in \mathbb{N}$ . Although this fact was not proved by Daley (and it took almost 20 years to be proved), he did show sufficient and necessary conditions for certain extinction when one of the two mating functions in Example 1 is in place. For both cases, Daley also assumed that the reproduction occurred following **Model A** (see (1.11)). This result, that we now state, was then proved in the setting of a general offspring distribution  $(p_{j,k})_{j,k \in \mathbb{N}}$  by Karlin and Kaplan in [KK73], and it is in this way that we state it. Whilst Daley's theorems were proved using analytical tools by iterations of functions and Cauchy's Integral Formula, Karlin and Kaplan approach is a more probabilistic one, using PGFs and a supermartingale technique.

### Extinction for Daley's examples

1. Consider first the model of completely promiscuous mating (see [Dal68], Theorem 1, and [KK73], Theorem 1), and assume that every mating unit gives offspring according to a probability distribution  $(p_{j,k})_{j,k \in \mathbb{N}}$ . Consider a random vector  $(X, Y)$  with the same law as the offspring distribution and assume that  $\mathbb{E}(X) < +\infty$  and that  $\mathbb{P}(Y > 0) > 0$ . Then, the process is extinct almost surely starting from any initial condition if and only if  $\mathbb{E}(X) \leq 1$ . That is, the mean number of females produced by a single mating unit is less or equal to one. We emphasize that the law of reproduction for the males does not play any role other than to be almost surely different from zero.



2. For the model of perfect fidelity with promiscuous mating (see [Dal68], Theorem 2, and [KK73], Theorem 4) and, using the same notation that in the previous case, assume that  $\mathbb{E}(X) < +\infty$  and in addition  $\mathbb{E}(Y) < +\infty$ . Then, the process is extinct almost surely starting from any initial condition if and only if  $\min\{\mathbb{E}(X), d\mathbb{E}(Y)\} \leq 1$ . For example, in the case when  $d = 1$  (perfect fidelity mating), the mean number of females **and** males needs to be greater than one to ensure a positive probability that the process persist indefinitely.

The first attempt to obtain results for a general version of this process came from Hull in [Hul82], where he considers the case of a *superadditive* mating function. That is,  $\xi : \mathbb{N}^2 \rightarrow \mathbb{N}$  is such that  $\xi(x, y) = \xi(0, y) = \xi(x, 0) = 0$  for all  $x, y \in \mathbb{N}$ , and that in addition,

$$\xi(x_1 + x_2, y_1 + y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2), \text{ for all } x_1, x_2, y_1, y_2 \in \mathbb{N}. \quad (1.13)$$

The intuition here is that if we consider a population of  $x_1 + x_2$  females and  $y_1 + y_2$  males, then, separating this into two subpopulations of sizes  $(x_1, y_1)$  and  $(x_2, y_2)$  will not increase the number of mating units that they originally formed. One can check that both mating functions proposed by Daley (Example 1) are in fact superadditive. In particular, any superadditive function is also a *common sense mating function* in the sense of Alsmeyer, so this is a stronger condition than the one presented by Daley.

Hull proved that, defining  $m = \mathbb{E}(Z_1 \mid Z_0 = 1)$ , then a necessary condition for certain extinction is  $m \leq 1$  (see [Hul82], Corollary). The converse is not necessarily true, as Hull pointed out himself. In fact, consider the following mating function

$$\xi(x, y) = \begin{cases} x + y - 1 & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{otherwise,} \end{cases} \quad (1.14)$$

which is superadditive. Assume in addition that the process follows the **Model A** (see (1.11)), with  $\alpha > 0$  and  $p_3 = 1$  (then  $p_n = 0$  for all  $n \neq 3$ ). Since  $p_0 = 0$ , the process is extinct if and only if at a given generation there are only females or only males. Then,

$$\mathbb{P}(Z_{n+1} = 0 \mid Z_n = k) = (1 - \alpha)^{3k} + \alpha^{3k}.$$

Hence, using again that  $p_0 = 0$ , for all  $\alpha \in (0, 1)$ ,

$$\mathbb{P}(Z_n > 0, \text{ for all } n \in \mathbb{N} \mid Z_0 = 1) \geq \prod_{\ell \in \mathbb{N}^*} \left(1 - \left((1 - \alpha)^{3\ell} - \alpha^{3\ell}\right)\right) > 0.$$

On the other hand,

$$m = \mathbb{E}(Z_1 \mid Z_0 = 1) = 2(1 - (1 - \alpha)^3 - \alpha^3).$$

Noting that  $m \searrow 0$  as  $\alpha \rightarrow 1$ , one can choose  $\alpha$  close enough to 1 and such that  $m \leq 1$ .

Nevertheless, for this particular counterexample, Bruss showed in [Bru84] that

$$\lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_1 \mid Z_0 = k)}{k} = 3.$$

The idea is that, although for the process with mating function defined in (1.14) the mean number of mating units produced by a single mating unit is smaller than one, this is not the case when the initial condition is larger. In fact, as Bruss points out  $\mathbb{E}(Z_1 \mid Z_0 = k) > k$  for all values of  $k \geq 2$ . In other words, when the population is large, the process has an *average* unit reproduction mean greater than one. We remark that Bruss's reasoning is consistent with Daley's intuition in [Dal68] of studying the mean number of mating units for large initial population. Bruss defined in [Bru84] the *mean growth rates* (although this name came later in a future work of Daley, Hull and Taylor), which plays a key role in the forthcoming results.

## 1.2.2 A general result for certain extinction

We define for  $k \in \mathbb{N}$  the quantity

$$r_k = \frac{\mathbb{E}(Z_1 \mid Z_0 = k)}{k}, \quad (1.15)$$

describing the mean growth rate starting from  $k$  mating units. Using these rates, the first general extinction result was proved in [DHT86] for superadditive mating functions. Their main result concerning certain extinction starts with the following fact.

**Proposition 5.** [see [DHT86], Lemma 2.2] *In a bisexual branching process with superadditive mating function the mean growth rates satisfy that the limit*

$$r = \lim_{k \rightarrow +\infty} r_k = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_1 \mid Z_0 = k)}{k} \quad (1.16)$$

is well defined and equal to

$$\sup_{k \geq 1} \frac{\mathbb{E}(Z_1 \mid Z_0 = k)}{k}.$$

The quantity  $r$  is often referred to as the asymptotic growth rate of the process  $(Z_n)_{n \in \mathbb{N}}$ .

The main tool to prove the previous assertion is the fact that, if  $\xi$  is a superadditive function, then  $(\mathbb{E}(Z_1 \mid Z_0 = k))_{k \in \mathbb{N}}$  forms a superadditive sequence, and so Proposition 5 follows from Fekete's Lemma. This result was proved alongside a long list of monotonicity properties for the process  $(Z_n)_{n \in \mathbb{N}}$  (see [DHT86], Proposition 2.1)

*Remark 1.* Proposition 5 only ensures that  $r$  is well defined, but it does not guarantee that this value is finite. In fact, for many examples this value turns out to be equal to  $+\infty$ . The following example of mating function defined by Alsmeyer in [Als02] is a clear case where  $\xi$  is a superadditive mating function with  $r = +\infty$ .

**Example 2. [Bilateral promiscuous mating]** Consider the case where both females and males practice a promiscuous mating. That is, at every generation, any possible combination of pairs of females and males forms a mating unit. This corresponds to consider the mating function

$$\xi(x, y) = xy.$$

Although this is by no means a realistic model, it is a clear example of a superadditive function where  $r = +\infty$ .

To avoid trivial cases, for the rest of the section we assume that the mating function and the offspring distribution are such that the non-triviality condition holds:

$$\mathbb{P}(Z_1 = j \mid Z_0 = j) < 1, \text{ for all } j \in \mathbb{N}.$$

It can be proved that this condition implies the following extinction-explosion dichotomy

$$\mathbb{P}(Z_n \xrightarrow{n \rightarrow +\infty} 0) + \mathbb{P}(Z_n \xrightarrow{n \rightarrow +\infty} +\infty) = 1,$$

which is a classic condition that is also true in the asexual case.

The quantity  $r$  relates the behaviour of the process  $(Z_n)_{n \in \mathbb{N}}$  when the population is large, as stated in the following result (see Remark 2 below)

**Lemma 6.** [see [DHT86], Lemma 2.3] For a sequence  $\{(X_i, Y_i)\}_{i \in \mathbb{N}}$  of i.i.d. bivariate random variables with finite expectations and a mating function  $\xi$  that is superadditive, one has

$$\frac{1}{k} \xi \left( \sum_{i=1}^k X_i, \sum_{i=1}^k Y_i \right) \xrightarrow[k \rightarrow +\infty]{a.s.} \lim_{j \rightarrow +\infty} \frac{\xi(j\mathbb{E}(X), j\mathbb{E}(Y))}{j}. \quad (1.17)$$

If we denote  $Z_1(k)$  the first generation of a bisexual Galton-Watson process with mating function  $\xi$ , offspring distribution given by the law of  $(X, Y)$  and  $Z_0 = k$ , then (1.17) can be read as

$$\frac{Z_1(k)}{k} \xrightarrow[k \rightarrow +\infty]{a.s.} \lim_{j \rightarrow +\infty} \frac{\xi(j\mathbb{E}(X), j\mathbb{E}(Y))}{j}.$$

*Remark 2.* Although it is not explicitly said in [DHT86], the convergence in (1.17) holds also in  $L^1$ , which allows to conclude that

$$r = \lim_{j \rightarrow +\infty} \frac{\xi(j\mathbb{E}(X), j\mathbb{E}(Y))}{j}, \quad (1.18)$$

in the case where  $r < +\infty$ . We explicitly proved the  $L^1$  convergence in [FVZ22] for a more general model, by showing that the sequence  $(Z_1(k))_{k \in \mathbb{N}}$  is uniformly integrable. We also proved that the alternative definition given by (1.18) holds even when  $r = +\infty$ . The details of this convergence can be found in Chapter 3, Theorem 22, where a more general Law of Large Numbers is proved.

This second way of computing  $r$  is a more easy way to calculate its value for different mating functions. In particular for Daley's mating functions given in Example 1 we have that:

- For the completely promiscuous mating where  $\xi(x, y) = x\mathbb{1}_{y>0}$ , and assuming that  $\mathbb{E}(Y) > 0$ , one has that  $r = \mathbb{E}(X)$ .
- For the perfect fidelity with promiscuous mating one has  $r = \min\{\mathbb{E}(X), d\mathbb{E}(Y)\}$ .

Combining this with Daley's conditions for certain extinction, one has that for these examples,  $r \leq 1$  is a necessary and sufficient condition for certain extinction. This result can be generalized for any superadditive mating function.

**Theorem 7** (see [DHT86], Theorem 3.1). For a bisexual Galton-Watson process with superadditive mating function, define

$$q_j = \mathbb{P}(\exists n \in \mathbb{N}, Z_n = 0 \mid Z_0 = j), \quad (1.19)$$

the probability of extinction starting from  $j \in \mathbb{N}$  mating units. Then

$$q_j = 1 \text{ for all } j \in \mathbb{N} \iff r \leq 1.$$

As in the asexual case, a process such that  $r < 1$  is called *subcritical*, if  $r = 1$  is called *critical* and for  $r > 1$  is called *supercritical*.

### 1.2.3 The probability of extinction

Once the question of conditions for certain extinction is answered (at least in the superadditive case) a next natural question is whether explicit expressions for the probability of extinction can be found (that is, for the values of  $q_j$ ). We recall that the big difficulty for this is the lack of the branching property in the general superadditive model. In particular, due to this reason, the

same approach using probability generating functions (PGFs), which gives a simple equation for the probability of extinction in the asexual case, cannot be applied in the bisexual one. The first attempts to understand these probabilities comes at the end of the work of [DHT86], where lower and upper bounds are given. Some preliminary results are explored in [FV92, Hu193], but no general answer is provided.

In this section we give some insights to the work of Alsmeyer and Rösler [AR96, AR02] on the extinction probability for the completely promiscuous mating in the very particular case of **Model B** (see (1.12)).

Let us now define  $(X_n)_{n \in \mathbb{N}}$  a Galton-Watson process with a killing rate. That is, consider  $\kappa : \mathbb{N}^* \rightarrow [0, 1]$  a function and such that  $(X_n)_{n \in \mathbb{N}}$  evolves as a classical Galton-Watson process with transition matrix given by  $\mathbf{M} = (m_{i,j})_{i,j \in \mathbb{N}}$  and at every generation  $n$ , the process is sent to  $\{0\}$  with probability  $\kappa(X_n)$ . The function  $\kappa$  is known as the killing rate of the process. The process  $(X_n)_{n \in \mathbb{N}}$  has transition matrix  $\mathbf{N} = (n_{i,j})_{i,j \in \mathbb{N}}$  given by

$$n_{i,j} = \kappa(i)\mathbf{1}_{j=0} + (1 - \kappa(i))m_{i,j}.$$

With the help of the previous computations, we have the following result.

**Lemma 8.** [see [AR96], Lemma 2.2 and 2.3] Consider  $(X_n)_{n \in \mathbb{N}}$  a supercritical Galton-Watson process with a killing rate  $\kappa$ . Define for  $j \in \mathbb{N}$ ,  $q_j$  the probability of extinction for  $(X_n)_{n \in \mathbb{N}}$  under  $\{X_0 = j\}$ . Then,  $q_j$  is the unique solution of

$$q_j = \kappa(j) + (1 - \kappa(j))(Mq)_j \tag{1.20}$$

with  $q_0 = 1$  and  $q_{+\infty} := \lim_{k \rightarrow +\infty} q_k = 0$ .

Note that in particular, if we consider  $(Z_n)_{n \in \mathbb{N}}$  a bisexual branching process with a completely promiscuous mating that follows **Model B**, then  $(Z_n)_{n \in \mathbb{N}}$  distributes as a Galton-Watson process with killing rate  $\kappa(i) = (p_0^Y)^i$  and transition matrix that of a classical Galton-Watson process with offspring distribution  $(p_j^X)_{j \in \mathbb{N}}$ . Thus, (1.20) gives us an equation that, when solved, gives the probability of extinction for  $(Z_n)_{n \in \mathbb{N}}$ . This is the only result known so far where the probability of extinction can be express as a solution of a simple equation. The problem of finding the probability of extinction for a general model remains to this day a challenging open question.

### 1.2.4 Convergence properties

Consider  $(Z_n)_{n \in \mathbb{N}}$  a bisexual branching process with superadditive mating and  $r$  the asymptotic growth rate defined in (1.16). As in the asexual case, we look at the question regarding the growth of the population in the event of survival. The first results of this type are due to Bagley in [Bag86], although it is applied for the particular case of perfect fidelity mating and **Model A** is assumed in place. Some years later, González and Molina presented several results concerning this subject (see [GM96, GM97b]).

We start by setting

$$W_n = \frac{Z_n}{r^n}, \text{ for all } n \in \mathbb{N}.$$

We remark that

$$\begin{aligned} \mathbb{E}(W_{n+1} \mid \mathcal{Z}_n) &= \frac{1}{r^{n+1}} \mathbb{E}(Z_{n+1} \mid \mathcal{Z}_n) \\ &\leq \frac{rZ_n}{r^{n+1}} \\ &= W_n, \end{aligned}$$

and so  $(W_n)_{n \in \mathbb{N}}$  forms a non-negative supermartingale, thus almost surely convergent to some random variable  $W$ . This implies that the process  $(Z_n)_{n \in \mathbb{N}}$  grows at a rate at most equal to  $r$ , and at a rate equal to  $r$  in the event where  $0 < W < +\infty$ . To give some intuition on the assumptions that have to be put in place to ensure the non-degeneracy of  $W$ , we first show an approach given by Alsmeyer in [Als02], which helps to give some motivation for the results to follow.

Start by defining the process  $(V_n)_{n \in \mathbb{N}}$  given by  $V_0 = Z_0$ , and for  $n \in \mathbb{N}^*$ ,

$$V_n = \frac{Z_n}{r_{Z_0} \cdot \dots \cdot r_{Z_{n-1}}}.$$

We claim that  $(V_n)_{n \in \mathbb{N}}$  is a positive martingale. In fact,

$$\begin{aligned} \mathbb{E}(V_{n+1} | \mathcal{F}_n) &= \frac{1}{r_{Z_0} \cdot \dots \cdot r_{Z_n}} \mathbb{E}(Z_{n+1} | Z_n) \\ &= \frac{r_{Z_n} Z_n}{r_{Z_1} \cdot \dots \cdot r_{Z_n}} \\ &= V_n. \end{aligned}$$

Hence,  $(V_n)_{n \in \mathbb{N}}$  converges almost surely to a random variable  $V$ . The relation with  $(W_n)_{n \in \mathbb{N}}$  is as follows:

$$W_n = V_n \prod_{k=0}^{n-1} \frac{r_{Z_k}}{r},$$

and taking  $k \rightarrow +\infty$ ,

$$W = V \prod_{n \in \mathbb{N}} \frac{r_{Z_n}}{r}.$$

Then, the non-degeneracy of  $W$  depends on the limit  $V$  and also on the convergence of the infinite product on the right. For the latter, note that we can write it as

$$\prod_{n \in \mathbb{N}} \frac{r_{Z_n}}{r} = \prod_{n \in \mathbb{N}} \left( 1 - \left( \frac{r - r_{Z_n}}{r} \right) \right).$$

Hence, the almost sure convergence of the product is equivalent to the almost sure convergence of the series

$$\sum_{n \in \mathbb{N}} (r - r_{Z_n}),$$

and thus, it is not surprising that an assumption that has to be put in place is a fast convergence of the mean growth rates  $(r_j)_{j \in \mathbb{N}}$  to its limit  $r$ .

For  $k \in \mathbb{N}$ , define

$$\varepsilon_k = r - r_k. \tag{1.21}$$

It is clear that  $\varepsilon_k \geq 0$  for all  $k \in \mathbb{N}$  and that  $\varepsilon_k \rightarrow 0$  as  $k \rightarrow +\infty$ , thank to Proposition 5. The following theorem is inspired by a similar work done by Klebaner (see [Kle84]) on the study of limit theorems for a model of Population-Size-Dependent branching processes.

**Theorem 9** (see [GM96], Theorem 2.2). *Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a positive, monotonic and non-increasing function, such that for all  $n \in \mathbb{N}$ ,  $\varepsilon_n \leq f(n)$ . In addition, suppose that  $xf(x)$  is concave in  $\mathbb{R}_+$ . If  $f$  is such that*

$$\sum_{n \in \mathbb{N}} \frac{f(n)}{n} < +\infty, \tag{1.22}$$

then  $\lim_{n \rightarrow +\infty} \mathbb{E}(W_n | Z_0 = i) > 0$ , for all  $i$  such that  $q_i < 1$ , where  $q_i$  is defined in (1.19).

Using an argument given by Klebaner in [Kle85], it is shown that a sufficient condition for the existence of a function such as the one described in the theorem is that the sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  is monotonic non-increasing and such that

$$\sum_{k \in \mathbb{N}} \frac{\varepsilon_k}{k} < +\infty.$$

The idea behind the assumption (1.22) is to ensure a fast convergence of  $(r_k)_{k \in \mathbb{N}}$  towards  $r$ .

Next in line is to add sufficient conditions to ensure a convergence in  $L^1$ . This together with the previous theorem will ensure that  $W$  is a non-degenerate random variable. For that define for  $k \in \mathbb{N}$

$$R_k = \mathbb{E}(|Z_1 - kr_k| \mid Z_0 = k).$$

**Theorem 10.** [see [GM96], Theorem 2.4] *Suppose that  $(\varepsilon_k)_{k \in \mathbb{N}}$  and  $(R_k/k)_{k \in \mathbb{N}}$  are monotonic non-increasing sequences with*

$$\sum_{n \in \mathbb{N}} \frac{\varepsilon_n}{n} < +\infty \text{ and } \sum_{n \in \mathbb{N}} \frac{R_n}{n^2} < +\infty. \quad (1.23)$$

*Then  $(W_n)_{n \in \mathbb{N}}$  converges in  $L^1$  towards  $W$ , and  $W$  is non-degenerate in 0.*

The intuition behind (1.23) is that, to ensure a convergence in  $L^1$  towards  $W$ , in addition to having  $(r_k)_{k \in \mathbb{N}}$  converging fast enough towards  $r$ , we have a control over the convergence of the  $L^1$  norm of  $Z_n$  as the initial condition grows towards infinity.

Given the relationship between the process of females and males  $(F_n, M_n)_{n \in \mathbb{N}}$  with that of the mating units  $(Z_n)_{n \in \mathbb{N}}$  it is not a surprise that both  $(F_n)_{n \in \mathbb{N}}$  and  $(M_n)_{n \in \mathbb{N}}$  converge after proper rescaling (see [GM96], Theorem 3.1): if we denote  $\mathbb{E}(X)$  (resp.  $\mathbb{E}(Y)$ ) the expectation for the number of females (resp. males) given by a single mating unit, then

$$\frac{F_n}{r^{n-1}} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \mathbb{E}(X)W \text{ and } \frac{M_n}{r^{n-1}} \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \mathbb{E}(Y)W.$$

We remark that the fact that we need to divide by  $r^{n-1}$  instead of  $r^n$  comes from the way how we enumerate our process (the  $n$ -th generation of mating units gives birth to the  $(n+1)$ -th generation of females and males). In addition, if the convergence of  $(Z_n)_{n \in \mathbb{N}}$  is in  $L^1$ , then the convergence of  $(F_n)_{n \in \mathbb{N}}$  and  $(M_n)_{n \in \mathbb{N}}$  holds also in  $L^1$ .

Sufficient and necessary conditions for a convergence in  $L^2$  are discussed in [GM97b], although those details and results will not be discussed here (the conditions are mainly of the same type as in Theorem 10, but with extra conditions on the variance of the process as the initial population grows to infinity). Similar results for the convergence of the total population up to generation  $n$ ,  $\sum_{k=0}^n Z_k$  are investigated in [GM97a, GM97c].

We finish this part with an  $X \log X$ -type condition for  $L^1$  convergence. Although in [Bag86] this result was established for the perfect fidelity model and in [GM96] for the completely promiscuous one, the result that we cite comes from [GM98] and it is a more general version that encompasses the two previous ones.

Consider  $(F_n)_{n \in \mathbb{N}}$  the process of females and define

$$D_n = \frac{F_n}{r^{n-1}} - \mathbb{E}(X)W_n.$$

**Theorem 11** ([GM98], Theorem 1). *Suppose that*

1. The sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$  is monotonic non-increasing and such that

$$\sum_{n \in \mathbb{N}} \frac{\varepsilon_n}{n} < +\infty.$$

2. For all  $n \in \mathbb{N}$ ,  $D_n \geq 0$ .

Then, the sequence  $\left(\frac{F_n}{r^{n-1}}\right)_{n \in \mathbb{N}}$  converges in  $L^1$  to a non-degenerate limit in 0 if and only if  $\mathbb{E}(X \log X) < +\infty$ .

Of course, a similar result can be stated for the process of males.

*Remark 3.* We remark that although an  $L \log L$ -type condition is presented here for the convergence in  $L^1$ , it does not imply that the events  $\{W = 0\}$  and  $\{Z_n \rightarrow 0\}$  coincide. The (almost sure) equality is proved under an  $L \log L$ -type condition and some extra assumptions in Chapter 4 (see Section 4.2).

### 1.2.5 Other variations of single-type bisexual models

In the last decades, several variation of this model have been studied. We will not work with this kind of models within this manuscript, but we cite some references for the interested reader. These variations include models such as processes in random or varying environment ([Ma06], [MM09], [MMR03]), processes with immigration ([GMM00], [GM96], [MX06]), processes with mating function depending on the number of couples ([MMR02], [MMR06], [XW05]) and more recently, processes with population-size-dependent and random mating ([JMM17], [MM19]). For general models of single-type bisexual populations, its scaling limits has been studied in [BCMSM23]. The interested reader can also consult the surveys of Hull [Hul03], Alsmeyer [Als02] and Molina [Mol10] for a wide description of the work accomplished on this family of processes.

## 1.3 Existing results in multi-type bisexual models

A natural question that follows is whether the previous results for the bisexual branching process can be extended to a multi-type version, in the same way that has been shown for the asexual process. In words of Hull [Hul03] (p. 336):

*“Multi-type processes will be the most relevant topic for future research associated with the bisexual process. We must develop an extinction theory for traits that are inherited from both parents(...). It has been said that the 20th century was the century of physics and the 21st century will be the age of biology. The bisexual Galton-Watson branching process may be the most relevant model to use in questions associated with the human genome. But first, a multi-type bisexual branching process needs to be developed to model classic Mendellian genetics.”*

So far, the only attempts to develop a multi-type model for the bisexual branching process that we can find in the literature are two extensions of the two mating functions proposed by Daley for the single-type case (see Example 1). We now give some insight into these two models.

### Multi-type completely promiscuous mating

The first process is proposed by Karlin and Kaplan in [KK73], and it is an extension to higher dimension of the completely promiscuous mating model. This process is defined as follows. Consider  $q_f, q_m \in \mathbb{N}$  and a  $(q_f + q_m)$ -dimensional multi-type Galton-Watson branching process (in the classical way of asexual models)  $(X_{n,1}, \dots, X_{n,q_f}, Y_{n,1}, \dots, Y_{n,q_m})_{n \in \mathbb{N}}$ . The idea is that

the first  $q_f$  types represent different types of females and the remaining types are  $q_m$  different types of males. We assume that only the females produce offspring. For  $1 \leq k \leq q_f$  we denote by  $L_k$  a random vector defined on  $\mathbb{N}^{q_f+q_m}$  and with probability distribution  $(p_k(v))_{v \in \mathbb{N}^{q_f+q_m}}$ . This probability distribution  $p_k$  represents the offspring of a female of type  $k$ . We assume the process has the following properties:

- (1)  $X_{0,i} = 1$  for some  $1 \leq i \leq q_f$  and  $Y_{0,j} = 1$  for all  $1 \leq j \leq q_m$ .
- (2) The random vectors  $(L_k)_{1 \leq k \leq q_f}$  satisfy:
  - (a) For all  $1 \leq i \leq q_f$ ,

$$\max_{q_f < j \leq q_f + q_m} \mathbb{P}(L_{i,j} = 0) < 1.$$

In other words, all types of females can produce any type of male with positive probability.

- (b) The matrix  $H \in \mathbb{R}_+^{q_f \times q_f}$  given for all  $1 \leq i, k \leq q_f$  by

$$H_{i,k} = \mathbb{E}(L_{i,k})$$

is positively regular. The matrix  $H$  corresponds to the expectation matrix of the process formed only by the females of every generation.

With these assumptions, we have the following result.

**Theorem 12** (see [KK73], Theorem 3). *Suppose that assumptions (1) and (2) hold. Define the  $q_f$ -dimensional vector process  $(Z_n)_{n \in \mathbb{N}}$*

$$(Z_{n,1}, \dots, Z_{n,q_f}) = (X_{n,1}, \dots, X_{n,q_f}) \prod_{m=0}^{n-1} \min_{1 \leq j \leq q_m} \{Y_{m,j}, 1\}.$$

*That is, we consider the multi-type branching process of females  $(X_{n,1}, \dots, X_{n,q_f})_{n \in \mathbb{N}}$  killed at the first instant when there is absence of one of the types of males.*

*A necessary and sufficient condition for certain extinction of  $(Z_n)_{n \in \mathbb{N}}$  is that  $\rho^* \leq 1$ , where  $\rho^*$  is the largest positive eigenvalue of  $H$ .*

Karlin and Kaplan did not provide a proof of their result, but instead proved it for the single-type case and left the proof of the previous Theorem for the reader. For completeness, we have extended their proof for the single-type case to the multi-type one in Appendix A.

## Multi-type bisexual process for the evolution of Y-linked genes

The second model was first introduced by Hull in [Hul98] where he used a two-sex model to rediscuss the original question proposed by Galton regarding the extinction of surnames in British aristocracy. This model was reconsidered later in [GHMM06] to study the evolution of a population with Y-linked genes. This model has been modified in several papers in the future. We present first the general model, and then we reference the different kinds of results present in the literature.

It is known that in the human population (and in some animal populations) the sex of the individuals is determined by a pair of chromosomes ( $X$  and  $Y$ ); females having chromosomes  $XX$  and males  $XY$ . The idea is to describe a process that models the evolution of a population



having a characteristic carried on the  $Y$  chromosome. The model can be described as follows. There is one type of female (since there is no  $Y$  chromosome present), denominated with the letter  $F$ , and two types of males. One type of male has allele  $A$ , and is denominated as  $M^A$ , and the second one has allele  $a$ , namely  $M^a$ . A mating unit can be formed by a female and a male with allele  $A$ , in which case the mating unit will be called of type  $Z^A$ . It can also be formed by a female and a male with allele  $a$ , in which case we denote it  $Z^a$ . We define the process  $(Z_n^A, Z_n^a)_{n \in \mathbb{N}}$  by

1. We start with a given initial condition  $(Z_0^A, Z_0^a) = (i, j) \in \mathbb{N}^2$ .
2. Every mating unit gives, independently, birth to a certain (random) number of individuals according to a probability distribution  $(p_k^A)_{k \in \mathbb{N}}$  for the  $Z^A$  mating units and  $(p_k^a)_{k \in \mathbb{N}}$  for the  $Z^a$  mating units.
3. Every descendent is a female (resp. male) with probability  $\alpha \in (0, 1)$  (resp.  $1 - \alpha$ ) independently of the others and of the type of their parents. In other words, the number of females of the  $n$ -th generation formed by  $(Z_n^A, Z_n^a)$  mating units is given by

$$F_{n+1} = \sum_{\ell=1}^{Z_n^A} f_{n,\ell}^A + \sum_{\ell=1}^{Z_n^a} f_{n,\ell}^a,$$

and of males given by

$$M_{n+1}^A = \sum_{\ell=1}^{Z_n^A} m_{n,\ell}^A, \quad M_{n+1}^a = \sum_{\ell=1}^{Z_n^a} m_{n,\ell}^a,$$

where  $(f_{n,\ell}^A, m_{n,\ell}^A)_{\ell, n \in \mathbb{N}}$  and  $(f_{n,\ell}^a, m_{n,\ell}^a)_{\ell, n \in \mathbb{N}}$  are two independent sequences of i.i.d. random variables with distribution given by

$$\mathbb{P}((f_{1,1}^A, m_{1,1}^A) = (j, k)) = \binom{j+k}{j} \alpha^j (1-\alpha)^k p_{j+k}^A, \quad \text{for all } j, k \in \mathbb{N},$$

and similarly for  $(f_{1,1}^a, m_{1,1}^a)$ .

Two types of mating are studied in the literature and are multi-type versions of the perfect fidelity mating presented by Daley (see Example 1).

- (a) The first case was introduced by González et al. in [GHMM06] and considers that the allele present in every male (its genotype) has an effect on its physical characteristics (its phenotype). Moreover, the  $a$  allele is considered pernicious or of a negative character, so  $M^A$ -type males secure mating first. Then,  $M^a$ -type males will mate if there are females available. Perfect fidelity mating is the only class of mating considered. In other words, given  $(F_{n+1}, M_{n+1}^A, M_{n+1}^a)$ , then  $(Z_{n+1}^A, Z_{n+1}^a)$  are determined by

$$Z_{n+1}^A = \min\{F_{n+1}, M_{n+1}^A\}, \quad Z_{n+1}^a = \min\{\max\{0, F_{n+1} - M_{n+1}^A\}, M_{n+1}^a\}.$$

- (b) The second case was introduced by González et al. in [GMM09] and considers that the allele present in every male has no effect on its physical characteristics. Then the mating goes as follows:

- If  $F_{n+1} \geq M_{n+1}^A + M_{n+1}^a$ , then all males mate with one female. The  $(n+1)$ -th generation of mating units then is  $(Z_{n+1}^A, Z_{n+1}^a) = (M_{n+1}^A, M_{n+1}^a)$ .
- If  $F_{n+1} < M_{n+1}^A + M_{n+1}^a$ , then all the females mate and they choose the allele of their partner. Since the allele has no impact in physical characteristics, this choice is made randomly. Thus, the number of  $Z^A$  mating units in the  $(n+1)$ -th generation is an hypergeometric distribution of parameters  $F_{n+1}$ ,  $M_{n+1}^A + M_{n+1}^a$  and  $M_{n+1}^A$ , i.e.

$$\mathbb{P}(Z_{n+1}^A = k \mid F_{n+1}, M_{n+1}^A, M_{n+1}^a) = \frac{\binom{M_{n+1}^A}{k} \binom{M_{n+1}^a}{F_{n+1}-k}}{\binom{M_{n+1}^A + M_{n+1}^a}{F_{n+1}}}.$$

Once the number of  $Z^A$ -type mating units are determined, the rest of the females mate with  $M^a$ -type males, giving as a result

$$Z_{n+1}^a = F_{n+1} - Z_{n+1}^A.$$

For the model with the mating mechanism presented in (a), some extinction conditions were given in [GHMM06], and long time behaviour properties were studied in [GMM08]. The second model is studied in [GMM09, AGM11].

## 1.4 Two results on continuous time two-sex populations

So far, continuous time bisexual processes constitute one of the less investigated topics in the two-sex branching process literature. Although one very general model was proposed by Molina and Yanev in [MY03], no results have been proved for this model. In this section, we present two results for two very particular models of continuous time bisexual processes. The first, due to Karlin and Kaplan [KK73], and the second one to Asmussen [Asm80].

The result of Karlin and Kaplan corresponds to an age dependent version of the completely promiscuous mating model (see Example 1). The process can be described as a two-type continuous time branching process  $(X_t, Y_t)_{t \geq 0}$ , where  $X_t$  (resp.  $Y_t$ ) stands for the number of females (resp. of males) alive at time  $t \geq 0$ . We assume that the following conditions on the process hold.

1.  $X_0 = Y_0 = 1$ .
2. The lifetime of a female (resp. of a male) is a random variable whose probability distribution function is  $F_X$  (resp.  $F_Y$ ). It is assumed that both lifetime probability distributions have finite means. All lifetimes are independent of each other.
3. When a female dies it is removed from the population and it is replaced by a random number of new females and males according to a probability distribution  $(p_{i,j})_{i,j \in \mathbb{N}}$ . When a male dies it is removed from the population and no replacement takes place.
4. The offspring probability distribution are such that

$$\sum_{i \in \mathbb{N}} p_{i,0} < 1 \text{ and } \sum_{j \in \mathbb{N}} p_{0,j} + p_{1,j} < 1.$$

In other words, there is a positive probability of giving birth to at least one male at every time. The same holds to giving birth to more than one female.

**Theorem 13** (see [KK73], Theorem 2). Assume that  $(X_t, Y_t)_{t \geq 0}$  is a two-type continuous time branching process following (1) – (4). Define the process  $(Z_t)_{t \geq 0}$  by

$$Z_t = X_t \min\{Y_t, 1\}.$$

Set  $m = \sum_{i,j \in \mathbb{N}} ip_{i,j}$  the expected number of females in the offspring distribution. Then

$$\mathbb{P} \left( \lim_{t \rightarrow +\infty} Z_t = 0 \right) = 1 \iff m \leq 1.$$

The second model for a continuous time process came seven years later with the work of Asmussen. It corresponds to a continuous time pure birth branching process with two sexes. The idea for this process is the following: consider a population  $(X_t, Y_t)_{t \geq 0}$  evolving in continuous time. The transitions can only happen from  $(x, y) \in \mathbb{N}^2$  towards  $(x + 1, y)$  or  $(x, y + 1)$ . No death is considered. The evolution of the process is as follows: consider a function  $R : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  called the *marriage function* (Asmussen’s version of our mating function). Consider also two parameters  $\lambda_f, \lambda_m > 0$ . If at a given time  $t \geq 0$  the process is at the state  $(X_t, Y_t) \in \mathbb{N}^2$ , then it jumps to the state  $(X_t + 1, Y_t)$  at a rate  $\lambda_f R(X_t, Y_t)$  and to  $(X_t, Y_t + 1)$  at a rate  $\lambda_m R(X_t, Y_t)$ .

Of course, since no death takes place this process diverges towards infinity. The idea is to describe the asymptotic behaviour when  $t \rightarrow +\infty$ . In order to do so, the following assumptions are put in place:

- (a) The marriage function has the form

$$R(x, y) = (x + y)h \left( \frac{y}{x + y} \right),$$

for a certain function  $h : [0, 1] \rightarrow \mathbb{R}_+$ . In other words, for a fixed proportion of males, the marriage function is linear in the total population.

- (b) The function  $h$  is positive on  $(0, 1)$ :  $h(s) > 0$  for all  $0 < s < 1$ .

- (c) Define  $p = \frac{\lambda_m}{\lambda_f + \lambda_m}$  (i.e. the probability that a jump occurs due to the birth of one male). There exists  $c > 0$ ,  $q \in (0, 1]$  and an interval  $I \in (0, 1)$  containing a neighbourhood of  $p$ , such that

$$|h(s_1) - h(s_2)| \leq c|s_1 - s_2|^q, \text{ for all } s_1, s_2 \in I.$$

Under these assumptions, Asmussen proved two results regarding the asymptotic behaviour of the process. The first one, stated here below, corresponds to the convergence of the proportion of females and males and the total population (properly rescaled).

**Theorem 14** (see [Asm80], Theorem 2). Suppose that assumptions (a) – (c) hold. Set

$$P_t^Y = \frac{Y_t}{X_t + Y_t} \text{ and } T_t = X_t + Y_t.$$

In other words, for  $t \geq 0$ ,  $P_t^Y$  and  $T_t$  represent the proportion of males and the total population at time  $t$ , respectively. Then, there exists a real random variable  $W$  with  $\mathbb{P}(0 < W < +\infty) = 1$  and such that

$$P_t^Y = \frac{\lambda_m}{\lambda_f + \lambda_m} + o(1), \quad T_t = e^{\lambda t} W + o(e^{\lambda t}),$$

almost surely as  $t \rightarrow +\infty$ , for  $\lambda = R(\lambda_f, \lambda_m)$ .

Note that as a direct consequence, one obtains that if  $P_t^X$  corresponds to the proportion of females at time  $t \geq 0$ , then almost surely

$$P_t^X \xrightarrow[t \rightarrow +\infty]{} \frac{\lambda_f}{\lambda_f + \lambda_m}.$$

We finish with Asmussen's second result, which is a Central Limit Theorem for the processes  $(P_t^Y)_{t \geq 0}$  and  $(T_t)_{t \geq 0}$ .

**Theorem 15** (see [Asm80], Theorem 3). *Suppose that assumptions (1) – (3) are in place and in addition suppose that for  $p = \frac{\lambda_m}{\lambda_f + \lambda_m}$ ,*

$$h(x) = h(p) + (x - p)h'(p) + O((x - p)^2) \text{ as } x \rightarrow p.$$

Denote for  $\lambda = R(\lambda_f, \lambda_m)$  and  $t \geq 0$ ,

$$A_t = \frac{1}{\sqrt{W}e^{-\lambda t}}(P_t^Y - p),$$

and

$$B_t = \frac{1}{\sqrt{W}e^{\lambda t}}(T_t - We^{\lambda t}).$$

Then,

1. The limiting distribution of  $(A_t, B_t)_{t \geq 0}$  exists and it is the two-dimensional normal distribution with mean zero and covariance matrix

$$\begin{pmatrix} \gamma_1^2 & \rho \\ \rho & \gamma_2^2 \end{pmatrix} = \begin{pmatrix} p(1-p) & -p(1-p)\frac{h'(p)}{h(p)} \\ -p(1-p)\frac{h'(p)}{h(p)} & 1 + 2p(2-p)\left(\frac{h'(p)}{h(p)}\right)^2 \end{pmatrix}.$$

2. For all  $(\alpha, \beta) \neq (0, 0)$  and

$$C_t = \alpha A_t + \beta B_t, \quad \sigma^2 = \alpha^2 \gamma_1^2 + \beta^2 \gamma_2^2 + 2\alpha\beta\rho,$$

we have that

$$\limsup_{t \rightarrow +\infty} \frac{C_t}{\sqrt{2\sigma^2 \log t}} = 1, \quad \liminf_{t \rightarrow +\infty} \frac{C_t}{\sqrt{2\sigma^2 \log t}} = -1, \quad a.s..$$

## Chapter 2

# Construction of a Superadditive Multi-type Bisexual Model

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In this chapter, we introduce a general model for a multi-type bisexual Galton-Watson process. This model encompasses the ones presented in the previous chapter, in Sections 1.1 and 1.2. The idea is to consider a population that can be classified into two main classes (females and males) and that evolves in discrete time. Each one of these classes can be sub-divided into smaller (but finite) sub-classes. In other words, we have different types of females and different types of males. The number of types of females is not necessarily the same number for the types of males. As in the single-type case, mating units are formed through the application of a mating function. Only mating functions can carry reproduction, giving birth to new females and males, and this cycle repeats indefinitely. In this chapter, we first formally introduce the model in Section 2.1 to then discuss in Section 2.2.1 the different assumptions that will be put in place for the different results presented in the chapters to follow, together with some examples. Properties of this process are presented in Sections 2.3.1 and 2.3.2.

### 2.1 The Model

Consider three strictly positive integers  $p, q_f$  and  $q_m$  that will be fixed throughout the entire chapter. We construct a multi-type bisexual branching process modeling a population that can be divided into  $q_f$  types of females and  $q_m$  types of males. They come together with the purpose of reproduction and can form mating units of  $p$  different types. Given  $(z_{0,1}, \dots, z_{0,p}) \in \mathbb{N}^p$  an initial condition, where, for  $i \in \{1, \dots, p\}$ ,  $z_{0,i}$  stands for the number of mating units of type  $i$  in the zero-th generation, we construct the process  $(Z_n)_{n \in \mathbb{N}}$  as the continuing iteration of the following steps:

1. For each  $i \in \{1, \dots, p\}$ , consider a probability distribution taking values on  $\mathbb{N}^{q_f} \times \mathbb{N}^{q_m}$  given by

$$p_i := (p_i(v^f, v^m))_{v^f \in \mathbb{N}^{q_f}, v^m \in \mathbb{N}^{q_m}}. \quad (2.1)$$

If at a given generation  $n \in \mathbb{N}$  we have a configuration  $Z_n = (Z_{n,1}, \dots, Z_{n,p}) \in \mathbb{N}^p$  of mating units, then each mating unit of type  $i$  begets independently of each other a certain random number of females and males of each type and according to the probability distribution  $p_i$ . Then, for  $v^f = (v_1^f, \dots, v_{q_f}^f) \in \mathbb{N}^{q_f}$  and  $v^m = (v_1^m, \dots, v_{q_m}^m) \in \mathbb{N}^{q_m}$ ,  $p_i(v^f, v^m) \in [0, 1]$  is the probability that a mating unit of type  $i$  has exactly  $v_1^f$  females of type 1,  $v_1^m$  males of type 1 and so on.

2. The total number of females of type  $j \in \{1, \dots, q_f\}$  and of males of type  $k \in \{1, \dots, q_m\}$  in the  $(n+1)$ -th generation will be the sum over all the females of type  $j$  (respectively of males of type  $k$ ) begotten by all the mating units in the  $n$ -th generation. This can be expressed as follows.

$$F_{n+1,j} = \sum_{i=1}^p \sum_{\ell=1}^{Z_{n,i}} X_{i,j}^{(\ell,n)} \quad \text{and} \quad M_{n+1,k} = \sum_{i=1}^p \sum_{\ell=1}^{Z_{n,i}} Y_{i,k}^{(\ell,n)}, \quad (2.2)$$

where for  $i \in \{1, \dots, p\}$  and  $j \in \{1, \dots, q_f\}$ ,  $X_{i,j}^{(\ell,n)}$  represents the number of females of type  $j$  produced by the  $\ell$ -th couple of type  $i$  in the  $n$ -th generation. Analogously, for  $i \in \{1, \dots, p\}$  and  $k \in \{1, \dots, q_m\}$ ,  $Y_{i,k}^{(\ell,n)}$  represents the number of males of type  $k$  produced by the  $\ell$ -th couple of type  $i$  in the  $n$ -th generation. Since every reproduction occurs independently, then for every  $i \in \{1, \dots, p\}$  the sequence

$$\left( (X_{i,1}^{(\ell,n)}, \dots, X_{i,q_f}^{(\ell,n)}), (Y_{i,1}^{(\ell,n)}, \dots, Y_{i,q_m}^{(\ell,n)}) \right)_{\ell, n \in \mathbb{N}}$$

forms an i.i.d. family of random vectors with probability distribution given by  $p_i$ . In the following,  $(X_i, Y_i)$  will always denote a random vector with this probability distribution.

3. As in the single-type case, the mating units in the  $(n+1)$ -th generation are produced with the help of a mating function  $\xi : \mathbb{N}^{q_f} \times \mathbb{N}^{q_m} \rightarrow \mathbb{N}^p$ , obtaining

$$(Z_{n+1,1}, \dots, Z_{n+1,p}) = \xi((F_{n+1,1}, \dots, F_{n+1,q_f}), (M_{n+1,1}, \dots, M_{n+1,q_m})).$$

We call the process  $(Z_n)_{n \in \mathbb{N}}$  the multi-type bisexual Galton-Watson branching process. Considering the empty sum as zero,  $(Z_n)_{n \in \mathbb{N}}$  forms a discrete time Markov Chain on  $\mathbb{N}^p$ . Imposing  $\xi(0) = 0$ , then  $\{0\}$  is an absorbing state.

One can check that by choosing  $p = q_f = q_m = 1$  one recovers the single-type bisexual process presented in Section 1.2. This model also encompasses the two multi-type models presented in the literature and explained in Section 1.3 (see more details in Section 2.2 below).

### A simplification on the notation

We remark that the fact that a given individual is either female or male, can be encoded as a type in the population. In other words, considering  $q = q_f + q_m$ , we can set for any  $n \in \mathbb{N}$  the vector  $W_n \in \mathbb{N}^q$  given by

$$W_n = (F_{n,1}, \dots, F_{n,q_f}, M_{n,1}, \dots, M_{n,q_m}).$$

We say that  $W_n$  represents the vector of *individuals* at the  $n$ -th generation. Considering the following slightly change in the notation we can simplify the description of the process, which in turn will simplify the computations on the proofs of the results to follow.

- For  $v \in \mathbb{N}^q$ , we denote  $p_i(v) := p_i((v_1, \dots, v_{q_f}), (v_{q_f+1}, \dots, v_{q_f+q_m}))$  and in the same way  $\xi(v) := \xi((v_1, \dots, v_{q_f}), (v_{q_f+1}, \dots, v_{q_f+q_m}))$ .
- In the following, for every  $i \in \{1, \dots, p\}$ , we consider a random vector on  $\mathbb{N}^q$  with distribution probability  $(p_i(v))_{v \in \mathbb{N}^q}$  denoted  $V_i = (V_{i,1}, \dots, V_{i,q})$ . We recall that for all  $i \in \{1, \dots, p\}$ ,

$$(V_{i,1}, \dots, V_{i,q}) \stackrel{(d)}{=} (X_{i,1}, \dots, X_{i,q_f}, Y_{i,1}, \dots, Y_{i,q_m}).$$

With all this notation in mind we can describe the multi-type bisexual branching process as follows. Given  $Z_0 = z_0 \in \mathbb{N}^p$ , then for all  $n \in \mathbb{N}$  and all  $j \in \{1, \dots, q\}$ .

$$W_{n+1,j} = \sum_{i=1}^p \sum_{k=1}^{Z_{n,i}} V_{i,j}^{(k,n)}, \quad \text{with } Z_n = \xi(W_{n,1}, \dots, W_{n,q}), \quad (2.3)$$

where, for all  $i \in \{1, \dots, p\}$ ,  $(V_i^{(k,n)})_{k,n \in \mathbb{N}}$  forms an i.i.d. family of random vectors with the same distribution as  $V_i$ . In the following we use this notation for the process. When we need to go back to the original notation presented in (2.2) (as it will be the case in the examples to follow) we will say that the model is presented in the *bisexual population setting*.

We remark that the number of sexes presented in our population is in fact arbitrary. In particular, dividing the  $q$  types of individuals in a larger number of classes we can extend this definition to a *multi-sexual* process, as they appear for instance for several plants species (see [BK04] and [BLVD<sup>+</sup>11, BT12]).

We stress the fact that for the main part of our results, and unless the contrary is clearly stated, we consider that  $\xi$  is a deterministic function. Some examples of cases where  $\xi$  is a random function will be presented later in Chapter 6.

*Remark 4.* Since our model we considers only deterministic mating functions, we will see that the model of multi-type bisexual process defined by Karlin and Kaplan, and presented in Section 1.3, is encompassed by our model. However, the second multi-type bisexual model presented in the previous chapter, in Section 1.3, considers a random mating function, and thus it can not be seen as an example of our model.

## Basic assumptions on the offspring distribution

We present now the basic assumptions on the offspring distribution that will be in place. This assumptions are considered to be always in place unless the contrary is clearly stated.

**Assumption 1.** The random vectors  $V_1, \dots, V_p$  are  $p$  mutually independent random vectors with values on  $\mathbb{N}^q$ . This in particular implies that mating units reproduce independently of each others. Note that for  $i \in \{1, \dots, p\}$  and  $j_1, j_2 \in \{1, \dots, q\}$  with  $j_1 \neq j_2$ , the random variables  $V_{i,j_1}$  and  $V_{i,j_2}$  are not necessary independent.

**Assumption 2.** We assume that all the offspring distributions are integrable. In other words, for all  $i \in \{1, \dots, p\}$ ,  $j \in \{1, \dots, q\}$   $V_{i,j} \in L^1$ . This allows us to define the matrix  $\mathbb{V} \in \mathbb{R}^{p,q}$  given by

$$\mathbb{V}_{i,j} = \mathbb{E}(V_{i,j}) \text{ for all } i \in \{1, \dots, p\}, j \in \{1, \dots, q\}. \quad (2.4)$$

Finally, we assume that, for all  $j \in \{1, \dots, q\}$ ,  $\sum_{i=1}^p \mathbb{V}_{i,j} > 0$ . That is, every type of individual  $j \in \{1, \dots, q\}$  is produced by at least one type of mating unit  $i \in \{1, \dots, p\}$ .

We remark that we are able to recover all the models presented in Chapter 1, by choosing appropriately the parameters  $p, q$  and  $\xi$ .

1. If we choose  $p = q = 1$  and  $\xi : \mathbb{N} \rightarrow \mathbb{N}$  the identity function on  $\mathbb{N}$ , then we recover a process that has the same distribution as the asexual single-type Galton-Watson process as the one presented in Section 1.1.1. In this case  $\mathbb{V}$  is a real number, corresponding to the mean of the offspring distribution, denoted  $m$  in the previous chapter.
2. If we now choose  $p = q > 1$  and again  $\xi : \mathbb{N}^p \rightarrow \mathbb{N}^p$  the identity function, but now on  $\mathbb{N}^p$ , then the process has the same distribution as a multi-type asexual Galton-Watson process (see Section 1.1.2). For this model,  $\mathbb{V}$  corresponds to the expectation matrix of the process, denoted  $M$  in the previous chapter.
3. Finally, as we stated before, by choosing  $p = 1$  and  $q = 2$  (and no restrictions on  $\xi$ ) we recover the single-type bisexual Galton-Watson process of Section 1.2. In this case  $\mathbb{V}$  is just the expectation of the offspring distribution. With the notation of the previous chapter,  $\mathbb{V} = (\mathbb{E}(X), \mathbb{E}(Y))$ .

## 2.2 Assumptions and Examples

In this section we discuss about three different assumptions that can be put in place in the results to follow in the next chapters. This section is divided in two parts.

In the first part we present the first assumption, which is the superadditivity of the mating function. This assumption is treated separately from the rest because it will be assumed in all the chapters up to Chapter 5. The case of a bisexual process with superadditive mating is the most general case treated in the single-type case with deterministic mating function. The extension of this assumption to the multi-dimensional case seems reasonable since, in one hand it is an easy assumption to verify, and in the other it allows us to focus specially in the case where the population is large.

The second and third assumptions are in the second part of this section. They are presented separately because they are not supposed to be in place all the time. For this reason, we will clearly state when we assume that one or the other hold before stating any of our results. These two assumptions are the primitivity of the process, which in simple words means that every type is reachable from another if the population is large enough. This hypothesis, although simple at first, contributes with a rich structure and allows us to treat the extinction problem as an eigenvalue problem, like in the asexual case.

Our third assumption is the transitivity of the process, which is a classic hypothesis in the branching process theory. Some conditions under which this assumption holds are also discussed.

### 2.2.1 Superadditivity assumption and the asymptotic growth rate

We start with a first assumption on the mating function of the process.

**Assumption 3** (Superadditivity). We suppose that the mating function  $\xi$  of the process is (componentwise) superadditive. That is, for all  $i \in \{1, \dots, p\}$ ,

$$\xi_i(x_1 + x_2) \geq \xi_i(x_1) + \xi_i(x_2), \quad \forall x_1, x_2 \in \mathbb{N}^q. \quad (2.5)$$

The intuition for this type of processes is that two populations of individuals form a bigger number of mating units of each type together rather than separate. We extend then the idea of a



superadditive bisexual Galton-Watson process to the multi-type case, which was first introduced for the single-type case by Hull in [Hul82].

We now define  $\mathfrak{M} : \mathbb{R}_+^p \rightarrow (\mathbb{R}_+ \cup \{+\infty\})^p$  by

$$\mathfrak{M} = \lim_{r \rightarrow +\infty} \frac{\xi(kz\mathbb{V})}{k} = \sup_{k \geq 1} \frac{\xi(kz\mathbb{V})}{k}, \quad (2.6)$$

where  $\xi$  is any superadditive extension of  $\xi$  to  $\mathbb{R}_+^q$  (see Remark 6 below) and  $\mathbb{V}$  is the matrix defined in (2.4). To give a better intuition on who is  $\mathfrak{M}$ , note that for  $z \in \mathbb{N}^p$ , we have

$$\mathfrak{M}(z) = \lim_{k \rightarrow +\infty} \frac{\xi(\mathbb{E}(W_1 | Z_0 = kz))}{k}.$$

Thus, in this case  $\mathfrak{M}$  corresponds to the quantity of mating units that are produced by the average number of individuals in the offspring, when the size of the initial population grows in a fixed direction. We then use any extension of  $\xi$ , allowing us to define this function on  $\mathfrak{M}$ . We emphasize that the limit is well defined and equal to the supremum according to Fekete's Lemma thanks to the superadditivity of  $\xi$ . We stress the fact that  $\mathfrak{M}$  may take the value  $+\infty$  in some or all of its components and in our results it is not assumed that the function is finite unless the contrary is clearly stated. One can see the similarity of  $\mathfrak{M}$  and the asymptotic growth rate  $r$  in the single-type case. Although the definition of  $r$  given in [Hul82] is

$$r = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_1 | Z_0 = k)}{k},$$

we recall that it has an alternative definition (see (1.18) in Chapter 1), which is equivalent to the way  $\mathfrak{M}$  is defined here.

*Remark 5.* We will prove in Chapter 3 that in fact  $\mathfrak{M}$  is equal to

$$\mathfrak{M}(z) = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_1 | Z_0 = \lfloor kz \rfloor)}{k}.$$

Thus, recovering the original definition for the asymptotic growth rate given by (1.16) in the single-type case, but now in a multi-dimensional setting.

*Remark 6.* A superadditive function  $\xi$  on  $\mathbb{N}^q$  can always be extended to a superadditive function on  $\mathbb{R}_+^q$  (for instance setting  $x \in \mathbb{R}_+^q \rightarrow \xi(\lfloor x \rfloor)$ ) and it will appear that  $\mathfrak{M}$  does not depend on the choice of this extension (see Proposition 20).

**Example 3.** We can compute the function  $\mathfrak{M}$  for the single-type bisexual branching process and for asexual branching models. In each one of these cases we obtain:

1. If  $p = 1$  and  $q = 2$ , so that our process is the single-type bisexual branching process, we have for all  $z \in \mathbb{N}$ ,

$$\mathfrak{M}(z) = rz,$$

with  $r$  the asymptotic growth rate (1.16).

2. If  $p = q$  and  $\xi$  is the identity function, so that the process distributes as a multi-type (if  $p > 1$ ) or single-type (with  $p = 1$ ) asexual branching process. Then for every  $z \in \mathbb{N}^p$ ,

$$\mathfrak{M}(z) = z\mathbb{V}.$$

Thus, the functional  $\mathfrak{M}$  corresponds in this case to the expectation matrix of the process, which is used to study its asymptotic behaviour according to its largest eigenvalue.

We remark that in previous cases  $\mathfrak{M}$  turns out to be the operator that we have to study for deciding if there exists certain extinction. We also emphasize the fact that in all these cases the function  $\mathfrak{M}$  is a linear function. As we will see, in the general setting it is not the case, although it turns out to be concave (see Proposition 19). In order to analyse the behaviour of the function  $\mathfrak{M}$ , we will make use of a concave Perron-Frobenius theory and, more precisely, of the results developed by Krause [Kra94] which will ensure the existence of eigenelements for this function. This eigenelements will play the same role that they play in the case of linear operators in the asexual case. More details are treated in Section 2.3.1.

### Examples in the context of bisexual populations

The following are examples of multi-type processes which are written in terms of a bisexual population, recovering all the original notation given in (2.2). We recall also that we write  $X_i$  (resp.  $Y_i$ ) for the female (resp. male) offspring distribution of a mating unit of type  $i$ . We define the matrices  $\mathbb{X} \in \mathbb{R}_+^{p \times q_f}$  and  $\mathbb{Y} \in \mathbb{R}_+^{p \times q_m}$  given by

$$\mathbb{X}_{i,j} = \mathbb{E}(X_{i,j}), \mathbb{Y}_{i,k} = \mathbb{E}(Y_{i,k}), \text{ for } 1 \leq i \leq p, 1 \leq j \leq q_f, 1 \leq k \leq q_m.$$

For each of these examples, we compute the associated function  $\mathfrak{M}$ .

We start with two examples which are extensions to a higher dimension of the two first models presented by Daley in [Dal68]. These two models will work as our canonical examples, in the sense that we will constantly come back to them in the future to analyze our different results.

**Example 4** (Perfect fidelity with promiscuous mating). Consider the case where  $p = q_m = q_f$ , a vector  $(d_1, \dots, d_p) \in \mathbb{N}^p$  and the mating function given for every  $i \in \{1, \dots, p\}$  by,

$$\xi_i((x_1, \dots, x_p), (y_1, \dots, y_p)) = \min\{x_i, d_i y_i\},$$

which is an extension of the case of perfect fidelity with promiscuous mating presented by Daley in [Dal68] (see Example 1) to the multi-dimensional case. In this case, we have for all  $i \in \{1, \dots, p\}$ ,

$$\frac{\min\{k(z\mathbb{X})_i, kd_i(z\mathbb{Y})_i\}}{k} = \min\{(z\mathbb{X})_i, d_i(z\mathbb{Y})_i\}, \forall k \geq 1,$$

Hence the function  $\mathfrak{M}$  takes the form

$$\mathfrak{M}(z) = \min\{z\mathbb{X}, Dz\mathbb{Y}\}, \tag{2.7}$$

where  $D \in \mathbb{R}_+^{p \times p}$  is the diagonal matrix formed by  $(d_1, \dots, d_p)$ , and the minimum is applied componentwise.

*Remark 7.* As in the single-type case, we refer to the previous example as perfect fidelity mating in the case where  $d_i = 1$  for all  $i \in \{1, \dots, p\}$ .

**Example 5** (Completely promiscuous mating). This case was previously studied by Karlin and Kaplan [KK73] and it corresponds to the case where the number of mating units is equal to the number of females present in every generation (in particular this implies that we have  $p = q_f$ ) given the condition that there is at least one male of each type present in every generation. In other words, we are considering the mating function

$$\xi((x_1, \dots, x_p), (y_1, \dots, y_{q_m})) = (x_1, \dots, x_p) \prod_{i=1}^{q_m} \mathbb{1}_{\{y_i > 0\}}.$$

The function  $\mathfrak{M}$  in this case corresponds to

$$\mathfrak{M}(z) = z\mathbb{X}\mathbb{1}_{\{zY>0\}}.$$

We remark that our model encompasses also processes where there is only one type of mating unit, but several types of females and males, as is the case for our next example (thus generalizing the concept of single-type process).

**Example 6** (Single-type model with multiple mating strategies). Consider a single-type bisexual branching process (in this case “single-type” needs to be interpreted as “there is only one type of mating unit”). Each mating unit reproduces independently according to the distribution of a given random vector  $(X, Y)$ . As in the classic single-type model, given that we have  $Z_n \in \mathbb{N}$  mating units in the  $n$ -th generation, we can compute the offspring they beget as

$$F_{n+1} = \sum_{k=1}^{Z_n} X^{(k)}, \quad M_{n+1} = \sum_{k=1}^{Z_n} Y^{(k)}.$$

Assume that we have  $N$  superadditive mating functions  $\xi_1, \dots, \xi_N$ , where  $N \in \mathbb{N}$  is fixed and for all  $i \in \{1, \dots, N\}$ ,  $\xi_i : \mathbb{N}^2 \rightarrow \mathbb{N}$ . Every mating function can be thought as a different strategy for mating.

Consider two vectors of non-negative entries

$$(\alpha_1, \dots, \alpha_N) \text{ and } (\beta_1, \dots, \beta_N), \text{ with } \sum_{i=1}^N \alpha_i = \sum_{i=1}^N \beta_i = 1.$$

Then, a female (resp. a male) chooses the mating strategy  $\xi_i$  with probability  $\alpha_i$  (resp.  $\beta_i$ ). Denote  $f_{n+1,i}$  (resp.  $m_{n+1,i}$ ) the number of females (resp. of males) that chooses the mating strategy  $\xi_i$ . Then, the number of mating units that they form is

$$Z_{n+1} = \sum_{i=1}^N \xi_i(f_{n+1,i}, m_{n+1,i}) = \xi((f_{n+1,1}, \dots, f_{n+1,N}), (m_{n+1,1}, \dots, m_{n+1,N})),$$

where  $\xi : \mathbb{N}^N \times \mathbb{N}^N \rightarrow \mathbb{N}$  is the function given by

$$\xi(x, y) = \sum_{i=1}^N \xi_i(x_i, y_i), \quad \forall x, y \in \mathbb{N}^N.$$

Since all the functions  $(\xi_i)_{1 \leq i \leq N}$  are superadditive,  $\xi$  is superadditive. We then can compute  $\mathfrak{M} : \mathbb{N} \rightarrow \mathbb{N}$  for the function  $\xi$  as follows. Define for  $i \in \{1, \dots, N\}$ ,

$$r_i = \lim_{k \rightarrow +\infty} \frac{\xi_i(k\alpha_i \mathbb{E}(X), k\beta_i \mathbb{E}(Y))}{k}.$$

Then for all  $z \in \mathbb{N}$ ,

$$\mathfrak{M}(z) = rz,$$

where  $r = \sum_{i=1}^N r_i$ .

### 2.2.2 Primitivity and Transitivity

We establish two more assumptions. We recall that, unlike Assumption 3, these assumptions are not always assumed and every time they are used will be clearly stated.

Our second assumption relates the probability of attaining any type of mating units starting from any initial condition, as long as this is large enough. In the case of asexual processes this assumption corresponds to the irreducibility and aperiodicity of the expectation matrix. This assumption is essential to guarantee the existence of eigenelements of  $\mathfrak{M}$ .

We start with the following definition.

**Definition 1.** We say that the process  $(Z_n)_{n \in \mathbb{N}}$  is primitive if for all  $i \in \{1, \dots, p\}$ , there exist  $n_i, k_i \in \mathbb{N}$  big enough such that for all  $m \geq n_i$ ,

$$\mathbb{E}(Z_m \mid Z_0 = k_i e_i) > 0$$

where  $e_i$  is the  $i$ -th canonical vector in  $\mathbb{R}^p$ .

In other words, a primitive process is such that any type of mating unit is reachable as long as the initial condition is large enough.

**Assumption 4 (Primitivity).** We suppose that the process is primitive.

As we will see later on, this assumption has a large repercussion on the properties of  $\mathfrak{M}$  (see Section 3.1.3 in Chapter 3).

We now state our third assumption, which is the transitivity of the process.

**Assumption 5 (Transitivity).** We assume that the process is transient, which implies in particular that the norm of the process holds the following explosion-extinction dichotomy.

$$\mathbb{P}\left(\lim_{n \rightarrow \infty} |Z_n| \in \{0, +\infty\} \mid Z_0 = z\right) = 1, \text{ for all } z \in \mathbb{N}^p.$$

Assumption 5 is a classical assumption in the branching process theory. The following two cases are some simple examples where it holds:

1. For all  $i \in \{1, \dots, p\}$ ,  $\mathbb{P}(W_1 = 0 \mid Z_0 = e_i) > 0$ . Since all the mating units reproduce independently, this is equivalent to  $\mathbb{P}(W_1 = 0 \mid Z_0 = z) > 0$  for all  $z \in \mathbb{N}^p$ . Recalling that  $\xi(0) = 0$ , this means that  $\mathbb{P}(Z_1 = 0 \mid Z_0 = z) > 0$  for all  $z \in \mathbb{N}^p$ , which makes every state connected to  $\{0\}$ , and so there are no recurrent states, which makes Assumption 5 true.
2. In the context of a bisexual population, if  $\xi$  is such that the very reasonable assumption  $\xi(x, 0) = \xi(0, y) = 0$  for all  $x \in \mathbb{N}^{q_f}$  and  $y \in \mathbb{N}^{q_m}$  holds (that is, if there is one of the sexes absent in the population, then there are no mating units formed). Then, if for all  $i \in \{1, \dots, p\}$ ,  $\mathbb{P}(F_1 = 0 \mid Z_0 = e_i) > 0$  we obtain that Assumption 5 holds using a similar argument as in the previous point. Equivalently, the same is true if for all  $i \in \{1, \dots, p\}$ ,  $\mathbb{P}(M_1 = 0 \mid Z_0 = e_i) > 0$ .

In the following proposition we have one more case where the transience assumption holds.

**Proposition 16.** *Assume that the process is strongly primitive, that is Assumption 4 is satisfied with  $k_i = 1$  for all  $i \in \{1, \dots, p\}$ . In addition assume that there exists  $\ell \in \{1, \dots, p\}$ ,  $\mathbb{P}(|Z_1| = 2 \mid Z_0 = e_\ell) > 0$ . Then, the process is transient.*

Note that the (not strong) primitivity of the process is not sufficient. In fact, choosing  $q = p = 2$  and

$$\xi(x, y) = \left( \left\lfloor \frac{y}{2} \right\rfloor, x \right), \quad V_1 \sim \frac{1}{2} \delta_{(2,0)} + \frac{1}{2} \delta_{(0,2)}, \quad V_2 \sim \delta_{(0,1)}$$

leads to a (not strongly) primitive branching process, satisfying  $\mathbb{P}(|Z_1|=2 \mid Z_0 = (1,0)) = 1/2 > 0$  and such that  $\{(1,0), (0,2)\}$  is a recurrent class.

To prove Proposition 16, we first state and prove an auxiliary lemma, which is a consequence of the superadditivity of  $\xi$ .

**Lemma 17.** *For all  $n \in \mathbb{N}$  and  $z_0, \tilde{z}_0, z_1, \tilde{z}_1 \in \mathbb{N}^p$ ,*

$$\begin{aligned} \mathbb{P}(Z_n \geq z_1 + \tilde{z}_1 \mid Z_0 \geq z_0 + \tilde{z}_0) &\geq \mathbb{P}(Z_n \geq z_1 + \tilde{z}_1 \mid Z_0 = z_0 + \tilde{z}_0) \\ &\geq \mathbb{P}(Z_n \geq z_1 \mid Z_0 = z_0) \times \mathbb{P}(Z_n \geq \tilde{z}_1 \mid Z_0 = \tilde{z}_0). \end{aligned}$$

*Proof.* The first inequality is a direct consequence of the fact that  $\xi$  is an increasing function, so we deal only with the second inequality. Let  $(Z_n(z))_{n \in \mathbb{N}}$  for  $z \in \mathbb{N}^p$  denotes a process with  $Z_0 = z$ . Let first prove by induction that  $Z_n(z_0 + \tilde{z}_0)$  stochastically dominates  $Z_n(z_0) + \tilde{Z}_n(\tilde{z}_0)$ , where  $\tilde{Z}_n$  is an independent copy of  $Z$  (that is generated by an independent copy  $(\tilde{V}^{(k,n)})_{k,n \in \mathbb{N}}$  of the family  $(V^{(k,n)})_{k,n \in \mathbb{N}}$ ). The result is trivial for  $n = 0$ . Let us assume that it is true for some  $n \in \mathbb{N}$ . Then by superadditivity of  $\xi$  and the fact that  $(V^{(k,n)})_{k,n \in \mathbb{N}}$  and  $(\tilde{V}^{(k,n)})_{k,n \in \mathbb{N}}$  are independent families of i.i.d. copies of  $V$ ,

$$\begin{aligned} \mathbb{P}(Z_{n+1} \geq z_1 + \tilde{z}_1 \mid Z_0 = z_0 + \tilde{z}_0) &= \mathbb{P} \left( \xi \left( \sum_{i=1}^p \sum_{k=1}^{Z_{n,i}(z_0 + \tilde{z}_0)} V_{i,\cdot}^{(k,n+1)} \right) \geq z_1 + \tilde{z}_1 \right) \\ &\geq \mathbb{P} \left( \xi \left( \sum_{i=1}^p \sum_{k=1}^{Z_{n,i}(z_0)} V_{i,\cdot}^{(k,n+1)} \right) + \xi \left( \sum_{i=1}^p \sum_{k=Z_{n,i}(z_0)+1}^{Z_{n,i}(z_0) + \tilde{Z}_{n,i}(\tilde{z}_0)} V_{i,\cdot}^{(k,n+1)} \right) \geq z_1 + \tilde{z}_1 \right) \\ &= \mathbb{P} \left( Z_{n+1}(z_0) + \tilde{Z}_{n+1}(\tilde{z}_0) \geq z_1 + \tilde{z}_1 \right). \end{aligned}$$

Then  $Z_{n+1}(z_0 + \tilde{z}_0)$  stochastically dominates  $Z_{n+1}(z_0) + \tilde{Z}_{n+1}(\tilde{z}_0)$ . The conclusion then follows from the independence of  $Z_{n+1}(z_0)$  and  $\tilde{Z}_{n+1}(\tilde{z}_0)$  and the last inequality which holds for all  $n \geq 0$ .  $\square$

We now prove Proposition 16

*Proof of Proposition 16.* We proceed by contradiction. If this assumption of transitivity is not satisfied, then there exists a non empty recurrent states class in  $\mathbb{N}^p \setminus \{0\}$ . We can choose  $z = (z_1, \dots, z_p) \in \mathbb{N}^p \setminus \{0\}$  with the minimal size in its class, so that  $\mathbb{P}(|Z_n| \geq |z| \mid Z_0 = z) = 1$  for all  $n \geq 1$ . We prove that  $2z$  is reachable from  $z$ , that is, there exist  $n \geq 1$  such that  $\mathbb{P}(Z_n \geq 2z \mid Z_0 = z) > 0$ . Once this is proved, we deduce from Lemma 17 that  $\mathbb{P}(|Z_n| \geq 2|z| \mid Z_0 \geq 2z) \geq \mathbb{P}(|Z_n| \geq |z| \mid Z_0 = z)^2 = 1$ , which implies that  $\mathbb{P}(\exists n, Z_n = z \mid Z_0 \geq 2z) = 0$  and contradicts the fact that  $z$  is recurrent. We thus deduce that there is no recurrent state, except the absorbing state  $\{0\}$ , and Assumption 5 is satisfied.

Let us prove that  $2z$  is reachable from  $z$  (in fact, we prove that every population can be doubled). As the process is strongly primitive, for all  $i \in \{1, \dots, p\}$ , there exists  $n_i$  such that for all  $m \geq n_i$ ,  $\mathbb{P}(Z_{m,\ell} \geq 1 \mid Z_0 = e_i) > 0$ . Let  $m = \max_{1 \leq i \leq p} n_i$ , then for all  $i \in \{1, \dots, p\}$ ,

$$\mathbb{P}(Z_m \geq e_\ell \mid Z_0 = e_i) > 0. \quad (2.8)$$

Moreover, as  $\mathbb{P}(|Z_1|=2 \mid Z_0 = e_\ell) > 0$ , there exist  $j_1, j_2 \in \{1, \dots, p\}$  such that

$$\mathbb{P}(Z_1 \geq e_{j_1} + e_{j_2} \mid Z_0 = e_\ell) > 0. \quad (2.9)$$

By strongly primitive assumption, for  $n = \max\{n_{j_1}, n_{j_2}\}$ ,

$$\mathbb{P}(Z_n \geq e_i \mid Z_0 = e_{j_1}) > 0 \quad \text{and} \quad \mathbb{P}(Z_n \geq e_i \mid Z_0 = e_{j_2}) > 0,$$

hence, by Lemma 17,

$$\mathbb{P}(Z_n \geq 2e_i \mid Z_0 = e_{j_1} + e_{j_2}) \geq \mathbb{P}(Z_n \geq e_i \mid Z_0 = e_{j_1}) \mathbb{P}(Z_n \geq e_i \mid Z_0 = e_{j_2}) > 0. \quad (2.10)$$

Finally,

$$\begin{aligned} \mathbb{P}(Z_{n+m+1} \geq 2e_i \mid Z_0 = e_i) &\geq \mathbb{P}(Z_{n+m+1} \geq 2e_i \mid Z_{m+1} \geq e_{j_1} + e_{j_2}) \\ &\quad \times \mathbb{P}(Z_{m+1} \geq e_{j_1} + e_{j_2} \mid Z_m \geq e_\ell) \\ &\quad \times \mathbb{P}(Z_m \geq e_\ell \mid Z_0 = e_i) \end{aligned}$$

then, from (2.8), (2.9), (2.10) and the Markov property,  $\mathbb{P}(Z_{n+m+1} \geq 2e_i \mid Z_0 = e_i) > 0$  for all  $i \in \{1, \dots, p\}$ , and we conclude by Lemma 17 that  $\mathbb{P}(Z_{n+m+1} \geq 2z \mid Z_0 = z) > 0$ .  $\square$

## 2.3 First Properties of $\mathfrak{M}$

This section is devoted to study the first properties of  $\mathfrak{M}$ . In the first subsection we tackle the question of existence of eigenelements for  $\mathfrak{M}$ . In the second subsection, we prove that the value of  $\mathfrak{M}$  does not depend on the chosen extension for the mating function.

### 2.3.1 Existence of Eigenelements

Consider  $A$  a real strictly positive  $N \times N$  matrix. A well-known result that goes back to Perron [Per07] states that

$$\lim_{n \rightarrow \infty} \frac{A^n x}{\rho^n} = c(x)v, \quad \forall x \in \mathbb{R}_+^p,$$

where  $\rho$  is the greatest eigenvalue of  $A$  with  $v$  its corresponding eigenvector and  $c$  is a suitable function. This result and its consequences are among the main tools used to study the asymptotic behaviour of the asexual multi-type Galton-Watson process, applied to the expectation matrix associated with the process. In this section we state similar results: a theorem that goes back to Ulrich Krause [Kra94] that provides us with the necessary tools to study the asymptotic behaviour of the multi-type bisexual branching process. For that, consider first the following definition.

**Definition 2.** A function  $M : \mathbb{R}_+^p \rightarrow \mathbb{R}_+^p$  is said to be

1. Concave if

$$M(\alpha x + (1 - \alpha)y) \geq \alpha M(x) + (1 - \alpha)M(y),$$

for all  $\alpha \in [0, 1]$  and all  $x, y \in \mathbb{R}_+^p$ .

2. Positively homogeneous if for all  $\alpha > 0$ ,  $M(\alpha x) = \alpha M(x)$  for all  $x \in \mathbb{R}_+^p$ .
3. Primitive if there exists  $n_0 \geq 1$  such that  $M^m(x) > 0$  for all  $m \geq n_0$  and  $x \in \mathbb{R}_+^p \setminus \{0\}$ .

And now we state the main theorem of this section, which is a combination of results from two articles [Kra94, Kra01]

**Theorem 18** (See [Kra94], Section 4 and [Kra01], Theorem 9). *Consider  $M : \mathbb{R}_+^p \rightarrow \mathbb{R}_+^p$  a concave, primitive and positively homogeneous function. Then,*

1. *The eigenvalue problem  $M(z) = \lambda z$  has a unique solution  $(\lambda^*, z^*) \in \mathbb{R} \times S^*$ , with  $\lambda^* > 0$ . If  $(\lambda, x) \in \mathbb{R} \times (\mathbb{R}_+^p \setminus \{0\})$  is another solution of the problem, then it must hold that  $x = rz^*$  for some  $r > 0$  and  $\lambda = \lambda^*$ .*
2. *The function  $L : (\mathbb{R}_+)^p \rightarrow \mathbb{R}_+ z^*$  given by  $L(x) = \lim_{k \rightarrow \infty} \frac{M^k(x)}{(\lambda^*)^k}$  exists on  $\mathbb{R}_+^p$  and we have  $L(x) = \mathcal{P}(x)z^*$  where  $\mathcal{P} : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  is a concave and positively homogeneous mapping with  $\mathcal{P}(x) > 0$  for all  $x \in \mathbb{R}_+^p \setminus \{0\}$ .*
3.  *$\lim_{k \rightarrow \infty} \frac{M^k(x)}{|M^k(x)|} = z^*$  for all  $x \in \mathbb{R}_+^p \setminus \{0\}$ .*
4.  *$\lim_{k \rightarrow \infty} \frac{|M^{k+1}(x)|}{|M^k(x)|} = \lambda^* = \lim_{k \rightarrow \infty} |M^k(x)|^{\frac{1}{k}}$  for all  $x \in \mathbb{R}_+^p \setminus \{0\}$ .*
5. *The convergence toward  $L(x) = \mathcal{P}(x)z^*$  is uniform on  $x \in S$ .*

Now we turn our attention to the function  $\mathfrak{M}$ . We prove that it is concave and positively homogenous.

**Proposition 19.** *The function  $\mathfrak{M}$  is positively homogeneous and concave.*

*Proof.* Let  $\alpha > 0$ , then

$$\mathfrak{M}(\alpha z) = \lim_{k \rightarrow +\infty} \frac{\xi(\alpha k z \mathbb{V})}{k} = \alpha \lim_{k \rightarrow +\infty} \frac{\xi(\alpha k z \mathbb{V})}{\alpha k} = \alpha \mathfrak{M}(z),$$

and so  $\mathfrak{M}$  is positively homogeneous. Using this and the fact that  $\xi$  (and hence  $\mathfrak{M}$ ) is a superadditive function, we deduce that  $\mathfrak{M}$  is a concave function.  $\square$

We will prove in Section 3.1.3 of Chapter 3 that when the process is primitive, then the function  $\mathfrak{M}$  is primitive, thus, allowing us to use Theorem 18 with this function and finding its eigenelements  $(\lambda^*, z^*)$ .

### 2.3.2 Extension of the Mating Function

In this section we prove that we can define  $\mathfrak{M}$  on  $\mathbb{R}_+^p$  by using any superadditive extension of  $\xi$ . Having a definition of  $\mathfrak{M}$  for all vectors on  $\mathbb{R}_+^p$  is crucial to apply Krause's Theorem. To do so, we prove that the value of  $\mathfrak{M}$  depends only on the value of the mating function for integer entries. In addition, we prove some results on the convergence of the function  $\xi$  towards  $\mathfrak{M}$ . As a corollary, we obtain the uniform convergence in the case where  $\mathfrak{M}$  is a finite and continuous function.

**Proposition 20.** *For all  $z \in \mathbb{R}_+^p$ , we have*

$$\mathfrak{M}(z) = \lim_{r \rightarrow +\infty} \frac{\xi(\lfloor rz \mathbb{V} \rfloor)}{r} = \sup_{r \geq 1} \frac{\xi(\lfloor rz \mathbb{V} \rfloor)}{r}. \quad (2.11)$$

In addition, for any  $i \in \{1, \dots, p\}$  and for any compact set  $K \subset S$  such that  $\mathfrak{M}_i$  is continuous and bounded on  $K$ , we have

$$\sup_{z \in K} \left| \mathfrak{M}_i(z) - \frac{\xi_i(rz\mathbb{V})}{r} \right| \xrightarrow{r \rightarrow +\infty} 0, \quad (2.12)$$

and, for any  $i \in \{1, \dots, p\}$  and for any compact set  $K \subset S$  such that  $\mathfrak{M}_i$  is infinite on  $K$ , we have

$$\inf_{z \in K} \frac{\xi_i(rz\mathbb{V})}{r} \xrightarrow{r \rightarrow +\infty} +\infty. \quad (2.13)$$

We emphasize that, in general, if  $\mathfrak{M}_i$  is continuous on a compact subset  $K$  of  $S$ , then there exist two disjoint compact subsets  $K_1$  and  $K_2$  of  $S$  such that  $K = K_1 \cup K_2$ , with  $\mathfrak{M}_i$  bounded on  $K_1$  and infinite on  $K_2$  (note that  $K_1$  or  $K_2$  may be empty). Indeed, on the one hand, by concavity of  $\mathfrak{M}_i$ ,  $\mathfrak{M}_i^{-1}(\{+\infty\})$  is an open subset of  $S$  and thus  $K_1 = K \cap \mathfrak{M}_i^{-1}([0, +\infty))$  is a compact subset of  $S$  on which  $\mathfrak{M}_i$  is continuous and thus bounded. On the other hand, by continuity of  $\mathfrak{M}_i$ ,  $K_2 = K \cap \mathfrak{M}_i^{-1}(\{+\infty\})$  is also compact.

*Proof of Proposition 20.* For the first assertion consider  $z \in \mathbb{R}_+^p$ ,  $u \in \{0, 1\}^p$  given by  $u_i = \mathbb{1}_{\{(z\mathbb{V})_i \neq 0\}}$  and let  $n \in \mathbb{N}^*$  be such that  $z\mathbb{V} \geq \frac{1}{n}u$ . Then, for all  $r > 0$ ,

$$\frac{\xi(\lfloor (r+n)z\mathbb{V} \rfloor)}{r+n} \geq \frac{\xi(\lfloor rz\mathbb{V} + u \rfloor)}{r+n} \geq \frac{\xi(rz\mathbb{V})}{r} \frac{r}{r+n}.$$

Taking the limit when  $r \rightarrow +\infty$ , we conclude that

$$\mathfrak{M}(z) \leq \lim_{r \rightarrow +\infty} \frac{\xi(\lfloor rz\mathbb{V} \rfloor)}{r}.$$

The reverse inequality is direct using the fact that  $\xi$  is non-decreasing in all its components, which concludes the proof of the first equality in (2.11). The second equality is a consequence of Fekete's Lemma and the fact that  $r \mapsto \xi(\lfloor rz\mathbb{V} \rfloor)$  is superadditive.

For the second part, take  $i \in \{1, \dots, p\}$  and let  $K$  be a compact subset of  $S$  such that  $\mathfrak{M}_i$  is bounded continuous on  $K$ . Since  $\mathfrak{M}_i(z) \geq \frac{\xi_i(rz\mathbb{V})}{r}$  for all  $z \in S$  and  $r > 0$ , we only have to prove that

$$\limsup_{r \rightarrow +\infty} \sup_{z \in K} \left( \mathfrak{M}_i(z) - \frac{\xi_i(rz\mathbb{V})}{r} \right) \leq 0.$$

Assume the contrary. Then there exist  $\varepsilon > 0$  and two sequences  $(z_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}}$  and  $(r_n)_{n \in \mathbb{N}} \in (0, +\infty)^{\mathbb{N}}$  such that  $r_n \nearrow +\infty$  and

$$\frac{\xi_i(r_n z_n \mathbb{V})}{r_n} \leq \mathfrak{M}_i(z_n) - \varepsilon.$$

Since  $K$  is compact, there exists, up to a subsequence,  $z_\infty \in K$  such that  $z_n \rightarrow z_\infty$ . In particular, for all  $\delta \in (0, 1)$ , there exists  $n_{\delta, \varepsilon}$  such that, for all  $n \geq n_{\delta, \varepsilon}$ ,  $z_n \geq (1 - \delta)z_\infty$  and  $\mathfrak{M}_i(z_n) \leq \mathfrak{M}_i(z_\infty) + \varepsilon/2$  and hence

$$\frac{\xi_i((1 - \delta)r_n z_\infty \mathbb{V})}{r_n} \leq \mathfrak{M}_i(z_\infty) - \varepsilon/2.$$



By definition of  $\mathfrak{M}_i$  and Proposition 19, the left hand side converges to  $\mathfrak{M}_i((1 - \delta)z_\infty) = (1 - \delta)\mathfrak{M}_i(z_\infty)$  when  $n \rightarrow +\infty$ , and hence

$$(1 - \delta)\mathfrak{M}_i(z_\infty) \leq \mathfrak{M}_i(z_\infty) - \varepsilon/2.$$

Since this is true for all  $\delta > 0, \varepsilon > 0$  and since  $\mathfrak{M}_i(z_\infty) < +\infty$  by assumption, this is a contradiction. We thus proved (2.12).

The proof of (2.13) is similar. Take  $i \in \{1, \dots, p\}$  and let  $K$  be a compact subset of  $S$  such that  $\mathfrak{M}_i$  is infinite on  $K$ . Assume that it does not hold true. Then there exist  $A > 0$  and two sequences  $(z_n)_{n \in \mathbb{N}} \in K^{\mathbb{N}}$  and  $(r_n)_{n \in \mathbb{N}} \in (0, +\infty)^{\mathbb{N}}$  such that  $r_n \nearrow +\infty$  and

$$\frac{\xi_i(r_n z_n \mathbb{V})}{r_n} \leq A.$$

This implies that, up to a subsequence,  $z_n$  converges to  $z_\infty \in K$  and that, for any  $\delta \in (0, 1)$ ,

$$(1 - \delta)\mathfrak{M}_i(z_\infty) \leq A.$$

But  $\mathfrak{M}_i$  is equal to  $+\infty$  on  $K$  and hence we obtained  $+\infty \leq A$ , which is a contradiction. This concludes the proof of Proposition 20.  $\square$

We state now a consequence of the last proposition regarding the uniform convergence towards  $\mathfrak{M}$  in the case where this is a finite and continuous function on  $S$ . By taking  $K = S$  in (2.12), and since  $\mathfrak{M}$  is positively homogenous (see Proposition 19) we have the following immediate result.

**Corollary 21.** *Assume  $\mathfrak{M}$  is finite and continuous over  $S$ . Then, we have*

$$\left| \frac{\mathfrak{M}(z)}{|z|} - \frac{\xi(z\mathbb{V})}{|z|} \right| \xrightarrow{|z| \rightarrow +\infty} 0.$$



# Chapter 3

## Laws of Large Numbers

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In this chapter we introduce two main results for a multi-type bisexual branching process with a superadditive mating function. We start with a strong law of large numbers for a large initial population. This result relates the behaviour of the process when the population is large with the iterates of the function  $\mathfrak{M}$  introduced in the previous chapter. The statement of the theorem and its proof are in Section 3.1.1. Additional results in the case when the function  $\mathfrak{M}$  takes an infinite value are discussed in Section 3.1.2. As a consequence, we obtain another characterization of this function in term of the mean growth rates of the process in Section 3.1.3. This allows to prove that, under suitable assumptions, the function  $\mathfrak{M}$  is primitive. Hence, leveraging on the concave Perron-Frobenius theory from the precedent chapter, the existence of eigenelements is established.

For our second result, and with the extra assumption that the process is primitive and transient, we prove a weak law of large numbers for the complete trajectory. The result and its proof are in Section 3.2. The consequences of this result are the main topic for discussion in Chapter 4.

### 3.1 Law of Large Numbers for a large initial population

Our first main result is a law of large numbers which relates  $\mathfrak{M}$  with the behaviour of  $(Z_n)_{n \in \mathbb{N}}$  in a large initial population setting. For this result, we assume that  $\xi$  is a superadditive function (Assumption 3), which allows us to ensure that  $\mathfrak{M}$  is well defined. We also assume that  $\mathfrak{M}$  is finite. We state our theorem and then we give an extra result in the case where  $\mathfrak{M}$  takes an infinite value.

We set  $\mathfrak{M}^n = \mathfrak{M} \circ \dots \circ \mathfrak{M}$  composed  $n$  times with the convention that  $\mathfrak{M}^0$  is the identity function. We start by stating the theorem and its proof. Later on, we discuss some consequences of this theorem. We finish this section with some simulations.

### 3.1.1 Statement and proof of the LLN

**Theorem 22** (Law of large numbers). *Let  $(z_m)_{m \geq 1}$  be a random sequence in  $\mathbb{N}^p$  and  $z_\infty \in \mathbb{R}_+^p \setminus \{0\}$  a deterministic value such that  $z_m \sim_{m \rightarrow +\infty} m z_\infty$  almost surely. For all  $m \geq 1$ , denote by  $(Z_n^m)_{n \geq 0}$  a multi-type bisexual Galton-Watson branching process with common mating function and offspring distribution, but with initial configuration  $Z_0^m = z_m$ . Define  $\mathfrak{M}$  as in (2.6) and assume it is finite over  $S$ . Then, for all  $n \geq 0$ ,*

$$Z_n^m \sim_{m \rightarrow +\infty} m \mathfrak{M}^n(z_\infty) \text{ almost surely.}$$

*If in addition  $(z_m/m)_{m \geq 1}$  is independent of the random variables  $V_{i,j}^{(k,n)}$  and uniformly integrable, then  $Z_n^m/m$  converges to  $\mathfrak{M}^n(z_\infty)$  in  $L^1$ .*

In order to prove this theorem, we introduce first an auxiliary lemma and its proof.

**Lemma 23.** *Let  $(z_k)_{k \geq 0}$  be a random sequence in  $\mathbb{N}^p$  such that  $z_k \sim_{k \rightarrow +\infty} k z_\infty \in \mathbb{R}_+^p$  almost surely. We have*

$$\frac{1}{k} \xi \left( \sum_{i=1}^p \sum_{m=1}^{z_{k,i}} V_{i,\cdot}^{(m)} \right) \rightarrow \mathfrak{M}(z_\infty),$$

*almost surely when  $k \rightarrow +\infty$ .*

The proof of this lemma is inspired by [DHT86].

*Proof of Lemma 23.* Since all the variables  $V_{i,j}$  are integrable, then thanks to the strong law of large numbers, we have that for all  $1 \leq i \leq p$ ,  $1 \leq j \leq q$ ,

$$\frac{1}{n} \sum_{m=1}^n V_{i,j}^{(m,1)} \xrightarrow[n \rightarrow +\infty]{a.s.} \mathbb{V}_{i,j}.$$

Assume first that  $z_{\infty,i} > 0$ . In this case, since  $z_{k,i} \rightarrow +\infty$  almost surely we deduce that

$$\frac{1}{z_{k,i}} \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \xrightarrow[k \rightarrow +\infty]{a.s.} \mathbb{V}_{i,j}$$

and hence

$$\frac{1}{k z_{\infty,i}} \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \xrightarrow[k \rightarrow +\infty]{a.s.} \mathbb{V}_{i,j}.$$

Fix  $0 < \varepsilon < \min_{i,j/\mathbb{V}_{i,j} \neq 0} \mathbb{V}_{i,j}$ . Hence, with probability one there exists  $k_0$  (random) such that if  $k \geq k_0$ , then

$$k z_{\infty,i} (\mathbb{V}_{i,j} - \varepsilon) \leq \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \leq k z_{\infty,i} (\mathbb{V}_{i,j} + \varepsilon).$$

Assume now that  $z_{\infty,i} = 0$ . Then, almost surely, there exists  $k_0$  such that for all  $k \geq k_0$ ,  $z_{k,i} = 0$ , so that the last inequality also holds true.

We consider again the general case  $z_\infty \geq 0$ . Summing on  $i$  we obtain that, almost surely, there exists  $k_0$  such that, for all  $k \geq k_0$  and all  $j \leq q$ ,

$$\sum_{\substack{i=1 \\ \mathbb{V}_{i,j} \neq 0}}^p k z_{\infty,i} (\mathbb{V}_{i,j} - \varepsilon) \leq \sum_{i=1}^p \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \leq \sum_{\substack{i=1 \\ \mathbb{V}_{i,j} \neq 0}}^p k z_{\infty,i} (\mathbb{V}_{i,j} + \varepsilon),$$

where we used the fact that  $V_{i,j}^{(m,1)} = 0$  almost surely if  $\mathbb{V}_{i,j} = 0$ .

Define the matrices  $\mathbb{V}_+^\varepsilon := \left( (\mathbb{V}_{i,j} + \varepsilon) \mathbf{1}_{\mathbb{V}_{i,j} \neq 0} \right)_{1 \leq i \leq p, 1 \leq j \leq q}$  and  $\mathbb{V}_-^\varepsilon := \left( (\mathbb{V}_{i,j} - \varepsilon) \mathbf{1}_{\mathbb{V}_{i,j} \neq 0} \right)_{1 \leq i \leq p, 1 \leq j \leq q}$ . Since the function  $\xi$  is superadditive, in particular it is non decreasing. Hence, we get

$$\frac{\xi(k z_\infty \mathbb{V}_-^\varepsilon)}{k} \leq \frac{1}{k} \xi \left( \left( \sum_{i=1}^p \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \right)_{1 \leq j \leq q} \right) \leq \frac{\xi(k z_\infty \mathbb{V}_+^\varepsilon)}{k}. \quad (3.1)$$

Let define

$$\delta = \frac{\varepsilon}{\min_{i,j/\mathbb{V}_{i,j} \neq 0} \mathbb{V}_{i,j}},$$

and note that  $\delta < 1$  thanks to our choice of  $\varepsilon$ .

We note that

$$\limsup_{k \rightarrow +\infty} \frac{\xi(k z_\infty \mathbb{V}_+^\varepsilon)}{k} = \limsup_{k \rightarrow +\infty} \frac{1}{k} \xi \left( \left( \sum_{\substack{i=1 \\ \mathbb{V}_{i,j} \neq 0}}^p k z_{\infty,i} \left( 1 + \frac{\varepsilon}{\mathbb{V}_{i,j}} \right) \mathbb{V}_{i,j} \right)_{1 \leq j \leq q} \right) \quad (3.2)$$

$$\leq \lim_{k \rightarrow +\infty} \frac{1}{k} \xi \left( \left( \sum_{i=1}^p k z_{\infty,i} (1 + \delta) \mathbb{V}_{i,j} \right)_{1 \leq j \leq q} \right) \quad (3.3)$$

$$= (1 + \delta) \mathfrak{M}(z_\infty), \quad (3.4)$$

and similarly,  $\liminf_{k \rightarrow +\infty} \frac{\xi(k z_\infty \mathbb{V}_-^\varepsilon)}{k} \geq (1 - \delta) \mathfrak{M}(z_\infty)$ .

Hence, taking  $k \rightarrow +\infty$  in (3.1), we obtain

$$(1 - \delta) \mathfrak{M}(z_\infty) \leq \liminf_{k \rightarrow +\infty} \frac{1}{k} \xi \left( \left( \sum_{i=1}^p \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \right)_{1 \leq j \leq q} \right) \leq (1 + \delta) \mathfrak{M}(z_\infty) \quad (3.5)$$

Finally, taking  $\varepsilon \rightarrow 0$ , then  $\delta$  goes to 0 and we conclude the desired result.  $\square$

*Remark 8.* The proof of Lemma 23 does not use directly the fact that  $\xi$  is a superadditive function, but only a non-decreasing function. Nevertheless, the superadditive assumption guarantees that  $\mathfrak{M}$  is well defined. If we deal with a case where  $\xi$  is only non-decreasing but  $\mathfrak{M}$  cannot be defined, one can still get estimates on the behaviour of the population following the same reasoning as in the proof recently presented. In fact, if one defines

$$\mathfrak{M}_{\inf}(z) = \liminf_{k \rightarrow +\infty} \frac{\xi(kz\mathbb{V})}{k} \quad \text{and} \quad \mathfrak{M}_{\sup}(z) = \limsup_{k \rightarrow +\infty} \frac{\xi(kz\mathbb{V})}{k},$$

then proceeding as in (3.1) and then as in (3.2) one can deduce that almost surely

$$\begin{aligned} \mathfrak{M}_{\inf}(z_\infty) &\leq \liminf_{k \rightarrow +\infty} \frac{1}{k} \xi \left( \left( \sum_{i=1}^p \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \right)_{1 \leq j \leq q} \right) \\ &\leq \limsup_{k \rightarrow +\infty} \frac{1}{k} \xi \left( \left( \sum_{i=1}^p \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \right)_{1 \leq j \leq q} \right) \\ &\leq \mathfrak{M}_{\sup}(z_\infty). \end{aligned}$$

We now proceed with the proof of Theorem 22.

*Proof of Theorem 22.* We first prove the almost sure convergence in Step 1, and then the  $L^1$  convergence in Step 2.

*Step 1. Almost sure convergence.* The result is trivial for  $n = 0$ . By Lemma 23, we have

$$\frac{Z_1^m}{m} = \frac{1}{m} \xi \left( \sum_{i=1}^p \sum_{k=1}^{z_{m,i}} V_{i,\cdot}^{(1,k)} \right) \xrightarrow[m \rightarrow \infty]{a.s.} \mathfrak{M}(z_\infty). \quad (3.6)$$

If  $\mathfrak{M}_\ell(z_\infty) > 0$  for some  $\ell \in \{1, \dots, p\}$ , then this proves that  $Z_{1,\ell}^m \sim_{m \rightarrow +\infty} m \mathfrak{M}_\ell(z_\infty)$  almost surely. If  $\mathfrak{M}_\ell(z_\infty) = 0$ , then  $\xi_\ell(k z_\infty \mathbb{V})$  vanishes for all  $k \geq 1$  and hence  $Z_{1,\ell} \mathbb{1}_{Z_0 \leq C z_\infty} = 0$  almost surely for all  $C > 0$ . Since  $z_m \sim_{m \rightarrow +\infty} m z_\infty$ , we deduce that there exists a (random)  $m_0 \geq 1$  such that, for all  $m \geq m_0$ ,  $Z_{1,\ell}^m = 0$ . Thus we proved that  $Z_1^m \sim_{m \rightarrow +\infty} m \mathfrak{M}(z_\infty)$  almost surely, which proves the result when  $n = 1$ .

Assume now that  $Z_n^m \sim_{m \rightarrow +\infty} m \mathfrak{M}^n(z_\infty)$  a.s. for some  $n \geq 1$ . Then the previous step with  $z_m = Z_n^m$  entails that

$$Z_{n+1}^m \sim_{m \rightarrow \infty} m \mathfrak{M}(\mathfrak{M}^n(z_\infty)) = m \mathfrak{M}^{n+1}(z_\infty) \quad \text{a.s.}$$

This concludes the proof of the first assertion in Theorem 22.

*Step 2. Convergence in  $L^1$ .* We prove now the  $L^1$ -convergence. Denote  $\mathbf{1}_q \in \mathbb{N}^q$ ,  $\mathbf{1}_q = (1, \dots, 1)$ , and fix  $z_0 \in \mathbb{N}^p$  such that  $z_0 \mathbb{V} \geq \mathbf{1}_q$ . Consider the multi-type bisexual branching process with initial position  $Z_0^m = z_m$ ,  $m \geq 1$ , and denote by  $W_1^m$  the number of children in the first generation. We have  $W_1^m \leq |W_1^m| \mathbf{1}_q \leq |W_1^m| z_0 \mathbb{V}$ , and so using the second equality in (2.6),

$$Z_1^m = \xi(W_1^m) \leq \xi(|W_1^m| z_0 \mathbb{V}) \leq \mathfrak{M}(|W_1^m| z_0).$$

Using Proposition 19, we deduce that

$$Z_1^m \leq |W_1^m| \mathfrak{M}(z_0). \quad (3.7)$$

By assumption, the random vector

$$U^{(m)} := \lfloor z_m/m \rfloor + 1$$

is uniformly integrable, and we have  $z_m \leq m U^{(m)}$  almost surely, so that

$$0 \leq |W_1^m| \leq \left| \sum_{i=1}^p \sum_{k=1}^{m U_i^{(m)}} V_{i,\cdot}^{(k,1)} \right|. \quad (3.8)$$

Since  $U^{(m)}$  is independent from the other terms, we have

$$\begin{aligned} & \mathbb{E} \left( \left| \frac{1}{m} \sum_{j=1}^q \sum_{i=1}^p \sum_{k=1}^{mU_i^{(m)}} (V_{i,j}^{(k,1)} - \mathbb{V}_{i,j}) \right| \right) \\ & \leq \sum_{u \in (\mathbb{N} \setminus \{0\})^p} \sum_{j=1}^q \sum_{i=1}^p \frac{1}{mu_i} \mathbb{E} \left( \left| \sum_{k=1}^{mu_i} (V_{i,j}^{(k,1)} - \mathbb{V}_{i,j}) \right| \right) |u| \mathbb{P} (U^{(m)} = u). \end{aligned} \quad (3.9)$$

Using the law of large numbers, we deduce that, for each  $i \in \{1, \dots, p\}$  and  $j \in \{1, \dots, q\}$ ,

$$\frac{1}{m} \mathbb{E} \left( \left| \sum_{k=1}^m (V_{i,j}^{(k,1)} - \mathbb{V}_{i,j}) \right| \right) \xrightarrow{m \rightarrow +\infty} 0.$$

In particular, this ensures that the family

$$f_m := \max_{u \in (\mathbb{N} \setminus \{0\})^p} \sum_{j=1}^q \sum_{i=1}^p \frac{1}{mu_i} \mathbb{E} \left( \left| \sum_{k=1}^{mu_i} (V_{i,j}^{(k,1)} - \mathbb{V}_{i,j}) \right| \right)$$

converges to 0 when  $m \rightarrow +\infty$ . Since the family  $(U^{(m)})_{m \geq 0}$  is uniformly integrable, we deduce by (3.9) that

$$\mathbb{E} \left( \left| \frac{1}{m} \sum_{j=1}^q \sum_{i=1}^p \sum_{k=1}^{mU_i^{(m)}} (V_{i,j}^{(k,1)} - \mathbb{V}_{i,j}) \right| \right) \leq f_m \mathbb{E} (|U^{(m)}|) \xrightarrow{m \rightarrow +\infty} 0.$$

In particular, since  $(U^{(m)})_{m \geq 0}$  is uniformly integrable and thus  $\left( \frac{1}{m} \sum_{j=1}^q \sum_{i=1}^p \sum_{k=1}^{mU_i^{(m)}} \mathbb{V}_{i,j} \right)_{m \geq 0}$  is uniformly integrable, this shows that  $\left( \frac{1}{m} \sum_{j=1}^q \sum_{i=1}^p \sum_{k=1}^{mU_i^{(m)}} V_{i,j}^{(k,1)} \right)_{m \geq 0}$  is uniformly integrable. By inequalities (3.8) and (3.7), this entails that  $(Z_1^m/m)_{m \geq 1}$  is uniformly integrable too. Now, since we also proved that  $Z_1^m/m$  converges almost surely to  $\mathfrak{M}(z_\infty)$ , this implies that  $Z_1^m/m$  converges in  $L^1$  to  $\mathfrak{M}(z_\infty)$ .

As above, the result for general  $n \geq 1$  derives by iteration, which concludes the proof of Theorem 22.  $\square$

### 3.1.2 A generalization of the Law of Large Numbers

Consider  $\vec{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p$  and define the function  $\mathfrak{N}(\vec{\alpha}, \cdot) : \mathbb{R}_+^p \rightarrow \mathbb{R}_+^p$  where for  $i \in \{1, \dots, p\}$ , the  $i$ -th entry of  $\mathfrak{N}(\vec{\alpha}, \cdot)$  is given by

$$\mathfrak{N}_i(\vec{\alpha}, z) = \lim_{k \rightarrow +\infty} \frac{\xi_i(kz\mathbb{V})}{k^{\alpha_i}} \quad (3.10)$$

For instance,  $\mathfrak{N}(\mathbf{1}_p, \cdot) = \mathfrak{M}(\cdot)$ , with  $\mathbf{1}_p = (1, \dots, 1)$ . The function  $\mathfrak{N}(\vec{\alpha}, \cdot)$  allows us to give a similar result as Theorem 22 that encompasses the case when some (or all) components of  $\mathfrak{M}$  are not finite (see for instance Example 7 below). This result is stated in the following corollary.

**Corollary 24.** *Let  $(z_m)_{m \geq 1}$  be a random sequence in  $\mathbb{N}^p$  and  $z_\infty \in \mathbb{R}_+^p \setminus \{0\}$  a deterministic value such that  $\frac{z_m}{m} \rightarrow z_\infty$  almost surely as  $m \rightarrow +\infty$ . For all  $m \geq 1$ , denote by  $(Z_n^m)_{n \geq 0}$  a multi-type*

bisexual Galton-Watson branching process with common mating function and offspring distribution, but with initial configuration  $Z_0^m = z_m$ . Assume that there exists  $\vec{\alpha} = (\alpha_1, \dots, \alpha_p) \in \mathbb{R}_+^p \setminus \{0\}$  such that  $\mathfrak{N}(\vec{\alpha}, z) < +\infty$  for all  $z \in S$ . Then for all  $n \in \mathbb{N}$ , and all  $i \in \{1, \dots, p\}$ ,

$$\frac{Z_{n,i}^m}{m^{\alpha_i^n}} \xrightarrow{m \rightarrow +\infty} \mathfrak{N}^n(\vec{\alpha}, z_\infty) \text{ almost surely.}$$

We emphasize that  $\mathfrak{N}(\alpha, z)$  is not a positively homogeneous function as is the case of  $\mathfrak{M}$ . In particular, for any  $c > 0$ , we have that

$$\mathfrak{N}(\vec{\alpha}, cz) = (c^{\alpha_1} \mathfrak{N}_1(\vec{\alpha}, z), \dots, c^{\alpha_p} \mathfrak{N}_p(\vec{\alpha}, z))$$

which makes natural that in Corollary 24  $(Z_{n,i}^m)_{m \in \mathbb{N}}$  grows towards  $\mathfrak{N}(\alpha, z)$  at rate  $m^{\alpha_i^n}$ .

*Proof of Corollary 24.* We proceed using an induction argument. The proof is similar to those of Lemma 23 and Theorem 22 so we only state the main differences. The case  $n = 0$  is direct so we first focus on  $n = 1$ . We follow the proof of Lemma 23, which remains equal to that of Lemma 23 up until equation (3.1), where we instead divide by  $k_i^\alpha$ . This implies that now equation (3.5) reads (if we express it componentwise)

$$(1 - \delta)^{\alpha_i} \mathfrak{N}_i(\vec{\alpha}, z_\infty) \leq \liminf_{k \rightarrow +\infty} \frac{1}{k^{\alpha_i}} \xi_i \left( \left( \sum_{i=1}^p \sum_{m=1}^{z_{k,i}} V_{i,j}^{(m,1)} \right)_{1 \leq j \leq q} \right) \leq (1 + \delta)^{\alpha_i} \mathfrak{N}_i(\vec{\alpha}, z_\infty). \quad (3.11)$$

Hence, by taking  $\delta \rightarrow 0$  we complete this case. For the induction step, suppose that the statement holds for some  $n \in \mathbb{N}$ , and then, applying the change of variable  $k = m^{\alpha^n}$  and considering the same reasoning but with the sequence  $z_k = Z_n^k$ , we obtain

$$\frac{(Z_{n+1,i}^m)}{(m^{\alpha^{(n+1)}})} = \frac{(Z_{n+1,i}^m)}{(m^{\alpha^n})^\alpha} \xrightarrow{m \rightarrow +\infty} \mathfrak{N}_i(\vec{\alpha}, \mathfrak{N}^n(\vec{\alpha}, z_\infty)) = \mathfrak{N}_i^{n+1}(\vec{\alpha}, z_\infty) \text{ a.s.,}$$

and the proof follows. □

*Remark 9.* We observe that in general the statement

$$Z_{n,i}^m \sim_{m \rightarrow +\infty} (m^{\alpha^n}) \mathfrak{N}(\vec{\alpha}, z_\infty)$$

does not hold since the fact that  $\mathfrak{N}(\vec{\alpha}, z_\infty) = 0$  does not imply that  $Z_{n,i}$  is zero in finite time almost surely.

We consider the following example of a model where  $\mathfrak{M}$  is not finite.

**Example 7** (Bilateral completely promiscuous mating). We consider a model in the context of two-sex multi-type population where  $p = q_m = q_f$ . We assume that the mating function is given by

$$\xi((x_1, \dots, x_p), (y_1, \dots, y_p)) = (x_1 y_1, \dots, x_p y_p).$$

We have that  $\mathfrak{M}(z) = +\infty$  for all  $z \in \mathbb{R}_+^p \setminus \{0\}$  but

$$\mathfrak{N}(\mathbf{21}_p, z) = ((z\mathbb{X})_1(z\mathbb{Y})_1, \dots, (z\mathbb{X})_p(z\mathbb{X})_p),$$

where  $\mathbf{1}_p = (1, \dots, 1)$ .



### 3.1.3 A second characterization of $\mathfrak{M}$

One of the first consequences of Theorem 22 is that, by setting for  $z \in \mathbb{R}_+$  the sequence  $z_m = \lfloor mz \rfloor$ , for all  $m \in \mathbb{N}$ , we have the following corollary.

**Corollary 25.** *We have for all  $n \in \mathbb{N}$  and all  $z \in \mathbb{R}_+^p$ ,*

$$\mathfrak{M}^n(z) = \lim_{m \rightarrow +\infty} \frac{\mathbb{E}(Z_n \mid Z_0 = \lfloor mz \rfloor)}{m} = \sup_{m \geq 1} \frac{\mathbb{E}(Z_n \mid Z_0 = \lfloor mz \rfloor)}{m}. \quad (3.12)$$

Thus, the function  $\mathfrak{M}$  extends to the multi-type case the *asymptotic growth rate* introduced in the single-type case by Bruss in [Bru84] and used by Daley et al. in [DHT86] to study the extinction conditions for the process. Note also that, in the situation where  $z_m = \lfloor mz \rfloor$ , one can adapt the proof of Klebaner [Kle93] to obtain convergence in law in Theorem 22, as detailed in [Ada16]. However, almost sure and  $L^1$  convergence obtained in Theorem 22 are needed in the proofs of the results to come later in this chapter and in the next one.

*Remark 10.* The  $L^1$  convergence is a new result in the single-type case. In particular, it provides the first formal proof of the fact that the two definitions of  $r$  in (1.16) and (1.18) are equivalent (a property implicitly used in [DHT86]).

To prove the corollary, we start with the state and proof of the following auxiliary lemma.

**Lemma 26.** *For all  $n \in \mathbb{N}$ , the function  $z \in \mathbb{N}^p \mapsto \mathbb{E}(Z_n \mid Z_0 = z)$  is superadditive.*

*Proof of Lemma 26.* We prove this by an induction argument over  $n \in \mathbb{N}$ . For the case  $n = 1$ , we prove an even stronger result which will be helpful later. We prove that for any superadditive function  $f : \mathbb{N}^p \rightarrow \mathbb{R}_+^p$ , the function

$$z \in \mathbb{N}^p \rightarrow \mathbb{E}(f(Z_1) \mid Z_0 = z),$$

is superadditive. Consider  $z^1, z^2 \in \mathbb{N}^p$ , then

$$\begin{aligned} \mathbb{E}(f(Z_1) \mid Z_0 = z^1 + z^2) &= \mathbb{E}\left(f(\xi((W_{1,j})_{1 \leq j \leq q})) \mid Z_0 = z^1 + z^2\right) \\ &= \mathbb{E}\left(f\left(\xi\left(\left(\sum_{i=1}^p \sum_{k=1}^{z_i^1 + z_i^2} V_{i,j}^{(k)}\right)_{1 \leq j \leq q}\right)\right)\right) \\ &\geq \mathbb{E}\left(f\left(\xi\left(\left(\sum_{i=1}^p \sum_{k=1}^{z_i^1} V_{i,j}^{(k)}\right)_{1 \leq j \leq q}\right)\right) + \xi\left(\left(\sum_{i=1}^p \sum_{k=z_i^1+1}^{z_i^1+z_i^2} V_{i,j}^{(k)}\right)_{1 \leq j \leq q}\right)\right) \\ &\geq \mathbb{E}\left(f\left(\xi\left(\left(\sum_{i=1}^p \sum_{k=1}^{z_i^1} V_{i,j}^{(k)}\right)_{1 \leq j \leq q}\right)\right)\right) + \mathbb{E}\left(f\left(\xi\left(\left(\sum_{i=1}^p \sum_{k=1}^{z_i^2} V_{i,j}^{(k)}\right)_{1 \leq j \leq q}\right)\right)\right) \\ &= \mathbb{E}\left(f(Z_1) \mid Z_0 = z^1\right) + \mathbb{E}\left(f(Z_1) \mid Z_0 = z^2\right). \end{aligned}$$

Assume now that this result is true for some  $n \in \mathbb{N}$ . Then for  $n + 1$  we have, using the Markov property, that

$$\mathbb{E}(Z_{n+1} \mid Z_0 = z^1 + z^2) = \mathbb{E}(\mathbb{E}_{Z_n}(\widetilde{Z}_1) \mid Z_0 = z^1 + z^2),$$

where  $(\tilde{Z}_n)_{n \in \mathbb{N}}$  is an independent copy of  $(Z_n)_{n \in \mathbb{N}}$ . Consider now the function  $z \in \mathbb{N}^p \mapsto g(z) = \mathbb{E}(\tilde{Z}_1 \mid \tilde{Z}_0 = z)$ . Using the result for  $n = 1$  we know that  $g$  is a superadditive function. Hence, using the induction hypothesis, we have

$$\begin{aligned} \mathbb{E}(Z_{n+1} \mid Z_0 = z^1 + z^2) &= \mathbb{E}(g(Z_n) \mid Z_0 = z^1 + z^2) \\ &\geq \mathbb{E}(g(Z_n) \mid Z_0 = z^1) + \mathbb{E}(g(Z_n) \mid Z_0 = z^2) \\ &= \mathbb{E}(Z_{n+1} \mid Z_0 = z^1) + \mathbb{E}(Z_{n+1} \mid Z_0 = z^2). \end{aligned}$$

The proof is then complete.  $\square$

We now are in place to prove Corollary 25.

*Proof of Corollary 25.* Theorem 22 yields the first equality in Corollary 25 when  $\mathfrak{M}$  takes finite values. For the second one, it is a classical consequence of superadditive sequences, which is the case thanks to Lemma 26 and the fact that, for all  $z \in \mathbb{N}^p$ , the function  $k \mapsto \lfloor kz \rfloor$  is also superadditive. In the situation where  $\mathfrak{M}$  is not finite valued, we consider the vector  $\mathbf{1}_p = (1, \dots, 1) \in \mathbb{N}^p$  and introduce the superadditive function

$$\hat{\xi}(x) = |x| \mathbf{1}_p.$$

Then, for all  $\alpha \in \mathbb{N}$ , we define the superadditive mating function

$$\xi_{(\alpha)}(x) = \min\{\xi(x), \alpha \hat{\xi}(x)\} := \left( \min\{\xi(x)_i, \alpha \hat{\xi}(x)_i\} \right)_{1 \leq i \leq p},$$

and we denote by  $\mathfrak{M}_{(\alpha)}$  the function associated if we consider  $\xi_{(\alpha)}$  as mating function with the same offspring distribution as the original process. We can check that  $\mathfrak{M}_{(\alpha)}(z) = \min\{\mathfrak{M}(z), \alpha \hat{\xi}(z\mathbb{V})\}$  and so we obtain that  $\mathfrak{M}_{(\alpha)}(z) \nearrow \mathfrak{M}(z)$  as  $\alpha \rightarrow +\infty$ , for all  $z \in \mathbb{R}_+^p$ . Since clearly  $\xi_{(\alpha)} \nearrow \xi$ , using the Monotone Convergence Theorem, for all  $z \in \mathbb{N}^p$  and  $i \in \{1, \dots, p\}$ ,

$$\mathbb{E}_{(\alpha)}(Z_1 \mid Z_0 = z) \xrightarrow{\alpha \rightarrow +\infty} \mathbb{E}(Z_1 \mid Z_0 = z), \quad (3.13)$$

where  $\mathbb{E}_{(\alpha)}$  is the probability law associated to the process with mating function  $\xi_{(\alpha)}$ . In particular, using Corollary 25 for the finite valued  $\mathfrak{M}_{(\alpha)}$ , for all  $z \in \mathbb{R}_+^p$ ,

$$\begin{aligned} \mathfrak{M}(z) &= \sup_{\alpha > 0} \mathfrak{M}_{(\alpha)}(z) \\ &= \sup_{\alpha > 0} \sup_{m \geq 1} \frac{\mathbb{E}_{(\alpha)}(Z_1 \mid Z_0 = \lfloor mz \rfloor)}{m} \\ &= \sup_{m \geq 1} \sup_{\alpha > 0} \frac{\mathbb{E}_{(\alpha)}(Z_1 \mid Z_0 = \lfloor mz \rfloor)}{m} \\ &= \sup_{m \geq 1} \frac{\mathbb{E}(Z_1 \mid Z_0 = \lfloor mz \rfloor)}{m}. \end{aligned}$$

Since  $m \mapsto \mathbb{E}(Z_1 \mid Z_0 = \lfloor mz \rfloor)$  defines a superadditive sequence, we deduce that

$$\mathfrak{M}(z) = \lim_{m \rightarrow +\infty} \frac{\mathbb{E}(Z_1 \mid Z_0 = \lfloor mz \rfloor)}{m},$$

which concludes the proof of Corollary 25.  $\square$

*Remark 11.* We remark that thanks to the new characterization of  $\mathfrak{M}$  given in (3.12) we have an upper bound for the expectation of the process. For all  $z \in \mathbb{N}^p$ ,

$$\mathfrak{M}^n(z) \geq \mathbb{E}(Z_n \mid Z_0 = z), \text{ for all } n \in \mathbb{N}. \quad (3.14)$$

We now turn our attention to the primitivity of the function  $\mathfrak{M}$ . We recall that in the last chapter we prove that  $\mathfrak{M}$  is a concave and positively homogeneous function (see Proposition 19). Adding the primitivity of this operator will enable us to apply the concave Perron-Frobenius theory detailed in Section 2.3.1.

For the rest of this section we suppose that Assumption 4 is in place. That is,  $(Z_n)_{n \in \mathbb{N}}$  is a primitive process.

**Proposition 27.** *The function  $\mathfrak{M}$  is a primitive function.*

*Proof.* Since  $(Z_n)_{n \in \mathbb{N}}$  is primitive, we can find  $N, k \in \mathbb{N}$  big enough so that for all  $i \in \{1, \dots, p\}$  and  $m \geq N$  we have that  $\mathbb{E}(Z_m \mid Z_0 = ke_i) > 0$ . Hence, for  $m \geq N$  and  $z \in \mathbb{N}^p \setminus \{0\}$ , using (3.14) and the superadditivity of  $\xi$  and then of  $z \mapsto \mathbb{E}(Z_m \mid Z_0 = z)$ ,

$$k\mathfrak{M}^m(z) = \mathfrak{M}^m(kz) \geq \mathbb{E}(Z_m \mid Z_0 = kz) \geq \sum_{i=1}^p z_i \mathbb{E}(Z_m \mid Z_0 = ke_i) > 0,$$

and so  $\mathfrak{M}^m(z) > 0$ , which concludes the proof.  $\square$

Thanks to Proposition 27, we have the following result which is a direct consequence of Theorem 18 on Chapter 2.

**Proposition 28.** *Assume that  $(Z_n)_{n \in \mathbb{N}}$  is primitive, then there exists a unique  $\lambda^* > 0$  and a unique  $z^* \in S^*$  such that*

$$\mathfrak{M}(z^*) = \lambda^* z^*.$$

We finish this section with a last proposition which is a consequence of the primitivity of  $\mathfrak{M}$ .

**Proposition 29.** *Assume that  $(Z_n)_{n \in \mathbb{N}}$  is primitive. We have  $\inf_{z \in S} |\mathfrak{M}(z)| > 0$  and, for all compact subset  $K \subset S^*$  and for all  $i \in \{1, \dots, p\}$ ,  $\inf_{z \in K} \mathfrak{M}_i(z) > 0$ .*

*Proof.* We start by proving the first assertion. Since  $\mathfrak{M}$  is primitive by Proposition 27, there exists  $n_0 \geq 1$  such that  $\mathfrak{M}^{n_0}(e_i) > 0$  for all  $i \in \{1, \dots, p\}$ . In particular,  $\mathfrak{M}(e_i) \neq 0$  for all  $i \in \{1, \dots, p\}$ . Using the concavity of  $\mathfrak{M}$ , we deduce that

$$\inf_{z \in S} |\mathfrak{M}(z)| \geq \inf_{z \in S} \left| \sum_{i=1}^p z_i \mathfrak{M}(e_i) \right| \geq \min_{i \in \{1, \dots, p\}} |\mathfrak{M}(e_i)| > 0.$$

Let us now prove the second assertion. For  $z \in K$ , if  $\mathfrak{M}_i(z) = +\infty$ , by Proposition 19,  $\mathfrak{M}_i(z') = +\infty$  for all  $z' \in K$  and the result follows directly. If  $\mathfrak{M}_i(z) < +\infty$ , since  $\mathfrak{M}_i$  is concave, it is locally Lipschitz on  $S^*$  and hence  $z \mapsto \mathfrak{M}_i(z)$  is continuous on the compact set  $K$ . It is thus sufficient to prove the result for any fixed  $z \in S^*$ . For this, we simply observe that, for any two  $z, z' \in S^*$ , we have, using the fact that  $z \mapsto \mathfrak{M}_i(z)$  is positively homogeneous and increasing,

$$\mathfrak{M}_i(z) \geq \frac{\min_{j \in \{1, \dots, p\}} z_j}{\max_{j \in \{1, \dots, p\}} z'_j} \mathfrak{M}_i(z').$$

Hence  $z \mapsto \mathfrak{M}_i(z)$  is either null or positive on  $S^*$ . Since  $\mathfrak{M}$  is primitive by Proposition 27,  $z \mapsto \mathfrak{M}_i(z)$  is not null (take for instance  $z = \mathfrak{M}^{n_0}(e_i)$ ), which concludes the proof.  $\square$

### 3.1.4 Simulations

In this section we present simulations to illustrate the results in Theorem 22 for two cases of different mating functions, both in dimension  $p = 2$ . For both cases, for  $k \in \mathbb{N}$ ,  $Z_n^k$  denotes a multi-type bisexual branching process at time  $n \in \mathbb{N}$  and issued from  $Z_0 = z_k$ , where  $(z_k)_{k \in \mathbb{N}}$  is a random sequence such that for every  $k$ ,  $z_k = (z_{k,1}, z_{k,2})$  is distributed as

$$z_{k,1} \sim \text{Binom}(k, p) \text{ and } z_{k,2} = k - z_{k,1} \text{ a.s.},$$

where  $p = 0.6$ . We have that, thanks to the strong law of large numbers that

$$z_k \underset{k \rightarrow +\infty}{\sim} k(p, 1 - p). \tag{3.15}$$

In what follows, we simulate  $Z_n^k$  for different values of  $n$  and  $k$  and for two different mating functions. We start with the perfect fidelity mating (see Example 4) and then we move to the bilateral promiscuous mating (see Example 7). For both cases, and to simplify the computations, we assume that all offspring distributions are independent and distributed according to a Poisson distribution. We recall also that we denote  $\mathbb{X}$  and  $\mathbb{Y}$  the expectation matrices for the offspring of females and males, respectively. Hence, given  $\{Z_n = (z_1, z_2)\}$ , the number of females (resp. of males) of type  $j \in \{1, 2\}$  produced in the next generation is distributed according to a Poisson random variable of parameter  $z_1 \mathbb{X}_{1,j} + z_2 \mathbb{X}_{2,j}$  (resp.  $z_1 \mathbb{Y}_{1,j} + z_2 \mathbb{Y}_{2,j}$ ).

For both processes we use the expectation matrices

$$\mathbb{X} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \quad \mathbb{Y} = \begin{pmatrix} 1 & 1.5 \\ 2 & 2 \end{pmatrix}.$$

#### Multi-type perfect fidelity mating

For this case we consider the  $p = q_f = q_m = 2$  and  $d_1 = d_2 = 1$ . Using Theorem 22 and (3.15), we have that for all  $n \in \mathbb{N}$

$$\frac{Z_n^k}{k} \underset{k \rightarrow +\infty}{\longrightarrow} \mathfrak{M}^n(p, 1 - p),$$

where  $\mathfrak{M}(z) = \min\{z\mathbb{X}, z\mathbb{Y}\}$ .

We simulate  $10^5$  independent copies of  $\frac{Z_n^k}{k}$  for different values of  $n \in \mathbb{N}$  and  $k \in \mathbb{N}$ . For every value of  $k$  and  $n$  simulated we trace the histogram for the sample. As an example, we start considering  $k = 1$  and  $n = 1$ . In both Figure 3.1 and Figure 3.2 in page 69, we have also marked with a red line the value of  $\mathfrak{M}(p, 1 - p)$  corresponding in this case to (1.4, 1.7). We also trace the empirical mean of the sample. We appreciate that, since  $k = 1$  we do not approximate well the value of  $\mathfrak{M}(p, 1 - p)$ , the sample is far from concentrating around this value. We also note that since  $k = 1$  we are starting with only one mating unit, making a large proportion of the simulations equal to 0 after just one step.

In the following we make  $k$  take the values of 5, 50 and 500. We also let  $n$  take the values 1, 5, 10 and 20 and trace the histogram for  $10^5$  simulations. We show the results for only the mating units of type 1, since for type 2 the results are equivalent. In every figure we also trace the corresponding value of  $\mathfrak{M}_1^n(p, 1 - p)$  and the empirical mean of the sample.

In Figure 3.3 in page 70, the values of  $k$  increase from left to right whilst the values of  $n$  do the same from up to bottom. We appreciate that as  $k$  increases, for all values of  $n$ , the sample gathers more closer to the value of  $\mathfrak{M}^n(p, 1 - p)$ . In the tables below, we illustrate the values of the standard deviation for all the samples used in the histograms. We appreciate that the standard deviation decreases with the value of  $k$ .

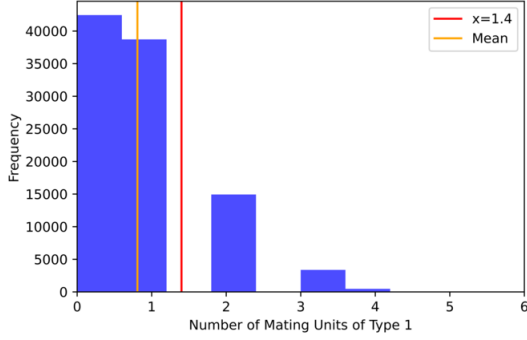


Figure 3.1: Histogram for Type 1 with  $n = k = 1$

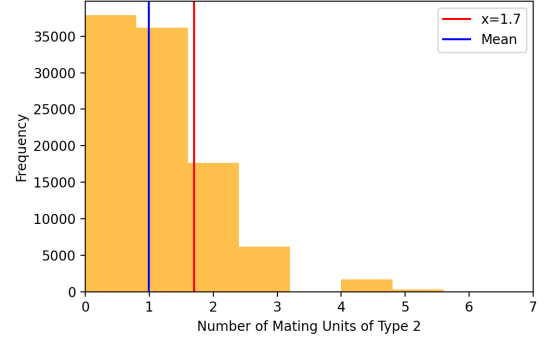


Figure 3.2: Histogram for Type 2 with  $n = k = 1$

Value of $k$	Mean	Std. Dev.
1	0.8	0.85
5	1.26	0.47
50	1.39	0.18
500	1.39	0.05
50000	1.39	0.005

Table 3.1:  $n = 1$ ,  $\mathfrak{M}(p, 1 - p) = 1.4$

Value of $k$	Mean	Std. Dev.
1	62.7	70.3
5	134.9	47.6
50	167.6	17.2
500	171.9	5.4
50000	172.4	0.56

Table 3.2:  $n = 5$ ,  $\mathfrak{M}(p, 1 - p) = 172.4$

Value of $k$	Mean	Std. Dev.
1	24555	27759
5	52891	18635
50	65863	6750.7
500	67601	2157
50000	67754	220.3

Table 3.3:  $n = 10$ ,  $\mathfrak{M}(p, 1 - p) = 67753.8$

Value of $k$	Mean	Std. Dev.
1	$37 \times 10^8$	$42 \times 10^8$
5	$81 \times 10^8$	$28 \times 10^8$
50	$101 \times 10^8$	$10 \times 10^8$
500	$104.4 \times 10^8$	$33 \times 10^7$
50000	$104.6 \times 10^8$	$34 \times 10^6$

Table 3.4:  $n = 20$ ,  $\mathfrak{M}(p, 1 - p) \approx 104.6 \times 10^8$

To observe how the process approaches its corresponding limiting value for larger values of  $k$ , we finish by simulating the value of  $\frac{Z_n^k}{k}$  for values of  $k \in \{1, \dots, 500000\}$  and trace the corresponding graph. For every graph, 20 simulation of the process were used.

In Figure 3.4 in page 71, we appreciate how the value of  $\frac{Z_n^k}{k}$  approaches that of  $\mathfrak{M}^n(p, 1 - p)$  as  $k$  increases.

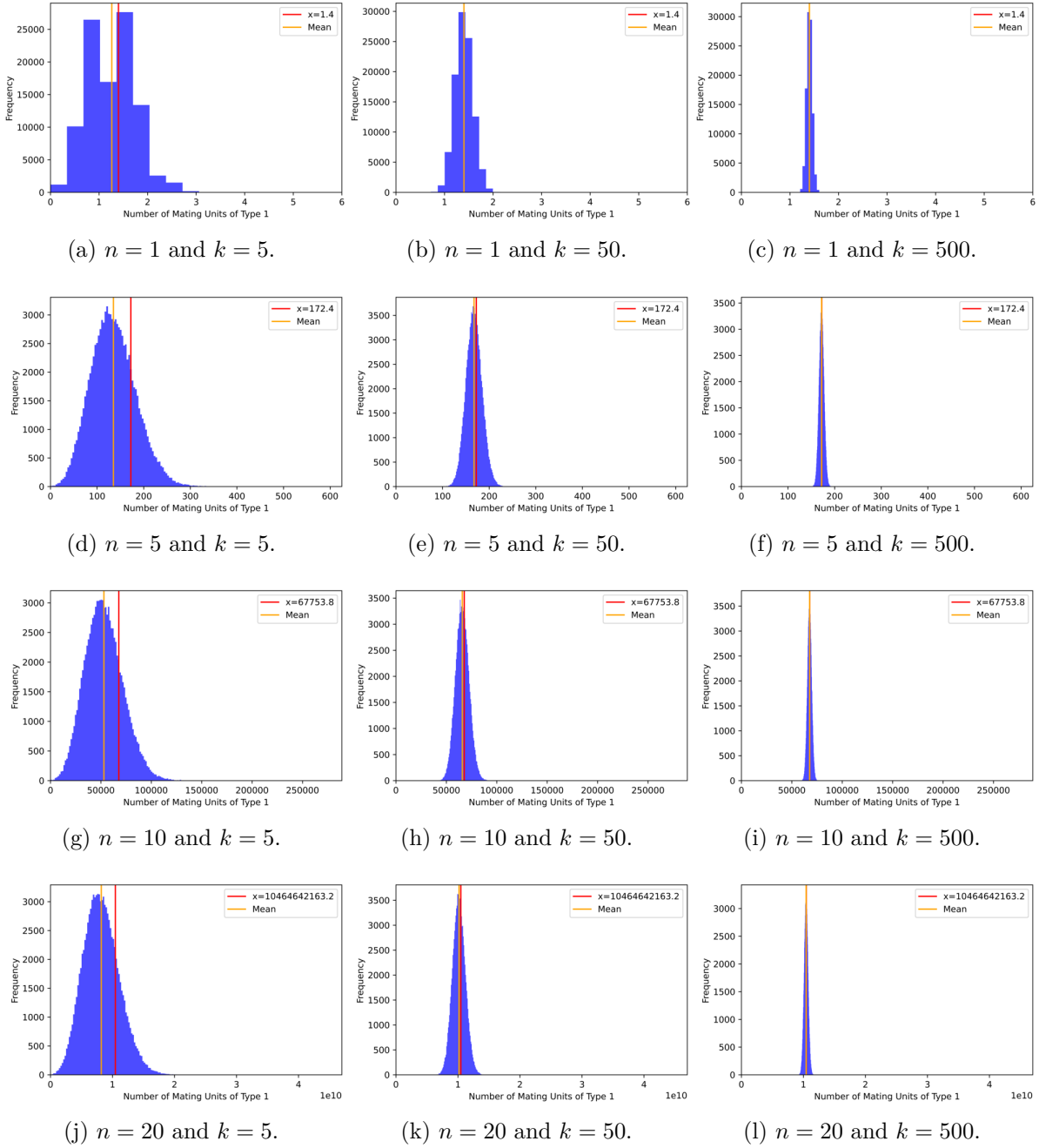
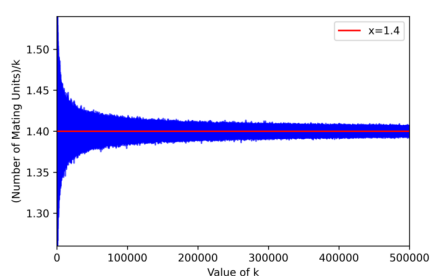
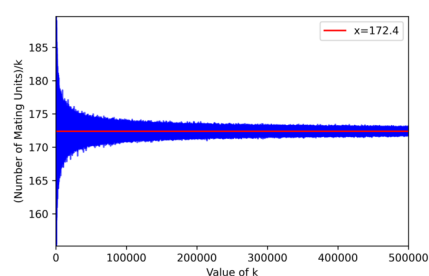


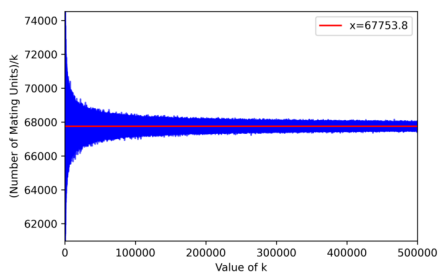
Figure 3.3: Frequency of the value of  $\frac{Z_{n,1}^k}{k}$  for different parameters  $n$  and  $k$  for  $10^5$  simulations of the two-dimensional perfect fidelity mating model starting with  $z_0$  a binomial random variable.



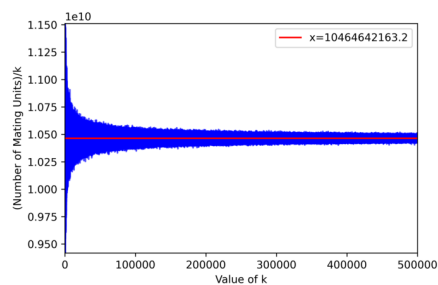
(a)  $n = 1$ .



(b)  $n = 5$ .



(c)  $n = 10$ .



(d)  $n = 20$ .

Figure 3.4: Representation for 20 simulations of  $k \rightarrow \frac{Z_{n,1}^k}{k}$  with different values of  $n$  for the two-dimensional perfect fidelity mating model starting with  $z_0$  a binomial random variable.

### Multi-type bilateral promiscuous mating

For this case we also consider  $p = q_f = q_m = 2$ . Corollary 24 and (3.15) allows us to conclude that

$$\frac{Z_n^k}{k^{2^n}} \xrightarrow[k \rightarrow +\infty]{} \mathfrak{N}^n(2, (p, 1-p)),$$

where for  $z = (z^1, z^2)$ ,  $\mathfrak{N}(2, z) = ((z\mathbb{X})_1(z\mathbb{Y})_1, (z\mathbb{X})_2(z\mathbb{Y})_2)$

For this model we simulate the generations  $n \in \{1, 2, 3\}$  for the values of  $k \in \{1, 10, 100, 1000, 10000\}$ . We illustrate only the mating units of type 1, since for type 2 we obtain similar results.

We appreciate in Figure 3.5 in page 73 that we still have convergence towards the desired value, although it takes a higher value of  $k$  to approach it. This is due mainly to the fastest rate of grow of the process.

## 3.2 Trajectorial Law of Large Numbers

In this section we seek to relate the behaviour of  $\mathfrak{M}$  with the complete trajectory of the process. To accomplish this, we assume for the rest of the chapter that Assumptions 4 and 5 are in place. That is,  $(Z_n)_{n \in \mathbb{N}}$  forms a primitive and transient Markov process.

Using Theorem 18, we can find the eigenelements of  $\mathfrak{M}$ , namely  $\lambda^* > 0$  and  $z^* \in \mathbb{R}_+^p$ . We state a trajectorial law of large numbers for the case where  $\lambda^* > 1$ . As a consequence, and using the additional results of Krause from Theorem 18, we prove that in the event of survival the asymptotic profile of the process converges to  $z^*$ .

Combining this result with those in Chapter 4, we obtain a trajectorial law of large numbers for the general case where  $\lambda^*$  is not necessarily greater than one. This result is stated in Corollary 31.

### 3.2.1 Statement of the Trajectorial LLN

In the following theorem, we deal with the behaviour of the whole process when  $\lambda^* > 1$ . In addition we prove that, on the non-extinction event  $\{Z_n \neq 0, \forall n \geq 0\}$ , the process will almost surely follow the direction of the eigenvector  $z^*$ . We emphasize that, for this result, we do not assume any  $V \log V$  type condition. As far as we know, this result is new even in the single-type case.

In what follows, given  $x, y \in \mathbb{R}^p$ , we denote

$$[x, y] := \{z \in \mathbb{R}^p, x \leq z \leq y\}.$$

**Theorem 30** (Trajectorial Law of Large Numbers). *Assume that  $\mathfrak{M}$  is finite over  $S$ . Let  $\lambda^*$  and  $z^*$  given by Theorem 18 and assume that  $\lambda^* > 1$ . Then there exists  $n_0 \geq 1$  such that, for all  $\varepsilon \in (0, 1)$  and all  $\eta \in (0, 1)$ , there exists  $r > 0$  such that, for all  $|z| \geq r$ ,*

$$\mathbb{P}(Z_{n_0} \neq 0 \text{ and } \forall n \geq n_0, Z_{n+1} \in [(1-\varepsilon)\mathfrak{M}(Z_n), (1+\varepsilon)\mathfrak{M}(Z_n)] \mid Z_0 = z) \geq 1 - \eta. \quad (3.16)$$

*In addition, on the non-extinction event  $\{Z_n \neq 0, \forall n \geq 0\}$ , and up to a  $\mathbb{P}(\cdot \mid Z_0 = z)$ -negligible event, for all  $k \geq 0$ ,*

$$\lim_{n \rightarrow +\infty} \frac{Z_{n+k}}{|Z_n|} = (\lambda^*)^k z^*. \quad (3.17)$$

Let us go a little deeper on the statements of this theorem:



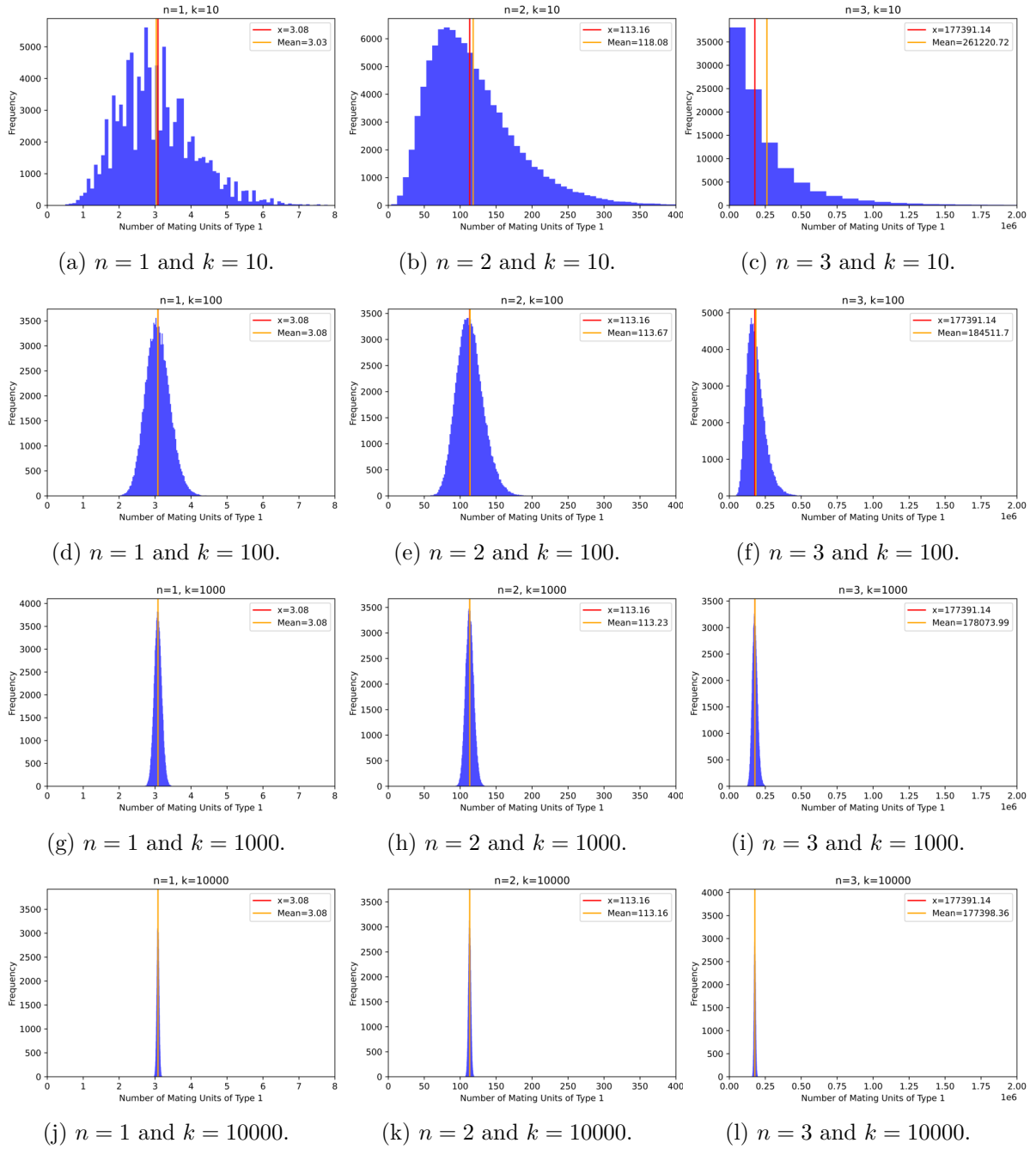


Figure 3.5: Frequency of the value of  $\frac{Z_{n,1}^k}{k}$  for different parameters  $n$  and  $k$  for  $10^5$  simulations of the two-dimensional bilateral promiscuous model starting with  $z_0$  a binomial random variable.

1. For (3.16) we first remark that  $n_0$  does not depend on the parameters  $\varepsilon, \eta$  or  $r$ . In fact, it is a deterministic constant that depends only on  $\mathfrak{M}$ . It will be clear from the proof that it is in fact chosen as the minimal  $n \in \mathbb{N}$  such that  $\mathfrak{M}^n(z) > 0$  for all  $z \in \mathbb{N}^p \setminus \{0\}$ , which exists thanks to the primitivity of  $\mathfrak{M}$ .

2. The theorem states that with high probability (as close to one as desired) we have  $Z_{n_0} \neq 0$  and

$$(1 - \varepsilon)\mathfrak{M}(Z_n) \leq Z_{n+1} \leq (1 + \varepsilon)\mathfrak{M}(Z_n), \text{ for all } n \geq n_0. \quad (3.18)$$

Thus,  $\varepsilon \in (0, 1)$  is a parameter that can be chosen to control the relative distance between  $\mathfrak{M}(Z_n)$  and  $Z_{n+1}$ . The parameter  $\eta \in (0, 1)$  allows to have control on the value of this probability.

3. Once all the parameters are chosen, the theorem provides (3.16) for all  $z \in \mathbb{N}^p$  with  $|z| \geq r$ . We emphasize the fact that only the norm of  $z$  needs to be large to obtain the result, and no lower bound on any specific component of  $z$  is required. This fact is thanks to the primitivity of the process.

4. If one iterates (3.18)  $N$  times, for  $N \in \mathbb{N}$  large enough, one obtains that for all  $n \geq n_0$

$$(1 - \varepsilon)^N \mathfrak{M}^N(Z_n) \leq Z_{n+N}.$$

Thus, using the primitivity of the function  $\mathfrak{M}$ , one can prove that if  $\lambda^* > 1$ , then there is a positive probability that the process survives indefinitely. This fact is discussed in more details in Chapter 4.

5. The result in (3.17) is a consequence of a combination of (3.16) and the properties of  $\mathfrak{M}$  provided by Krause's result. In particular by taking  $k = 0$ , one concludes that on the event of survival

$$\lim_{n \rightarrow +\infty} \frac{Z_n}{|Z_n|} = z^*$$

almost surely. Thus, in the long time limit, the proportions of the mating units present in the population are given by the entries of  $z^*$ . Hence, we recover a known result for asexual branching processes but now in the bisexual setting.

It will be proved in Chapter 4 that the almost sure extinction of the process is equivalent to  $\lambda^* \leq 1$ . On the event of extinction,  $\mathfrak{M}(Z_n)$  vanishes for  $n$  large enough almost surely, which entails that  $Z_{n+1} \in [(1 - \varepsilon)\mathfrak{M}(Z_n), (1 + \varepsilon)\mathfrak{M}(Z_n)]$  for all  $n$  large enough. For  $\lambda^* > 1$  (so that extinction is not almost sure), this also holds true with probability one, as proved alongside Theorem 30. We thus obtain

**Corollary 31.** *Assume that  $(Z_n)_{n \in \mathbb{N}}$  is primitive and transient, and that  $\mathfrak{M}$  is finite over  $S$ . Then, for all  $\varepsilon \in (0, 1)$  and all  $z \in \mathbb{N}^p$ ,*

$$\mathbb{P}(\exists N \geq 0 \text{ such that, } \forall n \geq N, Z_{n+1} \in [(1 - \varepsilon)\mathfrak{M}(Z_n), (1 + \varepsilon)\mathfrak{M}(Z_n)] \mid Z_0 = z) = 1. \quad (3.19)$$

### 3.2.2 Proofs

In order to prove Theorem 30, we first prove an auxiliary result for the case where the initial condition is strictly positive. In other words, all types of mating units are present at the zero-th generation. More precisely we show that for all  $\varepsilon \in (0, 1)$ ,  $\delta \in (0, 1/p]$ ,

$$\lim_{\substack{|z| \rightarrow +\infty \\ z \in U_\delta}} \mathbb{P}(\forall n \in \mathbb{N}, Z_{n+1} \in [(1 - \varepsilon)\mathfrak{M}(Z_n), (1 + \varepsilon)\mathfrak{M}(Z_n)]) = 1, \quad (3.20)$$

where  $U_\delta$  is the set given by

$$U_\delta = \{z \in \mathbb{N}^p : z \geq |z|\delta \mathbf{1}_p\}, \quad (3.21)$$

where we recall that  $\mathbf{1}_p = (1, \dots, 1) \in \mathbb{N}^p$  and for all  $a, b \in \mathbb{R}_+^p$ ,  $[a, b] := \{z \in \mathbb{R}_+^p, a \leq z \leq b\}$ . We remark that, for  $\delta > 0$ ,  $U_\delta$  is non-empty if and only if  $\delta \in (0, 1/p]$ .

This result is in Lemma 35 and is based on Lemmas 32, 33 and 34 for which we introduce the following additional notation.

For any  $\varepsilon \in (0, 1)$  and  $n \geq 1$ , we consider the sequence of events

$$A_n^\varepsilon := \{\forall i \in \{1, \dots, n\}, Z_i \in [(1 - \varepsilon)\mathfrak{M}(Z_{i-1}), (1 + \varepsilon)\mathfrak{M}(Z_{i-1})]\},$$

or simply  $A_n$  when there is no risk of ambiguity. We also set  $A_0^\varepsilon = \Omega$ .

Then, we prove that for any initial value, either the process goes to extinction or it reaches a set  $U_\delta$  in finite time with probability one. This is proved in Lemma 36.

Both results then lead to the proof of Theorem 30.

**Lemma 32.** *Assume that  $\mathfrak{M}$  is bounded on  $S$ . For any  $\delta \in (0, 1/p]$  and  $\varepsilon \in (0, 1)$ , there exists  $c_0 > 0$  such that for all  $z \in U_\delta$  and all  $n \in \mathbb{N}$ ,*

$$\mathbb{P}(A_n^\varepsilon \mid Z_0 = z) \geq 1 - \sum_{i=1}^n c_0 \mathbb{E}\left(\mathbb{1}_{A_{i-1}^\varepsilon} f(|Z_{i-1}|) \mid Z_0 = z\right).$$

where

$$f(x) = x \sum_{i=1}^p \sum_{j=1}^q \mathbb{P}(V_{i,j} > x) + \sum_{i=1}^p \sum_{j=1}^q \frac{\mathbb{E}(V_{i,j}^2 \mathbb{1}_{V_{i,j} \leq x})}{x}.$$

*Proof.* We prove this lemma in two steps.

*Step 1.* We first consider the case  $n = 1$ . That is, we prove that for  $\delta \in (0, 1/p]$  and  $\varepsilon \in (0, 1)$ , there exists  $c_1$  such that, for  $z \in U_\delta$ ,

$$\mathbb{P}(Z_1 \in [(1 - \varepsilon)\mathfrak{M}(z), (1 + \varepsilon)\mathfrak{M}(z)] \mid Z_0 = z) \geq 1 - c_1 |z| \sum_{i=1}^p \sum_{j=1}^q \mathbb{P}(V_{i,j} > |z|) - c_1 \sum_{i=1}^p \sum_{j=1}^q \frac{\mathbb{E}(V_{i,j}^2 \mathbb{1}_{V_{i,j} \leq |z|})}{|z|}. \quad (3.22)$$

For  $Z_0 = z$ , we have  $Z_1 = \xi(W_{1,1}, \dots, W_{1,q})$  with  $W_{1,j} = \sum_{i=1}^p \sum_{k=1}^{z_i} V_{i,j}^{(k,1)}$  for  $1 \leq j \leq q$ . Fix  $\delta_1 \in (0, 1)$  and  $r_1 > 0$  (depending on  $\delta_1$ ) such that, for all  $z \in \mathbb{N}^p$  with  $|z| \geq r_1$ ,

$$(1 - \delta_1)\mathbb{V}_{i,j} \leq \mathbb{V}_{i,j}^{\leq |z|} := \mathbb{E}(V_{i,j} \mathbb{1}_{V_{i,j} \leq |z|}) \leq (1 + \delta_1)\mathbb{V}_{i,j}.$$

For all  $|z| \geq r_1$  with  $z \in U_\delta$ , we have

$$\begin{aligned} \mathbb{P}\left((W_{1,1}, \dots, W_{1,q}) \geq (1 - \delta_1)^2 z \mathbb{V} \mid Z_0 = z\right) \\ \geq \mathbb{P}\left((W_{1,1}, \dots, W_{1,q}) \geq (1 - \delta_1) z \mathbb{V}^{\leq |z|} \mid Z_0 = z\right) \\ \geq 1 - \sum_{j=1}^q \mathbb{P}\left(W_{1,j} < (1 - \delta_1) \sum_{i=1}^p z_i \mathbb{V}_{i,j}^{\leq |z|} \mid Z_0 = z\right) \end{aligned}$$

but for  $j \in \{1, \dots, q\}$ ,

$$\begin{aligned}
 \mathbb{P}(W_{1,j} < (1 - \delta_1) \sum_{i=1}^p z_i \mathbb{V}_{i,j}^{\leq |z|} \mid Z_0 = z) &\leq \mathbb{P}(\exists i \in \{1, \dots, p\}, k \in \{1, \dots, z_i\} \text{ s.t. } V_{i,j}^{(k,1)} > |z| \mid Z_0 = z) \\
 &\quad + \mathbb{P}\left(\sum_{i=1}^p \sum_{k=1}^{z_i} \left(V_{i,j}^{(k,1)} \mathbb{1}_{V_{i,j}^{(k,1)} \leq |z|} - \mathbb{V}_{i,j}^{\leq |z|}\right) < -\delta_1 \sum_{i=1}^p z_i \mathbb{V}_{i,j}^{\leq |z|} \mid Z_0 = z\right) \\
 &\leq \sum_{i=1}^p z_i \mathbb{P}(V_{i,j} > |z|) + \frac{\text{Var}\left(\sum_{i=1}^p \sum_{k=1}^{z_i} V_{i,j}^{(k,1)} \mathbb{1}_{V_{i,j}^{(k,1)} \leq |z|} \mid Z_0 = z\right)}{\delta_1^2 \left(\sum_{i=1}^p z_i \mathbb{V}_{i,j}^{\leq |z|}\right)^2} \\
 &\leq |z| \sum_{i=1}^p \mathbb{P}(V_{i,j} > |z|) + \frac{\sum_{i=1}^p \sum_{k=1}^{z_i} \text{Var}\left(V_{i,j}^{(k,1)} \mathbb{1}_{V_{i,j}^{(k,1)} \leq |z|} \mid Z_0 = z\right)}{\delta_1^2 \delta^2 (1 - \delta_1)^2 \left(\sum_{i=1}^p \mathbb{V}_{i,j}\right)^2 |z|^2} \\
 &\leq |z| \sum_{i=1}^p \mathbb{P}(V_{i,j} > |z|) + \frac{\sum_{i=1}^p \mathbb{E}\left(V_{i,j}^2 \mathbb{1}_{V_{i,j} \leq |z|}\right)}{\delta_1^2 \delta^2 (1 - \delta_1)^2 \left(\sum_{i=1}^p \mathbb{V}_{i,j}\right)^2 |z|},
 \end{aligned}$$

where we used the independence of the random variables  $V_{i,j}^{(k,1)}$ , the fact that  $z_i \geq \delta|z|$  for all  $i \in \{1, \dots, p\}$ , and  $\mathbb{V}_{i,j}^{\leq |z|} \geq (1 - \delta_1)\mathbb{V}_{i,j}$ . Proceeding similarly for the event  $\{(W_{1,1}, \dots, W_{1,q}) \leq (1 + \delta_1)^2 z \mathbb{V}\}$ , we deduce that there exists a constant  $c > 0$  such that

$$\mathbb{P}(W_1 \in [(1 - \delta_1)^2 z \mathbb{V}, (1 + \delta_1)^2 z \mathbb{V}] \mid Z_0 = z) \geq 1 - c|z| \sum_{i=1}^p \sum_{j=1}^q \mathbb{P}(V_{i,j} > |z|) - c \sum_{i=1}^p \sum_{j=1}^q \frac{\mathbb{E}(V_{i,j}^2 \mathbb{1}_{V_{i,j} \leq |z|})}{|z|},$$

where  $W_1 = (W_{1,1}, \dots, W_{1,q})$ .

If we now apply Proposition 20 with the compact set  $U_\delta \cap S$ , for all  $\varepsilon' > 0$ , there exists  $r_2 > 0$  such that if  $\ell > r_2(1 - \delta_1)^2$ ,

$$\left| \frac{\xi(\ell u \mathbb{V})}{\ell} - \mathfrak{M}(u) \right| \leq \varepsilon', \quad \forall u \in U_\delta \cap S,$$

and we deduce that for all  $z \in U_\delta$  such that  $|z| \geq r_2$ ,

$$\left| \frac{\xi((1 - \delta_1)^2 z \mathbb{V})}{(1 - \delta_1)^2 |z|} - \frac{\mathfrak{M}(z)}{|z|} \right| \leq \varepsilon'.$$

Hence, for all  $z \in U_\delta$  with  $|z| \geq r_2$ ,

$$\frac{\xi((1 - \delta_1)^2 z \mathbb{V})}{(1 - \delta_1)^2 |z|} \geq \mathfrak{M}(z/|z|) - \varepsilon'.$$

In addition, since  $\mathfrak{M}$  is concave on  $S$ , it is locally Lipschitz on  $S^*$  and in particular each of its components are uniformly bounded away from 0 on  $U_\delta \cap S \subset S^*$  by a constant  $m_1 > 0$  (which depends on  $\delta \in (0, 1/p]$ ). Hence, for all  $z \in U_\delta$  with  $|z| \geq r_2$ ,

$$\frac{\xi((1 - \delta_1)^2 z \mathbb{V})}{(1 - \delta_1)^2 |z|} \geq \mathfrak{M}(z/|z|)(1 - \varepsilon'/m_1).$$

Similarly, there exist  $r_3 > 0$  and  $m_2 > 0$  such that, for all  $z \in U_\delta$  with  $|z| \geq r_3$ ,

$$\frac{\xi((1 + \delta_1)^2 z \mathbb{V})}{(1 + \delta_1)^2 |z|} \leq \mathfrak{M}(z/|z|)(1 + \varepsilon'/m_2).$$

Hence, for  $z \in U_\delta$  with  $|z| \geq r_1 \vee r_2 \vee r_3$ ,

$$\begin{aligned} \{(W_{1,1}, \dots, W_{1,q}) \in [(1 - \delta_1)^2 z \mathbb{V}, (1 + \delta_1)^2 z \mathbb{V}]\} \\ \subset \{Z_1 \in [\mathfrak{M}(z)(1 - \varepsilon'/m_1)(1 - \delta_1)^2, \mathfrak{M}(z)(1 + \varepsilon'/m_2)(1 + \delta_1)^2]\}. \end{aligned}$$

Choosing  $\varepsilon'$  and  $\delta_1$  small enough, we deduce that there exists  $r_4 > 0$  such that, for all  $z \in U_\delta$  with  $|z| \geq r_4$ , (3.22) holds true for some constant  $c_1$ . Up to a change in the constant  $c_1$ , we deduce that this is true for all  $z \in U_\delta$ .

*Step 2.* We iterate now the result obtained in the previous step. We have, for all  $z \in U_\delta$ .

$$\mathbb{P}(Z_1 \in [(1 - \varepsilon)\mathfrak{M}(z), (1 + \varepsilon)\mathfrak{M}(z)] \mid Z_0 = z) \geq 1 - c_1 f(|z|)$$

for some constant  $c_1 > 0$ . Then, observing that  $Z_1 \in [(1 - \varepsilon)\mathfrak{M}(z), (1 + \varepsilon)\mathfrak{M}(z)]$  implies that  $Z_1 \in [|z|(1 - \varepsilon)\mathfrak{M}(z/|z|), (1 + \varepsilon)|z|\mathfrak{M}(z/|z|)]$  and hence that

$$Z_1/|Z_1| \geq \frac{(1 - \varepsilon)\mathfrak{M}(z/|z|)}{(1 + \varepsilon)|\mathfrak{M}(z/|z|)} \geq \delta'_1 \mathbf{1}_p, \quad (3.23)$$

with

$$\delta'_1 := \frac{(1 - \varepsilon) \min_{u \in U_\delta \cap S, i \in \{1, \dots, p\}} \mathfrak{M}_i(u)}{(1 + \varepsilon) \sup_{u \in S} |\mathfrak{M}(u)|}.$$

where  $\min_{u \in U_\delta \cap S, i \in \{1, \dots, p\}} \mathfrak{M}_i(u) > 0$  by Proposition 29. Now, applying the same reasoning as in *Step 1* but with  $\delta_1$  instead of  $\delta$ , we deduce that there exists a constant  $c_2 > 0$  such that, on the event  $\{Z_0 = z\}$ ,

$$\begin{aligned} \mathbb{P}(Z_2 \in [(1 - \varepsilon)\mathfrak{M}(Z_1), (1 + \varepsilon)\mathfrak{M}(Z_1)] \mid Z_1) &\geq \mathbf{1}_{Z_1 \in [(1 - \varepsilon)\mathfrak{M}(z), (1 + \varepsilon)\mathfrak{M}(z)]} (1 - c_2 f(|Z_1|)) \\ &= \mathbf{1}_{A_1^\varepsilon} (1 - c_2 f(|Z_1|)). \end{aligned}$$

And hence, using Markov's property at time 1,

$$\begin{aligned} \mathbb{P}(A_2^\varepsilon \mid Z_0 = z) &\geq \mathbb{E} \left( \mathbb{P}(Z_2 \in [(1 - \varepsilon)\mathfrak{M}(Z_1), (1 + \varepsilon)\mathfrak{M}(Z_1)] \mid Z_1) \mathbf{1}_{A_1^\varepsilon} \mid Z_0 = z \right) \\ &\geq \mathbb{P}(A_1^\varepsilon \mid Z_0 = z) - \mathbb{E} \left( \mathbf{1}_{A_1^\varepsilon} c_2 f(|Z_1|) \mid Z_0 = z \right) \\ &\geq 1 - c_1 f(|z|) - c_2 \mathbb{E} \left( \mathbf{1}_{A_1^\varepsilon} f(|Z_1|) \mid Z_0 = z \right) \end{aligned}$$

Iterating this procedure, we deduce that there exists a positive sequence  $(c_n)_{n \geq 1}$  such that, for all  $n \geq 1$ ,

$$\mathbb{P}(A_n^\varepsilon \mid Z_0 = z) \geq 1 - \sum_{i=1}^n c_i \mathbb{E} \left( \mathbf{1}_{A_{i-1}^\varepsilon} f(|Z_{i-1}|) \mid Z_0 = z \right). \quad (3.24)$$

According to Theorem 18 (5), there exists  $n_0$  such that

$$\sup_{u \in S} |(\lambda^*)^{-n_0} \mathfrak{M}^{n_0}(u) - \mathcal{P}(u)z^*| \leq \frac{1}{2} \inf_S \mathcal{P} \min_{i \in \{1, \dots, p\}} z_i^*,$$

so that  $\mathfrak{M}^{n_0}(u) \in [(\lambda^*)^{n_0} \mathcal{P}(u)z^*/2, 3(\lambda^*)^{n_0} \mathcal{P}(u)z^*/2]$ ,  $\forall u \in S$ . Then, on  $A_{n_0}^\varepsilon$ , we have

$$Z_{n_0} \in [(1 - \varepsilon)^{n_0} \mathfrak{M}^{n_0}(z), (1 + \varepsilon)^{n_0} \mathfrak{M}^{n_0}(z)] \subset [(1 - \varepsilon)^{n_0} (\lambda^*)^{n_0} \mathcal{P}(z)z^*/2, 3(1 + \varepsilon)^{n_0} (\lambda^*)^{n_0} \mathcal{P}(z)z^*/2],$$

so that (recall that  $|z^*| = 1$ )

$$Z_{n_0}/|Z_{n_0}| \geq \frac{(1 - \varepsilon)^{n_0} (\lambda^*)^{n_0} \mathcal{P}(z)z^*/2}{3(1 + \varepsilon)^{n_0} (\lambda^*)^{n_0} \mathcal{P}(z)/2} \geq \min z^* \frac{(1 - \varepsilon)^{n_0}}{3(1 + \varepsilon)^{n_0}} \mathbf{1}_p,$$

that is  $Z_{n_0} \in U_{\delta'}$  with  $\delta' := \min z^* \frac{(1 - \varepsilon)^{n_0}}{3(1 + \varepsilon)^{n_0}}$  which does not depend on  $\delta$ . By the same argument, on the event  $A_{n_0+1}^\varepsilon$  we have  $Z_{n_0+1} \in U_{\delta'}$  as  $Z_1 \in U_{\delta'_1}$  by (3.23). Iterating this procedure, we more generally obtain that on  $A_{n_0+k}^\varepsilon$  we have  $Z_{n_0+k} \in U_{\delta'}$  for all  $k \geq 0$ . Then we deduce from Step 1 that there exists  $c'$  (which depends on  $\delta'$  but does not depend on  $\delta$ ) such that for all  $k \geq 0$

$$\mathbb{P}(A_{n_0+k+1}^\varepsilon \mid Z_0, \dots, Z_{n_0+k}) \geq \mathbf{1}_{A_{n_0+k}^\varepsilon} (1 - c' f(|Z_{n_0+k}|)).$$

In the same way as for (3.24), we then have, for all  $n \geq n_0 + 1$ ,

$$\mathbb{P}(A_n^\varepsilon \mid Z_0 = z) \geq 1 - \sum_{i=1}^{n_0} c_i \mathbb{E} \left( \mathbf{1}_{A_{i-1}^\varepsilon} f(|Z_{i-1}|) \mid Z_0 = z \right) - \sum_{i=n_0+1}^n c' \mathbb{E} \left( \mathbf{1}_{A_{i-1}^\varepsilon} f(|Z_{i-1}|) \mid Z_0 = z \right)$$

and the lemma holds for  $c_0 = \max_{i \in \{1, \dots, n_0\}} c_i \vee c'$ . □

We prove now a useful auxiliary lemma.

**Lemma 33.** *Let  $(x_n)_{n \geq 0}$  be a positive sequence such that, for some  $\alpha > 1$  and  $c > 0$ ,*

$$x_n \geq c\alpha^{n-k} x_k, \quad \forall n \geq k \geq 0. \quad (3.25)$$

Then

$$\sum_{n \geq 0} f(x_n) \leq \frac{2\alpha}{c(\alpha - 1)} \sum_{i=1}^p \sum_{j=1}^q \mathbb{E} \left( \frac{V_{i,j}^2}{(cx_0) \vee V_{i,j}} \right)$$

with  $f$  defined in Lemma 32.

*Proof.* We first consider the first part and then the second part of  $f(x_n)$ , for each  $i \in \{1, \dots, p\}$  and  $j \in \{1, \dots, q\}$ .

We have, using Fubini's Theorem,

$$\sum_{n \geq 0} x_n \mathbb{P}(V_{i,j} > x_n) = \mathbb{E} \left( \sum_{n \geq 0} x_n \mathbf{1}_{V_{i,j} > x_n} \right) \leq \mathbb{E} \left( \mathbf{1}_{V_{i,j} > \min_{n \in \mathbb{N}} x_n} \sum_{n=0}^{N_{i,j}} x_n \right),$$

where  $N_{i,j} := \max\{n \geq 0, x_n < V_{i,j}\}$ , with the convention that  $\max \emptyset = -1$  (note that, since  $\alpha > 1$ ,  $x_n \rightarrow +\infty$  so that  $N_{i,j} < +\infty$ ). Inequality (3.25) entails that  $\min_{n \in \mathbb{N}} x_n \geq cx_0$  and that, for all  $n \leq N_{i,j}$ ,  $x_n \leq \alpha^{n-N_{i,j}} x_{N_{i,j}}/c \leq \alpha^{n-N_{i,j}} V_{i,j}/c$ . We deduce that

$$\begin{aligned} \sum_{n \geq 0} x_n \mathbb{P}(V_{i,j} > x_n) &\leq \mathbb{E} \left( \mathbb{1}_{V_{i,j} > cx_0} V_{i,j} \sum_{n=0}^{N_{i,j}} \alpha^{n-N_{i,j}}/c \right) \\ &\leq \mathbb{E} \left( \mathbb{1}_{V_{i,j} > cx_0} V_{i,j} \frac{\alpha}{c(\alpha-1)} \right) \\ &\leq \frac{\alpha}{c(\alpha-1)} \mathbb{E} \left( \frac{V_{i,j}^2}{(cx_0) \vee V_{i,j}} \right), \end{aligned}$$

where we used the fact that  $\mathbb{1}_{V_{i,j} > cx_0} \leq \frac{V_{i,j}}{(cx_0) \vee V_{i,j}}$  almost surely.

Using again Fubini's Theorem, we have

$$\sum_{n \geq 0} \frac{\mathbb{E} \left( V_{i,j}^2 \mathbb{1}_{V_{i,j} \leq x_n} \right)}{x_n} = \mathbb{E} \left( V_{i,j}^2 \sum_{n \geq 0} \frac{\mathbb{1}_{V_{i,j} \leq x_n}}{x_n} \right) \leq \mathbb{E} \left( V_{i,j}^2 \sum_{n=N'_{i,j}}^{+\infty} \frac{1}{x_n} \right),$$

where  $N'_{i,j} = \min\{n \geq 0, x_n \geq V_{i,j}\} = N_{i,j} + 1$ . Inequality (3.25) entails that, for all  $n \geq N'_{i,j}$ ,  $1/x_n \leq \alpha^{N'_{i,j}-n}/(cx_{N'_{i,j}})$ . Hence, using the fact that  $x_{N'_{i,j}} \geq V_{i,j}$  by definition of  $N'_{i,j}$ , we obtain

$$\sum_{n \geq 0} \frac{\mathbb{E} \left( V_{i,j}^2 \mathbb{1}_{V_{i,j} \leq x_n} \right)}{x_n} \leq \mathbb{E} \left( V_{i,j}^2 \frac{1}{x_{N'_{i,j}} \vee V_{i,j}} \sum_{n=N'_{i,j}}^{+\infty} \frac{\alpha^{N'_{i,j}-n}}{c} \right) \leq \mathbb{E} \left( \frac{V_{i,j}^2}{(cx_0) \vee V_{i,j}} \frac{\alpha}{c(\alpha-1)} \right),$$

where we used the fact that  $x_{N'_{i,j}} \geq cx_0$  by (3.25).

Summing over  $i \in \{1, \dots, p\}$  and  $j \in \{1, \dots, q\}$ , this concludes the proof of Lemma 33.  $\square$

Now we state a second auxiliary lemma, where we prove that for all  $z \in \mathbb{N}^p$ , the sequence  $(\mathfrak{M}^n(z))_{n \in \mathbb{N}}$  holds the property (3.25).

**Lemma 34.** *Assume that  $\mathfrak{M}$  is bounded on  $S$  and  $\lambda^* > 1$ . There exists a constant  $c_0 \in (0, 1]$  and  $\lambda \in (1, \lambda^*)$  such that, for all  $z \in \mathbb{N}^p$  and all  $n \geq 1$  and  $k \in \{0, \dots, n\}$ ,*

$$|\mathfrak{M}^n(z)| \geq c_0 \lambda^{n-k} |\mathfrak{M}^k(z)|.$$

*Proof.* Let  $\delta \in (0, 1)$  such that  $\lambda := \frac{1-\delta}{1+\delta} \lambda^* > 1$ . Fix  $z \in \mathbb{N}^p \setminus \{0\}$ . If there exists  $x > 0$  such that  $z \in [(1-\delta)xz^*, (1+\delta)xz^*]$ , then  $|z| \leq |(1+\delta)xz^*| = (1+\delta)x$ , and, since  $\mathfrak{M}$  is increasing and positively homogeneous,

$$|\mathfrak{M}(z)| \geq |\mathfrak{M}((1-\delta)xz^*)| = |(1-\delta)x\mathfrak{M}(z^*)| = |(1-\delta)x\lambda^*z^*| = (1-\delta)x\lambda^* \geq \lambda|z|. \quad (3.26)$$

Moreover, Theorem 18 (5) entails that there exists  $n_0 \geq 1$  such that, for any  $z \in S$  and  $n \geq n_0$ ,

$$\mathfrak{M}^n(z) \in [(1-\delta)(\lambda^*)^n C(z)z^*, (1+\delta)(\lambda^*)^n C(z)z^*],$$

so that, according to (3.26), for  $n \geq k \geq n_0$

$$|\mathfrak{M}^{n+1}(z)| \geq \lambda |\mathfrak{M}^n(z)| \geq \lambda^{n+1-k} |\mathfrak{M}^k(z)| \geq \lambda^{n+1-n_0} |\mathfrak{M}^{n_0}(z)|,$$

By homogeneity of  $\mathfrak{M}$ , this extends to all  $z \in \mathbb{N}^p$ .

For all  $n \in \{0, \dots, n_0\}$  and all  $z \in (\mathbb{N} \setminus \{0\})^p$ , we have, for all  $k \in \{0, \dots, n\}$ ,

$$\begin{aligned} |\mathfrak{M}^{n+1}(z)| &\geq \left| \mathfrak{M} \left( \frac{\mathfrak{M}^n(z)}{|\mathfrak{M}^n(z)|} \right) \right| |\mathfrak{M}^n(z)| \geq \inf_S |\mathfrak{M}| |\mathfrak{M}^n(z)| \geq (\inf_S |\mathfrak{M}|)^{n+1-k} |\mathfrak{M}^k(z)| \\ &\geq c_1^{n+1-k} \lambda^{n+1-k} |\mathfrak{M}^k(z)|, \end{aligned}$$

where  $c_1 = \lambda^{-1} \inf_S |\mathfrak{M}|$ . Setting  $c_0 = 1 \wedge c_1^{n_0}$  concludes the proof of Lemma 34.  $\square$

We are now in position to compute the limit (3.20).

**Lemma 35.** *Assume that  $\mathfrak{M}$  is bounded on  $S$  and  $\lambda^* > 1$ . For any  $\delta \in (0, 1/p]$  and  $\varepsilon \in (0, 1)$ , we have*

$$\mathbb{P} \left( \bigcap_{n \geq 1} A_n^\varepsilon \mid Z_0 = z \right) \xrightarrow[|z| \rightarrow +\infty, z \in U_\delta]{} 1.$$

*Proof.* Take  $c_0 > 0$  and  $\lambda \in (1, \lambda^*)$  from Lemma 34. We assume without loss of generality that  $\alpha := (1 - \varepsilon)\lambda > 1$ . For all  $i \geq k \geq 1$ , on the event  $A_i$ , we have then

$$|Z_i| \geq |(1 - \varepsilon)^k \mathfrak{M}^k(Z_{i-k})| \geq c_0 \lambda^k (1 - \varepsilon)^k |Z_{i-k}| = c_0 \alpha^k |Z_{i-k}|.$$

Hence, according to Lemma 33, almost surely, on the event  $\{Z_0 = z\}$ ,

$$\sum_{n \geq 1} \mathbb{1}_{A_{n-1}} f(|Z_{n-1}|) \leq \frac{2\alpha}{c_0(\alpha - 1)} \sum_{i=1}^p \sum_{j=1}^q \mathbb{E} \left( \frac{V_{i,j}^2}{(c_0|z|) \vee V_{i,j}} \right).$$

We deduce that

$$\sum_{n \geq 1} \mathbb{E} (\mathbb{1}_{A_{n-1}} f(|Z_{n-1}|) \mid Z_0 = z) \leq \frac{2\alpha}{c_0(\alpha - 1)} \sum_{i=1}^p \sum_{j=1}^q \mathbb{E} \left( \frac{V_{i,j}^2}{(c_0|z|) \vee V_{i,j}} \right).$$

Letting  $n \rightarrow +\infty$  in Lemma 32, we obtain that there exists  $c'_0 > 0$  such that, for all  $z \geq \delta|z| \mathbf{1}_p$  (recall that  $A_{n-1} \supset A_n$  for all  $n \geq 1$ ),

$$\begin{aligned} \mathbb{P} \left( \bigcap_{n \geq 1} A_n \mid Z_0 = z \right) &= \lim_{n \rightarrow +\infty} \mathbb{P} \left( \bigcap_{k=1}^n A_k \mid Z_0 = z \right) \\ &\geq 1 - c'_0 \sum_{n \geq 1} \mathbb{E} (\mathbb{1}_{A_{n-1}} f(|Z_{n-1}|) \mid Z_0 = z) \\ &\geq 1 - \frac{2c'_0 \alpha}{c_0(\alpha - 1)} \sum_{i,j} \mathbb{E} \left( \frac{V_{i,j}^2}{(c_0|z|) \vee V_{i,j}} \right). \end{aligned}$$

But  $V_{i,j}$  is integrable for all  $i, j$  and hence, by dominated convergence theorem,

$$\mathbb{E} \left( \frac{V_{i,j}^2}{(c_0|z|) \vee V_{i,j}} \right) \xrightarrow[|z| \rightarrow +\infty]{} 0.$$

This concludes the proof of Lemma 35.  $\square$



Lemma 35 is useful for large starting values  $z$  such that  $z \geq \delta|z|\mathbf{1}_p$ . In order to use it for all initial positions, we show that such values are eventually reached by the process in finite time.

**Lemma 36.** *Assume that  $\mathfrak{M}$  is bounded on  $S$ . There exist  $\delta_0 \in (0, 1/p]$  and  $n_0 \geq 1$  such that, for all  $r > 0$ , for all  $\eta \in (0, 1)$ , there exists  $\rho > 0$  such that*

$$\inf_{z \in \mathbb{N}^p, |z| \geq \rho} \mathbb{P}(\tau_{\delta_0, r} \leq n_0 \mid Z_0 = z) \geq 1 - \eta,$$

where

$$\tau_{\delta_0, r} = \inf\{n \geq 0, Z_n \geq |Z_n|\delta_0\mathbf{1}_p \text{ and } |Z_n| \geq r\}.$$

In addition,

$$\inf_{z \in \mathbb{N}^p} \mathbb{P}(\tau_{\delta_0, r} \wedge T_0 < +\infty \mid Z_0 = z) = 1,$$

with  $T_0 := \inf\{n \geq 0, |Z_n| = 0\}$  the extinction time of the process.

*Proof.* Let  $n_0 \geq 1$  and  $\delta_1 > 0$  such that, for all  $i \in \{1, \dots, p\}$ ,  $\mathfrak{M}^{n_0}(e_i) > \delta_1\mathbf{1}_p$ . Such  $n_0$  and  $\delta_1$  exist as  $\mathfrak{M}$  is primitive by Proposition 27. In order to prove the first inequality of the lemma, we show that there exists  $\alpha > 0$  such that, for all  $\eta \in (0, 1)$ , there exists  $k_1 > 0$  such that

$$\mathbb{P}(Z_{n_0} \geq |z|\delta_2\mathbf{1}_p \mid Z_0 = z) \geq 1 - \eta/2, \quad \forall |z| \geq k_1 \quad (3.27)$$

with  $\delta_2 := \delta_1/2p$ , and there exists  $k_2 > 0$  such that

$$\mathbb{P}(\alpha|z| > |Z_{n_0}| \mid Z_0 = z) \geq 1 - \eta/2 \quad \forall |z| \geq k_2. \quad (3.28)$$

Once this is proved, we set  $\delta_0 = \delta_2/\alpha$  (which does not depend on  $\eta$ ). Then for  $r > 0$ , setting  $\rho = k_1 \vee k_2 \vee r/\delta_2p$ , we obtain from the general property  $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$  that, for all  $z$  such that  $|z| \geq \rho$ ,

$$\begin{aligned} \mathbb{P}(\tau_{\delta_0, r} \leq n_0 \mid Z_0 = z) &\geq \mathbb{P}(Z_{n_0} \geq |Z_{n_0}|\delta_0\mathbf{1}_p \text{ and } |Z_{n_0}| \geq r \text{ and } |Z_{n_0}| < \alpha|z| \mid Z_0 = z) \\ &\geq \mathbb{P}(Z_{n_0} \geq \alpha|z|\delta_0\mathbf{1}_p \text{ and } |Z_{n_0}| < \alpha|z| \mid Z_0 = z) \\ &\geq \mathbb{P}(Z_{n_0} \geq |z|\delta_2\mathbf{1}_p \mid Z_0 = z) + \mathbb{P}(|Z_{n_0}| < \alpha|z| \mid Z_0 = z) - 1 \\ &\geq 1 - \eta, \end{aligned}$$

where we used for the second inequality that  $|\alpha|z|\delta_0\mathbf{1}_p| \geq r$  for all  $|z| \geq \rho$ .

Let us prove (3.27) and (3.28). By definition of  $n_0 \geq 1$  and  $\delta_1 > 0$  and by Theorem 22, we deduce that there exists  $k_0 \geq 1$  such that, for all  $k \geq k_0$  and all  $i \in \{1, \dots, p\}$ ,

$$\mathbb{P}(Z_{n_0} \geq k\delta_1\mathbf{1}_p/2 \mid Z_0 = ke_i) \geq (1 - \eta/2)^{1/p}.$$

In Chapter 2, Lemma 17 we proved that for  $z_0, \tilde{z}_0, z_1, \tilde{z}_1 \in \mathbb{N}^p$  we have,

$$\mathbb{P}(Z_n \geq z_1 + \tilde{z}_1 \mid Z_0 = z_0 + \tilde{z}_0) \geq \mathbb{P}(Z_n \geq z_1 \mid Z_0 = z_0) \times \mathbb{P}(Z_n \geq \tilde{z}_1 \mid Z_0 = \tilde{z}_0).$$

Iterating this result, we obtain

$$\begin{aligned} \mathbb{P}\left(Z_{n_0} \geq \sum_{i=1}^p \mathbf{1}_{z_i \geq k_0} z_i \delta_1 \mathbf{1}_p / 2 \mid Z_0 = z\right) &\geq \prod_{i=1}^p \mathbb{P}(Z_{n_0} \geq \mathbf{1}_{z_i \geq k_0} z_i \delta_1 \mathbf{1}_p / 2 \mid Z_0 = z_i e_i) \\ &\geq \prod_{i=1}^p (1 - \eta/2)^{1/p} = 1 - \eta/2. \end{aligned}$$

If in addition  $|z| \geq k_1 := pk_0$ , then

$$\frac{\delta_1}{2} \sum_{i=1}^p \mathbb{1}_{z_i \geq k_0} z_i \geq \frac{\delta_1}{2} (|z| - (p-1)k_0) = \frac{\delta_1}{2} |z| \left( 1 - \frac{(p-1)k_0}{|z|} \right) \geq \frac{\delta_1}{2p} |z| = \delta_2 |z|$$

and (3.27) holds.

Set  $\alpha = 2|\mathfrak{M}^{n_0}(\mathbf{1}_p)|$  (note that by definition of  $n_0$ , we have  $\mathfrak{M}^{n_0}(\mathbf{1}_p) > 0$ ). By superadditivity, we have

$$\mathbb{P}(|Z_{n_0}| \geq \alpha|z| \mid Z_0 = z) \leq \mathbb{P}(|Z_{n_0}| \geq \alpha|z| \mid Z_0 = |z|\mathbf{1}_p).$$

Now Theorem 22 entails that

$$\mathbb{P}(|Z_{n_0}| \geq \alpha m \mid Z_0 = m\mathbf{1}_p) = \mathbb{P}\left(\frac{|Z_{n_0}|}{m} \geq 2|\mathfrak{M}^{n_0}(\mathbf{1}_p)| \mid Z_0 = m\mathbf{1}_p\right) \xrightarrow{m \rightarrow +\infty} 0.$$

Using this, we conclude that for all  $\eta \in (0, 1)$ , there exists  $k_2 > 0$  such that, for all  $|z| \geq k_2$ ,

$$\mathbb{P}(|Z_{n_0}| \geq \alpha|z| \mid Z_0 = z) \leq \eta/2.$$

and (3.28) holds, which concludes the proof of the first part of the lemma.

In addition, setting  $\tau'_\rho = \inf\{n \geq 0, |Z_n| \geq \rho\}$ , we have for all starting point in  $z \in \mathbb{N}^p \setminus \{0\}$  such that  $|z| \leq \rho$ ,  $\mathbb{P}(\tau'_\rho \wedge T_0 < +\infty \mid Z_0 = z) = 1$  (by transience and because  $Z_n \in \mathbb{N}^p$  almost surely), we deduce that, for all  $\eta \in (0, 1)$ , there exists  $n_1 \geq 1$  such that

$$\mathbb{P}(\tau'_\rho \wedge T_0 \leq n_1 \mid Z_0 = z) \geq 1 - \eta, \quad \forall |z| \leq \rho.$$

Using the strong Markov property at time  $\tau'_\rho \wedge T_0$ , we deduce that

$$\mathbb{P}(\tau_{\delta_0, r} \wedge T_0 \leq n_0 + n_1 \mid Z_0 = z) \geq (1 - \eta)^2, \quad \forall z \in \mathbb{N}^p \setminus \{0\}.$$

Since this is true for all  $\eta \in (0, 1)$  this concludes the proof of Lemma 36.  $\square$

We are now in position to conclude the proof of Theorem 30. We prove the first part of the theorem, then we deduce Corollary 31 and finally we prove the second part of the theorem thanks to Corollary 31.

*Proofs of Theorem 30 and Corollary 31.* Consider  $\delta_0 \in (0, 1/p]$  and  $n_0 \in \mathbb{N}^*$  given by Lemma 36. According to Lemma 35, for all  $\varepsilon \in (0, 1)$ , for all  $\eta \in (0, 1)$ , there exists  $r > 0$  such that, for all  $z \in U_{\delta_0}$  with  $|z| \geq r$ ,

$$\mathbb{P}(\forall n \geq 0, Z_{n+1} \in [(1 - \varepsilon)\mathfrak{M}(Z_n), (1 + \varepsilon)\mathfrak{M}(Z_n)] \mid Z_0 = z) \geq 1 - \eta/2. \quad (3.29)$$

In addition, Lemma 36 entails that there exists  $\rho > 0$  such that, for all  $|z| \geq \rho$ ,

$$\mathbb{P}(\tau_{\delta_0, r} \leq n_0 \mid Z_0 = z) \geq 1 - \eta/2,$$

and hence, using the strong Markov property, we deduce that for all  $|z| \geq \rho$ ,

$$\mathbb{P}(\tau_{\delta_0, r} \leq n_0 \text{ and } \forall n \geq \tau_{\delta_0, r}, Z_{n+1} \in [(1 - \varepsilon)\mathfrak{M}(Z_n), (1 + \varepsilon)\mathfrak{M}(Z_n)] \mid Z_0 = z) \geq 1 - \eta.$$

Finally, this implies that

$$\mathbb{P}(Z_{n_0} \neq 0 \text{ and } \forall n \geq n_0, Z_{n+1} \in [(1 - \varepsilon)\mathfrak{M}(Z_n), (1 + \varepsilon)\mathfrak{M}(Z_n)] \mid Z_0 = z) \geq 1 - \eta.$$

This concludes the proof of the first part of Theorem 30.

We now prove Corollary 31. For this part we use the fact that if  $\lambda^* \leq 1$ , then the process is extinct almost surely, that will be proved in Chapter 4. We remark that this Corollary is not for any result in Chapter 4.

The result holds true trivially if  $z = 0$ . Hence, using the Markov property at time  $\tau_{\delta_0, r} \wedge T_0$ , we deduce from (3.29) and the fact that  $\mathbb{P}(\tau_{\delta_0, r} \wedge T_0 < +\infty) = 1$ , that for all  $z \in \mathbb{N}^p$ ,

$$\begin{aligned} \mathbb{P}\left(\tau_{\delta_0, r} \wedge T_0 < +\infty \text{ and } \forall n \geq \tau_{\delta_0, r} \wedge T_0 \text{ and } Z_{n+1} \in [(1-\varepsilon)\mathfrak{M}(Z_n), (1+\varepsilon)\mathfrak{M}(Z_n)] \mid Z_0 = z\right) \\ \geq 1 - \eta/2. \end{aligned}$$

Hence,

$$\mathbb{P}(\exists N \geq 0 \text{ such that, } \forall n \geq N, Z_{n+1} \in [(1-\varepsilon)\mathfrak{M}(Z_n), (1+\varepsilon)\mathfrak{M}(Z_n)] \mid Z_0 = z) \geq 1 - \eta/2.$$

Since the left hand term does not depend on  $\eta$ , this concludes the proof of Corollary 31.

For the second part of Theorem 30, we simply observe that,  $\mathbb{P}(\cdot \mid Z_0 = z)$ -almost surely, for all  $\varepsilon > 0$ , there exists  $N \geq 0$  (random) such that for all  $m, k \geq 0$ , for all  $n \geq N$ ,  $Z_{n+m} \in [(1-\varepsilon)^m \mathfrak{M}^m(Z_n), (1+\varepsilon)^m \mathfrak{M}^m(Z_n)]$ , hence

$$\begin{aligned} \frac{Z_{n+m+k}}{|Z_{n+m}|} &\in \left[ \frac{(1-\varepsilon)^{m+k} \mathfrak{M}^{m+k}(Z_n)}{(1+\varepsilon)^m |\mathfrak{M}^m(Z_n)|}, \frac{(1+\varepsilon)^{m+k} \mathfrak{M}^{m+k}(Z_n)}{(1-\varepsilon)^m |\mathfrak{M}^m(Z_n)|} \right] \\ &\subset \left[ \frac{(1-\varepsilon)^{m+k} \mathfrak{M}^{m+k}(Z_n/|Z_n|)}{(1+\varepsilon)^m |\mathfrak{M}^m(Z_n/|Z_n|)|}, \frac{(1+\varepsilon)^{m+k} \mathfrak{M}^{m+k}(Z_n/|Z_n|)}{(1-\varepsilon)^m |\mathfrak{M}^m(Z_n/|Z_n|)|} \right] \\ &\subset \left[ \frac{(1-\varepsilon)^{m+k}}{(1+\varepsilon)^m} \inf_{u \in S} \frac{\mathfrak{M}^{m+k}(u)}{|\mathfrak{M}^m(u)|}, \frac{(1+\varepsilon)^{m+k}}{(1-\varepsilon)^m} \sup_{u \in S} \frac{\mathfrak{M}^{m+k}(u)}{|\mathfrak{M}^m(u)|} \right], \end{aligned}$$

where the infimum and supremum should be understood component-wise. But, according to Theorem 18 (5),  $\frac{\mathfrak{M}^{m+k}(u)}{|\mathfrak{M}^m(u)|}$  converges uniformly in  $u \in S$  toward  $(\lambda^*)^k z^*$  when  $m \rightarrow +\infty$ , and hence, for all  $\varepsilon' > 0$ , choosing  $m$  large enough, we have, for all  $n \geq N$ ,

$$\frac{Z_{n+m+k}}{|Z_{n+m}|} \in \left[ \frac{(1-\varepsilon)^{m+k}}{(1+\varepsilon)^m} (1-\varepsilon') (\lambda^*)^k z^*, \frac{(1+\varepsilon)^{m+k}}{(1-\varepsilon)^m} (1+\varepsilon') (\lambda^*)^k z^* \right].$$

We deduce that (again the lim inf and lim sup should be understood component-wise)

$$\frac{(1-\varepsilon)^{m+k}}{(1+\varepsilon)^m} (1-\varepsilon') (\lambda^*)^k z^* \leq \liminf_{n \rightarrow +\infty} \frac{Z_{n+m+k}}{|Z_{n+m}|} \leq \limsup_{n \rightarrow +\infty} \frac{Z_{n+m+k}}{|Z_{n+m}|} \leq \frac{(1+\varepsilon)^{m+k}}{(1-\varepsilon)^m} (1+\varepsilon') (\lambda^*)^k z^*.$$

Taking, first the limit when  $\varepsilon \rightarrow 0$  ( $m$  depends on  $\varepsilon'$  but not on  $\varepsilon$ ), then the limit when  $\varepsilon' \rightarrow 0$ , concludes the proof of Theorem 30.  $\square$

### 3.2.3 Simulations

In this section we illustrate with simulations the convergence of the process  $\frac{Z_n}{|Z_n|}$  towards the eigenvector  $z^*$  of  $\mathfrak{M}$ . For this case, we consider the multi-type perfect fidelity model in dimension two. That is,  $p = q_m = q_f = 2$  and  $d = 1$ . We consider the initial condition  $z = (1, 2)$  for all our simulations.

In this case we have

$$\mathfrak{M}(z) = \min\{z\mathbb{X}, z\mathbb{Y}\}.$$

As in Section 3.1.4, we consider that all the offspring distributions are distributed according to an independent Poisson random variable of the corresponding parameter given by the expectation matrices,

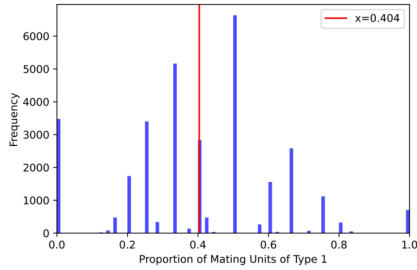
$$\mathbb{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0,8 \end{pmatrix}, \text{ and } \mathbb{Y} = \begin{pmatrix} 1 & 1,2 \\ 0,5 & 0,8 \end{pmatrix}.$$

For these values of  $\mathbb{X}$  and  $\mathbb{Y}$ , the eigenlements are approximated numeracally obtaining

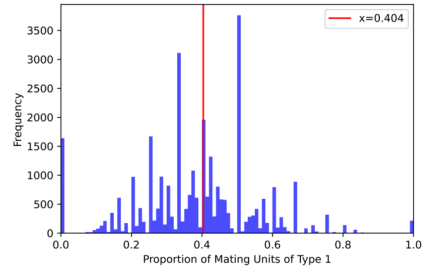
$$\lambda^* \approx 1.47 \text{ and } z^* \approx (0.4, 0.6).$$

We simulate  $10^5$  trajectories up to  $n = 40$  and use only those such that  $Z_{40} \neq 0$ . Then, we look at the trajectories at time  $n \in \{1, 5, 10, 15, 20, 25, 30, 40\}$ . We illustrate only the mating units of type 1, since for 2 the results are equivalent.

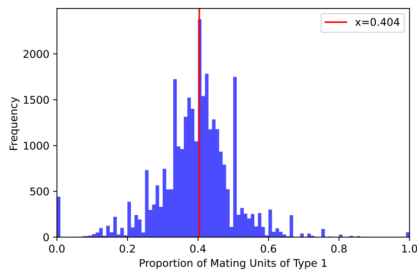
In Figure 3.6 in page 85, we appreciate the convergence towards the eigenvector as the value of  $n$  increases.



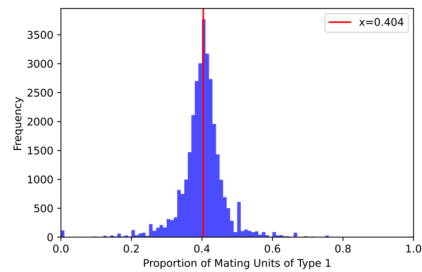
(a)  $n = 50$ .



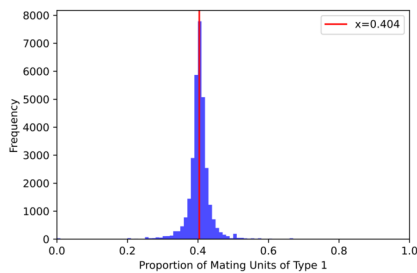
(b)  $n = 5$ .



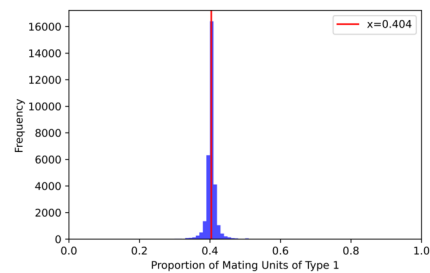
(c)  $n = 10$ .



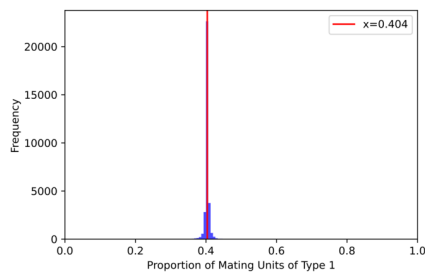
(d)  $n = 15$ .



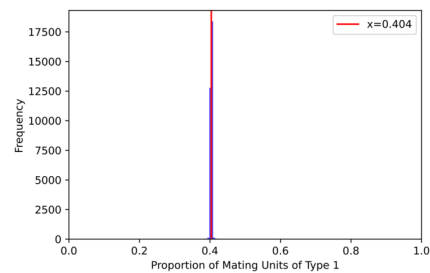
(e)  $n = 20$ .



(f)  $n = 25$ .



(g)  $n = 30$ .



(h)  $n = 40$ .

Figure 3.6: Dynamics on the variation of  $n$  for the renormalized process.



# Chapter 4

## Long Time Behaviour

### Contents

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In this chapter we explore the behaviour of a multi-type bisexual process  $(Z_n)_{n \in \mathbb{N}}$  with superadditive mating, in the limit when  $n$  grows to infinity. Throughout this chapter we suppose that Assumption 4 is in place (the process is primitive), and we denote by  $\lambda^*$  and  $z^*$  the eigenelements of the operator  $\mathfrak{M}$ . We explore two main results, both which are consequence of Theorem 30 in Chapter 3. First, using this theorem one can prove that if  $\lambda^* > 1$  then the process survives indefinitely with positive probability. In Section 4.1 we complete this picture by showing that if  $\lambda^* \leq 1$ , and under the assumption that the process is transient, then there is almost sure extinction in finite time. A corollary for the non-transient case is also exhibited. For our second main result, we study the behaviour in long time of the process, on the event of survival. Theorem 30 proves that the asymptotic profile  $\frac{Z_n}{|Z_n|}$  converges towards  $z^*$ . In Section 4.2 we give conditions that ensure that the process converges to  $z^*$  exponentially fast at a rate given by  $\lambda^*$ . We complete this result with an  $V \log V$ -type criteria for an  $L^1$  convergence.

### 4.1 Extinction - explosion of the process

In this section we are interested in finding conditions under which the process gets almost surely extinct for any initial state, i.e. if we define the probability of extinction as

$$q_z = \mathbb{P} \left( |Z_n| \xrightarrow[n \rightarrow +\infty]{} 0 \mid Z_0 = z \right) = 1, \tag{4.1}$$

then we seek conditions that ensure  $q_z = 1$  for all  $z \in \mathbb{N}^p$ .

On the other hand, we are also interested in the probability of explosion, i.e. the quantity

$$I_z = \mathbb{P}\left(|Z_n| \xrightarrow[n \rightarrow +\infty]{} +\infty \mid Z_0 = z\right). \quad (4.2)$$

We seek conditions under which  $I_z > 0$  for some  $z \in \mathbb{N}^p$ .

We start our study with case of a transient process, and then we move to the non-transient case. We finish with some examples and simulations.

#### 4.1.1 Conditions for almost sure extinction

We start with the case where the process is transient. We recall that in this case we have the extinction-explosion dichotomy. That is, for all  $z \in \mathbb{N}^p$ ,

$$\mathbb{P}(|Z_n| \xrightarrow[n \rightarrow +\infty]{} 0 \mid Z_0 = z) + \mathbb{P}(|Z_n| \xrightarrow[n \rightarrow +\infty]{} +\infty \mid Z_0 = z) = 1. \quad (4.3)$$

In terms of (4.1) and (4.2), it means that for all  $z \in \mathbb{N}^p$ ,  $I_z = 1 - q_z$ . Thus, we only focus on the value of  $q_z$ . The following theorem exhibits conditions for almost sure extinction in terms of  $\lambda^*$ , the eigenvalue of  $\mathfrak{M}$ .

**Theorem 37** (Almost sure extinction). *Assume that the process is transient and that  $\mathfrak{M}$  is finite over  $S$ . We have*

$$q_z = 1, \forall z \in \mathbb{N}^p \iff \lambda^* \leq 1.$$

*If  $\lambda^* > 1$  or if there exists  $z' \in \mathbb{N}^p$  such that one of the components of  $\mathfrak{M}(z')$  is not finite, then there exists  $r > 0$  such that, if  $|z| > r$ , then  $q_z < 1$ .*

In the following, a process such that  $\lambda^* \leq 1$  is called *subcritical*, if  $\lambda^* = 1$  it is called *critical* and if  $\lambda^* > 1$  it is called *supercritical*.

*Remark 12.* It will be useful in the proof to observe that, if we consider two superadditive functions  $\xi_1, \xi_2$  such that  $\xi_1(x) \leq \xi_2(x)$ ,  $\forall x \in \mathbb{N}^q$ , then a multi-type bisexual branching process with mating function  $\xi_1$  is stochastically dominated from above by a process with the same offspring distribution but with mating function  $\xi_2$ . We refer the reader to [DHT86] for a proof in the single-type case, since it is analogous in the multi-type one.

*Proof of Theorem 37.* In the following proof, we make use of Theorem 30 proved in the previous chapter. We divide the proof in two parts. We deal first with the case when  $\mathfrak{M}$  is finite over  $S$ , to then treat the case when it is not.

1. Assume  $\mathfrak{M}$  is finite over  $S$ .

- (a) Case when  $\lambda^* \leq 1$ : by assertion 2. in Theorem 18 (concave Perron-Frobenius Theorem), for all  $z \in \mathbb{N}^p$ ,  $(\mathfrak{M}^n(z))_{n \in \mathbb{N}}$  is a bounded sequence. From Corollary 25,  $\mathfrak{M}^n(z) \geq \mathbb{E}(Z_n \mid Z_0 = z)$ . Hence,  $\mathbb{E}(Z_n \mid Z_0 = z)$  is bounded for all  $n \in \mathbb{N}$  and so the probability that  $|Z_n|$  diverges to  $+\infty$  is zero. The conclusion is then given by Assumption 5 since then the extinction-explosion dichotomy (4.3) holds, making the probability of extinction equal to one.
- (b) Case when  $\lambda^* > 1$ : Theorem 30 entails that, for all  $\varepsilon \in (0, 1)$ , there exists  $n_0 \in \mathbb{N}$  and  $r > 0$  such that, if  $Z_0 = z \in \mathbb{N}^p$  with  $|z| > r$ , we have that with positive probability  $Z_{n_0} \neq 0$  and

$$Z_n \geq (1 - \varepsilon)^{n-n_0} \mathfrak{M}^{n-n_0}(Z_{n_0}), \text{ for all } n \geq n_0. \quad (4.4)$$

Since  $\mathfrak{M}$  is primitive, we have in particular that  $\mathfrak{M}(z) \neq 0$  for all  $z \neq 0$ . Hence,  $(1 - \varepsilon)^{n-n_0} \mathfrak{M}^{n-n_0}(Z_{n_0}) \neq 0$  (since  $\mathfrak{M}$  is primitive), and so we obtain  $q_z < 1$ .



2. Assume there exists  $z_0 \in S$  and  $i_0 \in \{1, \dots, p\}$  such that  $(\mathfrak{M}(z_0))_{i_0} = +\infty$ .

Consider the vector  $\mathbf{1}_p = (1, \dots, 1) \in \mathbb{N}^p$  and for all  $\alpha \in \mathbb{N}$  define the function

$$\xi_{(\alpha)}(x) = \min\{\xi(x), \alpha|x|\mathbf{1}_p\},$$

which is superadditive (as the minimum of two superadditive functions). We define  $\mathfrak{M}_{(\alpha)}$  the function associated to the process built with  $\xi_{(\alpha)}$  as its mating function and with the same offspring distribution as the original process. We can check that

$$\mathfrak{M}_{(\alpha)}(z) = \min\{\mathfrak{M}(z), \alpha|z|\mathbf{1}_p\},$$

and so we obtain that  $\mathfrak{M}_{(\alpha)}(z) \nearrow \mathfrak{M}(z)$  as  $\alpha \rightarrow +\infty$ , for all  $z \in \mathbb{R}_+^p$ . Note that in particular

$$\left(\mathfrak{M}_{(\alpha)}(z_0)\right)_{i_0} \xrightarrow{\alpha \rightarrow +\infty} +\infty. \quad (4.5)$$

Since clearly  $\xi_{(\alpha)} \nearrow \xi$ , using the Monotone Convergence Theorem, for all  $m \geq 1$  and  $i \in \{1, \dots, p\}$ ,

$$\mathbb{E}_{(\alpha)}(Z_m | Z_0 = ke_i) \xrightarrow{\alpha \rightarrow +\infty} \mathbb{E}(Z_m | Z_0 = ke_i). \quad (4.6)$$

where  $\mathbb{E}_{(\alpha)}$  is the probability law associated to the process with mating function  $\xi_{(\alpha)}$ .

By Assumption 4, there exists  $c_0 > 0, m \geq 1$  and  $k \geq 1$  such that for all  $i \in \{1, \dots, p\}$ ,

$$\mathbb{E}(Z_m | Z_0 = ke_i) \geq c_0 \mathbf{1}_p.$$

By (4.6), there exists  $\alpha_0 > 0$  (which depends on  $m$ ) such that for all  $\alpha > \alpha_0$  and all  $i \in \{1, \dots, p\}$ ,

$$\mathbb{E}_{(\alpha)}(Z_m | Z_0 = ke_i) \geq \frac{c_0 \mathbf{1}_p}{2} \geq \frac{c_0}{2 \max_{j \leq p} z_{0,j}} z_0.$$

This implies, by Corollary 25, that for all  $i \in \{1, \dots, p\}$ ,

$$\mathfrak{M}_{(\alpha)}^m(e_i) \geq \frac{c_0}{2k \max_{j \leq p} z_{0,j}} z_0.$$

Hence, by (4.5),

$$\left(\mathfrak{M}_{(\alpha)}^{m+1}(e_i)\right)_{i_0} \geq \frac{c_0}{2k \max_{j \leq p} z_{0,j}} \left(\mathfrak{M}_{(\alpha)}(z_0)\right)_{i_0} \xrightarrow{\alpha \rightarrow +\infty} +\infty.$$

This implies that

$$\inf_{z \in S} \left(\mathfrak{M}_{(\alpha)}^{m+1}(z)\right)_{i_0} \xrightarrow{\alpha \rightarrow +\infty} +\infty.$$

We remark that  $\mathfrak{M}_{(\alpha)}$  is concave, positively homogeneous and primitive. In addition  $|\mathfrak{M}_{(\alpha)}(z)| \leq p$ , for all  $z \in S$ , so  $\mathfrak{M}_{(\alpha)}$  is bounded on  $S$ , hence finite. Then, there exist  $\lambda_\alpha > 0$  and  $x_\alpha \in S$  such that

$$\mathfrak{M}_{(\alpha)}(x_\alpha) = \lambda_\alpha x_\alpha.$$

Using this we have that

$$\lambda_\alpha^{m+1} = |\mathfrak{M}_{(\alpha)}^{m+1}(x_\alpha)| \xrightarrow{\alpha \rightarrow +\infty} +\infty.$$

We conclude that there exists  $\alpha_0$  big enough such that  $\lambda_{\alpha_0} > 1$  and thanks to the previous computations the process with mating function  $\xi_{\alpha_0}$  survives indefinitely with positive probability. Since  $\xi_{(\alpha_0)} \leq \xi$ , this process is stochastically dominated by the original process, and so we can find  $r_{\alpha_0} > 0$  such that for all  $z \in \mathbb{N}^p$  with  $|z| > r_{\alpha_0}$ , we have  $q_z < 1$ .

□

*Remark 13.* Two things that one can deduce from the proof.

1. In the inequality in (4.4), as  $n$  goes to infinity one has that  $\mathfrak{M}^{n-n_0}(Z_{n_0})$  eventually has only strictly positive entries (since  $\mathfrak{M}$  is primitive). Then, in the case  $\lambda^* > 1$ , not only the number of mating units goes to infinity with positive probability, but all the types of mating units do it. This idea is coherent with the fact that, since the process is primitive, when the population is large enough all types of mating units can be formed.
2. Using Theorem 30, one has that for all  $\eta > 0$ , there exists  $r > 0$  such that if  $|z| > r$ , then  $q_z \leq \eta$ . Thus, setting a very large initial population, the probability of extinction can be as close to zero as desired.

In the non-transient case, a similar result can be established but only regarding the probability of explosion. Only for the following corollary, the process is not assumed to be transient.

**Corollary 38.** *Assume that  $\mathfrak{M}$  is finite over  $S$ . We have*

$$I_z = 0, \forall z \in \mathbb{N}^p \iff \lambda^* \leq 1.$$

*If  $\lambda^* > 1$  or if there exists  $z' \in \mathbb{N}^p$  such that one of the components of  $\mathfrak{M}(z')$  is not finite, then there exists  $z \in \mathbb{N}^p$  such that  $I_z > 0$ .*

*Proof.* The proof is very similar to the one of Theorem 37, so we state only the main differences. Assume  $\mathfrak{M}$  is finite on  $S$ . If  $\lambda^* \leq 1$ , the same reasoning allows us to conclude that  $I_z = 0$ .

On the other hand, if  $\lambda^* > 1$ , instead of using Theorem 30 we use Lemma 35 (which does not use the transitivity of the process). This lemma states that if  $\lambda^* > 1$ , then

$$\mathbb{P} \left( \bigcap_{n \geq 1} A_n^\varepsilon \mid Z_0 = z \right) \xrightarrow{|z| \rightarrow +\infty, z \in U_\delta} 1,$$

for

$$A_n^\varepsilon := \{\forall i \in \{1, \dots, n\}, Z_i \in [(1 - \varepsilon)\mathfrak{M}(Z_{i-1}), (1 + \varepsilon)\mathfrak{M}(Z_{i-1})]\},$$

and

$$U_\delta = \{z \in \mathbb{N}^p : z \geq |z|\delta \mathbf{1}_p\}.$$

Hence, the same reasoning used in the proof of Theorem 37 can be applied here but only for the initial conditions  $z_0 \in U_\delta$ . Then, there exists  $r > 0$  such that if  $|z| > r$  and  $z \in U_\delta$ ,  $I_z > 0$ .

The case where  $\mathfrak{M}$  is not finite is treated exactly as in the proof of Theorem 37. □

### 4.1.2 Examples

We first point out that the last theorem encompasses the well known extinction criteria for the asexual multi-type Galton-Watson process and single-type bisexual Galton-Watson process.

**Example 8** (Multi-type Galton-Watson process). We recall that this model corresponds to the case where  $p = q$  and the mating function is given by the identity function,  $\xi(x) = x$ , so that the process forms an asexual multi-type Galton-Watson process. In this case it is easy to see that  $\mathfrak{M}(z) = z\mathbb{V}$  and hence  $\mathfrak{M}$  is a linear function and  $\lambda^*$  is its greatest eigenvalue. We thus recover the well known fact that  $\lambda^* \leq 1$  is a necessary and sufficient condition for certain extinction.

**Example 9** (Single-type bisexual Galton-Watson process). The single-type bisexual Galton-Watson case corresponds to the case where  $p = 1$  and  $q = 2$ . In this case it is easy to see that  $\mathfrak{M}(z) = rz$  for  $r$  the asymptotic growth rate,  $z^* = 1$  and  $\lambda^* = r$ . We recover the fact that  $\lambda^* \leq 1$  is a necessary and sufficient condition for certain extinction (see [DHT86]).

We now turn our attention to the examples in the context of a multi-type bisexual population introduced in Section 2.2.1.

**Example 10** (Perfect fidelity mating function). We recall that the mating function corresponds to  $\xi(x, y) = \min\{x, y\}$  and that

$$\mathfrak{M}(z) = \min\{z\mathbb{X}, z\mathbb{Y}\}.$$

Let us discuss different particular instances of this model.

- If  $\mathbb{X} \leq \mathbb{Y}$  are aperiodic irreducible non-negative matrices, then  $\mathfrak{M}(z) = z\mathbb{X}$  and so  $\mathfrak{M}$  is a linear function and  $\lambda^*$  is its greatest eigenvalue. In the super-critical case (i.e.  $\lambda^* > 1$ ), the asymptotic profile of the types of the process in the non-extinction event is given by its positive left eigenvector. We thus observe that, despite the interaction between males and females, the extinction and growth characterization of the process is similar to the classical Galton-Watson case.
- The case  $\mathbb{X}$  proportional with  $\mathbb{Y}$  can be handled similarly: Let  $(X_{i,j})_{1 \leq i \leq p, 1 \leq j \leq p}$  and  $(Y_{i,j})_{1 \leq i \leq p, 1 \leq j \leq p}$  defined by

$$X_{i,j} = \sum_{\ell=1}^{U_{i,j}} \varepsilon_{i,j,\ell} \text{ and } Y_{i,j} = \sum_{i=\ell}^{U_{i,j}} (1 - \varepsilon_{i,j,\ell}),$$

where  $(U_{i,j})_{1 \leq i \leq p, 1 \leq j \leq p}$  is a random integrable array with mean  $\mathbf{U}$  and  $(\varepsilon_{i,j,\ell})_{1 \leq i \leq p, 1 \leq j \leq p, \ell \in \mathbb{N}}$  is an array of i.i.d.  $\{0, 1\}$  valued random variables independent from  $U$ . The variable  $U_{i,j}$  describes the number of children of type  $j$  from a mating unit of type  $i$  and  $\varepsilon_{i,j,\ell}$  determines if the  $\ell$ -th child is a female or a male. Note that in this example  $X_{i,j}$  and  $Y_{i,j}$  are not independent. Then, setting  $\alpha = \mathbb{P}(\varepsilon_{i,j,\ell} = 1)$ , we have

$$\mathbb{X} = \alpha \mathbf{U} \text{ and } \mathbb{Y} = (1 - \alpha) \mathbf{U}.$$

As a consequence,

$$\mathfrak{M}(z) = \min\{\alpha, 1 - \alpha\} z \mathbf{U}$$

and  $\mathfrak{M}$  is a linear function. Assume now that  $\mathbf{U}$  is an aperiodic irreducible non-negative matrix with greatest eigenvalue  $\lambda_{\mathbf{U}}$  and positive left eigenvector  $z_{\mathbf{U}}$ . Then  $\lambda^* = \min\{\alpha, 1 - \alpha\} \lambda_{\mathbf{U}}$  and, in the super-critical case, the asymptotic profile of the types of the process  $(Z_n)_{n \in \mathbb{N}}$  on the non-extinction event is given by  $z^* = z_{\mathbf{U}}$ .

- Let us now consider a non-linear case. Assume that  $\mathbb{X} = \alpha \mathbf{I}_p + \beta \mathbf{1}_p$  and  $\mathbb{Y} = \alpha' \mathbf{I}_p + \beta' \mathbf{1}_p$ , where  $\alpha, \alpha' \geq 0$  and  $\beta, \beta' > 0$  are constants,  $\mathbf{I}_p$  is the identity matrix of size  $p \times p$  and  $\mathbf{1}_p$  is the matrix of size  $p \times p$  filled with ones. Then, for all  $i \in \{1, \dots, p\}$ ,

$$\mathfrak{M}_i(z) = \min\{\alpha z_i + \beta |z|, \alpha' z_i + \beta' |z|\}.$$

Note that, for any permutation  $\sigma$  of  $\{1, \dots, p\}$ , we have

$$\mathfrak{M}(z_{\sigma(1)}, \dots, z_{\sigma(p)}) = (\mathfrak{M}_{\sigma(1)}(z), \dots, \mathfrak{M}_{\sigma(p)}(z)).$$

Hence  $z^*$  is stable by permutation of its components, so that  $z^* = (1/p, \dots, 1/p)$ . We deduce that

$$\lambda^* = |\mathfrak{M}(z^*)| = p \min\{\alpha/p + \beta, \alpha'/p + \beta'\} = \min\{\alpha + \beta p, \alpha' + \beta' p\}.$$

**Example 11** (Completely promiscuous mating function). We recall that in this case

$$\xi((x_1, \dots, x_p), (y_1, \dots, y_{q_m})) = (x_1, \dots, x_p) \prod_{i=1}^{q_m} \mathbb{1}_{\{y_i > 0\}}.$$

and thus,

$$\mathfrak{M}(z) = z\mathbb{X}\mathbb{1}_{\{z\mathbb{Y} > 0\}}.$$

We assume that  $\mathbb{X}$  is positively regular and that, for all  $1 \leq i \leq p$ ,  $1 \leq j \leq q_m$ ,  $\mathbb{Y}_{i,j} > 0$ , (this last condition ensures that the primitivity assumption holds) so that  $z\mathbb{Y} > 0$  for all  $z \in \mathbb{R}_+^p \setminus \{0\}$ . In particular  $\mathfrak{M}(z) = z\mathbb{X}$  for all  $z \in \mathbb{R}_+^p$ , which implies that the unique unitary positive eigenvector of  $\mathfrak{M}$  and its corresponding eigenvalue are the ones of  $\mathbb{X}$  given by the Perron-Frobenius Theorem. This result already appeared in [KK73].

**Example 12** (Single-type model with multiple mating strategies). We consider the Example 6. We recall that in this case  $\mathfrak{M}(z) = rz$ , with

$$r = \lim_{k \rightarrow +\infty} \sum_{i=1}^N \frac{\xi_i(k\alpha_i \mathbb{E}(X), k\beta_i \mathbb{E}(Y))}{k},$$

where

- $N$  is a fixed number and  $(\xi)_{1 \leq i \leq N}$  is a family of superadditive mating functions.
- $(\alpha_1, \dots, \alpha_N)$  and  $(\beta_1, \dots, \beta_N)$  are two vectors with  $\alpha_i, \beta_i \in [0, 1]$  for all  $i \in \{1, \dots, N\}$  and

$$\sum_{i=1}^N \alpha_i = \sum_{i=1}^N \beta_i = 1.$$

- $\mathbb{E}(X)$  (resp.  $\mathbb{E}(Y)$ ) is the expected number of females (resp. males) produced by a single mating unit.

Thanks to Theorem 37, we have that  $r \leq 1$  is a sufficient and necessary condition for certain extinction. As an example consider the particular case with  $N = 2$ , and consider the two mating strategies:

$$\xi_1(x, y) = x\mathbb{1}_{y > 0} \text{ and } \xi_2(x, y) = \min\{x, y\}.$$

Hence, the individuals in the population can chose between the completely promiscuous mating and the perfect fidelity mating. We assume that the probability of choosing one of the two is independent of the sex of the individual, so  $\alpha_1 = \beta_1$  (and then  $\alpha_2 = \beta_2$ ). For this case we obtain that

1. If  $\mathbb{E}(Y) \geq \mathbb{E}(X) > 0$ , then  $r = \mathbb{E}(X)$ .
2. If  $\mathbb{E}(X) > \mathbb{E}(Y) > 0$ , then  $r = \alpha_1 \mathbb{E}(X) + (1 - \alpha_1) \mathbb{E}(Y)$

This is coherent with the fact that the males only play a role in the case where the perfect fidelity mating is chosen. In the case where  $\mathbb{E}(X) > \mathbb{E}(Y)$ , a balance between the probability of choosing this mating and both expectations is required. If  $\mathbb{E}(X) \leq \mathbb{E}(Y)$ , then  $\alpha_1$  plays no role since when  $\mathbb{E}(X) > 1$ , then the process survives, regardless of the mating function that every individual chooses.

*Remark 14.* The function  $\mathfrak{M}(z)$  takes the form  $rz$  for any single-type process, regardless of the dimension of the process of individuals. Hence, we always obtain

$$r = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}(Z_1 | Z_0 = k)}{k} \leq 1$$

as a sufficient and necessary condition for almost sure extinction.

### 4.1.3 Simulations

In this section we simulate two versions of the perfect fidelity model in dimension 2. That is, we set  $p = q_f = q_m = 2$  and consider, as in the previous chapter, that all the offspring distributions are issued from Poisson random variables. This in particular ensures that the processes are transient since every state is connected to zero. For the first version we consider the expectation matrices  $\mathbb{X}_A$  and  $\mathbb{Y}_A$  given by

$$\mathbb{X}_A = \begin{pmatrix} 0 & 0.3 \\ 0.7 & 0.8 \end{pmatrix} \text{ and } \mathbb{Y}_A = \begin{pmatrix} 0.9 & 0.8 \\ 0.5 & 0.5 \end{pmatrix}.$$

With these values, we can approximate numerically the associated eigenvalue  $\lambda_A^*$ , obtaining  $\lambda_A^* \approx 1.008$ . For the second version, we consider the expectation matrices

$$\mathbb{X}_B = \begin{pmatrix} 0 & 0.3 \\ 0.7 & 0.8 \end{pmatrix} \text{ and } \mathbb{Y}_B = \begin{pmatrix} 0.9 & 0.7 \\ 0.5 & 0.5 \end{pmatrix}.$$

Note that the only difference is that  $\mathbb{Y}_A$  and  $\mathbb{Y}_B$  differ in the first line, second column by 0.1. With this, we numerically approximate the associated eigenvalue obtaining  $\lambda_B^* \approx 0.99$ .

For both processes we simulate 300 trajectories up to time  $n = 500$ , starting in both cases from  $z_0 = (250, 280)$ . Then, we plot the total number of mating units at time  $n$ , i.e.  $Z_{n,1} + Z_{n,2}$ .

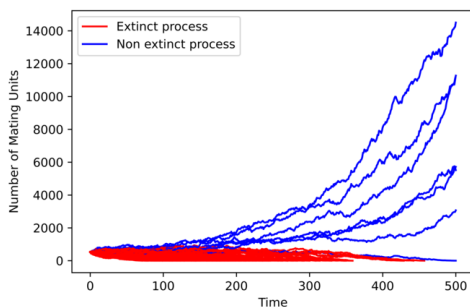


Figure 4.1: Process A.  $\lambda_A^* \approx 1,008$

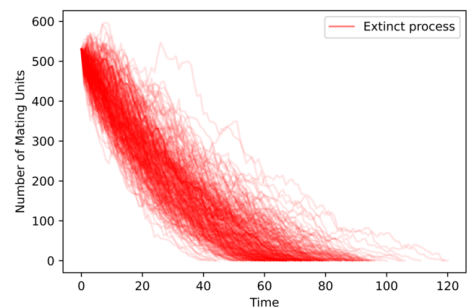


Figure 4.2:  $\lambda_B^* \approx 0,99$

According to Theorem 37, the process with matrices  $\mathbb{X}_A$  and  $\mathbb{Y}_A$  has a positive probability of survival. In Figure 4.1, we appreciate that some trajectories (the blue ones) move away from

zero and most of them start to exhibit an exponential growth. On the other hand, Theorem 37 states that the process with matrices  $\mathbb{X}_B$  and  $\mathbb{Y}_B$  gets almost surely extinct in finite time. In Figure 4.2 we check that all the trajectories are extinct around time  $n = 100$ .

## 4.2 Convergence Properties

In this section we are interested on the behaviour of the process  $(Z_n)_{n \in \mathbb{N}}$  on the event of survival. From Theorem 30 we know that the proportion of every type of mating units converges to  $z^*$ , the eigenvector of  $\mathfrak{M}$ . We complete this result by showing that the process grows at rate  $(\lambda^*)^n$  towards a non-negative random variable which has the same direction as  $z^*$ . We also show that a similar result holds for the process of individuals  $(W_n)_{n \in \mathbb{N}}$ . We prove that, when the limit is non-degenerate, the event of extinction coincides (almost surely) with the event where this limit is equal to zero. This last part is well known in the classical branching case. As far as we know, it is new even in the single-type bisexual branching case. We finish with conditions for an  $L^1$  convergence and some simulations.

In this whole section we assume that the process is primitive and transient.

**Theorem 39** (Asymptotic profile). *For all  $z \in \mathbb{N}^p$ , there exists a real non-negative random variable  $\mathcal{C}$  such that*

$$\frac{Z_n}{(\lambda^*)^n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}(\cdot | Z_0 = z) \text{ a.s.}} \mathcal{C}z^* \text{ and } \frac{W_n}{(\lambda^*)^n} \xrightarrow[n \rightarrow +\infty]{\mathbb{P}(\cdot | Z_0 = z) \text{ a.s.}} \frac{1}{\lambda^*} \mathcal{C}z^*\mathbb{V}, \quad (4.7)$$

with  $\lambda^*$  and  $z^*$  given by Theorem 18.

Assume in addition that  $\mathcal{C}$  is non-degenerate at 0 for some  $z \in \mathbb{N}^p$ , which means that  $\mathbb{P}(\mathcal{C} > 0 | Z_0 = z) > 0$  for some  $z \in \mathbb{N}^p$ . Then  $\mathbb{P}(\mathcal{C} > 0 | Z_0 = z) > 0$  for all  $z \in \mathbb{N}^p$  such that  $q_z < 1$  and, for all  $z \in \mathbb{N}^p$  and up to a  $\mathbb{P}(\cdot | Z_0 = z)$  negligible event,

$$\{\mathcal{C} = 0\} = \{\exists n \in \mathbb{N}, Z_n = 0\}.$$

### 4.2.1 Identification of an intrinsic supermartingale and proof of Theorem 39

In this section we prove Theorem 39. In order to do so, we start by introducing the process

$$C_n = \frac{\mathcal{P}(Z_n)}{(\lambda^*)^n}, \text{ for } n \geq 0, \quad (4.8)$$

where  $\mathcal{P}$  is the function defined by Theorem 18 and is given by

$$\mathcal{P}(z) = \lim_{n \rightarrow +\infty} \frac{|\mathfrak{M}^n(z)|}{(\lambda^*)^n}. \quad (4.9)$$

We recall that this function is well defined, concave and positively homogenous. A particular case of this function is when  $\mathfrak{M}$  turns out to be linear, and the form of  $\mathcal{P}$  depends on the dimension of the process.

- If  $\mathfrak{M}$  is a linear function defined on  $\mathbb{R}_+$  (i.e.  $p = 1$ ), then in this case  $\mathcal{P}$  is the identity function on  $\mathbb{R}_+$ .
- If  $\mathfrak{M}$  is a linear function defined on  $\mathbb{R}_+^p$ , then the classical Perron-Frobenius Theorem states that  $\mathcal{P}(z) = K \langle z^*, z \rangle$ , where  $z^*$  is the right eigenvector associated to the Perron-Frobenius root  $\lambda^*$ , and  $K > 0$  a suitable constant.

We remark that the proof of the first part of Theorem 39 relies on the convergence of  $(C_n)_{n \in \mathbb{N}}$ . We claim that if this sequence converges almost surely to some random variable  $\mathcal{C}$ , then the sequence  $\left(\frac{Z_n}{(\lambda^*)^n}\right)_{n \in \mathbb{N}}$  also converges. In fact, on the event of extinction the result is trivial. On the event of survival, as  $\mathcal{P}$  is positively homogeneous, we have

$$\frac{Z_n}{(\lambda^*)^n} = C_n \frac{Z_n}{|Z_n|} \frac{1}{\mathcal{P}\left(\frac{Z_n}{|Z_n|}\right)}.$$

Applying Theorem 30 and if  $C_n \rightarrow \mathcal{C}$  we deduce that

$$\frac{Z_n}{(\lambda^*)^n} \rightarrow \mathcal{C} z^* \frac{1}{\mathcal{P}(z^*)} = \mathcal{C} z^*, \text{ as } n \rightarrow +\infty,$$

obtaining the desired result. We also remark that the convergence of the sequence  $\left(\frac{Z_n}{(\lambda^*)^n}\right)_{n \in \mathbb{N}}$  is of the same type as the convergence of  $(C_n)_{n \in \mathbb{N}}$ .

The proof of Theorem 39 is divided in two parts. First, we prove the first statement, which, thanks to the previous computations, is proven by showing that  $(C_n)_{n \in \mathbb{N}}$  converges almost surely. Then, we prove the second statement regarding the almost sure equality

$$\{\mathcal{C} = 0\} = \{\exists n \in \mathbb{N}, Z_n = 0\}.$$

#### 4.2.1.1 Almost sure convergence

In this section, we prove the first statement of Theorem 39. The proof comes from the following result.

**Lemma 40.** *The sequence  $(C_n)_{n \in \mathbb{N}}$  is a supermartingale with respect to  $(\mathcal{F}_n)_{n \in \mathbb{N}}$ , the natural filtration of  $(Z_n)_{n \in \mathbb{N}}$ .*

*Proof.* First, from the definition of  $\mathcal{P}$  and Theorem 18, it follows that  $\mathcal{P} \circ \mathfrak{M} = \lambda^* \mathcal{P}$ . Then we recall that, by Corollary 25, we have for all  $n \in \mathbb{N}$ ,

$$\mathbb{E}(Z_{n+1} \mid \mathcal{F}_n) \leq \mathfrak{M}(Z_n).$$

Hence, Jensen's inequality together with the fact that  $\mathcal{P}$  is increasing (since  $\mathfrak{M}$  is) and concave imply that

$$\mathbb{E}(C_{n+1} \mid \mathcal{F}_n) = \frac{\mathbb{E}(\mathcal{P}(Z_{n+1}) \mid \mathcal{F}_n)}{(\lambda^*)^{n+1}} \leq \frac{\mathcal{P}(\mathbb{E}(Z_{n+1} \mid \mathcal{F}_n))}{(\lambda^*)^{n+1}} \leq \frac{\mathcal{P}(\mathfrak{M}(Z_n))}{(\lambda^*)^{n+1}} = \frac{\mathcal{P}(Z_n)}{(\lambda^*)^n} = C_n.$$

□

Since  $(C_n)_{n \in \mathbb{N}}$  is a non-negative supermartingale, it exists a non-negative random variable  $\mathcal{C}$  such that  $C_n \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \mathcal{C}$ .

We now prove the convergence of the process  $\left(\frac{W_n}{(\lambda^*)^n}\right)_{n \in \mathbb{N}}$  in Theorem 39. We adapt the proof for the single-type case presented in [GM96]

Let  $N_n = \frac{Z_n}{(\lambda^*)^n}$ ,  $S_n = \frac{W_n}{(\lambda^*)^n}$  and  $\tilde{\mathcal{F}}_n = \sigma(Z_0, V_{i,j}^{(k,\ell)}, 1 \leq i \leq p, 1 \leq j \leq q, k \in \mathbb{N}^*, 1 \leq \ell \leq n)$ .

We consider  $Z_0 = z$  and define for  $n \geq 1$ ,  $\hat{S}_n = (\hat{S}_{n,1}, \dots, \hat{S}_{n,q})$  with for  $j \in \{1, \dots, q\}$ ,

$$\hat{S}_{n,j} = \frac{1}{(\lambda^*)^n} \sum_{i=1}^p \sum_{k=1}^{Z_{n-1,i}} V_{i,j}^{(k,n)} \mathbf{1}_{V_{i,j}^{(k,n)} \leq (\lambda^*)^{n-1}}$$

Note that for  $j \in \{1, \dots, q\}$ ,

$$\mathbb{E}(\hat{S}_{n+1,j} | \tilde{\mathcal{F}}_n) = \frac{1}{\lambda^*} \sum_{i=1}^p N_{n,i} \mathbb{E}(V_{i,j} \mathbf{1}_{V_{i,j} \leq (\lambda^*)^n}). \quad (4.10)$$

Since  $\lambda^* > 1$ , we have that  $V_{i,j} \mathbf{1}_{V_{i,j} \leq (\lambda^*)^n} \rightarrow V_{i,j}$  almost surely as  $n \rightarrow +\infty$ . Hence, by the Monotone Convergence Theorem,  $\mathbb{E}(V_{i,j} \mathbf{1}_{V_{i,j} \leq (\lambda^*)^n}) \rightarrow \mathbb{E}(V_{i,j})$ . Then, taking  $n \rightarrow +\infty$ , we have that

$$\mathbb{E}(\hat{S}_{n+1} | \tilde{\mathcal{F}}_n) \xrightarrow{n \rightarrow +\infty} \frac{1}{\lambda^*} \mathcal{C}z^* \mathbb{V} \text{ a.s.} \quad (4.11)$$

We consider now the martingale given by

$$\left( \sum_{n=1}^m (\hat{S}_n - \mathbb{E}(\hat{S}_n | \tilde{\mathcal{F}}_{n-1})) \right)_{m \in \mathbb{N}}.$$

We have that for all  $j \in \{1, \dots, q\}$  and  $n \in \mathbb{N}$

$$\begin{aligned} & \text{Var}(\hat{S}_{n+1,j} - \mathbb{E}(\hat{S}_{n+1,j} | \tilde{\mathcal{F}}_n)) \\ &= \text{Var} \left( \frac{1}{(\lambda^*)^{n+1}} \sum_{i=1}^p \sum_{k=1}^{Z_{n,i}} \left( V_{i,j}^{(k,n)} \mathbf{1}_{V_{i,j}^{(k,n)} \leq (\lambda^*)^n} - \mathbb{E} \left( V_{i,j}^{(k,n)} \mathbf{1}_{V_{i,j}^{(k,n)} \leq (\lambda^*)^n} \right) \right) \right) \\ &= (\lambda^*)^{-2(n+1)} \sum_{i=1}^p \mathbb{E}(Z_{n,i}) \text{Var} \left( V_{i,j} \mathbf{1}_{V_{i,j} \leq (\lambda^*)^n} \right) \\ &\leq (\lambda^*)^{-n-2} \sum_{i=1}^p \frac{\mathfrak{M}_i^n(z)}{(\lambda^*)^n} \text{Var} \left( V_{i,j} \mathbf{1}_{V_{i,j} \leq (\lambda^*)^n} \right) \\ &\leq (\lambda^*)^{-n-2} \sum_{i=1}^p \frac{\mathfrak{M}_i^n(z)}{(\lambda^*)^n} \int_0^{+\infty} x^2 \mathbf{1}_{x \leq (\lambda^*)^n} dF_{i,j}(x), \end{aligned}$$

where  $F_{i,j}(x) = \mathbb{P}(V_{i,j} \leq x)$ .

Since  $\left( \frac{\mathfrak{M}_i^n(z)}{(\lambda^*)^n} \right)_{n \in \mathbb{N}}$  is convergent, then it is bounded, and so there exists  $C > 0$  such that

$$\begin{aligned} \sum_{n \in \mathbb{N}} \text{Var}(\hat{S}_{n+1,j} - \mathbb{E}(\hat{S}_{n+1,j} | \tilde{\mathcal{F}}_n)) &\leq C \sum_{i=1}^p \int_0^{+\infty} x^2 \sum_{n \in \mathbb{N}} \frac{1}{(\lambda^*)^n} \mathbf{1}_{x \leq (\lambda^*)^n} dF_{i,j}(x) \\ &= C \sum_{i=1}^p \int_0^{+\infty} x^2 O(x^{-1}) dF_{i,j}(x) < +\infty, \end{aligned}$$

and so applying the Martingale Convergence Theorem, we have that  $\sum_{n \in \mathbb{N}} (\hat{S}_{n+1,j} - \mathbb{E}(\hat{S}_{n+1,j} | \tilde{\mathcal{F}}_n))$  is convergent almost surely and in  $L^1$ . This implies that  $\hat{S}_{n+1,j} - \mathbb{E}(\hat{S}_{n+1,j} | \tilde{\mathcal{F}}_n) \rightarrow 0$  almost surely. Thanks to (4.11), we have that  $\hat{S}_{n+1} \rightarrow (\lambda^*)^{-1} \mathcal{C}z^* \mathbb{V}$ . We finish by proving that  $(S_n)_{n \in \mathbb{N}}$



is an equivalent sequence. Fix  $j \in \{1, \dots, q\}$

$$\begin{aligned}
\sum_{n \in \mathbb{N}} \mathbb{P}(S_{n+1,j} \neq \hat{S}_{n+1,j}) &= \sum_{n \in \mathbb{N}} \mathbb{E}(\mathbb{P}(\exists i \leq p, \exists k \leq Z_{n,i}, V_{i,j}^{(k,n)} > (\lambda^*)^n \mid \tilde{\mathcal{F}}_n)) \\
&\leq \sum_{n \in \mathbb{N}} \sum_{i=1}^p \mathbb{E} \left( \sum_{k=1}^{Z_{n,i}} \mathbb{P}(V_{i,j} > (\lambda^*)^n) \right) \\
&\leq C \sum_{i=1}^p \sum_{n \in \mathbb{N}} (\lambda^*)^n \mathbb{P}(V_{i,j} > (\lambda^*)^n) \\
&\leq C \sum_{i=1}^p \int_0^{+\infty} \sum_{n \in \mathbb{N}} (\lambda^*)^n \mathbf{1}_{x > (\lambda^*)^n} dF_{i,j}(x) \\
&= \sum_{i=1}^p \int_0^{+\infty} O(x) dF_{i,j}(x) < +\infty.
\end{aligned}$$

The conclusion then follows by the Borel-Cantelli lemma.

#### 4.2.1.2 Extinction vs $\{\mathcal{C} = 0\}$

In this section, we prove the second part of Theorem 39.

The inclusion  $\{\exists n \in \mathbb{N}, Z_n = 0\} \subset \{\mathcal{C} = 0\}$  is obvious. We then consider the case  $\lambda^* > 1$  and show that  $\{\forall n, Z_n \neq 0\} \subset \{\mathcal{C} > 0\}$ .

For all  $\varepsilon > 0$ , we set

$$\tau_\varepsilon = \inf \left\{ n \geq 0, \frac{\mathcal{P}(Z_n)}{(\lambda^*)^n} \leq \varepsilon \right\}.$$

By assumption  $\mathbb{P}(\mathcal{C} > 0 \mid Z_0 = z') > 0$  for some  $z' \in \mathbb{N}^p$ . For any  $z \in \mathbb{N}^p$  such that  $q_z < 1$ , one deduces from Lemma 36 that there exists  $n \geq 0$  such that  $\mathbb{P}(Z_n \geq z' \mid Z_0 = z) > 0$  (taking  $r$  large enough so that  $\delta_0 r \mathbf{1}_p \geq z'$ ). Hence, using the Markov property at time  $n$  and the fact that the mating function is non-decreasing, we deduce that

$$\mathbb{P}(\mathcal{C} > 0 \mid Z_0 = z) \geq \mathbb{P}(\mathcal{C} > 0 \mid Z_0 = z') \times \mathbb{P}(Z_n \geq z' \mid Z_0 = z) > 0.$$

In particular, this shows that  $\mathcal{C}$  is non-degenerate for all  $z \in \mathbb{N}^p$  such that  $q_z < 1$ . Hence, for all  $z$  such that  $q_z < 1$ , there exists  $\varepsilon_z > 0$  such that

$$\mathbb{P}(\tau_{\varepsilon_z} = +\infty \mid Z_0 = z) > 0.$$

In addition, Theorem 30 entails that there exists  $r_0 \in \mathbb{N}$  such that, for all  $|z| \geq r_0$ ,  $q_z < 1$ . Hence, setting

$$\varepsilon_0 = \min_{i=1, \dots, p} \varepsilon_{r_0 e_i}$$

and using the fact that, by superadditivity,  $\mathbb{P}(\tau_{\varepsilon_0} = +\infty \mid Z_0 = z)$  is increasing with  $z$ , we deduce that, for all  $|z| \geq pr_0$ ,

$$\begin{aligned}
\mathbb{P}(\tau_{\varepsilon_0} = +\infty \mid Z_0 = z) &\geq a_0 := \min_{i=1, \dots, p} \mathbb{P}(\tau_{\varepsilon_0} = +\infty \mid Z_0 = r_0 e_i) \\
&\geq \min_{i=1, \dots, p} \mathbb{P}(\tau_{\varepsilon_{r_0 e_i}} = +\infty \mid Z_0 = r_0 e_i) > 0.
\end{aligned}$$

We define  $\tau^0 = 0$ ,  $\tau^1 = \tau_{\varepsilon_0}$  and, for all  $n \geq 1$ ,

$$\tau^{n+1} = \inf\{n \geq \tau^n + 1, \frac{\mathcal{P}(Z_n)}{(\lambda^*)^n} \leq \varepsilon_0\}.$$

Then we obtain, for all  $|z| \geq pr_0$  and all  $n \geq 1$ ,

$$\begin{aligned} & \mathbb{P}(\{\tau^{n+1} < +\infty\} \cap \{|Z_m| \geq pr_0, \forall m \in [0, \tau^n]\} \mid Z_0 = z) \\ & \leq \mathbb{E} \left( 1_{\tau^n < +\infty, |Z_m| \geq pr_0, \forall m \in [0, \tau^n]} \mathbb{P}(\tau_{\varepsilon_0} < +\infty \mid Z_0 = z')_{|z'=Z_{\tau^n}} \mid Z_0 = z \right) \\ & \leq \mathbb{P}(\{\tau^n < +\infty\} \cap \{|Z_m| \geq pr_0, \forall m \in [0, \tau^{n-1}]\} \mid Z_0 = z) (1 - a_0) \\ & \leq \dots \leq (1 - a_0)^{n+1}. \end{aligned} \tag{4.12}$$

In addition, according to Theorem 22 and Theorem 30, for all  $\eta \in (0, 1)$ , there exists  $r_\eta > 0$  such that, for all  $|z| \geq r_\eta$ ,

$$\mathbb{P}(|Z_n| \geq pr_0, \forall n \geq 0 \mid Z_0 = z) \geq 1 - \eta.$$

Using this last inequality and (4.12), we deduce that, for all  $|z| \geq r_\eta$  and all  $n \geq 1$ ,

$$\begin{aligned} \mathbb{P}(\tau^{n+1} < +\infty \mid Z_0 = z) & \leq \mathbb{P}(\{\tau^{n+1} < +\infty\} \cap \{|Z_n| \geq pr_0, \forall n \in [0, \tau^n]\} \mid Z_0 = z) \\ & \quad + \mathbb{P}(\exists n \geq 0 \text{ such that } |Z_n| < pr_0 \mid Z_0 = z) \\ & \leq (1 - a_0)^{n+1} + \eta. \end{aligned}$$

Since  $\{\mathcal{C} = 0\} \subset \{\tau^{n+1} < +\infty, \forall n \geq 1\}$ , we deduce that, for all  $|z| \geq r_\eta$ ,

$$\mathbb{P}(\mathcal{C} = 0 \mid Z_0 = z) \leq \eta. \tag{4.13}$$

Denoting  $T_\eta := \inf\{n \geq 0, |Z_n| \geq r_\eta\}$ , we deduce from the transitivity assumption that  $\{T_\eta < +\infty\} \supset \{\forall n, Z_n \neq 0\}$ . Hence, for all  $z \in \mathbb{N}^p$ , using the strong Markov property at time  $T_\eta$  and then (4.13),

$$\begin{aligned} \mathbb{P}(\mathcal{C} = 0 \text{ and } \forall n, Z_n \neq 0 \mid Z_0 = z) & \leq \mathbb{P}(\mathcal{C} = 0 \text{ and } T_\eta < +\infty \mid Z_0 = z) \\ & = \mathbb{E} \left( 1_{T_\eta < +\infty} \mathbb{P} \left( \lim_{n \rightarrow +\infty} \frac{\mathcal{P}(Z_n)}{(\lambda^*)^{n+u}} = 0 \mid Z_0 = z' \right)_{|u=T_\eta, z'=Z_{T_\eta}} \mid Z_0 = z \right) \\ & = \mathbb{E} \left( 1_{T_\eta < +\infty} \mathbb{P}(\mathcal{C} = 0 \mid Z_0 = z')_{|z'=Z_{T_\eta}} \mid Z_0 = z \right) \leq \eta. \end{aligned}$$

Since the last inequality holds true for all  $\eta \in (0, 1)$ , we deduce that, for all  $z \in \mathbb{N}^p$ ,

$$\mathbb{P}(\mathcal{C} = 0 \text{ and } \forall n, Z_n \neq 0 \mid Z_0 = z) = 0.$$

#### 4.2.2 $L^1$ convergence and non-degeneracy of the limit.

A natural question that arises is to find conditions so that the previous convergence holds also in  $L^1$  and the limit is non-degenerate at 0. The following proposition deals with this question. The condition we present is inspired by the type of criteria that was first presented by Klebaner's article [Kle85] and that was then used in the single-type case by González and Molina [GM96].

**Proposition 41.** *Assume that  $\lambda^* > 1$ . If there exists a concave monotone increasing function  $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , such that for all  $y \in \mathbb{R}_+$ ,*

$$\sup_{z \in \mathbb{R}_+^p : \mathcal{P}(z)=y} \mathbb{E}(|\mathcal{P}(Z_1) - \mathcal{P}(\mathfrak{M}(\lfloor z \rfloor))| \mid Z_0 = \lfloor z \rfloor) \leq U(y), \quad (4.14)$$

with  $y \mapsto U(y)/y$  non-increasing and

$$\int_1^{+\infty} \frac{U(y)}{y^2} dy < +\infty,$$

then the convergence in Theorem 39 is in  $L^1$  and the random variable  $\mathcal{C}$  is non-degenerate at 0.

The intuition here is that if we look at the difference  $|C_{n+1} - C_n|$ , for  $n \in \mathbb{N}$ , we have,

$$|C_{n+1} - C_n| = \frac{|\mathcal{P}(Z_{n+1}) - \lambda^* \mathcal{P}(Z_n)|}{(\lambda^*)^{n+1}} = \frac{|\mathcal{P}(Z_{n+1}) - \mathcal{P}(\mathfrak{M}(Z_n))|}{(\lambda^*)^{n+1}},$$

where we used that for all  $z \in \mathbb{R}_+^p$ , thanks to the definition of  $\mathcal{P}$  (see (4.9)),

$$\mathcal{P}(\mathfrak{M}(z)) = \lim_{n \rightarrow +\infty} \frac{|\mathfrak{M}^{n+1}(z)|}{(\lambda^*)^n} = \lambda^* \mathcal{P}(z).$$

Hence, the idea is to use the hypothesis on Proposition 41 to control the difference  $|C_{n+1} - C_n|$  and prove that  $(C_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^1$ , thus convergent. We claim that this suffices to prove the convergence in  $L^1$  of the sequence  $\left(\frac{Z_n}{(\lambda^*)^n}\right)_{n \in \mathbb{N}}$ . In fact, consider the following result.

**Lemma 42.** *We have*

$$\inf_{z \in S} \mathcal{P}(z) > 0.$$

*Proof of Lemma 42.* Set

$$K_{\min} = \min_{i \in \{1, \dots, p\}} \mathcal{P}(e_i) > 0,$$

and consider  $z \in S$ . Then, since  $\mathcal{P}$  is positively homogeneous,

$$\mathcal{P}(z) = \sum_{i=1}^p z_i \mathcal{P}(e_i) \geq K_{\min} \sum_{i=1}^p z_i = K_{\min} > 0,$$

and the conclusion follows.  $\square$

We have that

$$\frac{Z_n}{\mathcal{P}(Z_n)} = \frac{Z_n}{|Z_n|} \frac{1}{\mathcal{P}\left(\frac{Z_n}{|Z_n|}\right)}$$

converges almost surely to  $z^*$ . By Lemma 42, it is also an almost surely bounded sequence, thus convergent in  $L^1$  towards  $z^*$  thanks to the Dominated Convergence Theorem. Finally, since

$$\frac{Z_n}{(\lambda^*)^n} = C_n \frac{Z_n}{\mathcal{P}(Z_n)},$$

we have that if  $C_n \xrightarrow[n \rightarrow +\infty]{L^1} \mathcal{C}$ , then we can conclude our claim.

We now turn our attention to the proof of Proposition 41, for which we make use of the following auxiliary lemma.

**Lemma 43.** [See [Kle84] - Lemma 2] Let  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a non-increasing function, such that  $x \mapsto xf(x)$  is non-decreasing and

$$\sum_{n=1}^{+\infty} \frac{f(n)}{n} < +\infty.$$

Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence of positive numbers satisfying for some  $m > 1$  and all  $n \geq 0$

$$|a_{n+1} - a_n| \leq a_n f(a_n m^n).$$

Then

- $\lim_{n \rightarrow +\infty} a_n = a$  exists,
- there exists a constant  $b_0$  depending only on the function  $f$  and  $m$  such that if  $a_0 > b_0$  then  $a > 0$ .

*Proof of Proposition 41.* We start our proof by noticing that, for all  $z \in \mathbb{N}^p$ ,

$$\begin{aligned} |\mathbb{E}(C_{n+1} | Z_0 = z) - \mathbb{E}(C_n | Z_0 = z)| &\leq \mathbb{E}(|C_{n+1} - C_n| | Z_0 = z) \\ &= \mathbb{E}(\mathbb{E}(|C_{n+1} - C_n| | \mathcal{F}_n) | Z_0 = z) \\ &= \frac{1}{(\lambda^*)^{n+1}} \mathbb{E}(\mathbb{E}(|\mathcal{P}(Z_{n+1}) - \lambda^* \mathcal{P}(Z_n)| | \mathcal{F}_n) | Z_0 = z) \\ &\leq \frac{1}{(\lambda^*)^{n+1}} \mathbb{E}(U(\mathcal{P}(Z_n)) | Z_0 = z), \end{aligned}$$

and so applying Jensen's inequality, since  $U$  is concave, we get

$$\mathbb{E}(|C_{n+1} - C_n| | Z_0 = z) \leq (\lambda^*)^{-1} \mathbb{E}(C_n | Z_0 = z) \frac{U(\mathbb{E}(\mathcal{P}(Z_n) | Z_0 = z))}{\mathbb{E}(\mathcal{P}(Z_n) | Z_0 = z)}. \quad (4.15)$$

From this inequality, we have that if we define

$$F(x) = \frac{U(x)}{\lambda^* x},$$

then, we obtain,

$$|\mathbb{E}(C_{n+1} | Z_0 = z) - \mathbb{E}(C_n | Z_0 = z)| \leq \mathbb{E}(C_n | Z_0 = z) F(\mathbb{E}(C_n | Z_0 = z) (\lambda^*)^n),$$

and we can apply Lemma 43 with  $f = F$  and  $m = \lambda^*$ . This implies that, for all  $z \in \mathbb{N}^p$ ,

$$c(z) := \lim_{n \rightarrow +\infty} \mathbb{E}(C_n | Z_0 = z)$$

exists. It also implies that there exists  $b_0$  such that if  $\mathcal{P}(z) \geq b_0$ , then  $c(z) > 0$ . Since  $\mathcal{P}$  is homogeneous and lower bounded away from 0 on  $S$ , we deduce that there exists  $r_0 > 0$  such that, for all  $z \in \mathbb{N}^p$  with  $|z| \geq r_0$ ,  $c(z) > 0$ . Since in addition  $c(z)$  is increasing with  $z$ , we deduce that

$$\underline{c} := \inf_{z \in \mathbb{N}^p, |z| \geq r_0} c(z) > 0.$$

If we now define

$$T = \inf\{n \in \mathbb{N} : |Z_n| \geq r_0\},$$

we have that, if  $z \in \mathbb{N}^p$  is such that  $q_z < 1$ , then  $\mathbb{P}(T < +\infty \mid Z_0 = z) \geq q_z > 0$  by transitivity assumption and so, applying the strong Markov property we obtain

$$\begin{aligned} c(z) &= \lim_{n \rightarrow \infty} \mathbb{E}(C_n \mid Z_0 = z) \\ &\geq \lim_{n \rightarrow +\infty} \mathbb{E} \left( \mathbb{E}(C_n \mid Z_0 = y)_{|y=Z_T} (\lambda^*)^{-T} \mathbf{1}_{T < +\infty} \mid Z_0 = z \right) \\ &\geq \underline{c} \mathbb{E} \left( (\lambda^*)^{-T} \mathbf{1}_{T < +\infty} \mid Z_0 = z \right) > 0. \end{aligned}$$

Now fix  $z \in \mathbb{N}^p$  such that  $q_z < 1$  and take  $\varepsilon$  such that  $c(z) - \varepsilon > 0$ . We can find  $N_0$  such that for all  $n \geq N_0$ ,

$$c(z) - \varepsilon \leq \mathbb{E}(C_n \mid Z_0 = z) \leq c(z) + \varepsilon.$$

Hence, using this in (4.15), we get that for all  $n \geq N_0$ , since  $x \rightarrow \hat{U}(x)/x$  is non-increasing,

$$\mathbb{E}(|C_{n+1} - C_n| \mid Z_0 = z) \leq (\lambda^*)^{-1} (c(z) + \varepsilon) \frac{U((c(z) + \varepsilon)(\lambda^*)^n)}{(c(z) - \varepsilon)(\lambda^*)^n},$$

and so we can find  $C > 0$  and  $\delta > 0$ , such that for all  $n \in \mathbb{N}$ ,

$$\mathbb{E}(|C_{n+1} - C_n| \mid Z_0 = z) \leq C \frac{U(\delta(\lambda^*)^n)}{(\lambda^*)^n}.$$

On the other hand, since the integral  $\int_1^{+\infty} \frac{U(y)}{y^2} dy$  is finite, we have that the series

$$\sum_{n \in \mathbb{N}} \frac{U(\delta(\lambda^*)^n)}{(\lambda^*)^n}$$

converges (see [Kle85]). This implies that  $(C_n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $L^1$ , which gives the  $L^1$  convergence.

Finally, we have that if  $z \in \mathbb{N}^p$  is such that  $q_z < 1$ , then

$$\mathbb{E}(C \mid Z_0 = z) = \lim_{n \rightarrow +\infty} \mathbb{E}(C_n \mid Z_0 = z) > 0,$$

which proves that the limit is non-degenerate at 0.  $\square$

### 4.2.3 $V \log V$ -type condition

In general in the branching process theory, a condition of type  $V \log V$  is sufficient to ensure the  $L^1$  convergence and the non-degeneracy of the limit (see Theorem 2 for the asexual case). In addition, the existence of the function  $U$  in the previous theorem may be difficult to check. In the following proposition we state sufficient conditions to ensure its existence, under a  $V \log V$ -type condition and extra assumptions on the functions  $\mathcal{P}$ ,  $\xi$  and  $\mathfrak{M}$ .

**Proposition 44.** *Assume that  $\mathfrak{M}$  is finite over  $S$ . In addition assume that both functions  $\mathcal{P}$  and  $\xi$  are Lipschitz, that  $\mathbb{E}(V_{i,j} \log V_{i,j}) < +\infty$  for all  $i \in \{1, \dots, p\}$ ,  $j \in \{1, \dots, q\}$ , and that there exist  $C, \alpha > 0$  such that*

$$\left| \frac{\xi(z\mathbb{V})}{|z|} - \frac{\mathfrak{M}(z)}{|z|} \right| \leq C |z|^{-\alpha}, \quad \forall z \in \mathbb{N}^p \setminus \{0\}. \quad (4.16)$$

Then, the condition (4.14) is satisfied.

*Remark 15.* Since originally  $\xi$  is only defined over  $\mathbb{N}^q$ , the statement “ $\xi$  is a Lipschitz function” must be interpreted as “there exists an extension of  $\xi$  from  $\mathbb{N}^q$  to  $\mathbb{R}_+^q$  that is Lipschitz”.

The previous conditions are not optimal. For instance, in the next section (see Proposition 47), we state that in the model of Example 5, (while there is no Lipschitz extension for  $\xi$  over all  $\mathbb{R}_+^q$ ) the  $V \log V$  condition is sufficient to ensure the  $L^1$  convergence to a non-degenerate random variable in (4.7).

To prove Proposition 44, we start by stating and proving a lemma that is useful for the proof and for the next section.

**Lemma 45.** *Consider  $p$  non-negative independent and integrable random variables  $X_1, \dots, X_p$ . Set  $z \in \mathbb{N}^p$  and  $\beta > 0$ , then*

$$\mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (X_i^{(k)} - \mathbb{E}(X_i)) \right| \right) \leq |z|^{\beta+1/2} + 2|z| \sum_{i=1}^p \int_{|z|^\beta}^{+\infty} x \, dF_i(x),$$

where  $(X_1^{(k)}, \dots, X_p^{(k)})_{k \in \mathbb{N}}$  are i.i.d. copies of  $(X_1, \dots, X_p)$ , and  $F_i(x) = \mathbb{P}(X_i \leq x)$ .

*Proof.* We have

$$\begin{aligned} \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (X_i^{(k)} - \mathbb{E}(X_i)) \right| \right) &\leq \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (X_i^{(k)} \mathbf{1}_{X_i^{(k)} \leq |z|^\beta} - \mathbb{E}(X_i \mathbf{1}_{X_i \leq |z|^\beta})) \right| \right) \\ &\quad + \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (X_i^{(k)} \mathbf{1}_{X_i^{(k)} > |z|^\beta} - \mathbb{E}(X_i \mathbf{1}_{X_i > |z|^\beta})) \right| \right). \end{aligned}$$

We bound the two expectations above separately. For the first one we have, setting  $Y_i^{(k)} = X_i^{(k)} \mathbf{1}_{X_i^{(k)} \leq |z|^\beta}$ ,

$$\begin{aligned} \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (Y_i^{(k)} - \mathbb{E}(Y_i^{(k)})) \right| \right)^2 &\leq \mathbb{E} \left( \left( \sum_{i=1}^p \sum_{k=1}^{z_i} (Y_i^{(k)} - \mathbb{E}(Y_i^{(k)})) \right)^2 \right) \\ &= \sum_{i=1}^p \sum_{k=1}^{z_i} \mathbb{E} \left( (Y_i^{(k)} - \mathbb{E}(Y_i^{(k)}))^2 \right) \\ &= \sum_{i=1}^p \sum_{k=1}^{z_i} \text{Var} \left( X_i^{(k)} \mathbf{1}_{X_i^{(k)} \leq |z|^\beta} \right) \leq \sum_{i=1}^p \sum_{k=1}^{z_i} |z|^{2\beta} \\ &= |z|^{2\beta+1}. \end{aligned}$$

For the second term, we have

$$\begin{aligned} \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (X_i^{(k)} \mathbf{1}_{X_i^{(k)} > |z|^\beta} - \mathbb{E}(X_i^{(k)} \mathbf{1}_{X_i^{(k)} > |z|^\beta})) \right| \right) &\leq 2 \sum_{i=1}^p \sum_{k=1}^{z_i} \mathbb{E} \left( X_i^{(k)} \mathbf{1}_{X_i^{(k)} > |z|^\beta} \right) \\ &= 2 \sum_{i=1}^p z_i \int_0^{+\infty} x \mathbf{1}_{x > |z|^\beta} \, dF_i(x) \\ &\leq 2|z| \sum_{i=1}^p \int_{|z|^\beta}^{+\infty} x \, dF_i(x), \end{aligned}$$

and the result follows.  $\square$

The following lemma is also a key ingredient of the proof.

**Lemma 46.** [See [Kle85] - Page 52] Let  $f$  be a positive function on  $[1, +\infty)$  such that  $x \mapsto \frac{f(x)}{x}$  is non-increasing and

$$\int_1^{+\infty} \frac{f(x)}{x^2} dx < +\infty.$$

Then there exists a monotone increasing function  $\hat{f}$  such that for all  $x \in \mathbb{R}_+$ ,  $\hat{f}(x) \geq f(x)$ ,  $x \rightarrow \hat{f}(x)/x$  is non-increasing,  $\int_1^{+\infty} (\hat{f}(x)/x^2) dx < +\infty$  and  $\hat{f}$  is concave on  $\mathbb{R}_+$ .

*Proof of Proposition 44.* Since  $\mathcal{P}$  is continuous, positive on  $\mathbb{R}_+^p \setminus \{0\}$  and positively homogeneous, there exist  $L_1, L_2 > 0$  such that, for all  $z \in \mathbb{R}_+^p$ ,

$$L_1|z| \leq \mathcal{P}(z) \leq L_2|z|. \quad (4.17)$$

Let us consider first  $z \in \mathbb{N}^p \setminus \{0\}$ . Since  $\mathcal{P}$  is Lipschitz, there exists  $K_1 > 0$  such that

$$\begin{aligned} \mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(z)| \mid Z_0 = z) &= \mathbb{E}(|\mathcal{P}(Z_1) - \mathcal{P}(\mathfrak{M}(z))| \mid Z_0 = z) \\ &\leq K_1 \mathbb{E}(|Z_1 - \mathfrak{M}(z)| \mid Z_0 = z) \\ &\leq K_1 \left( \mathbb{E}(|Z_1 - \xi(z\mathbb{V})| \mid Z_0 = z) + |z| \left| \frac{\xi(z\mathbb{V})}{|z|} - \frac{\mathfrak{M}(z)}{|z|} \right| \right). \end{aligned}$$

Using that  $\xi$  is also Lipschitz, there exists  $K_2 > 0$  such that

$$\begin{aligned} \mathbb{E}(|Z_1 - \xi(z\mathbb{V})| \mid Z_0 = z) &\leq K_2 \sum_{j=1}^q \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} V_{i,j}^{(k)} - (z\mathbb{V})_j \right| \right) \\ &= K_2 \sum_{j=1}^q \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (V_{i,j}^{(k)} - \mathbb{E}(V_{i,j}^{(k)})) \right| \right) \end{aligned}$$

Making use of Lemma 45 with  $\beta \in (0, 1/2)$  and applying (4.16), we obtain that there exists  $K_3$  such that for all  $z \in \mathbb{N}^p$ ,

$$\mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(z)| \mid Z_0 = z) \leq K_3 \left( |z|^{\beta+1/2} + |z| \sum_{i=1}^p \sum_{j=1}^q \int_{|z|^\beta}^{+\infty} x dF_{i,j}(x) + |z|^{1-\alpha} \right),$$

for  $\alpha > 0$  and  $F_{i,j}(x) = \mathbb{P}(V_{i,j} \leq x)$ .

To extend the previous bound to  $z \notin \mathbb{N}^p$ , note that if  $z \in \mathbb{R}_+^p$  with  $|z| > 2p$ , we have  $0 < \frac{1}{2}|z| \leq \lfloor |z| \rfloor \leq |z|$ . Hence there exist  $K_4, K_5 > 0$  such that for all  $z \in \mathbb{R}_+^p$  with  $|z| > 2p$ ,

$$\mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(\lfloor |z| \rfloor)| \mid Z_0 = \lfloor |z| \rfloor) \leq K_4 \left( \lfloor |z| \rfloor \sum_{i=1}^p \sum_{j=1}^q \int_{K_5 \lfloor |z| \rfloor^\beta}^{+\infty} x dF_{i,j}(x) + \lfloor |z| \rfloor^{\beta+1/2} + \lfloor |z| \rfloor^{1-\alpha} \right).$$

Finally, applying (4.17), we get that there exists  $C_1, C_2 > 0$ , such that for all  $z \in \mathbb{R}_+^p$  with  $|z| > 2p$

$$\mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(\lfloor |z| \rfloor)| \mid Z_0 = \lfloor |z| \rfloor) \leq C_1 \left( \mathcal{P}(z) \sum_{i=1}^p \sum_{j=1}^q \int_{C_2^\beta \mathcal{P}(z)^\beta}^{+\infty} x dF_{i,j}(x) + \mathcal{P}(z)^{\beta+1/2} + \mathcal{P}(z)^{1-\alpha} \right).$$

Now define

$$C_3 := \max_{z \in \mathbb{N}^p: |z| \leq 2p} \mathbb{E}(|\mathcal{P}(Z_1) - \mathcal{P}(\lfloor z \rfloor)| \mid Z_0 = z) < +\infty.$$

Then, for all  $z \in \mathbb{R}_+^p$ ,

$$\begin{aligned} & \mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(\lfloor z \rfloor)| \mid Z_0 = \lfloor z \rfloor) \\ & \leq C_1 \left( \mathcal{P}(z) \sum_{i=1}^p \sum_{j=1}^q \int_{C_2^\beta \mathcal{P}(z)^\beta}^{+\infty} x \, dF_{i,j}(x) + \mathcal{P}(z)^{\beta+1/2} + \mathcal{P}(z)^{1-\alpha} \right) + C_3. \end{aligned}$$

This implies that for all  $y \in \mathbb{R}_+$ ,

$$\sup_{z: \mathcal{P}(z)=y} \mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(\lfloor z \rfloor)| \mid Z_0 = \lfloor z \rfloor) \leq C_1 \left( y \sum_{i=1}^p \sum_{j=1}^q \int_{C_2^\beta y^\beta}^{+\infty} x \, dF_{i,j}(x) + y^{\beta+1/2} + y^{1-\alpha} \right) + C_3.$$

Now set  $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  given by

$$F(y) = C_1 \left( y \sum_{i=1}^p \sum_{j=1}^q \int_{C_2^\beta y^\beta}^{+\infty} x \, dF_{i,j}(x) + y^{\beta+1/2} + y^{1-\alpha} \right) + C_3.$$

Then we have on one hand that

$$\frac{F(y)}{y} = C_1 \left( \sum_{i=1}^p \sum_{j=1}^q \int_{C_2^\beta y^\beta}^{+\infty} x \, dF_{i,j}(x) + y^{\beta-1/2} + y^{-\alpha} \right) + \frac{C_3}{y},$$

defines a non-increasing function on  $(0, +\infty)$  since  $\beta \in (0, 1/2)$  and  $\alpha > 0$ . On the other hand,

$$\int_1^{+\infty} \frac{F(y)}{y^2} \, dy = C_1 \left( \sum_{i=1}^p \sum_{j=1}^q \int_1^{+\infty} \int_{C_2^\beta y^\beta}^{+\infty} \frac{x}{y} \, dF_{i,j}(x) \, dy + \int_1^{+\infty} \left( \frac{1}{y^{3/2-\beta}} + \frac{1}{y^{1+\alpha}} \right) \, dy \right) + \int_1^{+\infty} \frac{C_3}{y^2} \, dy.$$

Once again, since  $\alpha > 0$  and  $\beta \in (0, 1/2)$ , we only need to prove that the first integral is finite. In fact, applying Fubini's Theorem,

$$\int_1^{+\infty} \int_{C_2^\beta y^\beta}^{+\infty} \frac{x}{y} \, dF_{i,j}(x) \, dy \leq \int_0^{+\infty} x \int_1^{\frac{1}{C_2^\beta} x^{1/\beta}} \frac{dy}{y} \, dF_{i,j}(x) = \int_0^{+\infty} x O(\log x) \, dF_{i,j}(x),$$

where the last integral is finite, since  $\mathbb{E}(V_{i,j} \log V_{i,j}) < +\infty$  by assumption.

Applying Lemma 46, there exists a concave monotone increasing function  $U : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ , with  $y \rightarrow U(y)/y$  non-increasing and  $\int_1^{+\infty} \frac{U(y)}{y^2} \, dy < +\infty$ , and such that for all  $y \in \mathbb{R}_+$ ,

$$\sup_{z: \mathcal{P}(z)=y} \mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(\lfloor z \rfloor)| \mid Z_0 = \lfloor z \rfloor) \leq F(y) \leq U(y).$$

The proof is then complete. □



We finish with the following convergence for the models of completely promiscuous mating. This example does not satisfy the assumptions of Proposition 44 since  $\xi : \mathbb{N}^q \rightarrow \mathbb{N}^p$  does not have a Lipschitz extension.

**Proposition 47.** *Consider the model in Example 11. Assume in addition that  $\mathbb{E}(X_{i,j} \log X_{i,j})$  is finite for all  $i, j \in \{1, \dots, p\}$ . Then, the rescaled process  $Z_n/(\lambda^*)^n$  converges almost surely and in  $L^1$  to a non-degenerate random vector, with the same direction as  $z^*$ .*

*Proof of Proposition 47.* We recall that the case of the completely promiscuous mating corresponds to setting  $p = q_f$  and

$$\xi((x_1, \dots, x_p), (y_1, \dots, y_{q_m})) = (x_1, \dots, x_p) \prod_{j=1}^{q_m} \mathbb{1}_{y_j > 0}.$$

Hence, we have  $\mathfrak{M}(z) = z\mathbb{X}\mathbb{1}_{z\mathbb{Y} > 0}$ . Since, by assumption,  $\mathbb{Y}_{i,j} > 0$  for all  $i \in \{1, \dots, p\}$  and all  $j \in \{1, \dots, q_m\}$ , we have  $\mathfrak{M}(z) = z\mathbb{X}$ . Then, as in the multi-type asexual case

$$\mathcal{P}(z) = K\langle z^*, z \rangle,$$

where  $z^*$  is the positive right eigenvector of  $\mathbb{X}$  and  $K$  a suitable constant.

Consider  $z \in \mathbb{N}^p$ , then

$$\begin{aligned} \mathbb{E}(|\mathcal{P}(Z_1) - \lambda^*\mathcal{P}(z)| | Z_0 = z) &= K\mathbb{E}(|\langle z^*, Z_1 \rangle - \langle z^*, z\mathbb{X} \rangle| | Z_0 = z) \\ &= K\mathbb{E}(|\langle z^*, Z_1 - z\mathbb{X} \rangle| | Z_0 = z) \\ &\leq C_0 \sum_{j=1}^p \mathbb{E}(|Z_{1,j} - (z\mathbb{X})_j| | Z_0 = z), \end{aligned}$$

for some constant  $C_0 > 0$  and hence

$$\begin{aligned} \mathbb{E}(|\mathcal{P}(Z_1) - \lambda^*\mathcal{P}(z)| | Z_0 = z) &\leq C_0 \sum_{j=1}^p \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} X_{i,j}^{(k)} \mathbb{1}_{\forall \ell \leq q_m, M_{1,\ell} > 0} - \sum_{i=1}^p z_i \mathbb{X}_{i,j} \right| \right) \\ &\leq C_0 \sum_{j=1}^p \mathbb{E} \left( \left| \sum_{i=1}^p \sum_{k=1}^{z_i} (X_{i,j}^{(k)} - \mathbb{X}_{i,j}) \right| \right) \\ &\quad + C_0 \sum_{j=1}^p \sum_{i=1}^p \sum_{k=1}^{z_i} \mathbb{E} \left( X_{i,j}^{(k)} \mathbb{1}_{\exists \ell \leq q_m, M_{1,\ell} = 0} | Z_0 = z \right), \end{aligned}$$

with  $(X^{(k)})_{k \in \mathbb{N}}$  a family of i.i.d copies of  $X = (X_{i,j})_{1 \leq i, j \leq p}$ . For the second term, we have for  $i, j \in \{1, \dots, p\}$  and  $k \in \{1, \dots, z_i\}$  fixed

$$\begin{aligned} \mathbb{E} \left( X_{i,j}^{(k)} \mathbb{1}_{\exists \ell \leq q_m, M_{1,\ell} = 0} | Z_0 = z \right) &\leq \sum_{\ell=1}^{q_m} \mathbb{E} \left( X_{i,j}^{(k)} \mathbb{1}_{M_{1,\ell} = 0} | Z_0 = z \right) \\ &= \sum_{\ell=1}^{q_m} \mathbb{E} \left( X_{i,j}^{(k)} \prod_{i'=1}^p \prod_{k'=1}^{z_{i'}} \mathbb{1}_{Y_{i',\ell}^{(k')} = 0} \right) \\ &\leq \sum_{\ell=1}^{q_m} \mathbb{X}_{i,j} \prod_{i'=1}^p \prod_{\substack{k'=1 \\ (i',k') \neq (i,k)}}^{z_{i'}} \mathbb{P} \left( Y_{i',\ell}^{(k')} = 0 \right) \\ &\leq \mathbb{X}_{i,j} q_m \gamma^{|z| - 1}, \end{aligned}$$

with  $\gamma = \max_{i' \leq p, \ell \leq q_m} \mathbb{P}(Y_{i', \ell} = 0) \in [0, 1)$ . Then, applying Lemma 45 with  $\beta \in (0, 1/2)$ , we deduce that there exists  $C_1 > 0$  such that, for all  $z \in \mathbb{N}^p$ ,

$$\mathbb{E}(|\mathcal{P}(Z_1) - \lambda^* \mathcal{P}(z)| | Z_0 = z) \leq C_1 \left( |z| \sum_{i=1}^p \sum_{j=1}^q \int_{|z|^\beta}^{+\infty} x \, dF_{i,j}(x) + |z|^{\beta+1/2} + |z|^\gamma |z|^{-1} \right).$$

The result then follows as in the proof of Proposition 44. □

#### 4.2.4 Simulations

In this section we work again with the perfect fidelity model in dimension 2. We consider all the offspring distributions to be issued according to a Poisson distribution. We consider the expectation matrices

$$\mathbb{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0.8 \end{pmatrix} \text{ and } \mathbb{Y} = \begin{pmatrix} 1 & 1.2 \\ 0.5 & 0.8 \end{pmatrix}.$$

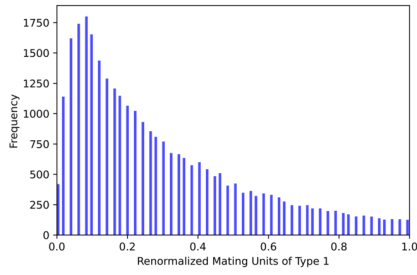
For this model we approximate numerically the associated eigenvalue of the function  $\mathfrak{M}$  obtaining  $\lambda^* \approx 1.47$ . We simulate  $10^5$  trajectories starting from  $z_0 = (1, 2)$  and compute  $\frac{Z_n}{(\lambda^*)^n}$  for the values of  $n \in \{10, 15, 20, 25, 30, 40, 50, 100\}$ .

In Figure 4.3 in page 4.3 we see the histograms approximating the density function of the variable  $\frac{Z_n}{(\lambda^*)^n}$ . We appreciate that the figures are more and more similar the higher is the value of  $n$ . To compute this similarity, we approximate the Wassersteins distance between the histograms, for all the consecutive values of  $n$ .

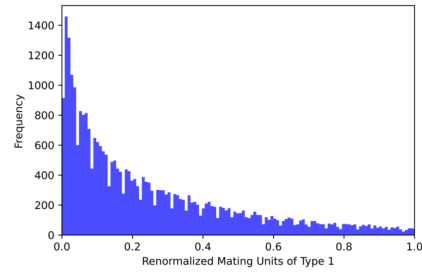
Values of $n$	Wassersteins Distance
$n = 10$ and $n = 15$	0,0453
$n = 15$ and $n = 20$	0,0056
$n = 20$ and $n = 25$	0,0016
$n = 25$ and $n = 30$	0,0014
$n = 30$ and $n = 40$	0,0009
$n = 40$ and $n = 50$	0,0001
$n = 50$ and $n = 100$	0

Table 4.1: Wassersteins distance for consecutive chosen values  $n$

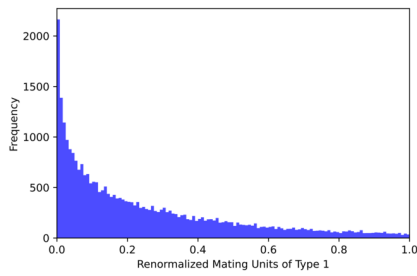
In Table 4.1 we see that the distance between the histograms decreases until obtaining a null distance when comparing the values of  $n = 50$  and  $n = 100$ . Thus, illustrating the convergence of the distribution of  $\frac{Z_n}{(\lambda^*)^n}$ .



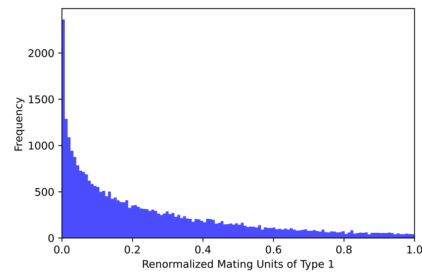
(a)  $n = 10$ .



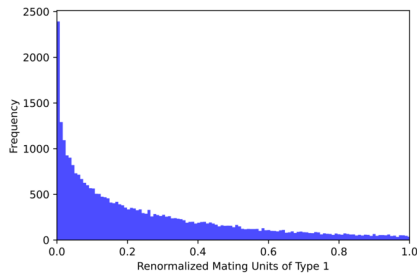
(b)  $n = 15$ .



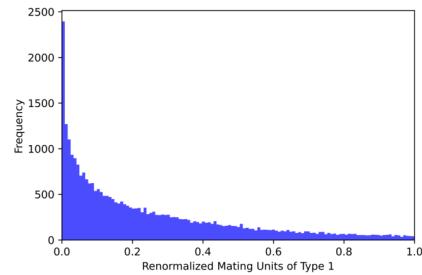
(c)  $n = 20$ .



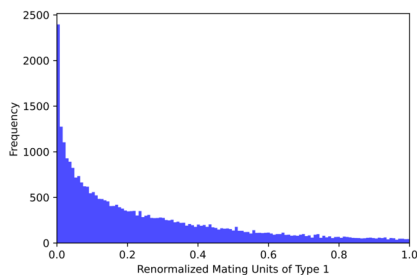
(d)  $n = 25$ .



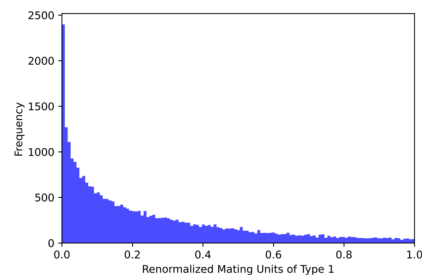
(e)  $n = 30$ .



(f)  $n = 40$ .



(g)  $n = 50$ .



(h)  $n = 100$ .

Figure 4.3: Approximation for the density functions of  $\frac{Z_{n,1}}{(\lambda^*)^n}$  for different values of  $n$ , computed with  $10^5$  simulations of the two-dimensional perfect fidelity mating model initialized with  $z_0 = (1, 2)$ .



# Chapter 5

## Quasi-limiting behaviour in the subcritical case

### Contents

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In this chapter we study the quasi-limiting behaviour of a multi-type bisexual branching process  $(Z_n)_{n \in \mathbb{N}}$ . We assume that the mating function is superadditive (Assumption 3) and that the process is primitive (Assumption 4). This ensures the existence of eigenelements for the function  $\mathfrak{M}$  given by Theorem 18. We also assume that the process is subcritical, meaning that the eigenvalue  $\lambda^*$  associated to the function  $\mathfrak{M}$  holds  $\lambda^* \leq 1$  (see Theorem 37). This problem has been studied and largely solved decades ago for asexual subcritical Galton Watson processes (see e.g. [Yag47, SVJ66, AV01]), but remains open for its bisexual counterpart.

For any probability measure  $\mu$  on  $\mathbb{N}^p$ , we use the notation  $\mathbb{P}_\mu$  and  $\mathbb{E}_\mu$  for the law and associated expectation of the multi-type bisexual Galton-Watson process with initial number of couples  $Z_0$  distributed according to  $\mu$ . As usual, we use the notations  $\mathbb{P}_z$  and  $\mathbb{E}_z$  when  $\mu = \delta_z$  for some  $z \in \mathbb{N}^p$ .

We are interested in conditions ensuring the existence of a quasi-stationary distribution for  $(Z_n)_{n \in \mathbb{N}}$ . We recall that a *quasi-stationary distribution* for this process is a probability measure  $\nu_{QS}$  on  $\mathbb{N}^p \setminus \{0\}$  such that, for all  $n \geq 0$ ,  $\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) > 0$  and

$$\mathbb{P}_{\nu_{QS}}(Z_n \in \cdot \mid Z_n \neq 0) = \nu_{QS}(\cdot).$$

It is known that (we refer the reader to the book [CMSM13] and to the surveys [vDP13, MV12] for general properties and further examples), for any quasi-stationary distribution  $\nu_{QS}$ , there exists  $\theta_{QS} \in (0, 1]$  such that

$$\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) = \theta_{QS}^n \text{ and } \mathbb{P}_{\nu_{QS}}(Z_n \in \cdot, Z_n \neq 0) = \theta_{QS}^n \nu_{QS}.$$

In what follows,  $\theta_{QS}$  is referred to as the *absorption parameter* of  $\nu_{QS}$ .

In Section 5.1 we prove that the process admits infinitely many quasi-stationary distributions. Then, under an additional polynomial moment assumption, we prove in Section 5.2 that the process admits a finitely generated family of quasi-stationary distributions. Finally, under an additional irreducibility assumption, we show in Section 5.3 that the process admits a unique quasi-limiting distribution for compactly supported initial distributions. Although this is similar to the classical Galton-Watson case,  $\lambda^*$  is not in general equal to the exponential survival rate of the process.

Throughout this chapter we add the extra assumption that the mating function is sub-affine.

**Assumption 6.** There exists  $\alpha, \beta \in \mathbb{R}_+^p$  such that

$$\alpha|x_1 + x_2| + \beta \geq \xi(x_1 + x_2) \geq \xi(x_1) + \xi(x_2), \quad \forall x_1, x_2 \in \mathbb{N}^q. \quad (5.1)$$

Assumption 6 implies in particular that  $\mathfrak{M}$  is bounded over  $S$ .

## 5.1 A criterion for the existence of infinitely many quasi-stationary distributions

In this section we prove that in the subcritical case,  $(Z_n)_{n \in \mathbb{N}}$  admits infinitely many quasi-stationary distributions. Our approach will be first to state and prove a general result ensuring the existence of infinitely many quasi-stationary distributions for Feller processes, and then apply it to our process.

### 5.1.1 General setting

Let  $(E, d)$  be a Polish space endowed with a  $\sigma$ -algebra  $\mathcal{E}$  that contains the topology induced by  $d$ . Let  $K$  be a sub-Markov kernel on  $E$ , which means that  $K$  is a non-negative kernel on  $E$  such that  $K(x, E) \leq 1$  for all  $x \in E$ . We assume that  $K$  has Feller regularity, meaning that the associated operator  $K : f \mapsto Kf := \int_E K(\cdot, dy)f(y)$  is a well defined functional from  $C_b(E)$  to  $C_b(E)$  (the set of continuous bounded functions on  $E$ ). We say that a probability measure  $\mu$  on  $E$  is a quasi-stationary distribution for  $K$  if, for some  $\lambda \in (0, 1]$ ,

$$\int_E \mu(dx)K(x, \cdot) = \lambda\mu.$$

Our aim is to prove that  $K$  admits infinitely many quasi-stationary distributions under mild assumptions.

In the following theorem, we say that a sequence  $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$  tends to infinity if it eventually leaves any compact set: for any compact subset  $L$  of  $E$ , there exists  $n_L \geq 0$  such that, for all  $n \geq n_L$ ,  $x_n \notin L$ . Similarly, we say that a real functional on  $E$  tends to infinity when  $x \rightarrow +\infty$  if, for all  $R > 0$ , there exists a compact subset  $L_R$  of  $E$  such that the functional is bounded below by  $R$  on  $E \setminus L_R$ . We also define the iterated kernels  $K^n$  as usual by

$$K^0(x, A) = \mathbf{1}_{x \in A}, \quad K^{\ell+1}(x, A) = \int_E K^\ell(x, dy)K(y, A) \quad \forall x \in E, A \in \mathcal{E}, \ell \geq 0.$$

**Theorem 48.** Assume that there exist  $\theta_1 > \theta_0 > 0$  and  $(x_n)_{n \in \mathbb{N}} \in E^{\mathbb{N}}$  tending to infinity such that

$$\sum_{\ell=0}^{\infty} \theta_0^{-\ell} K^\ell(x_n, E) < +\infty, \quad \forall n \geq 0 \quad (5.2)$$

and

$$\sum_{\ell=0}^{\infty} \theta_1^{-\ell} K^\ell(x, E) \xrightarrow{x \rightarrow +\infty} +\infty. \quad (5.3)$$

Then  $K$  admits an infinite family of quasi-stationary distributions  $(\mu_\lambda)_{\lambda \in [\theta_0, \theta_1)}$ , with  $\int_E \mu_\lambda(dx) K(x, \cdot) = \lambda \mu$  for all  $\lambda \in [\theta_0, \theta_1)$ .

*Proof of Theorem 48.* Fix  $\lambda \in [\theta_0, \theta_1)$  and consider, for all  $n \geq 0$ , the probability measure

$$\mu_n = \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, E)},$$

which is well defined since  $\lambda \geq \theta_0$ . We have, for all  $n \geq 0$ ,

$$\begin{aligned} \mu_n K &:= \int_E \mu_n(dx) K(x, \cdot) = \frac{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, E)} \\ &= \frac{\lambda \sum_{\ell=0}^{\infty} \lambda^{-(\ell+1)} K^{\ell+1}(x_n, \cdot)}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, E)} \\ &= \frac{\lambda \sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, \cdot) - \lambda \delta_x}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, E)} \\ &\leq \lambda \mu_n. \end{aligned} \quad (5.4)$$

We deduce that, for all  $\ell \geq 1$ ,

$$\mu_n K^\ell \leq \lambda^\ell \mu_n.$$

In particular, we obtain that

$$\int_E \mu_n(dx) \sum_{\ell=0}^{\infty} \theta_1^{-\ell} K^\ell(x, E) = \sum_{\ell=0}^{\infty} \theta_1^{-\ell} \mu_n K^\ell(E) \leq \sum_{\ell=0}^{\infty} \left(\frac{\lambda}{\theta_1}\right)^\ell = \frac{\theta_1}{\theta_1 - \lambda}.$$

Since we assumed that  $\sum_{\ell=0}^{\infty} \theta_1^{-\ell} K^\ell(x, E)$  goes to infinity when  $x \rightarrow \infty$ , this and Prokhorov's Theorem imply that the sequence of measures  $(\mu_n)_{n \in \mathbb{N}}$  is relatively compact for the weak convergence topology.

Let  $\mu_\infty$  be any adherent point of  $(\mu_n)_{n \in \mathbb{N}}$ , and observe that  $\mu_\infty$  is a probability measure. Since  $K$  is assumed to preserve bounded continuous functions, we deduce that, up to a subsequence,

$$\mu_\infty K = \lim_{n \rightarrow +\infty} \mu_n K.$$

But we computed (see (5.4))

$$\mu_n K = \lambda \mu_n - \frac{\lambda \delta_x}{\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, E)},$$

where, by assumption,

$$\sum_{\ell=0}^{\infty} \lambda^{-\ell} K^\ell(x_n, E) \geq \sum_{\ell=0}^{\infty} \theta_1^{-\ell} K^\ell(x_n, E) \xrightarrow{n \rightarrow +\infty} +\infty.$$

We deduce that  $\lim_{n \rightarrow +\infty} \mu_n K = \lim_{n \rightarrow +\infty} \lambda \mu_n$ , which is equal to  $\lambda \mu_\infty$ . We thus proved that  $\mu_\infty K = \lambda \mu_\infty$ , which means that  $\mu_\infty$  is a quasi-stationary distribution for  $K$  with absorption parameter  $\lambda$ .  $\square$

*Remark 16.* One easily checks from the proof that the condition that  $K$  preserves bounded continuous functions is stronger than necessary. Actually, it would be sufficient to assume that  $K$  sends a generating family of bounded continuous functions into a family of bounded continuous functions. For instance, it is sufficient to assume that  $K$  sends compactly supported continuous functions to bounded continuous functions.

*Remark 17.* One can easily extend this result to the general setting of non-conservative kernels  $K$ . More precisely, assume that there exists a positive function  $h$  on  $E$  such that  $Kh \leq ch$  for some  $c > 0$ . Then the kernel defined by

$$\widetilde{K}(x, A) = \frac{1}{ch(x)} \int_A K(x, dy)h(y), \quad \forall x \in E, \quad \forall A \in \mathcal{E},$$

is a sub-Markov kernel to which Theorem 48 may apply. Note that the assumptions would translate into: for any continuous function  $f : E \rightarrow \mathbb{R}$ ,

$$x \in E \mapsto \frac{1}{h(x)} \int_E K(x, dy)h(y)f(y)$$

is continuous; for all  $n \geq 0$ ,

$$\sum_{\ell=0}^{\infty} \theta_0^{-\ell} \int_E K^\ell(x_n, dy)h(y) < +\infty$$

and

$$\frac{1}{h(x)} \sum_{\ell=0}^{\infty} \theta_1^{-\ell} \int_E K^\ell(x_n, dy)h(y) \xrightarrow{x \rightarrow +\infty} +\infty.$$

In addition, the resulting quasi-stationary measures may not be a finite measure if  $h$  is not lower bounded (we only obtain that  $\int_E \mu_\lambda(dy)h(y) < +\infty$  for all  $\lambda \in [\theta_0, \theta_1)$ ). On the other hand, if  $h$  is lower bounded away from 0, then  $\mu_\lambda$  is a finite measure.

### 5.1.2 Application to the subcritical bisexual branching process

In this section we prove that Theorem 48 can be applied to the process  $(Z_n)_{n \in \mathbb{N}}$ . We recall that we are assuming Assumption 6, that is, the mating function  $\xi$  is sub-affine. Although as will appear in the proof, for Theorem 49 we do not use the fact that  $\xi$  is sub-affine, but the fact that this implies  $\mathfrak{M}$  is bounded over  $S$ .

**Theorem 49.** *If Assumption 6 holds true and  $\lambda^* < 1$ , then  $(Z_n)_{n \in \mathbb{N}}$  admits an infinite dimensional set of quasi-stationary distributions. More precisely, for any  $\theta \in (\lambda^*, 1)$ , there exists a quasi-stationary distribution with absorption parameter  $\theta$ .*

In the proof we make use of the function  $\mathcal{P} : \mathbb{N}^p \rightarrow \mathbb{R}_+$  given by Theorem 18. We recall that it corresponds to the function

$$\mathcal{P}(z) = \lim_{n \rightarrow +\infty} \frac{|\mathfrak{M}^n(z)|}{(\lambda^*)^n}.$$

*Proof of Theorem 49.* Set  $K(x, dy) = \mathbb{P}_x(Z_1 \in dy, Z_1 \neq 0)$  for all  $x \in E = \mathbb{N}^p \setminus \{0\}$ .

Using the fact that  $\mathcal{P}$  is lower bounded away from 0 on  $\mathbb{N}^p \setminus \{0\}$  (as  $\mathcal{P}$  is strictly positive and positively homogeneous) and  $K$  vanishes at 0, we deduce that there exists a constant  $c > 0$  such that, for all  $\ell \geq 1$  and all  $x \in \mathbb{N}^p$ ,

$$K^\ell(x, E) \leq \frac{1}{c} \mathbb{E}_x(\mathcal{P}(Z_\ell)).$$



In addition, in Chapter 4, Section 4.2.1, we show that  $((\lambda^*)^{-n}\mathcal{P}(Z_n))_{n \in \mathbb{N}}$  is a supermartingale so that, for any  $\theta_0 \in (\lambda^*, 1)$ ,

$$\sum_{\ell=0}^{\infty} \theta_0^{-\ell} K^\ell(x, E) \leq \frac{1}{c} \sum_{\ell=0}^{\infty} \theta_0^{-\ell} (\lambda^*)^\ell \mathcal{P}(x) < +\infty.$$

This shows that (5.2) holds true for any sequence  $(x_n)_{n \in \mathbb{N}}$ .

Since  $\mathfrak{M}$  is primitive, Theorem 22 in Chapter 3 implies that there exists  $n_0 \in \mathbb{N}$ , such that for all  $z \in \mathbb{N}^p \setminus \{0\}$ , there exists  $c_0 > 0$  and  $k_0 \geq 1$  such that, for all  $k \geq k_0$ ,

$$\mathbb{P}(Z_{n_0} \geq c_0(1, \dots, 1) \mid Z_0 = kz) > 0.$$

Applying this result for  $z$  taken in each element of the canonical basis of  $\mathbb{N}^p$  and using the superadditivity of  $\xi$ , one deduces that there exist uniform constants  $r > 0$  and  $c'_0 > 0$  such that

$$\inf_{|z| > r} \mathbb{P}(Z_{n_0} \geq c'_0(1, \dots, 1) \mid Z_0 = z) > 0.$$

Using the Markov property and by iteration, this implies that, for all  $n \geq 1$ ,

$$\mathbb{P}_z(Z_n = 0) \xrightarrow{|z| \rightarrow +\infty} 0.$$

In particular, we deduce that, for any  $\theta_1 \in (0, 1)$ ,

$$\sum_{\ell=0}^{\infty} \theta_1^{-\ell} \mathbb{P}_z(Z_\ell \neq 0) \xrightarrow{|z| \rightarrow +\infty} +\infty.$$

and hence (5.3) holds true.

We deduce that Theorem 48 applies to the subcritical bisexual branching process, which proves Theorem 49.  $\square$

## 5.2 Finite dimensional family of quasi-stationary distributions

In this section we consider the problem of existence of a finitely generated set of quasi-stationary distributions satisfying an integrability assumption and of convergence of conditional laws toward a quasi-stationary distribution. Since  $\mathfrak{M}$  is concave on  $\mathbb{R}_+^p$ , it is locally Lipschitz on  $S^*$ . We add the following regularity assumption on  $\mathfrak{M}$ , where we use the notation, for all  $I \subset \{1, \dots, q\}$ ,

$$\mathcal{Z}_I := \{x \in S \text{ such that } x^i > 0 \forall i \in I, x^i = 0 \forall i \notin I\}.$$

**Assumption 7.** For all  $I \neq \emptyset$ ,  $\mathfrak{M}$  is uniformly continuous over  $\mathcal{Z}_I$ .

We also consider the following moment condition, based on the *exponential absorption parameter*  $\theta_0 \in [0, 1]$ , defined by

$$\theta_0 = \sup_{z \in \mathbb{N}^p \setminus \{0\}} \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n = z) > 0\},$$

with the convention  $\sup \emptyset = 0$ .

**Assumption 8.** There exists  $\eta > 1$  such that  $(\lambda^*)^\eta < \theta_0$  and such that  $\mathbb{E}(V_{i,j}^\eta) < +\infty$  for all  $i, j$ .

We emphasize that Assumption 8 implies  $\lambda^* < 1$ . This assumption also requires implicitly that  $\theta_0 > 0$ , which is the case in many situations.

The following theorem is based on [CV22] and make use of the following function and measure spaces. Given a positive function  $g$  on  $\mathbb{N}^p \setminus \{0\}$ , we set

$$L^\infty(g) := \{f : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{R}, \|f/g\|_\infty < +\infty\},$$

endowed with the norm  $\|f\|_g = \|f/g\|_\infty$ , and

$$\mathcal{M}(g) := \{\mu \text{ non-negative measure on } \mathbb{N}^p \setminus \{0\} \text{ such that } \mu(g) < +\infty\}.$$

Finally, when Assumption 8 is enforced, we define, for all  $a \in (1, \eta)$  such that  $(\lambda^*)^a < \theta_0$ , the function  $Q_a : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{R}_+$  by

$$Q_a(z) = \mathcal{P}(z)^a.$$

**Theorem 50.** *We assume that Assumptions 6, 7 and 8 are in place and that the process  $(Z_n)_{n \in \mathbb{N}}$  is aperiodic. We fix  $a \in (1, \eta)$  such that  $(\lambda^*)^a < \theta_0$ .*

*Then there exist  $\ell \geq 1$  and a family of quasi-stationary distributions  $\nu_1, \dots, \nu_\ell \in \mathcal{M}(Q_a)$  for  $(Z_n)_{n \in \mathbb{N}}$  with absorption parameter  $\theta_0$  such that any quasi-stationary distribution  $\nu \in \mathcal{M}(Q_a)$  for  $(Z_n)_{n \in \mathbb{N}}$  with absorption parameter  $\theta_0$  is a convex combination of  $\nu_1, \dots, \nu_\ell$ . In addition, there exists a function  $j : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{N}$  and there exists, for each  $i \in \{1, \dots, \ell\}$ , a non-negative non-identically zero function  $\eta_i \in L^\infty(Q_a)$  such that, for all  $f \in L^\infty(Q_a)$ , all  $n \geq 1$  and all  $z \in \mathbb{N}^p \setminus \{0\}$ ,*

$$\left| \theta_0^{-n} n^{-j(z)} \mathbb{E}_z(f(Z_n) \mathbf{1}_{Z_n \neq 0}) - \sum_{i=1}^{\ell} \eta_i(z) \nu_i(f) \right| \leq \alpha_n Q_a(z) \|f\|_{Q_a}, \quad (5.5)$$

where  $\alpha_n$  goes to 0 when  $n \rightarrow +\infty$  which does not depend on  $f$  nor on  $z$  (but may depend on  $a$ ).

*In addition, the set of quasi-stationary distributions for  $(Z_n)_{n \in \mathbb{N}}$  in  $\mathcal{M}(Q_a)$  (not necessarily with absorption parameter  $\theta_0$ ) is included in the convex hull of a finite set of quasi-stationary distributions in  $\mathcal{M}(Q_a)$ .*

One can check that  $\theta_0 \leq \lambda^*$  and that  $\theta_0 = \lambda^*$  if and only if  $\mathcal{P}$  is linear, which is the case for instance if  $\mathfrak{M}$  is linear. When equality holds, our result only requires polynomial moments for exponents that are arbitrarily close to 1. In the situation where  $\theta_0 < \lambda^*$ , one can not choose  $\eta$  arbitrarily close to 1 in Assumption 8, since  $\eta > \frac{|\log \theta_0|}{|\log \lambda^*|}$ .

The proof of Theorem 50 is based on recent advances in the theory of quasi-stationary distributions, which allow to derive several properties from the existence of Foster-Lyapunov type criteria. The following proposition is thus one of the main results of the present chapter.

**Proposition 51.** *If Assumptions 6, 7 and 8 hold true, then, for any  $a \in (1, \eta)$  such that  $(\lambda^*)^a < \theta_0$ , we have*

$$\mathbb{E}(Q_a(Z_1) | Z_0 = z) \leq \theta_a Q_a(z) + C_a, \quad \forall z \in \mathbb{N}^p \setminus \{0\} \quad (5.6)$$

for some constants  $\theta_a \in [0, \theta_0)$  and  $C_a > 0$ .

We idea is to prove first Proposition 51 to then give the proof of Theorem 50. We start with the following technical lemma on the regularity of  $\mathcal{P}$ ,

**Lemma 52.** *Assume that Assumptions 6, 7 and 8 hold true. Then, for all  $I \subset \{1, \dots, q\}$ ,  $I \neq \emptyset$ , the operator  $\mathcal{P}$  restricted to  $\mathcal{Z}_I$  is uniformly continuous.*

*Proof of Lemma 52.* Since  $\mathcal{P} \circ \mathfrak{M}^{n_0} = (\lambda^*)^{n_0} \mathcal{P}$ , it is sufficient to prove that  $\mathcal{P} \circ \mathfrak{M}^{n_0}$  is uniformly continuous on  $\mathcal{Z}_I$ . But  $\mathfrak{M}^{n_0} \mathcal{Z}_I \subset \mathfrak{M}^{n_0} S$  which is relatively compact in  $(\mathbb{R}_+^*)^q$  (since  $\mathfrak{M}^{n_0}$  is primitive and bounded by assumption), and  $\mathcal{P}$  is concave and thus locally Lipschitz in  $(\mathbb{R}_+^*)^q$ , thus it is sufficient to prove that  $\mathfrak{M}^{n_0}$  is uniformly continuous on  $\mathcal{Z}^I$ .

Let us first observe that  $\mathfrak{M}$  is uniformly continuous on  $(\varepsilon + \mathbb{R}_+^*) \mathcal{Z}_I$  for any  $\varepsilon > 0$ , since  $\mathfrak{M}$  is uniformly continuous and lower bounded away from zero on  $\mathcal{Z}_I$ , and homogeneous. In order to conclude, it remains to prove that there exist  $J \subset \{1, \dots, q\}$  and  $\varepsilon > 0$  such that  $\mathfrak{M}(\mathcal{Z}_I) \subset (\varepsilon + \mathbb{R}_+^*) \mathcal{Z}_J$  (the result then follows by induction). Let  $x \in \mathcal{Z}_I$  and let  $J = \{i \in \{1, \dots, q\}, \mathfrak{M}(x)_i > 0\}$ . Then, for all  $y \in \mathcal{Z}_I$ , there exists  $\delta > 0$  such that  $\delta x \leq y$ , so that, using the fact that  $\mathfrak{M}$  is increasing and homogeneous,  $\mathfrak{M}(y) \geq \delta x$  and hence  $J \subset \{i \in \{1, \dots, q\}, \mathfrak{M}(y)_i > 0\}$ . Reciprocally,  $\{i \in \{1, \dots, q\}, \mathfrak{M}(y)_i > 0\} \subset J$ , so that  $\mathfrak{M}(y) \in \mathbb{R}_+ \mathcal{Z}_J$ . Since in addition  $\mathfrak{M}$  is primitive and bounded, we deduce that the norm of  $\mathfrak{M}$  is bounded from below on  $S$ , and hence that  $\mathfrak{M}(\mathcal{Z}_I) \subset (\varepsilon + \mathbb{R}_+^*) \mathcal{Z}_J$  for some  $\varepsilon > 0$ .  $\square$

We proceed now with Proposition 51

*Proof of Proposition 51.* This proof is divided in two steps: first we show that, for all  $z \in \mathbb{R}_+^p \setminus \{0\}$ ,

$$\limsup_{n \rightarrow +\infty} \frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mid Z_0 = \lfloor nz \rfloor] \leq (\lambda^*)^a Q_a(z); \quad (5.7)$$

Second, we extend this to a uniform convergence in  $z$  by using the regularity of  $\mathfrak{M}$ .

*Step 1.* We have, for any  $z \in \mathbb{R}_+^p \setminus \{0\}$  and  $\varepsilon > 0$ , by the definition of  $(Z_n)_{n \in \mathbb{N}}$  given by (2.3) and the fact that  $\mathcal{P}$  is positively homogenous (see Theorem 18)

$$\begin{aligned} \mathbb{E} [Q_a(Z_1) \mid Z_0 = \lfloor nz \rfloor] &= \mathbb{E} \left[ \left( \mathcal{P} \circ \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right)^a \right] \\ &= |n|^a \mathbb{E} \left[ \left( \mathcal{P} \left( \frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \right)^a \right] \\ &\leq |n|^a \mathbb{E} \left[ \left( \mathcal{P} \left( \varepsilon \mathfrak{M}(z) + \frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \right)^a \right] \end{aligned} \quad (5.8)$$

According to Lemma 23, we have

$$\frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \xrightarrow[n \rightarrow +\infty]{a.s.} \mathfrak{M}(z).$$

Let  $I \subset \{1, \dots, p\}$  be such that  $\mathfrak{M}(z) \in \mathbb{R}_+^* \mathcal{Z}_I$ , then  $\varepsilon \mathfrak{M}(z) + \frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \in \mathbb{R}_+^* \mathcal{Z}_I$  almost surely. Indeed, if  $\varepsilon \mathfrak{M}(z) + \frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \notin \mathbb{R}_+^* \mathcal{Z}_I$  with positive probability, then, with positive probability,

$$\frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \in \mathbb{R}_+^* \mathcal{Z}_J \text{ where } I \subsetneq J.$$

In particular, since there exists  $\ell$  large enough such that

$$\sum_{i=1}^p \sum_{k=1}^{\lfloor nz_i \rfloor} V_{i,\cdot}^{(k,1)} \leq \ell z \mathbb{V}$$

with arbitrary large probability, and since  $\xi$  is non-decreasing, we deduce that, for any  $\ell$  large enough,  $\frac{1}{\ell}\xi(\ell z \mathbb{V}) \in \mathcal{Z}_{J_\ell}$  with  $I \subsetneq J_\ell$ . Passing to the (non-decreasing) limit when  $\ell \rightarrow +\infty$ , we deduce that  $\mathfrak{M}(z) \notin \mathbb{R}_+^* \mathcal{Z}_I$ , which is not correct.

Since  $\mathcal{P}$  is uniformly continuous in  $(\varepsilon + \mathbb{R}_+^*) \mathcal{Z}_I$  by Lemma 52, we deduce that

$$\mathcal{P} \left( \varepsilon \mathfrak{M}(z) + \frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor n z_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \xrightarrow[n \rightarrow +\infty]{a.s.} \mathcal{P}((1 + \varepsilon) \mathfrak{M}(z)) = (1 + \varepsilon) \lambda^* \mathcal{P}(z).$$

In addition, since  $\xi$  is sub-affine, since  $\frac{1}{n} \sum_{i=1}^p \sum_{k=1}^{\lfloor n z_i \rfloor} V_{i,\cdot}^{(k,1)}$  is bounded in  $L^\eta$  and  $\mathcal{P}$  being positively homogeneous, we deduce that  $\mathcal{P} \left( \varepsilon \mathfrak{M}(z) + \frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor n z_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right)$  is also bounded in  $L^\eta$ , and we deduce from the dominated convergence theorem that

$$\mathbb{E} \left[ \left( \mathcal{P} \left( \varepsilon \mathfrak{M}(z) + \frac{1}{n} \xi \left( \sum_{i=1}^p \sum_{k=1}^{\lfloor n z_i \rfloor} V_{i,\cdot}^{(k,1)} \right) \right) \right)^a \right] \xrightarrow[n \rightarrow +\infty]{} (1 + \varepsilon)^a (\lambda^*)^a \mathcal{P}(z)^a.$$

Since this is true for all  $\varepsilon > 0$ , we deduce from (5.8) that (5.7) holds true. This concludes the first step of the proof.

*Step 2.* Let  $\varepsilon_1 > 0$  be a small number that will be fixed later,  $m_1 = m(\varepsilon_1) \geq 1$  an integer and  $z^1, \dots, z^{m_1} \in (1 + \varepsilon_1) \mathcal{Z}_I$ , be such that, for all  $z \in \mathcal{Z}_I$ , there exists  $k = k(z, \varepsilon_1) \in \{1, \dots, m_1\}$  such that  $z \leq z^k$  and  $|z - z^k| \leq 2\varepsilon_1$ . The existence of this finite family  $z^1, \dots, z^{m_1}$  easily follows from a compactness argument. Let  $n_1 = n_1(\varepsilon_1)$  be large enough so that, for all  $n \geq n_1$ , for all  $k \in \{1, \dots, m_1\}$ ,

$$\frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mid Z_0 = \lfloor n z^k \rfloor] \leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z^k)^a. \quad (5.9)$$

where such  $n_1$  exists from (5.7). Then, since  $Q_a$  is non-decreasing and since  $Z_1$  (with  $Z_0 = \lfloor n z \rfloor$ ) is stochastically non-decreasing with respect to  $z$ , we deduce that, for all  $n \geq n_1$  and all  $z \in \mathcal{Z}_I$ ,

$$\frac{1}{|n|^a} \mathbb{E} [Q_a(Z_1) \mid Z_0 = \lfloor n z \rfloor] \leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z^{k(z, \varepsilon_1)})^a \quad (5.10)$$

$$\leq (1 + \varepsilon_1) (\lambda^*)^a \mathcal{P}(z)^a \left( \sup_{x, y \in \mathcal{Z}_I, |x-y| \leq 2\varepsilon_1} \frac{\mathcal{P}(x)}{\mathcal{P}(y)} \right)^a. \quad (5.11)$$

Since  $\mathcal{P}$  is uniformly continuous on  $\mathcal{Z}_I$  and lower bounded away from 0 on  $\mathcal{Z}_I$ , we deduce that there exists  $\varepsilon_1$  small enough so that

$$\theta_a := (1 + \varepsilon_1)^{a+1} (\lambda^*)^a \left( \sup_{x, y \in \mathcal{Z}_I, |x-y| \leq 2\varepsilon_1} \frac{\mathcal{P}(x)}{\mathcal{P}(y)} \right)^a < \theta_0.$$

Since any  $z \in \mathbb{N}^p \setminus \{0\}$  can be written as  $|z| \frac{z}{|z|}$ , we deduce that, for any  $z$  with  $|z| \geq n_1$ ,

$$\frac{1}{|z|^a} \mathbb{E} [Q_a(Z_1) \mid Z_0 = z] \leq \theta_a \mathcal{P}(z/|z|)^a. \quad (5.12)$$

Finally, the homogeneity of  $\mathcal{P}$  allows us to conclude that Proposition 51 holds true.  $\square$

Finally, we prove Theorem 50 by means of Proposition 51.

*Proof of Theorem 50.* According to Proposition 51, for any  $a \in (1, \eta)$  such that  $(\lambda^*)^a < \theta_0$ , we have

$$\limsup_{|z| \rightarrow +\infty} \frac{\mathbb{E}_z(Q_a(Z_1)\mathbf{1}_{Z_1 \neq 0})}{Q_a(z)} \leq \theta_a < \theta_0.$$

Since  $Q_a(z) \rightarrow \infty$  when  $|z| \rightarrow +\infty$ , a straightforward adaptation of Theorem 5.1 in [CV22] (see also Remark 20 therein) shows that Theorem 4.1 in the same reference applies and hence (see also Remark 16 therein) the two first statements of Theorem 50 hold true, and it only remains to prove the last one.

More precisely, it remains to prove that the set of quasi-stationary distributions for  $(Z_n)_{n \in \mathbb{N}}$  has finite dimension. We observe that Theorem 4.1 in [CV22] also implies that  $j \equiv 0$  on the set  $E = \{x \in \mathbb{N}^p \setminus \{0\}, \sum_{i \in I} \eta_i(x) = 0\}$  and we claim that  $E$  is finite.

Indeed, assume the contrary and fix  $z_0 \in \mathbb{N}^p \setminus \{0\}$  such that  $\sum_{i=1}^{\ell} \eta_i(z_0) > 0$  and  $j(z_0) = 0$  (existence of such a point  $z_0$  is guaranteed by Proposition 2.3 in [CV22]). We observe that

$$\liminf_{n \rightarrow +\infty} \theta_0^{-n} \mathbb{P}_{z_0}(Z_n \neq 0) > 0.$$

Then, by primitivity of  $\mathfrak{M}$ , we have  $n_0 \in \mathbb{N}$  such that for any  $z$  large enough  $\mathbb{P}_z(Z_{n_0} \geq z_0) > 0$ . Hence, using the fact that  $E$  is not finite and hence contains arbitrarily large points, there exists  $z \in E$  and  $z'_0 \geq z_0$  such that  $\mathbb{P}_z(Z_{n_0} = z'_0) > 0$ . By super-additivity of  $\xi$ , we have  $\mathbb{P}_{z'_0}(Z_n \neq 0) \geq \mathbb{P}_z(Z_n \neq 0)$  and hence

$$\liminf_{n \rightarrow +\infty} \theta_0^{-n} \mathbb{P}_z(Z_{n_0+n} \neq 0) > 0.$$

This is not compatible with the definition of  $E$  and (5.5). We have thus proved by contradiction that  $E$  is finite.

Let  $\nu_{QS}$  be a quasi-stationary distribution for  $(Z_n)_{n \in \mathbb{N}}$  in  $\mathcal{M}(Q_a)$  and let  $\theta_{QS}$  be its absorption parameter. The first part of Theorem 50 shows that  $\theta_{QS} \leq \theta_0$ , and that if  $\theta_{QS} = \theta_0$  then  $\nu_{QS}$  is in the convex hull of the finite family  $\nu_1, \dots, \nu_{\ell}$ . Assume now that  $\theta_{QS} < \theta_0$ . Then (5.5) implies that  $\sum_{i=1}^{\ell} \nu_{QS}(\eta_i) = 0$  and hence the support of  $\nu_{QS}$  is included in  $E$ . Since  $E$  is finite and since  $E$  is a closed set for the Markov chain, we deduce, by usual finite dimensional linear algebra arguments, that  $\nu_{QS}$  belongs to the convex hull of any maximal family of positive left eigenvector of the transition matrix of  $(Z_n)_{n \in \mathbb{N}}$  restricted to  $E$ . This concludes the proof of the last assertion of Theorem 50.  $\square$

### 5.3 Existence of a unique quasi-stationary distribution

In this section we show that under the extra assumption that the process is irreducible and aperiodic, there exists a unique quasi-limiting distribution for compactly supported initial distributions. We recall that  $(Z_n)_{n \in \mathbb{N}}$  is irreducible if for all  $x, y \in \mathbb{N}^p \setminus \{0\}$ , there exists  $n \geq 0$  such that  $\mathbb{P}_x(Z_n = y) > 0$ . It is well known that under this assumption, for all  $z \in \mathbb{N}^p \setminus \{0\}$ ,

$$\theta_0 = \sup\{\theta > 0, \liminf_{n \rightarrow +\infty} \theta^{-n} \mathbb{P}_z(Z_n = z) > 0\},$$

which does not depend on  $z$ .

In the irreducible case, we have the following last result.

**Theorem 53.** Assume that Assumptions 6, 7 and 8 hold true and that the process  $(Z_n)_{n \in \mathbb{N}}$  is aperiodic and irreducible. We fix  $a \in (1, \eta)$  such that  $(\lambda^*)^a < \theta_0$ .

Then the process  $(Z_n)_{n \in \mathbb{N}}$  admits a unique quasi-stationary distribution  $\nu_{QS}$  in  $\mathcal{M}(Q_a)$ . Its absorption parameter  $\theta_{QS}$  is equal to  $\theta_0$  and there exists a unique positive function  $\eta_{QS}$  in  $L^\infty(Q_a)$ , such that

$$\mathbb{P}_{\nu_{QS}}(Z_n \neq 0) = \theta_0^n \text{ and } \mathbb{E}_z(\eta_{QS}(Z_n) \mathbf{1}_{Z_n \neq 0}) = \theta_0^n \eta_{QS}(z), \quad \forall n \geq 0, z \in \mathbb{N}^p \setminus \{0\} \quad (5.13)$$

and  $\eta_{QS}$  is lower bounded away from 0. In addition, for any probability measure  $\mu \in \mathcal{M}(Q_a)$  and any function  $f \in L^\infty(Q_a)$ , we have that there exist  $C > 0$  and  $\gamma \in (0, 1)$  such that

$$\left| \theta_0^{-n} \mathbb{E}_\mu(f(Z_n) \mathbf{1}_{Z_n \neq 0}) - \mu(\eta_{QS}) \nu_{QS}(f) \right| \leq C \gamma^n \mu(Q_a) \|f\|_{Q_a}, \quad \forall n \geq 0. \quad (5.14)$$

Moreover, for any probability measure  $\mu \in \mathcal{M}(Q_a)$  and any function  $f \in L^\infty(Q_a)$ ,

$$|\mathbb{E}_\mu(f(Z_n) | Z_n \neq 0) - \nu_{QS}(f)| \leq C \gamma^n \mu(Q_a) \|f\|_{Q_a}, \quad \forall n \geq 0. \quad (5.15)$$

Finally, for all probability measure  $\mu \in \mathcal{M}(\eta_{QS})$ ,

$$\mathbb{P}_\mu(Z_n \in \cdot | Z_n \neq 0) \xrightarrow[n \rightarrow +\infty]{TV} \nu_{QS}. \quad (5.16)$$

where  $\xrightarrow[n \rightarrow +\infty]{TV}$  stands for convergence in total variation.

In particular, (5.16) states that for all  $z \in \mathbb{N}^p$ , the conditional law  $\mathbb{P}_z(Z_n \in \cdot | Z_n \neq 0)$  converges in total variation to  $\nu_{QS}$ , and it does it exponentially fast, according to (5.15).

*Proof of Theorem 53.* Fix  $a \in (1, \eta)$  such that  $(\lambda^*)^a < \theta_0$ , and consider the Lyapunov type property with constants  $\theta_a$  and  $C_a$  from Proposition 51. We make use of Section 2 in [CV17]. Let us recall the main assumption of this reference, with the notations and settings of the present proof.

**Assumption (E).** There exist a positive integer  $n_1$ , positive real constants  $\theta_1, \theta_2, c_1, c_2, c_3$ , two functions  $\varphi_1, \varphi_2 : \mathbb{N}^p \setminus \{0\} \rightarrow \mathbb{R}_+$  and a probability measure  $\nu$  on a measurable subset  $K \subset \mathbb{N}^p \setminus \{0\}$  such that

(E1) (*Local Dobrushin coefficient*).  $\forall z \in K$ ,

$$\mathbb{P}_z(Z_{n_1} \in \cdot) \geq c_1 \nu(\cdot \cap K).$$

(E2) (*Global Lyapunov criterion*). We have  $\theta_1 < \theta_2$  and

$$\begin{aligned} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \varphi_1(z) &\geq 1, \quad \sup_{z \in K} \varphi_1(z) < \infty \\ \inf_{z \in K} \varphi_2(z) &> 0, \quad \sup_{z \in \mathbb{N}^p \setminus \{0\}} \varphi_2(z) \leq 1, \\ \mathbb{E}_z(\varphi_1(X_1)) &\leq \theta_1 \varphi_1(z) + c_2 \mathbf{1}_K(z), \quad \forall z \in \mathbb{N}^p \setminus \{0\} \\ \mathbb{E}_z(\varphi_2(X_1)) &\geq \theta_2 \varphi_2(z), \quad \forall z \in \mathbb{N}^p \setminus \{0\}. \end{aligned}$$

(E3) (*Local Harnack inequality*). We have

$$\sup_{n \in \mathbb{Z}_+} \frac{\sup_{y \in K} \mathbb{P}_y(Z_n \neq 0)}{\inf_{y \in K} \mathbb{P}_y(Z_n \neq 0)} \leq c_3$$

(E4) (*Aperiodicity*). For all  $z \in K$ , there exists  $n_4(z)$  such that, for all  $n \geq n_4(z)$ ,

$$\mathbb{P}_z(Z_n \in K) > 0.$$

In a first step, we check that Assumption E therein holds true for the process under consideration (in the following proof, (E), (E1), (E2), (E3) and (E4) refer to the assumptions in [CV17]). In a second step, we prove (5.13), (5.14) and (5.15).

*Step 1. Assumption (E) holds true.* Fix  $\theta_1 \in (\theta_a, \theta_0)$  and let  $r_1 \geq 1$  be large enough so that  $Q_a(z) \geq \frac{C_a}{\theta_1 - \theta_a}$  for all  $|z| \geq r_1$ , so that

$$\mathbb{E}_z(Q_a(Z_1)) \leq \theta_a Q_a(z) + C_a \leq \theta_1 Q_a(z), \quad \forall z \in \mathbb{N}^p \text{ such that } |z| \geq r_1. \quad (5.17)$$

Let us set  $K = \{z \in \mathbb{N}^p, |z| \leq r_1\}$  and  $\varphi_1 = Q_a$ . As in the above reference, we define

$$P_n f(z) = \mathbb{E}_z(f(Z_n) \mathbf{1}_{Z_n \neq 0}), \quad \forall z \in \mathbb{N}^p \setminus \{0\}.$$

We deduce from the irreducibility and aperiodicity property that there exists  $n_1 \geq 0$  such that

$$c_1 := \inf_{x, y \in K} \mathbb{P}_x(Z_{n_1} = y) > 0. \quad (5.18)$$

Setting  $\nu = \delta_{(1, \dots, 1)}$ , this entails property (E1).

We have  $\inf \varphi_1 = \inf Q_a \geq 1$  by Assumption (P), so that the first line of (E2) is satisfied. Moreover, according to (5.17), the third line of (E2) also holds true.

Choosing  $\theta_2 \in (\theta_1, \theta_0)$ , we deduce from the definition of  $\theta_0$ , the irreducibility and the aperiodicity of  $(Z_n)_{n \in \mathbb{N}}$  that

$$\theta_2^{-n} \mathbb{P}_{(1, \dots, 1)}(Z_n = (1, \dots, 1)) \xrightarrow{n \rightarrow +\infty} +\infty.$$

Hence, using (5.18) and the markov property at time  $n_1$ , we deduce that, for all  $z \in K$ ,

$$\theta_2^{-n} \mathbb{P}_z(Z_n \neq 0) \xrightarrow{n \rightarrow +\infty} +\infty.$$

In particular

$$\theta_2^{-n} \min_{i \in \{1, \dots, p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow{n \rightarrow +\infty} +\infty,$$

where  $e_i$  is the  $i^{\text{th}}$  element of the canonical basis of  $\mathbb{N}^p$ . Since the process  $(Z_n)_{n \in \mathbb{N}}$  is super-additive and since, for all  $z \in \mathbb{N}^p \setminus \{0\}$ , there exists  $i \in \{1, \dots, p\}$  such that  $z \geq e_i$ , we deduce that

$$\theta_2^{-n} \inf_{z \in \mathbb{N}^p \setminus \{0\}} \mathbb{P}_z(Z_n \neq 0) \geq \theta_2^{-n} \min_{i \in \{1, \dots, p\}} \mathbb{P}_{e_i}(Z_n \neq 0) \xrightarrow{n \rightarrow +\infty} +\infty.$$

Lemma 3.2 in [CV17] allows to conclude that there exists a function  $\varphi_2 : \mathbb{N}^p \setminus \{0\} \rightarrow [0, 1]$  such that  $\inf_{\mathbb{N}^p \setminus \{0\}} \varphi_2 > 0$  and  $P_1 \varphi_2 \geq \theta_2 \varphi_2$  on  $\mathbb{N}^p \setminus \{0\}$ . As a consequence the second and fourth lines of (E2) are satisfied.

For all  $x, y \in K$  and  $n \geq 0$ , we have, using the Markov property at time  $n_1$  and the fact that  $\mathbb{P}_x(Z_n \neq 0)$  is decreasing in  $n$ ,

$$\mathbb{P}_x(Z_n \neq 0) \geq \mathbb{P}_x(Z_{n \vee n_1} \neq 0) \geq \mathbb{P}_x(Z_{n_1} = y) \mathbb{P}_y(Z_{n \vee n_1 - n_1} \neq 0) \geq c_1 \mathbb{P}_y(Z_n \neq 0).$$

This proves (E3).

Finally (E4) holds true since we assumed that the process  $(Z_n)_{n \in \mathbb{N}}$  is aperiodic.

We have thus proved that Assumption (E) holds true for  $(Z_n)_{n \in \mathbb{N}}$  absorbed when it leaves  $\mathbb{N}^p \setminus \{0\}$ . This concludes the first step.

*Step 2. Conclusion of the proof.* Theorem 2.1 in [CV17] implies that there exists a unique quasi-stationary distribution  $\nu_{QS}$  in  $\mathcal{M}(Q_a)$  and that (5.15) holds true (using the fact that in our case,  $\varphi_2$  is lower bounded away from 0). Theorem 2.2 in this reference also implies that there exists an associated non-negative eigenfunction  $\eta_{QS}$  with eigenvalue  $\theta_{QS}$ , such that  $\inf_K \eta_{QS} > 0$  and

$$\theta_{QS}^{-n} \mathbb{P} \cdot (Z_n \neq 0) \xrightarrow[n \rightarrow +\infty]{L^\infty(Q_a)} \eta_{QS}. \quad (5.19)$$

Since the process is irreducible, for all  $z \in \mathbb{N}^p \setminus \{0\}$ , there exists  $n \geq 0$  such that  $\mathbb{P}_z(Z_n = (1, \dots, 1)) > 0$  and hence

$$\eta_{QS}(z) = \mathbb{E}_z(\eta_{QS}(Z_n) \mathbf{1}_{Z_n \neq 0}) \geq \eta_{QS}((1, \dots, 1)) \mathbb{P}_z(Z_n = (1, \dots, 1)) > 0,$$

so that  $\eta_{QS}$  is positive. We thus proved (5.13). Finally, since  $\xi$  is super-additive,  $\theta_{QS}^{-n} \mathbb{P}_z(Z_n \neq 0)$  and hence  $\eta_{QS}$  increases with  $z$ , so that  $\eta_{QS}$  is lower bounded away from 0.

Corollary 2.4 in [CV17] shows that (5.14) holds true.

Finally, (5.16) is an immediate consequence of Theorem 2.7 (iii) in [CV17]. This concludes the proof of Theorem 53.  $\square$



## Chapter 6

# On the unlimited growth of some two-sex population models

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This chapter focuses in the study of the probability of survival for certain models of bisexual branching processes that are not encompassed in the results given in the previous chapters. In particular, we are interested into giving sufficient conditions for survival for processes where the mating function can be a random function. This model can be described as a Markov process  $(W_n)_{n \in \mathbb{N}}$  defined on  $\mathbb{N}^q$  by

$$W_{n+1,j} = \sum_{i=1}^p \phi_i^{(n)}(W_n) \sum_{k=1} V_{i,j}^{(n,k)}, \text{ for all } j \in \{1, \dots, q\}, \quad (6.1)$$

where  $W_0 = w_0 \in \mathbb{N}^q$  is given and the random vectors  $(V_i)_{1 \leq i \leq q}$  hold the same assumptions of integrability and independence described in Chapter 2, Section 2.1, for the process with deterministic mating function, and are omitted here. We assume that the family  $(\phi^{(n)})_{n \in \mathbb{N}}$  is an i.i.d. collection of random functions with the same distribution as a given random function  $\phi : \mathbb{R}_+^q \rightarrow \mathbb{R}_+^p$ , independents of the offspring distribution, and such that

$$\mathbb{P}(\phi(x) \in \mathbb{N}^p) = 1, \text{ for all } x \in \mathbb{N}^q,$$

and

$$\mathbb{P}(\phi(0) = 0) = 1,$$

where  $\phi$  has the same law as  $\phi^{(1)}$ . In the following, whenever we discuss about this process,  $\phi$  will always be a function with the same law as the mating function. We remark that  $(W_n)_{n \in \mathbb{N}}$  is the process of individuals (and not of mating units, as we have seen in the deterministic case so far).

Our approach will be a general one that can be applied to discrete time Markov processes defined on a state space of finite dimension. To fix ideas, throughout this chapter  $(\mathcal{X}_n)_{n \in \mathbb{N}}$  will denote an arbitrary Markov process on  $\mathbb{R}_+^q$ . We denote by  $\mathbb{P}_x$  the probability measure  $\mathbb{P}(\cdot \mid \mathcal{X}_0 = x)$  and by  $\mathbb{E}_x$  its associated expectation.

We say that  $(\mathcal{X}_n)_{n \in \mathbb{N}}$  has unlimited growth with positive probability if

$$\mathbb{P}_x(g(\mathcal{X}_n) \xrightarrow[n \rightarrow +\infty]{} +\infty) > 0, \quad (6.2)$$

for some  $x \in \mathbb{R}_+^q$ , and where  $g : \mathbb{R}_+^q \rightarrow \mathbb{R}_+$  is some function such that for any sequence  $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_+^q$ ,

$$g(x_n) \rightarrow +\infty \implies |x_n| \rightarrow +\infty. \quad (6.3)$$

In this chapter we present two main results that exhibit sufficient conditions for unlimited growth. In Section 6.1 we give conditions for having an exponential growth towards infinity. That is, such that

$$\mathbb{P}_x(g(\mathcal{X}_n) \geq r^n, \text{ for all } n \in \mathbb{N}) > 0,$$

for some  $x \in \mathbb{R}_+^p$  and with  $r > 1$  and  $g$  a function satisfying (6.3). In Section 6.1.1.2 we apply this result to some models of bisexual branching processes with random mating function. We show an application of these results by describing an age structured bisexual branching process in Section 6.2. In Section 6.3 we give conditions to have a linear growth towards infinity. We finish by applying these results to some models of continuous time two-sex birth and death processes in Section 6.4.

## 6.1 Exponential growth towards infinity

We start this section with the following result from [GMM05].

**Theorem 54** (see [GMM05], Theorem 2). *For a vector  $\mu \in \mathbb{R}_+^q$  with strictly positive entries, define the function  $g_\mu : \mathbb{R}_+^q \rightarrow \mathbb{R}$  given by*

$$g_\mu(x) = \langle x, \mu \rangle^{-1} \mathbb{E}_x(\langle \mathcal{X}_1, \mu \rangle) - 1. \quad (6.4)$$

*If there exists  $\mu \in \mathbb{R}_+^q$  such that*

$$\liminf_{\|x\| \rightarrow +\infty} g_\mu(x) > 0, \quad (6.5)$$

*and in addition for some  $\delta > 0$ ,*

$$\mathbb{E}_x \left( |\langle \mathcal{X}_{n+1}, \mu \rangle - \mathbb{E}(\langle \mathcal{X}_{n+1}, \mu \rangle \mid \mathcal{X}_n)|^{1+\delta} \mid \mathcal{X}_n = x \right) = O(|x|^\delta). \quad (6.6)$$

*Then, there exists a constant  $K \geq 0$  such that,  $\mathbb{P}_x(|\mathcal{X}_n| \xrightarrow[n \rightarrow +\infty]{} +\infty) > 0$  if  $|x| > K$ .*

This result can be applied to a wide variety of multi-dimensional Markov Chains to exhibit a criteria for unlimited growth. As a simple example, consider a multi-type (asexual) Galton-Watson branching process. We recall that this corresponds to a process defined recursively by

$$X_{n+1,j} = \sum_{i=1}^q \sum_{k=1}^{X_{n,i}} V_{i,j}^{(k,n)}.$$

Set  $M$  to be the expectation matrix of the process (see (1.6)). Suppose that  $M$  is positively regular, which ensures the existence of a Perron-Frobenius root  $\rho^*$  with an associated left eigenvector  $u^* \in \mathbb{R}_+^q$ . Note that if we define  $g_{u^*}$  as in (6.4), we have for all  $x \in \mathbb{N}_+^p$

$$g_{u^*}(x) = \langle u^*, x \rangle^{-1} \langle u^*, xM \rangle - 1 = \rho^* - 1.$$

Then, we recover that if  $\rho > 1$ , the process grows towards infinity with positive probability. Nevertheless, Theorem 54 does not cover some other models, as is the case of the bisexual branching processes as the one presented in (6.1). As an example consider the case with  $q = 2$  and  $p = 1$  (single-type with random mating) with the extra assumption that

$$\mathbb{P}(\phi(x, 0) = \phi(0, y) = 0) = 1, \text{ for all } x, y \in \mathbb{N}.$$

For  $\mu = (\mu_1, \mu_2) \in \mathbb{R}_+^2$  with strictly positive entries, we compute the function  $g_\mu$  given by (6.4) for this process, obtaining

$$g_\mu(x, y) = \frac{\mathbb{E}(\phi(x, y))(\mathbb{E}(F)\mu_1 + \mathbb{E}(M)\mu_2)}{x\mu_1 + y\mu_2} - 1.$$

Then if  $x = 0$  we obtain, for any  $y \in \mathbb{N}$ ,  $g_\mu(0, y) = -1$ . Then,

$$\liminf_{|(x,y)| \rightarrow +\infty} g_\mu(x, y) \leq -1,$$

and (6.5) does not hold for any mating function  $\phi$ . The main issue here, is that Theorem 54 needs the process to grow in every direction, which is not the case for  $(W_n)_{n \in \mathbb{N}}$ . To overcome this issue, we change this approach for a more general one that allows to study the survival of a process that grows only on some directions, but that also encompasses this theorem.

Our first result consists in sufficient conditions for unlimited growth where we look at several functions of the process at the same time. This ensures that a function of the process grows exponentially fast towards infinity.

**Theorem 55** (Exponential Growth). *Assume that there exist  $N \in \mathbb{N}$  and  $N$  functions  $f_1, \dots, f_N$ , where for all  $i \in \{1, \dots, N\}$ ,  $f_i : \mathbb{R}_+^q \rightarrow \mathbb{R}_+$ . Define*

$$g(x) = \min_{i \in \{1, \dots, N\}} f_i(x),$$

and assume that the following statements hold:

1. For any sequence  $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_+^q$ ,

$$g(x_n) \xrightarrow{n \rightarrow +\infty} +\infty \implies |x_n| \xrightarrow{n \rightarrow +\infty} +\infty. \quad (6.7)$$

2. For all  $i \in \{1, \dots, N\}$ ,  $\mathbb{E}_x(f_i(\mathcal{X}_1)) < +\infty$ , and there exist  $r > 1$ ,  $m \in \mathbb{N}$  and  $\varepsilon \in (0, 1)$  such that for all  $n \geq m$ , and all  $x \in \mathbb{R}_+^q$ ,

$$g(x) \geq r^n \implies (1 - \varepsilon) \min_{i \in \{1, \dots, N\}} \mathbb{E}_x(f_i(\mathcal{X}_1)) \geq r^{n+1}. \quad (6.8)$$

3. There exist  $0 < \alpha < \delta$ , such that for all  $i \in \{1, \dots, N\}$ .

$$\mathbb{E}_x \left( \log(1 + |f_i(\mathcal{X}_1) - \mathbb{E}_x(f_i(\mathcal{X}_1))|)^{1+\delta} \right) = O(\log(1 + \mathbb{E}_x(f_i(\mathcal{X}_1)))^\alpha), \quad (6.9)$$

as  $|x| \rightarrow +\infty$ .

Then there exists  $K > 0$ , such that for all  $x \in \mathbb{R}_+^q$  such that  $g(x) \geq K$ , we have

$$\mathbb{P}_x(g(\mathcal{X}_n) \geq r^n, \text{ for all } n \in \mathbb{N}) > 0.$$

Thus, with positive probability,  $(g(\mathcal{X}_n))_{n \in \mathbb{N}}$  grows exponentially towards infinity and  $(\mathcal{X}_n)_{n \in \mathbb{N}}$  has unlimited growth.

Before going into the proof of this theorem, we discuss the different assumptions that are in place:

1. Assumption (6.7) allows to guarantee that (6.3) holds. Nevertheless, it is important to remark that if  $(\mathcal{X}_n)_{n \in \mathbb{N}}$  takes value on  $\mathbb{N}^q$  (as it is the case of a multi-type bisexual branching process) then this assumption is immediately verified, since the restriction of  $g$  to  $\mathbb{N}^q$  is a locally finite function and (6.7) holds directly.
2. The idea behind (6.8) is that all the functions  $f_i$  are growing exponentially in mean, which allows us to prove that with high probability  $f_i(X_1)$  will grow on the event  $\{\mathcal{X}_0 = x\}$ , for large values of  $g(x)$ . We emphasize that the value of  $\varepsilon > 0$  on the left hand side does not depend neither on the initial condition nor on the value of  $n \in \mathbb{N}$ , which may be difficult to verify (see Remark 18 below).
3. Whereas assumption (6.8) allows us to prove that the value of  $g(X_n)$  grows with positive probability after one step, hypothesis (6.9) allows us to prove that the value of this probability is high enough so that we can iterate this computation for the whole trajectory. Hence, obtaining a positive probability that this exponential growth is held in every step. It will be clear from the proof that this condition can be replaced by the following: assume that for  $0 < \alpha < \delta$ ,

$$\mathbb{E}_x \left( |f_i(X_1) - \mathbb{E}_x(f_i(X_1))|^\delta \right) = O(\mathbb{E}_x(f_i(X_1))^\alpha). \quad (6.10)$$

The way in that this assumption is presented in (6.9) will help make a comparison of this result with the one presented later regarding linear growth in Section 6.3

*Remark 18.* The following alternative hypothesis is a sufficient condition for (6.8) to hold. There exist  $A > 0$  and  $r' > 1$  such that for all  $x \in \mathbb{R}_+^q$ ,  $g(x) > A$  implies that

$$\min_{i \in \{1, \dots, N\}} \mathbb{E}_x(f_i(X_1)) \geq r'g(x). \quad (6.11)$$

Indeed, if we fix  $\delta \in (0, r' - 1)$  and set  $\varepsilon \in (0, 1)$  such that

$$\frac{r' - \delta}{1 - \varepsilon} \leq r'.$$

Then we can set  $N \in \mathbb{N}$  as the smallest integer such that  $(r' - \delta)^N \geq A$ . Thus, we have that for all  $n \geq N$ , if  $g(x) \geq (r' - \delta)^n$ , then

$$\mathbb{E}_x(f_i(X_1)) \geq r'g(x) \geq \frac{(r' - \delta)^{n+1}}{1 - \varepsilon},$$

and (6.8) follows with  $r = r' - \delta > 1$ .

*Remark 19.* Note that Theorem 54 can be deduced from Theorem 55. In fact, if (6.5) holds true for some direction  $\mu \in \mathbb{R}_+^q$  with strictly positive entries. Set

$$c := \liminf_{|x| \rightarrow +\infty} \left( \frac{\mathbb{E}_x(\langle \mathcal{X}_1, \mu \rangle)}{\langle x, \mu \rangle} - 1 \right) > 0.$$

Then, if we fix  $\varepsilon \in (0, c)$ , since  $\mu$  has strictly positive entries, there exists  $A > 0$  such that if  $\langle x, \mu \rangle \geq A$ , then

$$\mathbb{E}_x(\langle \mathcal{X}_1, \mu \rangle) \geq (c - \varepsilon + 1)\langle x, \mu \rangle.$$

In other words, we have that (6.11) holds with  $r' = c - \varepsilon + 1 > 1$ . Finally, (6.7) and (6.10) hold directly. We have shown that Theorem 55 holds with  $N = 1$  and  $f_1(x) = \langle x, \mu \rangle$ .

We proceed now with the proof.

*Proof of Theorem 55.* We divide this proof in two steps. We deal with one step of the process first, to then work with the complete trajectory.

*Step 1.* Let  $x \in \mathbb{R}_+^q$  such that  $g(x) \geq r^n$  for some  $n \geq m$ . We have that, from (6.8),

$$\begin{aligned} \mathbb{P}_x(g(\mathcal{X}_1) < r^{n+1}) &= \mathbb{P}_x(\exists i \in \{1, \dots, N\}, f_i(\mathcal{X}_1) < r^{n+1}) \\ &\leq \sum_{i=1}^N \mathbb{P}_x(f_i(\mathcal{X}_1) < r^{n+1}) \\ &\leq \sum_{i=1}^N \mathbb{P}_x(f_i(\mathcal{X}_1) < (1 - \varepsilon)\mathbb{E}_x(f_i(\mathcal{X}_1))) \\ &\leq \sum_{i=1}^N \mathbb{P}_x(|f_i(\mathcal{X}_1) - \mathbb{E}_x(f_i(\mathcal{X}_1))| > \varepsilon\mathbb{E}_x(f_i(\mathcal{X}_1))) \\ &\leq \sum_{i=1}^N \frac{\mathbb{E}_x(\log(1 + |f_i(\mathcal{X}_1) - \mathbb{E}_x(f_i(\mathcal{X}_1))|)^{1+\delta})}{\log(1 + \varepsilon\mathbb{E}_x(f_i(\mathcal{X}_1)))^{1+\delta}}, \end{aligned}$$

where we used the fact that  $y \in \mathbb{R}_+ \rightarrow \log(1 + y)^{1+\delta}$  is an increasing function, together with Markov's inequality. Hence, thanks to (6.8) and (6.9), we can find  $C_1, C_2, C > 0$  large enough such that

$$\begin{aligned} \mathbb{P}_x(g(\mathcal{X}_1) < r^{n+1}) &\leq \sum_{i=1}^N \frac{\mathbb{E}_x(\log(1 + |f_i(\mathcal{X}_1) - \mathbb{E}_x(f_i(\mathcal{X}_1))|)^{1+\delta})}{C_1 \log(1 + \mathbb{E}_x(f_i(\mathcal{X}_1)))^{1+\delta}} \\ &\leq \sum_{i=1}^N \frac{C_2}{\log(1 + \mathbb{E}_x(f_i(\mathcal{X}_1)))^{1+\delta-\alpha}} \\ &\leq \sum_{i=1}^N \frac{C_2}{\log(1 + r^{n+1})^{1+\delta-\alpha}} \\ &\leq \frac{NC_2}{\left( (n+1)\log(r) + \log\left(\frac{1}{r^{n+1}} + 1\right) \right)^{1+\delta-\alpha}} \\ &\leq \frac{C}{(n+1)^{1+\delta-\alpha}}, \end{aligned}$$

where in the third inequality we used again that  $\mathbb{E}_x(f_i(\mathcal{X}_1)) \geq r^{n+1}$  for all  $i \in \{1, \dots, N\}$ . We remark that we have used the fact that  $r > 1$ , thus  $\log(r) > 0$ . We remark that the same bound can be obtained (with a different constant  $C$ ) if (6.11) were in place instead. In summary, we have obtained, for  $\eta = \delta - \alpha > 0$  and all  $x \in \mathbb{R}_+^q$  with  $g(x) \geq r^n$ ,

$$\mathbb{P}_x(g(\mathcal{X}_1) < r^{n+1}) \leq \frac{C}{(n+1)^{1+\eta}}.$$

*Step 2.* Start by fixing  $m' \geq m$  such that  $\frac{C}{(m'+1)^{1+\eta}} < 1$  and set  $K = r^{m'}$ . Hence, for all  $n \in \mathbb{N}$ , and all  $x \in \mathbb{R}_+^q$ , by the previous step

$$\mathbb{P}_x\left(g(\mathcal{X}_{n+1}) < Kr^{n+1} \mid \mathcal{F}_n\right) \mathbb{1}_{g(\mathcal{X}_n) \geq Kr^n} \leq \frac{C}{(m'+n+1)^{1+\eta}} \mathbb{1}_{g(\mathcal{X}_n) \geq Kr^n}, \text{ a.s.} \quad (6.12)$$

Set  $A_n = \{g(\mathcal{X}_n) \geq Kr^n\}$ . Thus, (6.12) can be read as

$$\mathbb{P}(A_{n+1} \mid \mathcal{F}_n) \mathbb{1}_{A_n} \geq \left(1 - \frac{C}{(m'+n+1)^{1+\eta}}\right) \mathbb{1}_{A_n}, \text{ a.s.}$$

Then, for  $x \in \mathbb{R}_+^q$  with  $g(x) \geq K$ ,

$$\begin{aligned} \mathbb{P}_x\left(A_{n+1} \mid \bigcap_{k=0}^n A_k\right) &= \frac{\mathbb{E}_x\left(\mathbb{E}_x(\mathbb{1}_{A_{n+1}} \mid \mathcal{F}_n) \mathbb{1}_{\bigcap_{k=0}^n A_k}\right)}{\mathbb{P}_x\left(\bigcap_{k=0}^n A_k\right)} \\ &= \frac{\mathbb{E}_x\left(\mathbb{P}_x(A_{n+1} \mid \mathcal{F}_n) \mathbb{1}_{A_n} \mathbb{1}_{\bigcap_{k=0}^{n-1} A_k}\right)}{\mathbb{P}_x\left(\bigcap_{k=0}^n A_k\right)} \\ &\geq \frac{\mathbb{E}_x\left(\left(1 - \frac{C}{(m'+n+1)^{1+\eta}}\right) \mathbb{1}_{A_n} \mathbb{1}_{\bigcap_{k=0}^{n-1} A_k}\right)}{\mathbb{P}_x\left(\bigcap_{k=0}^n A_k\right)} \\ &= \left(1 - \frac{C}{(m'+n+1)^{1+\eta}}\right). \end{aligned}$$

As a consequence, since  $K > 1$ ,

$$\begin{aligned} \mathbb{P}_x(g(\mathcal{X}_n) \geq r^n, \text{ for all } n \in \mathbb{N}) &\geq \mathbb{P}_x(g(\mathcal{X}_n) \geq Kr^n, \text{ for all } n \in \mathbb{N}) \\ &= \lim_{\ell \rightarrow +\infty} \prod_{n=0}^{\ell} \mathbb{P}_x\left(A_{n+1} \mid \bigcap_{k=0}^n A_k\right) \\ &\geq \lim_{\ell \rightarrow +\infty} \prod_{n=0}^{\ell} \left(1 - \frac{C}{(m'+n+1)^{1+\eta}}\right) \\ &> 0, \end{aligned}$$

since  $\sum_{n \in \mathbb{N}} \frac{1}{(m' + n + 1)^{1+\eta}} < +\infty$ . Finally, thanks to (6.7),

$$\mathbb{P}_x(|\mathcal{X}_n| \xrightarrow[n \rightarrow +\infty]{} +\infty) > 0.$$

□

### 6.1.1 Examples of bisexual branching processes with random mating

In this section we seek to give conditions for survival with positive probability to some models of bisexual branching processes with random mating function (see (6.1)). We start with two corollaries that give sufficient conditions to imply Theorem 55. These results are based on the idea that the process can eventually grow in a deterministic direction, as it is the case in many examples of branching processes in general.

#### 6.1.1.1 Two useful corollaries

We start with a corollary for a process that grows eventually towards a deterministic direction  $y \in \mathbb{R}_+^q$ .

**Corollary 56.** *Assume that there exists  $\varepsilon \in (0, 1)$ ,  $y \in \mathbb{R}_+^q$ ,  $N \in \mathbb{N}$  and  $r > 1$  such that for all  $n \geq N$  and  $x \in \mathbb{R}_+^q$ ,*

$$x \geq r^n y \Rightarrow (1 - \varepsilon)\mathbb{E}_x(\mathcal{X}_1) \geq r^{n+1}y, \quad (6.13)$$

where the inequality holds componentwise.

In addition, assume that there exist  $0 < \alpha < \delta$  such that

$$\mathbb{E}_x \left( \log(1 + |\mathcal{X}_1 - \mathbb{E}_x(\mathcal{X}_1)|)^{1+\delta} \right) = O(\log(1 + \mathbb{E}_x(\mathcal{X}_1))^\alpha), \quad (6.14)$$

as  $|x| \rightarrow +\infty$ . Then, there exists  $K > 0$  such that if  $x \geq Ky$ ,

$$\mathbb{P}_x(|\mathcal{X}_n| \geq |y|r^n \text{ for all } n \in \mathbb{N}) > 0.$$

*Proof of Corollary 56.* Consider  $y \in \mathbb{R}_+^q$  for which (6.13) holds and define  $\mathcal{I} = \{i \in \{1, \dots, q\}, y_i > 0\}$ , the index of non-zero components of  $y$ . For every  $i \in \mathcal{I}$ , we set  $f_i : \mathbb{N}^q \rightarrow \mathbb{R}_+$  by

$$f_i(x) = \frac{x_i}{y_i}.$$

Denote  $g(x) = \min_{i \in \mathcal{I}} f_i(x)$  and we verify that all three assumptions of Theorem 55 hold.

1. Assumption (6.7) is directly verified due to the form of the function  $g$ .
2. If  $x \in \mathbb{R}_+^q$  is such that  $g(x) \geq r^n$  with  $n \geq N$  and  $r > 1$ , then by (6.13),

$$(1 - \varepsilon)\mathbb{E}_x(f_i(\mathcal{X}_1)) = \frac{(1 - \varepsilon)\mathbb{E}_x(\mathcal{X}_{1,i})}{y_i} \geq \frac{r^{n+1}y_i}{y_i} = r^{n+1},$$

and (6.8) holds.

We are in place to apply Theorem 55 (since the moments condition is assumed to be true), obtaining that there exists  $K > 0$  such that if  $g(x) \geq K$

$$\mathbb{P}_x(g(\mathcal{X}_n) \geq r^n, \text{ for all } n \in \mathbb{N}) > 0.$$

Noting that  $\{g(\mathcal{X}_n) \geq r^n\} \subseteq \{|\mathcal{X}_n| \geq |y|r^n\}$  we conclude that if  $x \geq Ky$ ,

$$\mathbb{P}_x(|\mathcal{X}_n| \geq |y|r^n, \text{ for all } n \in \mathbb{N}) > 0.$$

□

The following corollary is inspired by the results in bisexual processes with superadditive mating function given in Chapter 4, where the survival of the process can be determined in terms of an eigenvalue problem. This also can be stated as a sub-eigenvalue problem in the case where the expectation holds a monotonicity assumption, as stated in the following result.

**Corollary 57.** Define the function  $R_{\text{inf}}: \mathbb{R}_+^q \rightarrow \mathbb{R}_+^q$  given by

$$R_{\text{inf}}(x) := \liminf_{t \rightarrow +\infty} \frac{\mathbb{E}(\mathcal{X}_1 \mid \mathcal{X}_0 = \lfloor tx \rfloor)}{t}, \quad (6.15)$$

We assume that there exists  $x^* \in \mathbb{R}_+^q$  such that

1. For some  $\lambda^* > 1$  the inequality

$$R_{\text{inf}}(x^*) \geq \lambda^* x^* \quad (6.16)$$

holds.

2. For all  $x \in \mathbb{N}^p$  and all  $\eta \geq 1$

$$x \geq \eta x^* \Rightarrow \mathbb{E}(\mathcal{X}_1 \mid \mathcal{X}_0 = x) \geq \mathbb{E}(\mathcal{X}_1 \mid \mathcal{X}_0 = \lfloor \eta x^* \rfloor). \quad (6.17)$$

If in addition the moment condition (6.14) holds, then for all  $\gamma \in (0, \lambda^* - 1)$  there exists  $\widetilde{K} > 0$ , such that for all  $x \geq \widetilde{K}x^*$

$$\mathbb{P}_x(|\mathcal{X}_n| \geq \widetilde{K}(\lambda^* - \gamma)^n, \text{ for all } n \in \mathbb{N}) > 0.$$

We remark that the previous criteria encompasses the superadditive model defined in Chapter 2, in which case  $R_{\text{inf}} = \mathfrak{M}$ , the operator defined in (2.6). Corollary 57 then gives a tool to prove the survival of the process, for example, in the cases where the existence of eigenelements for  $\mathfrak{M}$  can not be guaranteed.

*Proof of Corollary 57.* We prove that the assumptions of Corollary 56 hold. For  $\gamma \in (0, \lambda^* - 1)$ , fix  $\varepsilon \in (0, 1)$  such that  $\lambda^* - \gamma < \lambda(1 - \varepsilon)$ . The inequality in (6.16) implies that there exists  $t_0$  such that for all  $t \geq t_0$ ,

$$(1 - \varepsilon)\mathbb{E}(\mathcal{X}_1 \mid \mathcal{X}_0 = \lfloor tx^* \rfloor) \geq (\lambda^* - \gamma)tx^*. \quad (6.18)$$

Note that thanks to the choice of  $\gamma$ , we have  $(\lambda^* - \gamma) > 1$ , so (6.18) holds for  $t = (\lambda^* - \gamma)^n t_0$  with any  $n \in \mathbb{N}$ . Consider now  $x \in \mathbb{R}_+^q$  with  $x \geq (\lambda^* - \gamma)^n t_0 x^*$ , for some  $n \in \mathbb{N}$ . Applying (6.17) and then (6.18), we obtain

$$\begin{aligned} (1 - \varepsilon)\mathbb{E}(\mathcal{X}_1 \mid \mathcal{X}_0 = x) &\geq (1 - \varepsilon)\mathbb{E}(\mathcal{X}_1 \mid \mathcal{X}_0 = \lfloor (\lambda^* - \gamma)^n t_0 x^* \rfloor) \\ &\geq (\lambda^* - \varepsilon)(\lambda^* - \varepsilon)^n t_0 x^* \\ &= (\lambda^* - \gamma)^{n+1} t_0 x^*. \end{aligned}$$



Thus, we have shown that Corollary 56 holds with  $y = t_0 x^*$ ,  $r = \lambda^* - \gamma$ . We deduce that there exists  $K > 0$  such that, if  $x \geq K t_0 x^*$ , then

$$\mathbb{P}_x(|\mathcal{X}_1| \geq t_0 |x^*| (\lambda^* - \gamma)^n, \text{ for all } n \in \mathbb{N}) > 0.$$

The conclusion follows by setting  $\widetilde{K} = t_0 \max\{K, |x^*|\}$ .  $\square$

### 6.1.1.2 Examples

We turn our attention to some examples of bisexual branching processes with random mating. In this section we consider  $(W_n)_{n \in \mathbb{N}}$  a process as the one described in (6.1) with random mating function  $\phi$ . We recall that

$$\mathbb{E}(W_1 | W_0 = w) = \mathbb{E}(\phi(w))\mathbb{V}, \quad (6.19)$$

where  $\mathbb{V}$  is the expectation matrix for the offspring distribution, defined for the deterministic case in Chapter 2 (see (2.4)).

**Example 13** (Increasing random function). In this example we consider a model of multi-type bisexual branching process with random mating and we seek to apply Corollary 57. We start by defining, for  $p, q \in \mathbb{N}$ , the space of functions

$$\mathbb{F}_I^{q,p} := \{f : \mathbb{R}_+^q \longrightarrow \mathbb{R}_+^p, f \text{ is an increasing function}\}, \quad (6.20)$$

and

$$\mathbb{F}_S^{q,p} := \{f : \mathbb{R}_+^q \longrightarrow \mathbb{R}_+^p, f \text{ is a superadditive function}\}. \quad (6.21)$$

If we define  $R_{\text{inf}}$  for this process as in (6.15), then we obtain thanks to (6.19),

$$R_{\text{inf}}(w) = \liminf_{k \rightarrow +\infty} \frac{\mathbb{E}(\phi(kw))}{k} \mathbb{V}.$$

Assume that  $\mathbb{P}(\phi \in \mathbb{F}_I^{q,p}) = 1$ , that is,  $\phi$  is almost surely an increasing function. Hence, for  $w_1 \geq w_2$ ,

$$\mathbb{E}(W_1 | W_0 = w_1) = \mathbb{E}(\phi(w_1))\mathbb{V} \geq \mathbb{E}(\phi(w_2))\mathbb{V} \geq \mathbb{E}(W_1 | W_0 = w_2),$$

and in particular, condition (6.17) is satisfied. Finally, assume that  $\phi$  is such that the process  $(W_n)_{n \in \mathbb{N}}$  holds moment condition (6.14). Then, if there exists  $\lambda^* > 1$  and  $w^* \in \mathbb{R}_+^q$  such that

$$R_{\text{inf}}(w^*) \geq \lambda^* w^*,$$

the process has a positive probability of survival. We remark also that for all  $w \in \mathbb{R}_+^q$ ,

$$R_{\text{inf}}(w) \geq \liminf_{k \rightarrow +\infty} \frac{\mathbb{E}\left(\phi(kw) \mathbf{1}_{\phi \in \mathbb{F}_S^{q,p}}\right)}{k} \mathbb{V}, \quad (6.22)$$

with equality if  $\mathbb{P}(\phi \in \mathbb{F}_S^{q,p}) = 1$ . We have that for all  $w \in \mathbb{R}_+^q$ , the function

$$k \mapsto \mathbb{E}\left(\phi(kw) \mathbf{1}_{\phi \in \mathbb{F}_S^{q,p}}\right) \mathbb{V}$$

is in fact superadditive, and so Fekete's lemma ensures the existence of the function  $\mathbb{M} : \mathbb{R}_+^q \longrightarrow (\mathbb{R}_+ \cup \{+\infty\})^q$  given for all  $w \in \mathbb{R}_+^q$  by

$$\mathbb{M}(w) = \lim_{k \rightarrow +\infty} \frac{\mathbb{E}\left(\phi(kw) \mathbf{1}_{\phi \in \mathbb{F}_S^{q,p}}\right)}{k} \mathbb{V}.$$

Then, thanks to (6.22), if there exist  $\rho^* > 0$  and  $x^* \in \mathbb{R}_+^q$  such that

$$\mathbb{M}(x^*) \geq \rho^* x^*,$$

then the process has a positive probability of survival. We finish by remarking that  $\mathbb{M}$  is a positively homogeneous and concave function. Then, if  $\mathbb{M}$  is a primitive function, by applying Theorem 18 (concave Perron-Frobenius Theorem), we can guarantee the existence of eigenelements of  $\mathbb{M}$ .

**Example 14** (Deterministic mating with random perturbation). The idea of this example is to add a random perturbation to a supercritical bisexual branching process with deterministic mating. The goal is to find conditions under which this new process stays supercritical despite this perturbation. To that end we consider  $(Z_n)_{n \in \mathbb{N}}$  a multi-type bisexual branching process on  $\mathbb{R}_+^p$ , as defined in Chapter 2, with superadditive deterministic mating function  $\xi : \mathbb{R}_+^q \rightarrow \mathbb{R}_+^p$ . We emphasize that, as in the previous chapters,  $Z_n$  stands for the number of mating units in the generation  $n$ . Since the function  $\xi$  is superadditive, we recall that the function

$$\mathfrak{M}(z) = \lim_{k \rightarrow +\infty} \frac{\xi(kz\mathbb{V})}{k} \quad (6.23)$$

is well defined (see (2.6)), where  $\mathbb{V}$  is the expectation matrix of the offspring distribution. We assume that this function is primitive so that there exists  $\lambda^* > 0$  and  $z^* \in S^*$ , its eigenelements given by Theorem 18. We assume  $\lambda^* > 1$  so  $(Z_n)_{n \in \mathbb{N}}$  is a supercritical process.

We define a second process  $(W_n)_{n \in \mathbb{N}}$  on  $\mathbb{R}_+^q$ , which is a multi-type bisexual branching process with random mating (see (6.1)) with the same offspring distribution as  $(Z_n)_{n \in \mathbb{N}}$  and with mating function given for  $w \in \mathbb{R}_+^q$  by

$$\phi(w) = \xi(w) + \varphi(w),$$

where  $\varphi : \mathbb{R}_+^q \rightarrow \mathbb{R}^p$  is a random function. We stress the fact that  $\varphi$  is not necessarily a positive function. The idea is that  $(W_n)_{n \in \mathbb{N}}$  is obtained by perturbing the mating function of  $(Z_n)_{n \in \mathbb{N}}$  and the perturbation is given by  $\varphi$ . We assume that for all  $w \in \mathbb{R}_+^q$ ,  $\mathbb{E}(|\varphi(w)|) < +\infty$ .

We add the following assumptions on the function  $\xi$  and  $\varphi$ .

**Assumption 9.** We assume that  $\xi$  and  $\varphi$  are such that the moments condition (6.14) holds for  $(W_n)_{n \in \mathbb{N}}$ .

For the following assumption, we set

$$w^* = z^*\mathbb{V} \in \mathbb{R}_+^q. \quad (6.24)$$

**Assumption 10.** We assume that there exists  $c \in (0, \lambda^* - 1)$  and  $r \in (1, \lambda^* - c)$  such that for all  $w \in \mathbb{R}_+^q$ ,

$$w \geq r^n w^* \implies \mathbb{E}(\varphi(w)) \geq -cr^n z^* \quad (6.25)$$

We seek to use these assumptions to apply Corollary 56 to this process. Since (6.14) holds by Assumption 9, in the following we prove (6.13). For that, we set  $r \in (1, \lambda^*)$  for which Assumption 10 holds.

We start by noticing that for  $w^*$  given by (6.24),

$$\lim_{k \rightarrow +\infty} \frac{\xi(kw^*)}{k} \mathbb{V} = \mathfrak{M}(z^*) \mathbb{V} = \lambda^* z^* \mathbb{V} = \lambda^* w^*.$$

Hence, if we fix  $\varepsilon \in (0, \lambda^* - 1)$  such that  $(\lambda^* - c - \varepsilon)(1 - \varepsilon) > r$ , there exists  $k_0 > 0$  such that for all  $k \geq k_0$ ,

$$\xi(kw^*)\mathbb{V} \geq (\lambda^* - \varepsilon)kw^* \quad (6.26)$$

The idea is to apply Corollary 56 with  $y = w^*$ . Fix  $N \in \mathbb{N}$  such that  $r^N \geq k_0$ , then (6.26) holds with  $k = r^n$  for all  $n \geq N$ . Consider  $n \geq N$  and  $w \in \mathbb{R}_+^q$  with  $w \geq r^n w^*$ . Using (6.26)

$$\begin{aligned} \mathbb{E}(W_1 | W_0 = w) &= ((\xi(w) + \mathbb{E}(\varphi(w)))\mathbb{V}) \\ &= \xi(w)\mathbb{V} + \mathbb{E}(\varphi(w))\mathbb{V} \\ &\geq \xi(r^n w^*)\mathbb{V} + \mathbb{E}(\varphi(w))\mathbb{V} \\ &\geq (\lambda^* - \varepsilon)r^n w^* + \mathbb{E}(\varphi(w))\mathbb{V} \\ &= \frac{r^{n+1}}{1 - \varepsilon}w^* + \left(\lambda^* - \varepsilon - \frac{r}{1 - \varepsilon}\right)r^n w^* + \mathbb{E}(\varphi(w))\mathbb{V}. \\ &= \frac{r^{n+1}}{1 - \varepsilon}w^* + \left(\left(\lambda^* - \varepsilon - \frac{r}{1 - \varepsilon}\right)r^n z^* + \mathbb{E}(\varphi(w))\right)\mathbb{V}, \end{aligned}$$

where in the first inequality we used the fact that  $\xi$  is an increasing function. Thanks to the choice of  $\varepsilon$ , we have that

$$\left(\lambda^* - \varepsilon - \frac{r}{1 - \varepsilon}\right) > c.$$

Thus, combining this with Assumption 10,

$$\left(\lambda^* - \varepsilon - \frac{r}{1 - \varepsilon}\right)r^n z^* + \mathbb{E}(\varphi(w)) > cr^n z^* + \mathbb{E}(\varphi(w)) \geq 0,$$

since  $w \geq r^n w^*$ .

We conclude that

$$(1 - \varepsilon)\mathbb{E}(W_1 | W_0 = w) \geq r^{n+1}w^*,$$

and all the assumptions of Corollary 56 are satisfied. Then, the process  $(W_n)_{n \in \mathbb{N}}$  has a positive probability of survival.

We remark that the condition in Assumption 10 states that if  $\lambda^* > 1$  is larger, then the expectation of  $\varphi(w)$  can be negative following the direction of  $z^*$ , but controlled with the value of  $r \in (1, \lambda^*)$ . This is consistent with the fact that  $(Z_n)_{n \in \mathbb{N}}$  grows in the direction of  $z^*$  (see Chapter 3), so this perturbation is compensated.

## 6.2 Application: Age structured two-sex branching process

In this section we define an age structured branching process for a two-sex population and we use the results from Section 6.1 to give sufficient conditions for survival. We start by stating the model, and we emphasize that for convenience we change the notation with respect to the previous processes that we have defined. For this model, we consider the following

- We consider a population of individuals formed by females and males. Each individual has an age between 1 and  $N$ , for  $N \in \mathbb{N}$ , so we set  $(W_n)_{n \in \mathbb{N}}$  a multi-type bisexual branching process on  $\mathbb{N}^{2N}$ . We write for  $n \in \mathbb{N}$ ,

$$W_n = (X_{n,1}, Y_{n,1}, \dots, X_{n,N}, Y_{n,N}),$$

where for  $i \in \{1, \dots, N\}$ ,  $X_{n,i}$  (resp.  $Y_{n,i}$ ) stands for the number of females (resp. of males) of age  $i$  the  $n$ -th generation.

- Individuals can form mating units of  $N^2$  types according to a random mating function  $\phi : \mathbb{R}_+^{2N} \rightarrow \mathbb{R}_+^{N \times N}$ , where for  $w \in \mathbb{N}^{2N}$  and  $i, j \in \{1, \dots, N\}$ ,  $\phi_{i,j}(w)$  stands for the number of mating units form by a female of age  $i$  and a male of age  $j$ . We say that a mating unit is of type  $(i, j)$ .
- A mating unit of type  $(i, j)$  will then give birth to a certain number of individuals of age 1 (or *newborns*) following the probability distribution of an integrable random variable  $N(i, j)$ .
- A newborn is a female with probability  $\alpha \in (0, 1)$  independent of all the rest.
- An individual of age  $i \in \{1, \dots, N_1\}$  at the  $n$ -th generation becomes an individual of age  $i + 1$  in generation  $n + 1$ .
- An individual of age  $N$  at the  $n$ -th generation, disappears from the population at generation  $n + 1$ .

Formally, we can describe the previous points as follows. Given  $W_0 = w_0 \in \mathbb{N}^{2N}$  we define for  $n \in \mathbb{N}$  the newborns at generation  $n + 1$  by

$$I(W_n) = \sum_{i,j=1}^N \sum_{k=1}^N \phi_{i,j}^{(n)}(W_n) N_{k,n}(i, j),$$

where  $(\phi^{(n)})_{n \in \mathbb{N}}$  are i.i.d. random functions with the same law of the mating function  $\phi$ , and  $(N_{k,n}(i, j))_{k,n \in \mathbb{N}}$  are i.i.d. random variable with the same distribution as  $N(i, j)$ , and independent of  $\phi$ . We then set

$$X_{n+1,1} = \sum_{i=1}^{I_{n+1}(W_n)} \chi_{n,i} = I_{n+1} - Y_{n+1,1},$$

where  $(\chi_{n,i})_{n,i \in \mathbb{N}}$  are all i.i.d random variables following the distribution of a Bernoulli( $\alpha$ ). Finally, we set

$$W_{n+1} = (X_{n+1,1}, Y_{n+1,1}, X_{n,1}, Y_{n,1}, \dots, X_{n,N-1}, Y_{n,N-1}).$$

In other words, given  $W_n = (x_1, y_1, \dots, x_N, y_N) \in \mathbb{N}^{2N}$ , we have  $W_{n+1} = (X_{n+1,1}, Y_{n+1,1}, x_1, y_1, \dots, x_{N-1}, y_{N-1})$ .

We assume that the mating function holds that for all  $w \in \mathbb{N}^{2N}$ ,  $\mathbb{P}(\phi(w) \in \mathbb{N}^{N \times N}) = 1$  and that  $\mathbb{P}(\phi(0) = 0) = 1$ .

### 6.2.1 Survival Criteria

In this subsection we use the results in Section 6.1, and in particular that of Corollary 57, to state a sufficient condition for survival for the process presented in the previous section. We start

We start by setting

$$I(w) = \sum_{i=1}^N \sum_{j=1}^N \phi_{i,j}(w) N_{k,1}(i, j).$$

That is  $I(w)$  has the distribution of the number of newborns produced by a population  $w \in \mathbb{N}^{2N}$ . We define for  $A, B \in \mathbb{R}_+^{N \times N}$ ,

$$\langle A, B \rangle = \sum_{i,j=1}^N A_{i,j} B_{i,j}, \quad (6.27)$$

then for all  $w \in \mathbb{N}^{2N}$ ,  $\mathbb{E}(I(w)) = \langle \Phi(w), \mathbb{N} \rangle$ , for  $\Phi_{i,j}(w) = \mathbb{E}(\phi_{i,j}(w))$  and  $\mathbb{N}_{i,j} = \mathbb{E}(N(i, j))$ .

Consider the function  $f_{\text{inf}} : \mathbb{R}_+^{2N} \rightarrow (\mathbb{R}_+ \cup \{+\infty\})$  given by

$$f_{\text{inf}}(w) = \liminf_{k \rightarrow +\infty} \left\langle \frac{\Phi(\lfloor kw \rfloor)}{k}, \mathbb{N} \right\rangle. \quad (6.28)$$

We assume that the mating function and the offspring distribution are such that  $\text{Var}(I(w)) < +\infty$  for all  $w \in \mathbb{N}^{2N}$ . We state the following survival criteria in terms of  $f_{\text{inf}}$ .

**Proposition 58.** *Assume that  $f_{\text{inf}}$  is an increasing function on  $\mathbb{R}_+^q$ . Define the function  $g_\alpha : (0, +\infty) \rightarrow \mathbb{R}_+$  given by,*

$$g_\alpha(\lambda) = f_{\text{inf}} \left( \alpha, 1 - \alpha, \frac{\alpha}{\lambda}, \frac{1 - \alpha}{\lambda}, \dots, \frac{\alpha}{\lambda^{N-1}}, \frac{1 - \alpha}{\lambda^{N-1}} \right). \quad (6.29)$$

*If there exists  $\lambda^* > 1$  such that  $g_\alpha(\lambda^*) \geq \lambda^*$ , then there is a positive probability of survival.*

*Proof.* We start by pointing out that if we define  $R_{\text{inf}}$  as in (6.15) in Corollary 57, then we have for all  $w = (x_1, y_1, \dots, x_N, y_N)$ ,

$$R_{\text{inf}}(w) = (\alpha f_{\text{inf}}(w), (1 - \alpha) f_{\text{inf}}(w), x_1, y_1, \dots, x_{N-1}, y_{N-1}).$$

Set  $w^* = \left( \alpha, 1 - \alpha, \frac{\alpha}{\lambda^*}, \frac{1 - \alpha}{\lambda^*}, \dots, \frac{\alpha}{(\lambda^*)^{N-1}}, \frac{1 - \alpha}{(\lambda^*)^{N-1}} \right)$ . We have that

$$R_{\text{inf}}(w^*) = \left( \alpha g_\alpha(\lambda^*), (1 - \alpha) g_\alpha(\lambda^*), \alpha, 1 - \alpha, \frac{\alpha}{\lambda^*}, \frac{1 - \alpha}{\lambda^*}, \dots, \frac{\alpha}{(\lambda^*)^{N-1}}, \frac{1 - \alpha}{(\lambda^*)^{N-1}} \right) \geq \lambda^* w^*.$$

This fact, together with the assumption that  $f_{\text{inf}}$  is increasing allow us to apply Corollary 57, which concludes the proof.  $\square$

The reason to define  $g_\alpha$  as in (6.29) is that the existence of  $\lambda^*$  holding  $g_\alpha(\lambda^*) \geq \lambda^*$  is also a necessary condition for the existence of a subeigenvector of  $R_{\text{inf}}$ , as shown in the proposition below. Thus, condition (6.29) is equivalent to (6.16) in Corollary 57.

**Proposition 59.** *Define  $R_{\text{inf}} : \mathbb{R}_+^q \rightarrow \mathbb{R}_+^q$  as*

$$R_{\text{inf}}(w) = (\alpha f_{\text{inf}}(w), (1 - \alpha) f_{\text{inf}}(w), x_1, y_1, \dots, x_{N-1}, y_{N-1}).$$

*Assume that there exists  $w \in \mathbb{R}_+^q$  and  $\lambda_w > 0$  such that  $R_{\text{inf}}(w) \geq \lambda_w w$ . Then, if we define  $g_\alpha$  as in (6.29), we have  $g_\alpha(\lambda_w) \geq \lambda_w$ .*

*Proof.* Write  $w = (x_1, y_1, \dots, x_N, y_N)$ . If  $R_{\text{inf}}(w) \geq \lambda_w w$ , we have that

$$f_{\text{inf}}(w) \geq \lambda_w \frac{x_1}{\alpha}, \quad f_{\text{inf}}(w) \geq \lambda_w \frac{y_1}{1 - \alpha}.$$

Set  $w_1 = \max \left\{ \frac{x_1}{\alpha}, \frac{y_1}{1 - \alpha} \right\}$ . We have that for all  $i \in \{2, \dots, N\}$ ,

$$x_i \leq \frac{x_1}{(\lambda_w)^{i-1}} \leq \frac{\alpha w_1}{(\lambda_w)^{i-1}}, \quad y_i \leq \frac{y_1}{(\lambda_w)^{i-1}} \leq \frac{(1 - \alpha) w_1}{(\lambda_w)^{i-1}}.$$

Thus, if we define

$$w^* = \left( \alpha w_1, (1 - \alpha) w_1, \frac{\alpha w_1}{\lambda_w}, \frac{(1 - \alpha) w_1}{\lambda_w}, \dots, \frac{\alpha w_1}{\lambda_w^{N-1}}, \frac{(1 - \alpha) w_1}{\lambda_w^{N-1}} \right),$$

then, since  $f_{\text{inf}}$  is an increasing function and  $w^* \geq w$ , we have,

$$f_{\text{inf}}(w^*) \geq f_{\text{inf}}(w) \geq \lambda_w w_1.$$

Finally, since for all  $\eta \in \mathbb{R}_+$  and all  $x \in \mathbb{R}_+^q$ ,  $f_{\text{inf}}(\eta x) = \eta f_{\text{inf}}(x)$ , we have that

$$g_\alpha(\lambda_w) = \frac{f_{\text{inf}}(w^*)}{w_1} \geq \lambda_w.$$

□

## 6.2.2 Example of Mating Function

We consider the following example of mating function, which maximize the expected number of newborns from one generation to the next.

We set  $F : \mathbb{R}_+^{2N} \rightarrow \mathbb{R}_+$  given by,

$$F(w) = \max_{z \in C(w)} \langle z, \mathbb{N} \rangle, \quad (6.30)$$

where for every  $w = (x_1, y_1, \dots, x_N, y_N) \in \mathbb{R}_+^{2N}$ , the set  $C(w) \subseteq \mathbb{R}_+^{N \times N}$  is given by,

$$C(w) = \left\{ z_{i,j} \in \mathbb{N}^{N \times N} : \sum_{k=1}^N z_{i,k} \leq x_i, \sum_{k=1}^N z_{k,j} \leq y_j, \forall i, j \in \{1, \dots, N\} \right\}. \quad (6.31)$$

The idea behind the definition of  $C(w)$  comes from the perfect fidelity mating (see Example 4). In fact if for  $i, j \in \{1, \dots, N\}$   $z_{i,j}$  represents the number of mating units formed by a female of age  $i$  and a male of age  $j$ , then  $\sum_{k=1}^N z_{i,k}$  is the total number of couples where the female has age  $i$ . Thus, the first restriction within  $C(w)$  forces this quantity to be less than or equal to the total number of females of age  $i$  in the population. The second restriction is analogous but for the males.

We set then our mating function  $\phi : \mathbb{R}_+^{2N} \rightarrow \mathbb{R}_+^{N \times N}$  as any function that holds that for all  $w \in \mathbb{N}^{2N}$ ,  $w = (x_1, y_1, \dots, x_N, y_N)$ ,

$$\mathbb{E}(I(w)) = \langle \Phi(w), \mathbb{N} \rangle = F(w). \quad (6.32)$$

For instance, one can set  $\phi(w)$  as a solution for the maximization problem given in (6.30), chosen uniformly at random. The idea is that the mating units are formed such that the mean number of newborns begotten is maximized.

**Proposition 60.** *The function  $F$  defined in (6.30) is superadditive.*

*Proof.* It is easy to check that, for all  $w^1, w^2 \in \mathbb{N}^{2N}$ , we have that  $C(w^1) + C(w^2) \subseteq C(w^1 + w^2)$ . Then,

$$\begin{aligned} F(w^1 + w^2) &= \max_{z \in C(w^1 + w^2)} \langle z, \mathbb{N} \rangle \\ &\geq \max_{z \in C(w^1) + C(w^2)} \langle z, \mathbb{N} \rangle \\ &= \max_{\substack{z^1 \in C(w^1) \\ z^2 \in C(w^2)}} \langle z^1, \mathbb{N} \rangle + \langle z^2, \mathbb{N} \rangle \\ &= \max_{z^1 \in C(w^1)} \langle z^1, \mathbb{N} \rangle + \max_{z^2 \in C(w^2)} \langle z^2, \mathbb{N} \rangle \\ &= F(w^1) + F(w^2). \end{aligned}$$

□

If we now define  $f_{\text{inf}}$  as in (6.28), one has thanks to Proposition 60 that

$$f_{\text{inf}}(w) = \lim_{k \rightarrow +\infty} \frac{F(kw)}{k},$$

where the limit is well defined because of the superadditivity. In the last proposition, we show that it is also finite, and we give an explicit expression for its value.

**Proposition 61.** *Define for all  $w \in \mathbb{R}_+^q$ , the set  $D(w)$  given by*

$$D(w) = \left\{ z \in \mathbb{R}_+^{N \times N} : \sum_{k=1}^N z_{i,k} \leq x_i, \sum_{k=1}^N z_{k,j} \leq y_j, \forall i, j \in \{1, \dots, N\} \right\}.$$

We have that

$$f_{\text{inf}} = \max_{w \in D(w)} \langle z, \mathbb{N} \rangle.$$

That is, we extend the maximization problem defining the function  $F$  to all positive real vectors.

*Proof.* Define  $G(w) = \max_{w \in D(w)} \langle z, \mathbb{N} \rangle$ . We prove that,

$$G(w) = \sup_{k \in \mathbb{N}} \frac{F(kw)}{k} = \lim_{k \rightarrow +\infty} \frac{F(kw)}{k}.$$

If  $w = 0$  the conclusion is direct. Consider  $w \in \mathbb{R}_+^{2N} \setminus \{0\}$  and fix  $u > 0$  such that  $uw_i > N$  for all  $w_i \neq 0$ . Thanks to our choice, we have that for all  $k \in \mathbb{N}$ ,  $k > u$ , if  $z \in D((k-u)w)$ , then  $[z] \in C(kw)$ . In other words, for all  $k \in \mathbb{N}$ ,  $k > u$ ,

$$(k-u) \max_{z \in D(w)} \langle z, \mathbb{V} \rangle = \max_{z \in D((k-u)w)} \langle z, \mathbb{V} \rangle \leq \max_{z \in C(kw)} \langle z, \mathbb{V} \rangle.$$

With this

$$G(w) \leq \frac{F(kw)}{k-u} = \frac{k}{k-u} \frac{F(kw)}{k}.$$

Taking  $k \rightarrow +\infty$ , we conclude the first inequality. For the second one we have that

$$\sup_{k \in \mathbb{N}} \frac{1}{k} \max_{z \in C(kw)} \langle z, \mathbb{V} \rangle \leq \sup_{k \in \mathbb{N}} \frac{1}{k} \max_{z \in D(kw)} \langle z, \mathbb{V} \rangle = \max_{z \in D(w)} \langle z, \mathbb{V} \rangle = G(w),$$

and the result follows. □

Thus, for this case, the survival criteria given by Proposition 58 can be stated as follows: there exists  $\lambda^* > 1$  such that

$$\max_{z \in D(\lambda^*)} \langle z, \mathbb{N} \rangle \geq \lambda^*,$$

where for  $\lambda > 0$ , the set  $D(\lambda)$  is given by

$$D(\lambda) = \left\{ z \in \mathbb{R}_+^{N \times N} : \sum_{k=1}^N z_{i,k} \leq \frac{\alpha}{\lambda^{i-1}}, \sum_{k=1}^N z_{k,j} \leq \frac{1-\alpha}{\lambda^{i-1}}, \forall i, j \in \{1, \dots, N\} \right\}.$$

### 6.3 Linear growth towards infinity

In this section we deal with a variation of Theorem 55, but for processes that grow at least linearly towards infinity. We use then, in Section 6.4, to give necessary and sufficient conditions for extinction for two models of continuous time birth and death process for two-sex populations.

We start by stating the main result of this section.

**Proposition 62.** *Assume that there exist  $N \in \mathbb{N}$  and  $N$  functions  $f_1, \dots, f_N$ , where for all  $i \in \{1, \dots, N\}$ ,  $f_i : \mathbb{R}_+^q \rightarrow \mathbb{R}_+$ . Define*

$$g(x) = \min_{i \in \{1, \dots, N\}} f_i(x),$$

and assume that the following statements hold:

1. For any sequence  $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}_+^q$ ,

$$g(x_n) \xrightarrow{n \rightarrow +\infty} +\infty \Rightarrow |x_n| \xrightarrow{n \rightarrow +\infty} +\infty \quad (6.33)$$

2. For all  $i \in \{1, \dots, N\}$ ,  $\mathbb{E}_x(f_i(\mathcal{X}_1)) < +\infty$ , and there exist  $c > 0$ ,  $m \in \mathbb{N}$  and  $\varepsilon \in (0, 1)$  such that for all  $n \geq m$ , and all  $x \in \mathbb{R}_+^q$ ,

$$g(x) \geq cn \Rightarrow (1 - \varepsilon) \min_{i \in \{1, \dots, N\}} \mathbb{E}_x(f_i(\mathcal{X}_1)) \geq c(n + 1). \quad (6.34)$$

3. There exist  $0 < \alpha < \delta$  such that for all  $i \in \{1, \dots, N\}$ ,

$$\mathbb{E}_x \left( |f_i(\mathcal{X}_1) - \mathbb{E}_x(f_i(\mathcal{X}_1))|^{1+\delta} \right) = O(\mathbb{E}_x(f_i(\mathcal{X}_1))^\alpha), \quad (6.35)$$

as  $|x| \rightarrow +\infty$ .

Then there exists  $K > 0$ , such that for all  $x \in \mathbb{R}_+^q$  such that  $g(x) \geq K$ , we have

$$\mathbb{P}_x(g(\mathcal{X}_n) \geq cn, \text{ for all } n \in \mathbb{N}) > 0.$$

In particular,  $\mathbb{P}_x(|X_n| \xrightarrow{n \rightarrow +\infty} +\infty) > 0$ , and the process has a linear (at least) unlimited growth with positive probability.

Note that the assumptions here are very similar to those of Theorem 55. The differences are in (6.34), where we now ask for a linear growth instead of an exponential one, and in (6.35), where we are far more restrictive in the moment condition. Although this last assumption may seem difficult to hold, it holds true for processes where the expectation of the increment is uniformly bounded and for discretization of continuous time models under mild assumptions (see examples below).

*Remark 20.* As in the case of Theorem 55, assumption 6.34 can be replaced with the following: there exist  $A, c > 0$  such that for all  $x \in \mathbb{R}_+^q$ ,

$$g(x) \geq A \Rightarrow \min_{i \in \{1, \dots, N\}} \mathbb{E}_x(f_i(\mathcal{X}_1)) \geq g(x) + c. \quad (6.36)$$



*Proof of Proposition (62).* The proof is very similar to that of Theorem 55 and so we only emphasize the main differences. Following the same reasoning as in *Step 1*, but instead using the function  $y \rightarrow y^{1+\delta}$  for Markov's inequality, we can conclude that if  $g(x) \geq cn$ ,

$$\begin{aligned} \mathbb{P}_x(g(X_1) < c(n+1)) &\leq \sum_{i=1}^N \frac{\mathbb{E}_x(|f_i(X_1) - \mathbb{E}_x(f_i(X_1))|^{1+\delta})}{\varepsilon^{1+\delta} \mathbb{E}_x(f_i(X_1))^{1+\delta}} \\ &\leq \sum_{i=1}^N \frac{\mathbb{E}_x(|f_i(X_1) - \mathbb{E}_x(f_i(X_1))|^{1+\delta})(1-\varepsilon)^{1+\delta-\alpha}}{\varepsilon^{1+\delta} \mathbb{E}_x(f_i(X_1))^\alpha (n+1)^{1+\delta-\alpha}} \end{aligned}$$

Hence, applying assumption (6.35), there exists  $C > 0$  such that

$$\mathbb{P}_x(g(X_1) < c(n+1)) \leq \frac{C}{(n+1)^{1+\eta}},$$

with  $\eta = \delta - \alpha > 0$ .

The conclusion follows as in *Step 2*, since  $\sum_{n \in \mathbb{N}} \frac{1}{n^{1+\eta}} < +\infty$ .  $\square$

### 6.3.1 Examples

Using Proposition 62 we show two examples of known results that can be deduced using Proposition 62: a random walk with a drift in discrete time and the (asexual) continuous time multi-type Galton-Watson branching process.

**Example 15** (Killed Random Walk with a Drift). Consider a sequence of i.i.d. random variables  $(V_n)_{n \in \mathbb{N}}$  with  $\mathbb{E}(|V_1|^{1+\alpha}) < +\infty$  for some  $\alpha > 0$ , and define the process  $(W_n)_{n \in \mathbb{N}}$  given by  $W_0 = w \in \mathbb{R}_+^*$  and for  $n \in \mathbb{N}$ ,

$$W_{n+1} = W_n + V_n.$$

Define  $T_0 = \inf\{n \in \mathbb{N}, W_n \leq 0\}$  the first time  $W$  exits  $\mathbb{R}_+$  and consider the process  $(W_{n \wedge T_0})_{n \in \mathbb{N}}$ . We consider the function  $g(x) = x$ , for which obviously (6.33) holds. In addition, note that for all  $w \in \mathbb{R}_+$ ,

$$\mathbb{E}_w(|W_1 - \mathbb{E}_w(W_1)|^{1+\alpha}) = \mathbb{E}(|V_1 - \mathbb{E}(V_1)|^{1+\alpha}) < +\infty,$$

which proves that (6.35) is true. Finally, if we suppose that  $\mathbb{E}(V_1) > 0$ , then we obtain that

$$\mathbb{E}_w(g(W_1)) = g(w) + \mathbb{E}(V_1).$$

Thus, applying Proposition 62 we obtain that  $\mathbb{P}_w(W_n \geq \mathbb{E}(V_1)n, \text{ for all } n \in \mathbb{N}) > 0$ , thus recovering the known result (see [Fel91]) that  $\mathbb{P}_w(T_0 < +\infty) < 1$  for this process.

**Example 16** (Continuous time multi-type Galton-Watson process). Consider a continuous time Markov process  $(X_t)_{t \geq 0}$  on  $\mathbb{N}^p$  such that for all  $i \in \{1, \dots, p\}$ , at rate  $\lambda_i$  an individual of type  $i$  is killed and replaced with a random vector  $V_i \in \mathbb{N}^p$ . Consider the sequence of stopping times  $(\tau_n)_{n \in \mathbb{N}}$  given by  $\tau_0 = 0$  and for all  $n \in \mathbb{N}$ ,

$$\tau_{n+1} = \inf\{t \geq \tau_n : X_t \neq X_{\tau_n}\}.$$

That is,  $(\tau_n)_{n \in \mathbb{N}}$  corresponds to the jump times of the process  $(X_t)_{t \geq 0}$ . Define the discrete time process  $(Y_n)_{n \in \mathbb{N}}$  given by  $Y_n = X_{\tau_n}$ . We have that for  $y \in \mathbb{N}^p$  and  $\mu \in \mathbb{R}_+^p$ ,

$$\mathbb{E}_y(\langle \mu, Y_1 \rangle) = \langle \mu, y \rangle + \frac{\langle \mu, yA \rangle}{\sum_{i=1}^p \lambda_i y_i},$$

where  $A$  is the  $p \times p$  matrix given for  $i, j \in \{1, \dots, p\}$  by  $A_{i,j} = \lambda_i(\mathbb{E}(V_{i,j}) - \mathbf{1}_{i=j})$ . We recover then the fact that the existence of a strictly positive eigenvalue  $\eta^*$  of  $A$ , associated with a positive eigenvector  $\mu^* \in \mathbb{R}_+^p$  is a sufficient condition for survival:

$$\mathbb{E}_y(\langle \mu^*, Y_1 \rangle) = \langle \mu^*, y \rangle + \eta^* \frac{\langle \mu^*, y \rangle}{\sum_{i=1}^p \lambda_i y_i} \geq \langle \mu^*, y \rangle + \eta^* \frac{\min_{i \in \{1, \dots, p\}} \mu_i^*}{\max_{i \in \{1, \dots, p\}} \lambda_i}.$$

Thus, Proposition 62 holds with  $f_1(y) = g(y) = \langle \mu^*, y \rangle$ .

## 6.4 Application: continuous time two-sex populations

In this section we will see how the previous results, stated for discrete time processes, can be applied to obtain extinction conditions for a continuous time version of a two-sex birth and death process.

We consider a continuous time process  $(X_t, Y_t)_{t \geq 0}$ , where  $X_t$  and  $Y_t$  represent the number of females and males, respectively, at time  $t \in \mathbb{R}_+$ , with the following transition probabilities: for all  $(x, y) \in \mathbb{N}_+^2$ , and all  $t, h > 0$ ,

$$\begin{aligned} & \mathbb{P}((X_{t+h}, Y_{t+h}) = (a, b) \mid (X_t, Y_t) = (x, y)) \\ &= \begin{cases} \phi_f(x, y)h + o(h) & \text{if } a = x + 1, b = y \text{ [birth of a female]} \\ \mu_f(x)h + o(h) & \text{if } a = x - 1, b = y \text{ [death of a female]} \\ \phi_m(x, y)h + o(h) & \text{if } a = x, b = y + 1 \text{ [birth of a male]} \\ \mu_m(y)h + o(h) & \text{if } a = x, b = y - 1 \text{ [death of a male]} \\ 1 - (\phi_f(x, y) + \phi_m(x, y) + \mu_f(x) + \mu_m(y))h + o(h) & \text{in any other case [no event]}, \end{cases} \end{aligned} \quad (6.37)$$

with  $\phi_f, \phi_m : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  the birth rates of females and males respectively and  $\mu_f, \mu_m : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  the death rates of females and males respectively.

Let us consider  $(T_n)_{n \in \mathbb{N}}$  the sequence of jump times of  $(X_t, Y_t)_{t \geq 0}$ , that is  $T_0 = 0$  and for all  $n \in \mathbb{N}$ ,

$$T_{n+1} = \begin{cases} \inf \{t > T_n, (X_t, Y_t) \neq (X_{T_n}, Y_{T_n})\}, & \text{if } (X_{T_n}, Y_{T_n}) \neq (0, 0) \\ T_n, & \text{if } (X_{T_n}, Y_{T_n}) = (0, 0). \end{cases}$$

Then the process  $(X_{T_n}, Y_{T_n})_{n \in \mathbb{N}}$  is a discrete Markov chain with absorption in  $\{(0, 0)\}$ , with transition probabilities

$$(X_{T_{n+1}}, Y_{T_{n+1}}) = \begin{cases} (X_{T_n} + 1, Y_{T_n}) & \text{with probability } \frac{\phi_f(x, y)}{\phi_f(x, y) + \phi_m(x, y) + \mu_f(x) + \mu_m(y)} \\ (X_{T_n} - 1, Y_{T_n}) & \text{with probability } \frac{\mu_f(x)}{\phi_f(x, y) + \phi_m(x, y) + \mu_f(x) + \mu_m(y)} \\ (X_{T_n}, Y_{T_n} + 1) & \text{with probability } \frac{\phi_m(x, y)}{\phi_f(x, y) + \phi_m(x, y) + \mu_f(x) + \mu_m(y)} \\ (X_{T_n}, Y_{T_n} - 1) & \text{with probability } \frac{\mu_m(y)}{\phi_f(x, y) + \phi_m(x, y) + \mu_f(x) + \mu_m(y)}, \end{cases} \quad (6.38)$$

if  $(X_{T_n}, Y_{T_n}) \neq (0, 0)$ , and with  $(X_{T_m}, Y_{T_m}) = (0, 0)$  almost surely for all  $m \geq n$  if  $(X_{T_n}, Y_{T_n}) = (0, 0)$ .

We will study extinction conditions for two specific examples in the next two sections applying Proposition 62 to (a subsequence of) the discrete Markov chain  $(X_{T_n}, Y_{T_n})_{n \in \mathbb{N}}$ . The first example, studied in Section 6.4.1, can be seen as a continuous extension of the completely promiscuous model of Example 5 with  $q_m = q_f = 1$ . It also can be seen as a particular case of a continuous

model treated in [KK73] in which conditions for survival and certain extinction have already been found. However, we give here a probabilistical proof using Proposition 62 although the proof in [KK73] is completely analytical. The second example, studied in Section 6.4.2), is a continuous time birth and death process inspired by the perfect fidelity model of Example 4. Up to our knowledge, the result about the extinction condition is new for this model.

### 6.4.1 Continuous time completely promiscuous model

The continuous time completely promiscuous model corresponds to the case where

$$\begin{cases} \phi_f(x, y) &= \lambda_f x \mathbb{1}_{y>0}, \\ \phi_m(x, y) &= \lambda_m x \mathbb{1}_{y>0}, \\ \mu_f(x) &= \mu_f x, \\ \mu_m(y) &= \mu_m y, \end{cases} \quad (6.39)$$

with  $\lambda_f, \lambda_m, \mu_f, \mu_m > 0$ . The model can be interpreted in the following way: each female can produce offspring if there is at least one male in the population (similarly to Example 5). In this case, offspring are produced at the rate  $\lambda_f + \lambda_m$  by each female. The probability that the offspring is a female is  $\lambda_f/(\lambda_f + \lambda_m)$ . The death rates  $\mu_f$  and  $\mu_m$  are linear in the number of females and males respectively, meaning that they describe natural death and neglect competition between individuals.

As stated above, we want to apply Proposition 62 to a subsequence of the discrete Markov chain  $(X_{T_n}, Y_{T_n})_{n \in \mathbb{N}}$  to recover the following result with a new probabilistic proof.

**Proposition 63.** *The process  $(X_t, Y_t)_{t \geq 0}$ , defined by (6.37)-(6.39) has a positive probability of survival if and only if  $\lambda_f > \mu_f$ .*

First of all, let us remark that the process of females is stochastically dominated by a linear birth and death process with birth rate  $\bar{\phi}_f(x) = \lambda_f x$  and death rate  $\mu_f(x) = \mu_f x$  (see [BM15], Chapter 2). Then, as this linear process goes to extinction almost surely if and only if  $\lambda_f \leq \mu_f$ , then we conclude that the process  $(X_{T_n}, Y_{T_n})_{n \in \mathbb{N}}$  also goes to extinction almost surely if  $\lambda_f \leq \mu_f$ . Let us now assume that  $\lambda_f > \mu_f$  and prove that the process  $(X_{T_n}, Y_{T_n})_{n \in \mathbb{N}}$  has a positive probability of survival.

Let us first explain why Proposition 62 can not be applied to the whole sequence  $(X_{T_n}, Y_{T_n})_{n \in \mathbb{N}}$ . From (6.38), we obtain for  $x, y \in \mathbb{N}^*$ ,

$$\mathbb{E}_{(x,y)}(X_{T_1}) = x + \frac{\lambda_f x - \mu_f x}{\lambda_f x + \lambda_m x + \mu_f x + \mu_m y}, \quad (6.40)$$

and

$$\mathbb{E}_{(x,y)}(Y_{T_1}) = y + \frac{\lambda_m x - \mu_m y}{\lambda_f x + \lambda_m x + \mu_f x + \mu_m y}. \quad (6.41)$$

We obtain that both expectations  $\mathbb{E}_{(x,y)}(X_{T_1})$  and  $\mathbb{E}_{(x,y)}(Y_{T_1})$  are lower bounded by the quantity

$$\min\{x, y\} + \frac{\min\{(\lambda_f - \mu_f)x, \lambda_m x - \mu_m y\}}{(\lambda_f + \lambda_m + \mu_f)x + \mu_m y}.$$

If we then seek to apply Proposition 62 with the functions  $f_1(x, y) = x$ ,  $f_2(x, y) = y$  and  $g(x, y) = \min\{x, y\}$ , we face two problems:

1. We have to be careful with the quantity  $\lambda_m y - \mu_m x$ , and consider only the cases when this one is positive. In order to do so, we consider  $K \in \mathbb{N}$  such that  $\frac{1}{K} < \frac{\lambda_m}{\mu_m}$ . Then, if  $y < \frac{1}{K}x$  we have  $\lambda_m x - \mu_m y > 0$ .
2. We need the quantity  $c(x, y) = \frac{\min\{(\lambda_f - \mu_f)x, \lambda_m x - \mu_m y\}}{(\lambda_f + \lambda_m + \mu_f)x + \mu_m y}$  to be uniformly bounded from below for all  $x, y \in \mathbb{N}$  large enough for a constant  $c > 0$ . Note that this is not true since, for example, for  $x$  fixed and  $y \rightarrow +\infty$ , we have  $c(x, y) \rightarrow 0$ . To overcome this issue we will modify the sequence of stopping times, and we look at both processes only when they are close. When they are separated, we will only look at the minimum between the two, which has an increasing mean, as exposed in the previous point.

Let  $K \in \mathbb{N}^*$  be large enough so that  $\lambda_m K - \mu_m > 0$  and  $L > 1$ . We define a sequence of stopping times  $(\tau_n)_{n \in \mathbb{N}}$ , given by  $\tau_0 = 0$  and

$$\tau_{n+1} = \begin{cases} \inf\{t > \tau_n, (X_t, Y_t) \neq (X_{\tau_n}, Y_{\tau_n})\} = \inf\{T_i > \tau_n, i \in \mathbb{N}\} & \text{if } \left| \frac{1}{K}X_{\tau_n} - Y_{\tau_n} \right| < L \\ \inf\{t > \tau_n, Y_t \neq Y_{\tau_n}\} \wedge \inf\left\{t > \tau_n, \left| \frac{1}{K}X_t - Y_t \right| < L\right\} & \text{if } \frac{1}{K}X_{\tau_n} - Y_{\tau_n} \geq L \\ \inf\{t > \tau_n, X_t \neq X_{\tau_n}\} \wedge \inf\left\{t > \tau_n, \left| \frac{1}{K}X_t - Y_t \right| < L\right\} & \text{if } Y_{\tau_n} - \frac{1}{K}X_{\tau_n} \geq L. \end{cases} \quad (6.42)$$

That is, as long as  $\left| \frac{1}{K}X_t - Y_t \right| < L$ , the sequence  $(\tau_n)_{n \in \mathbb{N}}$  coincides with the sequence of jump times  $(T_n)_{n \in \mathbb{N}}$ . When  $\left| \frac{1}{K}X_t - Y_t \right| \geq L$  the sequence  $(\tau_n)_{n \in \mathbb{N}}$  coincides with the jump times of the smaller process among  $\left( \frac{1}{K}X_t \right)_{t \geq 0}$  and  $(Y_t)_{t \geq 0}$ .

Define the discrete time process  $(\tilde{X}_n, \tilde{Y}_n)_{n \in \mathbb{N}}$  given by  $(\tilde{X}_n, \tilde{Y}_n) = \left( \frac{1}{K}X_{\tau_n}, Y_{\tau_n} \right)$ . For  $x, y \in \mathbb{N}$  we define the probability measure  $\tilde{\mathbb{P}}_{\left(\frac{1}{K}x, y\right)}$  by

$$\tilde{\mathbb{P}}_{\left(\frac{1}{K}x, y\right)}(\cdot) = \mathbb{P}\left(\cdot \mid (\tilde{X}_0, \tilde{Y}_0) = \left(\frac{1}{K}x, y\right)\right),$$

and denote by  $\tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}$  the associated expectation.

We emphasize that, since  $L > 1$ , the sequence of stopping times that we just defined ensures that whenever  $|\tilde{X}_n - \tilde{Y}_n| \geq L$ , we have that if  $\tilde{X}_n < \tilde{Y}_n$  then almost surely  $\tilde{X}_{n+1} \leq \tilde{Y}_{n+1}$ . Analogously, when  $\tilde{Y}_n < \tilde{X}_n$ , then almost surely  $\tilde{Y}_{n+1} \leq \tilde{X}_{n+1}$ .

With the aim of proving that assumptions of Proposition 62 hold, we start by showing that for all  $\varepsilon > 0$ , there exists  $z > 0$  such that if  $\frac{x}{K} \geq z$  and  $y \geq z$ , then

$$\min\left\{\tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(\tilde{X}_1), \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(\tilde{Y}_1)\right\} \geq \min\left\{\frac{1}{K}x, y\right\} + \frac{\min\{\lambda_m K - \mu_m, \lambda_f - \mu_f\}}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon. \quad (6.43)$$

Fix  $\varepsilon > 0$ . We compute the expectation of  $\tilde{X}_1$  and  $\tilde{Y}_1$  separating in cases according to the value of  $(\tilde{X}_0, \tilde{Y}_0) = \left(\frac{1}{K}x, y\right)$ , for  $(x, y) \in \mathbb{N}^2$ .

1. If  $\left| \frac{1}{K}x - y \right| < L$ , from (6.40) we have

$$\begin{aligned} \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(\tilde{X}_1) &= \frac{1}{K} \mathbb{E}_{(x, y)}(X_{\tau_1}) = \frac{1}{K} \mathbb{E}_{(x, y)}(X_{T_1}) \\ &= \frac{1}{K} \left( x + \frac{(\lambda_f - \mu_f)x}{(\lambda_m + \lambda_f + \mu_f)x + \mu_m y} \right) \\ &\geq \frac{1}{K}x + \frac{1}{K} \frac{(\lambda_f - \mu_f)x}{(\lambda_m + \lambda_f + \mu_f)x + \mu_m \left( L + \frac{1}{K}x \right)} \end{aligned}$$

The term on the right side in the last expression converges towards  $\frac{\lambda_f - \mu_f}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m}$  as  $x \rightarrow +\infty$ . Then, there exists  $z_1 > 0$  such that for  $x \geq Kz_1$ ,

$$\begin{aligned} \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(\tilde{X}_1) &\geq \frac{1}{K}x + \frac{\lambda_f - \mu_f}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon \\ &\geq \min \left\{ \frac{1}{K}x, y \right\} + \frac{\min\{\lambda_m K - \mu_m, \lambda_f - \mu_f\}}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon. \end{aligned}$$

Similarly for  $\tilde{Y}_1$ ,

$$\begin{aligned} \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(\tilde{Y}_1) &= y + \frac{\lambda_m x - \mu_m y}{(\lambda_f + \lambda_m + \mu_f)x + \mu_m y} \\ &\geq y + \frac{\lambda_m(Ky - LK) - \mu_m y}{(\lambda_f + \lambda_m + \mu_f)(Ky + LK) + \mu_m y} \\ &= y + \frac{(\lambda_m K - \mu_m)y - LK\lambda_m}{(\lambda_f + \lambda_m + \mu_f)(Ky + LK) + \mu_m y} \\ &\xrightarrow{y \rightarrow \infty} y + \frac{\lambda_m K - \mu_m}{\lambda_f + \lambda_m + \mu_f} \frac{1}{K} \end{aligned}$$

Again, there exists  $z_2 > 0$  such that if  $y > z_2$ , then

$$\begin{aligned} \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(\tilde{Y}_1) &\geq y + \frac{\lambda_m K - \mu_m}{(\lambda_f + \mu_f + \lambda_m)K + \mu_m} - \varepsilon \\ &\geq \min \left\{ \frac{1}{K}x, y \right\} + \frac{\min\{\lambda_m K - \mu_m, \lambda_f - \mu_f\}}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon. \end{aligned}$$

2. If  $\frac{1}{K}x - y \geq L$ , we have thanks to the choice of our sequence of stopping times  $(\tau_n)_{n \in \mathbb{N}}$  that  $\tilde{X}_1 \geq \tilde{Y}_1$  a.s. Then, we bound only the expectation of  $\tilde{Y}_1$ . As  $y < \frac{1}{K}x$ , the sequence  $\tau_n$  contains the first jump of  $y$ ,  $\tilde{Y}_1 = Y_{\tau_1} \in \{y, y - 1, y + 1\}$  and

$$\begin{aligned} \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(\tilde{Y}_1) &= y + \tilde{\mathbb{P}}_{\left(\frac{1}{K}x, y\right)}(\tilde{Y}_1 = y + 1) - \tilde{\mathbb{P}}_{\left(\frac{1}{K}x, y\right)}(\tilde{Y}_1 = y - 1) \\ &= y + \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(1_{\tilde{Y}_1 = y + 1} - 1_{\tilde{Y}_1 = y - 1}) \\ &= y + \tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}\left(\tilde{\mathbb{E}}_{\left(\frac{1}{K}x, y\right)}(1_{\tilde{Y}_1 = y + 1} - 1_{\tilde{Y}_1 = y - 1} \mid \tilde{X}_1)\right) \end{aligned}$$

The time  $\tau_1$  corresponds either to the first time when  $\tilde{X}_1 - y = \frac{1}{K}X_t - y < L$  and then  $\tilde{Y}_1 = Y_{\tau_1} = y$ , or to the first jump time of  $(Y_t)_{t \geq 0}$  and then  $\tilde{X}_1 - y = \frac{1}{K}X_{\tau_1} - y > L$ . Therefore

$$\begin{aligned} \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)}(\tilde{Y}_1) &= y + \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)} \left( \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)} \left( 1_{\tilde{Y}_1=y+1} - 1_{\tilde{Y}_1=y-1} \mid \tilde{X}_1 \right) 1_{\tilde{X}_1 > L+y} \right) \\ &= y + \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)} \left( \frac{\lambda_m K \tilde{X}_1 - \mu_m y}{\lambda_m K \tilde{X}_1 + \mu_m y} 1_{\tilde{X}_1 > L+y} \right) \end{aligned}$$

First, let us remark that the function  $x \rightarrow \mathbb{P}_{(x, y)}(Y_{\tau_1} = y)$  is decreasing for  $x \geq L + y$ . In fact on the event  $\{Y_{\tau_1} = y\}$ , the process  $(X_t)_{t \geq 0}$  will reach  $L + y - 1$  before the first jump of  $(Y_t)_{t \geq 0}$ . It will also reach all values between  $X_0 = x$  and  $L + y - 1$  before the first jump of  $(Y_t)_{t \geq 0}$ . Then, for  $x > x' \geq L + y$ , starting from  $x$ , on the event  $\{Y_{\tau_1} = y\}$ , the process  $(X_t)_{t \geq 0}$  has to reach first  $x'$  then  $L + y - 1$  before the first jump of  $(Y_t)_{t \geq 0}$ . Hence, by the Markov property

$$\begin{aligned} \mathbb{P}_{(x, y)}(Y_{\tau_1} = y) &= \mathbb{P}_{(x, y)} \left( \left\{ \inf\{t > 0, X_t = x'\} < \inf\{t > 0, Y_t \neq y\} \right\} \cap \{Y_{\tau_1} = y\} \right) \\ &= \mathbb{P}_{(x, y)} \left( \inf\{t > 0, X_t = x'\} < \inf\{t > 0, Y_t \neq y\} \right) \mathbb{P}_{(x', y)}(Y_{\tau_1} = y) \\ &\leq \mathbb{P}_{(x', y)}(Y_{\tau_1} = y). \end{aligned}$$

We then deduce that the function  $x \mapsto \mathbb{P}_{(x, y)}(\tilde{Y}_1 \neq y) = P_{(x, y)}(\tilde{X}_1 > L + y)$  is increasing. Moreover, we remark that for fixed  $y \in \mathbb{N}$ ,  $a \mapsto \frac{\lambda_m a - \mu_m y}{\lambda_m a + \mu_m y}$  is an increasing function. Hence,

$$\begin{aligned} \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)}(\tilde{Y}_1) &\geq y + \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)} \left( \frac{\lambda_m K - \mu_m}{\lambda_m K + \mu_m} 1_{\tilde{X}_1 > L+y} \right) \\ &= y + \frac{\lambda_m K - \mu_m}{\lambda_m K + \mu_m} \tilde{\mathbb{P}}_{(\frac{1}{K}x, y)}(\tilde{X}_1 > L + y) \\ &\geq y + \frac{\lambda_m K - \mu_m}{\lambda_m K + \mu_m} \mathbb{P}_{(K(y+L), y)}(Y_{\tau_1} \neq y) \\ &\geq y + \frac{\lambda_m K - \mu_m}{\lambda_m K + \mu_m} \mathbb{P}_{(K(y+L), y)}(Y_{T_1} \neq y) \\ &\geq y + \frac{\lambda_m K - \mu_m}{\lambda_m K + \mu_m} \frac{\lambda_m K(y+L) + \mu_m y}{(\lambda_f + \lambda_m + \mu_f)K(y+L) + \mu_m y} \\ &\xrightarrow{y \rightarrow +\infty} y + \frac{\lambda_m K - \mu_m}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} \end{aligned}$$

Finally, we can find  $z_3 > 0$  such that if  $y \geq z_3$ , we obtain

$$\begin{aligned} \min \left\{ \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)}(\tilde{X}_1), \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)}(\tilde{Y}_1) \right\} &\geq y + \frac{\lambda_m K - \mu_m}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon. \\ &\geq \min \left\{ \frac{1}{K}x, y \right\} + \frac{\min\{\lambda_m K - \mu_m, \lambda_f - \mu_f\}}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon. \end{aligned}$$

3. The last case when  $y - \frac{1}{K}x \geq L$  is done in an analogous way, noting again that under these assumptions,  $\tilde{X}_1 \leq \tilde{Y}_1$  almost surely, so we only need to focus on the expectation of  $\tilde{X}_1$ . This is treated similarly as in the previous case, obtaining for  $\frac{1}{K}x \geq z_4$ , some  $z_4 > 0$ ,

$$\min \left\{ \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)}(\tilde{X}_1), \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)}(\tilde{Y}_1) \right\} \geq \min \left\{ x, \frac{1}{K}y \right\} + \frac{\min\{\lambda_m K - \mu_m, \lambda_f - \mu_f\}}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon.$$

Taking  $z = \max\{z_1, z_2, z_3, z_4\}$ , we obtain (6.43).

Let now justify that Proposition 62 can be applied to the process  $(\tilde{X}_n, \tilde{Y}_n)_{n \in \mathbb{N}}$  with  $f_1(x, y) = \min\{x, y + L + 1\}$  and  $f_2(x, y) = \min\{x + L + 1, y\}$ , for which  $g(x, y) = \min\{x, y\}$ . The first assumption (6.33) holds by definition of  $g$ . The assumption (6.34) is a consequence of (6.43) and (6.36) in Remark 20, so we only have to deal with the moments condition (6.35).

We study the value of  $f_i(\tilde{X}_1, \tilde{Y}_1)$  for  $i \in \{1, 2\}$ , starting from  $(\tilde{X}_0, \tilde{Y}_0) = (\frac{1}{K}x, y)$  with  $x, y \in \mathbb{N}$ . As the sequence of the definition of the jump times  $(\tau_n)_n$ , we have  $f_i(\tilde{X}_1, \tilde{Y}_1) \geq \min\{\tilde{X}_1, \tilde{Y}_1\} \geq \min\{\frac{1}{K}x, y\} - 1$ . Moreover, we have  $f_i(\tilde{X}_1, \tilde{Y}_1) \leq \min\{\tilde{X}_1 + L + 1, \tilde{Y}_1 + L + 1\} = \min\{\tilde{X}_1, \tilde{Y}_1\} + L + 1$ . In addition,

1. If  $|\frac{1}{K}x - y| < L$ , then  $\tilde{X}_1 \leq \frac{1}{K}x + 1$  and  $\tilde{Y}_1 \leq y + 1$  almost surely;
2. If  $y - \frac{1}{K}x \geq L$  then  $\tilde{X}_1 \leq \tilde{Y}_1$ , hence  $\min\{\tilde{X}_1, \tilde{Y}_1\} = \tilde{X}_1 \leq \frac{1}{K}(x + 1) \leq \min\{\frac{1}{K}x, y\} + 1$  almost surely.
3. If  $\frac{1}{K}x - y \geq L$ , analogously to the previous case, we obtain  $\min\{\tilde{X}_1, \tilde{Y}_1\} \leq \min\{\frac{1}{K}x, y\} + 1$  almost surely.

We then have for  $i \in \{1, 2\}$  and for all  $x, y \in \mathbb{N}$ ,

$$\min \left\{ \frac{x}{K}, y \right\} - (L + 2) \leq \min \left\{ \frac{1}{K}x, y \right\} - 1 \leq f_i(\tilde{X}_1, \tilde{Y}_1) \leq \min \left\{ \frac{x}{K}, y \right\} + (L + 2).$$

Hence, for  $i \in \{1, 2\}$  and all  $\alpha > 0$ ,

$$\sup_{(x, y) \in \mathbb{N}^2 \setminus \{(0, 0)\}} \mathbb{E}_{(\frac{1}{K}x, y)} \left( \left| f_i(\tilde{X}_1, \tilde{Y}_1) - \tilde{\mathbb{E}}_{(\frac{1}{K}x, y)}(f_i(\tilde{X}_1, \tilde{Y}_1)) \right|^{1+\alpha} \right) \leq (2L + 4)^{1+\alpha} < +\infty.$$

We obtain that (6.35) is true for any value of  $\alpha > 0$ . We are in place to apply Proposition 62, obtaining that for all  $K > 0$  such that  $\lambda_m K - \mu_m > 0$ ,  $L > 1$  and  $\varepsilon > 0$ , there exists  $z > 0$  such that if  $x \geq Kz$  and  $y \geq z$ , then

$$\mathbb{P}_{(\frac{1}{K}x, y)} \left( \min\{\tilde{X}_n, \tilde{Y}_n\} \geq \left( \frac{\min\{\lambda_m K - \mu_m, \lambda_f - \mu_f\}}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m} - \varepsilon \right) n, \text{ for all } n \in \mathbb{N} \right) > 0. \quad (6.44)$$

We have, thanks to the choice of the stopping times  $(\tau_n)_{n \in \mathbb{N}}$ , we have that, for all  $x, y \in \mathbb{N}$  and up to a  $\mathbb{P}_{(x, y)}$ -negligible event, the following chain of inclusions

$$\begin{aligned} \left\{ \min\{\tilde{X}_n, \tilde{Y}_n\} \xrightarrow{n \rightarrow +\infty} +\infty \right\} &\subseteq \left\{ \min \left\{ \frac{X_{\tau_n}}{K}, Y_{\tau_n} \right\} \xrightarrow{n \rightarrow +\infty} \right\} \\ &\subseteq \left\{ X_{\tau_n} \xrightarrow{n \rightarrow +\infty} +\infty, Y_{\tau_n} \xrightarrow{n \rightarrow +\infty} +\infty \right\} \\ &\subseteq \left\{ X_{T_n} \xrightarrow{n \rightarrow +\infty} +\infty, Y_{T_n} \xrightarrow{n \rightarrow +\infty} +\infty \right\} \\ &\subseteq \left\{ X_t \xrightarrow{t \rightarrow +\infty} +\infty, Y_t \xrightarrow{t \rightarrow +\infty} +\infty \right\}. \end{aligned}$$

Thus, we conclude that there exists  $z \geq 0$  and  $K > 0$  such that, if  $X_0 \geq Kz$  and  $Y_0 \geq z$ , then the process  $(X_t, Y_t)_{t \geq 0}$  has an unlimited growth with positive probability. Since for the process  $(X_t, Y_t)_{t \geq 0}$ , the set  $(\mathbb{N} \setminus \{0\})^2$  contains only one class of communication (the set itself), we conclude that for all  $x, y \in \mathbb{N}^*$ ,

$$\mathbb{P}_{(x,y)}(X_t \xrightarrow[t \rightarrow +\infty]{} +\infty, Y_t \xrightarrow[t \rightarrow +\infty]{} +\infty) > 0. \quad (6.45)$$

*Remark 21.* Note that the function

$$K \in \mathbb{R}_+ \longrightarrow \frac{\min\{\lambda_m K - \mu_m, \lambda_f - \mu_f\}}{(\lambda_f + \lambda_m + \mu_f)K + \mu_m}$$

is increasing for  $K$  such that  $\lambda_m K - \mu_m < \lambda_f - \mu_f$  and decreasing when  $\lambda_m K - \mu_m > \lambda_f - \mu_f$ . This implies that the best lower bound that (6.44) can provide about the asymptotic growth of the population is with the value of  $K = \frac{\lambda_f - \mu_f + \mu_m}{\lambda_m}$ , for which  $\lambda_m K - \mu_m = \lambda_f - \mu_f$ .

*Remark 22.* Although the fact that  $\lambda_f > \mu_f$  is a sufficient and necessary condition for the survival of the process can be deduced from Theorem 2 in [KK73], the fact on the event of survival both  $(X_t)_{t \geq 0}$  and  $(Y_t)_{t \geq 0}$  grow towards infinity with positive probability (as stated in (6.45)) is a new result.

## 6.4.2 Continuous time perfect fidelity model

The continuous time perfect fidelity model corresponds to the model (6.37) with

$$\begin{cases} \phi_f(x, y) &= \lambda_f \min\{x, y\}, \\ \phi_m(x, y) &= \lambda_m \min\{x, y\}, \\ \mu_f(x) &= \mu_f x, \\ \mu_m(y) &= \mu_m y, \end{cases} \quad (6.46)$$

with  $\lambda_f, \lambda_m, \mu_f, \mu_m > 0$ . Contrary to the continuous time completely promiscuous model of Section 6.4.1, the birth rates of females and males are linear in  $\min\{x, y\}$ , modeling the fact that males and females have at most one partner (similarly to Example 4).

By the same argument that for the continuous time completely promiscuous model, the processes of female is stochastically dominated by a linear birth and death process with birth rate  $\bar{\phi}_f(x) = \lambda_f x$  and death rate  $\mu_f(x) = \mu_f x$ , and then the process goes to extinction almost surely if  $\lambda_f \leq \mu_f$ . By the same argument on the male population, the process also goes to extinction almost surely if  $\lambda_m \leq \mu_m$ .

Let assume that  $\lambda_f > \mu_f$  and that  $\lambda_m > \mu_m$ . In the same way as in Section 6.4.1, let set  $L > 1$  and let define the sequence of stopping times  $(\tau_n)_{n \in \mathbb{N}}$  given by  $\tau_0 = 0$  and for all  $n \in \mathbb{N}$

$$\tau_{n+1} = \begin{cases} \inf\{t > \tau_n, (X_t, Y_t) \neq (X_{\tau_n}, Y_{\tau_n})\} & \text{if } |X_{\tau_n} - Y_{\tau_n}| < L \\ \inf\{t > \tau_n, Y_t \neq Y_{\tau_n}\} \wedge \inf\{t > \tau_n, |X_t - Y_t| < L\} & \text{if } X_{\tau_n} - Y_{\tau_n} \geq L \\ \inf\{t > \tau_n, X_t \neq X_{\tau_n}\} \wedge \inf\{t > \tau_n, |X_t - Y_t| < L\} & \text{if } Y_{\tau_n} - X_{\tau_n} \geq L. \end{cases}$$

and define the discrete time process defined for all  $n \in \mathbb{N}$  such that  $\tau_n < +\infty$  by  $(\tilde{X}_n, \tilde{Y}_n) = (X_{\tau_n}, Y_{\tau_n})$ . Then proceeding in a similar way as in the previous proof, we can demonstrate that for all  $\varepsilon > 0$  there exists  $z > 0$  such that if  $\min\{x, y\} \geq z$ ,

$$\min\left\{\mathbb{E}_{(x,y)}(\tilde{X}_1), \mathbb{E}_{(x,y)}(\tilde{Y}_1)\right\} \geq \min\{x, y\} + \frac{\min\{\lambda_f - \mu_f, \lambda_m - \mu_m\}}{\lambda_f + \lambda_m + \mu_f + \mu_m} - \varepsilon. \quad (6.47)$$



Using the functions  $h_1(x, y) = \min\{x, y + L + 1\}$  and  $h_2 = \{x + L + 1, y\}$  as in the previous case and Proposition 62, we can conclude that

$$\mathbb{P}_{(x,y)} \left( \min\{X_{\tau_n}, Y_{\tau_n}\} \geq \left( \frac{\min\{\lambda_f - \mu_f, \lambda_m - \mu_m\}}{\lambda_f + \lambda_m + \mu_f + \mu_m} - \varepsilon \right) n, \text{ for all } n \in \mathbb{N} \right) > 0.$$

By the same irreducibility argument as for the continuous time completely promiscuous model, it leads to a positive survival probability in the case  $\lambda_f > \mu_f$  and that  $\lambda_m > \mu_m$ . Finally, we obtain the extinction condition stated in the following proposition.

**Proposition 64.** *The process  $(X_t, Y_t)_{t \geq 0}$ , defined by (6.37)-(6.46) has a positive probability of survival if and only if  $\min\{\lambda_f - \mu_f, \lambda_m - \mu_m\} > 0$ .*



## Appendix A

# A proof for the multi-type process of Karlin and Kaplan

In this Appendix, we prove a theorem stated by Karlin and Kaplan in [KK73]. We state the theorem keeping the author's notation.

**Theorem 65.** Consider  $k_1, k_2 \in \mathbb{N}$ ,  $T_f = \{1, \dots, k_1\}$ ,  $T_m = \{1, \dots, k_2\}$  and a  $(k_1 + k_2)$ -multitype branching process  $(F_1(n), \dots, F_{k_1}(n), M_1(n), \dots, M_{k_2}(n))_{n \in \mathbb{N}}$  fulfilling the following conditions

(i)  $F_i(0) = 1$ , for some  $i \in T_f$ .  
 $M_j(0) = 1$  for all  $j \in T_m$ .

(ii) (a) Males have no decedents.  
(b) Let

$$X_{ij} = \#\{\text{type } j \text{ females produced by a type } i \text{ female}\}, \quad i, j \in T_f.$$

$$Y_{ij} = \#\{\text{type } j \text{ males produced by a type } i \text{ female}\}, \quad i \in T_f, j \in T_m.$$

Then,

- $\mathbb{P}(Y_{ij} = 0) < 1$ ,  $\forall i \in T_f, j \in T_m$ .
- The matrix  $H$  given by

$$h_{ij} = \mathbb{E}(X_{ij}), \quad i, j \in T_f$$

is irreducible and aperiodic.

Consider the vector process

$$\vec{W}(n) = (W_1(n), \dots, W_{k_1}(n)) = (F_1(n), \dots, F_{k_1}(n)) \prod_{m=1}^n \min_{j \in T_m} (M_j(m), 1).$$

Then, a sufficient and necessary condition for certain extinction of the process  $\vec{W}(n)$  is that  $\rho \leq 1$ , where  $\rho$  is the largest positive eigenvalue of  $H$ .

We adapt the proof that the authors give for the single-type case.

*Proof.* (1) *Case of certain extinction.*

First notice that  $\forall i \in T_f, \forall n \in \mathbb{N}$ , we have

$$W_i(n) \leq F_i(n), \text{ a.s.}$$

Then, since when  $\rho \leq 1$ ,  $\vec{F}(n) \rightarrow \vec{0}$  a.s. as  $n \rightarrow \infty$ , we have that  $\vec{W}(n) \rightarrow \vec{0}$  a.s. when  $n \rightarrow \infty$ .

(2) *Case of non-certain extinction*

We will prove that  $\exists N \in \mathbb{N}, \exists \eta > 1$  s.t., for any initial contion of the process

$$\lim_{n \rightarrow \infty} \mathbb{P} \left( W_i(Nj) \geq \eta^j, \forall i \in T_f, \forall j \leq n \right) > 0.$$

To achieve that, first notice that since  $H$  is irreducible and aperiodic, we have the estimation

$$H^m \sim \rho^m u_2 u_1 \text{ as } m \rightarrow \infty,$$

where  $u_1$  is a left eigenvector and  $u_2$  is a right eigenvector, both with positive coordinates. Then, since  $\rho > 1$ , we can fix  $N$  big enough so that  $m_i := (H^N)_{ii} > 1$  for all  $i \in T_f$ .

Now, we can iterate Bayes's rule and use the Markov property on  $\vec{W}$  to the probability inside the limit and obtain

$$\prod_{j=0}^{n-1} \mathbb{P} \left( W_i(N(j+1)) \geq \eta^{j+1}, \forall i \in T_f | W_i(Nj) \geq \eta^j, \forall i \in T_f \right).$$

We will estimate the probability inside the product form below by dividing the  $\beta_i = |W_i(Nj)| \geq \eta^j$  females of each type into two groups.

First, to simplify calculations, take  $\beta = \min \beta_i$ . That is, we assume that we have at least  $\beta$  females of each type at the  $j$ -th generation. Now we divide:

- Take a fraction  $[\delta\beta]$  of each type of female. These females will be in charge of creating at least one male of each type at every generation, for  $N$  generations. This ensures the process survives after the  $N$  generations.
- The remaining  $\beta - [\delta\beta]$  females of each type will be in charge of producing at least  $\eta^{j+1}$  females after  $N$  generations.

With this, using the independence between different females, we get:

$$\mathbb{P} \left( W_i(N(j+1)) \geq \eta^{j+1}, \forall i \in T_f | W_i(Nj) \geq \eta^j, \forall i \in T_f \right) \geq B_j C_j,$$

where  $B_j$  is the event carried out by the first group and  $C_j$  by the second group. In other words:

$$B_j = \mathbb{P}([\delta\beta] \text{ females produce at least one male of each type in every generation, for } N \text{ generations}),$$

---

$C_j = \mathbb{P}(\beta - [\delta\beta] \text{ females produce at least } \eta^{j+1} \text{ females after } N \text{ generations}).$

To simplify notation we call

$$R = [\delta\beta], \quad L' = \beta - [\delta\beta], \quad L = [(1 - \delta)\eta^j].$$

Now, we estimate  $B_j$  and  $C_j$ .

- (1) We estimate  $C_j$  first. Using independence of different females, we will consider the following case: a female will be produced by a female of the same type. With this we get:

$$\begin{aligned} C_j &= \mathbb{P}(L' \text{ females of each type produce together at least } \eta^{j+1} \text{ females} \\ &\quad \text{of each type after } N \text{ generations}) \\ &\geq D_1 \dots D_{k_1} \end{aligned}$$

where  $D_i = \mathbb{P}(L' \text{ females of type } i \text{ produce at least } \eta^{j+1} \text{ females of type } i \text{ after } N \text{ generations}).$  If we call  $Z^i$  the number of females of type  $i$  produced by a female of type  $i$  after  $N$  generations, then

$$\begin{aligned} D_i &= \mathbb{P}\left(\sum_{\ell=1}^{L'} Z_\ell^i \geq \eta^{j+1}\right) \\ &\geq \mathbb{P}\left(\sum_{\ell=1}^L Z_\ell^i \geq \eta^{j+1}\right), \end{aligned}$$

where  $(Z_\ell^i)_{\ell \geq 1}$  are i.i.d random variables with the same distribution as  $Z^i$ . Now, using the fact that

$$\mathbb{E}(\vec{F}(n) | \vec{F}(0) = v_0) = v_0 H^n,$$

we get that  $\mathbb{E}\left(\sum_{\ell=1}^L Z_\ell^i\right) = L(H^N)_{ii} = Lm_i$ . For the following, we will assume that the variables  $Z^i$  have finite variance. Now, subtracting  $Lm_i$  from both sides, since  $L = [(1 - \delta)\eta^j] \geq (1 - \delta)\eta^j$ , we get:

$$\begin{aligned} D_i &\geq \mathbb{P}\left(\sum_{\ell=1}^L Z_\ell^i - Lm_i \geq \eta^{j+1} - Lm_i\right) \\ &\geq \mathbb{P}\left(\sum_{\ell=1}^L Z_\ell^i - Lm_i \geq \eta^j(\eta - (1 - \delta)m_i)\right). \end{aligned}$$

Since  $m_i > 1$ ,  $\forall i \in T_f$ , we can find  $\eta > 1$  and  $\delta > 0$  small enough so that

$$\eta - (1 - \delta)m_i < 0, \quad \forall i \in T_f.$$

We call  $-\epsilon_i = \eta - (1 - \delta)m_i$  for  $\epsilon_i > 0$ , and obtain

$$\begin{aligned}
 D_i &\geq \mathbb{P}\left(\sum_{\ell=1}^L Z_\ell^i - Lm_i \geq -\epsilon_i \eta^j\right) \\
 &\geq \mathbb{P}\left(\left|\sum_{\ell=1}^L Z_\ell^i - Lm_i\right| \leq \epsilon_i \eta^j\right) \\
 &\geq 1 - \frac{d_i}{\eta^j},
 \end{aligned}$$

where, in the last step, we used Chebyshev's inequality and  $d_i > 0$ .

If  $Z_i$  has infinite variance, we can truncate them properly, and obtain the same result, as in Theorem 1.

And so, we have proven that  $C_j \geq 1 - a_j$  with  $\sum a_j < \infty$ .

- (2) Now we estimate  $B_j$  from below. Once again, using independence between different females, we will consider the case where the females of a type have to produce at least one male of their same type for every generation during  $N$  generations. In other words,

$$\begin{aligned}
 B_j &= \mathbb{P}(R \text{ females of each type produce together at least} \\
 &\quad \text{one male of each type in every generation,} \\
 &\quad \text{for } N \text{ generations}) \\
 &\geq E_1 \dots E_{k_1}
 \end{aligned}$$

where  $E_i = \mathbb{P}(R \text{ females of type } i \text{ produce at least one male of type } i \text{ in every generation, for } N \text{ generations})$ .

We get that

$$\begin{aligned}
 E_i &\geq \sum_{\ell=1}^N \mathbb{P}(R \text{ females of type } i \text{ produce at least one male of type } i \\
 &\quad \text{after } \ell \text{ generations}) - (N - 1) \\
 &= 1 - \sum_{\ell=1}^N \gamma_\ell^R \\
 &\geq 1 - \sum_{j=1}^N \gamma_\ell^{\delta \eta^j}
 \end{aligned}$$

with  $\gamma_{i,j} = \mathbb{P}(\text{A female of type } i \text{ produce no males of type } i \text{ after } j \text{ generations})$ . So we get the estimation

$$E_i \geq 1 - \Gamma_i^{\eta^j}.$$

Hence, we get  $B_j \geq 1 - b_j$ , where  $\sum b_j < \infty$ .

Now, since  $\sum a_j, \sum b_j < \infty$ , we get

$$\lim_{n \rightarrow \infty} \mathbb{P}\left(W_i(Nj) \geq \eta^j, \forall i \in T_f, \forall j \leq n\right) \geq \lim_{n \rightarrow \infty} \prod_{j=0}^{n-1} (1 - a_j)(1 - b_j) > 0.$$

□

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## Résumé

Le processus de Galton-Watson bi-sexué, introduit par Daley, est une extension du processus de Galton-Watson classique, décrivant l'accouplement de femelles et de mâles, formant des couples capables de se reproduire. Des propriétés telles que des conditions d'extinction et le comportement asymptotique ont été étudiées au cours des dernières années dans le cas uni-type, mais les versions multi-types n'ont été traitées que dans certains cas particuliers. Dans cette thèse, nous développons une version multidimensionnelle générale du modèle de Daley, où nous considérons différents types de femelles et de mâles, qui s'accouplent selon une "fonction d'appariement". Nous supposons que cette fonction est superadditive, ce qui signifie simplement que deux groupes de femelles et de mâles forment un plus grand nombre de couples ensemble plutôt que séparément. L'une des principales difficultés dans l'étude de ce processus est que, contrairement au processus de Galton-Watson classique (asexué), l'opérateur associé au processus n'est pas linéaire mais concave. Pour surmonter ce problème, nous utilisons une théorie de Perron-Frobenius pour les opérateurs concaves, qui garantit l'existence d'éléments propres pour notre opérateur. Dans cette thèse, nous démontrons des lois des grands nombres et établissons des conditions nécessaires et suffisantes pour l'extinction presque sûre du processus. Nous étudions également, à travers l'identification d'une surmartingale, la convergence du processus en temps long dans le cas surcritique ainsi que l'existence de distributions quasi-stationnaires pour le régime sous-critique. Enfin, quelques extensions à des modèles avec une fonction d'appariement aléatoire et des modèles en temps continu sont traitées.

**Mots-clés:** Processus de branchement avec interaction, Critère d'extinction, Loi des grands nombres, Comportement malthusien, Opérateurs concaves, Distributions quasi-stationnaires

## Abstract

The bisexual Galton-Watson process, introduced by Daley, is an extension of the classical Galton-Watson process, but taking into account the mating of females and males, which form couples that can accomplish reproduction. Properties such as extinction conditions and asymptotic behaviour have been studied in the past years in the single-type case, but multi-type versions have only been treated in some particular cases. In this thesis we deal with a general multi-dimensional version of Daley's model, where we consider different types of females and males, which mate according to a "mating function". We consider that this function is superadditive, which in simple words implies that two groups of females and males will form a larger number of couples together rather than separate. One of the main difficulties in the study of this process is the absence of a linear operator that is the key to understand its behavior in the asexual case, but in our case it turns out to be only concave. To overcome this issue, we use a concave Perron-Frobenius theory which ensures the existence of eigen-elements for some concave operators. Using this tool, we find a necessary and sufficient condition for almost sure extinction as well as laws of large numbers. We also study the convergence of the process in the long-time through the identification of a supermartingale, and the existence of quasi-stationary distributions for the subcritical regime. Finally, some extensions to models with random mating function and models in continuous time are considered.

**Keywords:** Branching processes with interaction, Extinction criterion, Law of large numbers, Malthusian behaviour, Concave operators, Quasi-stationary distributions

