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Well-posedness and stabilization of coupled hyperbolic equations involving Timoshenko, Rao-Nakra and Bresse systems by various types of controls

Zeinab Mohamad-Ali

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Well-posedness and stabilization of coupled hyperbolic equations involving Timoshenko, Rao-Nakra and Bresse systems by various types of controls

Auteur
Zeinab MOHAMAD ALI

Directeurs
Dr. Aissa GUESMIA
Prof. Bassam MOURAD

Thèse

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Membres du jury :

Directeurs de thèse	M. Aissa GUESMIA	Maître de conférences, Université de Lorraine, France
	M. Bassam MOURAD	Professeur, Université Libanaise, Liban
Rapporteurs	M. Ahmed BCHATNIA	Professeur, Université de Tunis El Manar, Tunisie
	M. Stéphane GERBI	Maître de conférences, Université Savoie Mont Blanc, France
Examineurs externes	Mme. Cristina PIGNOTTI présidente du jury	Professeur, Università Dell'Aquila, Italie
	Mme. Judith VANCOSTE-NOBLE	Maîtresse de conférences, Université Paul Sabatier Toulouse III, France
Examineurs internes	Mme. Julie VALEIN	Maîtresse de conférences, Université de Lorraine, France
	M. Ali WEHBE	Professeur, Université Libanaise, Liban

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Abstract

This thesis is devoted to study the well-posedness and stabilization of some locally coupled systems.

First, we study the well-posedness and stability of a one-dimensional Timoshenko system with localized internal fractional kelvin-Voigt damping in a bounded domain. We investigate three cases : the first one, when the damping is localized in the bending moment, the second case when the damping is localized in the shear stress, we prove that the system is well posed in the sense of semigroup theory and its energy decays polynomially with rate t^{-1} in both cases. While, when the fractional Kelvin-Voigt is acting on the shear stress and the bending moment simultaneously, we show that the system is well posed in the sense of semigroup theory and polynomially stable with energy decay rate of type $t^{\frac{-4}{2-\alpha}}$, provided that the two dampings are acting in the same sub-interval.

Second, we consider the generalized Rao-Nakra beam equation. The system consists of four wave equations for the longitudinal displacements and the shear angle of the top and bottom layers and one Euler-Bernoulli beam equation for the transversal displacement. We start by proving that the system is well posed in the sense of semigroup theory. Then, we study the stability problem. First, we show that the analytic stability holds when all the displacements are globally damped through Kelvin-Voigt damping. Second, we consider the case where the local damping acts only on the shear angle displacements of the top and bottom layers, and we obtain sufficient conditions for the system to be strongly stable. Using frequency domain arguments combined with the multiplier method, we prove that the energy of the system decays polynomially with rate $t^{-\frac{1}{2}}$.

Finally, we investigate the stability of a Bresse-type system in the whole line with a frictional damping working only on the first equation (vertical displacement). Our objectives are proving some stability and non-stability results depending on the parameters in the system. More precisely, we prove that, in some cases, the system is polynomially stable, and in some other cases, the solution does not converge to zero at all. The proofs are based on the energy method and Fourier analysis combined with some well chosen weight functions.

Keywords : C_0 -semigroup, Timoshenko system, Rao-Nakra system, Bresse system, Frequency domain approach, Multiplier method, Strong stability, Analytic stability, Polynomial stability, Kelvin-Voigt damping, Fractional derivative, Frictional damping, Cauchy problem, Asymptotic behavior, Energy method, Fourier analysis.

Résumé

Cette thèse est consacrée à l'étude de l'existence, l'unicité et la régularité des solutions, et la stabilisation de certains systèmes localement couplés.

Tout d'abord, nous étudions l'existence, l'unicité et la régularité des solutions et la stabilité d'un système de Timoshenko unidimensionnel avec un amortissement fractionnaire interne localisé de Kelvin-Voigt dans un domaine borné. Nous étudions trois cas : le premier, lorsque l'amortissement est localisé dans le moment de flexion, le deuxième lorsque l'amortissement est localisé dans la contrainte de cisaillement, nous prouvons que le système est bien posé au sens de la théorie des semigroupes et l'énergie du système décroît polynomialement avec le taux t^{-1} dans les deux cas. En revanche, lorsque le Kelvin-Voigt fractionnaire agit simultanément sur la contrainte de cisaillement et le moment de flexion, nous montrons que le système est bien posé au sens de la théorie des semigroupes et il est polynomialement stable, où le taux de décroissance de l'énergie est de type $t^{\frac{-4}{2-\alpha}}$, à condition que les deux amortissements agissent dans le même sous-intervalle.

Deuxièmement, nous considérons l'équation de la poutre de Rao-Nakra généralisée. Le système se compose de quatre équations d'ondes pour les déplacements longitudinaux et l'angle de cisaillement des couches supérieure et inférieure et d'une équation de poutre d'Euler-Bernoulli pour le déplacement transversal. On commence par montrer que le système est bien posé au sens de la théorie des semigroupes. Ensuite, on traite la question de la stabilité. Tout d'abord, nous montrons que la stabilité analytique est assurée lorsque tous les déplacements sont globalement amortis par l'amortissement de Kelvin-Voigt. Ensuite, nous considérons le cas où l'amortissement local n'agit que sur les déplacements de l'angle de cisaillement des couches supérieure et inférieure, et nous obtenons des conditions suffisantes pour que le système soit fortement stable. En utilisant la méthode fréquentielle combinée avec la méthode des multiplicateurs, on montre que l'énergie du système décroît polynomialement avec un taux $t^{-\frac{1}{2}}$.

Enfin, nous étudions la stabilité d'un système de type Bresse dans la ligne entière avec un amortissement par frottement en travaillant uniquement sur la première équation (déplacement vertical). Nos objectifs sont de prouver certains résultats de stabilité et de non-stabilité en fonction des paramètres du système. Plus précisément, nous prouvons que, dans certains cas, le système est polynomialement stable, et dans d'autres cas, la solution ne converge pas du tout vers zéro. Les preuves sont basées sur la méthode de l'énergie et l'analyse de Fourier combinées avec certaines fonctions de poids bien choisies.

Mots-clés : C_0 -semigroupe, Système de Timoshenko, Système de Rao-Nakra, Système de Bresse, Méthode fréquentielle, Méthode des multiplicateurs, Stabilité forte, Stabilité analytique, Stabilité polynomiale, Amortissement de Kelvin-Voigt, Dérivée fractionnaire, Amortissement Frictionnel, Problème de Cauchy, Comportement asymptotique, Méthode de l'énergie, Analyse de Fourier.

List of symbols and notations

\mathbb{R}	The set of real numbers.
\mathbb{R}_+	The set of non negative real numbers.
\mathbb{R}^*	The set of non zero real numbers.
\mathbb{N}	The set of natural numbers.
\mathbb{N}^*	The set of non zero natural numbers.
\mathbb{Z}	The set of integer numbers.
\mathbb{Z}^*	The set of non zero integer numbers.
\mathbb{Q}	The set of rational numbers.
\mathbb{Q}_+	The set of non negative rational numbers.
\mathbb{Q}^*	The set of non zero rational numbers.
\mathbb{C}	The set of complex numbers.
i	The imaginary unit.
\Re	The real part.
\Im	The imaginary part.
L^p	The Lebesgue space.
H^m	The sobolev space.
C^0	The space of continuous function.
C^k	The space of k continuously differentiable functions, $k \in \mathbb{N}^*$.
$ \cdot $	The modulus.
$\ \cdot\ $	The norm.
\max	The maximum.
\min	The minimum.
\sup	The supreme.
\inf	The infimum.
$f_{yy} = \partial_{yy} f$	The second partial derivative of f with respect of y .
$\partial_t^{\alpha, \eta}$	Fractional Derivative.
$g'(s)$	The derivative of g with respect to s .
\det	The determinant.
\hat{f}	The Fourier transform of f .
\hat{f}^{-1}	The reverse Fourier transform of f .
C_c^∞	The space of smooth functions with compact support.
H^{-1}	The dual space of H_0^1 .
V'	The dual space of V .
∂_x^j	The partial derivative of order j with respect of x , $j \in \mathbb{N}$.
C_0^∞	The space of smooth functions whith compact support.

Asymptotic Notations

Let f and g be two functions defined on some subset of the real numbers, we define

- $f(x) = O(g(x))$ as $x \rightarrow \infty$ to mean that there exists a positive number M and a real number x_0 such that

$$|f(x)| \leq M |g(x)| \quad \forall x \geq x_0.$$

- $f(x) = o(g(x))$ as $x \rightarrow \infty$ to mean that

$$\lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| = 0.$$

- $f \sim g$ as $x \rightarrow \infty$ to mean that

$$\lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| = 1.$$

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Introduction

1 Introduction in English

1.1 Context and physical motivation

Many natural phenomena can be described by partial differential equations, and the control of such equations is a relatively new and active field of study.

For the control of PDE, which forms the framework of this thesis, the studied models take into account the variations and spatial distributions of variables that reflect the state of the system. These issues are then posed in the context of infinite-dimensional dynamic systems.

The field of control theory in partial differential equations has expanded thanks to the introduction and study of more practical models. It proves advantageous in various domains including engineering, biology, computer science and biomedical research.

1.1.1 Control theory

Control theory is a branch of engineering and mathematics that deals with the analysis and design of systems that can be controlled or regulated to behave in a desired manner.

The primary objective of control theory is to develop mathematical models that describe the dynamics of a system and to design control strategies that can influence the system behavior. These strategies are typically implemented using feedback mechanisms, where the system output is measured and used to adjust the control inputs in real-time.

The "stabilization problem", which consists of determining a control law that brings the system to equilibrium from any starting point, is one of the most popular control problems. This problem has attracted and continues to attract many researchers, and many techniques can be used to solve the stabilization problem for dynamical systems. In practice, the stabilization problem is the most important in the design phase of any control system because it provides the desired performance to the designed control system. This theory is applied in a wide range of scientific and technical disciplines.

For example, it can be used for noise reduction in systems such as engines or electronic devices. It can also be applied to structural vibration, aiming to attenuate or eliminate undesirable oscillations. Furthermore, control and stabilization theory is used to study waves and seismic earthquakes, with the aim of predicting and minimizing the adverse effects of these natural phenomena. Also, it is beneficial in many other areas such as biology, biomedical research, robotic systems and computer science.

In summary, control theory is a field that enables us to understand, analyze and manipulate the behavior of dynamic systems through feedback mechanisms.

The primary aim of this research is to investigate the well-posedness and asymptotic behavior of the solutions of the *Timoshenko, Bresse and Rao-Nakra beam systems* (models).

One of the most commonly used theories is the beam theory.

1.1.2 Beam theory

The stabilization of flexible beams has become a significant area of research, given their widespread utilization in various applications, including aerospace and civil structures.

Beams are certainly the most commonly used mathematical models by structural engineers and play a crucial role in their various construction projects. Due to their simplicity and effectiveness, beams have become the cornerstone of structural design, allowing engineers to analyze and understand the behavior of different types of structures quickly and accurately.

Analyzing beams models under internal or external loads is crucial for advancing engineering practices, as beams are commonly employed in the design of flexible structures found in various fields, including structural and mechanical engineering. Examples of such applications include bridges, roads, buildings, airplanes, robots and many more.



Figure 1.1 – Highway beam bridges construction.

Source: <https://www.steel-bridges.com/highway-bridge-composite-beam.html>



Figure 1.2 – Beam of an airplane.

Source: <https://slideplayer.fr/slide/13960149/>

In the following section we present several beam theory models:

1.1.3 Some beam models

The term "beam" designates an object whose length is large compared to the transverse dimensions. The beam theory is a model used in the field of material resistance. This theory, in case of small deflections, is classified into two main groups. The first one (Euler-Bernoulli theory) ignores the shear deformations whereas the other (Timoshenko theory) takes this effect into account.

Euler-Bernoulli beam theory:

This theory is the oldest and simplest classical theory for beam bending. It simplifies the linear theory of elasticity and allows for the calculation of load-carrying capacity and deflection characteristics of beams. It specifically applies to the scenario where beams experience small deflections and are subjected solely to lateral loads. This theory assumes that the cross-section of the beam remains perpendicular to the neutral axis, even after deformation. The Euler-Bernoulli beam theory is suitable for thin beams. By ignoring the effects of shear deformation and rotatory inertia, it is thus a special case of Timoshenko beam theory.

Timoshenko beam theory:

In 1921, physicist and engineer Stephen Timoshenko developed the Timoshenko beam model in [102]. Unlike the Euler-Bernoulli beam, the mathematical model of the Timoshenko beam takes into consideration both the effects of shear deformation and rotational bending, making it suitable for describing the behavior of thick beams. This model is particularly useful in cases where the beam is subjected to large transverse loads such as torsion loads. Applications of the Timoshenko beam can be found in various engineering disciplines including structural and mechanical engineering, such as bridges, road constructions, buildings, aircraft and numerous other fields. The mathematical model of the transverse vibration of a Timoshenko beam can be described by the following system of coupled partial differential equations (for details see, e.g., [103, 104]):

$$\begin{cases} \rho_1 u_{tt} - k_1 (u_x + y)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 y_{tt} - k_2 y_{xx} + k_1 (u_x + y) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.1)$$

where t is the time variable, x is the space coordinate along the beam, of length L , in its equilibrium position. With $\rho_1 := \rho A$, $\rho_2 := \rho I$, $k_1 := KAG$, $k_2 := EI$, where ρ the density in mass per unit volume, A the cross-section area, I the area moment of inertia with respect to the central axis of bending, K the shear coefficient for measuring the stiffness of materials ($K < 1$), E the Young modulus of elasticity, G the shear modulus or modulus of rigidity. Functions $S := k_1 (u_x + y)$ and $M := k_2 y_x$ stand for the bending moment and the shear force, respectively. The functions u and y denote the transverse displacement and the rotation angle of a filament of the beam, respectively. See, for example [46, 53, 54, 55, 56, 57, 58] and the references therein for various stability results of (1.1) under (finite or infinite) memories or frictional dampings. For unbounded domain (\mathbb{R} instead of $(0, L)$), we refer reader to [53] and the references therein.

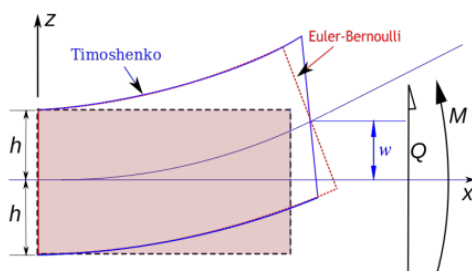


Figure 1.3 – Deformation of a Timoshenko beam (blue) compared with that of an Euler-Bernoulli beam (red). <https://upload.wikimedia.org/wikipedia/commons/thumb/5/57/TimoshenkoBeam.svg/1200px-TimoshenkoBeam.svg.png?20150802215617>

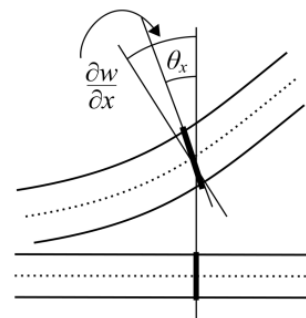


Figure 1.4 – Deformation of a Timoshenko beam. The normal rotates by an amount θ_x which is not equal to $\frac{dw}{dx}$, contrarily to the Euler-Bernoulli hypothesis (normal remains normal). https://upload.wikimedia.org/wikipedia/commons/thumb/e/e4/Plate_theory.svg/800px-Plate_theory.svg.png

Bresse beam theory:

The Bresse systems, named after the French engineer Jacques Antoine Charles Bresse, are a well-known set of systems and methodologies in the field of civil engineering. One of the notable contributions of Bresse was in the construction of bridges. He introduced the concept of metal arch bridges, which offered advantages in terms of cost, structural integrity, and aesthetics. Bresse's designs for metal arch bridges revolutionized bridge engineering and were widely adopted in France and other parts of the world.

In 1993-1994, Lagnese et al. in [70, 71] developed a general model of 3 dimensional nonlinear elastic beams. A special case of this model represents a linear model coupling 3 wave equations. This specific model describes the motion of a planar elastic beam when subjected to small deformations. It is the Bresse system also known as the circular arch beam, originally derived by Bresse, in 1859, in [28] and is given by the following equations:

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.2)$$

where $N := k_0(\omega_x - l\varphi)$ is the axial force, $Q := k(\varphi_x + \psi + l\omega)$ is the shear force and $M := b\psi_x$ is the bending moment. The functions φ , ω and ψ denote the vertical, longitudinal, and shear angle displacements, respectively. Furthermore, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_0 = EA$, $k = k'GA$, $b = EI$ and $l = R^{-1}$, where ρ denotes the density, E is the elastic modulus, G is the shear modulus, k' is the shear factor, A is the cross-sectional area, I is the second moment of area of the cross-section and R is the radius of the curvature of the beam, and we assume that all this quantities are positive. We note that when $R \rightarrow +\infty$, then $l \rightarrow 0$ and the Bresse model reduces to well-known Timoshenko beam equations. Thus, the Bresse system is more general than the Timoshenko where the longitudinal displacement ω is not considered and assuming zero initial curvature. We refer the readers to, for example, [1, 2, 25, 39, 42, 49, 50, 51, 52] and the references therein for the stability of (1.2) in bounded domains using different control mechanisms.

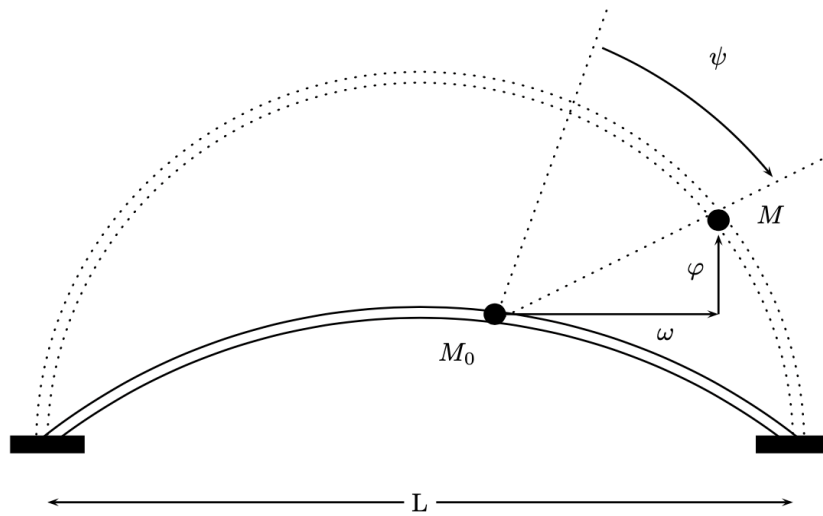


Figure 1.5 – After deformation the particle M_0 of the beam take the position M .

Source: [33]

Rao-Nakra beam theory:

The classical sandwich beam is an engineering model for a three-layer beam consisting of two stiff face plates and a much more compliant inner core layer. Numerous models of this three-layered laminated beam and plate have been suggested over the past six decades [82, 91, 111].

The Rao-Nakra beam is a mathematical model used to analyze the dynamic behavior of beams subjected to dynamic loading. It is named after K. Rama Rao and Birendra Kumar Nakra, who developed this beam theory in the field of structural engineering. The Rao-Nakra beam theory provides a refined approach to analyze the response of beams under dynamic loads, taking into account the effects of both shear deformation and rotary inertia. Unlike simpler beam theories, such as Euler-Bernoulli or Timoshenko beam theories, the Rao-Nakra beam theory considers the coupling between axial, bending, and torsional deformations, making it suitable for analyzing a wider range of beam structures.

The model proposed by Rao and Nakra [91] incorporates rotary inertia in each layer and longitudinal inertia, in addition to transverse inertia. This model assumes that continuous piecewise linear displacements are present throughout the cross-sections, and it applies the Kirchhoff hypothesis to the face plates. The modeling includes consideration of transverse, longitudinal and rotational inertial forces.

The Rao-Nakra model is derived by using Timoshenko beam assumptions for the core layer, a “noslip” assumption for the displacements along the interface and Euler-Bernoulli beam assumptions for the face plate layers. In 1999, Liu-Trogon-Yong [73] developed the following generalized Rao-Nakra beam model, where the shear effect of the bottom and top layer are taken into account:

$$\begin{cases} \rho_1 h_1 u_{tt}^1 - E_1 h_1 u_{xx}^1 - \tau = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_1 I_1 y_{tt}^1 - E_1 I_1 y_{xx}^1 - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - E_3 h_3 u_{xx}^3 + \tau = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 I_3 y_{tt}^3 - E_3 I_3 y_{xx}^3 - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+. \end{cases} \quad (1.3)$$

Here, u^1, y^1, u^3 and y^3 are, respectively, the longitudinal displacement and shear angle of the top and bottom layers; ω is the transverse displacement of the beam and τ is the shear stress in the core layer defined by

$$\tau = -u^1 - \frac{h_1}{2} y^1 + h_2 \omega_x + u^3 - \frac{h_3}{2} y^3.$$

The physical parameters h_i, ρ_i, E_i, G_i and I_i are positive and represent the thickness, density, Young’s modulus, shear modulus and moments of inertia of the i th layer, for $i = 1, 2, 3$, respectively. In addition, $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$, $EI = E_1 I_1 + E_3 I_3$ and the shear modulus $G_2 = \frac{E_2}{2(1+\nu)}$ with $-1 < \nu < \frac{1}{2}$ being the Poisson ratio (see figure 1.6).

If we neglect the rotatory inertia and the transverse shear of the top and bottom layers, equations (1.3)₂ and (1.3)₅ are reduced to the familiar Euler-Bernoulli hypothesis $y^1 - \omega_x = y^3 - \omega_x = 0$.

Several models were derived from (1.3) and studied in the literature by neglecting some components and/or parameters and/or considering some connections between them, such as the Rao-Nakra sandwich beam [91]

$$\begin{cases} \rho_1 h_1 u_{tt}^1 - E_1 h_1 u_{xx}^1 - \frac{G_2}{h_2} (-u^1 + u^3 + \alpha \omega_x) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - E_3 h_3 u_{xx}^3 + \frac{G_2}{h_2} (-u^1 + u^3 + \alpha \omega_x) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + EI \omega_{xxxx} - \frac{G_2 \alpha}{h_2} (-u^1 + u^3 + \alpha \omega_x)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.4)$$

which is obtained when we consider the core material to be linearly elastic, i.e. $\tau = 2G_2\gamma$, with the shear strain

$$\gamma = \frac{1}{2h_2} \left(-u^1 + u^3 + \alpha\omega_x \right) \quad \text{and} \quad \alpha = h_2 + \frac{1}{2} (h_1 + h_3),$$

where the shear modulus $G_2 = \frac{E_2}{2(1+\nu)}$ with $-1 < \nu < \frac{1}{2}$ being the Poisson ratio.

The laminated beam [65]

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ I_p(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - u_x) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ 3I_p s_{tt} - 3D s_{xx} + 3G(\psi - \omega_x) + 4\mu s + 4\delta s_t = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.5)$$

which is obtained when the extensional motion of the bottom and top layers is neglected.

The Mead-Markus model [82]

$$\begin{cases} \rho h\omega_{tt} + EI\omega_{xxxx} - G_2\alpha\gamma = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ 2h_2\gamma_{xx} - G_2d\gamma - \alpha\omega_{xxx} = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.6)$$

which is obtained if the extensional forces in the top and bottom layers are also neglected and which can be simplified into a six-order PDE for ω .

For the well-posedness and stability of (1.4), (1.5) and (1.6), we refer the readers, for example, to [19, 20, 53, 59, 62, 73, 92, 105] and the references therein. In case of unbounded domain, see for example [40, 41, 45, 47] and the references therein.

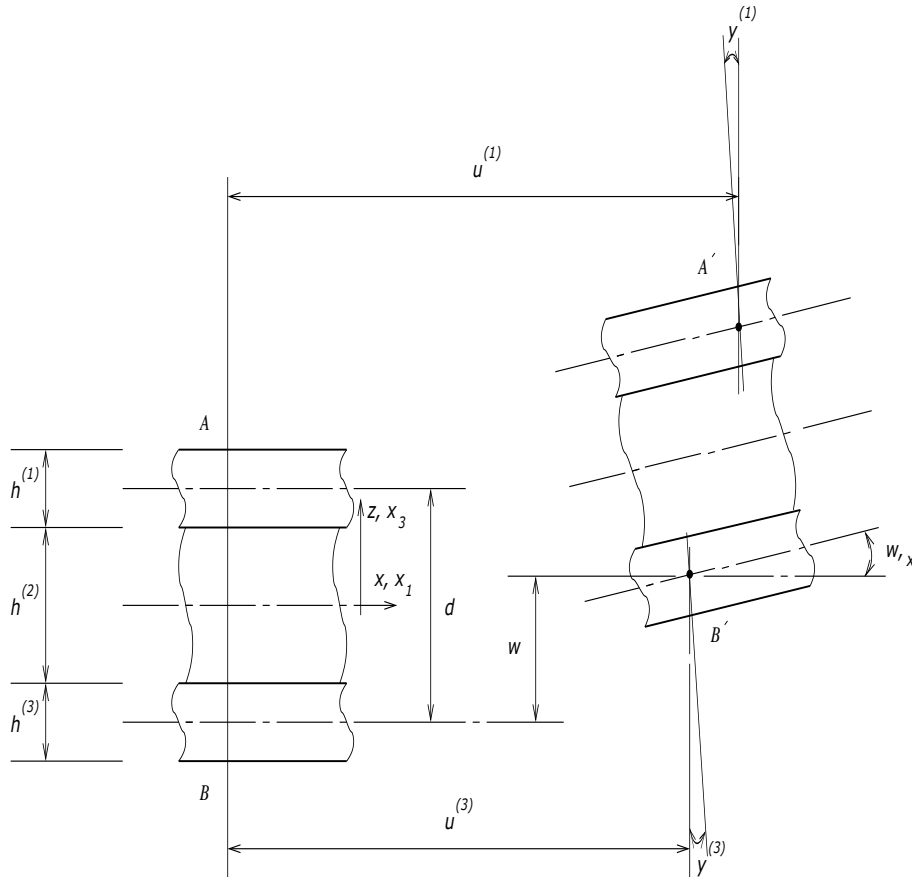


Figure 1.6 – Model describing the generalized Rao-Nakra beam.
Source: [10]

In the upcoming section, we delve into the crucial role played by damping mechanisms and cite specific types, emphasizing that the chosen ones "Viscous damping, Kelvin-Voigt damping and Fractional damping" are central to our investigation.

1.2 Damping mechanisms

Mechanical structures like beams and plates play a crucial role in modern life, and their vibration characteristics are extensively examined by numerous researchers. However, these vibrations can be undesirable due to their potential to cause damage and destruction. In order to restrain those vibrations, several dampings can be added such as frictional (viscous), Kelvin-Voigt, time delay, past history (infinite memory), finite memory, heat conduction (coupling with some heat equations) and fractional dampings.

Viscoelasticity refers to the property of materials that display both viscous and elastic behavior when subjected to deformation. In the context of vibration studies, there are various mathematical models used to represent physical damping. The two most common types of damping encountered are linear viscous damping and Kelvin-Voigt damping, both of which fall under the category of proportional damping. Viscous damping is typically associated with external friction forces, such as air resistance acting on vibrating structures, hence it is referred to as "external damping". On the other hand, Kelvin-Voigt damping arises from the internal friction within the material of vibrating structures, earning it the name "internal damping".

Fractional damping has received considerable attention in recent years and has found widespread applications across various scientific domains. The fractional-order type of damping is not only significant from a theoretical perspective but also holds importance for practical applications. It naturally emerges in a wide range of physical, chemical, biological, and ecological phenomena, as evident in reference [80] and the extensive bibliography provided therein. Fractional damping is utilized to effectively describe memory and hereditary properties of diverse materials and processes, enhancing our understanding and modeling of complex systems in various fields.

The concept of indirect damping mechanisms was first introduced by Russell in reference [95]. Since then, it has continued to captivate the interest of numerous researchers. In particular, the focus on cases where only one equation within a coupled system is damped has led to the emergence of the "indirect" stabilization problems. These problems were initiated and explored in [15, 16, 17] and have since been subject to further investigation by various authors, as evidenced by studies in [18, 31, 38, 43, 76], and other comprehensive references.

1.3 Stability theory

Stability issues have motivated the introduction of new mathematical concepts in engineering, particularly, in control engineering. Stability theory has long been of interest to mathematicians and astronomers, and has had a stimulating impact in these fields.

The aim of stabilization is to attenuate vibrations by retro-action (feedback), it therefore consists of guaranteeing the decrease of the energy of the solutions towards 0 more or less quickly by a dissipation mechanism.

More precisely, the stabilization problem in which we are interested most consists of determining the asymptotic behavior of the energy E , studying its limit in order to determine whether this limit is zero or not, and if this limit is zero, to give an estimate of its speed of decay towards zero.

There are several type of stability that can be studied. The first type consists of simply analyzing the decay of the energy of the solutions towards zero, i.e.

$$E(t) \longrightarrow 0, \quad \text{when } t \longrightarrow +\infty.$$

This is called strong stabilization.

For the second, we are interested in the fastest decay of energy, that is to say when it tends towards 0 exponentially, i.e.

$$E(t) \leqslant Ce^{-\delta t}, \quad \forall t \geqslant 0,$$

where C and δ are positive constants with C which depends on the initial data.

As for the third, we study intermediate situations in which the decay of the solutions is not exponential, but of the polynomial type, for example,

$$E(t) \leqslant Ct^{-\alpha}, \quad \forall t > 0,$$

where C and α are positive constants with C which depends on the initial data.

1.4 Objectives and results of the thesis

Over the past few decades, extensive research efforts have been dedicated to investigating the existence of solutions on a local and global scale, as well as studying the long-term behavior of solutions for various problems.

In this respect, the aim of this dissertation is to study the well-posedness and stabilization of some locally coupled systems with different kinds of local discontinuous dampings on a bounded domain by using the semigroup theory and the energy method. We also study the stabilization of a Bresse system with a frictional damping in the whole line \mathbb{R} by using the Fourier analysis.

We now turn to a detailed presentation of the systems studied in this thesis and the main results obtained. These results will be presented in greater detail in Chapters 3, 4 and 5.

1.4.1 Preliminaires

In this part, we recall some well known results on the semigroup theory; especially the results and theorems related to the strong, polynomial and analytic stability of a C_0 -semigroup.

1.4.2 Polynomial stability of a transmission problem involving Timoshenko systems with fractional Kelvin-Voigt damping (Chapitre 3)

In this part, we consider the well-posedness and stabilization of a one-dimensional Timoshenko system with localized internal fractional kelvin-Voigt damping in a bounded domain under fully Dirichlet or Dirichlet-Neumann boundary conditions. The system is governed by the following partial differential equations:

$$\begin{cases} \rho_1 u_{tt} - [k_1(u_x + y) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y)]_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 y_{tt} - [k_2 y_x + D_2(x)\partial_t^{\alpha,\eta} y_x]_x + k_1(u_x + y) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+ \end{cases} \quad (1.7)$$

subject to the following initial conditions:

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in (0, L), \\ y(x, 0) = y_0(x), & y_t(x, 0) = y_1(x), & x \in (0, L) \end{cases} \quad (1.8)$$

and the following boundary conditions:

$$u(0, t) = y(0, t) = u(L, t) = y(L, t) = 0, \quad t \in \mathbb{R}_+ \quad (1.9)$$

or

$$u(0, t) = y_x(0, t) = u(L, t) = y_x(L, t) = 0, \quad t \in \mathbb{R}_+. \quad (1.10)$$

The coefficients ρ_1 , ρ_2 , k_1 and k_2 are positive constants, η is a non-negative constant and α is in $(0, 1)$. We assume that there exists $0 \leq a_1 < b_1 \leq L$, $0 \leq a_2 < b_2 \leq L$ and two positive constants d_1 and d_2 such that

$$D_1(x) = \begin{cases} d_1, & x \in (a_1, b_1), \\ 0, & x \in (0, a_1) \cup (b_1, L) \end{cases} \quad \text{and} \quad D_2(x) = \begin{cases} d_2, & x \in (a_2, b_2), \\ 0, & x \in (0, a_2) \cup (b_2, L). \end{cases}$$

The Caputo fractional derivative $\partial_t^{\alpha, \eta}$ of order $\alpha \in (0, 1)$ with respect to time variable t is defined by

$$[D^{\alpha, \eta} \omega](x, t) = \partial_t^{\alpha, \eta} \omega(x, t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(x, s) ds, \quad (1.11)$$

where Γ denotes the Gamma function. We shall study the well-posedness and stability of system (1.7)-(1.8) with boundary conditions (1.9) or (1.10) by distinguishing three cases. In the first one, we consider a fully damped system; *i.e.* the two equations are effectively damped. However, in the other two cases, we assume that the system is partially damped; *i.e.* only one of these equations is effectively damped. To this end, we introduce the following assumptions on the damping coefficients $D_j(x)$; $j = 1, 2$:

$$D_1 \neq 0 \quad \text{and} \quad D_2 \neq 0 \quad (A_1)$$

or

$$D_1 = 0 \quad \text{and} \quad D_2 \neq 0 \quad (A_2)$$

or

$$D_1 \neq 0 \quad \text{and} \quad D_2 = 0. \quad (A_3)$$

First, we need to reformulate the system into an augmented model. For this purpose, we recall Theorem 2 stated in [81].

Theorem 1.1. Let $\alpha \in (0, 1)$, $\eta \geq 0$ and $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ (defined almost everywhere on \mathbb{R}). The relationship between the input V and the output O of the following system:

$$\omega_t(x, \xi, t) + (\xi^2 + \eta) \omega(x, \xi, t) - \mu(\xi) V(x, t) = 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+, \quad (1.12)$$

$$\omega(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R}, \quad (1.13)$$

$$O(x, t) - \kappa(\alpha) \int_{\mathbb{R}} \mu(\xi) \omega(x, \xi, t) d\xi = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+ \quad (1.14)$$

is given by

$$O = I^{1-\alpha, \eta} V, \quad (1.15)$$

where

$$[I^{\alpha, \eta} V](x, t) = \int_0^t \frac{(t - \tau)^{\alpha-1} e^{-\eta(t-\tau)}}{\Gamma(\alpha)} V(x, \tau) d\tau \quad \text{and} \quad \kappa(\alpha) = \frac{\sin(\alpha\pi)}{\pi}.$$

By taking the inputs

$$V_1(x, t) = \sqrt{D_1(x)}(u_{xt}(x, t) + y_t(x, t))$$

and

$$V_2(x, t) = \sqrt{D_2(x)}y_{xt}(x, t)$$

in Theorem 1.1, we deduce that system (1.7)-(1.8) with boundary conditions (1.9) or (1.10) may be recast into the following augmented model:

$$\rho_1 u_{tt} - \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi \right)_x = 0, \quad (1.16)$$

$$\rho_2 y_{tt} - \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi, t) d\xi \right)_x \quad (1.17)$$

$$+ k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi = 0,$$

$$\omega_t^1(x, \xi, t) + (\xi^2 + \eta) \omega^1(x, \xi, t) - \sqrt{D_1(x)}(u_{xt} + y_t) \mu(\xi) = 0, \quad (1.18)$$

$$\omega_t^2(x, \xi, t) + (\xi^2 + \eta) \omega^2(x, \xi, t) - \sqrt{D_2(x)} y_{tx} \mu(\xi) = 0, \quad (1.19)$$

where $(x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+$, with the following initial conditions:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, L), \quad (1.20)$$

$$y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, L), \quad (1.21)$$

$$\omega^1(x, \xi, 0) = \omega^2(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R} \quad (1.22)$$

and with fully Dirichlet boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0 \quad (1.23)$$

or with Dirichlet-Neumann boundary conditions

$$u(0) = u(L) = y_x(0) = y_x(L) = 0, \quad \omega^2(0, \xi, t) = \omega^2(L, \xi, t) = 0. \quad (1.24)$$

We define the energy of the system (1.16)-(1.22) to either the boundary conditions (1.23) or (1.24) by

$$E(t) = \frac{1}{2} \int_0^L \left(\rho_1 |u_t|^2 + \rho_2 |y_t|^2 + k_1 |u_x + y|^2 + k_2 |y_x|^2 \right) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} \left(|\omega^1|^2 + |\omega^2|^2 \right) d\xi dx.$$

Let $(u, u_t, y, y_t, \omega^1, \omega^2)$ be a regular solution of the system (1.16)-(1.24). By some direct computations, we deduce that the energy E satisfies the following estimation:

$$E'(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) \left(|\omega^1|^2 + |\omega^2|^2 \right) d\xi dx. \quad (1.25)$$

Since $\alpha \in (0, 1)$, then $\kappa(\alpha) > 0$ and so $E'(t) \leq 0$, which implies that the system (1.16)-(1.22) subject to either the boundary conditions (1.23) or (1.24) is dissipative in the sense that its energy is decreasing with respect to time variable t .

We proceed by introducing the Hilbert energy spaces \mathcal{H}_1 and \mathcal{H}_2 by

$$\mathcal{H}_1 = \left(H_0^1(0, L) \times L^2(0, L) \right)^2 \times \mathcal{W}^2,$$

$$\mathcal{H}_2 = H_0^1(0, L) \times L^2(0, L) \times H_*^1(0, L) \times L^2(0, L) \times \mathcal{W}^2,$$

where

$$\mathcal{W} = L^2((0, L) \times \mathbb{R})$$

and

$$H_*^1(0, L) = \left\{ f \in H^1(0, L) \mid \int_0^L f dx = 0 \right\}.$$

It is easy to check that the space $H_*^1(0, L)$ is Hilbert spaces over \mathbb{C} equipped with the norm

$$\|u\|_{H_*^1(0, L)}^2 = \|u_x\|^2,$$

where $\|\cdot\|$ denotes the usual norm of $L^2(0, L)$. Both energy spaces \mathcal{H}_1 and \mathcal{H}_2 are equipped with the inner product defined by

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}_j} = & \rho_1 \int_0^L v \tilde{v} dx + \rho_2 \int_0^L z \tilde{z} dx + k_1 \int_0^L (u_x + y) (\overline{\tilde{u}_x + \tilde{y}}) dx \\ & + k_2 \int_0^L y_x \tilde{y}_x dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\omega^1 \overline{\tilde{\omega}^1} + \omega^2 \overline{\tilde{\omega}^2}) d\xi dx, \end{aligned}$$

for all $U = (u, v, y, z, \omega^1, \omega^2)$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}, \tilde{\omega}^1, \tilde{\omega}^2)$ in \mathcal{H}_j , $j = 1, 2$. We use $\|\cdot\|_{\mathcal{H}_j}$ to denote the corresponding norm. Now, we define the following unbounded linear operators \mathcal{A}_j on \mathcal{H}_j ($j = 1, 2$):

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} \mid v, z \in H_0^1(0, L), \\ \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \in L^2(0, L), \\ \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \in L^2(0, L), \\ -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1(x)} (v_x + z) \mu(\xi), \quad |\xi| \omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2(x)} z_x \mu(\xi), \quad |\xi| \omega^2 \in \mathcal{W} \end{array} \right\},$$

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} \mid v \in H_0^1(0, L), z \in H_*^1(0, L), \\ \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \in L^2(0, L), \\ \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \in L^2(0, L), \\ -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1(x)} (v_x + z) \mu(\xi), \quad |\xi| \omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2(x)} z_x \mu(\xi), \quad |\xi| \omega^2 \in \mathcal{W}, \\ y_x(0, t) = y_x(L, t) = 0 \end{array} \right\}$$

and

$$\mathcal{A}_j U^\top = \begin{pmatrix} v \\ \frac{1}{\rho_1} \left(\kappa_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \\ z \\ \frac{1}{\rho_2} \left(\kappa_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x - \frac{\kappa_1}{\rho_2} (u_x + y) - \frac{\kappa(\alpha)}{\rho_2} \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \\ -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1(x)} (v_x + z) \mu(\xi) \\ -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2(x)} z_x \mu(\xi) \end{pmatrix},$$

for all $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ and $j = 1, 2$. If U is a regular solution of system (1.16)-(1.22), then we transform this system into a first order evolution equation on the Hilbert space \mathcal{H}_j

$$\begin{cases} U_t(x, t) = \mathcal{A}_j U(x, t), \\ U(x, 0) = U_0(x), \end{cases} \quad (1.26)$$

where

$$U_0(x) = (u_0(x), u_1(x), y_0(x), y_1(x), 0, 0)$$

with $j = 1, 2$ corresponding to the boundary conditions (1.23) and (1.24), respectively.

Then we prove that the operator \mathcal{A}_j is m-dissipative in the energy space \mathcal{H}_j . Therefore, thanks to Lumer-Phillips theorem (see [88]), we deduce that \mathcal{A}_j generates a C_0 -semigroup of contraction $e^{t\mathcal{A}_j}$ in \mathcal{H}_j , $j = 1, 2$. Then the solution of the evolution problem (1.26) admits the following representation:

$$U(t) = e^{t\mathcal{A}_j} U_0, \quad t \geq 0,$$

and therefore, problem (1.26) is well-posed and we have the following result:

Theorem 1.2. For all $U_0 \in \mathcal{H}_j$, problem (1.26) admits a unique weak solution

$$U \in C(\mathbb{R}_+; \mathcal{H}_j).$$

Moreover, if $U_0 \in D(\mathcal{A}_j)$, then problem (1.26) admits a unique strong solution

$$U \in C(\mathbb{R}_+; D(\mathcal{A}_j)) \cap C^1(\mathbb{R}_+; \mathcal{H}_j).$$

Remark 1.3. All previous results still valid even when the Timoshenko System (1.7) is considered with only one fractional Kelvin-Voigt damping; i.e. $D_1 \equiv 0$ or $D_2 \equiv 0$.

Second, we use a general criteria of Arendt-Batty [22] to show the strong stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$ associated to the Timoshenko System (1.16)-(1.22). Our main result is the following theorem:

Theorem 1.4. Assume that either (A₁), (A₂) or (A₃) holds. Then, the C_0 -semigroup $e^{t\mathcal{A}_j}$ is strongly stable in the energy space \mathcal{H}_j in the sense that

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} = 0, \quad \forall U_0 \in \mathcal{H}_j, \quad j = 1, 2.$$

The argument for Theorem 1.4 relies on the subsequent lemmas.

Lemma 1.5. Assume that assumption (A_3) holds. Then, we have

$$\ker(i\lambda I - \mathcal{A}_j) = \{0\}, \quad \forall \lambda \in \mathbb{R}, \quad j = 1, 2.$$

Lemma 1.6. Assume that $\eta = 0$ and assumption (A_3) holds. Then, the operator $-\mathcal{A}_j$ is not invertible, and consequently, $0 \in \sigma(\mathcal{A}_j)$, $j = 1, 2$.

Lemma 1.7. Assume that assumption (A_3) holds and either $(\eta, \lambda) \in \mathbb{R}_+^* \times \mathbb{R}$ or $[\eta = 0 \text{ and } \lambda \in \mathbb{R}^*]$. Then, $i\lambda I - \mathcal{A}_j$ is surjective, $j = 1, 2$.

Thus, using Lemma 1.5, we directly deduce that \mathcal{A}_j has non pure imaginary eigenvalues. According to Lemmas 1.5, 1.6 and 1.7 and with the help of the closed graph theorem of Banach, we deduce that

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$$

if $\eta > 0$, and

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$$

if $\eta = 0$. Thus, we get the conclusion by applying Theorem 2.7 of Arendt and Batty.

Third, we study the polynomial stability of the system (1.16)-(1.22) when $\eta > 0$ by considering three cases: in the first one, the damping is effective only in the bending moment. However, in the second one, the damping is effective only in the shear stress. Finally, in the third case, the damping is present in both shear stress and bending moment. For this purpose, we will use a frequency domain approach with a multiplier method, namely, we will use Theorem 2.10. Our main results in this part is the following theorems:

Theorem 1.8. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (1.27)$$

Theorem 1.9. Assume that $\eta > 0$ and assumption (A_3) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (1.28)$$

For the third case we assume that the following assumption holds:

$$D_1 \neq 0, \quad D_2 \neq 0, \quad a_1 = a_2 = a \quad \text{and} \quad b_1 = b_2 = b. \quad (A_4)$$

Theorem 1.10. Assume that $\eta > 0$ and assumption (A_4) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:

$$E(t) \leq \frac{c}{t^{\frac{4}{2-\alpha}}} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (1.29)$$

1.4.3 Polynomial stability of the generalized Rao-Nakra beam with local Kelvin-Voigt damping (Chapitre 4)

In this part, we investigate the well-posedness and stabilization of the generalized Rao-Nakra beam equation. The system consists of four wave equations for the longitudinal displacements and the shear angle of the top and bottom layers and one Euler-Bernoulli beam equation for the transversal displacement. So, in this chapter, we are interested in studying the asymptotic behavior of the solution

of the following generalized Rao-Nakra beam model with Kelvin-Voigt damping and under various boundary conditions:

$$\left\{ \begin{array}{l} \rho_1 h_1 u_{tt}^1 - (E_1 h_1 u_x^1 + D_1(x) u_{tx}^1)_x - \tau = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_1 I_1 y_{tt}^1 - (E_1 I_1 y_x^1 + D_2(x) y_{tx}^1)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + (EI \omega_{xx} + D_3(x) \omega_{txx})_{xx} - G_1 h_1 (\omega_x + y^1)_x \\ \quad - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - (E_3 h_3 u_x^3 + D_4(x) u_{tx}^3)_x + \tau = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 I_3 y_{tt}^3 - (E_3 I_3 y_x^3 + D_5(x) y_{tx}^3) - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+. \end{array} \right. \quad (1.30)$$

System (1.30) is subjected to the following initial conditions:

$$\left\{ \begin{array}{l} u^1(x, 0) = u_0^1(x), \quad u_t^1(x, 0) = u_1^1(x), \quad x \in (0, L), \\ u^3(x, 0) = u_0^3(x), \quad u_t^3(x, 0) = u_1^3(x), \quad x \in (0, L), \\ \omega(x, 0) = \omega_0(x), \quad \omega_t(x, 0) = \omega_1(x), \quad x \in (0, L), \\ y^1(x, 0) = y_0^1(x), \quad y_t^1(x, 0) = y_1^1(x), \quad x \in (0, L), \\ y^3(x, 0) = u_0^3(x), \quad y_t^3(x, 0) = u_1^3(x), \quad x \in (0, L). \end{array} \right. \quad (1.31)$$

For the boundary conditions, we assume that variables u^1, u^3, y^1 and y^3 satisfy one of the following boundary conditions:

Dirichlet boundary conditions for all waves

$$u^1(0, t) = u^1(L, t) = u^3(0, t) = u^3(L, t) = y^1(0, t) = y^1(L, t) = y^3(0, t) = y^3(L, t) = 0. \quad (DD)$$

Neumann boundary conditions for all the waves

$$u_x^1(0, t) = u_x^1(L, t) = u_x^3(0, t) = u_x^3(L, t) = y_x^1(0, t) = y_x^1(L, t) = y_x^3(0, t) = y_x^3(L, t) = 0. \quad (NN)$$

And ω satisfies one of the following boundary conditions:

The clamped boundary condition

$$\omega(0, t) = \omega(L, t) = \omega_x(0, t) = \omega_x(L, t) = 0. \quad (FBC)$$

The hanged boundary condition

$$\omega(0, t) = \omega(L, t) = \omega_{xx}(0, t) = \omega_{xx}(L, t) = 0. \quad (HBC)$$

We suppose that there exists $0 < a < b < L$ and five positive constants d_i , for $i = 1, \dots, 5$, such that

$$D_i(x) = \begin{cases} d_0, & x \in (a, b), \\ 0, & x \in (0, a) \cup (b, L). \end{cases}$$

First, we begin by study the well-posedness of our system using the semigroup approach. Let $(u^1, u^3, \omega, y^1, y^3)$ be a regular solution of system (1.30)-(1.31). Its associated energy is given by

$$\begin{aligned} E(t) = & \frac{1}{2} \left(\rho_1 h_1 \int_0^L |u_t^1|^2 dx + E_1 h_1 \int_0^L |u_x^1|^2 dx + \rho_3 h_3 \int_0^L |u_t^3|^2 dx + E_3 h_3 \int_0^L |u_x^3|^2 dx \right. \\ & + \rho h \int_0^L |\omega_t|^2 dx + EI \int_0^L |\omega_{xx}|^2 dx + \rho_1 I_1 \int_0^L |y_t^1|^2 dx + E_1 I_1 \int_0^L |y_x^1|^2 dx \\ & + \rho_3 I_3 \int_0^L |y_t^3|^2 dx + E_3 I_3 \int_0^L |y_x^3|^2 dx + G_1 h_1 \int_0^L |\omega_x + y^1|^2 dx + \int_0^L |\tau|^2 dx \\ & \left. + G_3 h_3 \int_0^L |\omega_x + y^3|^2 dx \right). \end{aligned} \quad (1.32)$$

A straightforward computation gives that

$$\begin{aligned} \frac{d}{dt}E(t) &= - \int_0^L D_1(x)|u_{tx}^1|^2 dx - \int_0^L D_2(x)|y_{tx}^1|^2 dx - \int_0^L D_3(x)|\omega_{txx}|^2 dx \\ &\quad - \int_0^L D_4(x)|u_{tx}^3|^2 dx - \int_0^L D_5(x)|y_{tx}^3|^2 dx \\ &\leq 0. \end{aligned} \quad (1.33)$$

Thus, system (1.30)-(1.31) subject to either the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)] is dissipative in the sense that its energy is a non increasing function with respect to the time variable t .

Next, we introduce the following Hilbert energy spaces: for the boundary conditions (DD) and (FBC), set the energy space as

$$\mathcal{H}_1 = \left(H_0^1(0, L) \times L^2(0, L) \right)^2 \times H_0^2(0, L) \times L^2(0, L) \times \left(H_0^1(0, L) \times L^2(0, L) \right)^2.$$

For the boundary conditions (NN) and (HBC), set the energy space as

$$\mathcal{H}_2 = \left(H_*^1(0, L) \times L^2(0, L) \right)^2 \times (H^2(0, L) \cap H_0^1(0, L)) \times L^2(0, L) \times \left(H_*^1(0, L) \times L^2(0, L) \right)^2,$$

where

$$H_*^1(0, L) = \left\{ f \in H^1(0, L) \mid \int_0^L f dx = 0 \right\}.$$

Both spaces \mathcal{H}_1 and \mathcal{H}_2 are equipped with the inner product which induces the energy norm

$$\begin{aligned} \|U\|_{\mathcal{H}_j}^2 &= \| (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \|_{\mathcal{H}_j}^2 \\ &= E_1 h_1 \|u_x^1\|^2 + \rho_1 h_1 \|v^1\|^2 + E_3 h_3 \|u_x^3\|^2 + \rho_3 h_3 \|v^3\|^2 dx + EI \|\omega_{xx}\|^2 + \rho h \|\psi\|^2 \\ &\quad + E_1 I_1 \|y_x^1\|^2 + \rho_1 I_1 \|z^1\|^2 + E_3 I_3 \|y_x^3\|^2 + \rho_3 I_3 \|z^3\|^2 + G_1 h_1 \|\omega_x + y^1\|^2 \\ &\quad + \|\tau\|^2 + G_3 h_3 \|\omega_x + y^3\|^2, \quad j = 1, 2; \end{aligned} \quad (1.34)$$

here $\|\cdot\|$ denotes the norm of $L^2(0, L)$.

Now, we define the following linear unbounded operators $\mathcal{A}_j : D(\mathcal{A}_j) \subset \mathcal{H}_j \rightarrow \mathcal{H}_j$:

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in \mathcal{H}_1 \\ v^1, v^3, z^1, z^3 \in H_0^1(0, L), \psi \in H_0^2(0, L), \\ \left(E_1 h_1 u_x^1 + D_1(x) v_x^1 \right)_x \in L^2(0, L), \left(E_3 h_3 u_x^3 + D_4(x) v_x^3 \right)_x \in L^2(0, L), \\ \left(E_1 I_1 y_x^1 + D_2(x) z_x^1 \right)_x \in L^2(0, L), \left(E_3 I_3 y_x^3 + D_5(x) z_x^3 \right)_x \in L^2(0, L), \\ (EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} \in L^2(0, L) \end{array} \right\},$$

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in \mathcal{H}_2 \\ v^1, v^3, z^1, z^3 \in H_*^1(0, L), \psi \in H^2(0, L) \cap H_0^1(0, L), \\ \left(E_1 h_1 u_x^1 + D_1(x) v_x^1 \right)_x \in L^2(0, L), u_x^1(0, t) = u_x^1(L, t) = 0, \\ \left(E_3 h_3 u_x^3 + D_4(x) v_x^3 \right)_x \in L^2(0, L), u_x^3(0, t) = u_x^3(L, t) = 0, \\ \left(E_1 I_1 y_x^1 + D_2(x) z_x^1 \right)_x \in L^2(0, L), y_x^1(0, t) = y_x^1(L, t) = 0, \\ \left(E_3 I_3 y_x^3 + D_5(x) z_x^3 \right)_x \in L^2(0, L), y_x^3(0, t) = y_x^3(L, t) = 0, \\ (EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} \in L^2(0, L), \omega_{xx}(0, t) = \omega_{xx}(L, t) = 0 \end{array} \right\}$$

and

$$\mathcal{A}_j \begin{pmatrix} u^1 \\ v^1 \\ y^1 \\ z^1 \\ \omega \\ \psi \\ u^3 \\ v^3 \\ y^3 \\ z^3 \end{pmatrix} = \begin{pmatrix} v^1 \\ (\rho_1 h_1)^{-1} [(E_1 h_1 u_x^1 + D_1(x) v_x^1)_x + \tau] \\ z^1 \\ (\rho_1 I_1)^{-1} [(E_1 I_1 y_x^1 + D_2(x) z_x^1)_x + \frac{h_1}{2} \tau - G_1 h_1 (\omega_x + y^1)] \\ \psi \\ (\rho h)^{-1} [-(EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} + G_1 h_1 (\omega_x + y^1)_x + G_3 h_3 (\omega_x + y^3)_x + h_2 \tau_x] \\ v^3 \\ (\rho_3 h_3)^{-1} [(E_3 h_3 u_x^3 + D_4(x) v_x^3)_x - \tau] \\ z^3 \\ (\rho_3 I_3)^{-1} [(E_3 I_3 y_x^3 + D_5(x) z_x^3)_x + \frac{h_3}{2} \tau - G_3 h_3 (\omega_x + y^3)] \end{pmatrix}, \quad (1.35)$$

for all $U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in D(\mathcal{A}_j)$ and $j = 1, 2$. If U is a regular solution of system (1.30)-(1.31), then the system can be rewritten as evolution equation on the Hilbert space \mathcal{H}_j in the abstract form

$$\begin{cases} U_t(x, t) = \mathcal{A}_j U(x, t), \\ U(x, 0) = U_0(x), \end{cases} \quad (1.36)$$

where

$$U_0 = (u_0^1, u_1^1, y_0^1, y_1^1, \omega_0, \omega_1, u_0^3, u_1^3, y_0^3, y_1^3)$$

with $j = 1, 2$ corresponding to the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)], respectively.

Then, we prove that the operator \mathcal{A}_j is m-dissipative in \mathcal{H}_j , $j = 1, 2$. Thus, according to Lumer-Phillips theorem(see [88]), we deduce that \mathcal{A}_j generates a C_0 -semigroup of contractions $(e^{t\mathcal{A}_j})_{t \geq 0}$ in \mathcal{H}_j and then the problem is well-posed in \mathcal{H}_j . Hence, we have the following result:

Theorem 1.11. Let $U_0 \in \mathcal{H}_j$, then system (1.36) admits a unique weak solution U satisfying

$$U \in C^0(\mathbb{R}_+, \mathcal{H}_j).$$

Moreover, if $U_0 \in D(\mathcal{A}_j)$, then problem (1.36) admits a unique strong solution U satisfying

$$U \in C^1(\mathbb{R}_+, \mathcal{H}_j) \cap C^0(\mathbb{R}_+, D(\mathcal{A}_j)).$$

Second, we study the stability of our system and prove that its energy converges to zero when t goes to infinity for all initial data in \mathcal{H}_j . Since the resolvent of \mathcal{A}_j is not compact, we have to discuss the full spectrum on the imaginary axis. We use a general criteria of Arendt-Batty theorem in [24] which states that, in a reflexive Banach space, a C_0 -semigroup $e^{t\mathcal{A}_j}$ is strongly stable if $e^{t\mathcal{A}_j}$ is bounded, \mathcal{A}_j has no eigenvalues on the imaginary axis and $\sigma(\mathcal{A}_j) \cap i\mathbb{R}$ is countable. Before we state our main result, we need the following stability conditions:

$$D_i > d_0 > 0 \quad \text{for every } x \in (0, L), \quad i = 1, \dots, 5, \quad (\text{SSC1})$$

$$D_i(x) \geq d_0 > 0 \quad \text{in } (a, b), \quad i = 2, 5, \quad G_1 \neq G_3 \quad \text{and} \quad \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}. \quad (\text{SSC2})$$

So, we will prove the following stability result:

Theorem 1.12. Assume that condition (SSC1) or (SSC2) holds. Then, the C_0 -semigroup of contractions $(e^{t\mathcal{A}_j})_{t \geq 0}$ is strongly stable on the energy space \mathcal{H}_j , $j = 1, 2$; that is

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} = 0, \quad \forall U_0 \in \mathcal{H}_j.$$

To prove Theorem (1.12), we need the following two lemmas:

Lemma 1.13. Assume that condition (SSC2) hold, then we have

$$\ker(i\lambda I - \mathcal{A}_j) = \{0\}, \quad \forall \lambda \in \mathbb{R}, \quad j = 1, 2,$$

where I is the identity operator.

Lemma 1.14. Assume that condition (SSC2) hold, then, for all $\lambda \in \mathbb{R}$, we have

$$\mathcal{R}(i\lambda I - \mathcal{A}_j) = \mathcal{H}_j, \quad j = 1, 2.$$

Thus, using Lemma 1.13, we directly deduce that \mathcal{A}_j has non pure imaginary eigenvalues. According to Lemma 1.14 and with the help of the closed graph theorem of Banach, we deduce that $\sigma(\mathcal{A}_j) \cap i\mathbb{R} = \emptyset$. Thus, we get the conclusion by applying Theorem 2.7 of Arendt and Batty.

Now, we need to study the energy decay rate by using a frequency domain approach combined with piecewise multiplier technique in two cases depending on the regularity of the coefficient functions D_i , $i = 1, \dots, 5$, the localization of their supports and their number.

In the case when the five dampings are globally distributed, we prove an analytic stability. Our result is the following:

Theorem 1.15. Assume that condition (SSC1) holds and $D_i \in L^\infty(0, L)$, $i = 1, \dots, 5$. Then, the C_0 -semigroup $(e^{t\mathcal{A}_j})_{t \geq 0}$ is analytic in \mathcal{H}_j , for $j = 1, 2$.

In the case of only two local damping D_2 and D_5 acting on the shear angle displacements of the top and bottom layers, i.e. assumption (SSC2) holds, we show that the energy of the system decays polynomially. We prove the following theorem:

Theorem 1.16. Assume that condition (SSC2) holds. The C_0 -semigroup $(e^{t\mathcal{A}_j})_{t \geq 1}$ is polynomially stable; more precisely, there exists constant $C > 0$ such that for every $U_0 \in D(\mathcal{A}_j)$, we have

$$E(t) \leq \frac{C}{t^{\frac{1}{2}}}, \quad \forall t > 0, \quad \forall U_0 \in D(\mathcal{A}_j). \quad (1.37)$$

1.4.4 Some stability and non-stability results for a Cauchy problem of Bresse-type with frictional damping on the vertical displacement (Chapitre 5)

In this part, we study the stability of a Bresse-type system in the whole line \mathbb{R} with frictional damping working only on the first equation (vertical displacement). The system is governed by the following partial differential equations:

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k^2 l(\omega_x - l\varphi) + \gamma\varphi_t = 0, \\ \psi_{tt} - a^2 \psi_{xx} - (\varphi_x - \psi - l\omega) = 0, \\ \omega_{tt} - k^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) = 0, \end{cases} \quad (1.38)$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)(x), \quad (1.39)$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ and a, l, γ and k are positive constants. The functions $\omega(x, t)$, $\varphi(x, t)$ and $\psi(x, t)$ are, respectively, the longitudinal displacements, the vertical displacement of the beam and the rotation angle of the linear filaments material.

Indeed, stability proofs for Bresse systems in bounded domains typically rely on energy method by using in particular, Poincaré's inequality, some sobolev injections and the imposition of appropriate boundary conditions. However, the focus of this paper is on the unbounded situation where the domain is \mathbb{R} . Our proofs are based on some estimates for the Fourier image of the solution, Plancherel Theorem, as well as on a suitable linear combination of series associated to energy estimates. Thus, in order to establish the decay rate of the Bresse system (1.38)-(1.39), we have to transform the original problem to a first-order (with respect to the variable t) system, defining new variables. Then, we apply the energy method in the Fourier space to prove some point wise estimates which will help in the proof of the decay estimates.

First, we write system (1.38)-(1.39) as a first order Cauchy system of the form

$$\begin{cases} U_t + AU_x + LU = 0, & x \in \mathbb{R}, t > 0, \\ U(x, 0) = U_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.40)$$

where A is a real symmetric matrix and L is non-negative definite matrix. To this end, we introduce the following variables:

$$v = \varphi_x - \psi - l\omega, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad \phi = k(\omega_x - l\varphi) \quad \text{and} \quad \eta = \omega_t. \quad (1.41)$$

Consequently, system (1.38)-(1.39) can be rewritten into the following first order system:

$$\begin{cases} v_t - u_x + y + l\eta = 0, \\ u_t - v_x - lk\phi + \gamma u = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v = 0, \\ \phi_t - k\eta_x + lku = 0, \\ \eta_t - k\phi_x - lv = 0, \end{cases} \quad (1.42)$$

and the initial conditions takes the form

$$(v, u, z, y, \phi, \eta)(x, 0) = (v_0, u_0, z_0, y_0, \phi_0, \eta_0)(x).$$

System (1.42) is equivalent to system (1.40) with

$$U = \begin{pmatrix} v \\ u \\ z \\ y \\ \phi \\ \eta \end{pmatrix}, A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & -k & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & l \\ 0 & \gamma & 0 & 0 & -lk & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & lk & 0 & 0 & 0 & 0 \\ -l & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$U_0 = (v_0, u_0, z_0, y_0, \phi_0, \eta_0)^t.$$

By taking the Fourier transform of (1.40), with respect to the space variable x , we obtain the following Cauchy problem of a first order system in the Fourier space:

$$\begin{cases} \hat{U}_t + i\xi A \hat{U} + L \hat{U} = 0, & \xi \in \mathbb{R}, t > 0, \\ \hat{U}(\xi, 0) = \hat{U}_0, & \xi \in \mathbb{R}. \end{cases} \quad (1.43)$$

The solution of (1.43) is given by

$$\hat{U}(\xi, t) = e^{t\hat{\Phi}(\xi)}\hat{U}_0(\xi), \quad (1.44)$$

where

$$\hat{\Phi}(\xi) = -(i\xi A + L). \quad (1.45)$$

The first equation of system (1.43) can be rewritten into the following form (that is, the Fourier transform of (1.42)):

$$\begin{cases} \hat{v}_t - i\xi\hat{u} + \hat{y} + l\hat{\eta} = 0, \\ \hat{u}_t - i\xi\hat{v} - lk\hat{\phi} + \gamma\hat{u} = 0, \\ \hat{z}_t - ai\xi\hat{y} = 0, \\ \hat{y}_t - ai\xi\hat{z} - \hat{v} = 0, \\ \hat{\phi}_t - ki\xi\hat{\eta} + lk\hat{u} = 0, \\ \hat{\eta}_t - ki\xi\hat{\phi} - l\hat{v} = 0. \end{cases} \quad (1.46)$$

Now, let us define the energy functional associated with (1.43) by

$$\hat{E}(\xi, t) = \frac{1}{2}\|\hat{U}(\xi, t)\|^2 = \frac{1}{2}(|\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\phi}|^2 + |\hat{\eta}|^2)(\xi, t). \quad (1.47)$$

Let $(\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\phi}, \hat{\eta})$ be the solution of (1.43), then the energy \hat{E} is a non-increasing function with respect to t and satisfies, for all $t \geq 0$ and $\xi \in \mathbb{R}$,

$$\frac{d}{dt}\hat{E}(\xi, t) = -\gamma|\hat{u}(\xi, t)|^2. \quad (1.48)$$

Thus, (1.43) is a dissipative system.

The goal is to present and show some stability and non-stability results for (1.43) depending on the following conditions:

$$(a - k)(k - 1) > 0 \quad \text{and} \quad \max \left\{ \frac{l[l^2(a^2 - 1)(k^2 - 1) + k^2]}{l^2(a^2 - k^2)(k^2 - 1) + k^2}, \frac{l(a^2 - 1)}{a^2 - k^2} \right\} < \frac{l^2a^2 + 1}{la^2} \quad (1.49)$$

and

$$a = k = \frac{\sqrt{l^2 + 1}}{l}. \quad (1.50)$$

Second, we show an estimate for the Fourier image of the solution. This estimate will play the key role in proving Theorem 1.18 below.

Theorem 1.17. Let \hat{U} be the solution of (1.43).

1. Assume that (1.49) holds. Then there exist $c, \tilde{c} > 0$ such that

$$\|\hat{U}(\xi, t)\|^2 \leq \tilde{c}e^{-cf(\xi)t} \|\hat{U}_0(\xi)\|^2, \quad \xi \in \mathbb{R}, \quad t \in \mathbb{R}_+, \quad (1.51)$$

where

$$f(\xi) = \frac{\xi^2}{\xi^4 + \xi^2 + 1}. \quad (1.52)$$

2. Assume that (1.50) holds. Then $\|\hat{U}(\xi, t)\|$ does not converge to zero when time t goes to infinity.

Our main result in this chapter is stated in the following theorem:

Theorem 1.18. Assume that (1.49) holds. Let $N, \ell \in \mathbb{N}^*$ such that $\ell \leq N$, $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$ and U be the solution of (1.40). Then, for any $j = 0, \dots, N - \ell$, there exists $c_0 > 0$ such that

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0(1+t)^{-1/4-j/2} \|U_0\|_{L^1(\mathbb{R})} + c_0(1+t)^{-\ell/2} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}_+. \quad (1.53)$$

2 Introduction en Français

2.1 Contexte et motivation physique

De nombreux phénomènes naturels peuvent être décrits par des équations aux dérivées partielles et le contrôle de ces équations est un domaine d'étude relativement nouveau et actif.

Dans le cas du contrôle des EDP, qui constitue le cadre de cette thèse, les modèles étudiés prennent en compte les variations et les distributions spatiales des variables qui reflètent l'état du système. Ces questions sont alors posées dans le contexte de systèmes dynamiques en dimension infinie.

Le domaine de la théorie du contrôle des équations aux dérivées partielles s'est élargi grâce à l'introduction et à l'étude de modèles plus pratiques. Il s'avère avantageux dans divers domaines, notamment l'ingénierie, la biologie, l'informatique et la recherche biomédicale.

2.1.1 Théorie du contrôle

La théorie du contrôle est une branche de l'ingénierie et des mathématiques qui traite de l'analyse et de la conception de systèmes qui peuvent être contrôlés ou régulés pour se comporter de la manière souhaitée.

L'objectif principal de la théorie du contrôle est de développer des modèles mathématiques qui décrivent la dynamique d'un système et de concevoir des stratégies de contrôle qui peuvent influencer le comportement du système. Ces stratégies sont généralement mises en œuvre à l'aide de mécanismes de rétroaction, où la sortie du système est mesurée et utilisée pour ajuster les entrées de contrôle en temps réel.

Le "problème de stabilisation", qui consiste à déterminer une loi de contrôle qui amène le système à l'équilibre à partir de n'importe quel point de départ, est l'un des problèmes des contrôles les plus populaires. Ce problème a attiré et continue d'attirer de nombreux chercheurs en contrôle, et de nombreuses techniques peuvent être utilisées pour résoudre le problème de stabilisation des systèmes dynamiques. Dans la pratique, le problème de stabilisation est le plus important dans la phase de conception de tout système de contrôle car il fournit les performances souhaitées au système de contrôle conçu. Cette théorie est appliquée dans un large éventail de disciplines scientifiques et techniques. Par exemple, elle peut être utilisée pour la réduction du bruit dans des systèmes tels que les moteurs ou les appareils électroniques. Elle peut également être appliquée aux vibrations structurelles, dans le but d'atténuer ou d'éliminer les oscillations indésirables. En outre, la théorie du contrôle et de la stabilisation est utilisée pour étudier les vagues et les tremblements de terre, dans le but de prédire et de minimiser les effets néfastes de ces phénomènes naturels. Elle est également utile dans de nombreux autres domaines tels que la biologie, la recherche biomédicale, les systèmes robotiques et l'informatique.

En résumé, la théorie du contrôle est un domaine qui nous permet de comprendre, d'analyser et de manipuler le comportement des systèmes dynamiques par le biais de mécanismes de rétroaction.

L'objectif principal de cette recherche est d'étudier l'existence, l'unicité et la régularité des solutions et leur comportement asymptotique pour les systèmes de poutres de *Timoshenko, Bresse et Rao-Nakra* (modèles).

L'une des théories les plus couramment utilisées est la théorie des poutres.

2.1.2 Théorie des poutres

La stabilisation des poutres flexibles est devenue un domaine de recherche important, étant donné leur utilisation répandue dans diverses applications, notamment dans l'aérospatiale et les structures civiles.

Les poutres sont certainement les modèles mathématiques les plus utilisés par les ingénieurs structurels et jouent un rôle crucial dans leurs différents projets de construction. En raison de leur simplicité et de leur efficacité, les poutres sont devenues la pierre angulaire de la conception structurelle, permettant aux ingénieurs d'analyser et de comprendre le comportement de différents types de structures avec rapidité et précision.

L'analyse des modèles de poutres soumises à des charges internes ou externes est cruciale pour faire progresser les pratiques d'ingénierie, car les poutres sont couramment utilisées dans la conception de structures flexibles dans divers domaines, y compris l'ingénierie structurelle et mécanique. Parmi les exemples d'applications, citons les ponts, les routes, les bâtiments, les avions, les robots et bien d'autres encore.



Figure 1.7 – Construction de ponts routiers à poutres.

Source: <https://www.steel-bridges.com/highway-bridge-composite-beam.html>



Figure 1.8 – Poutre d'un avion.

Source: <https://slideplayer.fr/slide/13960149/>

Dans la section suivante, nous présentons plusieurs modèles de théorie des faisceaux :

2.1.3 Certains modèles de poutres

Le terme "poutre" désigne un objet dont la longueur est importante par rapport aux dimensions transversales. La théorie des poutres est un modèle utilisé dans le domaine de la résistance des matériaux. Cette théorie, dans le cas de petites déflexions, est classée en deux groupes principaux. La première ignore les déformations de cisaillement tandis que l'autre prend en compte cet effet.

Théorie d'Euler-Bernoulli :

Cette théorie est la plus ancienne et la plus simple des théories classiques de la flexion des poutres. Elle simplifie la théorie linéaire de l'élasticité et permet de calculer la capacité de charge et les caractéristiques de flexion des poutres. Elle s'applique spécifiquement au scénario dans lequel les poutres subissent de petites déformations et sont soumises uniquement à des charges latérales. Cette théorie suppose que la section transversale de la poutre reste perpendiculaire à l'axe neutre, même après déformation. La théorie des poutres d'Euler-Bernoulli convient aux poutres minces. En ignorant les effets de la déformation en cisaillement et de l'inertie rotative, elle est donc un cas particulier de la

théorie des poutres de Timoshenko.

Théorie de Timoshenko :

En 1921, le physicien et ingénieur Stephen Timoshenko a développé le modèle de faisceau de Timoshenko dans [102]. Contrairement à la poutre Euler-Bernoulli, le modèle mathématique de la poutre Timoshenko prend en considération à la fois les effets de la déformation en cisaillement et de la flexion rotative, ce qui le rend apte à décrire le comportement des poutres épaisses. Ce modèle est particulièrement utile dans les cas où la poutre est soumise à des charges transversales importantes, telles que les charges de torsion. Les applications de la poutre de Timoshenko peuvent être trouvées dans diverses disciplines de l'ingénierie, y compris l'ingénierie structurelle et mécanique, telles que les ponts, les constructions routières, les bâtiments, les avions et de nombreux autres domaines.

Le modèle mathématique de la vibration transversale d'une poutre de Timoshenko peut être décrit par le système suivant d'équations aux dérivées partielles couplées (pour plus de détails, voir, par exemple, [103, 104]) :

$$\begin{cases} \rho_1 u_{tt} - k_1 (u_x + y)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 y_{tt} - k_2 y_{xx} + k_1 (u_x + y) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.54)$$

où t est la variable temps, x est la coordonnée spatiale le long de la poutre, de longueur L , dans sa position d'équilibre. Avec $\rho_1 := \rho A$, $\rho_2 := \rho I$, $k_1 := KAG$, $k_2 := EI$, où ρ la densité en masse par unité de volume, A l'aire de la section transversale, I le moment d'inertie de l'aire par rapport à l'axe central de flexion, K le coefficient de cisaillement pour mesurer la rigidité des matériaux ($K < 1$), E le module d'élasticité de Young, G le module de cisaillement ou module de rigidité. Les fonctions $S := k_1 (u_x + y)$ et $M := k_2 y_x$ représentent le moment de flexion et l'effort de cisaillement. Les fonctions u et y représentent respectivement le déplacement transversal et l'angle de rotation d'un filament de la poutre. Voir, par exemple [46, 53, 54, 53, 55, 56, 57, 58] et les références qui y figurent pour divers résultats de stabilité de (1.54) sous l'effet de mémoires (finies ou infinies) ou d'amortissements par frottement. Pour un domaine non borné (\mathbb{R} au lieu de $(0, L)$), nous renvoyons le lecteur à [53] et les références qui y figurent.

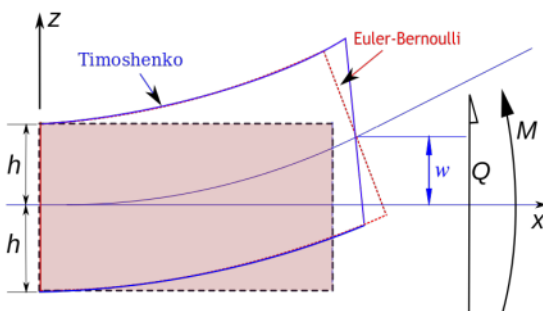


Figure 1.9 – Déformation d'une poutre de Timoshenko (bleu) comparée à celle d'une poutre d'Euler-Bernoulli (rouge).

Source: <https://upload.wikimedia.org/wikipedia/commons/thumb/5/57/TimoshenkoBeam.svg/1200px-TimoshenkoBeam.svg.png?20150802215617>

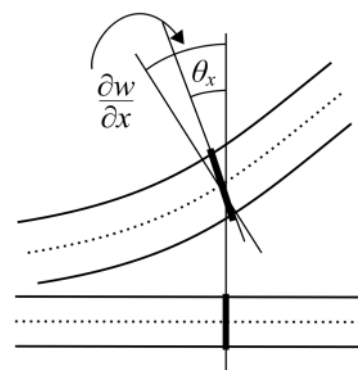


Figure 1.10 – Déformation d'une poutre de Timoshenko. La normale tourne d'une quantité θ_x qui n'est pas égale à $\frac{dw}{dx}$, contrairement à l'hypothèse d'Euler-Bernoulli (la normale reste normale).

Source: https://upload.wikimedia.org/wikipedia/commons/thumb/e/4/Plate_theory.svg/800px-Plate_theory.svg.png

Théorie de Bresse :

Les systèmes de Bresse, du nom de l'ingénieur français Jacques Antoine Charles Bresse, sont un ensemble de systèmes et de méthodologies bien connus dans le domaine du génie civil. L'une des contributions notables de Bresse a été la construction de ponts. Il a introduit le concept de ponts métalliques en arc, qui offraient des avantages en termes de coût, d'intégrité structurelle et d'esthétique. Les ponts métalliques en arc conçus par Bresse ont révolutionné l'ingénierie des ponts et ont été largement adoptés en France et dans d'autres parties du monde.

En 1993-1994, Lagnese et al. dans [70, 71] ont développé un modèle général de poutres élastiques non linéaires à 3 dimensions. Un cas particulier de ce modèle représente un modèle linéaire couplant 3 équations d'ondes. Ce modèle spécifique décrit le mouvement d'une poutre élastique plane lorsqu'elle est soumise à de petites déformations. Il s'agit du système de Bresse, également connu sous le nom de la poutre en arc circulaire, dérivé à l'origine par Bresse, en 1859, dans [28] et est donné par les équations suivantes :

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\omega)_x - k_0 l(\omega_x - l\varphi) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\omega) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_1 \omega_{tt} - k_0(\omega_x - l\varphi)_x + kl(\varphi_x + \psi + l\omega) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.55)$$

où $N := k_0(\omega_x - l\varphi)$ est la force axiale, $Q := k(\varphi_x + \psi + l\omega)$ est la force de cisaillement et $M := b\psi_x$ est le moment de flexion. Les fonctions φ , ω et ψ désignent respectivement les déplacements vertical, longitudinal et l'angle de cisaillement. En outre, $\rho_1 = \rho A$, $\rho_2 = \rho I$, $k_0 = EA$, $k = k'GA$, $b = EI$ et $l = R^{-1}$, où ρ représente la densité, E est le module d'élasticité, G est le module de cisaillement, k' est le facteur de cisaillement, A est l'aire de la section transversale, I est le second moment de l'aire de la section transversale et R est le rayon de courbure de la poutre, et nous supposons que toutes ces quantités sont positives. Nous notons que lorsque $R \rightarrow +\infty$, alors $l \rightarrow 0$ et le modèle de Bresse se réduit aux équations bien connues de la poutre de Timoshenko. Ainsi, le système de Bresse est plus général que celui de Timoshenko où le déplacement longitudinal ω n'est pas pris en compte et où la courbure initiale est supposée nulle. Nous renvoyons les lecteurs, par exemple, à [1, 2, 25, 39, 42, 49, 50, 51, 52] et les références qui y figurent pour la stabilité de (1.55) dans des domaines bornés en utilisant différents mécanismes de contrôle.

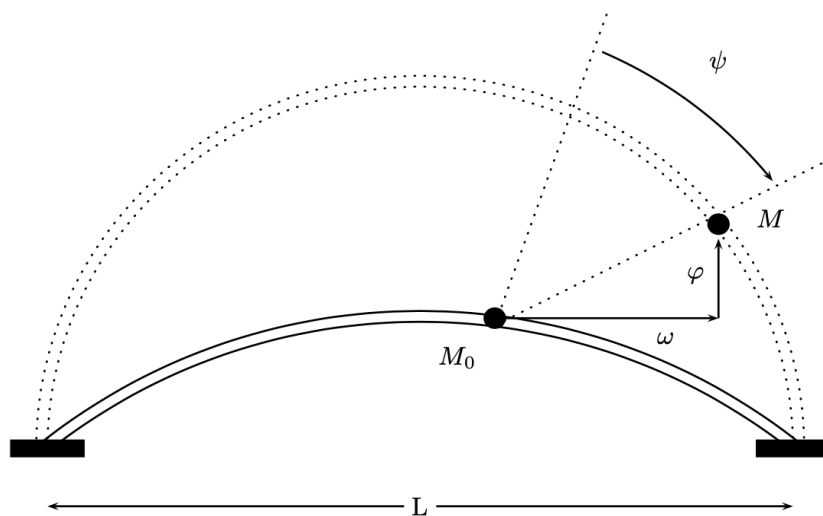


Figure 1.11 – After deformation the particle M_0 of the beam take the position M .
Source: [33]

Poutre Rao-Nakra :

La poutre sandwich classique est un modèle d'ingénierie pour une poutre à trois couches composée de deux plaques frontales rigides et d'une couche centrale intérieure beaucoup plus souple. De nombreux modèles de cette poutre et plaque stratifiée à trois couches ont été proposés au cours des six dernières décennies [82, 91, 111].

La poutre Rao-Nakra est un modèle mathématique utilisé pour analyser le comportement dynamique des poutres soumises à une charge dynamique. Elle porte le nom de K. Rama Rao et Birendra Kumar Nakra, qui ont développé cette théorie des poutres dans le domaine de l'ingénierie structurelle. La théorie des poutres de Rao-Nakra fournit une approche raffinée pour analyser la réponse des poutres sous des charges dynamiques, en prenant en compte les effets de la déformation en cisaillement et de l'inertie rotative. Contrairement aux théories plus simples, telles que celles d'Euler-Bernoulli ou de Timoshenko, la théorie des poutres de Rao-Nakra prend en compte le couplage entre les déformations axiales, de flexion et de torsion, ce qui la rend adaptée à l'analyse d'un plus grand nombre de structures de poutres.

Le modèle proposé par Rao et Nakra [91] incorpore l'inertie de rotation dans chaque couche et l'inertie longitudinale, en plus de l'inertie transversale. Ce modèle suppose que des déplacements linéaires continus par morceaux sont présents dans toutes les sections transversales et il applique l'hypothèse de Kirchhoff aux plaques frontales. La modélisation tient compte des forces d'inertie transversales, longitudinales et de rotation.

Le modèle de Rao-Nakra est dérivé en utilisant des hypothèses de poutre de Timoshenko pour la couche centrale, une hypothèse de "non-glissement" pour les déplacements le long de l'interface et des hypothèses de poutre d'Euler-Bernoulli pour les couches de la plaque frontale. En 1999, Liu-Trogon-Yong [73] a développé le modèle de poutre de Rao-Nakra généralisé suivant, où l'effet de cisaillement des couches inférieure et supérieure est pris en compte :

$$\begin{cases} \rho_1 h_1 u_{tt}^1 - E_1 h_1 u_{xx}^1 - \tau = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_1 I_1 y_{tt}^1 - E_1 I_1 y_{xx}^1 - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - E_3 h_3 u_{xx}^3 + \tau = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 I_3 y_{tt}^3 - E_3 I_3 y_{xx}^3 - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+. \end{cases} \quad (1.56)$$

Ici, u^1, y^1, u^3 et y^3 sont, respectivement, le déplacement longitudinal et l'angle de cisaillement des couches supérieure et inférieure ; ω est le déplacement transversal de la poutre et τ est la contrainte de cisaillement dans la couche centrale définie par

$$\tau = -u^1 - \frac{h_1}{2} y^1 + h_2 \omega_x + u^3 - \frac{h_3}{2} y^3.$$

Les paramètres physiques h_i, ρ_i, E_i, G_i et I_i sont positifs et représentent l'épaisseur, la densité, le module de Young, le module de cisaillement et les moments d'inertie de la $i^{\text{ème}}$ couche, pour $i = 1, 2, 3$, respectivement. En outre, $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$, $EI = E_1 I_1 + E_3 I_3$ et le module de cisaillement $G_2 = \frac{E_2}{2(1+\nu)}$ avec $-1 < \nu < \frac{1}{2}$ étant le coefficient de Poisson (voir figure 1.12).

Si l'on néglige l'inertie de rotation et le cisaillement transversal des couches supérieure et inférieure, les équations (1.56)₂ et (1.56)₅ se réduisent à l'hypothèse familière d'Euler-Bernoulli $y^1 - \omega_x = y^3 - \omega_x = 0$.

Plusieurs modèles ont été dérivés de (1.56) et étudiés dans la littérature en négligeant certains composants et/ou paramètres et/ou en considérant certaines connexions entre eux, comme la poutre sand-

wich de Rao-Nakra [91] :

$$\begin{cases} \rho_1 h_1 u_{tt}^1 - E_1 h_1 u_{xx}^1 - \frac{G_2}{h_2} (-u^1 + u^3 + \alpha \omega_x) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - E_3 h_3 u_{xx}^3 + \frac{G_2}{h_2} (-u^1 + u^3 + \alpha \omega_x) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + EI \omega_{xxxx} - \frac{G_2 \alpha}{h_2} (-u^1 + u^3 + \alpha \omega_x)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.57)$$

qui est obtenu lorsque nous considérons que le matériau du noyau est linéairement élastique, c'est-à-dire $\tau = 2G_2\gamma$, avec la déformation de cisaillement

$$\gamma = \frac{1}{2h_2} (-u^1 + u^3 + \alpha \omega_x) \quad \text{and} \quad \alpha = h_2 + \frac{1}{2} (h_1 + h_3),$$

où le module de cisaillement $G_2 = \frac{E_2}{2(1+\nu)}$ avec $-1 < \nu < \frac{1}{2}$ étant le coefficient de Poisson.

La poutre laminée [65]:

$$\begin{cases} \rho \omega_{tt} + G(\psi - \omega_x)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ I_p(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - u_x) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ 3I_p s_{tt} - 3D s_{xx} + 3G(\psi - \omega_x) + 4\mu s + 4\delta s_t = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.58)$$

qui est obtenue lorsque le mouvement d'extension des couches inférieure et supérieure est négligé.

Le modèle Mead-Markus [82]:

$$\begin{cases} \rho h \omega_{tt} + EI \omega_{xxxx} - G_2 \alpha \gamma = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ 2h_2 \gamma_{xx} - G_2 d \gamma - \alpha \omega_{xxx} = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (1.59)$$

que l'on obtient si l'on néglige également les forces d'extension dans les couches supérieure et inférieure et qui peut être simplifié en une EDP d'ordre six pour ω .

Pour le bien posé et la stabilité de (1.57), (1.58) et (1.59), nous renvoyons les lecteurs, par exemple, à [19, 20, 53, 59, 62, 73, 92, 105] et les références qui y figurent. Pour un domaine non borné, voir [40, 41, 45, 47] et leurs références.

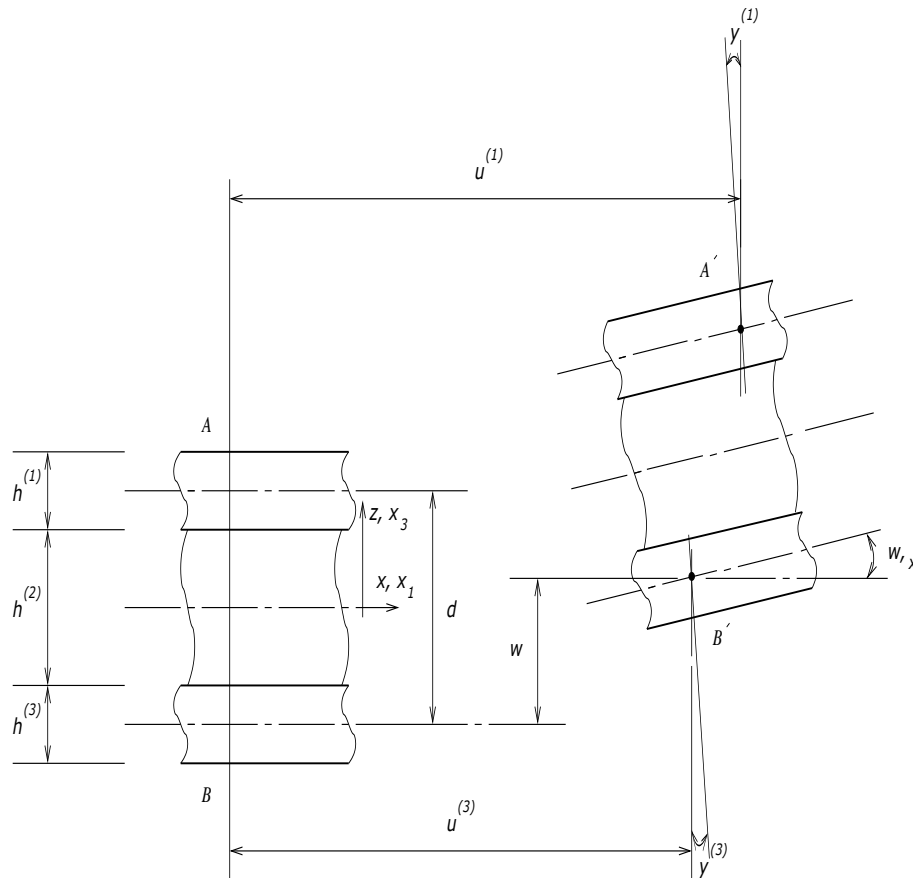


Figure 1.12 – Model describing the generalized Rao-Nakra beam.
Source: [10]

Dans la section suivante, nous approfondissons le rôle crucial joué par les mécanismes d'amortissement et citons des types spécifiques, en soulignant que ceux qui ont été choisis, à savoir l'amortissement visqueux, l'amortissement de Kelvin-Voigt et l'amortissement fractionnaire, sont au cœur de notre étude.

2.2 Mécanismes d'amortissement

Les structures mécaniques telles que les poutres et les plaques jouent un rôle crucial dans la vie moderne, et leurs caractéristiques vibratoires sont étudiées de manière approfondie par de nombreux chercheurs. Cependant, ces vibrations peuvent être indésirables en raison de leur capacité à causer des dommages et des destructions. Afin de limiter ces vibrations, plusieurs amortissements peuvent être ajoutés, tels que les amortissements par friction (visqueux), Kelvin-Voigt, les délais, l'historique (mémoire infinie), la mémoire finie, la conduction thermique (couplage avec certaines équations de la chaleur) et les amortissements fractionnaires.

La viscoélasticité est la propriété des matériaux qui présentent un comportement à la fois visqueux et élastique lorsqu'ils sont soumis à une déformation. Dans le contexte des études sur les vibrations, divers modèles mathématiques sont utilisés pour représenter l'amortissement physique. Les deux types d'amortissement les plus courants sont l'amortissement visqueux linéaire et l'amortissement de Kelvin-Voigt, qui entrent tous deux dans la catégorie de l'amortissement proportionnel. L'amortissement visqueux est généralement associé à des forces de frottement externes, telles que la résistance de

l'air agissant sur des structures vibrantes, d'où son nom "d'amortissement externe". En revanche, l'amortissement Kelvin-Voigt résulte du frottement interne dans le matériau des structures vibrantes, ce qui lui vaut le nom "d'amortissement interne".

L'amortissement fractionnaire a fait l'objet d'une attention considérable ces dernières années et a trouvé de nombreuses applications dans divers domaines scientifiques. Le type d'amortissement d'ordre fractionnaire n'est pas seulement significatif d'un point de vue théorique, il est également important pour les applications pratiques. Il apparaît naturellement dans un large éventail de phénomènes physiques, chimiques, biologiques et écologiques, comme le montrent la référence [80] et la vaste bibliographie qu'elle contient. L'amortissement fractionnaire est utilisé pour décrire efficacement les propriétés de mémoire et d'hérédité de divers matériaux et processus, améliorant ainsi notre compréhension et notre modélisation de systèmes complexes dans divers domaines.

Le concept de mécanismes d'amortissement indirect a été introduit pour la première fois par Russell dans la référence [95]. Depuis lors, il a continué à captiver l'intérêt de nombreux chercheurs. En particulier, l'accent mis sur les cas où une seule équation d'un système couplé est amortie a conduit à l'émergence des problèmes de stabilisation "indirecte". Ces problèmes ont été initiés et explorés dans [15, 16, 17] et ont depuis fait l'objet d'études plus approfondies de la part de divers auteurs, comme en témoignent les études réalisées en [18, 31, 38, 43, 76], et d'autres références complètes.

2.3 Théorie de la stabilité

Les questions de stabilité ont motivé l'introduction de nouveaux concepts mathématiques dans l'ingénierie, en particulier dans l'ingénierie de contrôle. La théorie de la stabilité intéresse depuis longtemps les mathématiciens et les astronomes et a eu un impact stimulant dans ces domaines.

La stabilisation a pour but d'atténuer les vibrations par rétro-action, elle consiste donc à garantir la décroissance de l'énergie des solutions vers 0 plus ou moins rapidement par un mécanisme de dissipation.

Plus précisément, le problème de stabilisation auquel nous nous intéressons consiste à déterminer le comportement asymptotique de l'énergie E , à étudier sa limite afin de déterminer si cette limite est nulle ou non, et si cette limite est nulle, à donner une estimation de sa vitesse de décroissance de l'énergie vers zéro.

Plusieurs types de stabilité peuvent être étudiés. Le premier type consiste simplement à analyser la décroissance de l'énergie des solutions vers zéro, c'est-à-dire

$$E(t) \longrightarrow 0, \quad \text{quand } t \longrightarrow +\infty.$$

C'est ce qu'on appelle la stabilisation forte.

Pour la seconde, on s'intéresse à la décroissance la plus rapide de l'énergie, c'est-à-dire lorsqu'elle tend vers 0 de manière exponentielle, soit

$$E(t) \leq C e^{-\delta t}, \quad \forall t \geq 0,$$

où C et δ sont des constantes positives avec C qui dépend des données initiales.

Quant au troisième, nous étudions des situations intermédiaires dans lesquelles la décroissance des solutions n'est pas exponentielle, mais de type polynomial, par exemple,

$$E(t) \leq C t^{-\alpha}, \quad \forall t > 0,$$

où C et α sont des constantes positives avec C qui dépend des données initiales.

2.4 Objectifs et résultats de la thèse

Au cours des dernières décennies, des efforts de recherche considérables ont été consacrés à l'étude de l'existence de solutions à l'échelle locale et mondiale, ainsi qu'à l'étude du comportement à long terme des solutions pour différents problèmes.

A cet égard, l'objectif de cette thèse est d'étudier le bien posé et la stabilisation de certains systèmes localement couplés avec différents types d'amortissements locaux discontinus sur un domaine borné en utilisant la théorie des semigroupes et la méthode de l'énergie. Nous étudions également la stabilisation du système de Bresse avec un amortissement frictionnel dans la ligne entière \mathbb{R} en utilisant l'analyse de Fourier.

Nous allons maintenant présenter en détail les systèmes étudiés dans cette thèse et les principaux résultats obtenus. Ces résultats seront présentés plus en détail dans les chapitres 3, 4 et 5.

2.4.1 Préliminaires

Dans cette partie, nous rappelons quelques résultats bien connus sur les semigroupes, en particulier les résultats et théorèmes relatifs à la stabilité forte, exponentielle, polynomiale et analytique d'un C_0 -semigroupe.

2.4.2 Polynomial stability of a transmission problem involving Timoshenko systems with fractional Kelvin-Voigt damping (Chapitre 3)

Dans cette partie, nous considérons le bien posé et la stabilisation d'un système de Timoshenko unidimensionnel avec un amortissement fractionnaire interne localisé de Kelvin-Voigt dans un domaine borné sous des conditions aux limites entièrement Dirichlet ou Dirichlet-Neumann. Le système est régi par les équations aux dérivées partielles suivantes :

$$\begin{cases} \rho_1 u_{tt} - [k_1(u_x + y) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y)]_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 y_{tt} - [k_2 y_x + D_2(x)\partial_t^{\alpha,\eta} y_x]_x + k_1(u_x + y) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+ \end{cases} \quad (1.60)$$

sous les conditions initiales suivantes :

$$\begin{cases} u(x, 0) = u_0(x), & u_t(x, 0) = u_1(x), & x \in (0, L), \\ y(x, 0) = y_0(x), & y_t(x, 0) = y_1(x), & x \in (0, L) \end{cases} \quad (1.61)$$

et les conditions aux bords suivantes :

$$u(0, t) = y(0, t) = u(L, t) = y(L, t) = 0, \quad t \in \mathbb{R}_+ \quad (1.62)$$

et

$$u(0, t) = y_x(0, t) = u(L, t) = y_x(L, t) = 0, \quad t \in \mathbb{R}_+. \quad (1.63)$$

Les coefficients ρ_1 , ρ_2 , k_1 et k_2 sont des constantes positives, η est une constante non négative et α est dans $(0, 1)$. Nous supposons qu'il existe $0 \leq a_1 < b_1 \leq L$, $0 \leq a_2 < b_2 \leq L$ et deux constantes positives d_1 et d_2 telles que

$$D_1(x) = \begin{cases} d_1, & x \in (a_1, b_1), \\ 0, & x \in (0, a_1) \cup (b_1, L) \end{cases} \quad \text{and} \quad D_2(x) = \begin{cases} d_2, & x \in (a_2, b_2), \\ 0, & x \in (0, a_2) \cup (b_2, L). \end{cases}$$

La dérivée fractionnaire de Caputo $\partial_t^{\alpha,\eta}$ d'ordre $\alpha \in (0, 1)$ par rapport à la variable temps t est définie par

$$[D^{\alpha,\eta}\omega](x, t) = \partial_t^{\alpha,\eta}\omega(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(x, s) ds, \quad (1.64)$$

où Γ désigne la fonction Gamma. Nous allons étudier le bien posé et la stabilité du système (1.60)-(1.61) avec les conditions aux limites (1.62) ou (1.63) en distinguant trois cas. Dans le premier, nous considérons un système entièrement amorti; c'est à dire les deux équations sont effectivement amorties. Par contre, dans les deux autres cas, nous supposons que le système est partiellement amorti; *i.e.* seule une de ces équations est effectivement amortie. A cette fin, nous introduisons les hypothèses suivantes sur les coefficients d'amortissement $D_j(x); j = 1, 2$:

$$D_1 \neq 0 \quad \text{and} \quad D_2 \neq 0 \quad (A_1)$$

ou

$$D_1 = 0 \quad \text{and} \quad D_2 \neq 0 \quad (A_2)$$

ou

$$D_1 \neq 0 \quad \text{and} \quad D_2 = 0. \quad (A_3)$$

Tout d'abord, nous devons reformuler le système dans un modèle augmenté. Pour ce faire, nous rappelons le théorème 2 énoncé dans [81].

Theorem 2.1. Soit $\alpha \in (0, 1)$, $\eta \geq 0$ et $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ (définie presque partout sur \mathbb{R}). La relation entre l'entrée V et la sortie O du système suivant :

$$\omega_t(x, \xi, t) + (\xi^2 + \eta) \omega(x, \xi, t) - \mu(\xi)V(x, t) = 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+, \quad (1.65)$$

$$\omega(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R}, \quad (1.66)$$

$$O(x, t) - \kappa(\alpha) \int_{\mathbb{R}} \mu(\xi) \omega(x, \xi, t) d\xi = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+ \quad (1.67)$$

est donnée par

$$O = I^{1-\alpha,\eta}V, \quad (1.68)$$

where

$$[I^{\alpha,\eta}V](x, t) = \int_0^t \frac{(t-\tau)^{\alpha-1} e^{-\eta(t-\tau)}}{\Gamma(\alpha)} V(x, \tau) d\tau \quad \text{and} \quad \kappa(\alpha) = \frac{\sin(\alpha\pi)}{\pi}.$$

En prenant les entrées

$$V_1(x, t) = \sqrt{D_1(x)}(u_{xt}(x, t) + y_t(x, t))$$

et

$$V_2(x, t) = \sqrt{D_2(x)}y_{xt}(x, t)$$

dans le Théorème 2.1, on en déduit que le système (1.60)-(1.61) avec des conditions aux bords (1.62)

ou (1.63) peut être transformé en un modèle augmenté :

$$\rho_1 u_{tt} - \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi \right)_x = 0, \quad (1.69)$$

$$\rho_2 y_{tt} - \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi, t) d\xi \right)_x \quad (1.70)$$

$$+ k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi = 0,$$

$$\omega_t^1(x, \xi, t) + (\xi^2 + \eta) \omega^1(x, \xi, t) - \sqrt{D_1(x)} (u_{xt} + y_t) \mu(\xi) = 0, \quad (1.71)$$

$$\omega_t^2(x, \xi, t) + (\xi^2 + \eta) \omega^2(x, \xi, t) - \sqrt{D_2(x)} y_{tx} \mu(\xi) = 0, \quad (1.72)$$

avec les conditions initiales suivantes :

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, L), \quad (1.73)$$

$$y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, L), \quad (1.74)$$

$$\omega^1(x, \xi, 0) = \omega^2(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R} \quad (1.75)$$

et avec des conditions aux bords entièrement de type Dirichlet

$$u(0) = u(L) = y(0) = y(L) = 0 \quad (1.76)$$

ou avec des conditions aux bords de type Dirichlet-Neumann

$$u(0) = u(L) = y_x(0) = y_x(L) = 0, \quad \omega^2(0, \xi, t) = \omega^2(L, \xi, t) = 0. \quad (1.77)$$

Nous définissons l'énergie du système (1.69)-(1.75) avec les conditions aux limites (1.76) ou (1.77) par

$$E(t) = \frac{1}{2} \int_0^L \left(\rho_1 |u_t|^2 + \rho_2 |y_t|^2 + k_1 |u_x + y|^2 + k_2 |y_x|^2 \right) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} \left(|\omega^1|^2 + |\omega^2|^2 \right) d\xi dx.$$

Soit $(u, u_t, y, y_t, \omega^1, \omega^2)$ une solution régulière du système (1.69)-(1.77). Par quelques calculs directs, nous déduisons que l'énergie E satisfait l'estimation suivante :

$$E'(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) \left(|\omega^1|^2 + |\omega^2|^2 \right) d\xi dx. \quad (1.78)$$

Puisque $\alpha \in (0, 1)$, alors $\kappa(\alpha) > 0$ et donc $E'(t) \leq 0$, ce qui implique que le système (1.69)-(1.75) soumis aux conditions aux bords (1.76) ou (1.77) est dissipatif dans le sens où son énergie est décroissante par rapport à la variable temps t .

Nous procédons à l'introduction des espaces énergétiques de Hilbert \mathcal{H}_1 et \mathcal{H}_2 donnés par

$$\mathcal{H}_1 = \left(H_0^1(0, L) \times L^2(0, L) \right)^2 \times \mathcal{W}^2,$$

$$\mathcal{H}_2 = H_0^1(0, L) \times L^2(0, L) \times H_*^1(0, L) \times L^2(0, L) \times \mathcal{W}^2,$$

où

$$\mathcal{W} = L^2((0, L) \times \mathbb{R})$$

et

$$H_*^1(0, L) = \left\{ f \in H^1(0, L) \mid \int_0^L f dx = 0 \right\}.$$

Il est facile de vérifier que l'espace $H_*^1(0, L)$ est un espace de Hilbert sur \mathbb{C} équipé de la norme

$$\|u\|_{H_*^1(0,L)}^2 = \|u_x\|^2,$$

où $\|\cdot\|$ dénote la norme usuelle de $L^2(0, L)$. Les deux espaces d'énergie \mathcal{H}_1 et \mathcal{H}_2 sont équipés du produit intérieur défini par

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}_j} = & \rho_1 \int_0^L v \bar{\tilde{v}} dx + \rho_2 \int_0^L z \bar{\tilde{z}} dx + k_1 \int_0^L (u_x + y) (\overline{\tilde{u}_x + \tilde{y}}) dx \\ & + k_2 \int_0^L y_x \bar{\tilde{y}_x} dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \left(\omega^1 \bar{\omega^1} + \omega^2 \bar{\omega^2} \right) d\xi dx, \end{aligned}$$

pour tous $U = (u, v, y, z, \omega^1, \omega^2)$ et $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}, \tilde{\omega}^1, \tilde{\omega}^2)$ dans \mathcal{H}_j , $j = 1, 2$. Nous utilisons $\|\cdot\|_{\mathcal{H}_j}$ pour désigner la norme correspondante. Nous définissons maintenant les opérateurs linéaires non bornés suivants \mathcal{A}_j sur \mathcal{H}_j ($j = 1, 2$):

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} \mid v, z \in H_0^1(0, L), \\ \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \in L^2(0, L), \\ \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \in L^2(0, L), \\ -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1(x)} (v_x + z) \mu(\xi), \quad |\xi| \omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2(x)} z_x \mu(\xi), \quad |\xi| \omega^2 \in \mathcal{W} \end{array} \right\},$$

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} \mid v \in H_0^1(0, L), z \in H_*^1(0, L), \\ \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \in L^2(0, L), \\ \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \in L^2(0, L), \\ -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1(x)} (v_x + z) \mu(\xi), \quad |\xi| \omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2(x)} z_x \mu(\xi), \quad |\xi| \omega^2 \in \mathcal{W}, \\ y_x(0, t) = y_x(L, t) = 0 \end{array} \right\}$$

et

$$\mathcal{A}_j U^\top = \begin{pmatrix} v \\ \frac{1}{\rho_1} \left(\kappa_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \\ z \\ \frac{1}{\rho_2} \left(\kappa_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x - \frac{\kappa_1}{\rho_2} (u_x + y) - \frac{\kappa(\alpha)}{\rho_2} \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \\ -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1(x)} (v_x + z) \mu(\xi) \\ -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2(x)} z_x \mu(\xi) \end{pmatrix},$$

pour tous $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ et $j = 1, 2$. Si U est une solution régulière du système (1.69)-(1.75), alors nous transformons ce système en une équation d'évolution du premier ordre sur l'espace d'Hilbert \mathcal{H}_j

$$\begin{cases} U_t(x, t) = \mathcal{A}_j U(x, t), \\ U(x, 0) = U_0(x), \end{cases} \quad (1.79)$$

où

$$U_0(x) = (u_0(x), u_1(x), y_0(x), y_1(x), 0, 0)$$

avec $j = 1, 2$ correspondant aux conditions aux bords (1.76) and (1.77), respectively.

Nous prouvons ensuite que l'opérateur \mathcal{A}_j est m-dissipatif dans l'espace d'énergie \mathcal{H}_j . Par conséquent, grâce au théorème de Lumer-Phillips (voir [88]), nous déduisons que \mathcal{A}_j génère un C_0 -semigroupe de contraction $e^{t\mathcal{A}_j}$ dans \mathcal{H}_j , $j = 1, 2$. Alors la solution du problème d'évolution (1.79) admet la représentation suivante :

$$U(t) = e^{t\mathcal{A}_j} U_0, \quad t \geq 0,$$

et donc le problème (1.79) est bien posé et nous avons le résultat suivant :

Theorem 2.2. Pour tout $U_0 \in \mathcal{H}_j$, le problème (1.79) admet une unique solution faible

$$U \in C(\mathbb{R}_+; \mathcal{H}_j).$$

En outre, si $U_0 \in D(\mathcal{A}_j)$, alors le problème (1.79) admet une unique solution forte

$$U \in C(\mathbb{R}_+; D(\mathcal{A}_j)) \cap C^1(\mathbb{R}_+; \mathcal{H}_j).$$

Remark 2.3. Tous les résultats précédents restent valables même lorsque le système de Timoshenko (1.60) est considéré avec un seul amortissement fractionnaire de Kelvin-Voigt; i.e. $D_1 \equiv 0$ ou $D_2 \equiv 0$.

Deuxièmement, nous utilisons un critère général d'Arendt-Batty [22] pour montrer la stabilité forte du C_0 -semigroupe $e^{t\mathcal{A}_j}$ associé au système de Timoshenko (1.69)-(1.75). Notre résultat principal est le théorème suivant :

Theorem 2.4. Supposons que (A_1) , (A_2) ou (A_3) soit valide. Alors, le C_0 -semigroupe $e^{t\mathcal{A}_j}$ est fortement stable dans l'espace d'énergie \mathcal{H}_j au sens où

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} = 0, \quad \forall U_0 \in \mathcal{H}_j, \quad j = 1, 2.$$

L'argument du théorème 2.4 repose sur les lemmes suivants.

Lemma 2.5. Supposons que l'hypothèse (A_3) tienne. Dans ce cas, nous avons

$$\ker(i\lambda I - \mathcal{A}_j) = \{0\}, \quad \forall \lambda \in \mathbb{R}, \quad j = 1, 2.$$

Lemma 2.6. Supposons que $\eta = 0$ et que l'hypothèse (A_3) soit vérifiée. Alors, l'opérateur $-\mathcal{A}_j$ n'est pas inversible, et par conséquent, $0 \in \sigma(\mathcal{A}_j)$, $j = 1, 2$.

Lemma 2.7. Supposons que l'hypothèse (A_3) vérifiée et que $(\eta, \lambda) \in \mathbb{R}_+^* \times \mathbb{R}$ ou $[\eta = 0 \text{ et } \lambda \in \mathbb{R}^*]$. Alors, $i\lambda I - \mathcal{A}_j$ est surjectif, $j = 1, 2$.

Ainsi, en utilisant le lemme 2.5, nous déduisons directement que \mathcal{A}_j a des valeurs propres imaginaires non pures. D'après les Lemmas 2.5, 2.6 et 2.7 et à l'aide du théorème du graphe fermé de Banach, nous déduisons que

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \phi$$

si $\eta > 0$, et

$$\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$$

si $\eta = 0$. On obtient donc la conclusion en appliquant le théorème 2.7 d'Arendt et Batty.

Troisièmement, nous étudions la stabilité polynomiale du système (1.69)-(1.75) lorsque $\eta > 0$ en considérant trois cas : dans le premier, l'amortissement n'est effectif que sur le moment de flexion. Cependant, dans le deuxième cas, l'amortissement n'est effectif que sur la contrainte de cisaillement. Enfin, dans le troisième cas, l'amortissement est présent à la fois sur la contrainte de cisaillement et le moment de flexion. Pour ce faire, nous utiliserons une approche basée sur le domaine des fréquences avec une méthode de multiplicateur, c'est-à-dire que nous utiliserons le Théorème 2.10. Nos principaux résultats dans cette partie sont les théorèmes suivants :

Theorem 2.8. Supposons que $\eta > 0$ et que l'hypothèse (A₂) soit vérifiée. Alors, pour $j = 1, 2$, il existe $c > 0$ tel que, pour chaque $U_0 \in D(\mathcal{A}_j)$, l'estimation d'énergie suivante est valable :

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (1.80)$$

Theorem 2.9. Supposons que $\eta > 0$ et que l'hypothèse (A₃) soit vérifiée. Alors, pour $j = 1, 2$, il existe $c > 0$ tel que, pour chaque $U_0 \in D(\mathcal{A}_j)$, l'estimation d'énergie suivante est valable :

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (1.81)$$

Pour le troisième cas, nous supposons que l'hypothèse suivante est valable :

$$D_1 \neq 0, \quad D_2 \neq 0, \quad a_1 = a_2 = a \quad \text{et} \quad b_1 = b_2 = b. \quad (A_4)$$

Theorem 2.10. Supposons que $\eta > 0$ et que l'hypothèse (A₄) soit vérifiée. Alors, pour $j = 1, 2$, il existe $c > 0$ tel que, pour chaque $U_0 \in D(\mathcal{A}_j)$, l'estimation d'énergie suivante est valable :

$$E(t) \leq \frac{c}{t^{\frac{4}{2-\alpha}}} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (1.82)$$

2.4.3 Polynomial stability of the generalized Rao-Nakra beam with local Kelvin-Voigt damping (Chapitre 4)

Dans cette partie, nous étudions le bien posé et la stabilité de l'équation de la poutre de Rao-Nakra généralisée. Le système se compose de quatre équations d'ondes pour les déplacements longitudinaux et l'angle de cisaillement des couches supérieure et inférieure et d'une équation de poutre d'Euler-Bernoulli pour le déplacement transversal. Dans ce chapitre, nous nous intéressons donc à l'étude du comportement asymptotique de la solution du modèle de poutre de Rao-Nakra généralisé suivant, avec amortissement de Kelvin-Voigt et sous diverses conditions aux limites :

$$\left\{ \begin{array}{l} \rho_1 h_1 u_{tt}^1 - (E_1 h_1 u_x^1 + D_1(x) u_{tx}^1)_x - \tau = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_1 I_1 y_{tt}^1 - (E_1 I_1 y_x^1 + D_2(x) y_{tx}^1)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + (EI \omega_{xx} + D_3(x) \omega_{txx})_{xx} - G_1 h_1 (\omega_x + y^1)_x \\ \quad - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - (E_3 h_3 u_x^3 + D_4(x) u_{tx}^3)_x + \tau = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_3 I_3 y_{tt}^3 - (E_3 I_3 y_x^3 + D_5(x) y_{tx}^3)_x - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+. \end{array} \right. \quad (1.83)$$

Le système (1.83) est soumis aux conditions initiales suivantes :

$$\begin{cases} u^1(x, 0) = u_0^1(x), & u_t^1(x, 0) = u_1^1(x), & x \in (0, L), \\ u^3(x, 0) = u_0^3(x), & u_t^3(x, 0) = u_1^3(x), & x \in (0, L), \\ \omega(x, 0) = \omega_0(x), & \omega_t(x, 0) = \omega_1(x), & x \in (0, L), \\ y^1(x, 0) = y_0^1(x), & y_t^1(x, 0) = y_1^1(x), & x \in (0, L), \\ y^3(x, 0) = u_0^2(x), & y_t^3(x, 0) = u_1^3(x), & x \in (0, L). \end{cases} \quad (1.84)$$

Pour les conditions aux limites, nous supposons que les variables u^1 , u^3 , y^1 et y^3 satisfont à l'une des conditions aux limites suivantes :

Conditions Dirichlet aux bords pour toutes les ondes

$$u^1(0, t) = u^1(L, t) = u^3(0, t) = u^3(L, t) = y^1(0, t) = y^1(L, t) = y^3(0, t) = y^3(L, t) = 0. \quad (DD)$$

Conditions Neumann aux bords pour toutes les ondes

$$u_x^1(0, t) = u_x^1(L, t) = u_x^3(0, t) = u_x^3(L, t) = y_x^1(0, t) = y_x^1(L, t) = y_x^3(0, t) = y_x^3(L, t) = 0. \quad (NN)$$

Et ω satisfait l'une des conditions aux limites suivantes :

La condition aux bords de serrage

$$\omega(0, t) = \omega(L, t) = \omega_x(0, t) = \omega_x(L, t) = 0. \quad (FBC)$$

La condition aux bords suspendue

$$\omega(0, t) = \omega(L, t) = \omega_{xx}(0, t) = \omega_{xx}(L, t) = 0. \quad (HBC)$$

Nous supposons qu'il existe $0 < a < b < L$ et cinq constantes positives d_i , pour $i = 1, \dots, 5$, telles que

$$D_i(x) = \begin{cases} d_0, & x \in (a, b), \\ 0, & x \in (0, a) \cup (b, L). \end{cases}$$

Tout d'abord, nous commençons par étudier l'existence, l'unicité et la régularité de notre système en utilisant l'approche des semigroupes. Soit $(u^1, u^3, \omega, y^1, y^3)$ une solution régulière du système (1.83)-(1.84). Son énergie associée est donnée par

$$\begin{aligned} E(t) = & \frac{1}{2} \left(\rho_1 h_1 \int_0^L |u_t^1|^2 dx + E_1 h_1 \int_0^L |u_x^1|^2 dx + \rho_3 h_3 \int_0^L |u_t^3|^2 dx + E_3 h_3 \int_0^L |u_x^3|^2 dx \right. \\ & + \rho h \int_0^L |\omega_t|^2 dx + EI \int_0^L |\omega_{xx}|^2 dx + \rho_1 I_1 \int_0^L |y_t^1|^2 dx + E_1 I_1 \int_0^L |y_x^1|^2 dx \\ & + \rho_3 I_3 \int_0^L |y_t^3|^2 dx + E_3 I_3 \int_0^L |y_x^3|^2 dx + G_1 h_1 \int_0^L |\omega_x + y^1|^2 dx + \int_0^L |\tau|^2 dx \\ & \left. + G_3 h_3 \int_0^L |\omega_x + y^3|^2 dx \right). \end{aligned} \quad (1.85)$$

Un calcul simple permet d'obtenir le résultat suivant

$$\begin{aligned} \frac{d}{dt} E(t) = & - \int_0^L D_1(x) |u_{tx}^1|^2 dx - \int_0^L D_2(x) |y_{tx}^1|^2 dx - \int_0^L D_3(x) |\omega_{txx}|^2 dx \\ & - \int_0^L D_4(x) |u_{tx}^3|^2 dx - \int_0^L D_5(x) |y_{tx}^3|^2 dx \\ \leq & 0. \end{aligned} \quad (1.86)$$

Ainsi, le système (1.83)-(1.84) sous réserve des conditions aux bords [(DD) et (FBC)] ou [(NN) et (HBC)] est dissipative dans le sens où son énergie est une fonction décroissante par rapport à la variable temps t .

Ensuite, nous introduisons les espaces d'énergie de Hilbert suivants : pour les conditions aux bords (DD) et (FBC), définir l'espace de l'énergie comme

$$\mathcal{H}_1 = \left(H_0^1(0, L) \times L^2(0, L) \right)^2 \times H_0^2(0, L) \times L^2(0, L) \times \left(H_0^1(0, L) \times L^2(0, L) \right)^2.$$

Pour les conditions aux bords (NN) et (HBC), définir l'espace de l'énergie comme

$$\mathcal{H}_2 = \left(H_*^1(0, L) \times L^2(0, L) \right)^2 \times \left(H^2(0, L) \cap H_0^1(0, L) \right) \times L^2(0, L) \times \left(H_*^1(0, L) \times L^2(0, L) \right)^2,$$

où

$$H_*^1(0, L) = \left\{ f \in H^1(0, L) \mid \int_0^L f dx = 0 \right\}.$$

Les deux espaces \mathcal{H}_1 et \mathcal{H}_2 sont munis du produit intérieur qui induit la norme d'énergie

$$\begin{aligned} \|U\|_{\mathcal{H}_j}^2 &= \| (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \|_{\mathcal{H}_j}^2 \\ &= E_1 h_1 \|u_x^1\|^2 + \rho_1 h_1 \|v^1\|^2 + E_3 h_3 \|u_x^3\|^2 + \rho_3 h_3 \|v^3\|^2 dx + EI \|\omega_{xx}\|^2 + \rho h \|\psi\|^2 \\ &\quad + E_1 I_1 \|y_x^1\|^2 + \rho_1 I_1 \|z^1\|^2 + E_3 I_3 \|y_x^3\|^2 + \rho_3 I_3 \|z^3\|^2 + G_1 h_1 \|\omega_x + y^1\|^2 \\ &\quad + \|\tau\|^2 + G_3 h_3 \|\omega_x + y^3\|^2, \quad j = 1, 2; \end{aligned} \tag{1.87}$$

ici $\|\cdot\|$ dénote la norme de $L^2(0, L)$.

Nous définissons maintenant les opérateurs linéaires non bornés suivants $\mathcal{A}_j : D(\mathcal{A}_j) \subset \mathcal{H}_j \rightarrow \mathcal{H}_j$:

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in \mathcal{H}_1 \\ v^1, v^3, z^1, z^3 \in H_0^1(0, L), \psi \in H_0^2(0, L), \\ \left(E_1 h_1 u_x^1 + D_1(x) v_x^1 \right)_x \in L^2(0, L), \left(E_3 h_3 u_x^3 + D_4(x) v_x^3 \right)_x \in L^2(0, L), \\ \left(E_1 I_1 y_x^1 + D_2(x) z_x^1 \right)_x \in L^2(0, L), \left(E_3 I_3 y_x^3 + D_5(x) z_x^3 \right)_x \in L^2(0, L), \\ (EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} \in L^2(0, L) \end{array} \right\},$$

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in \mathcal{H}_2 \\ v^1, v^3, z^1, z^3 \in H_*^1(0, L), \psi \in H^2(0, L) \cap H_0^1(0, L), \\ \left(E_1 h_1 u_x^1 + D_1(x) v_x^1 \right)_x \in L^2(0, L), u_x^1(0, t) = u_x^1(L, t) = 0, \\ \left(E_3 h_3 u_x^3 + D_4(x) v_x^3 \right)_x \in L^2(0, L), u_x^3(0, t) = u_x^3(L, t) = 0, \\ \left(E_1 I_1 y_x^1 + D_2(x) z_x^1 \right)_x \in L^2(0, L), y_x^1(0, t) = y_x^1(L, t) = 0, \\ \left(E_3 I_3 y_x^3 + D_5(x) z_x^3 \right)_x \in L^2(0, L), y_x^3(0, t) = y_x^3(L, t) = 0, \\ (EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} \in L^2(0, L), \omega_{xx}(0, t) = \omega_{xx}(L, t) = 0 \end{array} \right\}$$

et

$$\mathcal{A}_j \begin{pmatrix} u^1 \\ v^1 \\ y^1 \\ z^1 \\ \omega \\ \psi \\ u^3 \\ v^3 \\ y^3 \\ z^3 \end{pmatrix} = \begin{pmatrix} v^1 \\ (\rho_1 h_1)^{-1} [(E_1 h_1 u_x^1 + D_1(x) v_x^1)_x + \tau] \\ z^1 \\ (\rho_1 I_1)^{-1} [(E_1 I_1 y_x^1 + D_2(x) z_x^1)_x + \frac{h_1}{2} \tau - G_1 h_1 (\omega_x + y^1)] \\ \psi \\ (\rho h)^{-1} [-(EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} + G_1 h_1 (\omega_x + y^1)_x + G_3 h_3 (\omega_x + y^3)_x + h_2 \tau_x] \\ v^3 \\ (\rho_3 h_3)^{-1} [(E_3 h_3 u_x^3 + D_4(x) v_x^3)_x - \tau] \\ z^3 \\ (\rho_3 I_3)^{-1} [(E_3 I_3 y_x^3 + D_5(x) z_x^3)_x + \frac{h_3}{2} \tau - G_3 h_3 (\omega_x + y^3)] \end{pmatrix}, \quad (1.88)$$

pour tous $U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in D(\mathcal{A}_j)$ et $j = 1, 2$. Si U est une solution régulière du système (1.83)-(1.84), alors le système peut être réécrit comme une équation d'évolution sur l'espace de Hilbert \mathcal{H}_j sous la forme abstraite suivante :

$$\begin{cases} U_t(x, t) = \mathcal{A}_j U(x, t), \\ U(x, 0) = U_0(x), \end{cases} \quad (1.89)$$

où

$$U_0 = (u_0^1, u_1^1, y_0^1, y_1^1, \omega_0, \omega_1, u_0^3, u_1^3, y_0^3, y_1^3)$$

avec $j = 1, 2$ correspondant aux conditions aux bords [(DD) et (FBC)] ou [(NN) et (HBC)], respectivement.

Ensuite, nous prouvons que l'opérateur \mathcal{A}_j est m-dissipatif dans \mathcal{H}_j , $j = 1, 2$. Ainsi, selon Lummer-Phillips théorème (see [88]), on en déduit que \mathcal{A}_j génère un C_0 -semigroupe de contractions $(e^{t\mathcal{A}_j})_{t \geq 0}$ dans \mathcal{H}_j et le problème est alors bien posé dans \mathcal{H}_j . Nous obtenons donc le résultat suivant :

Theorem 2.11. Soit $U_0 \in \mathcal{H}_j$, alors le système (1.89) admet une unique solution faible U satisfaisant

$$U \in C^0(\mathbb{R}_+, \mathcal{H}_j).$$

En outre, si $U_0 \in D(\mathcal{A}_j)$, alors le problème (1.89) admet une unique solution forte U satisfaisant

$$U \in C^1(\mathbb{R}_+, \mathcal{H}_j) \cap C^0(\mathbb{R}_+, D(\mathcal{A}_j)).$$

Deuxièmement, nous étudions la stabilité de notre système et prouvons que son énergie converge vers zéro lorsque t va à l'infini pour toutes les données initiales dans \mathcal{H}_j . Comme le résolvant de \mathcal{A}_j n'est pas compact, nous devons étudier le spectre complet sur l'axe imaginaire. Nous utilisons un critère général du théorème d'Arendt-Batty dans [24] qui stipule que, dans un espace de Banach réflexif, un C_0 -semigroupe $e^{t\mathcal{A}_j}$ est fortement stable si $e^{t\mathcal{A}_j}$ est borné, \mathcal{A}_j n'a pas de valeurs propres sur l'axe imaginaire et $\sigma(\mathcal{A}_j) \cap i\mathbb{R}$ est dénombrable. Avant d'énoncer notre résultat principal, nous avons besoin des conditions de stabilité suivantes :

$$D_i > d_0 > 0 \quad \text{pour tout } x \in (0, L), \quad i = 1, \dots, 5, \quad (\text{SSC1})$$

$$D_i(x) \geq d_0 > 0 \quad \text{in } (a, b), \quad i = 2, 5, \quad G_1 \neq G_3 \quad \text{et} \quad \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}. \quad (\text{SSC2})$$

Nous allons donc prouver le résultat de stabilité suivant :

Theorem 2.12. Supposons que la condition (SSC1) ou (SSC2) soit vérifiée. Alors, le C_0 -semigroupe de contractions $(e^{t\mathcal{A}_j})_{t \geq 0}$ est fortement stable dans l'espace d'énergie \mathcal{H}_j , $j = 1, 2$; c'est-à-dire

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} = 0, \quad \forall U_0 \in \mathcal{H}_j.$$

Pour démontrer le Théorème (2.12), on a besoin des deux lemmes suivants :

Lemma 2.13. *Supposons que la condition (SSC2) soit vérifiée, on a alors*

$$\ker(i\lambda I - \mathcal{A}_j) = \{0\}, \quad \forall \lambda \in \mathbb{R}, \quad j = 1, 2,$$

où I est l'opérateur identité.

Lemma 2.14. *Supposons que la condition (SSC2) soit vérifiée, alors, pour tout $\lambda \in \mathbb{R}$, on a*

$$\mathcal{R}(i\lambda I - \mathcal{A}_j) = \mathcal{H}_j, \quad j = 1, 2.$$

Ainsi, en utilisant le Lemme 2.13, on en déduit directement que \mathcal{A}_j n'a pas des valeurs propres imaginaires pures. Selon le Lemme 2.14 et à l'aide du théorème du graphe fermé de Banach, nous déduisons que $\sigma(\mathcal{A}_j) \cap i\mathbb{R} = \emptyset$. On obtient donc la conclusion en appliquant le Théorème 2.7 d'Arendt et de Batty.

Il s'agit maintenant d'étudier le taux de décroissance de l'énergie en utilisant une approche basée sur le domaine des fréquences combinée avec la technique du multiplicateur par morceaux dans deux cas qui dépendent de la régularité des fonctions de coefficient D_i , $i = 1, \dots, 5$, de la localisation de leurs supports et de leur nombre.

Dans le cas où les cinq amortissements sont globalement distribués, nous prouvons une stabilité analytique. Notre résultat est le suivant :

Theorem 2.15. Supposons que la condition (SSC1) soit vérifiée et $D_i \in L^\infty(0, L)$, $i = 1, \dots, 5$. Alors, le C_0 -semigroupe $(e^{t\mathcal{A}_j})_{t \geq 0}$ est analytique dans \mathcal{H}_j , pour $j = 1, 2$.

Dans le cas de seulement deux amortissements locaux D_2 et D_5 agissant sur les déplacements de l'angle de cisaillement des couches supérieure et inférieure, c'est-à-dire que l'hypothèse (SSC2) est valable, nous montrons que l'énergie du système décroît de façon polynomiale. Nous prouvons le théorème suivant :

Theorem 2.16. Supposons que la condition (SSC2) soit vérifiée. Le C_0 -semigroupe $(e^{t\mathcal{A}_j})_{t \geq 1}$ est polynomialement stable ; plus précisément, il existe une constante $C > 0$ telle que pour tout U_0 dans $D(\mathcal{A}_j)$, nous avons

$$E(t) \leq \frac{C}{t^{\frac{1}{2}}}, \quad \forall t > 0, \quad \forall U_0 \in D(\mathcal{A}_j). \quad (1.90)$$

2.4.4 Some stability and non-stability results for a Cauchy problem of Bresse-type with frictional damping on the vertical displacement (Chapitre 5)

Dans cette partie, nous étudions la stabilité d'un système de type Bresse sur toute la ligne \mathbb{R} avec un amortissement par frottement n'agissant que sur la première équation (déplacement vertical). Le

système est régi par les équations aux dérivées partielles suivantes :

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k^2l(\omega_x - l\varphi) + \gamma\varphi_t = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi - l\omega) = 0, \\ \omega_{tt} - k^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) = 0, \end{cases} \quad (1.91)$$

avec les données initiales

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)(x), \quad (1.92)$$

où $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ et a, l, γ et k sont des constantes positives. Les fonctions $\omega(x, t)$, $\varphi(x, t)$ et $\psi(x, t)$ sont, respectivement, le déplacement longitudinal, le déplacement vertical et l'angle de rotation du matériau des filaments linéaires.

En effet, les preuves de stabilité pour les systèmes de Bresse dans des domaines bornés reposent généralement sur la méthode de l'énergie en utilisant en particulier l'inégalité de Poincaré, quelques injections de Sobolev et l'imposition de conditions aux limites appropriées. Cependant, ce chapitre se concentre sur la situation où le domaine est \mathbb{R} (donc non borné). Nos preuves sont basées sur des estimations de l'image de Fourier de la solution, le théorème de Plancherel, ainsi que sur une combinaison linéaire appropriée de séries associées à des estimations d'énergie. Ainsi, afin d'établir le taux de décroissance du système Bresse (1.91)-(1.92), nous devons transformer le problème original en système du premier ordre (par rapport à la variable t) en définissant de nouvelles variables. Ensuite, nous appliquons la méthode de l'énergie dans l'espace de Fourier pour prouver quelques estimations ponctuelles qui nous aideront à prouver les estimations de décroissance.

Tout d'abord, nous écrivons le système (1.91)-(1.92) comme un système de Cauchy du premier ordre de la forme

$$\begin{cases} U_t + AU_x + LU = 0, & x \in \mathbb{R}, t > 0, \\ U(x, 0) = U_0(x), & x \in \mathbb{R}, \end{cases} \quad (1.93)$$

où A est une matrice symétrique réelle et L une matrice définie non négative. À cette fin, nous introduisons les variables suivantes :

$$v = \varphi_x - \psi - l\omega, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad \phi = k(\omega_x - l\varphi) \quad \text{and} \quad \eta = \omega_t. \quad (1.94)$$

Par conséquent, le système (1.91)-(1.92) peut être réécrite dans le système du premier ordre suivant :

$$\begin{cases} v_t - u_x + y + l\eta = 0, \\ u_t - v_x - lk\phi + \gamma u = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v = 0, \\ \phi_t - k\eta_x + lku = 0, \\ \eta_t - k\phi_x - lv = 0, \end{cases} \quad (1.95)$$

et les conditions initiales prennent la forme

$$(v, u, z, y, \phi, \eta)(x, 0) = (v_0, u_0, z_0, y_0, \phi_0, \eta_0)(x).$$

Système (1.95) est équivalent au système (1.93) avec

$$U = \begin{pmatrix} v \\ u \\ z \\ y \\ \phi \\ \eta \end{pmatrix}, A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & -k & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & l \\ 0 & \gamma & 0 & 0 & -lk & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & lk & 0 & 0 & 0 & 0 \\ -l & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

et

$$U_0 = (v_0, u_0, z_0, y_0, \phi_0, \eta_0)^t.$$

En prenant la transformée de Fourier de (1.93), par rapport à la variable d'espace x , on obtient le problème de Cauchy suivant d'un système du premier ordre dans l'espace de Fourier :

$$\begin{cases} \hat{U}_t + i\xi A \hat{U} + L \hat{U} = 0, & \xi \in \mathbb{R}, t > 0, \\ \hat{U}(\xi, 0) = \hat{U}_0, & \xi \in \mathbb{R}. \end{cases} \quad (1.96)$$

La solution de (1.96) est donnée par

$$\hat{U}(\xi, t) = e^{t\hat{\Phi}(\xi)} \hat{U}_0(\xi), \quad (1.97)$$

où

$$\hat{\Phi}(\xi) = -(i\xi A + L). \quad (1.98)$$

La première équation du système (1.96) peut être réécrite sous la forme suivante (c'est-à-dire la transformée de Fourier de (1.95)) :

$$\begin{cases} \hat{v}_t - i\xi \hat{u} + \hat{y} + l\hat{\eta} = 0, \\ \hat{u}_t - i\xi \hat{v} - lk\hat{\phi} + \gamma\hat{u} = 0, \\ \hat{z}_t - ai\xi \hat{y} = 0, \\ \hat{y}_t - ai\xi \hat{z} - \hat{v} = 0, \\ \hat{\phi}_t - ki\xi \hat{\eta} + lk\hat{u} = 0, \\ \hat{\eta}_t - ki\xi \hat{\phi} - l\hat{v} = 0. \end{cases} \quad (1.99)$$

Définissons maintenant la fonctionnelle énergétique associée à (1.96) par

$$\hat{E}(\xi, t) = \frac{1}{2} \|\hat{U}(\xi, t)\|^2 = \frac{1}{2} (|\hat{v}|^2 + |\hat{u}|^2 + |\hat{z}|^2 + |\hat{y}|^2 + |\hat{\phi}|^2 + |\hat{\eta}|^2)(\xi, t). \quad (1.100)$$

Soit $(\hat{v}, \hat{u}, \hat{z}, \hat{y}, \hat{\phi}, \hat{\eta})$ la solution de (1.96), alors l'énergie \hat{E} est une fonction décroissante par rapport à t et satisfait, pour tous $t \geq 0$ et $\xi \in \mathbb{R}$,

$$\frac{d}{dt} \hat{E}(\xi, t) = -\gamma |\hat{u}(\xi, t)|^2. \quad (1.101)$$

Ainsi, (1.96) est un système dissipatif.

L'objectif est de présenter et de montrer quelques résultats de stabilité et de non-stabilité pour (1.96) en fonction des conditions suivantes :

$$(a - k)(k - 1) > 0 \quad \text{and} \quad \max \left\{ \frac{l[l^2(a^2 - 1)(k^2 - 1) + k^2]}{l^2(a^2 - k^2)(k^2 - 1) + k^2}, \frac{l(a^2 - 1)}{a^2 - k^2} \right\} < \frac{l^2 a^2 + 1}{l a^2} \quad (1.102)$$

and

$$a = k = \frac{\sqrt{l^2 + 1}}{l}. \quad (1.103)$$

Ensuite, nous montrons une estimation de l'image de Fourier de la solution. Cette estimation jouera un rôle clé dans la démonstration du Théorème 2.18 ci-dessous.

Theorem 2.17. Soit \hat{U} la solution de (1.96).

1. Supposons que (1.102) est vérifié. Alors, il existe $c, \tilde{c} > 0$ tels que

$$\|\hat{U}(\xi, t)\|^2 \leq \tilde{c} e^{-cf(\xi)t} \|\hat{U}_0(\xi)\|^2, \quad \xi \in \mathbb{R}, t \in \mathbb{R}_+, \quad (1.104)$$

où

$$f(\xi) = \frac{\xi^2}{\xi^4 + \xi^2 + 1}. \quad (1.105)$$

2. Supposons que (1.103) est vérifié. Alors, $\|\widehat{U}(\xi, t)\|$ ne converge pas vers zéro lorsque le temps t converge vers l'infini.

Notre principal résultat dans ce chapitre est énoncé dans le théorème suivant :

Theorem 2.18. Supposons que (1.102) est satisfaite. Soit $N, \ell \in \mathbb{N}^*$ tels que $\ell \leq N$, $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$ et U est la solution de (1.93). Alors, pour tout $j = 0, \dots, N - \ell$, il existe $c_0 > 0$ tel que

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/4-j/2} \|U_0\|_{L^1(\mathbb{R})} + c_0 (1+t)^{-\ell/2} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}_+. \quad (1.106)$$

3 List of works collected in the thesis

This thesis consists of three chapters:

- Chapitre 3: Polynomial stability of a transmission problem involving Timoshenko systems with fractional Kelvin-Voigt damping. In: *Mathematical Methods in the Applied Sciences* [60] (doi: <https://doi.org/10.1002/mma.8960>, Aissa Guesmia, **Zeinab Mohamad Ali**, Ali Wehbe and Wael Youssef).
- Chapitre 4: Polynomial stability of the generalized Rao-Nakra beam with local Kelvin-Voigt damping. (In preparation, Aissa Guesmia, **Zeinab Mohamad Ali** and Ali Wehbe).
- Chapitre 5: Some stability and non-stability results for a Cauchy problem of Bresse-type with frictional damping on the vertical displacement. (In preparation, Aissa Guesmia, **Zeinab Mohamad Ali**, Ali Wehbe and Wael Youssef).

Preliminaires

As the analysis in this thesis is based on the semigroup and spectral analysis theories, in this chapter, we will present some well-known results about semigroups, including some theorems about strong, exponential, and polynomial stability of a C_0 -semigroup. All the theorems are stated without proofs, but the relevant references are given. The reader may skip this chapter in the first reading, then refer to it as a reference of related. For more details, see [22, 24, 26, 29, 32, 66, 67, 68, 74, 78, 89].

1 Bounded and unbounded linear operators

We start this chapter by giving some well known results about bounded and unbounded operators. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proofs ([29, 67, 68]).

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces over \mathbb{C} , and H will always denote a Hilbert space equipped with the scalar product $\langle \cdot, \cdot \rangle_H$ and the corresponding norm $\|\cdot\|_H$.

A linear operator $T : E \rightarrow F$ is a transformation which maps linearly E in F , that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall u, v \in E \text{ and } \forall \alpha, \beta \in \mathbb{C}.$$

Definition 2.1. A linear operator $T : E \rightarrow F$ is said to be bounded if there exists $C \geq 0$ such that

$$\|Tu\|_F \leq C\|u\|_E, \quad \forall u \in E.$$

The set of all bounded linear operators from E into F is denoted by $\mathcal{L}(E, F)$. Moreover, the set of all bounded linear operators from E into E is denoted by $\mathcal{L}(E)$.

Definition 2.2. A bounded operator $T \in \mathcal{L}(E, F)$ is said to be compact if for each sequence $(x_n)_{n \in \mathbb{N}} \in E$ with $\|x_n\|_E = 1$ for each $n \in \mathbb{N}$, the sequence $(Tx_n)_{n \in \mathbb{N}}$ has a subsequence which converges in F .

The set of all compact operators from E into F is denoted by $\mathcal{K}(E, F)$. For simplicity one writes $\mathcal{K}(E) = \mathcal{K}(E, E)$.

Definition 2.3. Let $T \in \mathcal{L}(E, F)$, we define

- Range of T by

$$\mathcal{R}(T) = \{Tu : u \in E\} \subset F.$$

- Kernel of T by

$$\ker(T) = \{u \in E : Tu = 0\} \subset E.$$

Theorem 1.1. (Fredholm alternative see Theorem 6.6 in [29]). If $T \in \mathcal{K}(E)$, then

- $\ker(I - T)$ is of a finite dimension (I is the identity operator on E).
- $\mathcal{R}(I - T)$ is closed.
- $\ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$.

Definition 2.4. An unbounded linear operator T from E into F is a pair $(T, D(T))$, consisting of a subspace $D(T) \subset E$ (called the domain of T) and a linear transformation.

$$T : D(T) \subset E \mapsto F.$$

In the case when $E = F$ then we say $(T, D(T))$ is an unbounded linear operator on E . If $D(T) = E$, then $T \in \mathcal{L}(E, F)$.

Definition 2.5. Let $T : D(T) \subset E \mapsto F$ be an unbounded linear operator.

- The range of T is defined by

$$\mathcal{R}(T) = \{Tu : u \in D(T)\} \subset F.$$

- The Kernel of T is defined by

$$\ker(T) = \{u \in D(T) : Tu = 0\} \subset E.$$

- The graph of T is defined by

$$G(T) = \{(u, Tu) : u \in D(T)\} \subset E \times F.$$

Definition 2.6. A map T is said to be closed if $G(T)$ is closed in $E \times F$. The closedness of an unbounded linear operator T can be characterized as follows

if $u_n \in D(T)$ such that $u_n \rightarrow u$ in E and $Tu_n \rightarrow v$ in F , then $u \in D(T)$ and $Tu = v$.

Definition 2.7. Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator.

- The resolvent set of T is defined by

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

- The resolvent of T is defined by

$$R(\lambda, T) = \{(\lambda I - T)^{-1} : \lambda \in \rho(T)\}.$$

- The spectrum set of T is the complement of the resolvent set in \mathbb{C} , denoted by

$$\sigma(T) = \mathbb{C} / \rho(T).$$

Definition 2.8. Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator. We can split the spectrum $\sigma(T)$ of T into three disjoint sets given by

- The punctuel spectrum of T which is defined by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\}\};$$

in this case, λ is called an eigenvalue of T .

- The continuous spectrum of T which is defined by

$$\sigma_c(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) = \{0\}, \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded}\}.$$

- The residual spectrum of T which is defined by

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) = 0 \text{ and } \mathcal{R}(\lambda I - T) \text{ is not dense in } F\}.$$

Definition 2.9. Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator and let λ be an eigenvalue of A . A non-zero element $e \in E$ is called a generalized eigenvector of T associated with the eigenvalue λ , if there exists $n \in \mathbb{N}^*$ such that

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0.$$

If $n = 1$, then e is called an eigenvector.

Definition 2.10. Let $T : D(T) \subset E \mapsto F$ be a closed unbounded linear operator. We say that T has compact resolvent if there exist $\lambda_0 \in \rho(T)$ such that $(\lambda_0 I - T)^{-1}$ is compact.

Theorem 1.2. (see [67]). Let $(T, D(T))$ be a closed unbounded linear operator on H , then the space $(D(T), \|\cdot\|_{D(T)})$ where $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H, \forall u \in D(T)$, is a Banach space.

Theorem 1.3. (see [68]). Let $(T, D(T))$ be a closed unbounded linear operator on H , then $\rho(T)$ is an open set of \mathbb{C} .

2 Semigroups

2.1 Existence, uniqueness and regularity of solution

In this section, we introduce some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t &= \mathcal{A}U, \quad t > 0, \\ U(0) &= U_0, \end{cases} \quad (2.1)$$

where \mathcal{A} is the infinitesimal generator of a C_0 -semigroup $S(t)$ over a Hilbert space H . Let's start by basic definitions and theorems (see [29, 88]).

Let $(X, \|\cdot\|_X)$ be a Banach space and H be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$.

Definition 2.11. A family $(S(t))_{t \geq 0}$ of bounded linear operators in X is called a strong linear continuous semigroup (in short, a C_0 -semigroup) if

- $S(0) = I$ (I is the identity operator on X).
- $S(t + s) = S(t)S(s), \quad \forall t, s \geq 0$.
- For each $u \in H$, $S(t)u$ is continuous in t on $[0, +\infty[$.

Sometimes we also denote $S(t)$ by e^{tA} .

Definition 2.12. For a semigroup $(S(t))_{t \geq 0}$, we define an linear operator \mathcal{A} with domain $D(\mathcal{A})$ consisting of points u such that the limit

$$u\mathcal{A} := \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t}, \quad u \in D(\mathcal{A})$$

exists. Then \mathcal{A} is called the infinitesimal generator of the semigroup $(S(t))_{t \geq 0}$.

Proposition 2.13. (See Theorem 2.2 in [88]). Let $(S(t))_{t \geq 0}$ be a C_0 -semigroup in X . Then there exist a constant $M \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq M e^{\omega t}, \quad \forall t \geq 0.$$

If $\omega = 0$ then the corresponding semigroup is uniformly bounded. Moreover, if $M = 1$, then $(S(t))_{t \geq 0}$ is said to be a C_0 -semigroup of contractions.

Definition 2.14. An unbounded linear operator $(\mathcal{A}, D(\mathcal{A}))$ on H , is said to be dissipative if

$$\Re \langle \mathcal{A}u, u \rangle_H \leq 0, \quad \forall u \in D(\mathcal{A}).$$

Definition 2.15. An unbounded linear operator $(\mathcal{A}, D(\mathcal{A}))$ on X , is said to be m-dissipative if

- \mathcal{A} is a dissipative operator.
- $\exists \lambda_0 > 0$ such that $\mathcal{R}(\lambda_0 I - \mathcal{A}) = X$.

Theorem 2.1. (See Theorem 4.5 in [88]). Let \mathcal{A} be a m-dissipative operator, then

- $\mathcal{R}(\lambda I - \mathcal{A}) = X, \quad \forall \lambda > 0.$
- $]0, \infty[\subseteq \rho(\mathcal{A}).$

Theorem 2.2. (Hille-Yosida; see Theorem 3.1 in [88]). An unbounded linear operator $(\mathcal{A}, D(\mathcal{A}))$ on X is the infinitesimal generator of a C_0 -semigroup of contractions $(S(t))_{t \geq 0}$ if and only if

- \mathcal{A} is closed and $\overline{D(\mathcal{A})} = X$.
- The resolvent set $\rho(\mathcal{A})$ of \mathcal{A} contains \mathbb{R}^+ , and for all $\lambda > 0$,

$$\|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(X)} \leq \lambda^{-1}.$$

Theorem 2.3. (Lumer-Phillips; see Theorem 4.3 in [88]). Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on X , with dense domain $D(\mathcal{A})$ in X . The operator \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions if and only if it is a m-dissipative operator.

Theorem 2.4. (see Theorem 4.6 in [88]). Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on X . If \mathcal{A} is dissipative with $\mathcal{R}(I - \mathcal{A}) = X$, and X is reflexive, then $\overline{D(\mathcal{A})} = X$.

Corollary 2.16. Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on H . The operator \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions if and only if \mathcal{A} is a m-dissipative operator.

Theorem 2.5. Let \mathcal{A} be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space H . If \mathcal{A} is dissipative and $0 \in \rho(\mathcal{A})$, then \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions on H .

Theorem 2.6. (Hille-Yosida; see Theorem 7.4 in [29]). Let $(\mathcal{A}, D(\mathcal{A}))$ be an unbounded linear operator on H . Assume that \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions $(S(t))_{t \geq 0}$.

1. For $U_0 \in D(\mathcal{A})$, the problem (2.1) admits a unique strong solution

$$U(t) = S(t)U_0 \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, H).$$

2. For $U_0 \in H$, the problem (2.1) admits a unique weak solution

$$U(t) \in C^0(\mathbb{R}^+, H).$$

2.2 Stability

In this section, we introduce some definitions about strong, exponential, polynomial and analytic stability of a C_0 -semigroup. Then, we give some results about the stability of C_0 -semigroup. For more details see [22, 24, 26, 32, 66, 74, 78, 89].

Let $(X, \|\cdot\|_X)$ be a Banach space and H be a Hilbert space equipped with the inner product $\langle \cdot, \cdot \rangle_H$ and the induced norm $\|\cdot\|_H$.

Definition 2.17. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on X . We say that the C_0 -semigroup $(S(t))_{t \geq 0}$ is

- Strongly stable if

$$\lim_{t \rightarrow +\infty} \|S(t)u\|_X = 0, \quad \forall u \in X.$$

- Uniformly stable if

$$\lim_{t \rightarrow +\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$

- Exponentially stable if there exist two positive constants M and ϵ such that

$$\|S(t)u\|_X \leq Me^{-\epsilon t} \|u\|_X, \quad \forall t > 0, \forall u \in X.$$

- Polynomially stable if there exist two positive constants C and α such that

$$\|S(t)u\|_X \leq Ct^{-\alpha} \|u\|_X, \quad \forall t > 0, \forall u \in X.$$

The C_0 -semigroup $(S(t))_{t \geq 0}$ is said to be polynomially stable with optimal decay rate $t^{-\alpha}$ (with $\alpha > 0$) if it is polynomially stable with decay rate $t^{-\alpha}$ and, for any $\epsilon > 0$, the semigroup $(S(t))_{t \geq 0}$ does not decay at a rate $t^{-(\alpha+\epsilon)}$.

Proposition 2.18. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on X . The following statements are equivalent:

- $(S(t))_{t \geq 0}$ is uniformly stable.
- $(S(t))_{t \geq 0}$ is exponentially stable.

To show the strong stability of a C_0 -semigroup of contraction $(e^{tA})_{t \geq 0}$, we rely on the following result due to Arendt-Batty [22].

Theorem 2.7. (Arendt and Batty) Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on a reflexive Banach space X . If

- A has no pure imaginary eigenvalues,
- $\sigma(A) \cap i\mathbb{R}$ is countable,

then $S(t)$ is strongly stable.

Remark 2.8. If the resolvent $(I - T)^{-1}$ of T is compact, then $\sigma(T) = \sigma_p(T)$. Thus, the state of Theorem 2.7 lessens to $\sigma_p(A) \cap i\mathbb{R} = \emptyset$.

Next, when the C_0 -semigroup is strongly stable, we look for the necessary and sufficient conditions of exponential stability of a C_0 -semigroup. In case when the C_0 -semigroup is not exponentially stable, we may look for a polynomial one. In fact, exponential and polynomial stability results are obtained using different methods like: multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them. In this chapter, we will review only two methods. The following method is a frequency domain approach method which was introduced by Huang-Prüss (see [66, 89]).

Theorem 2.9. (Huang-Prüss) Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on H . $S(t)$ is uniformly stable if and only if

- $i\mathbb{R} \subset \rho(A)$.
- $\sup_{\beta \in \mathbb{R}} \|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty$.

Moreover, the following result is a frequency domain approach method which was introduced by Batty [24], Borichev [26] and Tomilov, Liu and Rao [74].

Theorem 2.10. (Batty, Borichev and Tomilov, Liu and Rao.) Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on H . If $i\mathbb{R} \subset \rho(A)$, then, for a fixed $\ell > 0$, the following conditions are equivalent:

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^\ell} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty, \quad (2.2)$$

$$\|S(t)U_0\|_H \leq \frac{C}{t^\ell} \|U_0\|_{D(A)} \quad \forall t > 0, U_0 \in D(A), \text{ for some } C > 0. \quad (2.3)$$

Also, the analytic property of a C_0 -semigroup of contraction $(S(t))_{t \geq 0}$ is characterized in the following theorem due to Arendt, Batty and Hieber [23].

Theorem 2.11. Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t \geq 0}$ on H . Assume that

$$i\mathbb{R} \subset \rho(A). \quad (A1)$$

Then, $(S(t))_{t \geq 0}$ is analytic if and only if

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{-1}} \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < \infty. \quad (A2)$$

Polynomial stability of a transmission problem involving Timoshenko systems with fractional Kelvin-Voigt damping

Résumé

Dans cette partie, nous étudions la stabilité d'un système de Timoshenko unidimensionnel avec un amortissement fractionnaire interne localisé de Kelvin-Voigt dans un domaine borné. Tout d'abord, nous reformulons le système dans un modèle augmenté et en utilisant un critère général d'Arendt-Batty, nous prouvons sa forte stabilité. Ensuite, nous étudions trois cas : le premier, lorsque l'amortissement est localisé dans le moment de flexion, le deuxième lorsque l'amortissement est localisé dans la contrainte de cisaillement, nous prouvons que l'énergie du système décroît polynomialement avec le taux t^{-1} dans les deux cas. Dans le troisième cas, le Kelvin-Voigt fractionnaire agit simultanément sur la contrainte de cisaillement et le moment de flexion. Nous montrons que le système est polynomialement stable avec un taux de décroissance de l'énergie de type $t^{\frac{-4}{2-\alpha}}$, à condition que les deux amortisseurs agissent dans le même sous-intervalle. La méthode est basée sur la méthode fréquentielle combinée à la technique des multiplicateurs.

Abstract

In this part, we study the stability of a one-dimensional Timoshenko system with localized internal fractional kelvin-Voigt damping in a bounded domain. First, we reformulate the system into an augmented model and, using a general criteria of Arendt-Batty, we prove the strong stability. Next, we investigate three cases: the first one when the damping is localized in the bending moment, the second case when the damping is localized in the shear stress, we prove that the energy of the system decays polynomially with rate t^{-1} in both cases. In the third case, the fractional Kelvin-Voigt is acting on the shear stress and the bending moment simultaneously. We show that the system is polynomially stable with energy decay rate of type $t^{\frac{-4}{2-\alpha}}$, provided that the two dampings are acting in the same sub-interval. The method is based on the frequency domain approach combined with the multiplier technique.

1 Introduction

We consider the Timoshenko beams subject to a feedback control that combines the fractional and the Kelvin-Voigt type. The fractional derivative here is of Caputo type and it is defined by:

$$[D^{\alpha,\eta}\omega](x,t) = \partial_t^{\alpha,\eta}\omega(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(x,s) ds, \quad (3.1)$$

where $\alpha \in (0, 1)$ is the order of the derivative, t is the time variable, $\eta \geq 0$ and Γ denotes the Gamma function. Fractional calculus includes various extensions of the usual definition of derivative from integer to real order. For mathematical description of the fractional derivative see [97]. In the recent years, many researches showed interest in problems involving fractional (or fractional Kelvin-Voigt) damping (see [7, 9, 11, 12, 14, 21, 69, 100]). Ammari et al. [21] studied the stabilization of a class of evolution systems with fractional damping. They showed a polynomial stability result under some consideration on the stability of an auxiliary dissipating system, also they gave some applications to the wave equations. Recently, in [69], the authors considered a system of coupled wave equations with a boundary control of fractional derivative type. They established an optimal decay result. Tahri et al. [100] studied the energy decay rate for a one-dimensional non-degenerate wave equation under a fractional control applied at the boundary. They proved the polynomial decay result with an estimation of the decay rates. Now, we mention some recent results that treated the stabilization of beams subject to fractional or Kelvin-Voigt damping. In [3] Contreras and Rivera considered the Timoshenko beam with localized Kelvin-Voigt dissipation distributed over two components: one of them with constitutive law of the type C^1 and the other with discontinuous law. The third component is simply elastic, where the viscosity is not effective. They showed that the system is exponentially stable if and only if the component with discontinuous constitutive law is not in the center of the beam. When the discontinuous component is in the middle, the solution decays polynomially. Recently, in [9], the authors studied the coupled wave and Euler-Bernoulli beam equations with one locally singular fractional Kelvin-Voigt damping acting only on one equation. The system considered is as follows

$$\left\{ \begin{array}{ll} u_{tt} - (au_x + d_1(x)\partial_t^{\alpha,\eta}u_x)_x = 0, & (x,t) \in (0,L) \times (0,\infty), \\ y_{tt} + (by_{xx} + d_2(x)\partial_t^{\alpha,\eta}y_{xx})_{xx} = 0, & (x,t) \in (-L,0) \times (0,\infty), \\ u(L,t) = y(-L,t) = y_x(-L,t) = 0, & t \in (0,\infty), \\ au_x(0,t) + by_{xxx}(0,t) = 0, y_{xx}(0,t) = 0, & t \in (0,\infty), \\ u(0,t) = y(0,t), & t \in (0,\infty), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in (0,L), \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x), & x \in (-L,0). \end{array} \right. \quad (3.2)$$

- If $d_1 = d_0\chi_{[\ell_0,\ell_1]}$ and $d_2 = 0$ where $0 < \ell_0 < \ell_1 < L$, the authors proved that the energy decays polynomially as $t^{\frac{-4}{2-\alpha}}$.
- If $d_2 = d_0\chi_{[-\ell_1,-\ell_0]}$ and $d_1 = 0$, the authors proved that the energy decays polynomially as $t^{\frac{-2}{3-\alpha}}$.

In [85] Oquendo and Roberto da Luz investigated the asymptotic behavior of the solutions of a Timoshenko beam with a fractional damping. The damping acts only in one of the equations and depends on a parameter $\theta \in [0, 1]$. Timoshenko systems with frictional or Kelvin-Voigt dampings are particular cases of this model. They proved that, for regular initial data, the semigroup of this system decays polynomially with rates that depend on θ and some relations between the structural parameters of the system. Moreover, they showed that the decay rates obtained are optimal and the only possibility to obtain exponential decay is when $\theta = 0$ and the wave propagation speeds of the equations coincide.

Zhao et al. [112] considered the following Timoshenko beam with Kelvin-Voigt damping:

$$\begin{cases} \rho_1 u_{tt} - [k_1(u_x + y)_x + D_1(u_{xt} - y_t)]_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 y_{tt} - (k_2 y_x + D_2 y_{xt})_x + k_1(u_x + y)_x + D_1(u_{xt} - y_t) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+. \end{cases} \quad (3.3)$$

They proved that the energy of the system (3.3) subjected to Dirichlet-Neumann boundary conditions is exponentially stable when the coefficient functions $D_1, D_2 \in C^{1,1}([0, L])$ and satisfy $D_1 \leq cD_2$ ($c > 0$). Next, Malacarne and Rivera [79] considered the Timoshenko system (3.3) under Dirichlet-Neumann boundary conditions. They showed that the system is analytic if and only if the damping is present in both the shear stress and the bending moment. Otherwise, the solution decays polynomially no matter where the damping is effective and that rate is optimal. Later, Tian and Zhang [101] considered the Timoshenko system under fully Dirichlet boundary conditions with locally or globally distributed Kelvin-Voigt damping when the coefficient functions $D_1, D_2 \in C([0, L])$. When the Kelvin-Voigt is globally distributed, they showed that the corresponding semigroup is analytic. Then, for their system with local Kelvin-Voigt damping, they analyzed the exponential and polynomial stability according to the properties of the coefficient functions D_1, D_2 . Next, Ghader and Wehbe [107] studied the stabilization of the following Timoshenko system with only one locally or globally distributed Kelvin-Voigt damping:

$$\begin{cases} \rho_1 u_{tt} - k_1(u_x + y)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 y_{tt} - (k_2 y_x + D y_{xt})_x + k_1(u_x + y) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+. \end{cases} \quad (3.4)$$

They established that the energy of the system (3.4) under fully Dirichlet or mixed boundary conditions decays polynomially.

To the best of our knowledge, the stabilization of Timoshenko system with one or two fractional Kelvin-Voigt damping has never been looked into yet. In particular, in the case, where only the first equation (equation of shear force) is effectively damped. In the present paper, we investigate the stability of the following Timoshenko system with fractional Kelvin-Voigt damping:

$$\begin{cases} \rho_1 u_{tt} - [k_1(u_x + y) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y)]_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \\ \rho_2 y_{tt} - [k_2 y_x + D_2(x)\partial_t^{\alpha,\eta} y_x]_x + k_1(u_x + y) + D_1(x)\partial_t^{\alpha,\eta}(u_x + y) = 0, & (x, t) \in (0, L) \times \mathbb{R}_+, \end{cases} \quad (3.5)$$

subject to the following initial conditions:

$$\begin{aligned} u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & x &\in (0, L), \\ y(x, 0) &= y_0(x), & y_t(x, 0) &= y_1(x), & x &\in (0, L). \end{aligned} \quad (3.6)$$

and the following boundary conditions:

$$u(0, t) = y(0, t) = u(L, t) = y(L, t) = 0, \quad t \in \mathbb{R}_+ \quad (3.7)$$

or

$$u(0, t) = y_x(0, t) = u(L, t) = y_x(L, t) = 0, \quad t \in \mathbb{R}_+. \quad (3.8)$$

The coefficients ρ_1, ρ_2, k_1 , and k_2 are positive constants, η is non-negative and $\alpha \in (0, 1)$. We assume that there exist $0 < a_1 < b_1 < L, 0 < a_2 < b_2 < L$ and two positive constants d_1 and d_2 , such that

$$D_1(x) = \begin{cases} d_1, & x \in (a_1, b_1), \\ 0, & x \in (0, a_1) \cup (b_1, L) \end{cases} \quad \text{and} \quad D_2(x) = \begin{cases} d_2, & x \in (a_2, b_2), \\ 0, & x \in (0, a_2) \cup (b_2, L). \end{cases}$$

We shall study the direct stability when $D_1 \neq 0$ and $D_2 \neq 0$. As well as the indirect stability when $D_1 = 0$ or $D_2 = 0$. In other words, we shall study the stability of system (3.5)-(3.6) with boundary

conditions (3.7) or (3.8) by distinguishing between three cases. In the first one, we consider a fully damped system *i.e.* the two equations are effectively damped. However, in the other two cases, we assume that the system is partially damped *i.e.* only one of these equations is effectively damped. To this end, we introduce the following assumptions on the damping coefficients $D_j(x); j = 1, 2$:

$$D_1 \neq 0 \quad \text{and} \quad D_2 \neq 0 \quad (\text{A}_1)$$

or

$$D_1 = 0 \quad \text{and} \quad D_2 \neq 0 \quad (\text{A}_2)$$

or

$$D_1 \neq 0 \quad \text{and} \quad D_2 = 0. \quad (\text{A}_3)$$

The remaining of this paper is organized as follows: In Subsection 2.1, we reformulate problem (3.5) into an augmented model and we prove the well-posedness of the problem by the semigroup approach. In Subsection 2.2, we show that our system is strongly stable in the sense that its energy converges to zero as t goes to infinity provided that any one of assumptions (A1), (A2) or (A3) holds. For this aim, a general criteria of Arendt-Batty is used. Moreover, using a frequency domain approach combining with a specific multiplier method, in Section 3 (respectively in Section ??), we prove that the energy of our system decays polynomially to zero like t^{-1} when the fractional Kelvin-Voigt damping is acting only on the bending moment equation (respectively only on the shear force equation). Finally, in the last section, we study the polynomial stability of our system when the fractional Kelvin-Voigt damping is present in both the shear stress and bending moment equations. We establish a polynomial energy decay rate of type $t^{\frac{-4}{2-\alpha}}$, provided that the two dampings are acting in the same sub-interval.

2 Well-Posedness and Strong Stability

2.1 Augmented model and well-posedness

For well-posedness of System (3.5)-(3.6) subjected to either boundary conditions (3.7) or (3.8), we recall Theorem 2 stated in [81].

Theorem 2.1. Let $\alpha \in (0, 1)$, $\eta \geq 0$ and $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ be the function defined almost everywhere on \mathbb{R} . The relationship between the input V and the output O of the following system:

$$\omega_t(x, \xi, t) + (\xi^2 + \eta) \omega(x, \xi, t) - \mu(\xi)V(x, t) = 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+, \quad (3.9)$$

$$\omega(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R}, \quad (3.10)$$

$$O(x, t) - \kappa(\alpha) \int_{\mathbb{R}} \mu(\xi) \omega(x, \xi, t) d\xi = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+ \quad (3.11)$$

is given by

$$O = I^{1-\alpha, \eta} V, \quad (3.12)$$

where

$$[I^{\alpha, \eta} V](x, t) = \int_0^t \frac{(t-\tau)^{\alpha-1} e^{-\eta(t-\tau)}}{\Gamma(\alpha)} V(x, \tau) d\tau \quad \text{and} \quad \kappa(\alpha) = \frac{\sin(\alpha\pi)}{\pi}.$$

Corollary 3.1. System (3.5)-(3.6) with boundary conditions (3.7) or (3.8) may be recast into the following augmented model

$$\rho_1 u_{tt} - \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi \right)_x = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \quad (3.13)$$

$$\begin{aligned} \rho_2 y_{tt} - \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi, t) d\xi \right)_x + k_1 (u_x + y) \\ + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+ \end{aligned} \quad (3.14)$$

$$\omega_t^1(x, \xi, t) + (\xi^2 + \eta) \omega^1(x, \xi, t) - \sqrt{D_1(x)} (u_{xt} + y_t) \mu(\xi) = 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+, \quad (3.15)$$

$$\omega_t^2(x, \xi, t) + (\xi^2 + \eta) \omega^2(x, \xi, t) - \sqrt{D_2(x)} y_{tx} \mu(\xi) = 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+. \quad (3.16)$$

System (3.13)-(3.16) is subjected to the following initial conditions:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, L), \quad (3.17)$$

$$y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, L), \quad (3.18)$$

$$\omega^1(x, \xi, 0) = \omega^2(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R}, \quad (3.19)$$

and fully Dirichlet boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0, \quad (3.20)$$

or with Dirichlet-Neumann boundary conditions

$$u(0) = u(L) = y_x(0) = y_x(L) = 0. \quad (3.21)$$

Proof. Using the same techniques used in [9] by taking

$$V_1(x, t) = \sqrt{D_1(x)} (u_{xt}(x, t) + y_t(x, t)) \quad \text{and} \quad V_2(x, t) = \sqrt{D_2(x)} y_{xt}(x, t),$$

we get system (3.13)-(3.21). \square

Now, let $(u, u_t, y, y_t, \omega^1, \omega^2)$ be a regular solution for the system (3.13)-(3.19) subjected to either the boundary conditions (3.20) or (3.21), then its associated energy is given by

$$E(t) = \frac{1}{2} \int_0^L \left(\rho_1 |u_t|^2 + \rho_2 |y_t|^2 + k_1 |u_x + y|^2 + k_2 |y_x|^2 \right) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} \left(|\omega^1|^2 + |\omega^2|^2 \right) d\xi dx.$$

Lemma 2.2. System (3.13)-(3.19) subjected to either the boundary conditions (3.20) or (3.21) is dissipative in the sense that it is energy is non-increasing function with respect to t and satisfies

$$E'(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) \left(|\omega^1|^2 + |\omega^2|^2 \right) d\xi dx \leq 0. \quad (3.22)$$

Proof. Let $(u, u_t, y, y_t, \omega^1, \omega^2)$ be a regular solution of (3.13)-(3.19), thus multiplying (3.13) and (3.14) by \bar{u}_t and \bar{y}_t , respectively, integrating over $(0, L)$, adding the resulting equations, then using the boundary conditions (3.20) or (3.21), we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^L \left(\rho_1 |u_t|^2 + \rho_2 |y_t|^2 + k_1 |u_x + y|^2 + k_2 |y_x|^2 \right) dx \\ + \Re \left(\kappa(\alpha) \int_0^L \sqrt{D_1(x)} (\bar{u}_{tx} + \bar{y}_t) \int_{\mathbb{R}} \mu(\xi) \omega^1 d\xi dx \right) \\ + \Re \left(\kappa(\alpha) \int_0^L \sqrt{D_2(x)} \bar{y}_{tx} \int_{\mathbb{R}} \mu(\xi) \omega^2 d\xi dx \right) = 0. \end{aligned} \quad (3.23)$$

Next, multiplying (3.15) and (3.16) by $\kappa(\alpha)\overline{\omega^1}$ and $\kappa(\alpha)\overline{\omega^2}$, respectively, integrating in $(0, L) \times \mathbb{R}$, then using the boundary condition (3.20) or (3.21), we get

$$\begin{aligned} & \frac{\kappa(\alpha)}{2} \frac{d}{dt} \int_0^L \int_{\mathbb{R}} |\omega^1(x, \xi)|^2 d\xi dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^1(x, \xi)|^2 d\xi dx \\ &= \Re \left(\kappa(\alpha) \int_0^L \sqrt{D_1(x)} (u_{xt} + y_t) \int_{\mathbb{R}} \mu(\xi) \overline{\omega^1}(x, \xi) d\xi dx \right), \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} & \frac{\kappa(\alpha)}{2} \frac{d}{dt} \int_0^L \int_{\mathbb{R}} |\omega^2(x, \xi)|^2 d\xi dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^2(x, \xi)|^2 d\xi dx \\ &= \Re \left(\kappa(\alpha) \int_0^L \sqrt{D_2(x)} y_{tx} \int_{\mathbb{R}} \mu(\xi) \overline{\omega^2}(x, \xi) d\xi dx \right). \end{aligned} \quad (3.25)$$

Finally, by adding (3.24), (3.25) and (3.23), we obtain (3.22). Since $\alpha \in (0, 1)$, then $\kappa(\alpha) > 0$, and therefore $E'(t) \leq 0$. The proof is thus complete. \square

Now, let us define the energy spaces \mathcal{H}_1 and \mathcal{H}_2 by:

$$\mathcal{H}_1 = \left(H_0^1(0, L) \times L^2(0, L) \right)^2 \times \mathcal{W}^2,$$

$$\mathcal{H}_2 = H_0^1(0, L) \times L^2(0, L) \times H_*^1(0, L) \times L^2(0, L) \times \mathcal{W}^2,$$

where

$$\mathcal{W} = L^2((0, L) \times \mathbb{R}),$$

and

$$H_*^1(0, L) = \left\{ f \in H^1(0, L) \mid \int_0^L f dx = 0 \right\}.$$

It is easy to check that the space H_*^1 is Hilbert spaces over \mathbb{C} equipped with the norm

$$\|u\|_{H_*^1(0, L)}^2 = \|u_x\|^2,$$

where $\|\cdot\|$ denotes the usual norm of $L^2(0, L)$. Both energy spaces \mathcal{H}_1 and \mathcal{H}_2 are equipped with the inner product defined by:

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}_j} &= \rho_1 \int_0^L v \tilde{v} dx + \rho_2 \int_0^L z \tilde{z} dx + k_1 \int_0^L (u_x + y) (\overline{\tilde{u}_x + \tilde{y}}) dx \\ &+ k_2 \int_0^L y_x \tilde{y}_x dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \left(\omega^1 \overline{\tilde{\omega}^1} + \omega^2 \overline{\tilde{\omega}^2} \right) d\xi dx, \end{aligned}$$

for all $U = (u, v, y, z, \omega^1, \omega^2)$ and $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z}, \tilde{\omega}^1, \tilde{\omega}^2)$ in \mathcal{H}_j , $j = 1, 2$. We use $\|U\|_{\mathcal{H}_j}$ to denote the corresponding norm. We now define the following unbounded linear operators \mathcal{A}_j on \mathcal{H}_j ($j = 1, 2$) by:

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} \mid v, z \in H_0^1(0, L), \\ \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \in L^2(0, L), \\ \left(k_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \in L^2(0, L), \\ -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1(x)} (v_x + z) \mu(\xi), \quad |\xi| \omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2(x)} z_x \mu(\xi), \quad |\xi| \omega^2 \in \mathcal{W} \end{array} \right\},$$

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega^1, \omega^2) \in \mathcal{H} \mid v \in H_0^1(0, L), z \in H_*^1(0, L), \\ \left(k_1(u_x + y) + \kappa(\alpha)\sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi)\omega^1(x, \xi)d\xi \right)_x \in L^2(0, L), \\ \left(k_2y_x + \kappa(\alpha)\sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi)\omega^2(x, \xi)d\xi \right)_x \in L^2(0, L), \\ -(\xi^2 + \eta)\omega^1(x, \xi) + \sqrt{D_1(x)}(v_x + z)\mu(\xi), \quad |\xi|\omega^1 \in \mathcal{W}, \\ -(\xi^2 + \eta)\omega^2(x, \xi) + \sqrt{D_2(x)}z_x\mu(\xi), \quad |\xi|\omega^2 \in \mathcal{W}, \\ y_x(0) = y_x(L) = 0, \end{array} \right\}$$

and

$$\mathcal{A}_j U^\top = \left(\begin{array}{c} \frac{1}{\rho_1} \left(\kappa_1(u_x + y) + \kappa(\alpha)\sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi)\omega^1(x, \xi)d\xi \right)_x \\ v \\ z \\ \frac{1}{\rho_2} \left(\kappa_2y_x + \kappa(\alpha)\sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi)\omega^2(x, \xi)d\xi \right)_x - \frac{\kappa_1}{\rho_2}(u_x + y) - \frac{\kappa(\alpha)}{\rho_2}\sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi)\omega^1(x, \xi)d\xi \\ -(\xi^2 + \eta)\omega^1(x, \xi) + \sqrt{D_1(x)}(v_x + z)\mu(\xi) \\ -(\xi^2 + \eta)\omega^2(x, \xi) + \sqrt{D_2(x)}z_x\mu(\xi) \end{array} \right),$$

for all $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$, for $j = 1, 2$.

Remark 2.3. The condition $|\xi|\omega^j \in \mathcal{W}$ is imposed to make

$$\int_{\mathbb{R}} (\xi^2 + \eta)|\omega^j(x, \xi)|^2 d\xi dx \in L^2(0, L) \quad \text{and} \quad \sqrt{D_j(x)} \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega^j(x, \xi) d\xi \in L^2(0, L), \quad (j = 1, 2).$$

In fact, if $|\xi|\omega^j \in \mathcal{W}$, we have

$$\begin{aligned} \left| \int_0^L \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega^j(x, \xi) d\xi dx \right| &\leq 2 \int_0^L \int_0^{+\infty} \xi^{\frac{2\alpha-1}{2}} |\omega^j(x, \xi)| d\xi dx \\ &\leq 2 \int_0^L \int_0^1 \xi^{\frac{2\alpha-1}{2}} |\omega^j(x, \xi)| d\xi dx + 2 \int_0^L \int_1^{+\infty} \xi^{\frac{2\alpha-1}{2}} |\omega^j(x, \xi)| d\xi dx. \end{aligned}$$

Since $0 < \alpha < 1$ and $\omega^j \in \mathcal{W}$, then by using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \int_0^L \int_0^1 \xi^{\frac{2\alpha-1}{2}} |\omega^j(x, \xi)| d\xi dx &\leq \left(\int_0^L \int_0^1 \xi^{2\alpha-1} d\xi dx \right)^{\frac{1}{2}} \left(\int_0^L \int_0^1 |\omega^j(x, \xi)|^2 d\xi dx \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{L}}{2\alpha} \left(\int_0^L \int_0^1 |\omega^j(x, \xi)|^2 d\xi dx \right)^{\frac{1}{2}} < +\infty. \end{aligned}$$

On the other hand, since $0 < \alpha < 1$ and $|\xi|\omega^j \in \mathcal{W}$, by using Cauchy-Schwarz inequality, we get

$$\begin{aligned} \int_0^L \int_1^{+\infty} \xi^{\frac{2\alpha-1}{2}} |\omega^j(x, \xi)| d\xi dx &= \int_0^L \int_1^{+\infty} \xi^{\frac{2\alpha-3}{2}} \xi |\omega^j(x, \xi)| d\xi dx \\ &\leq \left(\int_0^L \int_1^{+\infty} \xi^{2\alpha-3} d\xi dx \right)^{\frac{1}{2}} \left(\int_0^L \int_1^{+\infty} |\xi \omega^j(x, \xi)|^2 d\xi dx \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{L}}{2(1-\alpha)} \left(\int_0^L \int_1^{+\infty} |\xi \omega^j(x, \xi)|^2 d\xi dx \right)^{\frac{1}{2}} < +\infty. \end{aligned}$$

It follows that $\sqrt{D_j(x)} \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega^j(x, \xi) d\xi$ is well defined. Thus, the Timoshenko system (3.13)-(3.19) is transformed into a first order evolution equation on the Hilbert space \mathcal{H}_j

$$\begin{cases} U_t(x, t) = \mathcal{A}_j U(x, t), \\ U(x, 0) = U_0(x), \end{cases} \quad (3.26)$$

where

$$U_0(x) = (u_0(x), u_1(x), y_0(x), y_1(x), 0, 0)$$

with $j = 1, 2$ corresponding to the boundary conditions (3.20) and (3.21), respectively.

Lemma 2.4. *Let $0 < \alpha < 1$, $\eta \geq 0$, then the following integrals are well-defined:*

$$I(\eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1 + \xi^2 + \eta} d\xi \quad \text{and} \quad \tilde{I}_i(f_i, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{\xi^{\frac{2\alpha-1}{2}} f_i}{1 + \xi^2 + \eta} d\xi dx, \quad \text{for } i = 5 \text{ or } i = 6.$$

For the proof of $I(\eta, \alpha)$, see Lemma 2.1 in [9], and for the proof of $\tilde{I}_i(f_i, \eta, \alpha)$, see Proposition 2.4 in [9].

Proposition 3.2. *The unbounded linear operator \mathcal{A}_j is m -dissipative in the energy space \mathcal{H}_j , $j = 1, 2$.*

Proof. First, for all $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$, one has

$$\Re \langle \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) \left(|\omega^1(x, \xi)|^2 + |\omega^2(x, \xi)|^2 \right) d\xi dx \leq 0,$$

which implies that \mathcal{A}_j is dissipative. Here \Re is used to denote the real part of a complex number. We next prove the maximality of \mathcal{A}_j . Indeed, for $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}_j$, we prove the existence of $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$, the unique solution of the equation

$$(I - \mathcal{A}_j)U = F.$$

Equivalently, one must consider the system given by

$$u - v = f_1, \quad (3.27)$$

$$\rho_1 v - \left(\kappa_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x = \rho_1 f_2, \quad (3.28)$$

$$y - z = f_3, \quad (3.29)$$

$$\rho_2 z - \left(\kappa_2 y_x + \kappa(\alpha) \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x + \kappa_1 (u_x + y) \quad (3.30)$$

$$+ \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi = \rho_2 f_4,$$

$$(1 + \xi^2 + \eta) \omega^1(x, \xi) - \sqrt{D_1(x)} (v_x + z) \mu(\xi) = f_5(x, \xi), \quad (3.31)$$

$$(1 + \xi^2 + \eta) \omega^2(x, \xi) - \sqrt{D_2(x)} z_x \mu(\xi) = f_6(x, \xi). \quad (3.32)$$

From (3.27), (3.29), (3.31), (3.32) and the fact that $\eta \geq 0$, we get

$$v = u - f_1 \quad \text{and} \quad z = y - f_3, \quad (3.33)$$

$$\omega^1(x, \xi) = \frac{f_5(x, \xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{D_1(x)}\mu(\xi)u_x}{1 + \xi^2 + \eta} - \frac{\sqrt{D_1(x)}\mu(\xi)(f_1)_x}{1 + \xi^2 + \eta} + \frac{\sqrt{D_1(x)}\mu(\xi)y}{1 + \xi^2 + \eta} - \frac{\sqrt{D_1(x)}\mu(\xi)f_3}{1 + \xi^2 + \eta}, \quad (3.34)$$

$$\omega^2(x, \xi) = \frac{f_6(x, \xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{D_2(x)}\mu(\xi)y_x}{1 + \xi^2 + \eta} - \frac{\sqrt{D_2(x)}\mu(\xi)(f_3)_x}{1 + \xi^2 + \eta}. \quad (3.35)$$

Inserting (3.33), (3.34) and (3.35) in (3.28) and in (3.30), we get

$$\rho_1 u - \left(\kappa_1 (u_x + y) + D_1 u_x I + D_1 y I - D_1 (f_1)_x I - D_1 f_3 I + \sqrt{D_1} \tilde{I}_5 \right)_x = \rho_1 (f_1 + f_2), \quad (3.36)$$

$$\begin{aligned} \rho_2 y - \left(\kappa_2 y_x + D_2 y_x I - D_2 (f_3)_x I + \sqrt{D_2} \tilde{I}_6 \right)_x + \kappa_1 (u_x + y) + D_1 u_x I + D_1 y I \\ = \rho_2 (f_3 + f_4) + D_1 (f_1)_x I + D_1 f_3 I - \sqrt{D_1} \tilde{I}_5, \end{aligned} \quad (3.37)$$

with the following boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0, \quad (3.38)$$

or

$$u(0) = u(L) = y_x(0) = y_x(L) = 0 \quad (3.39)$$

where $I = I(\eta, \alpha)$ and $\tilde{I}_i = \tilde{I}_i(f_i, \eta, \alpha)$ for $i = 5, 6$, defined in Lemma 2.4. So, let $(\varphi, \psi) \in \mathcal{V}_j(0, L)$, where $\mathcal{V}_1(0, L) = H_0^1(0, L) \times H_0^1(0, L)$ and $\mathcal{V}_2(0, L) = H_0^1(0, L) \times H_*^1(0, L)$. Multiplying Equations (3.36) and (3.37) by $\bar{\varphi}$ and $\bar{\psi}$ respectively, and integrating over $(0, L)$, then we obtain the following variational Problem:

$$\begin{aligned} \int_0^L \left(\rho_1 u \bar{\varphi} + \rho_2 y \bar{\psi} + k_1 (u_x + y) (\overline{\varphi_x + \psi}) + k_2 y_x \bar{\psi}_x \right) dx + I(\eta, \alpha) \int_0^L D_1(x) (u_x + y) (\overline{\varphi_x + \psi}) dx \\ + I(\eta, \alpha) \int_0^L D_2(x) y_x \bar{\psi}_x dx = \int_0^L \left(\rho_1 (f_1 + f_2) \bar{\varphi} + \rho_2 (f_3 + f_4) \bar{\psi} \right) dx - \int_0^L \sqrt{D_1(x)} (\overline{\varphi_x + \psi}) \tilde{I}_5 dx \\ + I(\eta, \alpha) \int_0^L D_1(x) (f_1)_x (\overline{\varphi_x + \psi}) dx + I(\eta, \alpha) \int_0^L D_1(x) f_3 (\overline{\varphi_x + \psi}) dx \\ - \int_0^L \sqrt{D_2(x)} \bar{\psi}_x \tilde{I}_6 dx + I(\eta, \alpha) \int_0^L D_2(x) (f_3)_x \bar{\psi}_x dx, \quad \forall (\varphi, \psi) \in \mathcal{V}_j(0, L), \quad j = 1, 2. \end{aligned} \quad (3.40)$$

Using the fact that $I(\eta, \alpha) > 0$, we get that the left hand side of (3.40) is a sesquilinear continuous coercive form on $\mathcal{V}_j(0, L) \times \mathcal{V}_j(0, L)$, and the right hand side of (3.40) is an antilinear continuous form on $\mathcal{V}_j(0, L)$. Then, using the Lax-Milgram theorem, we deduce that there exists $(u, y) \in \mathcal{V}_j(0, L)$ unique solution of the variational problem (3.40). So, defining

$$v := u - f_1 \quad \text{and} \quad z := y - f_3, \quad (3.41)$$

$$\omega^1(x, \xi) := \frac{f_5(x, \xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{D_1}\mu(\xi)u_x}{1 + \xi^2 + \eta} - \frac{\sqrt{D_1}\mu(\xi)(f_1)_x}{1 + \xi^2 + \eta} + \frac{\sqrt{D_1(x)}\mu(\xi)y}{1 + \xi^2 + \eta} - \frac{\sqrt{D_1(x)}\mu(\xi)f_3}{1 + \xi^2 + \eta} \quad (3.42)$$

and

$$\omega^2(x, \xi) := \frac{f_6(x, \xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{D_2}\mu(\xi)y_x}{1 + \xi^2 + \eta} - \frac{\sqrt{D_2}\mu(\xi)(f_3)_x}{1 + \xi^2 + \eta}. \quad (3.43)$$

First, it is easy to see that $(v, z) \in \mathcal{V}_j(0, L)$, $j = 1, 2$. Next, using Equation (3.42), Lemma 2.4 and the fact that $\eta \geq 0$, $\alpha \in (0, 1)$, $f_5 \in \mathcal{W}$, we get

$$\begin{aligned}
 \int_0^L \int_{\mathbb{R}} |\omega^1(x, \xi)|^2 d\xi dx &\leq 5 \int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{(1 + \xi^2 + \eta)^2} d\xi dx \\
 &+ 5d_1 \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(1 + \xi^2 + \eta)^2} d\xi \int_0^L (|u_x|^2 + |(f_1)_x|^2 + |y|^2 + |f_3|^2) dx \\
 &\leq 5 \int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{(1 + \eta)^2} d\xi dx \\
 &+ 5d_1 \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(1 + \xi^2 + \eta)^2} d\xi \int_0^L (|u_x|^2 + |(f_1)_x|^2 + |y|^2 + |f_3|^2) dx \\
 &\leq \frac{5}{(1 + \eta)^2} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx \\
 &+ \frac{5d_1}{\kappa(\alpha)} I(\eta, \alpha) \int_0^L (|u_x|^2 + |(f_1)_x|^2 + |y|^2 + |f_3|^2) dx < \infty.
 \end{aligned}$$

It follows that $\omega^1(x, \xi) \in \mathcal{W}$. On the other hand, using equation (3.42), we get

$$\begin{aligned}
 \int_0^L \int_{\mathbb{R}} |\xi \omega^1(x, \xi)|^2 d\xi dx &\leq 5 \int_0^L \int_{\mathbb{R}} \frac{\xi^2 |f_5(x, \xi)|^2}{(1 + \xi^2 + \eta)^2} d\xi dx \\
 &+ 10d_1 \int_0^{+\infty} \frac{|\xi|^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} d\xi \int_0^L (|u_x|^2 + |(f_1)_x|^2 + |y|^2 + |f_3|^2) dx.
 \end{aligned} \tag{3.44}$$

By similar way used in Lemma [69, Lemma 2.1], it is easy to see that

$$\begin{aligned}
 \max_{\xi \in \mathbb{R}} \frac{\xi^2}{(1 + \xi^2 + \eta)^2} &= \frac{1}{4(1 + \eta)} < \frac{1}{4}. \\
 \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} &\underset{0}{\sim} \frac{\xi^{2\alpha+1}}{(1 + \eta)^2} \quad \text{and} \quad \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} \underset{+\infty}{\sim} \frac{1}{\xi^{3-2\alpha}}.
 \end{aligned}$$

It follows from equation (3.44) and the fact that $0 < \alpha < 1$,

$$\int_0^L \int_{\mathbb{R}} |\xi \omega^1(x, \xi)|^2 d\xi dx \leq \frac{5}{4} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx + c \int_0^L (|u_x|^2 + |(f_1)_x|^2 + |y|^2 + |f_3|^2) dx < \infty.$$

This implies that $|\xi| \omega^1(x, \xi) \in \mathcal{W}$. Similarly, we prove that $\omega^2(x, \xi)$, and $|\xi| \omega^2(x, \xi)$ belong to \mathcal{W} . It follows, from (3.42) and (3.43), that

$$\begin{aligned}
 -(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1}(v_x + z) \mu(\xi) &= \omega^1 - f_5(x, \xi) \in \mathcal{W}, \\
 -(\xi^2 + \eta) \omega^2(x, \xi) + \sqrt{D_2} z_x \mu(\xi) &= \omega^2 - f_6(x, \xi) \in \mathcal{W}.
 \end{aligned}$$

On the other hand, taking $\varphi \in C_c^\infty(0, L)$, $\psi \equiv 0$ in (3.40), integrating and using (3.41)-(3.43), we deduce that

$$\int_0^L \left[\rho_1 v - \left(\kappa_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x \right] \bar{\varphi} dx = \rho_1 \int_0^L f_2 \bar{\varphi} dx, \quad \forall \varphi \in C_c^\infty(0, L).$$

This implies that

$$\left(\kappa_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x = \rho_1 (v - f_2) \in L^2(0, L).$$

Similarly, by taking $\varphi \equiv 0$ and $\psi \in C_c^\infty(0, L)$ in (3.40), integrating and using (3.41)-(3.43), we obtain equation (3.30) and consequently

$$\left(k_2 y_x + \kappa(\alpha) \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \in L^2(0, L).$$

Therefore, $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ is solution of $(I - \mathcal{A}_j)U = F$. To conclude, we need to show the uniqueness of such a solution. So, let $U = (u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ be a solution of equation $(I - \mathcal{A}_j)U = F$ with $F = 0$, then we directly deduce that $U = 0$. The proof is thus complete. \square

Thanks to Lumer-Phillips theorem (see [88]), we deduce that \mathcal{A}_j generates a C_0 -semigroup of contraction $e^{t\mathcal{A}_j}$ in \mathcal{H}_j , $j = 1, 2$. Then the solution of the evolution problem (3.26) admits the following representation

$$U(t) = e^{t\mathcal{A}_j} U_0, \quad t \geq 0.$$

and therefore, problem (3.26) is well-posed and we have the following result.

Theorem 2.5. For all $U_0 \in \mathcal{H}_j$, problem (3.26) admits a unique weak solution

$$U \in C(\mathbb{R}_+; \mathcal{H}_j).$$

Moreover, if $U_0 \in D(\mathcal{A}_j)$, then Problem (3.26) admits a unique strong solution

$$U \in C(\mathbb{R}_+; D(\mathcal{A}_j)) \cap C^1(\mathbb{R}_+; \mathcal{H}_j).$$

Remark 2.6. All previous results still valid even when the Timoshenko system (3.5) is considered with only one fractional Kelvin-Voigt damping i.e. $D_1 \equiv 0$ or $D_2 \equiv 0$.

Remark 2.7. In the case of Dirichlet boundary conditions we can replace condition $0 < a_1 < b_1 < L$ and $0 < a_2 < b_2 < L$ by $0 \leq a_1 < b_1 \leq L$ and $0 \leq a_2 < b_2 \leq L$.

Now, we are able to study the strong stability of system (3.5).

2.2 Strong stability

In this part, we study the strong stability of system (3.13)-(3.19) with either of the boundary conditions (3.20) and (3.21), in which we distinguish between three cases. In the first case, we consider a fully dissipative system i.e. the two equations are effectively damped. However, in the other two cases, we assume that the system is partially dissipative i.e. only one equation is effectively damped. For this aim, we use a general criteria of Arendt-Batty [22] (see Theorem 2.7) to show the strong stability of the C_0 -semigroup $e^{t\mathcal{A}_j}$ associated to the Timoshenko system (3.13)-(3.19). Our main result is the following theorem.

Theorem 2.8. Assume that either (A_1) , (A_2) or (A_3) holds. Then, the C_0 -semigroup $e^{t\mathcal{A}_j}$ is strongly stable in the energy space \mathcal{H}_j in the sense that,

$$\lim_{t \rightarrow +\infty} \left\| e^{t\mathcal{A}_j} U_0 \right\|_{\mathcal{H}_j} = 0, \quad \forall U_0 \in \mathcal{H}_j, \quad j = 1, 2.$$

For brevity, we will show the proof of Theorem 2.8 under assumption (A₃) only, while the proof under assumptions (A₁) and (A₂) are left to the reader. System (3.13)-(3.16) becomes

$$\rho_1 u_{tt} - \left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi \right)_x = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \quad (3.45)$$

$$\rho_2 y_{tt} - k_2 y_{xx} + k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi, t) d\xi = 0, \quad (x, t) \in (0, L) \times \mathbb{R}_+, \quad (3.46)$$

$$\omega_t^1(x, \xi, t) + (\xi^2 + \eta) \omega^1(x, \xi, t) - \sqrt{D_1(x)} (u_{xt} + y_t) \mu(\xi) = 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+ \quad (3.47)$$

subject to the following initial conditions:

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, L), \quad (3.48)$$

$$y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, L), \quad (3.49)$$

$$\omega^1(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R} \quad (3.50)$$

with fully Dirichlet boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0 \quad (3.51)$$

or with Dirichlet-Neumann boundary conditions

$$u(0) = u(L) = y_x(0) = y_x(L) = 0. \quad (3.52)$$

The argument for Theorem 2.8 relies on the subsequent lemmas.

Lemma 2.9. *Assume that assumption (A₃) holds. Then, we have*

$$\ker(i\lambda I - \mathcal{A}_j) = \{0\}, \quad \forall \lambda \in \mathbb{R}, \quad j = 1, 2.$$

Proof. Let $U = (u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ and $\lambda \in \mathbb{R}$ such that

$$\mathcal{A}_j U = i\lambda U.$$

Equivalently, we have

$$v = i\lambda u, \quad (3.53)$$

$$\left(k_1 (u_x + y) + \kappa(\alpha) \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x = i\rho_1 \lambda v, \quad (3.54)$$

$$z = i\lambda y, \quad (3.55)$$

$$k_2 y_{xx} - k_1 (u_x + y) - \kappa(\alpha) \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi = i\rho_2 \lambda z, \quad (3.56)$$

$$-(\xi^2 + \eta) \omega^1(x, \xi) + \sqrt{D_1} (v_x + z) \mu(\xi) = i\lambda \omega^1(x, \xi). \quad (3.57)$$

With the following boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0, \quad \text{if } j = 1 \quad (3.58)$$

or

$$u(0) = y_x(0) = u(L) = y_x(L) = 0, \quad \text{if } j = 2. \quad (3.59)$$

First, a straightforward computation gives

$$0 = \Re \langle i\lambda U, U \rangle_{\mathcal{H}_j} = \Re \langle \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^1(x, \xi)|^2 d\xi dx,$$

consequently, we deduce that

$$\omega^1(x, \xi) = 0 \quad \text{a.e. in } (0, L) \times \mathbb{R}. \quad (3.60)$$

Combining (3.60) with (3.53)-(3.57) and using the definition of the function $D_1(x)$, we get

$$\lambda(u_x + y) = 0, \quad \text{over } (a_1, b_1), \quad (3.61)$$

$$k_1(u_x + y)_x + \rho_1 \lambda^2 u = 0, \quad \text{over } (0, L), \quad (3.62)$$

and

$$k_2 y_{xx} - k_1(u_x + y) + \rho_2 \lambda^2 y = 0, \quad \text{over } (0, L). \quad (3.63)$$

Here we will distinguish two cases.

Case1. If $\lambda = 0$:

From equations (3.53) and (3.55), we get

$$v = z = 0 \quad \text{on } (0, L).$$

1. If $j = 1$, using equations (3.62), (3.63) and the boundary conditions in (3.58) we can write u and y as

$$u = -\frac{a}{6}x^3 + \frac{aL}{4}x^2 - \frac{aL^2}{12}x \quad \text{and} \quad y = \frac{a}{2}x^2 - \frac{aL}{2}x \quad (3.64)$$

where a is a constant number to be determined. Now using (3.64) in (3.63) we get $a(k_2 + k_1 \frac{L^2}{12}) = 0$. Since $k_1, k_2 > 0$, we deduce that $a = 0$. Then we get $u = y = 0$. Hence, $U = 0$ over $(0, L)$. In this case, the proof is complete.

2. If $j = 2$, from (3.62), (3.63), the boundary conditions in (3.59) and the fact that $y \in H_*^1(0, L)$ (i.e., $\int_0^L y dx = 0$), we get

$$u = y = 0, \quad \text{over } (0, L),$$

therefore, $U = 0$, also in this case the proof is complete.

Case2. If $\lambda \neq 0$:

From equation (3.61), we get

$$(u_x + y) = 0 \quad \text{over } (a_1, b_1). \quad (3.65)$$

By using (3.65), (3.62) and the boundary conditions (3.58) or (3.59), we get

$$u = y = 0 \quad \text{over } (a_1, b_1). \quad (3.66)$$

Combining equations (3.62), (3.63) and (3.66), we get the following system

$$\begin{cases} k_1(u_x + y)_x + \rho_1 \lambda^2 u = 0, & \text{over } (0, L), \\ k_2 y_{xx} - k_1(u_x + y) + \rho_2 \lambda^2 y = 0, & \text{over } (0, L), \\ u = y = 0, & \text{over } (a_1, b_1). \end{cases} \quad (3.67)$$

According to the unique continuation theorem we get $U = 0$ over $(0, L)$. The proof is thus completed. \square

Lemma 2.10. Assume that $\eta = 0$ and assumption (A_3) holds. Then, the operator $-\mathcal{A}_j$ is not invertible, and consequently, $0 \in \sigma(\mathcal{A}_j)$, $j = 1, 2$.

Proof. Let $F = (\sin(\frac{\pi x}{L}), 0, 0, 0, 0) \in \mathcal{H}_j$, and assume that there exists $U = (u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ such that

$$-\mathcal{A}_j U = F.$$

It follows that

$$v = -\sin(\frac{\pi x}{L}) \text{ in } (0, L) \text{ and } \xi^2 \omega^1 + \frac{\pi}{L} \sqrt{D_1} \cos(\frac{\pi x}{L}) \mu(\xi) = 0.$$

Hence, we deduce that $\omega^1(x, \xi) = -\frac{\pi}{L} \xi^{\frac{2\alpha-5}{2}} \sqrt{D_1} \cos(\frac{\pi x}{L}) \notin \mathcal{W}$, which contradicts the fact that $U \in D(\mathcal{A}_j)$. Consequently, the operator $-\mathcal{A}_j$ is not invertible, as claimed. The proof is thus complete. \square

Lemma 2.11. *Let $0 < \alpha < 1$, $\eta \geq 0$ and $f_5(x, \xi) \in \mathcal{W}$. Assume that ($\eta > 0$ and $\lambda \in \mathbb{R}$) or ($\eta = 0$ and $\lambda \in \mathbb{R}^*$), then the following integrals:*

$$I_1(\lambda, \eta, \alpha) = i\lambda \kappa(\alpha) \int_{\mathbb{R}} \frac{\mu^2(\xi)}{i\lambda + \xi^2 + \eta} d\xi, \quad I_2(\lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{\mu(\xi)^2}{i\lambda + \xi^2 + \eta} d\xi$$

and

$$I_3(f_5, \lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{\mu(\xi) f_5(x, \xi)}{i\lambda + \xi^2 + \eta} d\xi$$

are well defined.

For the proof of Lemma 2.11, see [9, Lemma 2.2] and [69, Lemma 2.1].

Lemma 2.12. *Assume that assumption (A₃) holds and assume that either $(\eta, \lambda) \in \mathbb{R}_+^* \times \mathbb{R}$ or $\eta = 0$ and $\lambda \in \mathbb{R}^*$. Then, $i\lambda I - \mathcal{A}_j$ is surjective, $j = 1, 2$.*

Proof. Let $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}_j$, we must prove that there exists $U = (u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ such that

$$(i\lambda U - \mathcal{A}_j)U = F.$$

Equivalently, we have

$$i\lambda u - v = f_1, \quad (3.68)$$

$$i\lambda y - z = f_3, \quad (3.69)$$

$$\lambda^2 u + \frac{1}{\rho_1} \left(k_1(u_x + y) + \kappa(\alpha) \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x = -f_2 - i\lambda f_1, \quad (3.70)$$

$$\lambda^2 y + \frac{1}{\rho_2} k_2 y_{xx} - \frac{k_1}{\rho_2} (u_x + y) - \frac{\kappa(\alpha)}{\rho_2} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi = -f_4 - i\lambda f_3, \quad (3.71)$$

$$\omega^1(x, \xi) = \frac{f_5(x, \xi)}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{D_1} \mu(\xi) i\lambda u_x}{i\lambda + \xi^2 + \eta} - \frac{\sqrt{D_1} \mu(\xi) (f_1)_x}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{D_1(x)} \mu(\xi) i\lambda y}{i\lambda + \xi^2 + \eta} - \frac{\sqrt{D_1(x)} \mu(\xi) f_3}{i\lambda + \xi^2 + \eta}. \quad (3.72)$$

System (3.68)-(3.72) considered with fully Dirichlet boundary conditions (3.51) or with Dirichlet-Neumann boundary conditions (3.52). Now, inserting (3.72) in (3.70) and in (3.71), respectively, we get

$$\rho_1 \lambda^2 u + \left(\kappa_1 (u_x + y) + D_1 (u_x + y) I_1 - D_1 ((f_1)_x + f_3) I_2 + \sqrt{D_1} I_3 \right)_x = F_1, \quad (3.73)$$

$$\rho_2 \lambda^2 y + \kappa_2 y_{xx} - \kappa_1 (u_x + y) - D_1 (u_x + y) I_1 = F_2, \quad (3.74)$$

where

$$F_1 = -\rho_1 (f_2 + i\lambda f_1),$$

$$F_2 = -\rho_2(f_4 + i\lambda f_3) - D_1((f_1)_x + f_3)I_2(\lambda, \eta, \alpha) + \sqrt{D_1}I_3(\lambda, \eta, \alpha),$$

and $I_1 := I_1(\lambda, \eta, \alpha)$, $I_2 := I_2(\lambda, \eta, \alpha)$ and $I_3 := I_3(f_5, \lambda, \eta, \alpha)$ are defined in Lemma 2.11. System (3.73)-(3.74) considered with fully Dirichlet boundary conditions (3.51) or with Dirichlet-Neumann boundary conditions (3.52). Using Lemma 2.11, we get $I_1(\lambda, \eta, \alpha)$, $I_2(\lambda, \eta, \alpha)$ and $I_3(f_5, \lambda, \eta, \alpha)$ are well-defined, and $\Re(I_1(\lambda, \eta, \alpha)) > 0$.

Now, we distinguish two cases:

Case 1. $\eta > 0$ and $\lambda = 0$, then system (3.73)-(3.74) becomes

$$\begin{cases} -\left(\kappa_1(u_x + y) + \kappa(\alpha)\sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \left(\frac{f_5(x, \xi)}{\xi^2 + \eta} - \frac{\sqrt{D_1}\mu(\xi)((f_1)_x + f_3)}{\xi^2 + \eta} \right) d\xi \right)_x = \rho_1 f_2, \\ -\kappa_2 y_{xx} + \kappa_1(u_x + y) = \rho_2 f_4 - \kappa(\alpha)\sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \left(\frac{f_5(x, \xi)}{\xi^2 + \eta} - \frac{\sqrt{D_1}\mu(\xi)((f_1)_x + f_3)}{\xi^2 + \eta} \right) d\xi \end{cases}$$

with fully Dirichlet boundary conditions (3.51) or with Dirichlet-Neumann boundary conditions (3.52).

By applying Lax-Milgram theorem and using Lemma 2.11 it is easy to see that the above system has a unique strong solution $(u, y) \in \mathcal{V}_j(0, L)$, $j = 1, 2$. In this case the proof is complete.

Case 2. $\eta \geq 0$ and $\lambda \in \mathbb{R}^*$, then system (3.73)-(3.74) becomes

$$\begin{cases} \rho_1 \lambda^2 u + (\kappa_1(u_x + y) + D_1(u_x + y)I_1(\lambda, \eta, \alpha))_x = G_1, \\ \rho_2 \lambda^2 y + \kappa_2 y_{xx} - \kappa_1(u_x + y) - D_1(u_x + y)I_1(\lambda, \eta, \alpha) = G_2 \end{cases} \quad (3.75)$$

such that

$$G_1 = -\rho_1(f_2 + i\lambda f_1) + \left(D_1((f_1)_x + f_3)I_2(\lambda, \eta, \alpha) - \sqrt{D_1}I_3(f_5, \lambda, \eta, \alpha) \right)_x,$$

$$G_2 = -\rho_2(f_4 + i\lambda f_3) - D_1((f_1)_x + f_3)I_2(\lambda, \eta, \alpha) + \sqrt{D_1}I_3(f_5, \lambda, \eta, \alpha)$$

with fully Dirichlet boundary conditions (3.51) or with Dirichlet-Neumann boundary conditions (3.52). Now define the linear unbounded operators

$$\mathcal{L}_1 : \mathcal{V}_1(0, L) = H_0^1(0, L) \times H_0^1(0, L) \rightarrow H^{-1}(0, L) \times H^{-1}(0, L)$$

and

$$\mathcal{L}_2 : \mathcal{V}_2(0, L) = H_0^1(0, L) \times H_*^1(0, L) \rightarrow H^{-1}(0, L) \times (H_*^1(0, L))'$$

by

$$\mathcal{L}_j(u, y) = \left(-\frac{1}{\rho_1} (\kappa_1(u_x + y) + D_1(u_x + y)I_1(\lambda, \eta, \alpha))_x, -\frac{\kappa_2}{\rho_2} y_{xx} + \frac{\kappa_1}{\rho_2} (u_x + y) + \frac{1}{\rho_2} D_1(u_x + y)I_1(\lambda, \eta, \alpha) \right).$$

Let $\mathcal{U} = (u, y)$ and $\mathcal{F} = (G_1, G_2)$, then we transform system (3.75) with fully Dirichlet boundary conditions (3.51) or with Dirichlet-Neumann boundary conditions (3.52) into the following form:

$$(\lambda^2 \mathcal{I} - \mathcal{L}_j)\mathcal{U} = \mathcal{F}. \quad (3.76)$$

Since \mathcal{L}_1 is an isomorphism from $H_0^1(0, L) \times H_0^1(0, L)$ onto $H^{-1}(0, L) \times H^{-1}(0, L)$ and \mathcal{L}_2 is an isomorphism from $H_0^1(0, L) \times H_*^1(0, L)$ onto $H^{-1}(0, L) \times (H_*^1(0, L))'$ and \mathcal{I} is a compact operator from $H_0^1(0, L) \times H_0^1(0, L)$ onto $H^{-1}(0, L) \times H^{-1}(0, L)$ and from $H_0^1(0, L) \times H_*^1(0, L)$ onto $H^{-1}(0, L) \times (H_*^1(0, L))'$, then, using Fredholm's Alternative theorem, problem (3.76) admits a

unique solution in $H^{-1}(0, L) \times H^{-1}(0, L)$ (when $j = 1$), and in $H_0^1(0, L) \times H_\star^1(0, L)$ (when $j = 2$) if and only if $\lambda^2\mathcal{I} - \mathcal{L}_j$ is injective. For that purpose, let $\mathcal{U}_h = (u_h, y_h) \in \ker(\lambda^2\mathcal{I} - \mathcal{L}_j)$. Then, if we set

$$v_h = i\lambda u_h, \quad z_h = i\lambda y_h, \quad \text{and} \quad \omega_h^1 = \frac{i\lambda\sqrt{D_1}\mu(\xi)}{i\lambda + \xi^2 + \eta}(u_x + y),$$

we deduce that $U_h = (u_h, v_h, y_h, z_h, \omega_h^1) \in D(\mathcal{A}_j)$ is solution of

$$(i\lambda - \mathcal{A}_j)U_h = 0, \quad j = 1, 2.$$

It follows from Lemma 2.9, that $u_h = v_h = y_h = z_h = \omega_h^1 = 0$. This implies that equation (3.76) admits a unique solution (u, y) in $H^{-1}(0, L) \times H^{-1}(0, L)$ (when $j = 1$), and in $H_0^1(0, L) \times H_\star^1(0, L)$ (when $j = 2$) and, we have

$$\begin{aligned} (\kappa_1(u_x + y) + D_1(u_x + y)I_1(\lambda, \eta, \alpha))_x &\in L^2(0, L), \\ \kappa_2 y_{xx} - \kappa_1(u_x + y) - D_1(u_x + y)I_1(\lambda, \eta, \alpha) &\in L^2(0, L). \end{aligned}$$

Now, define $v := i\lambda u - f_1$, $z := i\lambda y - f_3$ and

$$\omega^1(x, \xi) = \frac{f_5(x, \xi)}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{D_1(x)}\mu(\xi)}{i\lambda + \xi^2 + \eta}(v_x + z). \quad (3.77)$$

It is easy to see that $\omega^1(x, \xi)$ and $|\xi|\omega^1(x, \xi) \in \mathcal{W}$. This implies that $U = (u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ is the unique solution of equation $(i\lambda I - \mathcal{A}_j)U = F$, $j = 1, 2$, and the proof is thus complete. \square

We are now in a position to conclude the proof of Theorem 2.8.

Proof of Theorem 2.8. Using Lemma 2.9, we directly deduce that \mathcal{A}_j has non pure imaginary eigenvalues. According to Lemmas 2.9, 2.10 and 2.12 and with the help of the closed graph theorem of Banach, we deduce that $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{\phi\}$ if $\eta > 0$ and $\sigma(\mathcal{A}) \cap i\mathbb{R} = \{0\}$ if $\eta = 0$. Thus, we get the conclusion by applying Theorem 2.7 of Arendt and Batty.

In the following sections, we aim to establish the polynomial stability of system (3.13)-(3.19) in three cases: In the first one, the damping is effective only in the bending moment. However, in the second one, the damping is effective only in the shear stress. Finally, in the third case, the damping is present in both the shear stress and the bending moment. For this purpose, we will use a frequency domain approach method, namely we will use Theorem 2.10.

3 Polynomial stability when the damping is effective in the bending moment

In this section, we study the polynomial stability of system (3.13)-(3.19) either in the boundary conditions (3.20) or (3.21) in the case $\eta > 0$, when the fractional Kelvin-Voigt damping is acting only on the bending moment equation, i.e assumption (A₂) holds. In this case, system (3.13)-(3.16) becomes

$$\rho_1 u_{tt} - k_1(u_x + y)_x = 0, (x, t) \in (0, L) \times \mathbb{R}_+, \quad (3.78)$$

$$\rho_2 y_{tt} - \left(k_2 y_x + \kappa(\alpha)\sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi)\omega^2(x, \xi, t)d\xi \right)_x + k_1(u_x + y) = 0, (x, t) \in (0, L) \times \mathbb{R}_+, \quad (3.79)$$

$$\omega_t^2(x, \xi, t) + (\xi^2 + \eta)\omega^2(x, \xi, t) - \sqrt{D_2(x)}y_{tx}\mu(\xi) = 0, (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+, \quad (3.80)$$

subject to the following initial conditions

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, L), \quad (3.81)$$

$$y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (0, L), \quad (3.82)$$

$$\omega^2(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R} \quad (3.83)$$

with fully Dirichlet boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0, \quad (3.84)$$

or with Dirichlet-Neumann boundary conditions

$$u(0) = u(L) = y_x(0) = y_x(L) = 0. \quad (3.85)$$

Our main result in this section is the following theorem.

Theorem 3.1. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (3.86)$$

According to Theorem 2.10 and by taking $\ell = 2$, the polynomial energy decay (3.86) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_j) \quad (H_1)$$

and

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{|\lambda|^2} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O(1), \quad (H_2)$$

are satisfied. Since condition (H_1) is already proved in Theorem 2.8 in the case $\eta > 0$. We will prove condition (H_2) by an argument of contradiction. For this purpose, suppose that (H_2) is false, then there exists

$$\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n^2)^\top)\} \subset \mathbb{R} \times D(\mathcal{A}_j)$$

with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_j} = \|(u_n, v_n, y_n, z_n, \omega_n^2)\|_{\mathcal{H}_j} = 1 \quad (3.87)$$

such that

$$\lambda_n^2 (i\lambda_n I - \mathcal{A}_j) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n})^\top \rightarrow 0 \quad \text{in } \mathcal{H}_j. \quad (3.88)$$

For simplicity, we drop the index n . Equivalently, from (3.88), we have

$$i\lambda u - v = \lambda^{-2} f_1 \quad \text{in } H_0^1(0, L), \quad (3.89)$$

$$i\lambda v - \frac{k_1}{\rho_1} (u_x + y)_x = \lambda^{-2} f_2 \quad \text{in } L^2(0, L), \quad (3.90)$$

$$i\lambda y - z = \lambda^{-2} f_3 \quad \text{in } \mathcal{O}_j(0, L), \quad (3.91)$$

$$i\lambda z - \frac{k_2}{\rho_2} \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x + \frac{k_1}{\rho_2} (u_x + y) = \lambda^{-2} f_4 \quad \text{in } L^2(0, L), \quad (3.92)$$

$$(i\lambda + \xi^2 + \eta) \omega^2 - i\lambda \sqrt{D_2} y_x \mu(\xi) = \lambda^{-2} (f_5 - \sqrt{D_2} (f_3)_x \mu(\xi)) \quad \text{in } \mathcal{W}, \quad (3.93)$$

where

$$\mathcal{O}_j(0, L) = \begin{cases} H_0^1(0, L), & \text{if } j = 1, \\ H_*^1(0, L), & \text{if } j = 2. \end{cases}$$

By inserting (3.89) in (3.90) and (3.91) in (3.92), we deduce that

$$\lambda^2 u + \frac{k_1}{\rho_1} (u_x + y)_x = -(\lambda^{-2} f_2 + i\lambda^{-1} f_1), \quad (3.94)$$

$$\begin{aligned} \lambda^2 y + \frac{k_2}{\rho_2} \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \\ - \frac{k_1}{\rho_2} (u_x + y) = -(\lambda^{-2} f_4 + i\lambda^{-1} f_3). \end{aligned} \quad (3.95)$$

From the above system, $\|U\|_{\mathcal{H}_j} = 1$ and $\|F\|_{\mathcal{H}_j} = o(1)$, we remark that

$$\begin{cases} \|u\| = O(|\lambda|^{-1}), \quad \|y\| = O(|\lambda|^{-1}), \quad \|u_{xx}\| = O(|\lambda|), \\ \left\| \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \right\| = O(|\lambda|). \end{cases} \quad (3.96)$$

Our main goal is to find a contradiction with (3.87) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 3.2. Let $\alpha \in (0, 1)$, $\eta > 0$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned} A_1 &:= A_1(\lambda, \eta, \alpha) = \int_{\mathbb{R}} \frac{|\xi|^{\alpha+\frac{1}{2}}}{(|\lambda| + \xi^2 + \eta)^2} d\xi = c_1 (|\lambda| + \eta)^{\frac{\alpha}{2} - \frac{5}{4}}, \\ A_2 &:= A_2(\lambda, \eta) = \left(\int_{\mathbb{R}} \frac{1}{(|\lambda| + \xi^2 + \eta)^2} d\xi \right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}} \frac{1}{(|\lambda| + \eta)^{\frac{3}{4}}} \end{aligned}$$

and

$$A_3 := A_3(\lambda, \eta) = \left(\int_{\mathbb{R}} \frac{\xi^2}{(|\lambda| + \xi^2 + \eta)^4} d\xi \right)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4} \frac{1}{(|\lambda| + \eta)^{\frac{5}{4}}},$$

where $c_1 = \int_1^\infty \frac{(y-1)^{\frac{\alpha}{2} - \frac{1}{4}}}{y^2} dy$.

For the proof of Lemma 3.2, see [7, Lemma B.5] and [69, Lemma 2.1].

Lemma 3.3. Assume that $\eta > 0$ and assumption (A₂) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following asymptotic behavior estimation:

$$\int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^2(x, \xi)|^2 d\xi dx = o(\lambda^{-2}). \quad (3.97)$$

Proof. Taking the inner product of (3.88) with U in \mathcal{H}_j , then using the fact that U is uniformly bounded in \mathcal{H}_j , we get

$$\begin{aligned} \kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^2(x, \xi)|^2 d\xi dx &= -\Re \left(\langle \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} \right) \\ &= \Re \left(\langle i\lambda U - \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} \right) = o(\lambda^{-2}). \end{aligned}$$

□

Lemma 3.4. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following asymptotic behavior estimation:

$$\int_{a_2}^{b_2} |y_x|^2 dx = o\left(\lambda^{-(3+\alpha)}\right). \quad (3.98)$$

Proof. From (3.93), we get

$$|\lambda| \sqrt{D_2(x)} |\xi|^{\frac{2\alpha-1}{2}} |y_x| \leq (|\lambda| + \xi^2 + \eta) |\omega^2(x, \xi)| + |\lambda|^{-2} |f_5(x, \xi)| + |\lambda|^{-2} \sqrt{D_2(x)} |\xi|^{\frac{2\alpha-1}{2}} |(f_3)_x|.$$

Multiplying the above inequality by $(\lambda + \xi^2 + \eta)^{-2} |\xi|$, integrating over \mathbb{R} with respect to the variable ξ , we obtain

$$\begin{aligned} |\lambda| \sqrt{D_2} |y_x| \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha+1}{2}}}{(\lambda + \xi^2 + \eta)^2} d\xi &\leq \int_{\mathbb{R}} \frac{|\xi \omega^2(x, \xi)|}{(\lambda + \xi^2 + \eta)} d\xi + |\lambda|^{-2} \int_{\mathbb{R}} \frac{|f_5(x, \xi)|}{(\lambda + \xi^2 + \eta)^2} d\xi \\ &+ |\lambda|^{-2} \sqrt{D_2} |(f_3)_x| \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha+1}{2}}}{(\lambda + \xi^2 + \eta)^2} d\xi. \end{aligned} \quad (3.99)$$

Next, applying the Cauchy-Schwarz inequality to (3.99), we obtain

$$\begin{aligned} A_1 \sqrt{D_2} |\lambda y_x| &\leq A_2 \left(\int_{\mathbb{R}} |\xi \omega^2(x, \xi)|^2 d\xi \right)^{\frac{1}{2}} + |\lambda|^{-2} A_3 \left(\int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &+ |\lambda|^{-2} A_1 \sqrt{D_2} |(f_3)_x|, \end{aligned} \quad (3.100)$$

where A_1, A_2 and A_3 are defined in Lemma 3.2. Using Young's inequality and the definition of the function $D_2(x)$ in (3.100), we arrive at

$$\begin{aligned} d_2 \int_{a_2}^{b_2} |\lambda|^2 |y_x|^2 dx &\leq 3 \frac{A_2^2}{A_1^2} \int_0^L \int_{\mathbb{R}} |\xi \omega^2(x, \xi)|^2 d\xi dx + 3 |\lambda|^{-4} \frac{A_3^2}{A_1^2} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx \\ &+ 3 |\lambda|^{-4} d_2 \int_{a_2}^{b_2} |(f_3)_x|^2 dx. \end{aligned}$$

It follows from Lemma 3.2 that

$$d_2 \int_{a_2}^{b_2} |\lambda|^2 |y_x|^2 dx \leq \frac{1}{c_1 (|\lambda| + \eta)^{\alpha-1}} \frac{o(1)}{\lambda^2} + \frac{1}{c_1 (|\lambda| + \eta)^\alpha} \frac{o(1)}{\lambda^4} + \frac{o(1)}{\lambda^4}.$$

Since $\alpha \in (0, 1)$, we have $\min\{1 + \alpha, 4 + \alpha, 4\} = 1 + \alpha$. Hence from the above equation, we get

$$\int_{a_2}^{b_2} |\lambda|^2 |y_x|^2 dx = \frac{o(1)}{\lambda^{1+\alpha}},$$

and so

$$\int_{a_2}^{b_2} |y_x|^2 dx = \frac{o(1)}{\lambda^{3+\alpha}}.$$

The proof is thus completed. \square

Lemma 3.5. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following asymptotic behavior estimation:

$$\int_{a_2}^{b_2} \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right|^2 dx = o(\lambda^{-2}).$$

Proof. Using the fact that $|P + Q|^2 \leq 2P^2 + 2Q^2$, we obtain

$$\begin{aligned} \int_{a_2}^{b_2} \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right|^2 dx &\leq 2 \int_{a_2}^{b_2} |y_x|^2 dx \\ &\quad + 2d_2 \frac{\kappa(\alpha)^2}{k_2^2} \int_{a_2}^{b_2} \left(\int_{\mathbb{R}} \frac{\mu(\xi) \sqrt{\xi^2 + \eta}}{\sqrt{\xi^2 + \eta}} \omega^2(x, \xi) d\xi \right)^2 dx \\ &\leq 2 \int_{a_2}^{b_2} |y_x|^2 dx + c_2 \int_{a_2}^{b_2} \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^2(x, \xi)|^2 d\xi dx, \end{aligned}$$

where $c_2 = 2d_2 \frac{\kappa(\alpha)^2}{k_2^2} A_4(\alpha, \eta)$ and $A_4(\alpha, \eta) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{|\xi|^2 + \eta} d\xi$. From [69, Lemma 2.1], we get A_4 is well defined. Using (3.97) and (3.98), we get our desired result. \square

Lemma 3.6. Assume that $\eta > 0$ and assumption (A₂) holds. Let $0 < \epsilon < \frac{b_2 - a_2}{2}$. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following asymptotic behavior estimation:

$$\int_{a_2+\epsilon}^{b_2-\epsilon} |\lambda y|^2 dx = \frac{O(1)}{\lambda^2}. \quad (3.101)$$

Proof. We define the function $\theta \in C_0^\infty(0, L)$ such that $0 \leq \theta(x) \leq 1$, for all $x \in (0, L)$ and

$$\theta(x) = \begin{cases} 1 & \text{if } x \in (a_2 + \epsilon, b_2 - \epsilon), \\ 0 & \text{if } x \in (0, L) \setminus (a_2, b_2). \end{cases} \quad (3.102)$$

First, multiplying equation (3.95) by $\theta \bar{y}$, integrating over $(0, L)$, then using the fact that $\lambda y, y$ are uniformly bounded in $L^2(0, L)$, $\|f_4\| = o(1)$ and $\|f_3\| = o(1)$, we obtain

$$\begin{aligned} \int_0^L \theta |\lambda y|^2 dx + \frac{k_2}{\rho_2} \int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x (\theta \bar{y}) dx &\quad (3.103) \\ - \frac{k_1}{\rho_2} \int_0^L \theta (u_x + y) \bar{y} dx &= o(\lambda^{-2}). \end{aligned}$$

Using integration by parts and the definition of $\theta(x)$, we get

$$\begin{aligned} &\frac{k_2}{\rho_2} \int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x (\theta \bar{y}) dx \\ &= - \frac{k_2}{\rho_2} \int_0^L \theta \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right) \bar{y}_x dx \\ &\quad - \frac{k_2}{\rho_2} \int_0^L \theta' \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right) \bar{y} dx. \end{aligned}$$

Now, using (3.98), Lemma 3.5, the definition of $D_2(x)$ and the fact that λy is bounded in $L^2(0, L)$, we get

$$\frac{k_2}{\rho_2} \int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x (\theta \bar{y}) dx = \frac{o(1)}{\lambda^2}. \quad (3.104)$$

On the other hand, using Young's inequality, we get

$$\frac{k_1}{\rho_2} |u_x + y| |\theta| |y| \leq \frac{k_1^2}{2\epsilon \rho_2^2} \theta \frac{|u_x + y|^2}{\lambda^2} + \frac{\epsilon \theta}{2} |\lambda y|^2.$$

Consequently, we obtain

$$\frac{k_1}{\rho_2} \int_0^L |\theta| |u_x + y| |y| dx \leq \frac{k_1^2}{2\varepsilon\rho_2^2} \int_0^L \theta \frac{|u_x + y|^2}{\lambda^2} dx + \frac{\varepsilon}{2} \int_0^L \theta |\lambda y|^2 dx. \quad (3.105)$$

Inserting (3.104) and (3.105) in (3.103) and using the fact that $u_x + y$ is uniformly bounded in $L^2(0, L)$, we arrive at

$$\left(1 - \frac{\varepsilon}{2}\right) \int_0^L \theta |\lambda y|^2 dx \leq \frac{o(1)}{\lambda^2} + \frac{O(1)}{\lambda^2}. \quad (3.106)$$

Then, for ε small enough, we have

$$0 < \int_0^L \theta |\lambda y|^2 dx \leq \frac{O(1)}{\lambda^2}. \quad (3.107)$$

From the above estimation and the definition of $\theta(x)$, we obtain (3.101). \square

Lemma 3.7. Assume that $\eta > 0$ and assumption (A₂) holds. Let $0 < \epsilon < \frac{b_2 - a_2}{2}$. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following asymptotic behavior estimation:

$$\int_{a_2 + \epsilon}^{b_2 - \epsilon} |u_x|^2 dx = o(1). \quad (3.108)$$

Proof. First, multiplying (3.95) by $-\theta \bar{u}_x$, integration by parts, using the fact that λu is bounded in $L^2(0, L)$, and $\|f_4\| = o(1)$, $\|f_3\| = o(1)$, $\|(f_3)_x\| = o(1)$, we obtain

$$\begin{aligned} & - \int_0^L \theta \lambda^2 y \bar{u}_x dx - \frac{k_2}{\rho_2} \int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x (\theta \bar{u}_x) dx \\ & + \frac{k_1}{\rho_2} \int_0^L \theta (u_x + y) \bar{u}_x dx = o(\lambda^{-2}). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \frac{k_1}{\rho_2} \int_0^L \theta |u_x|^2 dx &= - \frac{k_1}{\rho_2} \int_0^L \theta y \bar{u}_x dx + \int_0^L \theta \lambda^2 y \bar{u}_x dx \\ &+ \frac{k_2}{\rho_2} \int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x (\theta \bar{u}_x) dx \\ &+ \frac{o(1)}{\lambda^2}. \end{aligned} \quad (3.109)$$

Now, using the integration by parts, the fact that λu is uniformly bounded in $L^2(0, L)$, (3.101) and (3.98), we obtain

$$\int_0^L \theta \lambda^2 y \bar{u}_x dx = - \int_0^L \theta \lambda y_x \lambda \bar{u} dx - \int_0^L \theta' \lambda y \lambda \bar{u} dx = \frac{o(1)}{\lambda^{\frac{1+\alpha}{2}}}. \quad (3.110)$$

Using integration by parts, the fact that $\frac{\|u_{xx}\|}{\lambda} = O(1)$, u_x is uniformly bounded in $L^2(0, L)$ and Lemma 3.5, we find

$$\begin{aligned} & \frac{k_2}{\rho_2} \int_0^L \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x (\theta \bar{u}_x) dx = \\ & - \frac{k_2}{\rho_2} \int_0^L \theta \lambda \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right) \frac{1}{\lambda} \bar{u}_{xx} dx \\ & - \frac{k_2}{\rho_2} \int_0^L \theta' \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right) \bar{u}_x dx \\ & = o(1). \end{aligned} \quad (3.111)$$

Next, using (3.101) and the fact that u_x is bounded, we see that

$$\frac{k_1}{\rho_2} \int_0^L \theta y \bar{u}_x dx = \frac{O(1)}{\lambda^2}. \quad (3.112)$$

Now, inserting (3.110), (3.111) and (3.112) in (3.109), it leads to

$$\frac{k_1}{\rho_2} \int_0^L \theta |u_x|^2 dx = o(1). \quad (3.113)$$

From the above estimation and the definition of $\theta(x)$, we obtain (3.108). \square

Lemma 3.8. Assume that $\eta > 0$ and assumption (A₂) holds. Let $0 < \epsilon < \frac{b_2 - a_2}{2}$. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following asymptotic behavior estimation:

$$\int_{a_2 + \epsilon}^{b_2 - \epsilon} |\lambda u|^2 dx = o(1). \quad (3.114)$$

Proof. First, multiplying equation (3.94) by $\theta \bar{u}$, integrating over $(0, L)$, using the fact that λu and u are uniformly bounded in $L^2(0, L)$, and the fact that $\|f_1\| = o(1)$, $\|f_2\| = o(1)$, we remark that

$$\int_0^L \theta |\lambda u|^2 dx + \frac{k_1}{\rho_1} \int_0^L (u_x + y)_x (\theta \bar{u}) dx = o(\lambda^{-2}). \quad (3.115)$$

Using integration by parts, we obtain

$$\frac{k_1}{\rho_1} \int_0^L (u_x + y)_x (\theta \bar{u}) dx = -\frac{k_1}{\rho_1} \int_0^L \theta (u_x + y) \bar{u}_x dx - \frac{k_1}{\rho_1} \int_0^L \theta' (u_x + y) \bar{u} dx,$$

inserting the above equation in (3.115), we get

$$\begin{aligned} \int_0^L \theta |\lambda u|^2 dx &= \frac{k_1}{\rho_1} \int_0^L \theta (u_x + y) \bar{u}_x dx + \frac{k_1}{\rho_1} \int_0^L \theta' (u_x + y) \bar{u} dx + o(\lambda^{-2}) \\ &= \frac{k_1}{\rho_1} \left(\int_0^L \theta |u_x|^2 dx + \int_0^L \theta y \bar{u}_x dx + \int_0^L \theta' u_x \bar{u} dx + \int_0^L \theta' y \bar{u} dx \right) + o(\lambda^{-2}). \end{aligned} \quad (3.116)$$

Now, using definition of $\theta(x)$, (3.101), (3.108) and the fact that u bounded in $L^2(0, L)$, we get (3.114). \square

From what precedes, Lemmas 3.3-3.8, we deduce that

$$\|U\|_{\mathcal{H}_j} = o(1), \quad \text{over } (a_2 + \epsilon, b_2 - \epsilon).$$

Lemma 3.9. Assume that $\eta > 0$ and assumption (A₂) holds. Let $\phi \in C^1([0, L])$ and $\phi(0) = \phi(L) = 0$ be a given function. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following behavior estimation:

$$\int_0^L \phi' \left(\rho_1 |\lambda u|^2 + k_1 |u_x|^2 + \rho_2 |\lambda y|^2 + k_2 \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right|^2 \right) dx = o(1).$$

Proof. First, multiplying equation (3.94) by $2\rho_1 \phi \bar{u}_x$, integrating over $(0, L)$, taking the real part, the fact that u_x is uniformly bounded in $L^2(0, L)$, $\|f_1\| = o(1)$ and $\|f_2\| = o(1)$, we obtain

$$\rho_1 \int_0^L \phi \frac{d}{dx} |\lambda u|^2 dx + k_1 \int_0^L \phi \frac{d}{dx} |u_x|^2 dx + \Re \left\{ 2k_1 \int_0^L \phi y_x \bar{u}_x dx \right\} = o(\lambda^{-1}). \quad (3.117)$$

Now, multiplying equation (3.95) by $2\rho_2\phi\left(\bar{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)$, integrating over $(0, L)$, taking the real part, then using the fact that $\left(y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2(x)}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right)$ is uniformly bounded in $L^2(0, L)$, $\|y\| = O(|\lambda|^{-1})$, $\|f_3\| = o(1)$ and $\|f_4\| = o(1)$, we obtain

$$\begin{aligned} & \Re\left\{2\rho_2\lambda^2\int_0^L\phi y\left(\bar{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\} \\ & + k_2\int_0^L\phi\frac{d}{dx}\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2dx \\ & - \Re\left\{2k_1\int_0^L\phi u_x\left(\bar{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\} \\ & - \underbrace{\Re\left\{2k_1\int_0^L\phi y\left(\bar{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\}}_{=o(1)} \tag{3.118} \\ & = \underbrace{\Re\left\{2\rho_2\int_0^L\phi\left(-\lambda^{-2}f_4 - i\lambda^{-1}f_3\right)\left(\bar{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\}}_{=o(\lambda^{-1})}. \end{aligned}$$

Moreover, by using the definition of $D_2(x)$ and Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$ and $\eta > 0$ and by using Lemma 3.3 and the fact that $u_x, \lambda y$ are uniformly bounded in $L^2(0, L)$, we obtain

$$\begin{cases} \Re\left\{2\rho_2\lambda^2\int_0^L\phi y\left(\bar{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\} = \rho_2\int_0^L\phi\frac{d}{dx}|\lambda y|^2dx + o(1), \\ -\Re\left\{2k_1\int_0^L\phi u_x\left(\bar{y}_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi\right)dx\right\} = -\Re\left\{2k_1\int_0^L\phi u_x\bar{y}_x dx\right\} + o(\lambda^{-1}). \end{cases}$$

Inserting the above equations in (3.118), we obtain

$$\begin{aligned} & \rho_2\int_0^L\phi\frac{d}{dx}|\lambda y|^2dx + k_2\int_0^L\phi\frac{d}{dx}\left|y_x + \frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\omega^2(x,\xi)d\xi\right|^2dx \tag{3.119} \\ & - \Re\left\{2k_1\int_0^L\phi u_x y_x dx\right\} = o(1). \end{aligned}$$

Adding (3.117) and (3.119), then using integration by parts, we obtain (3.9). The proof is thus complete. \square

Lemma 3.10. Assume that $\eta > 0$ and assumption (A_2) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^2) \in D(\mathcal{A}_j)$ of system (3.89)-(3.93) satisfies the following asymptotic behavior estimation:

$$\|U\|_{\mathcal{H}_j} = o(1). \tag{3.120}$$

Proof. Define the cut-off functions $\theta_1, \theta_2 \in C_0^\infty([0, L])$ by

$$\theta_1(x) = \begin{cases} 1 & \text{if } x \in (0, a_2 + \epsilon), \\ 0 & \text{if } x \in (b_2 - \epsilon, L), \\ \in [0, 1] & \text{elsewhere} \end{cases}$$

and

$$\theta_2(x) = \begin{cases} 1 & \text{if } x \in (b_2 - \epsilon, L), \\ 0 & \text{if } x \in (0, a_2 + \epsilon), \\ \in [0, 1] & \text{elsewhere.} \end{cases}$$

Take $\phi = x\theta_1$ in Lemma 3.9, then use the fact that $\|U\|_{\mathcal{H}_j} = o(1)$ in $(a_2 + \epsilon, b_2 - \epsilon)$ and $a_2 + \epsilon < a_2 + 2\epsilon < b_2 - \epsilon$, we have

$$\begin{aligned} & \int_0^L x\theta_1' \left(\rho_1|v|^2 + k_1|u_x|^2 + \rho_2|z|^2 + k_2 \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi)\omega^2(x, \xi)d\xi \right|^2 \right) dx \\ & + \int_0^L \theta_1 \left(\rho_1|v|^2 + k_1|u_x|^2 + \rho_2|z|^2 + k_2 \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi)\omega^2(x, \xi)d\xi \right|^2 \right) dx = o(1), \end{aligned} \quad (3.121)$$

therefore, we get

$$\int_0^{a_2+\epsilon} \left(\rho_1|v|^2 + k_1|u_x|^2 + \rho_2|z|^2 + k_2 \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi)\omega^2(x, \xi)d\xi \right|^2 \right) dx = o(1). \quad (3.122)$$

Moreover, using estimation (3.97), the definition of D_2 and (3.122), we observe that

$$\begin{aligned} \int_0^{a_2+\epsilon} |y_x|^2 dx & \leq 2 \int_0^{a_2+\epsilon} \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi)\omega^2(x, \xi)d\xi \right|^2 dx \\ & \quad + 2 \frac{d_2 \kappa(\alpha)^2}{k_2^2} \int_{a_2}^{a_2+\epsilon} \int_{\mathbb{R}} |\xi|^{2\alpha-1} |\omega^2(x, \xi)|^2 d\xi dx \\ & = o(1), \end{aligned} \quad (3.123)$$

using (3.122) and (3.123), we get

$$\|U\|_{\mathcal{H}_j} = o(1) \text{ on } (0, a_2 + \epsilon).$$

Similarly, by taking $\phi = (x - L)\theta_2$, we can prove $\|U\|_{\mathcal{H}_j} = o(1)$ on $(b_2 - \epsilon, L)$. Therefore,

$$\|U\|_{\mathcal{H}_j} = o(1) \text{ on } (0, L).$$

Thus, the proof is complete. \square

Proof of Theorem 3.1. For $j = 1, 2$, from Lemma 3.10, we get that $\|U\|_{\mathcal{H}_j} = o(1)$, which contradicts (3.87). This implies that

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{\lambda^2} \left\| (i\lambda I - \mathcal{A}_j)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_j)} < \infty.$$

The result follows from Theorem 2.10.

4 Polynomial stability when the damping is effective in shear force

In this section, we study the polynomial stability of system (3.45)-(3.50) with the boundary conditions (3.51) or (3.52) in the case $\eta > 0$, when the fractional Kelvin-Voigt damping is acting only on the shear force equation, i.e assumption (A₃) holds. Our main result in this section is the following theorem.

Theorem 4.1. Assume that $\eta > 0$ and assumption (A_3) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:

$$E(t) \leq \frac{c}{t} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (3.124)$$

According to Theorem 2.10 and by taking $\ell = 2$, the polynomial energy decay (3.124) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_j) \quad (H_3)$$

and

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{|\lambda|^2} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O(1) \quad (H_4)$$

are satisfied. Since condition (H_3) is already proved in Theorem 2.8 in the case $\eta > 0$. We will prove condition (H_4) by an argument of contradiction. For this purpose, suppose that (H_4) is false, then there exists

$$\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n^1)^\top)\} \subset \mathbb{R} \times D(\mathcal{A}_j)$$

with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_j} = \|(u_n, v_n, y_n, z_n, \omega_n^1)\|_{\mathcal{H}_j} = 1 \quad (3.125)$$

such that

$$\lambda_n^2 (i\lambda_n I - \mathcal{A}_j) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n})^\top \rightarrow 0 \quad \text{in } \mathcal{H}_j. \quad (3.126)$$

For simplicity, we drop the index n . Equivalently, from (3.126), we have

$$i\lambda u - v = \lambda^{-2} f_1 \quad \text{in } H_0^1(0, L), \quad (3.127)$$

$$i\lambda v - \frac{k_1}{\rho_1} \left((u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x = \lambda^{-2} f_2 \quad \text{in } L^2(0, L), \quad (3.128)$$

$$i\lambda y - z = \lambda^{-2} f_3 \quad \text{in } \mathcal{O}_j(0, L), \quad (3.129)$$

$$\begin{aligned} & i\lambda z - \frac{k_2}{\rho_2} y_{xx} + \frac{k_1}{\rho_2} (u_x + y) \\ & + \frac{\kappa(\alpha)}{\rho_2} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi = \lambda^{-2} f_4 \quad \text{in } L^2(0, L), \end{aligned} \quad (3.130)$$

$$\begin{aligned} (i\lambda + \xi^2 + \eta) \omega^1 - i\lambda \sqrt{D_1} (u_x + y) \mu(\xi) &= \lambda^{-2} [f_5 - \sqrt{D_1} \mu(\xi) ((f_1)_x + f_3)] \\ &\quad \text{in } \mathcal{W}, \end{aligned} \quad (3.131)$$

where

$$\mathcal{O}_j(0, L) = \begin{cases} H_0^1(0, L), & \text{if } j = 1, \\ H_*^1(0, L), & \text{if } j = 2. \end{cases}$$

By inserting (3.127) in (3.128) and (3.129) in (3.130), we obtain

$$\lambda^2 u + \frac{k_1}{\rho_1} \left((u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)_x = -[\lambda^{-2} f_2 + i\lambda^{-1} f_1], \quad (3.132)$$

$$\lambda^2 y + \frac{k_2}{\rho_2} y_{xx} - \frac{k_1}{\rho_2} (u_x + y) - \frac{\kappa(\alpha)}{\rho_2} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi = -[\lambda^{-2} f_4 + i\lambda^{-1} f_3]. \quad (3.133)$$

From the above system, $\|U\|_{\mathcal{H}_j} = 1$ and $\|F\|_{\mathcal{H}_j} = o(1)$, we remark that

$$\begin{cases} \|u\| = O(|\lambda|^{-1}), \quad \|y\| = O(|\lambda|^{-1}), \quad \|y_{xx}\| = O(|\lambda|), \\ \left\| \left((u_x + y) + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right)_x \right\| = O(|\lambda|). \end{cases} \quad (3.134)$$

Our main goal is to find a contradiction with (3.125) such as $\|U_n\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several Lemmas.

Lemma 4.2. Assume that $\eta > 0$ and assumption (A₃) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^1(x, \xi)|^2 d\xi dx = o(\lambda^{-2}). \quad (3.135)$$

Proof. Taking the inner product of (3.126) with U in \mathcal{H}_j , then using the fact that U is uniformly bounded in \mathcal{H}_j , we get

$$\begin{aligned} \kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^1(x, \xi)|^2 d\xi dx &= -\Re \left(\langle \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} \right) \\ &= \Re \left(\langle i\lambda U - \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} \right) = o(\lambda^{-2}). \end{aligned}$$

□

Lemma 4.3. Assume that $\eta > 0$ and assumption (A₃) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\int_{a_1}^{b_1} |u_x + y|^2 dx = o(\lambda^{-(3+\alpha)}). \quad (3.136)$$

Proof. From (3.131), we get

$$\begin{aligned} |\lambda| \sqrt{D_1(x)} |\xi|^{\frac{2\alpha-1}{2}} |u_x + y| &\leq (|\lambda| + \xi^2 + \eta) |\omega^1(x, \xi)| \\ &\quad + |\lambda|^{-2} |f_5(x, \xi)| + |\lambda|^{-2} \sqrt{D_1(x)} |\xi|^{\frac{2\alpha-1}{2}} |(f_1)_x + f_3|. \end{aligned}$$

Multiplying the above equation by $(\lambda + \xi^2 + \eta)^{-2} |\xi|$, integrating over \mathbb{R} and proceeding in a similar way as in Lemma 3.3 (Section 3), we get our desired estimation (3.136). Thus, the proof is complete. □

Lemma 4.4. Assume that $\eta > 0$ and assumption (A₃) holds. Then, for $j = 1, 2$, the solution $U \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\int_{a_1}^{b_1} \left| (u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right|^2 dx = o(\lambda^{-2}).$$

Proof. using the fact that $|P + Q|^2 \leq 2P^2 + 2Q^2$, we obtain

$$\begin{aligned} &\int_{a_1}^{b_1} \left| (u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right|^2 dx \\ &\leq 2 \int_{a_1}^{b_1} |u_x + y|^2 dx + 2d_1 \frac{\kappa(\alpha)^2}{k_1^2} \int_{a_1}^{b_1} \left(\int_{\mathbb{R}} \frac{\mu(\xi) \sqrt{\xi^2 + \eta}}{\sqrt{\xi^2 + \eta}} \omega^1(x, \xi) d\xi \right)^2 dx \\ &\leq 2 \int_{a_1}^{b_1} |u_x + y|^2 dx + c_3 \int_{a_1}^{b_1} \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^1(x, \xi)|^2 d\xi dx, \end{aligned}$$

where $c_3 = 2d_1 \frac{\kappa(\alpha)^2}{k_1^2} A_4(\alpha, \eta)$ and $A_4(\alpha, \eta)$ is defined in Lemma 3.5.

Using (3.135) and (3.136), we get our desired result. Hence, the proof is complete. □

Lemma 4.5. Assume that $\eta > 0$ and assumption (A_3) holds. Let $0 < \epsilon < \frac{b_1 - a_1}{2}$. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\int_{a_1 + \epsilon}^{b_1 - \epsilon} |\lambda u|^2 dx = \frac{o(1)}{\lambda^2}. \quad (3.137)$$

Proof. We define the function $\theta_3 \in C_0^\infty(0, L)$ such that $0 \leq \theta_3(x) \leq 1$, for all $x \in (0, L)$, by

$$\theta_3(x) = \begin{cases} 1 & \text{if } x \in (a_1 + \epsilon, b_1 - \epsilon), \\ 0 & \text{if } x \in (0, L) \setminus (a_1, b_1). \end{cases} \quad (3.138)$$

First, multiplying (3.132) by $\theta_3 \bar{u}$, integrating over $(0, L)$, using the fact that λu and u are uniformly bounded in $L^2(0, L)$ and $\|f_1\| = o(1), \|f_2\| = o(1)$, we obtain

$$\int_0^L \theta_3 |\lambda u|^2 dx + \frac{k_1}{\rho_1} \int_0^L \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right]_x (\theta_3 \bar{u}) dx = o(\lambda^{-2}). \quad (3.139)$$

Using integration by parts on the second term of the left hand side of (3.139), we get

$$\begin{aligned} & \frac{k_1}{\rho_1} \int_0^L \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right]_x (\theta_3 \bar{u}) dx \\ &= -\frac{k_1}{\rho_1} \int_0^L \theta_3 \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right] \bar{u}_x dx \\ & \quad - \frac{k_1}{\rho_1} \int_0^L \theta_3' \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right] \bar{u} dx. \end{aligned}$$

Now, using (3.136), Lemma 4.4 and the fact that λy is bounded in $L^2(0, L)$, we get

$$\begin{aligned} & \frac{k_1}{\rho_1} \int_0^L \theta_3 \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right] \bar{u}_x dx \\ &= \frac{k_1}{\rho_1} \int_0^L \theta_3 \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right] (\bar{u}_x + \bar{y}) dx \\ & \quad - \frac{k_1}{\rho_1} \int_0^L \theta_3 \frac{1}{\lambda} \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right] (\lambda \bar{y}) dx \\ &= \frac{o(1)}{\lambda^2}. \end{aligned} \quad (3.140)$$

Thanks to Lemma 4.4 and the fact that λu is uniformly bounded in $L^2(0, L)$, we have

$$\frac{k_1}{\rho_1} \int_0^L \theta_3' \frac{1}{\lambda} \left[(u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right] \lambda \bar{u} dx = \frac{o(1)}{\lambda^2}. \quad (3.141)$$

Now, inserting (3.140) and (3.141) in (3.139), we get

$$\int_0^L \theta_3 |\lambda u|^2 dx = \frac{o(1)}{\lambda^2}. \quad (3.142)$$

From the above estimation and the definition of $\theta_3(x)$, we obtain (3.137). Thus, the proof is complete. \square

Lemma 4.6. Assume that $\eta > 0$ and assumption (A_3) holds. Let $0 < \epsilon < \frac{b_1 - a_1}{2}$, then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\int_{a_1 + \epsilon}^{b_1 - \epsilon} |\lambda y|^2 dx = o(1). \quad (3.143)$$

Proof. First, multiplying (3.133) by $\theta_3(\overline{u_x + y})$, integrating over $(0, L)$, using integration by parts, the fact that $\lambda u, \lambda y$ are uniformly bounded in $L^2(0, L)$ and $\|f_3\| = o(1), \|f_4\| = o(1)$, we obtain

$$\begin{aligned} & \int_0^L \theta_3 \lambda^2 y(\overline{u_x + y}) dx + \frac{k_2}{\rho_2} \int_0^L \theta_3 y_{xx}(\overline{u_x + y}) dx - \frac{k_1}{\rho_2} \int_0^L \theta_3 |u_x + y|^2 dx \\ & - \frac{\kappa(\alpha)}{\rho_2} \int_0^L \theta_3 \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi (\overline{u_x + y}) dx = o(\lambda^{-2}). \end{aligned} \quad (3.144)$$

Using (3.135), (3.136) and the fact that $\frac{1}{\lambda} \|y_{xx}\| = O(1)$, we get

$$\begin{cases} \frac{k_2}{\rho_2} \int_0^L \theta_3 \frac{1}{\lambda} y_{xx} \lambda (\overline{u_x + y}) dx = \frac{o(1)}{\lambda^{\frac{1+\alpha}{2}}}, \\ \frac{k_1}{\rho_2} \int_0^L \theta_3 |u_x + y|^2 dx = \frac{o(1)}{\lambda^{3+\alpha}}, \\ \frac{\kappa(\alpha)}{\rho_2} \int_0^L \theta_3 \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi (\overline{u_x + y}) dx = \frac{o(1)}{\lambda^{\frac{5+\alpha}{2}}}. \end{cases} \quad (3.145)$$

Inserting (3.145) in (3.144) and since $0 < \alpha < 1$, we get

$$\int_0^L \theta_3 \lambda^2 y \overline{u_x} dx + \int_0^L \theta_3 |\lambda y|^2 dx = \frac{o(1)}{\lambda^{\frac{1+\alpha}{2}}}. \quad (3.146)$$

Using the integration by parts, we get

$$\int_0^L \theta_3 \lambda^2 y \overline{u_x} dx = - \int_0^L \theta_3' \lambda y \lambda \overline{u} dx - \int_0^L \theta_3 \lambda y_x \lambda \overline{u} dx. \quad (3.147)$$

Now, using (3.137) and the fact that λy and y_x are uniformly bounded in $L^2(0, L)$, we get

$$\int_0^L \theta_3 \lambda^2 y \overline{u_x} dx = o(1). \quad (3.148)$$

Inserting (3.148) in (3.146), we get

$$\int_0^L \theta_3 |\lambda y|^2 dx = o(1).$$

From the above estimation and the definition of $\theta_3(x)$, we obtain (3.143). Thus, the proof is complete. \square

Lemma 4.7. Assume that $\eta > 0$ and assumption (A₃) holds. Let $0 < \epsilon < \frac{b_1 - a_1}{2}$. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\int_{a_1+\epsilon}^{b_1-\epsilon} |y_x|^2 dx = o(1). \quad (3.149)$$

Proof. First, multiplying (3.133) by $\theta_3 \overline{y}$, integrating over $(0, L)$, using integration by parts, using the fact that λy is bounded in $L^2(0, L)$, $\|f_3\| = o(1)$ and $\|f_4\| = o(1)$, we obtain

$$\begin{aligned} & \int_0^L \theta_3 \lambda^2 |y|^2 dx + \frac{k_2}{\rho_2} \int_0^L \theta_3 y_{xx} \overline{y} dx - \frac{k_1}{\rho_2} \int_0^L \theta_3 (u_x + y) \overline{y} dx \\ & - \frac{\kappa(\alpha)}{\rho_2} \int_0^L \theta_3 \sqrt{D_1(x)} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \overline{y} dx = o(\lambda^{-2}). \end{aligned} \quad (3.150)$$

Using integration by parts for the second term of the left hand side, we get

$$\frac{k_2}{\rho_2} \int_0^L \theta_3 y_{xx} \overline{y} dx = - \frac{k_2}{\rho_2} \int_0^L \theta_3' y_x \overline{y} dx - \frac{k_2}{\rho_2} \int_0^L \theta_3 |y_x|^2 dx. \quad (3.151)$$

Inserting (3.151) in (3.150), using (3.143), (3.135) and the fact that y_x and $u_x + y$ are uniformly bounded in $L^2(0, L)$, we get

$$\frac{k_2}{\rho_2} \int_0^L \theta_3 |y_x|^2 dx = o(1). \quad (3.152)$$

From the above estimation and the definition of $\theta_3(x)$, we obtain (3.149). Thus, the proof is complete. \square

From what precedes, from Lemmas 4.2-4.7, we deduce that

$$\|U\|_{\mathcal{H}_j} = o(1) \quad \text{over} \quad (a_1 - \epsilon, b_1 - \epsilon).$$

Lemma 4.8. Assume that $\eta > 0$ and assumption (A₃) holds. Let $h \in C^1([0, L])$ and $h(0) = h(L) = 0$ be a given function. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\int_0^L h' \left(\rho_1 |\lambda u|^2 + k_2 |y_x|^2 + \rho_2 |\lambda y|^2 + k_1 \left| u_x + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right|^2 \right) dx = o(1).$$

Proof. Let $S := u_x + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi$, from Lemma 4.2, the definition of $D_1(x)$ and the fact that u_x is uniformly bounded in $L^2(0, L)$, we get S is uniformly bounded in $L^2(0, L)$. First, multiplying (3.132) by $2\rho_1 h \bar{S}$, integrating over $(0, L)$, taking the real part, then using the fact that $\|f_1\| = o(1)$ and $\|f_2\| = o(1)$, we obtain

$$\begin{aligned} & \Re \left\{ \rho_1 \int_0^L 2h \lambda^2 u \bar{S} dx \right\} + \Re \left\{ k_1 \int_0^L h y_x \bar{S} dx \right\} + k_1 \int_0^L h (|S|^2)_x dx \\ &= \underbrace{\Re \left\{ -2\rho_1 \int_0^L h (\lambda^{-2} f_2 + i \lambda^{-1} f_1) \bar{S} dx \right\}}_{o(\lambda^{-1})}. \end{aligned} \quad (3.153)$$

Moreover, from the definition of S and the definition of $D_1(x)$ and from using Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$ and $\eta > 0$, Lemma 4.2 and the fact that $y_x, \lambda u$ are uniformly bounded in $L^2(0, L)$, we obtain

$$\left\{ \begin{aligned} \Re \left\{ 2k_1 \int_0^L h y_x \bar{S} dx \right\} &= \Re \left\{ 2k_1 \int_0^L h y_x \bar{u}_x dx \right\} + \underbrace{\Re \left\{ 2\kappa(\alpha) \sqrt{d_1} \int_{a_1}^{b_1} h y_x \int_{\mathbb{R}} \mu(\xi) \overline{\omega^1(x, \xi)} d\xi dx \right\}}_{=o(\lambda^{-1})}, \\ \Re \left\{ \rho_1 \int_0^L 2h \lambda^2 u \bar{S} dx \right\} &= \rho_1 \int_0^L h (|\lambda u|^2)_x dx + \underbrace{\Re \left\{ 2\rho_1 \frac{\kappa(\alpha)}{k_1} \sqrt{d_1} \int_{a_1}^{b_1} h \lambda^2 u \int_{\mathbb{R}} \mu(\xi) \overline{\omega^1(x, \xi)} d\xi dx \right\}}_{=o(1)}. \end{aligned} \right.$$

Inserting the above estimations in (3.153), we get

$$\rho_1 \int_0^L h (|\lambda u|^2)_x dx + k_1 \int_0^L h (|S|^2)_x dx + \Re \left\{ 2k_1 \int_0^L h y_x \bar{u}_x dx \right\} = o(1). \quad (3.154)$$

Now, multiplying (3.133) by $2\rho_2 h \bar{y}_x$, integrating over $(0, L)$, taking the real part, then using Lemma 4.2, the fact that y_x is uniformly bounded in $L^2(0, L)$, $\|y\| = O(|\lambda|^{-1})$, $\|f_3\| = o(1)$ and $\|f_4\| = o(1)$,

we obtain

$$\begin{aligned}
 & \rho_2 \int_0^L h (|\lambda y|^2)_x dx + k_2 \int_0^L h (|y_x|^2)_x dx - \Re \left\{ 2k_1 \int_0^L h u_x \overline{y_x} dx \right\} \\
 & - \underbrace{\Re \left\{ k_1 \int_0^L 2h y \overline{y_x} dx \right\}}_{=o(1)} - \underbrace{\Re \left\{ 2\kappa(\alpha) d_1 \int_{a_1}^{b_1} h \overline{y_x} \int_{\mathbb{R}} \mu(\xi) \overline{\omega^1(x, \xi)} d\xi dx \right\}}_{=o(\lambda^{-1})} \\
 & = \underbrace{\Re \left\{ -2\rho_2 \int_0^L h (\lambda^{-2} f_4 + i\lambda^{-1} f_4) \overline{y_x} dx \right\}}_{=o(\lambda^{-1})}.
 \end{aligned} \tag{3.155}$$

Adding (3.154) and (3.155), then using integration by parts, we obtain

$$\int_0^L h' \left(\rho_1 |\lambda u|^2 + k_2 |y_x|^2 + \rho_2 |\lambda y|^2 + k_1 \left| u_x + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right|^2 \right) dx = o(1). \tag{3.156}$$

The proof is thus complete. \square

Lemma 4.9. Assume that $\eta > 0$ and assumption (A_3) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1) \in D(\mathcal{A}_j)$ of system (3.127)-(3.131) satisfies the following asymptotic behavior estimation:

$$\|U\|_{\mathcal{H}_j} = o(1). \tag{3.157}$$

Proof. Define the cut-off functions $\theta_4, \theta_5 \in C_0^\infty([0, L])$ by

$$\theta_4(x) = \begin{cases} 1 & \text{if } x \in (0, a_1 + \epsilon), \\ 0 & \text{if } x \in (b_1 - \epsilon, L), \\ \in [0, 1] & \text{elsewhere} \end{cases}$$

and

$$\theta_5(x) = \begin{cases} 1 & \text{if } x \in (b_1 - \epsilon, L), \\ 0 & \text{if } x \in (0, a_1 + \epsilon), \\ \in [0, 1] & \text{elsewhere.} \end{cases}$$

First, by taking $\phi = x\theta_4$ in Lemma 4.8 and proceeding in a similar way as in Lemma 3.10, we get $\|U\|_{\mathcal{H}_j} = o(1)$ on $(0, a_1 + \epsilon)$. Moreover, by taking $\phi = (x - L)\theta_5$, we can prove $\|U\|_{\mathcal{H}_j} = o(1)$ on $(b_1 - \epsilon, L)$. Thus, the proof is complete. \square

Proof of Theorem 4.1. From Lemma 4.9 we get that $\|U\|_{\mathcal{H}_j} = o(1)$, which contradicts (3.125). This implies that

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{\lambda^2} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} < \infty.$$

The result follows from Theorem 2.10.

5 Polynomial stability when the damping is effective in shear force and bending moment

In this section, we study the polynomial stability of system (3.13)-(3.19) in the case $\eta > 0$, when the fractional Kelvin-Voigt damping is present in both shear stress and bending moment equations provided that the following assumption

$$D_1 \neq 0, \quad D_2 \neq 0, \quad a_1 = a_2 = a \quad \text{and} \quad b_1 = b_2 = b \quad (\text{A}_4)$$

hold. Our main result in this section is the following theorem.

Theorem 5.1. Assume that $\eta > 0$ and assumption (A₄) holds. Then, for $j = 1, 2$, there exists $c > 0$ such that, for every $U_0 \in D(\mathcal{A}_j)$, the following energy estimation holds:

$$E(t) \leq \frac{c}{t^{2-\alpha}} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad t > 0. \quad (3.158)$$

According to Theorem 2.10 and by taking $\ell = 1 - \frac{\alpha}{2}$, the polynomial energy decay (3.158) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_j) \quad (\text{H}_5)$$

and

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{|\lambda|^{1-\frac{\alpha}{2}}} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} = O(1), \quad (\text{H}_6)$$

are satisfied. In the case $\eta > 0$, according to Theorem 2.8, condition (H₅) is proved. Now, we will prove condition (H₆) by an argument of contradiction. For this purpose, suppose that (H₆) is false, then there exists

$$\left\{ (\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n^1, \omega_n^2)^\top) \right\} \subset \mathbb{R} \times D(\mathcal{A}_j)$$

with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_j} = \|(u_n, v_n, y_n, z_n, \omega_n^1, \omega_n^2)\|_{\mathcal{H}_j} = 1 \quad (3.159)$$

such that

$$\lambda_n^{1-\frac{\alpha}{2}} (i\lambda_n I - \mathcal{A}) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}, f_{6,n})^\top \rightarrow 0 \quad \text{in } \mathcal{H}_j. \quad (3.160)$$

For simplicity, we drop the index n . Equivalently, from (3.160), we have

$$i\lambda u - v = \lambda^{\frac{\alpha}{2}-1} f_1 \quad \text{in } H_0^1(0, L), \quad (3.161)$$

$$i\lambda v - \frac{k_1}{\rho_1} (S_{d_1})_x = \lambda^{\frac{\alpha}{2}-1} f_2 \quad \text{in } L^2(0, L), \quad (3.162)$$

$$i\lambda y - z = \lambda^{\frac{\alpha}{2}-1} f_3 \quad \text{in } \mathcal{O}_j(0, L), \quad (3.163)$$

$$i\lambda z - \frac{k_2}{\rho_2} (S_{d_2})_x + \frac{k_1}{\rho_2} (u_x + y) \quad (3.164)$$

$$+ \frac{\kappa(\alpha)\sqrt{D_1}}{\rho_2} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi = \lambda^{\frac{\alpha}{2}-1} f_4 \quad \text{in } L^2(0, L),$$

$$\begin{aligned} (i\lambda + \xi^2 + \eta) \omega^1 - i\lambda \sqrt{D_1} (u_x + y) \mu(\xi) &= \lambda^{\frac{\alpha}{2}-1} \left[f_5 - \sqrt{D_1} \mu(\xi) ((f_1)_x + f_3) \right] \\ &\quad \text{in } \mathcal{W}, \end{aligned} \quad (3.165)$$

$$\begin{aligned} (i\lambda + \xi^2 + \eta) \omega^2 - i\lambda \sqrt{D_2} y_x \mu(\xi) &= \lambda^{\frac{\alpha}{2}-1} \left[f_6 - \sqrt{D_2} (f_3)_x \mu(\xi) \right] \\ &\quad \text{in } \mathcal{W}, \end{aligned} \quad (3.166)$$

where

$$\mathcal{O}_j(0, L) = \begin{cases} H_0^1(0, L), & \text{if } j = 1, \\ H_*^1(0, L), & \text{if } j = 2 \end{cases}$$

and

$$S_{d_1} = \left((u_x + y) + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right) \text{ and } S_{d_2} = \left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right).$$

Here we will check the condition (H_6) by finding a contradiction with (3.159) by showing $\|U\|_{\mathcal{H}_j} = o(1)$. For clarity, we divide the proof into several lemmas.

Lemma 5.2. Assume that $\eta > 0$ and assumption (A_4) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (3.161)-(3.166) satisfies the following asymptotic behavior estimations:

$$\int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^1(x, \xi)|^2 d\xi dx = o(\lambda^{\frac{\alpha}{2}-1}) \text{ and } \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^2(x, \xi)|^2 d\xi dx = o(\lambda^{\frac{\alpha}{2}-1}), \quad (3.167)$$

$$\int_a^b |y_x|^2 dx = o(\lambda^{-\frac{\alpha}{2}-2}) \text{ and } \int_a^b |z_x|^2 dx = o(\lambda^{-\frac{\alpha}{2}}), \quad (3.168)$$

$$\int_a^b |u_x + y|^2 dx = o(\lambda^{-\frac{\alpha}{2}-2}) \text{ and } \int_a^b |v_x + z|^2 dx = o(\lambda^{-\frac{\alpha}{2}}), \quad (3.169)$$

$$\int_a^b |S_{d_1}|^2 = o(\lambda^{\frac{\alpha}{2}-1}) \text{ and } \int_a^b |S_{d_2}|^2 = o(\lambda^{\frac{\alpha}{2}-1}). \quad (3.170)$$

Proof. First, taking the inner product of (3.160) with U in \mathcal{H}_j , then using the fact that U is uniformly bounded in \mathcal{H}_j , we get

$$\begin{aligned} \kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) \left(|\omega^2(x, \xi)|^2 + |\omega^1(x, \xi)|^2 \right) d\xi dx &= -\Re \left(\langle \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} \right) \\ &= \Re \left(\langle i\lambda U - \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} \right) = o(\lambda^{\frac{\alpha}{2}-1}). \end{aligned}$$

estimation (3.167).

From (3.166) and proceeding in a similar way as in Lemma 3.3 (Section 3), we get the first estimation of (3.168). Now, we proof the second estimation of (3.168). From (3.163), we have

$$z_x = i\lambda y_x - \lambda^{\frac{\alpha}{2}-1} (f_3)_x.$$

It follows that

$$\|z_x\|_{L^2(a,b)} \leq \|\lambda y_x\|_{L^2(a,b)} + |\lambda|^{\frac{\alpha}{2}-1} \|(f_3)_x\|_{L^2(a,b)} \leq \frac{o(1)}{\lambda^{\frac{\alpha}{4}}} + \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}.$$

Since $\alpha \in (0, 1)$, we have $\min\{\frac{\alpha}{4}, 1 - \frac{\alpha}{2}\} = \frac{\alpha}{4}$, hence, from the above equation, we get

$$\int_a^b |z_x|^2 dx = o(\lambda^{-\frac{\alpha}{2}}).$$

From (3.165) and proceeding in a similar way as in Lemma 4.3 (Section 4), we get the first estimation of (3.169). Now, we proof the second estimation of (3.169). From (3.161) and (3.163), we have

$$v_x = i\lambda u_x - \lambda^{\frac{\alpha}{2}-1} (f_1)_x \quad \text{and} \quad z = i\lambda y - \lambda^{\frac{\alpha}{2}-1} f_3.$$

It follows that

$$\|v_x + z\|_{L^2(a,b)} \leq \|\lambda(u_x + y)\|_{L^2(a,b)} + |\lambda|^{\frac{\alpha}{2}-1} \|(f_1)_x + f_3\|_{L^2(a,b)} \leq \frac{o(1)}{\lambda^{\frac{\alpha}{4}}} + \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}.$$

Since $\alpha \in (0, 1)$, we get

$$\int_a^b |v_x + z|^2 dx = o(\lambda^{-\frac{\alpha}{2}}).$$

Finally, the proof of estimations (3.170) can follow using similar computations as in Section 3 (Lemma 3.5) and Section 4 (Lemma 4.4). \square

Lemma 5.3. Assume that $\eta > 0$, assumption (A₄) holds. Let $g \in C^1([a, b])$ such that

$$g(b) = -g(a) = 1, \max_{x \in (a, b)} |g(x)| = c_g \text{ and } \max_{x \in (a, b)} |g'(x)| = c'_g,$$

where c_g and c'_g are strictly positive constants. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (3.161)-(3.166) satisfies the following asymptotic behavior estimations:

$$|S_{d_2}(b)|^2 + |S_{d_2}(a)|^2 \leq \frac{\rho_2}{2k_2} \lambda^{1+\frac{\alpha}{2}} \int_a^b |z|^2 dx + o(1) \quad (3.171)$$

and

$$|z(b)|^2 + |z(a)|^2 \leq \left(\frac{\rho_2}{2k_2} \lambda^{1-\frac{\alpha}{2}} + 2c'_g \right) \int_a^b |z|^2 dx + \frac{o(1)}{\lambda}. \quad (3.172)$$

Proof. First, we will prove equation (3.171). Multiplying (3.164) by $-2\frac{\rho_2}{k_2} g \overline{S_{d_2}}$ and integrating over (a, b) , we get

$$\begin{aligned} |S_{d_2}(b)|^2 + |S_{d_2}(a)|^2 &= \int_a^b g' |S_{d_2}|^2 dx + \Re \left(2\frac{\rho_2 i \lambda}{k_2} \int_a^b g z \overline{S_{d_2}} dx \right) + \Re \left(2\frac{k_1}{k_2} \int_a^b g (u_x + y) \overline{S_{d_2}} dx \right) \\ &\quad + \Re \left(2\frac{\kappa(\alpha)}{k_2} \int_a^b g \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \overline{S_{d_2}} dx \right) - \Re \left(\frac{2\rho_2}{k_2} \lambda^{\frac{\alpha}{2}-1} \int_a^b g f_4 \overline{S_{d_2}} dx \right), \end{aligned}$$

consequently,

$$\begin{aligned} |S_{d_2}(b)|^2 + |S_{d_2}(a)|^2 &\leq c'_g \int_a^b |S_{d_2}|^2 dx + 2\frac{\rho_2 \lambda}{k_2} c_g \int_a^b |z| |S_{d_2}| dx \\ &\quad + 2\frac{k_1}{k_2} c_g \int_a^b |u_x + y| |S_{d_2}| dx \\ &\quad + 2\frac{\kappa(\alpha)}{k_2} \sqrt{d_1} c_g \int_a^b \left| \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right| |S_{d_2}| dx \\ &\quad + \frac{2\rho_2}{k_2} \lambda^{\frac{\alpha}{2}-1} c_g \int_a^b |f_4| |S_{d_2}| dx. \end{aligned} \quad (3.173)$$

Using Young's inequality and second estimation of (3.170), we obtain

$$\begin{aligned} \frac{2\rho_2 \lambda c_g}{k_2} \int_a^b |z| |S_{d_2}| dx &\leq \frac{\rho_2 \lambda^{1+\frac{\alpha}{2}}}{2k_2} \int_a^b |z|^2 dx + \frac{2\rho_2 \lambda^{1-\frac{\alpha}{2}} c_g^2}{k_2} \int_a^b |S_{d_2}|^2 dx \\ &\leq \frac{\rho_2 \lambda^{1+\frac{\alpha}{2}}}{2k_2} \int_a^b |z|^2 dx + o(1). \end{aligned} \quad (3.174)$$

Using Cauchy-Schwarz inequality, first estimations of (3.167) and (3.169), second estimation of (3.170) and using the fact that $\|f_4\| = o(1)$, we obtain

$$\begin{cases} \int_a^b |u_x + y| |S_{d_2}| dx = \frac{o(1)}{\lambda^{\frac{3}{2}}}, & \int_a^b |S_{d_2}|^2 dx = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}, \\ \int_a^b \left| \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right| |S_{d_2}| dx = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}, & \lambda^{\frac{\alpha}{2}-1} \int_a^b |f_4| |S_{d_2}| dx = \frac{o(1)}{\lambda^{\frac{3}{2}-\frac{3\alpha}{4}}}. \end{cases} \quad (3.175)$$

Inserting (3.174) and (3.175) in (3.173), and since $\alpha \in (0, 1)$, we obtain

$$|S_{d_2}(b)|^2 + |S_{d_2}(a)|^2 \leq \frac{\rho_2 \lambda^{1+\frac{\alpha}{2}}}{2k_2} \int_a^b |z|^2 dx + o(1). \quad (3.176)$$

Now, we will prove (3.172). From equation (3.163), we have

$$z_x = i\lambda y_x - \lambda^{\frac{\alpha}{2}-1}(f_3)_x. \quad (3.177)$$

Multiplying Equation (3.177) by $2g\bar{z}$ and integrating over (a, b) , then taking the real part, we get

$$|z(b)|^2 + |z(a)|^2 = \int_a^b g'|z|^2 dx + \Re \left\{ 2i\lambda \int_a^b g y_x \bar{z} dx \right\} - \Re \left\{ 2\lambda^{\frac{\alpha}{2}-1} \int_a^b g(f_3)_x \bar{z} dx \right\}.$$

Then,

$$|z(b)|^2 + |z(a)|^2 \leq c_{g'} \int_a^b |z|^2 dx + 2c_g \lambda \int_a^b |y_x| |z| dx + 2c_g \lambda^{\frac{\alpha}{2}-1} \int_a^b |(f_3)_x| |z| dx. \quad (3.178)$$

Using Young's inequality, we have

$$2c_g \lambda |y_x| |z| \leq \frac{\rho_2}{2k_2} \lambda^{1-\frac{\alpha}{2}} |z|^2 + \frac{2k_2 c_g^2}{\rho_2} \lambda^{1+\frac{\alpha}{2}} |y_x|^2$$

and

$$2c_g \lambda^{\frac{\alpha}{2}-1} |(f_3)_x| |z| \leq c_{g'} |z|^2 + \frac{c_g^2}{c_{g'}} \lambda^{-2+\alpha} |(f_3)_x|^2.$$

Using the above inequalities, first estimation of (3.168) and the fact that $\|(f_3)_x\| = o(1)$, then equation (3.178) becomes

$$|z(b)|^2 + |z(a)|^2 \leq \left(\frac{\rho_2}{2k_2} \lambda^{1-\frac{\alpha}{2}} + 2c_{g'} \right) \int_a^b |z|^2 dx + \frac{o(1)}{\lambda}.$$

□

Lemma 5.4. Assume that $\eta > 0$ and assumption (A_4) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (3.161)-(3.166) satisfies the following asymptotic behavior estimations:

$$\int_a^b |z|^2 dx = \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}} \quad \text{and} \quad \int_a^b |y|^2 dx = \frac{o(1)}{\lambda^{3+\frac{\alpha}{2}}}. \quad (3.179)$$

Proof. Multiplying Equation (3.164) by $-i\lambda^{-1}\bar{z}$ and integrating over (a, b) , then taking the real part, we get

$$\begin{aligned} \int_a^b |z|^2 dx = & -\frac{k_2}{\rho_2} \lambda^{-1} \Re \left\{ i \int_a^b (S_{d_2})_x \bar{z} dx \right\} + \frac{k_1}{\rho_2} \lambda^{-1} \Re \left\{ i \int_a^b (u_x + y) \bar{z} dx \right\} \\ & + \frac{\kappa(\alpha)}{\rho_2} \lambda^{-1} \Re \left\{ i \int_a^b \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \bar{z} dx \right\} - \lambda^{\frac{\alpha}{2}-2} \Re \left\{ i \int_a^b f_4 \bar{z} dx \right\}, \end{aligned}$$

consequently,

$$\begin{aligned} \int_a^b |z|^2 dx \leq & \frac{k_2}{\rho_2} \lambda^{-1} \left| \int_a^b (S_{d_2})_x \bar{z} dx \right| + \frac{k_2}{\rho_2} \lambda^{-1} |S_{d_2}(b)| |z(b)| \\ & + \frac{k_2}{\rho_2} \lambda^{-1} |S_{d_2}(a)| |z(a)| + \frac{k_1}{\rho_2} \lambda^{-1} \left| \int_a^b (u_x + y) \bar{z} dx \right| \\ & + \frac{\kappa(\alpha)}{\rho_2} \lambda^{-1} \left| \int_a^b \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \bar{z} dx \right| \\ & + \lambda^{\frac{\alpha}{2}-2} \int_a^b |f_4| |z| dx. \end{aligned} \quad (3.180)$$

Using Cauchy-Schwarz inequality, the fact that z is uniformly bounded in $L^2(0, L)$ and $\|f_4\| = o(1)$, we get

$$\lambda^{\frac{\alpha}{2}-2} \int_a^b |f_4| |z| dx = o(\lambda^{\frac{\alpha}{2}-2}). \quad (3.181)$$

Using Cauchy-Schwarz inequality, the second estimations in (3.168) and (3.170), we get

$$\frac{k_2}{\rho_2} \lambda^{-1} \left| \int_a^b (S_{d_2}) \bar{z}_x dx \right| = \frac{o(1)}{\lambda^{\frac{3}{2}}}. \quad (3.182)$$

Using Cauchy-Schwarz inequality, the first estimation in (3.169) and the fact that z is uniformly bounded in $L^2(0, L)$, we get

$$\frac{k_1}{\rho_2} \lambda^{-1} \left| \int_a^b (u_x + y) \bar{z} dx \right| = \frac{o(1)}{\lambda^{2+\frac{\alpha}{4}}}. \quad (3.183)$$

Using Young's inequality, Cauchy-Schwarz inequality and the first estimation of (3.167), we get

$$\begin{aligned} & \frac{\kappa(\alpha)}{\rho_2} \lambda^{-1} \left| \int_a^b \sqrt{D_1} \bar{z} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi dx \right| \\ & \leq \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}} \int_a^b |z|^2 dx + \frac{\kappa(\alpha)^2 d_1}{2\rho_2^2} \lambda^{-1-\frac{\alpha}{2}} \int_a^b \left(\int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \right)^2 dx \\ & \leq \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}} \int_a^b |z|^2 dx + \frac{\kappa(\alpha)^2 d_1}{2\rho_2^2} \lambda^{-1-\frac{\alpha}{2}} \int_a^b \left(\int_{\mathbb{R}} \frac{\mu(\xi) \sqrt{\xi^2 + \eta}}{\sqrt{\xi^2 + \eta}} \omega^1(x, \xi) d\xi \right)^2 dx \\ & \leq \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}} \int_a^b |z|^2 dx + c_4 \lambda^{-1-\frac{\alpha}{2}} \int_a^b \int_{\mathbb{R}} (\xi^2 + \eta) |\omega^1(x, \xi)|^2 d\xi dx \\ & \leq \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}} \int_a^b |z|^2 dx + \frac{o(1)}{\lambda^2}, \end{aligned} \quad (3.184)$$

where $c_4 = \frac{\kappa(\alpha)^2 d_1}{2\rho_2^2} \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{|\xi|^2 + \eta} d\xi$. Since $0 < \alpha < 1$ and $\eta > 0$, then c_4 is well defined.

Inserting estimations (3.181)-(3.184) in (3.180), we get

$$\begin{aligned} (1 - \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}}) \int_a^b |z|^2 dx & \leq \frac{k_2}{2\rho_2} \lambda^{-1+\frac{\alpha}{2}} (|z(a)|^2 + |z(b)|^2) \\ & \quad + \frac{k_2}{2\rho_2} \lambda^{-1-\frac{\alpha}{2}} (|S_{d_2}(a)|^2 + |S_{d_2}(b)|^2) \\ & \quad + \frac{o(1)}{\lambda^{\frac{3}{2}}} + \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}}. \end{aligned} \quad (3.185)$$

Now, inserting estimations (3.171) and (3.172) in (3.185), we get

$$\left(\frac{1}{2} - \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}} - \frac{k_2}{\rho_2} c'_g \lambda^{-1+\frac{\alpha}{2}} \right) \int_a^b |z|^2 dx \leq \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}} + \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}} + \frac{o(1)}{\lambda^{\frac{3}{2}}}. \quad (3.186)$$

Since $0 < \alpha < 1$, then $\min(2 - \frac{\alpha}{2}, 1 + \frac{\alpha}{2}, \frac{3}{2}) = 1 + \frac{\alpha}{2}$. Consequently,

$$\left(\frac{1}{2} - \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}} - \frac{k_2}{\rho_2} c'_g \lambda^{-1+\frac{\alpha}{2}} \right) \int_a^b |z|^2 dx \leq \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}. \quad (3.187)$$

Since $|\lambda| \rightarrow +\infty$, for λ large enough, we get

$$0 < \left(\frac{1}{2} - \frac{1}{2} \lambda^{-1+\frac{\alpha}{2}} - \frac{k_2}{\rho_2} c'_g \lambda^{-1+\frac{\alpha}{2}} \right) \int_a^b |z|^2 dx \leq \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}, \quad (3.188)$$

hence, the first asymptotic estimate of (3.179) holds. Then inserting the first asymptotic estimate of (3.179) in (3.163), we get the second estimate of (3.179). Thus, the proof is complete. \square

Lemma 5.5. Assume that $\eta > 0$ and assumption (A_4) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (3.161)-(3.166) satisfies the following asymptotic behavior estimations:

$$|S_{d_1}(b_1)|^2 + |S_{d_1}(a_2)|^2 \leq \frac{\rho_1}{2k_1} \lambda^{1+\frac{\alpha}{2}} \int_a^b |v|^2 dx + o(1) \quad (3.189)$$

and

$$|v(b_1)|^2 + |v(a_2)|^2 \leq \left(\frac{\rho_1}{2k_1} \lambda^{1-\frac{\alpha}{2}} + 2c'_g \right) \int_a^b |v|^2 dx + \frac{o(1)}{\lambda}. \quad (3.190)$$

Proof. First, we will prove equation (3.189). Multiplying (3.162) by $-2\frac{\rho_1}{k_1} g \overline{S_{d_1}}$ and integrating over (a, b) , we get

$$\begin{aligned} |S_{d_1}(b)|^2 + |S_{d_1}(a)|^2 &= \int_a^b g' |S_{d_1}|^2 dx + \Re \left\{ 2\frac{\rho_1 i \lambda}{k_1} \int_a^b g v \overline{S_{d_1}} dx \right\} \\ &\quad - \Re \left\{ \frac{2\rho_1}{k_1} \lambda^{\frac{\alpha}{2}-1} \int_a^b g f_2 \overline{S_{d_1}} dx \right\}, \end{aligned} \quad (3.191)$$

consequently,

$$\begin{aligned} |S_{d_1}(b)|^2 + |S_{d_1}(a)|^2 &\leq c'_g \int_a^b |S_{d_1}|^2 dx + 2\frac{\rho_1 \lambda}{k_1} c_g \int_a^b |v| |S_{d_1}| dx \\ &\quad + \frac{2\rho_1}{k_1} \lambda^{\frac{\alpha}{2}-1} c_g \int_a^b |f_2| |S_{d_1}| dx. \end{aligned} \quad (3.192)$$

Using Young's inequality and first estimation of (3.170), we obtain

$$\begin{aligned} \frac{2\rho_1 \lambda c_g}{k_1} \int_a^b |v| |S_{d_1}| dx &\leq \frac{\rho_1 \lambda^{1+\frac{\alpha}{2}}}{2k_1} \int_a^b |v|^2 dx + \frac{2\rho_1 \lambda^{1-\frac{\alpha}{2}} c_g^2}{k_1} \int_a^b |S_{d_1}|^2 dx \\ &\leq \frac{\rho_1 \lambda^{1+\frac{\alpha}{2}}}{2k_1} \int_a^b |v|^2 dx + o(1). \end{aligned} \quad (3.193)$$

Using Cauchy-Schwarz inequality, first estimation of (3.170) and using the fact that $\|f_2\| = o(1)$, we obtain

$$\begin{cases} \int_a^b |S_{d_1}|^2 dx = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}, \\ \lambda^{\frac{\alpha}{2}-1} \int_a^b |f_2| |S_{d_1}| dx = \frac{o(1)}{\lambda^{\frac{3}{2}-\frac{3\alpha}{4}}}. \end{cases} \quad (3.194)$$

Inserting (3.193) and (3.194) in (3.192) and since $0 < \alpha < 1$, we get

$$|S_{d_1}(b)|^2 + |S_{d_1}(a)|^2 \leq \frac{\rho_1 \lambda^{1+\frac{\alpha}{2}}}{2k_1} \int_a^b |v|^2 dx + o(1). \quad (3.195)$$

Now, we will prove (3.190). From equation (3.161), we have

$$v_x = i\lambda u_x - \lambda^{\frac{\alpha}{2}-1} (f_1)_x. \quad (3.196)$$

Multiplying equation (3.196) by $2g\bar{v}$ and integrating over (b, a) , then taking the real part, we get

$$|v(b)|^2 + |v(a)|^2 = \int_a^b g' |v|^2 dx + \Re \left\{ 2i\lambda \int_a^b g u_x \bar{v} dx \right\} - \Re \left\{ 2\lambda^{\frac{\alpha}{2}-1} \int_a^b g (f_1)_x \bar{v} dx \right\}.$$

Then,

$$|v(b)|^2 + |v(a)|^2 \leq c_{g'} \int_a^b |v|^2 dx + 2c_g \lambda \int_a^b |u_x| |v| dx + 2c_g \lambda^{\frac{\alpha}{2}-1} \int_a^b |(f_1)_x| |v| dx. \quad (3.197)$$

Using Young's inequality, we have

$$2c_g \lambda |u_x| |v| \leq \frac{\rho_1}{2k_1} \lambda^{1-\frac{\alpha}{2}} |v|^2 + \frac{2k_1 c_g^2}{\rho_1} \lambda^{1+\frac{\alpha}{2}} |u_x|^2 \quad \text{and} \quad 2c_g \lambda^{\frac{\alpha}{2}-1} |(f_1)_x| |v| \leq c_{g'} |v|^2 + \frac{c_g^2}{c_{g'}} \lambda^{\alpha-2} |(f_1)_x|^2. \quad (3.198)$$

Using equation (3.198) and the fact that $\|(f_1)_x\| = o(1)$, then equation (3.197) becomes

$$\begin{aligned} |v(b)|^2 + |v(a)|^2 &\leq \left(\frac{\rho_1}{2k_1} \lambda^{1-\frac{\alpha}{2}} + 2c_{g'} \right) \int_a^b |v|^2 dx \\ &\quad + \frac{2k_1 c_g^2}{\rho_1} \lambda^{1+\frac{\alpha}{2}} \int_a^b |u_x|^2 dx + \frac{o(1)}{\lambda^{2-\alpha}}. \end{aligned} \quad (3.199)$$

Using the first estimation of (3.169) and the second estimation of (3.179), we get

$$\begin{aligned} \lambda^{1+\frac{\alpha}{2}} \int_a^b |u_x|^2 dx &\leq 2\lambda^{1+\frac{\alpha}{2}} \int_a^b |u_x + y|^2 dx + 2\lambda^{1+\frac{\alpha}{2}} \int_a^b |y|^2 dx \\ &\leq \frac{o(1)}{\lambda} + \frac{o(1)}{\lambda^2} \\ &\leq \frac{o(1)}{\lambda}. \end{aligned} \quad (3.200)$$

Now, inserting (3.200) in (3.199) and $0 < \alpha < 1$, we get

$$|v(b)|^2 + |v(a)|^2 \leq \left(\frac{\rho_1}{2k_1} \lambda^{1-\frac{\alpha}{2}} + 2c_{g'} \right) \int_a^b |v|^2 dx + \frac{o(1)}{\lambda}. \quad (3.201)$$

□

Lemma 5.6. Assume that $\eta > 0$ and assumption (A₄) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (3.161)-(3.166) satisfies the following asymptotic behavior estimations:

$$\int_a^b |v|^2 dx = \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}} \quad \text{and} \quad \int_a^b |u|^2 dx = \frac{o(1)}{\lambda^{3+\frac{\alpha}{2}}}. \quad (3.202)$$

Proof. Multiplying Equation (3.162) by $-i\lambda^{-1}\bar{v}$ and integrating over (a, b) , then taking the real part, we get

$$\int_a^b |v|^2 dx = -\frac{k_1}{\rho_1} \lambda^{-1} \Re \left\{ i \int_a^b (S_{d_1})_x \bar{v} dx \right\} - \lambda^{-2+\frac{\alpha}{2}} \Re \left\{ i \int_a^b f_2 \bar{v} dx \right\},$$

consequently,

$$\begin{aligned} \int_a^b |v|^2 dx &\leq \frac{k_1}{\rho_1} \lambda^{-1} \left| \int_a^b (S_{d_1})_x \bar{v} dx \right| + \frac{k_1}{\rho_1} \lambda^{-1} |S_{d_1}(b)| |v(b)| \\ &\quad + \frac{k_1}{\rho_1} \lambda^{-1} |S_{d_1}(a)| |v(a)| + \lambda^{-2+\frac{\alpha}{2}} \int_a^b |f_2| |v| dx. \end{aligned} \quad (3.203)$$

From the fact that v is uniformly bounded in $L^2(0, L)$ and $\|f_2\| = o(1)$, we get

$$\lambda^{-2+\frac{\alpha}{2}} \int_a^b |f_2| |v| dx = o(\lambda^{-2+\frac{\alpha}{2}}). \quad (3.204)$$

Using the second estimation in (3.169), the first estimation in (3.179) and the first one in (3.170), we get

$$\begin{aligned} \frac{k_1}{\rho_1} \lambda^{-1} \left| \int_a^b (S_{d_1}) \bar{v}_x dx \right| &\leq \frac{k_1}{\rho_1} \lambda^{-1} \left| \int_a^b (S_{d_1}) (\overline{v_x + z}) dx \right| + \frac{k_1}{\rho_1} \lambda^{-1} \left| \int_a^b (S_{d_1}) \bar{z} dx \right| \\ &\leq \frac{o(1)}{\lambda^{\frac{3}{2}}} + \frac{o(1)}{\lambda^2} \\ &\leq \frac{o(1)}{\lambda^{\frac{3}{2}}}. \end{aligned} \quad (3.205)$$

Inserting (3.204) and (3.205) in (3.203), we get

$$\begin{aligned} \int_a^b |v|^2 dx &\leq \frac{k_1}{2\rho_1} \lambda^{-1+\frac{\alpha}{2}} (|v(b)|^2 + |v(a)|^2) \\ &\quad + \frac{k_1}{2\rho_1} \lambda^{-1-\frac{\alpha}{2}} (|S_{d_1}(b)|^2 + |S_{d_1}(a)|^2) \\ &\quad + \frac{o(1)}{\lambda^{\frac{3}{2}}} + \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}}. \end{aligned} \quad (3.206)$$

Now, inserting (3.189) and (3.190) in the above estimation, we get

$$\left(\frac{1}{2} - \frac{k_1}{\rho_1} c'_g \lambda^{-1+\frac{\alpha}{2}} \right) \int_a^b |v|^2 dx \leq \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}} + \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}} + \frac{o(1)}{\lambda^{\frac{3}{2}}}. \quad (3.207)$$

Since $0 < \alpha < 1$ and $|\lambda| \rightarrow +\infty$, for λ large enough, we get

$$0 < \left(\frac{1}{2} - \frac{k_1}{\rho_1} c'_g \lambda^{-1+\frac{\alpha}{2}} \right) \int_a^b |v|^2 dx \leq \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}. \quad (3.208)$$

Hence, the first asymptotic estimate of (3.202) holds. Inserting the first asymptotic estimate of (3.202) in (3.161), we get the second estimate of (3.202). Thus, the proof is complete. \square

From what precedes and from Lemma 5.2-5.6, we deduce that

$$\|U\|_{\mathcal{H}_j} = o(1) \quad \text{over } (a, b).$$

Lemma 5.7. Assume that $\eta > 0$ and assumption (A_4) holds. Let $h \in C^1([0, L])$ and $h(0) = h(L) = 0$ be a given function. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (3.161)-(3.166) satisfies the following asymptotic behavior estimation:

$$\int_0^L h' \left(\rho_1 |v|^2 + k_2 |S_{d_2}|^2 + \rho_2 |z|^2 + k_1 |S|^2 \right) dx = o(1),$$

where

$$S := u_x + \frac{\kappa(\alpha)}{k_1} \sqrt{D_1} \int_{\mathbb{R}} \mu(\xi) \omega^1(x, \xi) d\xi \quad \text{and} \quad S_{d_2} := y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi.$$

Proof. From Lemma 5.2, the definition of $D_1(x)$ and the fact that u_x is uniformly bounded in $L^2(0, L)$, we get S is uniformly bounded in $L^2(0, L)$. First, multiplying (3.162) by $2\rho_1 h \bar{S}$, integrating over $(0, L)$, taking the real part, and using the fact that $\|f_2\| = o(1)$, we obtain

$$\begin{aligned} &\Re \left\{ \rho_1 \int_0^L 2hi\lambda v \bar{S} dx \right\} - \Re \left\{ 2k_1 \int_0^L hy_x \bar{S} dx \right\} - k_1 \int_0^L h (|S|^2)_x dx \\ &= \underbrace{\Re \left\{ 2\rho_1 \int_0^L h \lambda^{\frac{\alpha}{2}-1} f_2 \bar{S} dx \right\}}_{o(\lambda^{\frac{\alpha}{2}-1})}. \end{aligned} \quad (3.209)$$

From equation (3.161), we have

$$i\lambda\bar{u}_x = -\bar{v}_x - \lambda^{\frac{\alpha}{2}-1}(\bar{f}_1)_x.$$

Then,

$$i\lambda\bar{S} = -\bar{v}_x - \lambda^{\frac{\alpha}{2}-1}(\bar{f}_1)_x + i\lambda\frac{\kappa(\alpha)}{k_1}\sqrt{D_1}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^1(x,\xi)}d\xi. \quad (3.210)$$

Moreover, from the definition of S and $D_1(x)$, Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$ and $\eta > 0$, Lemma 5.2, Lemma 5.6, equation (3.210), the fact that y_x and v are uniformly bounded in $L^2(0, L)$ and $\|(f_1)_x\| = o(1)$, we get

$$\left\{ \begin{array}{l} \Re \left\{ 2k_1 \int_0^L hy_x \bar{S} dx \right\} = \Re \left\{ 2k_1 \int_0^L hy_x \bar{u}_x dx \right\} + \underbrace{\Re \left\{ 2\kappa(\alpha)\sqrt{d_1} \int_a^b hy_x \int_{\mathbb{R}} \mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx \right\}}_{=\frac{o(1)}{\lambda^{\frac{3}{2}}}} \\ \Re \left\{ \rho_1 \int_0^L 2hi\lambda v \bar{S} dx \right\} = -\rho_1 \int_0^L h(|v|^2)_x dx - \underbrace{\lambda^{\frac{\alpha}{2}-1} 2\rho_1 \int_0^L hv(\bar{f}_1)_x dx}_{=\frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}} \\ \quad + \underbrace{\Re \left\{ 2\rho_1 \frac{\kappa(\alpha)}{k_1} \sqrt{d_1} \int_a^b hi\lambda v \int_{\mathbb{R}} \mu(\xi)\overline{\omega^1(x,\xi)}d\xi dx \right\}}_{=o(1)}. \end{array} \right.$$

Inserting the above estimations in equation (3.209), we obtain

$$-\rho_1 \int_0^L h(|v|^2)_x dx - k_1 \int_0^L h(|S|^2)_x dx - \Re \left\{ 2k_1 \int_0^L hy_x \bar{u}_x dx \right\} = o(1). \quad (3.211)$$

Now, multiplying (3.164) by $2\rho_2 h \bar{S}_{d_2}$, integrating over $(0, L)$, taking the real part, then using the fact that $\left(y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2(x)} \int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi \right)$ is uniformly bounded in $L^2(0, L)$, equation (3.167), equation (3.170), $\|y\| = O(|\lambda|^{-1})$ and $\|f_4\| = o(1)$, we obtain

$$\begin{aligned} & \Re \left\{ 2\rho_2 i\lambda \int_0^L hz \bar{S}_{d_2} dx \right\} - k_2 \int_0^L h \frac{d}{dx} \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi)\omega^2(x,\xi)d\xi \right|^2 dx \\ & + \Re \left\{ 2k_1 \int_0^L hu_x \bar{S}_{d_2} dx \right\} + \underbrace{\Re \left\{ 2k_1 \int_0^L hy \bar{S}_{d_2} dx \right\}}_{=\frac{o(1)}{\lambda}} \\ & + \underbrace{\Re \left\{ 2\kappa(\alpha) \int_0^L h\sqrt{D_1} \int_{\mathbb{R}} \mu(\xi)\omega^1(x,\xi)d\xi \bar{S}_{d_2} dx \right\}}_{=\frac{o(1)}{1-\frac{\alpha}{2}}} \\ & = \underbrace{\Re \left\{ 2\rho_2 \int_0^L h\lambda^{\frac{\alpha}{2}-1} f_4 \bar{S}_{d_2} dx \right\}}_{=\frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}}. \end{aligned} \quad (3.212)$$

From equation (3.163), we have

$$i\lambda\bar{y}_x = -\bar{z}_x - \lambda^{\frac{\alpha}{2}-1}(\bar{f}_3)_x.$$

Then,

$$i\lambda\bar{S}_{d_2} = -\bar{z}_x - \lambda^{\frac{\alpha}{2}-1}(\bar{f}_3)_x + i\lambda\frac{\kappa(\alpha)}{k_2}\sqrt{D_2}\int_{\mathbb{R}}\mu(\xi)\overline{\omega^2(x,\xi)}d\xi. \quad (3.213)$$

Moreover, from the definition of S_{d_2} and $D_2(x)$, Cauchy-Schwarz inequality, the fact that $0 < \alpha < 1$ and $\eta > 0$, Lemma 5.2, Lemma 5.4, equation (3.213), the fact that u_x and z are uniformly bounded in $L^2(0, L)$ and $\|(f_3)_x\| = o(1)$, we have

$$\left\{ \begin{array}{l} \Re \left\{ 2k_1 \int_0^L hu_x \overline{S_{d_2}} dx \right\} = \Re \left\{ 2k_1 \int_0^L hu_x \overline{y_x} dx \right\} + \underbrace{\Re \left\{ 2 \frac{k_1}{k_2} \kappa(\alpha) \sqrt{d_2} \int_a^b hu_x \int_{\mathbb{R}} \mu(\xi) \overline{\omega^2(x, \xi)} d\xi dx \right\}}_{= \frac{o(1)}{\lambda^{\frac{1}{2} - \frac{\alpha}{4}}}}, \\ \Re \left\{ \rho_2 \int_0^L 2hi\lambda z \overline{S_{d_2}} dx \right\} = -\rho_2 \int_0^L h (|z|^2)_x dx - \underbrace{\lambda^{\frac{\alpha}{2} - 1} 2\rho_1 \int_0^L h z \overline{(f_3)_x} dx}_{= \frac{o(1)}{\lambda^{1 - \frac{\alpha}{2}}}} \\ + \underbrace{\Re \left\{ \frac{2\rho_2 \kappa(\alpha)}{k_2} \sqrt{d_2} \int_a^b hi\lambda z \int_{\mathbb{R}} \mu(\xi) \overline{\omega^2(x, \xi)} d\xi dx \right\}}_{= o(1)}. \end{array} \right.$$

Inserting the above estimations in (3.212), we obtain

$$\begin{aligned} & -\rho_2 \int_0^L h \frac{d}{dx} |z|^2 dx - k_2 \int_0^L h \frac{d}{dx} \left| y_x + \frac{\kappa(\alpha)}{k_2} \sqrt{D_2} \int_{\mathbb{R}} \mu(\xi) \omega^2(x, \xi) d\xi \right|^2 dx \\ & + \Re \left\{ 2k_1 \int_0^L hu_x \overline{y_x} dx \right\} = o(1). \end{aligned} \quad (3.214)$$

Adding (3.211) and (3.214), we get our desired result. \square

Lemma 5.8. Assume that $\eta > 0$ and assumption (A_4) holds. Then, for $j = 1, 2$, the solution $(u, v, y, z, \omega^1, \omega^2) \in D(\mathcal{A}_j)$ of system (3.161)-(3.166) satisfies the following asymptotic behavior estimation:

$$\|U\|_{\mathcal{H}_j} = o(1).$$

Proof. Proceeding in a similar way as in Lemma 3.10, we get our desired result. \square

Proof of Theorem 5.1. From Lemma 5.8 we get that $\|U\|_{\mathcal{H}_j} = o(1)$, which contradicts (3.159). This implies that

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{\lambda^{1 - \frac{\alpha}{2}}} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} < \infty.$$

The result follows from Theorem 2.10.

6 Conclusion and comments

We have studied the stabilization of a one-dimensional Timoshenko system with localized internal fractional kelvin-Voigt damping via non-smooth coefficients. We proved the strong stability of the system using Arendt-Batty criteria. Polynomial stability results has been proved in three cases: The case of fractional kelvin-Voigt damping acting on the bending moment equation. We showed a polynomial energy decay rate of type t^{-1} . The case of fractional kelvin-Voigt damping acting on the shear force equation. We proved a polynomial energy decay rate of type t^{-1} . In the last case, the fractional kelvin-Voigt damping acting on the shear force and bending moment equations. We established a polynomial energy decay rate of type $t^{\frac{-4}{2-\alpha}}$, provided that the two dampings are acting in the same

sub-interval.

Thereby, we highlight the following important open problems: the *optimality* of the obtained decay rates and the *generalization* of our results to a Bresse system (see [13] for a Bresse system with local Kelvin-Voigt damping).

Polynomial stability of the generalized Rao-Nakra beam with local Kelvin-Voigt damping

Résumé

Dans cette partie, nous étudions le bien posé et la stabilité de l'équation de la poutre de Rao-Nakra généralisée. Le système se compose de quatre équations d'ondes pour les déplacements longitudinaux et l'angle de cisaillement des couches supérieure et inférieure et d'une équation de poutre d'Euler-Bernoulli pour le déplacement transversal. Tout d'abord, nous prouvons que le système est bien posé. Ensuite, nous montrons que la stabilité analytique se vérifie lorsque tous les déplacements sont globalement amortis par l'amortissement de Kelvin-Voigt. Troisièmement, nous établissons le cas où l'amortissement n'agit que sur les déplacements de l'angle de cisaillement des couches supérieure et inférieure. Dans ce cas, nous n'obtenons que des conditions suffisantes pour une forte stabilité. En utilisant la méthode fréquentielle combinée avec la méthode des multiplicateurs, nous prouvons que l'énergie du système décroît polynomialement avec un taux de décroissance $t^{-\frac{1}{2}}$.

Abstract

In this part, we focus on investigating the well-posedness and stabilization of the generalized Rao-Nakra beam equation. The system consists of four wave equations for the longitudinal displacements and the shear angle of the top and bottom layers and one Euler-Bernoulli beam equation for the transversal displacement. First, we prove that the system is well-posed. Second, we show that the analytic stability holds when all the displacements are globally damped through Kelvin-Voigt damping. Third, we establish the case where the damping acts only on the shear angle displacements of the top and bottom layers. In this case, we obtain only sufficient conditions for the strong stability. Using frequency domain arguments combined with the multiplier method, we prove that the energy of the system decays polynomially with a decay rate $t^{-\frac{1}{2}}$.

1 Introduction

The classical sandwich beam is an engineering model for a three-layer beam consisting of two stiff face plates and a much more compliant inner core layer. Numerous models of this three-layered

laminated beam and plate have been suggested over the past six decades [82, 91, 111]. Later, Liu-Trogon-Yong [73] developed the following generalized Rao-Nakra beam model, where the shear effect of the bottom and top layer are taken into account:

$$\begin{cases} \rho_1 h_1 u_{tt}^1 - E_1 h_1 u_{xx}^1 - \tau = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_1 I_1 y_{tt}^1 - E_1 I_1 y_{xx}^1 - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - E_3 h_3 u_{xx}^3 + \tau = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 I_3 y_{tt}^3 - E_3 I_3 y_{xx}^3 - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = 0 & \text{in } (0, L) \times \mathbb{R}_+. \end{cases} \quad (4.1)$$

Here, u^1, y^1, u^3 and y^3 are, respectively, the longitudinal displacement and shear angle of the top and bottom layers, ω is the transverse displacement of the beam and τ is the shear stress in the core layer defined by

$$\tau = -u^1 - \frac{h_1}{2} y^1 + h_2 \omega_x + u^3 - \frac{h_3}{2} y^3.$$

The physical parameters h_i, ρ_i, E_i, G_i and I_i are positive constants and represent the thickness, density, Young's modulus, shear modulus, and moments of inertia of the i th layer for $i = 1, 2, 3$, respectively. In addition, $\rho h = \rho_1 h_1 + \rho_2 h_2 + \rho_3 h_3$ and $EI = E_1 I_1 + E_3 I_3$.

If we neglected the rotatory inertia and the transverse shear of the top and bottom layers, equations (4.1)₂ and (4.1)₅ are reduced to the familiar Euler-Bernoulli hypothesis $y^1 - \omega_x = y^3 - \omega_x = 0$.

Several models were derived from (4.1) and studied in the literature by neglecting some components and/or parameters and/or considering some connections between them, such as the Rao-Nakra sandwich beam [91]:

$$\begin{cases} \rho_1 h_1 u_{tt}^1 - E_1 h_1 u_{xx}^1 - \frac{G_2}{h_2} (-u^1 + u^3 + \alpha \omega_x) = 0, \\ \rho_3 h_3 u_{tt}^3 - E_3 h_3 u_{xx}^3 + \frac{G_2}{h_2} (-u^1 + u^3 + \alpha \omega_x) = 0, \\ \rho h \omega_{tt} + EI \omega_{xxxx} - \frac{G_2 \alpha}{h_2} (-u^1 + u^3 + \alpha \omega_x)_x = 0, \end{cases} \quad (4.2)$$

which is obtained when we consider the core material to be linearly elastic, i.e. $\tau = 2G_2\gamma$ with the shear strain

$$\gamma = \frac{1}{2h_2} (-u^1 + u^3 + \alpha \omega_x) \quad \text{and} \quad \alpha = h_2 + \frac{1}{2} (h_1 + h_3),$$

where the shear modulus $G_2 = \frac{E_2}{2(1+\nu)}$ with $-1 < \nu < \frac{1}{2}$ being the Poisson ratio.

The laminated beam [65]:

$$\begin{cases} \rho \omega_{tt} + G(\psi - \omega_x)_x = 0, \\ I_p(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - u_x) = 0, \\ 3I_p s_{tt} - 3D s_{xx} + 3G(\psi - \omega_x) + 4\mu s + 4\delta s_t = 0, \end{cases} \quad (4.3)$$

which is obtained when the extensional motion of the bottom and top layers is neglected.

The Mead-Markus model [82]:

$$\begin{cases} \rho h \omega_{tt} + EI \omega_{xxxx} - G_2 \alpha \gamma = 0, \\ 2h_2 \gamma_{xx} - G_2 d \gamma - \alpha \omega_{xxx} = 0, \end{cases} \quad (4.4)$$

is obtained if the extensional forces in the top and bottom layers are also neglected, which can be simplified into a six-order PDE for ω .

For the well-posedness and stability of (4.2), (4.3) and (4.4), we refer the readers, for example, to [19, 20, 62, 73, 92, 105] and the references therein.

The boundary controllability problems of the Rao-Nakra beam, not only for the three-layer, but also for the multi-layer models, have been studied in a series of papers; see for example, [61, 64, 86, 90, 87]. Exact controllability results for the multi-layers Rao-Nakra plate system with locally distributed control in a neighborhood of a portion of the boundary were obtained by the Carleman estimates in [18, 63].

In 2013 (see [87]), the exponential stability of Rao-Nakra model (4.2) was obtained when standard boundary dampings are imposed on one end of the beam for all three layers. Later, in 2019, Wang [106] studied Rao-Nakra beam with boundary damping only on one end for two displacements and obtained that the semigroup generated by the system is polynomially stable of order 1/2.

In the recent years, many researchers showed interest in problem involving linear viscous damping and Kelvin-Voigt damping (see, for example, [4, 5, 6, 8, 13, 107, 108]). Viscous damping usually models external friction forces such as air resistance acting on the vibrating structures and is thus called "external damping", while Kelvin-Voigt damping originates from the internal friction of the material of the vibrating structures and thus called "internal damping".

In 2018 (see [72]), the authors considered the stability of the Rao-Nakra system (4.2) subject to viscous and/or kelvin-voigt damping. First, they showed that the system is unstable if only one damping is imposed on the beam equation, beyond this, the exponential stability holds when all three displacements are damped. Moreover, they showed the optimal polynomial stability of the system for seven different cases of damping locations and types when just two of the three equations are damped. In 2020 (see [77]), the authors proved certain orders of polynomial stability of the Rao-Nakra sandwich beam equation with various boundary conditions, when there is only one viscous damping acting either on the beam equation or one of the wave equations. For a few special cases, optimal orders are confirmed. They also studied the synchronization of the model with viscous damping on the transversal displacement. Their results reveal that the order of the polynomial decay rate is sensitive to various boundary conditions and to the damping locations. In 2021-2022, exponential stability of the Rao-Nakra sandwich beam with kelvin-voigt damping (or internal damping) and time delay was proved by the authers in [30, 94] respectively.

In 2021, the authers of [93] proved the exponential stability for a thermoelastic structure given by a Rao-Nakra sandwich beam.

In 2022, the authers of [36] studied the global well-posedness and exponential stability for a Rao-Nakra sandwich beam equation with time-varying weight and time-varying delay.

Recently, Liu and Akil considered in [10] the generalized Rao-Nakra beam equation, which consists of four wave equations for the longitudinal displacements and the shear angle of the top and bottom layers and one Euler-Bernoulli beam equation for the transversal displacement, with only two viscous dampings. The location of the damping are divided into two groups characterized by either both of the top and bottom layers are directly damped or otherwise. The system considered is as follows:

$$\left\{ \begin{array}{ll} \rho_1 h_1 u_{tt}^1 - E_1 h_1 u_{xx}^1 - \tau + a u_t^1 = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_1 I_1 y_{tt}^1 - E_1 I_1 y_{xx}^1 - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) + b y_t^1 = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + E I \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x + c \omega_t = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - E_3 h_3 u_{xx}^3 + \tau + d u_t^3 = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 I_3 y_{tt}^3 - E_3 I_3 y_{xx}^3 - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) + e y_t^3 = 0 & \text{in } (0, L) \times \mathbb{R}_+. \end{array} \right. \quad (4.5)$$

System (4.5) is subjected to the following initial conditions:

$$\begin{cases} u^1(x, 0) = u_0^1(x), & u_t^1(x, 0) = u_1^1(x), & x \in (0, L), \\ u^3(x, 0) = u_0^3(x), & u_t^3(x, 0) = u_1^3(x), & x \in (0, L), \\ \omega(x, 0) = \omega_0(x), & \omega_t(x, 0) = \omega_1(x), & x \in (0, L), \\ y^1(x, 0) = y_0^1(x), & y_t^1(x, 0) = y_1^1(x), & x \in (0, L), \\ y^3(x, 0) = u_0^2(x), & y_t^3(x, 0) = u_1^2(x), & x \in (0, L) \end{cases} \quad (4.6)$$

and the following boundary conditions:

$$u^1(0, t) = u^1(L, t) = u^3(0, t) = u^3(L, t) = y^1(0, t) = y^1(L, t) = y^3(0, t) = y^3(L, t) = 0 \quad (4.7)$$

and

$$\omega(0, t) = \omega(L, t) = \omega_x(0, t) = \omega_x(L, t) = 0. \quad (4.8)$$

Group one consists of three cases: $[a, d \neq 0 \text{ and } b = c = e = 0]$ or $[b, e \neq 0 \text{ and } a = c = d = 0]$ or $[a, e \neq 0 \text{ and } b = c = d = 0]$. Group two also consists of three cases: $[a, b \neq 0 \text{ and } c = d = e = 0]$ or $[a, c \neq 0 \text{ and } b = d = e = 0]$ or $[b, c \neq 0 \text{ and } a = d = e = 0]$. They obtained the necessary and sufficient conditions for the strong stability of group two. However, they only obtained sufficient conditions for strong stability and instability of some cases of group one. Furthermore, the polynomial stability of certain orders was proved for the cases of group two only; more precisely,

- If $(a, b > 0 \text{ and } c = e = d = 0)$, the authors proved that the energy decays polynomially as $t^{-\frac{2}{\ell}}$, where

$$\ell = \begin{cases} 3 & \text{if } \frac{E_1}{\rho_1} = \frac{E_3}{\rho_3}, \\ 5 & \text{if } \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}. \end{cases}$$

- If $(a, c > 0 \text{ and } b = e = d = 0 \text{ and } G_1 \neq G_3)$, the authors proved that the energy decays polynomially as $t^{-\frac{2}{\ell}}$, where

$$\ell = \begin{cases} 2 & \text{if } \left(\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3} \text{ and } \frac{G_3 h_3}{\rho_3 I_3} \neq \frac{G_1 h_1}{\rho_1 I_1} \right), \\ 6 & \text{if } \left(\frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3} \text{ or } \left(\frac{E_1}{\rho_1} = \frac{E_3}{\rho_3} \text{ and } \frac{G_3 h_3}{\rho_3 I_3} = \frac{G_1 h_1}{\rho_1 I_1} \right) \right). \end{cases}$$

- If $(b, c > 0 \text{ and } a = e = d = 0 \text{ and } \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3})$, the authors proved that the energy decays polynomially as $t^{-\frac{1}{3}}$.

As far as we know, no previous research has studied the stabilization of the generalized Rao-Nakra beam in the presence of local or global viscoelastic Kelvin-Voigt damping. So, in this paper, we are interested in studying the well-posedness and asymptotic behavior of the solution of the following generalized Rao-Nakra beam model with Kelvin-Voigt damping and under various boundary conditions:

$$\begin{cases} \rho_1 h_1 u_{tt}^1 - (E_1 h_1 u_x^1 + D_1(x) u_{tx}^1)_x - \tau = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_1 I_1 y_{tt}^1 - (E_1 I_1 y_x^1 + D_2(x) y_{tx}^1)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho h \omega_{tt} + (EI \omega_{xx} + D_3(x) \omega_{txx})_x - G_1 h_1 (\omega_x + y^1)_x \\ \quad - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 h_3 u_{tt}^3 - (E_3 h_3 u_x^3 + D_4(x) u_{tx}^3)_x + \tau = 0 & \text{in } (0, L) \times \mathbb{R}_+, \\ \rho_3 I_3 y_{tt}^3 - (E_3 I_3 y_x^3 + D_5(x) y_{tx}^3)_x - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = 0 & \text{in } (0, L) \times \mathbb{R}_+. \end{cases} \quad (4.9)$$

System (4.9) is subjected to the following initial conditions:

$$\begin{cases} u^1(x, 0) = u_0^1(x), u_t^1(x, 0) = u_1^1(x), & x \in (0, L), \\ u^3(x, 0) = u_0^3(x), u_t^3(x, 0) = u_1^3(x), & x \in (0, L), \\ \omega(x, 0) = \omega_0(x), \omega_t(x, 0) = \omega_1(x), & x \in (0, L), \\ y^1(x, 0) = y_0^1(x), y_t^1(x, 0) = y_1^1(x), & x \in (0, L), \\ y^3(x, 0) = u_0^2(x), y_t^3(x, 0) = u_1^2(x), & x \in (0, L). \end{cases} \quad (4.10)$$

For the boundary conditions, we assume that variables u^1, u^3, y^1 and y^3 satisfy one of the following boundary conditions:

Dirichlet boundary conditions for all waves

$$u^1(0, t) = u^1(L, t) = u^3(0, t) = u^3(L, t) = y^1(0, t) = y^1(L, t) = y^3(0, t) = y^3(L, t) = 0 \quad (\text{DD})$$

or Neumann boundary conditions for all the waves

$$u_x^1(0, t) = u_x^1(L, t) = u_x^3(0, t) = u_x^3(L, t) = y_x^1(0, t) = y_x^1(L, t) = y_x^3(0, t) = y_x^3(L, t) = 0, \quad (\text{NN})$$

where ω satisfies one of the following boundary conditions:

Clamped boundary condition

$$\omega(0, t) = \omega(L, t) = \omega_x(0, t) = \omega_x(L, t) = 0 \quad (\text{FBC})$$

or Hanged boundary condition

$$\omega(0, t) = \omega(L, t) = \omega_{xx}(0, t) = \omega_{xx}(L, t) = 0. \quad (\text{HBC})$$

We suppose that there exists $0 < a < b < L$ and five positive constants d_i , for $i = 1, \dots, 5$, such that

$$D_i(x) = \begin{cases} d_0, & x \in (a, b), \\ 0, & x \in (0, a) \cup (b, L). \end{cases}$$

In this work, we study the energy decay rate of system (4.9)-(4.10) subject to either the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)] in distinguished two cases. In the first case, we prove that the corresponding semigroup is analytic provided that our system is subject to five global Kelvin-Voigt damping. Then, in the presence of only two local damping D_2 and D_5 acting on the shear angle displacements of the top and bottom layers ($D_1 = D_3 = D_4 = 0$), we show that the energy of the system decays polynomially.

This paper is organized as follows: In section 2, we present the semigroup setting of the system for well-posedness results. In section 3, we show the strong stability. In Section 4, we prove the analytic stability when the five Kelvin-Voigt dampings are globally distributed. Later, section 5 is devoted to study the energy decay rate of our system where we prove that the energy has a polynomial decay rate of type $t^{-\frac{1}{2}}$.

2 Well-Posedness

In this part, by using semigroup theory, we give the Well-Posedness results for problem (4.9)-(4.10) subject to either the boundary conditions (DD) and (FBC) or (NN) and (HBC).

Let $(u^1, u^3, \omega, y^1, y^3)$ be a regular solution of system (4.9)-(4.10). Its associated energy is given by

$$\begin{aligned}
E(t) = & \frac{1}{2} \left(\rho_1 h_1 \int_0^L |u_t^1|^2 dx + E_1 h_1 \int_0^L |u_x^1|^2 dx + \rho_3 h_3 \int_0^L |u_t^3|^2 dx + E_3 h_3 \int_0^L |u_x^3|^2 dx \right. \\
& + \rho h \int_0^L |\omega_t|^2 dx + EI \int_0^L |\omega_{xx}|^2 dx + \rho_1 I_1 \int_0^L |y_t^1|^2 dx + E_1 I_1 \int_0^L |y_x^1|^2 dx \\
& + \rho_3 I_3 \int_0^L |y_t^3|^2 dx + E_3 I_3 \int_0^L |y_x^3|^2 dx + G_1 h_1 \int_0^L |\omega_x + y^1|^2 dx + \int_0^L |\tau|^2 dx \\
& \left. + G_3 h_3 \int_0^L |\omega_x + y^3|^2 dx \right). \tag{4.11}
\end{aligned}$$

A straightforward computation gives that

$$\begin{aligned}
\frac{d}{dt} E(t) = & - \int_0^L D_1(x) |u_{tx}^1|^2 dx - \int_0^L D_2(x) |y_{tx}^1|^2 dx - \int_0^L D_3(x) |\omega_{txx}|^2 dx \\
& - \int_0^L D_4(x) |u_{tx}^3|^2 dx - \int_0^L D_5(x) |y_{tx}^3|^2 dx \\
\leq & 0. \tag{4.12}
\end{aligned}$$

Thus, system (4.9)-(4.10) subject to either the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)] is dissipative in the sense that its energy is a non increasing function with respect to the time variable t . Now, we are in position to prove the existence and uniqueness of the solution of our system. For this aim, we introduce several Hilbert energy spaces.

For the boundary conditions [(DD) and (FBC)], set the energy space as

$$\mathcal{H}_1 = \left(H_0^1(0, L) \times L^2(0, L) \right)^2 \times H_0^2(0, L) \times L^2(0, L) \times \left(H_0^1(0, L) \times L^2(0, L) \right)^2.$$

For the boundary conditions [(NN) and (HBC)], set the energy space as

$$\mathcal{H}_2 = \left(H_*^1(0, L) \times L^2(0, L) \right)^2 \times (H^2(0, L) \cap H_0^1(0, L)) \times L^2(0, L) \times \left(H_*^1(0, L) \times L^2(0, L) \right)^2,$$

where

$$H_*^1(0, L) = \left\{ f \in H^1(0, L) \mid \int_0^L f dx = 0 \right\}.$$

Both spaces \mathcal{H}_1 and \mathcal{H}_2 are equipped with the inner product which induces the energy norm

$$\begin{aligned}
\|U\|_{\mathcal{H}_j}^2 = & \| (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \|_{\mathcal{H}_j}^2 \\
= & E_1 h_1 \|u_x^1\|^2 + \rho_1 h_1 \|v^1\|^2 + E_3 h_3 \|u_x^3\|^2 + \rho_3 h_3 \|v^3\|^2 + EI \|\omega_{xx}\|^2 + \rho h \|\psi\|^2 \\
& + E_1 I_1 \|y_x^1\|^2 + \rho_1 I_1 \|z^1\|^2 + E_3 I_3 \|y_x^3\|^2 + \rho_3 I_3 \|z^3\|^2 + G_1 h_1 \|\omega_x + y^1\|^2 \\
& + \|\tau\|^2 + G_3 h_3 \|\omega_x + y^3\|^2, \quad j = 1, 2, \tag{4.13}
\end{aligned}$$

here $\|\cdot\|$ denotes the norm of $L^2(0, L)$. It is clear that the above equality is equivalent norm on \mathcal{H}_j for $j = 1, 2$. We next define the following linear unbounded operators $\mathcal{A}_j : D(\mathcal{A}_j) \subset \mathcal{H}_j \rightarrow \mathcal{H}_j$:

$$D(\mathcal{A}_1) = \left\{ \begin{array}{l} U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in \mathcal{H}_1 \\ v^1, v^3, z^1, z^3 \in H_0^1(0, L), \psi \in H_0^2(0, L), \\ (E_1 h_1 u_x^1 + D_1(x) v_x^1)_x \in L^2(0, L), (E_3 h_3 u_x^3 + D_4(x) v_x^3)_x \in L^2(0, L), \\ (E_1 I_1 y_x^1 + D_2(x) z_x^1)_x \in L^2(0, L), (E_3 I_3 y_x^3 + D_5(x) z_x^3)_x \in L^2(0, L), \\ (EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} \in L^2(0, L) \end{array} \right\},$$

$$D(\mathcal{A}_2) = \left\{ U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in \mathcal{H}_2 \mid \right. \\ \left. \begin{aligned} &v^1, v^3, z^1, z^3 \in H_*^1(0, L), \psi \in H^2(0, L) \cap H_0^1(0, L), \\ &(E_1 h_1 u_x^1 + D_1(x) v_x^1)_x \in L^2(0, L), u_x^1(0, t) = u_x^1(L, t) = 0, \\ &(E_3 h_3 u_x^3 + D_4(x) v_x^3)_x \in L^2(0, L), u_x^3(0, t) = u_x^3(L, t) = 0, \\ &(E_1 I_1 y_x^1 + D_2(x) z_x^1)_x \in L^2(0, L), y_x^1(0, t) = y_x^1(L, t) = 0, \\ &(E_3 I_3 y_x^3 + D_5(x) z_x^3)_x \in L^2(0, L), y_x^3(0, t) = y_x^3(L, t) = 0, \\ &(EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} \in L^2(0, L), \omega_{xx}(0, t) = \omega_{xx}(L, t) = 0 \end{aligned} \right\}$$

and

$$\mathcal{A}_j \begin{pmatrix} u^1 \\ v^1 \\ y^1 \\ z^1 \\ \omega \\ \psi \\ u^3 \\ v^3 \\ y^3 \\ z^3 \end{pmatrix} = \begin{pmatrix} v^1 \\ (\rho_1 h_1)^{-1} [(E_1 h_1 u_x^1 + D_1(x) v_x^1)_x + \tau] \\ z^1 \\ (\rho_1 I_1)^{-1} [(E_1 I_1 y_x^1 + D_2(x) z_x^1)_x + \frac{h_1}{2} \tau - G_1 h_1 (\omega_x + y^1)] \\ \psi \\ (\rho h)^{-1} [-(EI \omega_{xx} + D_3(x) \psi_{xx})_{xx} + G_1 h_1 (\omega_x + y^1)_x + G_3 h_3 (\omega_x + y^3)_x + h_2 \tau_x] \\ v^3 \\ (\rho_3 h_3)^{-1} [(E_3 h_3 u_x^3 + D_4(x) v_x^3)_x - \tau] \\ z^3 \\ (\rho_3 I_3)^{-1} [(E_3 I_3 y_x^3 + D_5(x) z_x^3)_x + \frac{h_3}{2} \tau - G_3 h_3 (\omega_x + y^3)] \end{pmatrix}, \quad (4.14)$$

for all $U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in D(\mathcal{A}_j)$ and $j = 1, 2$. If U is a regular solution of system (4.9)-(4.10), then the system can be rewritten as evolution equation on the Hilbert space \mathcal{H}_j in the abstract form

$$\begin{cases} U_t(x, t) = \mathcal{A}_j U(x, t), \\ U(x, 0) = U_0(x), \end{cases} \quad (4.15)$$

where

$$U_0 = (u_0^1, u_1^1, y_0^1, y_1^1, \omega_0, \omega_1, u_0^3, u_1^3, y_0^3, y_1^3)$$

and $j = 1, 2$ corresponding to the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)], respectively.

Proposition 4.1. *The unbounded linear operator \mathcal{A}_j is m -dissipative in the hilbert space \mathcal{H}_j .*

Proof. We will prove proposition 4.1 in the case $D_1 = D_3 = D_4 = 0$ on $(0, L)$ and $D_2, D_5 \geq d_0 > 0$ on (a, b) . For all $U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in D(\mathcal{A}_j)$, we have

$$\Re(\langle \mathcal{A}_j U, U \rangle) = -d_0 \int_a^b |z_x^1|^2 dx - d_0 \int_a^b |z_x^3|^2 dx \leq 0,$$

which implies that \mathcal{A}_j is dissipative.

Now, we prove that \mathcal{A}_j is maximal. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}) \in \mathcal{H}_j$, we look for $U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in D(\mathcal{A}_j)$ unique solution of the equation

$$-\mathcal{A}_j U = F.$$

Equivalently, we consider the following system:

$$-v^1 = f_1, \quad (4.16)$$

$$-E_1 h_1 u_{xx}^1 - \tau = \rho_1 h_1 f_2, \quad (4.17)$$

$$-z^1 = f_3, \quad (4.18)$$

$$-\left(E_1 I_1 y_x^1 + D_2(x) z_x^1\right)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = \rho_1 I_1 f_4, \quad (4.19)$$

$$-\psi = f_5, \quad (4.20)$$

$$EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = \rho h f_6, \quad (4.21)$$

$$-v^3 = f_7, \quad (4.22)$$

$$-E_3 h_3 u_{xx}^3 + \tau = \rho_3 h_3 f_8, \quad (4.23)$$

$$-z^3 = f_9, \quad (4.24)$$

$$-\left(E_3 I_3 y_x^3 + D_5(x) z_x^3\right)_x - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = \rho_3 I_3 f_{10} \quad (4.25)$$

with the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)].

By inserting (4.18) in (4.19) and (4.24) in (4.25), system (4.16)-(4.25) implies that

$$-E_1 h_1 u_{xx}^1 - \tau = \rho_1 h_1 f_2, \quad (4.26)$$

$$-\left(E_1 I_1 y_x^1 - D_2(x) (f_3)_x\right)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = \rho_1 I_1 f_4, \quad (4.27)$$

$$EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = \rho h f_6, \quad (4.28)$$

$$-E_3 h_3 u_{xx}^3 + \tau = \rho_3 h_3 f_8, \quad (4.29)$$

$$-\left(E_3 I_3 y_x^3 - D_5(x) (f_9)_x\right)_x - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = \rho_3 I_3 f_{10}. \quad (4.30)$$

Let $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in \mathcal{V}_j(0, L)$, where

$$\mathcal{V}_1(0, L) := H_0^1(0, L) \times H_0^1(0, L) \times H_0^2(0, L) \times H_0^1(0, L) \times H_0^1(0, L)$$

and

$$\mathcal{V}_2(0, L) := H_0^1(0, L) \times H_*^1(0, L) \times (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L) \times H_*^1(0, L).$$

Multiplying equations (4.26), (4.27), (4.28), (4.29), (4.30) by $\overline{\phi_1}$, $\overline{\phi_2}$, $\overline{\phi_3}$, $\overline{\phi_4}$ and $\overline{\phi_5}$, respectively, inegrating over $(0, L)$, then using integration by parts, we obtain

$$E_1 h_1 \int_0^L u_x^1 \overline{\phi_{1x}} dx - \int_0^L \tau \overline{\phi_1} dx = \rho_1 h_1 \int_0^L f_2 \overline{\phi_1} dx, \quad (4.31)$$

$$\begin{aligned}
 & E_1 I_1 \int_0^L y_x^1 \overline{\phi_{2x}} dx - \frac{h_1}{2} \int_0^L \tau \overline{\phi_2} dx + G_1 h_1 \int_0^L (\omega_x + y^1) \overline{\phi_2} dx \\
 & = \rho_1 I_1 \int_0^L f_4 \overline{\phi_2} dx - \int_0^L D_2(x) (f_3)_x \overline{\phi_{2x}} dx,
 \end{aligned} \tag{4.32}$$

$$\begin{aligned}
 & EI \int_0^L \omega_{xx} \overline{\phi_{3xx}} dx + G_1 h_1 \int_0^L (\omega_x + y^1) \overline{\phi_{3x}} dx \\
 & + G_3 h_3 \int_0^L (\omega_x + y^3) \overline{\phi_{3x}} dx + h_2 \int_0^L \tau \overline{\phi_{3x}} dx = \rho h \int_0^L f_6 \overline{\phi_3} dx,
 \end{aligned} \tag{4.33}$$

$$E_3 h_3 \int_0^L u_x^3 \overline{\phi_{4x}} dx + \int_0^L \tau \overline{\phi_4} dx = \rho_3 h_3 \int_0^L f_8 \overline{\phi_4} dx, \tag{4.34}$$

$$\begin{aligned}
 & E_3 I_3 \int_0^L y_x^3 \overline{\phi_{5x}} dx - \frac{h_3}{2} \int_0^L \tau \overline{\phi_5} dx + G_3 h_3 \int_0^L (\omega_x + y^3) \overline{\phi_5} dx \\
 & = \rho_3 I_3 \int_0^L f_{10} \overline{\phi_5} dx - \int_0^L D_5(x) (f_9)_x \overline{\phi_{5x}} dx.
 \end{aligned} \tag{4.35}$$

Now, adding equations (4.31)-(4.35), we obtain the following variational problem:

$$a \left((u^1, y^1, \omega, u^3, y^3), (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \right) = L(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5), \tag{4.36}$$

for all $(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in \mathcal{V}_j(0, L)$, where

$$\begin{aligned}
 a \left((u^1, y^1, \omega, u^3, y^3), (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \right) & = E_1 h_1 \int_0^L u_x^1 \overline{\phi_{1x}} dx \\
 & + E_1 I_1 \int_0^L y_x^1 \overline{\phi_{2x}} dx + EI \int_0^L \omega_{xx} \overline{\phi_{3xx}} dx \\
 & + E_3 h_3 \int_0^L u_x^3 \overline{\phi_{4x}} dx + E_3 I_3 \int_0^L y_x^3 \overline{\phi_{5x}} dx \\
 & + G_1 h_1 \int_0^L (\omega_x + y^1) \overline{(\phi_{3x} + \phi_2)} dx \\
 & + G_3 h_3 \int_0^L (\omega_x + y^3) \overline{(\phi_{3x} + \phi_5)} dx \\
 & + \int_0^L \tau \left(-\overline{\phi_1} - \frac{h_1}{2} \overline{\phi_2} + h_2 \overline{\phi_{3x}} + \overline{\phi_4} - \frac{h_3}{2} \overline{\phi_5} \right) dx
 \end{aligned} \tag{4.37}$$

and

$$\begin{aligned}
 L(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) & = \rho_1 h_1 \int_0^L f_2 \overline{\phi_1} dx + \rho_1 I_1 \int_0^L f_4 \overline{\phi_2} dx \\
 & - \int_0^L D_2(x) (f_3)_x \overline{\phi_{2x}} dx + \rho h \int_0^L f_6 \overline{\phi_3} dx \\
 & + \rho_3 h_3 \int_0^L f_8 \overline{\phi_4} dx + \rho_3 I_3 \int_0^L f_{10} \overline{\phi_5} dx \\
 & - \int_0^L D_5(x) (f_9)_x \overline{\phi_{5x}} dx.
 \end{aligned} \tag{4.38}$$

We see that a is a sesquilinear, continuous and coercive form on $\mathcal{V}_j(0, L) \times \mathcal{V}_j(0, L)$ and L is an antilinear continuous form on $\mathcal{V}_j(0, L)$. Then, using Lax-Milgram Theorem, we deduce that there exists a unique $(u^1, y^1, \omega, u^3, y^3) \in \mathcal{V}_j(0, L)$ solution of the variational problem (4.36). By setting $-z^1 = f_3$ and $-z^3 = f_9$ and by applying the classical elliptic regularity we deduce that

$U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in D(\mathcal{A}_j)$ is the unique solution of $-\mathcal{A}_j U = F$ and consequently $0 \in \rho(\mathcal{A}_j)$, where $\rho(\mathcal{A}_j)$ denotes the resolvent set of \mathcal{A}_j . Then, \mathcal{A}_j is an isomorphism and since $\rho(\mathcal{A}_j)$ is open set of \mathbb{C} (see Theorem 6.7 (Chapter 3) in [68]), we easily get $R(\lambda I - \mathcal{A}_j) = \mathcal{H}_j$, for $\lambda > 0$. This, together with the dissipativeness of \mathcal{A}_j , imply that $D(\mathcal{A}_j)$ is dense in \mathcal{H}_j and that \mathcal{A}_j is m-dissipative in \mathcal{H}_j (see Theorems 4.5 and 4.6 in [88]). The proof is thus complete. \square

Thanks to Lummer-Phillips Theorem (see [88]), we deduce that \mathcal{A}_j generates a C_0 -semigroup of contractions $(e^{t\mathcal{A}_j})_{t \geq 0}$ in \mathcal{H}_j and therefore the solution of the Cauchy problem (4.15) admits the following representation:

$$U(t) = e^{t\mathcal{A}_j} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (4.15). Hence, we have the following result.

Theorem 2.1. Let $U_0 \in \mathcal{H}_j$, then system (4.15) admits a unique weak solution U satisfying

$$U \in C^0(\mathbb{R}_+, \mathcal{H}_j).$$

Moreover, if $U_0 \in D(\mathcal{A}_j)$, then problem (4.15) admits a unique strong solution U satisfying

$$U \in C^1(\mathbb{R}_+, \mathcal{H}_j) \cap C^0(\mathbb{R}_+, D(\mathcal{A}_j)).$$

3 Strong stability

Since the resolvent of \mathcal{A}_j is not compact, we have to discuss the full spectrum on the imaginary axis. We use a general criteria of Arendt-Batty theorem in [24] which states that in a reflexive Banach space a C_0 -semigroup $e^{t\mathcal{A}_j}$ is strongly stable if $e^{t\mathcal{A}_j}$ is bounded, \mathcal{A}_j has no eigenvalues on the imaginary axis and $\sigma(\mathcal{A}_j) \cap i\mathbb{R}$ is countable. Before we state our main result, we need the following stability conditions:

$$D_i > d_0 > 0, \quad \text{for every } x \in (0, L), \quad i = 1, \dots, 5, \quad (\text{SSC1})$$

$$D_i(x) \geq d_0 > 0 \quad \text{in } (a, b), \quad i = 2, 5, \quad G_1 \neq G_3 \quad \text{and} \quad \frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}. \quad (\text{SSC2})$$

So, we will prove the following result:

Theorem 3.1. Assume that condition (SSC1) or (SSC2) holds. Then, the C_0 -semigroup of contractions $(e^{t\mathcal{A}_j})_{t \geq 0}$ is strongly stable on the energy space \mathcal{H}_j , $j = 1, 2$; that is

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_j} U_0\|_{\mathcal{H}_j} = 0, \quad \forall U_0 \in \mathcal{H}_j.$$

We will prove Theorem 3.1 under condition (SSC2) only. For simplicity, we divide the proof of Theorem 3.1 into two lemmas.

Lemma 3.2. Assume that condition (SSC2) hold, we have

$$\ker(i\lambda I - \mathcal{A}_j) = \{0\}, \quad \forall \lambda \in \mathbb{R}, \quad j = 1, 2.$$

Proof. From Proposition 4.1, we have $0 \in \rho(\mathcal{A}_j)$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For this aim, suppose by contradiction that there exists $\lambda \in \mathbb{R}^*$ and $U \in D(\mathcal{A}_j) \setminus \{0\}$ such that

$$\mathcal{A}_j U = i\lambda U. \quad (4.39)$$

Equivalently, we have the following system:

$$v^1 = i\lambda u^1, \quad v^3 = i\lambda u^3, \quad \psi = i\lambda\omega, \quad z^1 = i\lambda y^1, \quad z^3 = i\lambda y^3 \quad (4.40)$$

and

$$\lambda^2 u^1 + (\rho_1 h_1)^{-1} \left[E_1 h_1 u_{xx}^1 + \tau \right] = 0, \quad (4.41)$$

$$\lambda^2 y^1 + (\rho_1 I_1)^{-1} \left[\left(E_1 I_1 y_x^1 + i\lambda D_2(x) y_x^1 \right)_x + \frac{h_1}{2} \tau - G_1 h_1 (\omega_x + y^1) \right] = 0, \quad (4.42)$$

$$\lambda^2 \omega + (\rho h)^{-1} \left[-EI \omega_{xxxx} + G_1 h_1 (\omega_x + y^1)_x + G_3 h_3 (\omega_x + y^3)_x + h_2 \tau_x \right] = 0, \quad (4.43)$$

$$\lambda^2 u^3 + (\rho_3 h_3)^{-1} \left[E_3 h_3 u_{xx}^3 - \tau \right] = 0, \quad (4.44)$$

$$\lambda^2 y^3 + (\rho_3 I_3)^{-1} \left[\left(E_3 I_3 y_x^3 + i\lambda D_5(x) y_x^3 \right)_x + \frac{h_3}{2} \tau - G_3 h_3 (\omega_x + y^3) \right] = 0. \quad (4.45)$$

A straightforward calculation gives

$$\begin{aligned} 0 &= \Re \left(\langle i\lambda U, U \rangle_{\mathcal{H}_j} \right) = \Re \left(\langle \mathcal{A}_j U, U \rangle_{\mathcal{H}_j} \right) = - \int_0^L D_2(x) |z_x^1|^2 dx - \int_0^L D_5(x) |z_x^3|^2 dx \\ &= -d_0 \int_a^b |z_x^1|^2 dx - d_0 \int_a^b |z_x^3|^2 dx = 0. \end{aligned}$$

Thus, we have

$$D_2(x) z_x^1 = 0 \quad \text{in } (0, L) \quad \text{and} \quad D_5(x) z_x^3 = 0 \quad \text{in } (0, L). \quad (4.46)$$

Consequently, From (4.40) we deduce that

$$y_x^1 = 0 \quad \text{in } (a, b) \quad \text{and} \quad y_x^3 = 0 \quad \text{in } (a, b). \quad (4.47)$$

$$\lambda^2 \tilde{u}^1 + (\rho_1 h_1)^{-1} \left[E_1 h_1 \tilde{u}_{xx}^1 + \tilde{\tau} \right] = 0 \quad \text{in } (a, b), \quad (4.48)$$

$$(\rho_1 I_1)^{-1} \left[\frac{h_1}{2} \tilde{\tau} - G_1 h_1 \tilde{\omega}_x \right] = 0 \quad \text{in } (a, b), \quad (4.49)$$

$$\lambda^2 \tilde{\omega} + (\rho h)^{-1} \left[-EI \tilde{\omega}_{xxxx} + G_1 h_1 \tilde{\omega}_{xx} + G_3 h_3 \tilde{\omega}_{xx} + h_2 \tilde{\tau}_x \right] = 0 \quad \text{in } (a, b), \quad (4.50)$$

$$\lambda^2 \tilde{u}^3 + (\rho_3 h_3)^{-1} \left[E_3 h_3 \tilde{u}_{xx}^3 - \tilde{\tau} \right] = 0 \quad \text{in } (a, b), \quad (4.51)$$

$$(\rho_3 I_3)^{-1} \left[\frac{h_3}{2} \tilde{\tau} - G_3 h_3 \tilde{\omega}_x \right] = 0 \quad \text{in } (a, b). \quad (4.52)$$

Now, we introduce the functions $\tilde{u}^i = u_x^i$, $\tilde{y}^i = y_x^i$, $\tilde{\tau} = \tau_x$ and $\tilde{\omega} = \omega_x$, for $i \in \{1, 3\}$. It follows from (4.47) that

$$\tilde{y}^1 = 0 \quad \text{in } (a, b) \quad \text{and} \quad \tilde{y}^3 = 0 \quad \text{in } (a, b), \quad (4.53)$$

and consequently, system (4.41)-(4.45) will be (after differentiating it with respect to x)

$$\lambda^2 \tilde{u}^1 + (\rho_1 h_1)^{-1} \left[E_1 h_1 \tilde{u}_{xx}^1 + \tilde{\tau} \right] = 0 \quad \text{in } (a, b), \quad (4.54)$$

$$(\rho_1 I_1)^{-1} \left[\frac{h_1}{2} \tilde{\tau} - G_1 h_1 \tilde{\omega}_x \right] = 0 \quad \text{in } (a, b), \quad (4.55)$$

$$\lambda^2 \tilde{\omega} + (\rho h)^{-1} \left[-EI \tilde{\omega}_{xxxx} + G_1 h_1 \tilde{\omega}_{xx} + G_3 h_3 \tilde{\omega}_{xx} + h_2 \tilde{\tau}_x \right] = 0 \quad \text{in } (a, b), \quad (4.56)$$

$$\lambda^2 \tilde{u}^3 + (\rho_3 h_3)^{-1} \left[E_3 h_3 \tilde{u}_{xx}^3 - \tilde{\tau} \right] = 0 \quad \text{in } (a, b), \quad (4.57)$$

$$(\rho_3 I_3)^{-1} \left[\frac{h_3}{2} \tilde{\tau} - G_3 h_3 \tilde{\omega}_x \right] = 0 \quad \text{in } (a, b). \quad (4.58)$$

From (4.55), (4.58) and (4.53), we get

$$\frac{1}{2}\tilde{\tau} - G_1\tilde{\omega}_x = 0 \quad \text{and} \quad \frac{1}{2}\tilde{\tau} - G_3\tilde{\omega}_x = 0 \quad \text{in} \quad (a, b). \quad (4.59)$$

Subtracting the two equations in (4.59) and using the fact that $G_3 \neq G_1$, we find

$$\tilde{\omega}_x = 0 \quad \text{in} \quad (a, b). \quad (4.60)$$

Using (4.59) and (4.60), we obtain

$$\tilde{\tau} = 0 \quad \text{in} \quad (a, b). \quad (4.61)$$

Using (4.56), (4.53), (4.60) and (4.61), we get

$$\lambda^2\tilde{\omega} = 0 \quad \text{in} \quad (a, b).$$

Since $\lambda \neq 0$, we get

$$\tilde{\omega} = 0 \quad \text{in} \quad (a, b). \quad (4.62)$$

Using the definition of $\tilde{\tau}$ and (4.53), (4.60) and (4.61), we get

$$\tilde{u}^1 = \tilde{u}^3 \quad \text{in} \quad (a, b). \quad (4.63)$$

Now, using (4.54), (4.57) and (4.61), we get

$$\lambda^2\tilde{u}^1 + \frac{E_1}{\rho_1}\tilde{u}_{xx}^1 = 0 \quad \text{and} \quad \lambda^2\tilde{u}^3 + \frac{E_3}{\rho_3}\tilde{u}_{xx}^3 = 0 \quad \text{in} \quad (a, b).$$

Subtracting the above two equations and using (4.63), we get

$$\left(\frac{E_1}{\rho_1} - \frac{E_3}{\rho_3}\right)\tilde{u}_{xx}^1 = 0 \quad \text{in} \quad (a, b). \quad (4.64)$$

Using the facts that $\frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}$ in (4.64), we obtain

$$\tilde{u}_{xx}^1 = \tilde{u}_{xx}^3 = 0 \quad \text{in} \quad (a, b). \quad (4.65)$$

From (4.54), (4.57), (4.61) and (4.65), we find

$$\lambda^2\tilde{u}^1 = \lambda^2\tilde{u}^3 = 0 \quad \text{in} \quad (a, b).$$

Since $\lambda \neq 0$, we get

$$\tilde{u}^1 = \tilde{u}^3 = 0 \quad \text{in} \quad (a, b). \quad (4.66)$$

Thus,

$$\tilde{U} = 0 \quad \text{in} \quad (a, b). \quad (4.67)$$

Now, after differentiating system (4.41)-(4.45) with respect to x , we can write it in $(0, a) \cup (b, L)$ as the following:

$$\lambda^2\tilde{u}^1 + (\rho_1 h_1)^{-1} \left[E_1 h_1 \tilde{u}_{xx}^1 + \tilde{\tau} \right] = 0, \quad (4.68)$$

$$\lambda^2\tilde{y}^1 + (\rho_1 I_1)^{-1} \left[E_1 I_1 \tilde{y}_{xx}^1 + \frac{h_1}{2}\tilde{\tau} - G_1 h_1 (\tilde{\omega}_x + \tilde{y}^1) \right] = 0, \quad (4.69)$$

$$\lambda^2\tilde{\omega} + (\rho h)^{-1} \left[-EI\tilde{\omega}_{xxxx} + G_1 h_1 (\tilde{\omega}_x + \tilde{y}^1)_x + G_3 h_3 (\tilde{\omega}_x + \tilde{y}^3)_x + h_2 \tilde{\tau}_x \right] = 0, \quad (4.70)$$

$$\lambda^2\tilde{u}^3 + (\rho_3 h_3)^{-1} \left[E_3 h_3 \tilde{u}_{xx}^3 - \tilde{\tau} \right] = 0, \quad (4.71)$$

$$\lambda^2\tilde{y}^3 + (\rho_3 I_3)^{-1} \left[E_3 I_3 \tilde{y}_{xx}^3 + \frac{h_3}{2}\tilde{\tau} - G_3 h_3 (\tilde{\omega}_x + \tilde{y}^3) \right] = 0. \quad (4.72)$$

Let $V = (\tilde{u}^1, \tilde{u}_x^1, \tilde{y}^1, \tilde{y}_x^1, \tilde{\omega}, \tilde{\omega}_x, \tilde{\omega}_{xx}, \tilde{\omega}_{xxx}, \tilde{u}^3, \tilde{u}_x^3, \tilde{y}^3, \tilde{y}_x^3)$. From (4.67) and the regularity of $\tilde{u}^1, \tilde{y}^1, \tilde{\omega}, \tilde{u}^3$ and \tilde{y}^3 , we have $V(a) = 0$. Now, system (4.68)-(4.72) can be written as

$$V_x = A_\lambda V \quad \text{in } (0, a), \quad (4.73)$$

where

$$A_\lambda = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_1 & 0 & K_2 & 0 & 0 & K_3 & 0 & 0 & K_4 & 0 & K_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ K_6 & 0 & K_7 & 0 & 0 & K_8 & 0 & 0 & K_9 & 0 & K_{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & K_{11} & 0 & K_{12} & K_{13} & 0 & K_{14} & 0 & 0 & K_{15} & 0 & K_{16} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ K_{17} & 0 & K_{18} & 0 & 0 & K_{19} & 0 & 0 & K_{20} & 0 & K_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ K_{22} & 0 & K_{23} & 0 & 0 & K_{24} & 0 & 0 & K_{25} & 0 & K_{26} & 0 \end{pmatrix} \quad (4.74)$$

and

$$\begin{aligned} K_1 &= \frac{-(\rho_1 h_1) \lambda^2 + 1}{E_1 h_1}, \quad K_2 = \frac{1}{2E_1}, \quad K_3 = \frac{-h_2}{E_1 h_1}, \quad K_4 = \frac{1}{E_1 h_1}, \quad K_5 = \frac{h_3}{2E_1 h_1}, \\ K_6 &= \frac{h_1}{2E_1 I_1}, \quad K_7 = \frac{-4(\rho_1 I_1) \lambda^2 + 4G_1 h_1 + h_1^2}{4E_1 I_1}, \quad K_8 = \frac{2G_1 h_1 - h_1 h_2}{2E_1 I_1}, \quad K_9 = \frac{-h_1}{2E_1 I_1}, \\ K_{10} &= \frac{h_1 h_3}{4E_1 I_1}, \quad K_{11} = \frac{-h_2}{EI}, \quad K_{12} = \frac{2G_1 h_1 - h_2 h_1}{2EI}, \quad K_{13} = \frac{(\rho h) \lambda^2}{EI}, \quad K_{14} = \frac{G_1 h_1 + G_3 h_3 + h_2^2}{EI}, \\ K_{15} &= \frac{h_2}{EI}, \quad K_{16} = \frac{2G_3 h_3 - h_2 h_3}{2EI}, \quad K_{17} = \frac{-1}{E_3 h_3}, \quad K_{18} = \frac{-h_1}{2E_3 h_3}, \quad K_{19} = \frac{h_2}{E_3 h_3}, \\ K_{20} &= \frac{-(\rho_3 h_3) \lambda^2 + 1}{E_3 h_3}, \quad K_{21} = \frac{-1}{2E_3}, \quad K_{22} = \frac{h_3}{2E_3 I_3}, \quad K_{23} = \frac{h_3 h_1}{4E_3 I_3}, \quad K_{24} = \frac{2G_3 h_3 - h_3 h_2}{2E_3 I_3}, \\ K_{25} &= \frac{-h_3}{2E_3 I_3}, \quad K_{26} = \frac{-4(\rho_3 I_3) \lambda^2 + 4G_3 h_3 + h_3^2}{4E_3 I_3}. \end{aligned}$$

Using ordinary differential equation theory, we deduce that system (4.73) has the unique trivial solution

$$V(x) = e^{A_\lambda(x-a)} V(a) = 0 \quad \text{in } (0, a). \quad (4.75)$$

The same argument as above leads us to prove that

$$V(x) = e^{A_\lambda(x-b)} V(b) = 0 \quad \text{in } (b, L), \quad (4.76)$$

consequently, we deduce that

$$\tilde{U} = 0 \quad \text{in } (0, a) \cup (b, L). \quad (4.77)$$

Finally, from (4.67) and (4.77), we obtain

$$\tilde{U} = 0 \quad \text{in } (0, L).$$

This gives that $U = C$, where C is a constant. Finally, from the boundary conditions, we deduce that $U = 0$ in $(0, L)$. Consequently, \mathcal{A} has no pure imaginary eigenvalues. The proof is thus complete. \square

Lemma 3.3. Assume that condition (SSC2) hold, for all $\lambda \in \mathbb{R}$, we have

$$\mathcal{R}(i\lambda I - \mathcal{A}_j) = \mathcal{H}_j, \quad j = 1, 2.$$

Proof. From Proposition 4.1, we have $0 \in \rho(\mathcal{A}_j)$. We still need to show the result for $\lambda \in \mathbb{R}^*$. For this aim, let $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10}) \in \mathcal{H}_j$, we want to find $U = (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3) \in D(\mathcal{A}_j)$ solution of

$$(i\lambda I - \mathcal{A}_j)U = F. \quad (4.78)$$

Equivalently, we have the following system:

$$i\lambda u^1 - v^1 = f_1, \quad (4.79)$$

$$\rho_1 h_1 i\lambda v^1 - E_1 h_1 u_{xx}^1 - \tau = \rho_1 h_1 f_2, \quad (4.80)$$

$$i\lambda y^1 - z^1 = f_3, \quad (4.81)$$

$$\rho_1 I_1 i\lambda z^1 - \left(E_1 I_1 y_x^1 + D_2(x) z_x^1 \right)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = \rho_1 I_1 f_4, \quad (4.82)$$

$$i\lambda \omega - \psi = f_5, \quad (4.83)$$

$$\rho h i\lambda \psi + EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = \rho h f_6, \quad (4.84)$$

$$i\lambda u^3 - v^3 = f_7, \quad (4.85)$$

$$\rho_3 h_3 i\lambda v^3 - E_3 h_3 u_{xx}^3 + \tau = \rho_3 h_3 f_8, \quad (4.86)$$

$$i\lambda y^3 - z^3 = f_9, \quad (4.87)$$

$$\rho_3 I_3 i\lambda z^3 - \left(E_3 I_3 y_x^3 + D_5(x) z_x^3 \right)_x - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = \rho_3 I_3 f_{10} \quad (4.88)$$

with the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)].

Inserting (4.79), (4.81), (4.83), (4.85) and (4.87) in (4.80), (4.82), (4.84), (4.86) and (4.88), we get

$$-\rho_1 h_1 \lambda^2 u^1 - E_1 h_1 u_{xx}^1 - \tau = h_1, \quad (4.89)$$

$$-\rho_1 I_1 \lambda^2 y^1 - \left(E_1 I_1 y_x^1 + i\lambda D_2(x) y_x^1 \right)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) = h_2, \quad (4.90)$$

$$-\rho \lambda^2 \omega + EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x = h_3, \quad (4.91)$$

$$-\rho_3 h_3 \lambda^2 u^3 - E_3 h_3 u_{xx}^3 + \tau = h_4, \quad (4.92)$$

$$-\rho_3 I_3 \lambda^2 y^3 - \left(E_3 I_3 y_x^3 + i\lambda D_5(x) y_x^3 \right)_x - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) = h_5, \quad (4.93)$$

where

$$\left\{ \begin{array}{l} h_1 := \rho_1 h_1 (f_1 + f_2), \in D'(0, L), \\ h_2 := \rho_1 I_1 (i\lambda f_3 + f_4) - [D_2(x)(f_3)_x]_x, \in D'(0, L), \\ h_3 := \rho h (f_5 + f_6), \in D'(0, L), \\ h_4 := \rho_3 h_3 (f_7 + f_8), \in D'(0, L), \\ h_5 := \rho_3 I_3 (i\lambda f_9 + f_{10}) - [D_5(x)(f_9)_x]_x, \in D'(0, L) \end{array} \right. \quad (4.94)$$

with the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)].

For all

$$(u^1, y^1, \omega, u^3, y^3) \in \mathcal{V}_1(0, L) := H_0^1(0, L) \times H_0^1(0, L) \times H_0^2(0, L) \times H_0^1(0, L) \times H_0^1(0, L)$$

for $j = 1$ and

$$(u^1, y^1, \omega, u^3, y^3) \in \mathcal{V}_2(0, L) := H_*^1(0, L) \times H_*^1(0, L) \times (H^2(0, L) \cap H_0^1(0, L)) \times H_*^1(0, L) \times H_*^1(0, L)$$

for $j = 2$, we define the linear operator \mathcal{L}_j by

$$\mathcal{L}_j \begin{pmatrix} u^1 \\ y^1 \\ \omega \\ u^3 \\ y^3 \end{pmatrix} = \begin{pmatrix} -E_1 h_1 u_{xx}^1 - \tau \\ -(E_1 I_1 y_x^1 + i\lambda D_2(x) y_x^1)_x - \frac{h_1}{2} \tau + G_1 h_1 (\omega_x + y^1) \\ EI \omega_{xxxx} - G_1 h_1 (\omega_x + y^1)_x - G_3 h_3 (\omega_x + y^3)_x - h_2 \tau_x \\ -E_3 h_3 u_{xx}^3 + \tau \\ -(E_3 I_3 y_x^3 + i\lambda D_5(x) y_x^3)_x - \frac{h_3}{2} \tau + G_3 h_3 (\omega_x + y^3) \end{pmatrix}. \quad (4.95)$$

Using Lax-Milgram Theorem, it is easy to see show that \mathcal{L}_j is an isomorphism from $\mathcal{V}_j(0, L)$ onto $(\mathcal{V}_j(0, L))'$. Let $\mathcal{U} = (u^1, y^1, \omega, u^3, y^3)$ and $\mathcal{F} = (h_1, h_2, h_3, h_4, h_5)$, then we transform system (4.89)-(4.93) with the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)] into the following form:

$$(I - \lambda^2 \mathcal{L}_j^{-1}) \mathcal{U} = \mathcal{L}_j^{-1} \mathcal{F}. \quad (4.96)$$

Since \mathcal{L}_j is an isomorphism from $\mathcal{V}_j(0, L)$ onto $(\mathcal{V}_j(0, L))'$ and I is a compact operator from $\mathcal{V}_j(0, L)$ onto $(\mathcal{V}_j(0, L))'$, then by using Fredholm's alternative Theorem, problem (4.96) admits a unique solution in $\mathcal{V}_j(0, L)$ if and only if the operator $(I - \lambda^2 \mathcal{L}_j^{-1})$ is injective. For that purpose, let $\tilde{\mathcal{U}} = (\tilde{u}^1, \tilde{y}^1, \tilde{\omega}, \tilde{u}^3, \tilde{y}^3) \in \ker(\lambda^2 I - \mathcal{L}_j)$. Then, we have $\lambda^2 \tilde{\mathcal{U}} - \mathcal{L}_j \tilde{\mathcal{U}} = 0$. It follows that

$$-\rho_1 h_1 \lambda^2 \tilde{u}^1 - E_1 h_1 \tilde{u}_{xx}^1 - \tilde{\tau} = 0, \quad (4.97)$$

$$-\rho_1 I_1 \lambda^2 \tilde{y}^1 - (E_1 I_1 \tilde{y}_x^1 + i\lambda D_2(x) \tilde{y}_x^1)_x - \frac{h_1}{2} \tilde{\tau} + G_1 h_1 (\tilde{\omega}_x + \tilde{y}^1) = 0, \quad (4.98)$$

$$-\rho \lambda^2 \tilde{\omega} + EI \tilde{\omega}_{xxxx} - G_1 h_1 (\tilde{\omega}_x + \tilde{y}^1)_x - G_3 h_3 (\omega_x + \tilde{y}^3)_x - h_2 \tilde{\tau}_x = 0, \quad (4.99)$$

$$-\rho_3 h_3 \lambda^2 \tilde{u}^3 - E_3 h_3 \tilde{u}_{xx}^3 + \tilde{\tau} = 0, \quad (4.100)$$

$$-\rho_3 I_3 \lambda^2 \tilde{y}^3 - (E_3 I_3 \tilde{y}_x^3 + i\lambda D_5(x) \tilde{y}_x^3)_x - \frac{h_3}{2} \tilde{\tau} + G_3 h_3 (\tilde{\omega}_x + \tilde{y}^3) = 0. \quad (4.101)$$

Multiplying equations (4.97)-(4.101) by $\overline{\tilde{u}^1}, \overline{\tilde{y}^1}, \overline{\tilde{\omega}}, \overline{\tilde{u}^3}, \overline{\tilde{y}^3}$, respectively, integrating over $(0, L)$ and taking the sum, then using by parts integration and the boundary conditions, and taking the imaginary part we get

$$d_0 \Im \left(i\lambda \int_a^b (|\tilde{y}_x^1|^2 + |\tilde{y}_x^3|^2) dx \right) = 0.$$

Thus, we get $\tilde{y}_x^1 = \tilde{y}_x^3 = 0$ in (a, b) . Now it is easy to see that, if $\tilde{\mathcal{U}}$ is a solution of system (4.97)-(4.101), then the vector \tilde{V} defined by

$$\tilde{V} = (\tilde{u}^1, i\lambda \tilde{u}^1, \tilde{y}^1, i\lambda \tilde{y}^1, \tilde{\omega}, i\lambda \tilde{\omega}, \tilde{u}^3, i\lambda \tilde{u}^3, \tilde{y}^3, i\lambda \tilde{y}^3)^T$$

belongs to $D(\mathcal{A}_j)$ and satisfies $i\lambda \tilde{V} - \mathcal{A}_j \lambda \tilde{V} = 0$. Using Lemma 3.2, we deduce that $\tilde{V} = 0$ and consequently $\text{Ker}(I - \lambda^2 \mathcal{L}_j^{-1}) = \{0\}$. Thanks to Fredholm's alternative, (4.96) admits a unique solution $\mathcal{U} \in \mathcal{V}_j$ and

$$\left(E_1 I_1 y_x^1 + i\lambda D_2(x) y_x^1 - D_2(x) (f_3)_x \right)_x \in L^2(0, L),$$

$$\left(E_3 I_3 y_x^3 + i\lambda D_5(x) y_x^3 - D_5(x) (f_9)_x \right)_x \in L^2(0, L).$$

Thus, by setting $z^1 = i\lambda y^1 - f_3$, $z^3 = i\lambda y^3 - f_9$ and a classical elliptic regularity arguments we conclude that $(i\lambda I - \mathcal{A}_j) = F$ admits a unique solution $U \in D(\mathcal{A}_j)$. The proof is thus complete. \square

Proof of Theorem 3.1. Using Lemma 3.2, we directly deduce that \mathcal{A}_j has non pure imaginary eigenvalues. According to Lemma 3.3 and with the help of the closed graph theorem of Banach, we deduce that $\sigma(\mathcal{A}_j) \cap i\mathbb{R} = \emptyset$. Thus, we get the conclusion by applying Theorem 2.7 of Arendt and Batty [22].

4 Analytic Stability

In this section, under the condition (SSC1), we will prove that the associated semi-group of the system (4.15) is analytic. For this aim, we will use the following result:

Theorem 4.1. Let $(\mathcal{S}(t))_{t \geq 0} = (e^{t\mathcal{A}_j})_{t \geq 0}$ be a C_0 -semigroup of contractions in a Hilbert space. Assume that

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_j) \quad (\text{AS1})$$

Then, $(e^{t\mathcal{A}_j})_{t \geq 0}$ is analytic if and only if

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} |\lambda| \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} < \infty. \quad (\text{AS2})$$

Now, we are ready to state our result.

Theorem 4.2. Assume that condition (SSC1) holds and $D_i \in L^\infty(0, L)$, $i = 1, \dots, 5$. Then, the C_0 -semigroup $(e^{t\mathcal{A}_j})_{t \geq 0}$ is analytic in \mathcal{H}_j , for $j = 1, 2$.

Proof. We know that condition (AS1) is verified. Our goal now is to prove that condition (AS2) is satisfied. For this aim, we proceed by a contradiction argument. Suppose that (AS2) does not hold, then there exist a sequence $(\lambda_n) \subset \mathbb{R}$ and a sequence $(U_n) \subset D(\mathcal{A}_j)$ such that

$$|\lambda_n| \longrightarrow +\infty, \quad \|U_n\|_{\mathcal{H}_j} = \|(u_n^1, v_n^1, u_n^3, v_n^3, \omega_n, \psi_n, y_n^1, z_n^1, y_n^3, z_n^3)^\top\|_{\mathcal{H}_j} = 1 \quad (4.102)$$

and

$$\lambda_n^{-1} (i\lambda_n U_n - \mathcal{A}_j U_n) = F := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}, f_{6,n}, f_{7,n}, f_{8,n}, f_{9,n}, f_{10,n})^\top \rightarrow 0 \text{ in } \mathcal{H}_j. \quad (4.103)$$

are satisfied. In what follows, we drop the index n for simplicity.

Step 1 (The dissipation). Taking the inner product of (4.103) with U in \mathcal{H}_j , then using the fact that U is uniformly bounded in \mathcal{H}_j , we get

$$\begin{aligned} \lambda^{-1} \int_0^L \left(D_1 |v_x^1|^2 + D_2 |z_x^1|^2 + D_3 |\psi_{xx}|^2 + D_4 |v_x^3|^2 + D_5 |z_x^3|^2 \right) dx &= -\lambda^{-1} \operatorname{Re}((i\lambda I - \mathcal{A}_j)U, U)_{\mathcal{H}_j} \\ &= o(1). \end{aligned}$$

It follows that

$$\begin{cases} \lambda^{-1/2} \|v_x^1\|_{L^2(0,L)} = o(1), & \lambda^{-1/2} \|v_x^3\|_{L^2(0,L)} = o(1), \\ \lambda^{-1/2} \|z_x^1\|_{L^2(0,L)} = o(1), & \lambda^{-1/2} \|z_x^3\|_{L^2(0,L)} = o(1), \\ \lambda^{-1/2} \|\psi_{xx}\|_{L^2(0,L)} = o(1). \end{cases} \quad (4.104)$$

Using Poincaré inequality, we get

$$\begin{cases} \lambda^{-1/2} \|v^1\|_{L^2(0,L)} = o(1), & \lambda^{-1/2} \|v^3\|_{L^2(0,L)} = o(1), \\ \lambda^{-1/2} \|z^1\|_{L^2(0,L)} = o(1), & \lambda^{-1/2} \|z^3\|_{L^2(0,L)} = o(1), \\ \lambda^{-1/2} \|\psi\|_{L^2(0,L)} = o(1). \end{cases} \quad (4.105)$$

Step 2 (Information on $u_x^1, y_x^1, \omega_{xx}, u_x^3$ and y_x^3). By detailing equation (4.103), we get the following system:

$$\lambda^{-1}(i\lambda u^1 - v^1) = f_1, \quad (4.106)$$

$$i\rho_1 h_1 v^1 - \lambda^{-1} \left[(E_1 h_1 u_x^1 + D_1 v_x^1)_x + \tau \right] = \rho_1 h_1 f_2, \quad (4.107)$$

$$\lambda^{-1}(i\lambda y^1 - z^1) = f_3, \quad (4.108)$$

$$i\rho_1 I_1 z^1 - \lambda^{-1} \left[(E_1 I_1 y_x^1 + D_2 z_x^1)_x + \frac{h_1}{2} \tau - G_1 h_1 (\omega_x + y^1) \right] = \rho_1 I_1 f_4, \quad (4.109)$$

$$\lambda^{-1}(i\lambda \omega - \psi) = f_5, \quad (4.110)$$

$$i\rho h \psi - \lambda^{-1} \left[-(EI\omega_{xx} + D_3 \psi_{xx})_{xx} + G_1 h_1 (\omega_x + y^1)_x + G_3 h_3 (\omega_x + y^3)_x + h_2 \tau_x \right] = \rho h f_6, \quad (4.111)$$

$$\lambda^{-1}(i\lambda u^3 - v^3) = f_7, \quad (4.112)$$

$$i\rho_3 h_3 v^3 - \lambda^{-1} \left[(E_3 h_3 u_x^3 + D_4 v_x^3)_x - \tau \right] = \rho_3 h_3 f_8, \quad (4.113)$$

$$\lambda^{-1}(i\lambda y^3 - z^3) = f_9, \quad (4.114)$$

$$i\rho_3 I_3 z^3 - \lambda^{-1} \left[(E_3 I_3 y_x^3 + D_5 z_x^3)_x + \frac{h_1}{2} \tau - G_3 h_3 (\omega_x + y^3) \right] = \rho_3 I_3 f_{10}. \quad (4.115)$$

By differentiating (4.106), (4.108), (4.110), (4.112), (4.114) with respect to the variable x , then using (4.104), (4.105) and the fact that $F \rightarrow 0$ in \mathcal{H}_j , we obtain

$$\begin{cases} \|u_x^1\|_{L^2(0,L)} = o(1), & \|u_x^3\|_{L^2(0,L)} = o(1), \\ \|y_x^1\|_{L^2(0,L)} = o(1), & \|y_x^3\|_{L^2(0,L)} = o(1), \\ \|\omega_{xx}\|_{L^2(0,L)} = o(1) \end{cases} \quad (4.116)$$

and

$$\begin{cases} \|u^1\|_{L^2(0,L)} = o(1), & \|u^3\|_{L^2(0,L)} = o(1), \\ \|y^1\|_{L^2(0,L)} = o(1), & \|y^3\|_{L^2(0,L)} = o(1), \\ \|\omega\|_{L^2(0,L)} = o(1). \end{cases} \quad (4.117)$$

Step 3 (Information on v^1, z^1, ψ, v^3 and z^3). Multiplying equation (4.107) by $-i\bar{v}^1$, then integrating over $(0, L)$ gives

$$\begin{aligned} \rho_1 h_1 \int_0^L |v^1|^2 dx - i\lambda^{-1} E_1 h_1 \int_0^L u_x^1 \bar{v}_x^1 dx - i\lambda^{-1} D_1 \int_0^L |v_x^1|^2 dx \\ - i\lambda^{-1} \int_0^L \tau \bar{v}^1 dx = -i\rho_1 h_1 \int_0^L f_2 \bar{v}^1 dx. \end{aligned} \quad (4.118)$$

Then, using (4.104) and (4.116), the fact that v^1 and τ are uniformly bounded in $L^2(0, L)$ and the fact that $\|f_2\|_{L^2(0,L)} = o(1)$, we find

$$\|v^1\|_{L^2(0,L)} = o(1). \quad (4.119)$$

Similarly, multiplying equation (4.113) by $-i\bar{v}^3$ and using the same arguments, we arrive at

$$\|v^3\|_{L^2(0,L)} = o(1). \quad (4.120)$$

Now, multiplying equation (4.109) by $-i\bar{z}^1$, then integrating over $(0, L)$ leads to

$$\begin{aligned} & \rho_1 I_1 \int_0^L |z^1|^2 dx - i\lambda^{-1} E_1 I_1 \int_0^L y_x^1 \cdot \bar{z}^1_x dx \\ & - i\lambda^{-1} D_2 \int_0^L |z^1_x|^2 dx - i\lambda^{-1} \frac{h_1}{2} \int_0^L \tau \bar{z}^1 dx \\ & - i\lambda G_1 h_1 \int_0^L (\omega_x + y^1) \bar{z}^1 dx = -i\rho_1 I_1 \int_0^L f_4 \bar{z}^1 dx. \end{aligned} \quad (4.121)$$

Using (4.104) and (4.116), the fact that z^1 , τ , $(\omega_x + y^1)$ are uniformly bounded in $L^2(0, L)$ and the fact that $\|f_4\|_{L^2(0,L)} = o(1)$, it appears that

$$\|z^1\|_{L^2(0,L)} = o(1). \quad (4.122)$$

Similarly, multiplying equation (4.115) by $-i\bar{z}^3$ and using the same arguments, we observe that

$$\|z^3\|_{L^2(0,L)} = o(1). \quad (4.123)$$

Finally, multiplying equation (4.111) by $-i\bar{\psi}$, then integrating over $(0, L)$ gives

$$\begin{aligned} & \rho h \int_0^L |\psi|^2 dx - i\lambda^{-1} EI \int_0^L \omega_{xx} \bar{\psi}_{xx} dx - i\lambda^{-1} D_3 \int_0^L |\psi_{xx}|^2 dx \\ & + i\lambda^{-1} Gh \int_0^L \omega_{xx} \bar{\psi} dx + i\lambda^{-1} G_1 h_1 \int_0^L y_x^1 \bar{\psi} dx + i\lambda^{-1} G_3 h_3 \int_0^L y_x^3 \bar{\psi} dx \\ & + i\lambda^{-1} h_2 \int_0^L \tau \bar{\psi}_x dx = -i\rho h \int_0^L f_6 \bar{\psi} dx. \end{aligned} \quad (4.124)$$

Using (4.104) and (4.116), the fact that ψ and τ are uniformly bounded in $L^2(0, L)$ and the fact that $\|f_6\|_{L^2(0,L)} = o(1)$, we find

$$\|\psi\|_{L^2(0,L)} = o(1). \quad (4.125)$$

Hence, we obtain $\|U_n\|_{\mathcal{H}_j} = o(1)$ which contradicts (4.102). Therefore, (AS2) holds and, consequently, we deduce the analytic stability of (4.15) in the case of five global dampings. \square

5 Polynomial Stability

In this section, we study the polynomial stability of system (4.9)-(4.10) subject to either the boundary conditions [(DD) and (FBC)] or [(NN) and (HBC)]. For this purpose, we will use a frequency domain approach. Our main result in this section is the following theorem.

Theorem 5.1. Assume that condition (SSC2) holds. The C_0 -semigroup $(e^{tA_j})_{t \geq 1}$ is polynomially stable; more precisely, there exists constant $C > 0$ such that for every $U_0 \in D(\mathcal{A}_j)$, we have

$$E(t) \leq \frac{C}{t^{\frac{1}{2}}} \|U_0\|_{D(\mathcal{A}_j)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_j). \quad (4.126)$$

Proof of Theorem 5.1. According to Theorem 2.10, the polynomial energy decay (4.126) hold if the following conditions:

$$i\mathbb{R} \subset \rho(\mathcal{A}_j) \quad (\text{H}_1)$$

and

$$\sup_{\lambda \in \mathbb{R}} \frac{1}{|\lambda|^4} \|(i\lambda I - \mathcal{A}_j)^{-1}\|_{\mathcal{L}(\mathcal{H}_j)} \leq O(1) \quad (\text{H}_2)$$

are satisfied. Since condition (H₁) is already proved. We will prove condition (H₂) by an argument of contradiction. For this purpose, suppose that (H₂) is false, then there exists

$$\{(\lambda_n, U_n := (u^1, v^1, y^1, z^1, \omega, \psi, u^3, v^3, y^3, z^3)^\top)\} \subset \mathbb{R} \times D(\mathcal{A}_j)$$

with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U\|_{\mathcal{H}_j} = \|(u^1, v^1, u^3, v^3, \omega, \psi, y^1, z^1, y^3, z^3)\|_{\mathcal{H}_j} = 1 \quad (4.127)$$

such that

$$\lambda_n^4 (i\lambda_n I - \mathcal{A}_j) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}, f_{6,n}, f_{7,n}, f_{8,n}, f_{9,n}, f_{10,n})^\top \rightarrow 0 \quad \text{in } \mathcal{H}_j. \quad (4.128)$$

For simplicity, we drop the index n . Equivalently, from (4.128), we have

$$i\lambda u^1 - v^1 = \lambda^{-4} f_1, \quad (4.129)$$

$$i\lambda v^1 - (\rho_1 h_1)^{-1} [E_1 h_1 u_{xx}^1 + \tau] = \lambda^{-4} f_2, \quad (4.130)$$

$$i\lambda y^1 - z^1 = \lambda^{-4} f_3, \quad (4.131)$$

$$i\lambda z^1 - (\rho_1 I_1)^{-1} \left[(E_1 I_1 y_x^1 + D(x) z_x^1)_x + \frac{h_1}{2} \tau - G_1 h_1 (\omega_x + y^1) \right] = \lambda^{-4} f_4, \quad (4.132)$$

$$i\lambda \omega - \psi = \lambda^{-4} f_5, \quad (4.133)$$

$$i\lambda \psi - (\rho h)^{-1} \left[-EI \omega_{xxxx} + G_1 h_1 (\omega_x + y^1)_x + G_3 h_3 (\omega_x + y^3)_x + h_2 \tau_x \right] = \lambda^{-\ell} f_6, \quad (4.134)$$

$$i\lambda u^3 - v^3 = \lambda^{-4} f_7, \quad (4.135)$$

$$i\lambda v^3 - (\rho_3 h_3)^{-1} [E_3 h_3 u_{xx}^3 - \tau] = \lambda^{-\ell} f_8, \quad (4.136)$$

$$i\lambda y^3 - z^3 = \lambda^{-4} f_9, \quad (4.137)$$

$$i\lambda z^3 - (\rho_3 I_3)^{-1} \left[(E_3 I_3 y_x^3 + D(x) z_x^3)_x + \frac{h_3}{2} \tau - G_3 h_3 (\omega_x + y^3) \right] = \lambda^{-4} f_{10}, \quad (4.138)$$

where

$$\tau = -u^1 + u^3 + h_2 \omega_x - \frac{h_1}{2} y^1 - \frac{h_3}{2} y^3. \quad (4.139)$$

From (4.129)-(4.138), we obtain the following system:

$$\lambda^2 u^1 + (\rho_1 h_1)^{-1} [E_1 h_1 u_{xx}^1 + \tau] = -\lambda^{-4} f_2 - i\lambda^{-3} f_1, \quad (4.140)$$

$$\lambda^2 y^1 + (\rho_1 I_1)^{-1} \left[(E_1 I_1 y_x^1 + D(x) z_x^1)_x + \frac{h_1}{2} \tau - G_1 h_1 (\omega_x + y^1) \right] = -\lambda^{-4} f_4 - i\lambda^{-3} f_3, \quad (4.141)$$

$$\lambda^2 \omega + (\rho h)^{-1} \left[-EI \omega_{xxxx} + G_h \omega_{xx} + G_1 h_1 y_x^1 + G_3 h_3 y_x^3 + h_2 \tau_x \right] = -\lambda^{-4} f_6 - i\lambda^{-3} f_5, \quad (4.142)$$

$$\lambda^2 u^3 + (\rho_3 h_3)^{-1} [E_3 h_3 u_{xx}^3 - \tau] = -\lambda^{-\ell} f_8 - i\lambda^{-\ell+1} f_7, \quad (4.143)$$

$$\lambda^2 y^3 + (\rho_3 I_3)^{-1} \left[(E_3 I_3 y_x^3 + D(x) z_x^3)_x + \frac{h_3}{2} \tau - G_3 h_3 (\omega_x + y^3) \right] = -\lambda^{-4} f_{10} - i\lambda^{-3} f_9, \quad (4.144)$$

where $G_h = G_1 h_1 + G_3 h_3$. For clarity, we divide the proof into several lemmas.

Lemma 5.2. *The solution $U \in D(\mathcal{A}_j)$ of system (4.129)-(4.138) satisfies the following asymptotic estimations:*

$$\int_a^b |z_x^1|^2 dx = \frac{o(1)}{\lambda^4}, \quad \int_a^b |z_x^3|^2 dx = \frac{o(1)}{\lambda^4}, \quad \int_a^b |y_x^1|^2 dx = \frac{o(1)}{\lambda^6}, \quad \int_a^b |y_x^3|^2 dx = \frac{o(1)}{\lambda^6} \quad \text{and} \quad \|\tau\| = \frac{O(1)}{\lambda^{\frac{1}{2}}}. \quad (4.145)$$

Proof. Taking the inner product of F with U in \mathcal{H}_j , then using the fact that U is uniformly bounded in \mathcal{H}_j , we get

$$d \int_a^b |z_x^1|^2 dx + d \int_a^b |z_x^3|^2 dx = -\Re(\langle \mathcal{A}U, U \rangle_{\mathcal{H}}) = \Re(\langle (i\lambda I - \mathcal{A})U, U \rangle_{\mathcal{H}}) = o(\lambda^{-4}). \quad (4.146)$$

Then, we obtain the first two estimations in (4.145). Now, using (4.146), (4.131), (4.137) and the facts that $f_3, f_9 \rightarrow 0$ in $H_0^1(0, L)$, we find

$$\int_a^b |y_x^1|^2 dx \leq \frac{2}{\lambda^2} \int_a^b |z_x^1|^2 dx + \frac{2}{\lambda^{10}} \int_a^b |(f_3)_x|^2 dx = \frac{o(1)}{\lambda^6}$$

and

$$\int_a^b |y_x^3|^2 dx \leq \frac{2}{\lambda^2} \int_a^b |z_x^3|^2 dx + \frac{2}{\lambda^{10}} \int_a^b |(f_9)_x|^2 dx = \frac{o(1)}{\lambda^6}.$$

Therefore the third and fourth estimations of (4.145) hold. Now, using the Gagliardo-Nirenberg interpolation inequality and the facts that $\lambda\omega$ and ω_{xx} are uniformly bounded in $L^2(0, L)$, we obtain

$$\|\omega_x\|_{L^2(0,L)} \leq \|\omega_{xx}\|_{L^2(0,L)}^{\frac{1}{2}} \|\omega\|_{L^2(0,L)}^{\frac{1}{2}} + \|\omega\|_{L^2(0,L)} \leq O(\lambda^{-\frac{1}{2}}). \quad (4.147)$$

Because $\lambda u^1, \lambda u^3, \lambda y^1, \lambda y^3$ are uniformly bounded in $L^2(0, L)$ and (4.147) in (4.139), then we get the last equation in (4.145). The proof of Lemma 5.2 has been completed. \square

For all $0 < \epsilon < \frac{b-a}{4}$, we fix the following cut-off function $\eta \in C_0^\infty(0, L)$ such that $0 \leq \eta(x) \leq 1$, for all $x \in (0, L)$ and

$$\eta(x) = \begin{cases} 1 & \text{if } x \in (a + \epsilon, b - \epsilon), \\ 0 & \text{if } x \in (0, L) \setminus (a, b). \end{cases} \quad (4.148)$$

Lemma 5.3. *The solution $U \in D(\mathcal{A}_j)$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimations:*

$$\int_a^b \eta |\lambda y^1|^2 dx = \frac{O(1)}{\lambda^2} \quad \text{and} \quad \int_a^b \eta |\lambda y^3|^2 dx = \frac{O(1)}{\lambda^2}. \quad (4.149)$$

Proof. Multiplying (4.141) by $\overline{\eta y^1}$, using integrating by parts on (a, b) , using the fact that λy^1 is uniformly bounded in $L^2(0, L)$ and the fact that $f_3, f_4 \rightarrow 0$ in $L^2(0, L)$, we arrive at

$$\begin{aligned} & \int_a^b \eta |\lambda y^1|^2 dx - \frac{E_1}{\rho_1} \int_a^b \eta |y_x^1|^2 dx - \frac{E_1}{\rho_1} \int_a^b \eta' y_x^1 \overline{y^1} dx \\ & - \frac{d}{\rho_1 I_1} \int_a^b \eta z_x^1 \overline{y^1} dx - \frac{d}{\rho_1 I_1} \int_a^b \eta' z_x^1 \overline{y^1} dx + \frac{h_1}{2\rho_1 I_1} \int_a^b \eta \tau \overline{y^1} dx \\ & - \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta \omega_x \overline{y^1} dx - \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta |y^1|^2 dx = \frac{o(1)}{\lambda^4}. \end{aligned} \quad (4.150)$$

Using Lemma 5.2, (4.139) and the fact that $\lambda\omega$, λu^1 , λu^3 , λy^1 , λy^3 are uniformly bounded in $L^2(0, L)$ in (4.150), it follows

$$\left\{ \begin{array}{l} -\frac{E_1}{\rho_1} \int_a^b \eta |y_x^1|^2 dx = \frac{o(1)}{\lambda^6}, \\ -\frac{E_1}{\rho_1} \int_a^b \eta' y_x^1 \bar{y}^1 dx = \frac{o(1)}{\lambda^4}, \\ -\frac{d}{\rho_1 I_1} \int_a^b \eta z_x^1 \bar{y}_x^1 dx = \frac{o(1)}{\lambda^5}, \\ -\frac{d}{\rho_1 I_1} \int_a^b \eta' z_x^1 \bar{y}^1 dx = \frac{o(1)}{\lambda^3}, \\ \frac{h_1}{2\rho_1 I_1} \int_a^b \eta \tau \bar{y}^1 dx = \frac{h_1}{2\rho_1 I_1} \int_a^b \eta (-u^1 + u^3 - \frac{h_1}{2} y^1 - \frac{h_3}{2} y^3) \bar{y}^1 dx + \frac{h_1}{2\rho_1 I_1} \int_a^b \eta h_2 \omega_x \bar{y}^1 dx \\ \quad = \frac{O(1)}{\lambda^2} - \frac{h_1 h_2}{2\rho_1 I_1} \int_a^b \eta \omega \bar{y}_x^1 dx - \frac{h_1 h_2}{2\rho_1 I_1} \int_a^b \eta' \omega \bar{y}^1 dx = \frac{O(1)}{\lambda^2} + \frac{o(1)}{\lambda^4}, \\ -\frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta \omega_x \bar{y}^1 dx = \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta \omega \bar{y}_x^1 dx + \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta' \omega \bar{y}^1 dx = \frac{o(1)}{\lambda^4} + \frac{O(1)}{\lambda^2}, \\ -\frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta |y^1|^2 dx = \frac{O(1)}{\lambda^2}. \end{array} \right. \quad (4.151)$$

Inserting (4.151) in (4.150) leads to the first estimation in (4.149). Similarly, multiplying (4.144) by $\eta \bar{y}^3$ and using the same arguments, we observe that the second estimation in (4.149) holds. The proof of Lemma 5.3 has been completed. \square

Lemma 5.4. *The solution $U \in D(\mathcal{A}_j)$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimation:*

$$\int_a^b \eta |\omega_x|^2 dx = \frac{O(1)}{\lambda}. \quad (4.152)$$

Proof. Multiplying (4.141) by $h_3 \rho_1 I_1 \eta \bar{\omega}_x$, using integrating by parts on (a, b) and the fact that $\|\omega_x\|_{L^2(0,L)} = O(\lambda^{-\frac{1}{2}})$, we get

$$\begin{aligned} & h_3 \rho_1 I_1 \lambda^2 \int_a^b \eta y^1 \bar{\omega}_x dx - E_1 I_1 h_3 \int_a^b \eta y_x^1 \bar{\omega}_{xx} dx - E_1 I_1 h_3 \int_a^b \eta' y_x^1 \bar{\omega}_x dx \\ & - h_3 d \int_a^b \eta z_x^1 \bar{\omega}_{xx} dx - h_3 d \int_a^b \eta' z_x^1 \bar{\omega}_x dx - G_1 h_1 h_3 \int_a^b \eta |\omega_x|^2 dx + \frac{h_1 h_3}{2} \int_a^b \eta \tau \bar{\omega}_x dx \\ & - G_1 h_1 h_3 I_1 \int_a^b \eta y^1 \bar{\omega}_x dx = -h_3 \rho_1 I_1 \lambda^{-4} \int_a^b \eta f_4 \bar{\omega}_x dx - h_3 \rho_1 I_1 \lambda^{-3} \int_a^b \eta f_3 \bar{\omega}_x dx. \end{aligned} \quad (4.153)$$

Using Lemma 5.2, Lemma 5.3, $\|\omega_x\|_{L^2(0,L)} = O(\lambda^{-\frac{1}{2}})$, the fact that ω_{xx} and $\lambda\omega$ are uniformly

bounded in $L^2(0, L)$ and $f_3, f_4, (f_3)_x \rightarrow 0$ in $L^2(0, L)$, we observe that (4.153) implies that

$$\left\{ \begin{array}{l} \left| \lambda^2 \int_a^b \eta y^1 \overline{\omega_x} dx \right| = \left| -\lambda^2 \int_a^b \eta' y^1 \overline{\omega} dx - \lambda^2 \int_a^b \eta y_x^1 \overline{\omega} dx \right| = \frac{O(1)}{\lambda} + \frac{o(1)}{\lambda^2}, \\ \left| \int_a^b \eta y_x^1 \overline{\omega_{xx}} dx \right| = \frac{o(1)}{\lambda^3}, \\ \left| \int_a^b \eta' y_x^1 \overline{\omega_x} dx \right| = \frac{o(1)}{\lambda^{\frac{7}{2}}}, \\ \left| \int_a^b \eta z_x^1 \overline{\omega_{xx}} dx \right| = \frac{o(1)}{\lambda^2}, \\ \left| \int_a^b \eta' z_x^1 \overline{\omega_x} dx \right| = \frac{o(1)}{\lambda^{\frac{3}{2}}}, \\ \left| \int_a^b \eta y^1 \overline{\omega_x} dx \right| = \left| -\int_a^b \eta y_x^1 \overline{\omega} dx - \int_a^b \eta' y^1 \overline{\omega} dx \right| = \frac{o(1)}{\lambda^4} + \frac{O(1)}{\lambda^3}, \\ \lambda^{-4} \int_a^b \eta f_4 \overline{\omega_x} dx = \frac{o(1)}{\lambda^{\frac{9}{2}}}, \\ -\lambda^{-3} \int_a^b \eta f_3 \overline{\omega_x} dx = \lambda^{-3} \int_a^b \eta (f_3)_x \overline{\omega} dx - \lambda^{-3} \int_a^b \eta' f_3 \overline{\omega} dx = \frac{o(1)}{\lambda^4}. \end{array} \right. \quad (4.154)$$

Inserting (4.154) in (4.153), we have

$$-G_1 h_1 h_3 \int_a^b \eta |\omega_x|^2 dx + \frac{h_1 h_3}{2} \int_a^b \eta \tau \overline{\omega_x} dx = \frac{O(1)}{\lambda}. \quad (4.155)$$

Similarly, multiplying (4.144) by $h_1 \rho_3 I_3 \eta \overline{\omega_x}$, and using the same technique, we obtain

$$-G_3 h_3 h_1 \int_a^b \eta |\omega_x|^2 dx + \frac{h_1 h_3}{2} \int_a^b \eta \tau \overline{\omega_x} dx = \frac{O(1)}{\lambda}. \quad (4.156)$$

Now, subtracting (4.155) and (4.156), we find

$$(G_3 - G_1) h_1 h_3 \int_a^b \eta |\omega_x|^2 dx = \frac{O(1)}{\lambda}.$$

Using the fact that $G_1 \neq G_3$ in the above estimation, we get

$$\int_a^b \eta |\omega_x|^2 dx = \frac{O(1)}{\lambda}. \quad (4.157)$$

The proof of Lemma 5.4 has been completed. \square

Lemma 5.5. *The solution $U \in D(\mathcal{A}_j)$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimation:*

$$\int_a^b \eta |\omega_{xx}|^2 dx = \frac{O(1)}{\lambda}. \quad (4.158)$$

Proof. First, using the Gagliardo-Nirenberg interpolation inequality and the facts that $\frac{\omega_{xxxx}}{\lambda}$ and ω_{xx} are uniformly bounded in $L^2(0, L)$, we get

$$\|\omega_{xxx}\|_{L^2(0,L)} \leq \|\omega_{xxxx}\|_{L^2(0,L)}^{\frac{1}{2}} \|\omega_{xx}\|_{L^2(0,L)}^{\frac{1}{2}} + \|\omega_{xx}\|_{L^2(0,L)} \leq O(\lambda^{\frac{1}{2}}). \quad (4.159)$$

Now, multiplying (4.141) by $h_3\rho_1 I_1 \eta \overline{\omega_{xxx}}$, using integrating by parts on (a, b) and the fact that $f_3, f_4 \rightarrow 0$ in $L^2(0, L)$ and $\|\omega_{xxx}\|_{L^2(0,L)} = O(\lambda^{\frac{1}{2}})$, then taking the real part, we find

$$\begin{aligned}
 & \Re \left\{ h_3 \rho_1 I_1 \lambda^2 \int_a^b \eta y^1 \overline{\omega_{xxx}} dx \right\} - \Re \left\{ E_1 I_1 h_3 \int_a^b \eta y_x^1 \overline{\omega_{xxxx}} dx \right\} \\
 & - \Re \left\{ E_1 I_1 h_3 \int_a^b \eta' y_x^1 \overline{\omega_{xxx}} dx \right\} - \Re \left\{ h_3 d \int_a^b \eta z_x^1 \overline{\omega_{xxxx}} dx \right\} \\
 & - \Re \left\{ h_3 d \int_a^b \eta' z_x^1 \overline{\omega_{xxx}} dx \right\} + \Re \left\{ G_1 h_1 h_3 \int_a^b \eta |\omega_{xx}|^2 dx \right\} \\
 & + \Re \left\{ G_1 h_1 h_3 \int_a^b \eta' \omega_x \overline{\omega_{xx}} dx \right\} + \Re \left\{ \frac{h_1 h_3}{2} \int_a^b \eta \tau \overline{\omega_{xxx}} dx \right\} \\
 & - \Re \left\{ G_1 h_1 h_3 I_1 \int_a^b \eta y^1 \overline{\omega_{xxx}} dx \right\} = \frac{o(1)}{\lambda^{\frac{5}{2}}}.
 \end{aligned} \tag{4.160}$$

Using Lemma 5.2, Lemma 5.3, $\|\omega_{xxx}\|_{L^2(0,L)} = O(\lambda^{\frac{1}{2}})$, $\|\omega_{xxxx}\|_{L^2(0,L)} = O(\lambda)$ and ω_{xx} is uniformly bounded in $L^2(0, L)$ in (4.160), we obtain

$$\left\{ \begin{aligned}
 & \Re \left\{ \lambda^2 \int_a^b \eta y^1 \overline{\omega_{xxx}} dx \right\} = \Re \left\{ -\lambda^2 \int_a^b \eta y_x^1 \overline{\omega_{xx}} dx - \lambda^2 \int_a^b \eta' y^1 \overline{\omega_{xx}} dx \right\} \\
 & = \frac{o(1)}{\lambda} + \Re \left\{ \lambda^2 \int_a^b \eta' y_x^1 \overline{\omega_{xx}} dx + \lambda^2 \int_a^b \eta'' y^1 \overline{\omega_{xx}} dx \right\} \\
 & = \frac{o(1)}{\lambda} + \frac{o(1)}{\lambda^{\frac{3}{2}}} + \Re \left\{ -\lambda^2 \int_a^b \eta'' y_x^1 \overline{\omega_{xx}} dx - \lambda^2 \int_a^b \eta''' y^1 \overline{\omega_{xx}} dx \right\} \\
 & = \frac{o(1)}{\lambda} + \frac{o(1)}{\lambda^{\frac{3}{2}}} + \frac{o(1)}{\lambda^2} + \frac{O(1)}{\lambda}, \\
 & \Re \left\{ \int_a^b \eta y_x^1 \overline{\omega_{xxxx}} dx \right\} = \frac{o(1)}{\lambda^2}, \\
 & \Re \left\{ \int_a^b \eta' y_x^1 \overline{\omega_{xxx}} dx \right\} = \frac{o(1)}{\lambda^{\frac{5}{2}}}, \\
 & \Re \left\{ \int_a^b \eta z_x^1 \overline{\omega_{xxxx}} dx \right\} = \frac{o(1)}{\lambda}, \\
 & \Re \left\{ \int_a^b \eta' z_x^1 \overline{\omega_{xxx}} dx \right\} = \frac{o(1)}{\lambda^{\frac{3}{2}}}, \\
 & \Re \left\{ \int_a^b \eta' \omega_x \overline{\omega_{xx}} dx \right\} = \frac{1}{2} \int_a^b \eta'' |\omega_x|^2 = \frac{O(1)}{\lambda}, \\
 & \Re \left\{ \int_a^b \eta y^1 \overline{\omega_{xxx}} dx \right\} = \Re \left\{ - \int_a^b \eta y_x^1 \overline{\omega_{xx}} dx - \int_a^b \eta' y^1 \overline{\omega_{xx}} dx \right\} = \frac{o(1)}{\lambda^3} + \frac{O(1)}{\lambda^2}.
 \end{aligned} \right. \tag{4.161}$$

Inserting (4.161) in (4.160), we get

$$G_1 h_1 h_3 \int_a^b \eta |\omega_{xx}|^2 dx + \Re \left\{ \frac{h_1 h_3}{2} \int_a^b \eta \tau \overline{\omega_{xxx}} dx \right\} = \frac{O(1)}{\lambda}. \tag{4.162}$$

Similarly, multiplying (4.144) by $h_1 \rho_3 I_3 \eta \overline{\omega_{xxx}}$, and using the same technique, we obtain

$$G_1 h_1 h_3 \int_a^b \eta |\omega_{xx}|^2 dx + \Re \left\{ \frac{h_1 h_3}{2} \int_a^b \eta \tau \overline{\omega_{xxx}} dx \right\} = \frac{O(1)}{\lambda}. \tag{4.163}$$

Now, subtracting (4.162) and (4.163), leads to

$$(G_3 - G_1) h_1 h_3 \int_a^b \eta |\omega_{xx}|^2 dx = \frac{O(1)}{\lambda}.$$

Using the fact that $G_1 \neq G_3$ in the above estimation, we get

$$\int_a^b \eta |\omega_{xx}|^2 dx = \frac{O(1)}{\lambda}. \quad (4.164)$$

The proof of Lemma 5.5 has been completed. \square

Lemma 5.6. *The solution $U \in D(\mathcal{A}_j)$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimation:*

$$\int_a^b \eta |\lambda \omega|^2 dx = \frac{O(1)}{\lambda}. \quad (4.165)$$

Proof. Multiplying (4.142) by $\eta \bar{\omega}$, using integrating by parts on (a, b) and the fact that $f_6, f_5 \rightarrow 0$ in $L^2(0, L)$ and $\lambda \omega$ is uniformly bounded in $L^2(0, L)$, then taking the real part, we find

$$\begin{aligned} & \int_a^b \eta |\lambda \omega|^2 dx - \frac{EI}{\rho h} \int_a^b \eta |\omega_{xx}|^2 dx - \Re \left\{ 2 \frac{EI}{\rho h} \int_a^b \eta' \omega_{xx} \bar{\omega}_x dx \right\} \\ & - \Re \left\{ \frac{EI}{\rho h} \int_a^b \eta'' \omega_{xx} \bar{\omega} dx \right\} + \Re \left\{ \frac{G_h}{\rho h} \int_a^b \eta \omega_{xx} \bar{\omega} dx \right\} + \Re \left\{ \frac{G_1 h_1}{\rho h} \int_a^b \eta y_x^1 \bar{\omega} dx \right\} \\ & + \Re \left\{ \frac{G_3 h_3}{\rho h} \int_a^b \eta y_x^3 \bar{\omega} dx \right\} + \Re \left\{ \frac{h_2}{\rho h} \int_a^b \eta \tau_x \bar{\omega} dx \right\} = \frac{o(1)}{\lambda^4}. \end{aligned} \quad (4.166)$$

Using Lemma 5.5, Lemma 5.4, Lemma 5.3, Lemma 5.2 and the fact that $\lambda \omega$ is uniformly bounded in $L^2(0, L)$, we get

$$\left\{ \begin{array}{l} \int_a^b \eta |\omega_{xx}|^2 dx = \frac{O(1)}{\lambda}, \\ \Re \left\{ \int_a^b \eta' \omega_{xx} \bar{\omega}_x dx \right\} = \frac{1}{2} \int_a^b \eta'' |\omega_x|^2 dx = \frac{O(1)}{\lambda}, \\ \Re \left\{ \int_a^b \eta'' \omega_{xx} \bar{\omega} dx \right\} = - \int_a^b \eta'' |\omega_x|^2 dx - \Re \left\{ \int_a^b \eta''' \omega_x \bar{\omega} dx \right\} \\ \quad = \frac{O(1)}{\lambda} + \frac{1}{2} \int_a^b \eta'''' |\omega|^2 dx = \frac{O(1)}{\lambda} + \frac{O(1)}{\lambda^2}, \\ \Re \left\{ \int_a^b \eta \omega_{xx} \bar{\omega} dx \right\} = - \int_a^b \eta |\omega_x|^2 dx - \Re \left\{ \int_a^b \eta' \omega_x \bar{\omega} dx \right\} \\ \quad = \frac{O(1)}{\lambda} + \frac{1}{2} \int_a^b \eta'' |\omega|^2 dx = \frac{O(1)}{\lambda} + \frac{O(1)}{\lambda^2}, \\ \Re \left\{ \int_a^b \eta y_x^1 \bar{\omega} dx \right\} = \frac{o(1)}{\lambda^4}, \\ \Re \left\{ \int_a^b \eta y_x^3 \bar{\omega} dx \right\} = \frac{o(1)}{\lambda^4}, \\ \Re \left\{ \int_a^b \eta \tau_x \bar{\omega} dx \right\} = \Re \left\{ - \int_a^b \eta \tau \bar{\omega}_x dx - \int_a^b \eta' \tau \bar{\omega} dx \right\} \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \Re \left\{ \int_a^b \eta \tau \overline{\omega_x} dx \right\} = \Re \left\{ \int_a^b \eta (-u^1 + u^3) \overline{\omega_x} dx \right\} + \Re \left\{ \int_a^b \eta \left(-\frac{h_1}{2} y^1 - \frac{h_3}{2} y^3 \right) \overline{\omega_x} dx \right\} + h_2 \int_a^b \eta |\omega_x|^2 dx \\ \quad = \frac{O(1)}{\lambda^{\frac{3}{2}}} + \Re \left\{ \int_a^b \left[\eta \left(\frac{h_1}{2} y_x^1 + \frac{h_3}{2} y_x^3 \right) \overline{\omega} + \eta' \left(\frac{h_1}{2} y^1 + \frac{h_3}{2} y^3 \right) \overline{\omega} \right] dx \right\} + \frac{O(1)}{\lambda} \\ \quad = \frac{O(1)}{\lambda}, \\ \Re \left\{ \int_a^b \eta' \tau \overline{\omega} dx \right\} = \Re \left\{ \int_a^b \eta' (-u^1 + u^3) \overline{\omega} dx \right\} + \Re \left\{ \int_a^b \eta' \left(-\frac{h_1}{2} y^1 - \frac{h_3}{2} y^3 \right) \overline{\omega} dx \right\} + \frac{h_2}{2} \int_a^b \eta'' |\omega|^2 dx \\ \quad = \frac{O(1)}{\lambda^2} + \frac{O(1)}{\lambda^3} + \frac{o(1)}{\lambda^{\frac{3}{2}}}. \end{array} \right.$$

Inserting the above estimations in (4.166), we obtain

$$\int_a^b \eta |\lambda \omega|^2 dx = \frac{O(1)}{\lambda}. \quad (4.167)$$

Then Lemma 5.6 holds. \square

Lemma 5.7. *The solution $U \in D(\mathcal{A})$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimation:*

$$\int_a^b \eta |u_x^1|^2 dx = o(1) \quad \text{and} \quad \int_a^b \eta |u_x^3|^2 dx = o(1). \quad (4.168)$$

Proof. Multiplying (4.140) by $\overline{\eta(E_1 I_1 y_x^1 + D(x) z_x^1)_x}$, using integrating by parts on (a, b) , the fact that $f_1, f_2 \rightarrow 0$ in $L^2(0, L)$ and $\lambda^{-1}(E_1 I_1 y_x^1 + D(x) z_x^1)_x$ is uniformly bounded in $L^2(0, L)$, we get

$$\begin{aligned} & -\lambda^2 \int_a^b \eta u_x^1 \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx - \lambda^2 \int_a^b \eta' u^1 \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx \\ & + \frac{E_1}{\rho_1} \int_a^b \eta u_{xx}^1 \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx - \frac{1}{\rho_1 h_1} \int_a^b \eta \tau_x \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx \\ & - \frac{1}{\rho_1 h_1} \int_a^b \eta' \tau \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx = \frac{o(1)}{\lambda^3}. \end{aligned} \quad (4.169)$$

Using the fact that λu^1 , u_x^1 and τ_x are uniformly bounded in $L^2(0, L)$ and Lemma 5.2, it appears that

$$\left\{ \begin{array}{l} \left| -\lambda^2 \int_a^b \eta u_x^1 \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx \right| = o(1), \\ \left| -\lambda^2 \int_a^b \eta' u^1 \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx \right| = \frac{o(1)}{\lambda}, \\ \left| -\frac{1}{\rho_1 h_1} \int_a^b \eta \tau_x \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx \right| = \frac{o(1)}{\lambda^2}, \\ \left| -\frac{1}{\rho_1 h_1} \int_a^b \eta' \tau \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx \right| = \frac{o(1)}{\lambda^{\frac{5}{2}}}. \end{array} \right. \quad (4.170)$$

Inserting (4.170) in (4.169), we get

$$\frac{E_1}{\rho_1} \int_a^b \eta u_{xx}^1 \overline{(E_1 I_1 y_x^1 + D(x) z_x^1)_x} dx = o(1). \quad (4.171)$$

Now, multiplying (4.141) by $\eta \overline{u_{xx}^1}$, using integrating by parts on (a, b) and the fact that $f_3, f_4, (f_3)_x \rightarrow 0$ in $L^2(0, L)$ and $\lambda^{-1} u_{xx}^1$ is uniformly bounded in $L^2(0, L)$, we obtain

$$\begin{aligned} & -\lambda^2 \int_a^b \eta y_x^1 \overline{u_x^1} dx - \lambda^2 \int_a^b \eta' y^1 \overline{u_x^1} dx + \frac{E_1}{\rho_1} \int_a^b \eta (E_1 I_1 y_x^1 + D(x) z_x^1)_x \overline{u_{xx}^1} dx \\ & - \frac{h_1}{2\rho_1 I_1} \int_a^b \eta \tau_x \overline{u_x^1} dx - \frac{h_1}{2\rho_1 I_1} \int_a^b \eta' \tau \overline{u_x^1} dx \\ & - \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta \omega_x \overline{u_{xx}^1} dx - \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta y^1 \overline{u_{xx}^1} dx = \frac{o(1)}{\lambda^3}. \end{aligned} \quad (4.172)$$

Using Lemma 5.2, Lemma 5.3, Lemma 5.4, Lemma 5.5 and the fact that $u_x^1, \lambda^{-1} u_{xx}^1$ are uniformly bounded in $L^2(0, L)$, we get

$$\left\{ \begin{array}{l} \left| -\lambda^2 \int_a^b \eta y_x^1 \overline{u_x^1} dx \right| = \frac{o(1)}{\lambda}, \\ \left| -\lambda^2 \int_a^b \eta' y^1 \overline{u_x^1} dx \right| = \left| \lambda^2 \int_a^b \eta' y_x^1 \overline{u_x^1} dx + \lambda^2 \int_a^b \eta'' y^1 \overline{u_x^1} dx \right| = \frac{o(1)}{\lambda^2} + \frac{O(1)}{\lambda}, \\ \left| -\frac{h_1}{2\rho_1 I_1} \int_a^b \eta' \tau \overline{u_x^1} dx \right| = \frac{O(1)}{\lambda^{\frac{1}{2}}}, \\ \left| -\frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta \omega_x \overline{u_{xx}^1} dx \right| = \left| \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta \omega_{xx} \overline{u_x^1} dx + \frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta' \omega_x \overline{u_x^1} dx \right| = \frac{O(1)}{\lambda^{\frac{1}{2}}} + \frac{o(1)}{\lambda^{\frac{1}{2}}}, \\ \left| -\frac{G_1 h_1}{\rho_1 I_1} \int_a^b \eta y^1 \overline{u_{xx}^1} dx \right| = \frac{O(1)}{\lambda}. \end{array} \right. \quad (4.173)$$

Inserting (4.173) in (4.172), we get

$$\frac{E_1}{\rho_1} \int_a^b \eta (E_1 I_1 y_x^1 + D(x) z_x^1)_x \overline{u_{xx}^1} dx - \frac{h_1}{2\rho_1 I_1} \int_a^b \eta \tau_x \overline{u_x^1} dx = \frac{O(1)}{\lambda^{\frac{1}{2}}}. \quad (4.174)$$

Using (4.171) and taking the real part we get

$$\Re \left(\int_a^b \eta \tau_x \overline{u_x^1} dx \right) = o(1). \quad (4.175)$$

Using the definition of τ in (4.175), we get

$$\begin{aligned} & - \int_a^b \eta |u_x^1|^2 dx + \Re \left(\int_a^b \eta u_x^3 \overline{u_x^1} dx \right) + \Re \left(h_2 \int_a^b \eta \omega_{xx} \overline{u_x^1} dx \right) - \Re \left(\frac{h_1}{2} \int_a^b \eta y_x^1 \overline{u_x^1} dx \right) \\ & - \Re \left(\frac{h_3}{2} \int_a^b \eta y_x^3 \overline{u_x^1} dx \right) = o(1). \end{aligned} \quad (4.176)$$

Now, using Lemma 5.2, Lemma 5.5 and using the fact that u_x^1 is uniformly bounded in $L^2(0, L)$, we get

$$- \int_a^b \eta |u_x^1|^2 dx + \Re \left(\int_a^b \eta u_x^3 \overline{u_x^1} dx \right) = o(1). \quad (4.177)$$

Similarly, multiplying (4.143) by $\eta \overline{(E_3 I_3 y_x^3 + D(x) z_x^3)_x}$ and (4.144) by $\eta \overline{u_{xx}^3}$ and using the same technique, we get

$$\int_a^b \eta |u_x^3|^2 dx - \Re \left(\int_a^b \eta u_x^1 \overline{u_{xx}^3} dx \right) = o(1). \quad (4.178)$$

Multiplying (4.140) by $\eta \bar{u}^3$, using integrating by parts on (a, b) , the fact that $f_1, f_2 \rightarrow 0$ in $L^2(0, L)$ and using the fact that λu^3 is uniformly bounded in $L^2(0, L)$, we get

$$\lambda^2 \int_a^b \eta u^1 \bar{u}^3 dx - \frac{E_1}{\rho_1} \int_a^b \eta u_x^1 \bar{u}_x^3 dx - \frac{E_1}{\rho_1} \int_a^b \eta' u_x^1 \bar{u}^3 dx + \frac{1}{\rho_1 h_1} \int_a^b \eta \tau \bar{u}^3 dx = \frac{o(1)}{\lambda^4}. \quad (4.179)$$

Now, using Lemma 5.2 and the fact that $\lambda u^3, u_x^1$ are uniformly bounded in $L^2(0, L)$, we get

$$\left\{ \begin{array}{l} \left| -\frac{E_1}{\rho_1} \int_a^b \eta' u_x^1 \bar{u}^3 dx \right| = \frac{O(1)}{\lambda}, \\ \left| \frac{1}{\rho_1 h_1} \int_a^b \eta \tau \bar{u}^3 dx \right| = \frac{O(1)}{\lambda^{\frac{3}{2}}}. \end{array} \right. \quad (4.180)$$

Inserting (4.180) in (4.179), we get

$$\lambda^2 \int_a^b \eta u^1 \bar{u}^3 dx - \frac{E_1}{\rho_1} \int_a^b \eta u_x^1 \bar{u}_x^3 dx = \frac{O(1)}{\lambda}. \quad (4.181)$$

Multiplying (4.143) by $\eta \bar{u}^1$, using integrating by parts over (a, b) and the fact that $f_7, f_8 \rightarrow 0$ in $L^2(0, L)$ and λu^1 is uniformly bounded in $L^2(0, L)$, we get

$$\lambda^2 \int_a^b \eta u^3 \bar{u}^1 dx - \frac{E_3}{\rho_3} \int_a^b \eta u_x^3 \bar{u}_x^1 dx - \frac{E_3}{\rho_3} \int_a^b \eta' u_x^3 \bar{u}^1 dx + \frac{1}{\rho_3 h_3} \int_a^b \eta \tau \bar{u}^1 dx = \frac{o(1)}{\lambda^4}. \quad (4.182)$$

Now, using Lemma 5.2 and the fact that $\lambda u^1, u_x^3$ are uniformly bounded in $L^2(0, L)$, we have

$$\left\{ \begin{array}{l} \left| -\frac{E_3}{\rho_3} \int_a^b \eta' u_x^3 \bar{u}^1 dx \right| = \frac{O(1)}{\lambda}, \\ \left| \frac{1}{\rho_3 h_3} \int_a^b \eta \tau \bar{u}^1 dx \right| = \frac{O(1)}{\lambda^{\frac{3}{2}}}. \end{array} \right. \quad (4.183)$$

Inserting (4.183) in (4.182), we get

$$\lambda^2 \int_a^b \eta u^3 \bar{u}^1 dx - \frac{E_3}{\rho_3} \int_a^b \eta u_x^3 \bar{u}_x^1 dx = \frac{O(1)}{\lambda}. \quad (4.184)$$

Subtracting (4.181) and (4.184) and taking the real part, we obtain

$$\left(\frac{E_1}{\rho_1} - \frac{E_3}{\rho_3} \right) \Re \left(\int_a^b \eta u_x^1 \bar{u}_x^3 dx \right) = \frac{O(1)}{\lambda}.$$

Using the fact that $\frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}$ in the above estimation, leads to

$$\Re \left(\int_a^b \eta u_x^1 \bar{u}_x^3 dx \right) = \frac{O(1)}{\lambda}. \quad (4.185)$$

Inserting (4.185) in (4.177) and (4.178), we get

$$\int_a^b \eta |u_x^1|^2 dx = o(1) \quad (4.186)$$

and

$$\int_a^b \eta |u_x^3|^2 dx = o(1). \quad (4.187)$$

This follows that Lemma 5.7 is satisfied. \square

Lemma 5.8. *The solution $U \in D(\mathcal{A})$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimations:*

$$\int_a^b \eta |\lambda u^1|^2 dx = o(1) \quad \text{and} \quad \int_a^b \eta |\lambda u^3|^2 dx = o(1). \quad (4.188)$$

Proof. Multiplying (4.140) and (4.143), respectively, by $\eta \bar{u}^1$ and $\eta \bar{u}^3$, using integrating by parts over (a, b) and the fact that $f_1, f_2, f_7, f_8 \rightarrow 0$ in $L^2(0, L)$ and λu^1 and λu^3 are uniformly bounded in $L^2(0, L)$, we arrive to

$$\int_a^b \eta |\lambda u^1|^2 dx - \frac{E_1}{\rho_1} \int_a^b \eta |u_x^1|^2 dx - \frac{E_1}{\rho_1} \int_a^b \eta' u_x^1 \bar{u}^1 dx + \frac{1}{\rho_1 h_1} \int_a^b \eta \tau \bar{u}^1 dx = o(\lambda^{-4}) \quad (4.189)$$

and

$$\int_a^b \eta |\lambda u^3|^2 dx - \frac{E_3}{\rho_3} \int_a^b \eta |u_x^3|^2 dx - \frac{E_3}{\rho_3} \int_a^b \eta' u_x^3 \bar{u}^3 dx + \frac{1}{\rho_3 h_3} \int_a^b \eta \tau \bar{u}^3 dx = o(\lambda^{-4}). \quad (4.190)$$

Using (4.168) and Lemma 5.2, the desired result (4.188) holds. The proof of Lemma 5.8 has been completed. \square

Lemma 5.9. *Let $P \in C^3([0, L])$ be a function such that $P(0) = P(L) = 0$. The solution $U \in D(\mathcal{A})$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimation:*

$$\begin{aligned} & - \int_0^L P' \left(\rho_1 h_1 |\lambda u^1|^2 + E_1 h_1 |u_x^1|^2 + \rho_3 h_3 |\lambda u^3|^2 + E_3 h_3 |u_x^3|^2 \right) dx \\ & - \int_0^L P' \left(\rho_1 I_1 |\lambda y^1|^2 + E_1 I_1 |y_x^1|^2 + \rho_3 I_3 |\lambda y^3|^2 + E_3 I_3 |y_x^3|^2 \right) dx \\ & - \int_0^L P' \left(\rho h |\lambda \omega|^2 + 3EI |\omega_{xx}|^2 + G_h |\omega_x|^2 \right) dx = o(1). \end{aligned} \quad (4.191)$$

Proof. First, multiplying (4.140) and (4.143) by $2P(\rho_1 h_1) \bar{u}_x^1$ and $2P(\rho_3 h_3) \bar{u}_x^3$, respectively, integrating over $(0, L)$, taking the real part and using the fact that u_x^1, u_x^3 are uniformly bounded in $L^2(0, L)$ and $f_1, f_2, f_7, f_8 \rightarrow 0$ in $L^2(0, L)$, we get

$$\rho_1 h_1 \int_0^L P \left(|\lambda u^1|^2 \right)_x dx + E_1 h_1 \int_0^L P \left(|u_x^1|^2 \right)_x dx + \Re \left(2 \int_0^L P \tau \bar{u}_x^1 dx \right) = o(\lambda^{-3}) \quad (4.192)$$

and

$$\rho_3 h_3 \int_0^L P \left(|\lambda u^3|^2 \right)_x dx + E_3 h_3 \int_0^L P \left(|u_x^3|^2 \right)_x dx - \Re \left(2 \int_0^L P \tau \bar{u}_x^3 dx \right) = o(\lambda^{-3}). \quad (4.193)$$

Let $S^1 := E_1 I_1 y_x^1 + D(x) z_x^1$ and $S^2 := E_3 I_3 y_x^3 + D(x) z_x^3$. From Lemma 5.2, the definition of $D(x)$ and the fact that $y^1(x)$ and $y^3(x)$ are uniformly bounded in $L^2(0, L)$, we see that S_1 and S_2 are uniformly bounded in $L^2(0, L)$. Now, multiplying (4.141) and (4.144) by $2P(\rho_1 I_1)(E_1 I_1)^{-1} \bar{S}^1$ and $2P(\rho_3 I_3)(E_1 I_1)^{-1} \bar{S}^2$, respectively, integrating over $(0, L)$, taking the real part, then using the fact that $f_3, f_4, f_9, f_{10} \rightarrow 0$ in $L^2(0, L)$, we get

$$\begin{aligned} & \Re \left(\frac{2\rho_1 I_1 \lambda^2}{E_1 I_1} \int_0^L P y^1 \bar{S}^1 dx \right) + (E_1 I_1)^{-1} \int_0^L P \left(|S^1|^2 \right)_x dx + \Re \left(\frac{h_1}{E_1 I_1} \int_0^L P \tau \bar{S}^1 dx \right) \\ & - \Re \left(\frac{2G_1 h_1}{E_1 I_1} \int_0^L P (\omega_x + y^1) \bar{S}^1 dx \right) = o(\lambda^{-3}) \end{aligned} \quad (4.194)$$

and

$$\begin{aligned} & \Re \left(\frac{2\rho_3 I_3 \lambda^2}{E_3 I_3} \int_0^L P y^3 \overline{S^2} dx \right) + E_3 I_3 \int_0^L P \left(|S^2|^2 \right)_x dx + \Re \left(h_3 \int_0^L P \tau \overline{S^2} dx \right) \\ & - \Re \left(2G_3 h_3 \int_0^L P (\omega_x + y^3) \overline{S^2} dx \right) = o(\lambda^{-3}). \end{aligned} \quad (4.195)$$

From the definition of S^1, S^2 and $D(x)$, Lemma 5.2 and the fact that λy^1 and λy^3 are uniformly bounded in $L^2(0, L)$, we observe that

$$\left\{ \begin{aligned} & \Re \left(\frac{2\rho_1 I_1 \lambda^2}{E_1 I_1} \int_0^L P y^1 \overline{S^1} dx \right) = \rho_1 I_1 \int_0^L P \left(|\lambda y^1|^2 \right)_x dx + \Re \left(\frac{2\rho_1 I_1 \lambda^2}{E_1 I_1} d \int_a^b P y^1 \overline{z_x^1} dx \right) \\ & = \rho_1 I_1 \int_0^L P \left(|\lambda y^1|^2 \right)_x dx + \frac{o(1)}{\lambda}, \\ & \Re \left(\frac{2\rho_3 I_3 \lambda^2}{E_3 I_3} \int_0^L P y^3 \overline{S^2} dx \right) = \rho_3 I_3 \int_0^L P \left(|\lambda y^3|^2 \right)_x dx + \Re \left(\frac{2\rho_3 I_3 \lambda^2}{E_3 I_3} d \int_a^b P y^3 \overline{z_x^3} dx \right) \\ & = \rho_3 I_3 \int_0^L P \left(|\lambda y^3|^2 \right)_x dx + \frac{o(1)}{\lambda}. \end{aligned} \right. \quad (4.196)$$

Inserting (4.196) in (4.194) and (4.195), we find

$$\begin{aligned} & \rho_1 I_1 \int_0^L P \left(|\lambda y^1|^2 \right)_x dx + (E_1 I_1)^{-1} \int_0^L P \left(|S^1|^2 \right)_x dx + \Re \left(\frac{h_1}{E_1 I_1} \int_0^L P \tau \overline{S^1} dx \right) \\ & - \Re \left(\frac{2G_1 h_1}{E_1 I_1} \int_0^L P (\omega_x + y^1) \overline{S^1} dx \right) = \frac{o(1)}{\lambda} \end{aligned} \quad (4.197)$$

and

$$\begin{aligned} & \rho_3 I_3 \int_0^L P \left(|\lambda y^3|^2 \right)_x dx + (E_3 I_3)^{-1} \int_0^L P \left(|S^2|^2 \right)_x dx + \Re \left(\frac{h_3}{E_3 I_3} \int_0^L P \tau \overline{S^2} dx \right) \\ & - \Re \left(\frac{2G_3 h_3}{E_3 I_3} \int_0^L P (\omega_x + y^3) \overline{S^2} dx \right) = \frac{o(1)}{\lambda}. \end{aligned} \quad (4.198)$$

Now, multiplying (4.142) by $2P(\rho h)\overline{\omega_x}$, integrating over $(0, L)$, taking the real part, then using integration by parts and the fact that ω_x is uniformly bounded in $L^2(0, L)$ and the fact that $f_4, f_5 \rightarrow 0$ in $L^2(0, L)$, we find

$$\begin{aligned} & \rho h \int_0^L P \left(|\lambda \omega|^2 \right)_x dx + 3EI \int_0^L P \left(|\omega_{xx}|^2 \right)_x dx - \Re \left(2EI \int_0^L P'' \omega_{xx} \overline{\omega_x} dx \right) \\ & + G_h \int_0^L P \left(|\omega_x|^2 \right)_x dx + \Re \left(2G_1 h_1 \int_0^L P y_x^1 \overline{\omega_x} dx \right) + \Re \left(2G_3 h_3 \int_0^L P y_x^3 \overline{\omega_x} dx \right) \\ & - \Re \left(2h_2 \int_0^L P \tau \overline{\omega_{xx}} dx \right) - \Re \left(2h_2 \int_0^L P' \tau \overline{\omega_x} dx \right) = o(\lambda^{-3}). \end{aligned} \quad (4.199)$$

Adding (4.192), (4.193), (4.197), (4.198) and (4.199), using the definition of τ and $D(x)$, Lemma 5.2, $\|\omega_x\|_{L^2(0,L)} = O(\lambda^{-\frac{1}{2}})$, $\|\tau\|_{L^2(0,L)} = O(\lambda^{-\frac{1}{2}})$ and the fact that $\lambda y^1, \lambda y^3, y_x^3, y_x^1$ and ω_{xx} are

uniformly bounded in $L^2(0, L)$, we obtain

$$\begin{aligned}
& \rho_1 h_1 \int_0^L P \left(|\lambda u^1|^2 \right)_x dx + E_1 h_1 \int_0^L P \left(|u_x^1|^2 \right)_x dx + \rho_3 h_3 \int_0^L P \left(|\lambda u^3|^2 \right)_x dx \\
& + E_3 h_3 \int_0^L P \left(|u_x^3|^2 \right)_x dx + \rho_1 I_1 \int_0^L P \left(|\lambda y^1|^2 \right)_x dx + (E_1 I_1)^{-1} \int_0^L P \left(|S^1|^2 \right)_x dx \\
& + \rho_3 I_3 \int_0^L P \left(|\lambda y^3|^2 \right)_x dx + (E_3 I_3)^{-1} \int_0^L P \left(|S^2|^2 \right)_x dx + \rho h \int_0^L P \left(|\lambda \omega|^2 \right)_x dx \\
& + 3EI \int_0^L P \left(|\omega_{xx}|^2 \right)_x dx + G_h \int_0^L P \left(|\omega_x|^2 \right)_x dx - \underbrace{\int_0^L P \left(|\tau|^2 \right)_x dx}_{=\frac{o(1)}{\lambda}} \\
& - \underbrace{\Re \left(2EI \int_0^L P'' \omega_{xx} \overline{\omega_x} dx \right)}_{=\frac{o(1)}{\lambda^{\frac{1}{2}}}} - \underbrace{\Re \left(2h_2 \int_0^L P' \tau \overline{\omega_x} dx \right)}_{=\frac{o(1)}{\lambda}} \\
& - \underbrace{\Re \left(2G_1 h_1 \int_0^L P y^1 \overline{y_x^1} dx \right)}_{=\frac{o(1)}{\lambda}} - \underbrace{\Re \left(2G_3 h_3 \int_0^L P y^3 \overline{y_x^3} dx \right)}_{=\frac{o(1)}{\lambda}} = \frac{o(1)}{\lambda}.
\end{aligned} \tag{4.200}$$

Thus, we obtain our desired result (4.191). \square

Now, we fix the cut-off functions $\theta_1, \theta_2 \in C_0^\infty([0, L])$ satisfying $0 \leq \theta_1(x) \leq 1$, $0 \leq \theta_2(x) \leq 1$, for all $x \in [0, L]$, and

$$\theta_1(x) = \begin{cases} 1 & \text{if } x \in (0, a + \epsilon), \\ 0 & \text{if } x \in (a + 2\epsilon, L) \end{cases} \quad \text{and} \quad \theta_2(x) = \begin{cases} 0 & \text{if } x \in (0, b - 2\epsilon), \\ 1 & \text{if } x \in (b - \epsilon, L). \end{cases}$$

Lemma 5.10. *The solution $U \in D(\mathcal{A}_j)$ of system (4.129)-(4.138) satisfies the following asymptotic behavior estimation:*

$$\|U\|_{\mathcal{H}_j} = o(1). \tag{4.201}$$

Proof. First, take $P = x\theta_1(x)$ in (4.191), we get

$$\begin{aligned}
& - \int_0^L (\theta_1 + x\theta_1') \left(\rho_1 h_1 |\lambda u^1|^2 + E_1 h_1 |u_x^1|^2 + \rho_3 h_3 |\lambda u^3|^2 + E_3 h_3 |u_x^3|^2 \right) dx \\
& - \int_0^L (\theta_1 + x\theta_1') \left(\rho_1 I_1 |\lambda y^1|^2 + E_1 I_1 |S^1|^2 + \rho_3 I_3 |\lambda y^3|^2 + E_3 I_3 |S^2|^2 \right) dx \\
& - \int_0^L (\theta_1 + x\theta_1') \left(\rho h |\lambda \omega|^2 + 3EI |\omega_{xx}|^2 + G_h |\omega_x|^2 \right) dx = o(1).
\end{aligned} \tag{4.202}$$

Using Lemmas 5.2, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8 and the definition of θ_1 , we have

$$\begin{aligned}
& \int_0^{a+\epsilon} \left(\rho_1 h_1 |\lambda u^1|^2 + E_1 h_1 |u_x^1|^2 + \rho_3 h_3 |\lambda u^3|^2 + E_3 h_3 |u_x^3|^2 \right) dx \\
& + \int_0^{a+\epsilon} \left(\rho_1 I_1 |\lambda y^1|^2 + E_1 I_1 |S^1|^2 + \rho_3 I_3 |\lambda y^3|^2 + E_3 I_3 |S^2|^2 \right) dx \\
& + \int_0^{a+\epsilon} \left(\rho h |\lambda \omega|^2 + 2EI |\omega_{xx}|^2 + G_h |\omega_x|^2 \right) dx = o(1).
\end{aligned} \tag{4.203}$$

Similarly, by taking $P = (x - L)\theta_2$, we see that

$$\begin{aligned} & \int_{b+\epsilon}^L \left(\rho_1 h_1 |\lambda u^1|^2 + E_1 h_1 |u_x^1|^2 + \rho_3 h_3 |\lambda u^3|^2 + E_3 h_3 |u_x^3|^2 \right) dx \\ & + \int_{b+\epsilon}^L \left(\rho_1 I_1 |\lambda y^1|^2 + E_1 I_1 |S^1|^2 + \rho_3 I_3 |\lambda y^3|^2 + E_3 I_3 |S^2|^2 \right) dx \\ & + \int_{b+\epsilon}^L \left(\rho h |\lambda \omega|^2 + 2EI |\omega_{xx}|^2 + G_h |\omega_x|^2 \right) dx = o(1). \end{aligned} \quad (4.204)$$

Finally, using equations (4.203), (4.204) and Lemma 5.2, (4.201) follows. \square

Proof of Theorem 5.1. For $j = 1, 2$, from Lemma 5.10, we have $\|U\|_{\mathcal{H}_j} = o(1)$, which contradicts (4.127). This implies that

$$\sup_{\lambda \in \mathbb{R}} \left\| (i\lambda I - \mathcal{A}_j)^{-1} \right\|_{\mathcal{L}(\mathcal{H}_j)} = O(\lambda^4).$$

The result follows from Theorem 2.10.

6 Conclusion and comments

In this paper, we investigated the asymptotic stability of the generalized Rao-Nakra beam subject to Kelvin-Voigt damping and under various boundary conditions. First, we proved the well-posedness using a semigroup approach. Then, we analysed the strong stability in two cases using Arendt-Batty criteria. We summarize the obtained results as follows:

- When the five dampings are globally distributed, no condition on the system coefficients is needed for stability.
- When the shear angle displacements of the top and bottom layers are damped, we need $G_1 \neq G_3$ and $\frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}$ for stability.

Second, we studied the energy decay rate by using a frequency domain approach combined with piecewise multiplier technique in two cases depending on the regularity of the coefficient functions $D_i, i = 1, \dots, 5$, the localization of their supports and their numbers:

- In the case when the five dampings are globally distributed, we proved an analytic stability.
- In the case of only two local damping D_2 and D_5 acting on the shear angle displacements of the top and bottom layers, i.e assumption (SSC2) holds, we showed a polynomial energy decay rate of type $t^{-\frac{1}{2}}$.

In conclusion, we provide the subsequent observations.

In systems involving partially damped wave equations that are interconnected through velocity or displacement, such as the Timoshenko beam system or the Bresse beam system, it is widely recognized that damping proves to be more impactful when the wave speeds are equal. That's what we saw when authors in [4] established the exponential stability of the solution if the three waves have the same speed of wave propagation and they proved that the energy of the system decays polynomially otherwise (also, see the works [13, 99, 107, 110]). However, for the Rao-Nakra sandwich beam or its generalized version, the opposite is true:

The key to stabilization is to break up the symmetry. This is the observation that Akil and Liu, in [10], encountered during their study of the generalized Rao-Nakra beam subject to viscous damping:

- When both the longitudinal displacement of the top layer and the shear angle of the (top or bottom)

layer are damped, no condition on the system coefficients is needed for stability.

- When only the longitudinal displacement of the top/bottom layer is damped, we need $G_1 \neq G_3$ for stability.
- When only the shear displacement of the top/bottom layer is damped, we need $\frac{E_1}{\rho_1} \neq \frac{E_3}{\rho_3}$ for stability.

Thereby, it would be interesting to solve the stability question of system (4.1) with fractional viscoelastic damping.

Some stability and non-stability results for a Cauchy problem of Bresse-type with frictional damping on the vertical displacement

Résumé

Dans cette partie, nous étudions la stabilité d'un système de type Bresse dans la ligne entière \mathbb{R} avec un amortissement frictionnel travaillant uniquement sur la première équation (déplacement vertical). Nos objectifs sont de prouver certains résultats de stabilité et de non-stabilité en fonction des paramètres du système. Plus précisément, nous prouvons que, dans certains cas, le système est polynomialement stable, et dans d'autres, la solution ne converge pas du tout vers zéro. Les preuves sont basées sur la méthode de l'énergie et l'analyse de Fourier combinées avec des fonctions de poids bien choisies.

Abstract

In this part, we study the stability of a Bresse-type system in the whole line \mathbb{R} with frictional damping working only on the first equation (vertical displacement). Our objectives are proving some stability and non-stability results depending on the parameters in the system. More precisely, we prove that, in some cases, the system is polynomially stable, and in some other ones, the solution does not converge to zero at all. The proofs are based on the energy method and Fourier analysis combined with well chosen weight functions.

1 Introduction

In this paper, we investigate the stability of the Bresse system (known as the circular arch problem) in the whole space \mathbb{R} with frictional damping acting only on the first equation. Namely, we consider

$$\begin{cases} \varphi_{tt} - (\varphi_x - \psi - l\omega)_x - k^2 l(\omega_x - l\varphi) + \gamma\varphi_t = 0, \\ \psi_{tt} - a^2\psi_{xx} - (\varphi_x - \psi - l\omega) = 0, \\ \omega_{tt} - k^2(\omega_x - l\varphi)_x - l(\varphi_x - \psi - l\omega) = 0, \end{cases} \quad (5.1)$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t)(x, 0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1)(x), \quad (5.2)$$

where $(x, t) \in \mathbb{R} \times \mathbb{R}_+$ and a, l, γ and k are positive constants. The functions $\omega(x, t)$, $\varphi(x, t)$ and $\psi(x, t)$ are, respectively, the longitudinal displacements, the vertical displacement of the beam and the rotation angle of the linear filaments material.

Before stating our results, let us first mention some other results related to the problem we address.

Case of bounded domain: The initial boundary value problem associated to (5.1) has been considered by many researchers recently. Liu-Rao [75] considered the Bresse system with two different dissipative mechanism, where the authors proved that the solutions decay exponentially to zero if and only if the velocities of wave propagations are the same. Otherwise, the solutions decay polynomially to zero provided that the boundary conditions is of Dirichlet-Neumann-Neumann or Dirichlet-Dirichlet-Dirichlet type. Fatori and Rivera [35], showed that, in general, the Bresse system is not exponentially stable but the polynomial stability holds with rates that depend on the wave propagations and the regularity of the initial data. The Bresse system with one frictional damping acting on the rotation angle was considered by Alabau Boussouira et al. [27], where the authors proved that the Bresse system is exponentially stable if and only if the velocities of waves propagations are the same. Otherwise, when the velocities are not the same, the system is not exponentially stable and they proved that the solution in this case goes to zero polynomially. Also, Noun and Wehbe [84] extended the results in [27] by considering only one locally distributed damping. There are numbers of publications concerning the stabilization and decay rate of the Bresse system, we also direct the reader to the cited references [34, 39, 51, 52, 98, 109] and the references therein.

The above mentioned works about the time decay of solutions of the Bresse system are concerned with the case of the bounded domain where the semigroup approach and the Poincaré inequality play crucial roles.

Case of unbounded domain: For the Cauchy problem, there are only few results. The decay rate of the solution of the problem (5.1)-(5.2) has been first studied by Soufyane and Said-Houari in [96] where they investigated the relationship between the frictional damping terms, the wave speeds of propagation and their influence on the decay rate of the solution in presence of more than one frictional damping. Moreover, Tej-Eddine Ghoul, Moez Khenissi and Belkacem Said-Houari in [37] considered the Bresse system with frictional damping terms and proved some optimal decay results for the L^2 -norm of the solution. Also, they showed that the solution does not decay at all if we consider just one damping term acting on the second equation of the system. On the other hand, by considering one damping term alone acting on the third equation, they proved that this damping term is strong enough to stabilize the whole system, by presenting a new method that is based on the eigenvalues expansion.

For stability and non-stability results for other cauchy problems (laminated Timoshenko beam, Timoshenko beam, Rao-Nakra sandwich beam, coupled wave equations) under various control mechanisms (frictional dampings, infinite memories and heat conduction effects), see [41, 44, 45, 47, 53, 47].

Thus, the goal of this work is to investigate the Cauchy problem of the Bresse system with one frictional damping via the vertical displacement. The proof is based on the energy method combined with the Fourier analysis (by using the transformation in the Fourier space) and well chosen weight function.

This paper is organised as follows: in section 2, we state the problem. Section 3 is devoted to prove some formulas and lemmas which will be the key of our proofs. In section 4, under condition (5.22), we investigate the asymptotic behavior of $\widehat{U}(\xi, t)$ when t goes to infinity. However, under condition (5.23), we prove that the solution does not converge to zero when t goes to infinity. Finally, in section

5, we prove our polynomial stability estimates under condition (5.22).

Notations. Let $\hat{f} = \mathcal{F}[f]$ be the Fourier transform of f given by

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) := \int_{\mathbb{R}} f(x)e^{-i\xi x} dx,$$

and let $\mathcal{F}[f]^{-1}$ be the inverse Fourier transform of f defined by

$$\mathcal{F}[f]^{-1}(x) := \int_{\mathbb{R}} f(\xi)e^{i\xi x} d\xi.$$

For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R})$ the usual Lebesgue space on \mathbb{R} with the norm $\|\cdot\|_{L^p}$. For a non-negative integer s , $H^s = H^s(\mathbb{R})$ denotes the classical Sobolev space, equipped with the norm $\|\cdot\|_{H^s}$. Let $L^1_1 = L^1_1(\mathbb{R})$ denote the weighted L^1 space on \mathbb{R} with the norm

$$\|f\|_{L^1_1} = \int_{\mathbb{R}} (1 + |x|)|f(x)| dx.$$

Finally, for a non-negative integer k , $C^k(I; H^s)$ denotes the space of k -times continuously differentiable functions on the interval I with values in the Sobolev space H^s .

2 Statement of the problem

Let us first write system (5.1)-(5.2) as a first order Cauchy system of the form

$$\begin{cases} U_t + AU_x + LU = 0, & x \in \mathbb{R}, t > 0, \\ U(x, 0) = U_0(x), & x \in \mathbb{R}, \end{cases} \quad (5.3)$$

where A is a real symmetric matrix and L is non-negative definite matrix. To this end, we introduce the following variables:

$$v = \varphi_x - \psi - l\omega, \quad u = \varphi_t, \quad z = a\psi_x, \quad y = \psi_t, \quad \phi = k(\omega_x - l\varphi), \quad \eta = \omega_t,$$

$$U = (v, u, z, y, \phi, \eta)^T \quad \text{and} \quad U_0(x) = (v, u, z, y, \phi, \eta)^T(x, 0).$$

Then, using the three equations of (5.1), we get

$$\begin{cases} v_t - u_x + y + l\eta = 0, \\ u_t - v_x - lk\phi + \gamma u = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v = 0, \\ \phi_t - k\eta_x + lku = 0, \\ \eta_t - k\phi_x - lv = 0. \end{cases} \quad (5.4)$$

Consequently, system (5.1)-(5.2) can be rewritten in the form (5.3) with

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k \\ 0 & 0 & 0 & 0 & -k & 0 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & l \\ 0 & \gamma & 0 & 0 & -lk & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & lk & 0 & 0 & 0 & 0 \\ -l & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By taking the Fourier transform of (5.3), we obtain the following Cauchy problem of a first order system:

$$\begin{cases} \widehat{U}_t + i\xi A \widehat{U} + L \widehat{U} = 0, & \xi \in \mathbb{R}, t > 0, \\ \widehat{U}(\xi, 0) = \widehat{U}_0(\xi), & \xi \in \mathbb{R}. \end{cases} \quad (5.5)$$

The solution of (5.5) is given by

$$\widehat{U}(\xi, t) = e^{t\widehat{\Phi}(\xi)} \widehat{U}_0(\xi), \quad (5.6)$$

where

$$\widehat{\Phi}(\xi) = -(i\xi A + L). \quad (5.7)$$

The first equation of (5.5) can be rewritten into the following form (that is, the Fourier transform of (5.4)):

$$\begin{cases} \widehat{v}_t - i\xi \widehat{u} + \widehat{y} + l\widehat{\eta} = 0, \\ \widehat{u}_t - i\xi \widehat{v} - lk\widehat{\phi} + \gamma\widehat{u} = 0, \\ \widehat{z}_t - ai\xi \widehat{y} = 0, \\ \widehat{y}_t - ai\xi \widehat{z} - \widehat{v} = 0, \\ \widehat{\phi}_t - ki\xi \widehat{\eta} + lk\widehat{u} = 0, \\ \widehat{\eta}_t - ki\xi \widehat{\phi} - l\widehat{v} = 0. \end{cases} \quad (5.8)$$

Let us define the energy functional of (5.5) by

$$\widehat{E}(\xi, t) = \frac{1}{2} \|\widehat{U}(\xi, t)\|^2 = \frac{1}{2} (|\widehat{v}|^2 + |\widehat{u}|^2 + |\widehat{z}|^2 + |\widehat{y}|^2 + |\widehat{\phi}|^2 + |\widehat{\eta}|^2) (\xi, t). \quad (5.9)$$

Lemma 2.1. *Let \widehat{U} be the solution of (5.5). Then the energy \widehat{E} satisfies*

$$\frac{d}{dt} \widehat{E}(\xi, t) = -\gamma |\widehat{u}(\xi, t)|^2, \quad \xi \in \mathbb{R}, t \in \mathbb{R}_+. \quad (5.10)$$

Proof. Multiplying the equations of (5.8) by \bar{v} , \bar{u} , \bar{z} , \bar{y} , $\bar{\phi}$ and $\bar{\eta}$, respectively, adding the obtained equalities and taking the real part, we obtain (5.10). It is clear from (5.10) that \widehat{E} a non-increasing function, and so, (5.5) is a dissipative system. \square

3 Preliminaries

In this section, we prove some formulas and two lemmas which will be the key of our proofs. We start by proving the next nine formulas.

Multiplying (5.8)₄ by $i\xi \bar{z}$ and (5.8)₃ by $i\xi \bar{y}$, taking the real parts and subtracting the resulting equations, we get

$$\frac{d}{dt} \Re(i\xi \widehat{y} \bar{z}) + a\xi^2 (|\widehat{z}|^2 - |\widehat{y}|^2) = \Re(i\xi \widehat{v} \bar{z}). \quad (5.11)$$

Multiplying (5.8)₃ by $i\xi \bar{\eta}$ and (5.8)₆ by $-i\xi \bar{z}$, adding the results and taking the real part, we have

$$\frac{d}{dt} \Re(i\xi \widehat{z} \bar{\eta}) = -a\xi^2 \Re(\widehat{y} \bar{\eta}) + k\xi^2 \Re(\widehat{\phi} \bar{z}) - l\Re(i\xi \widehat{v} \bar{z}). \quad (5.12)$$

Multiplying the (5.8)₁ by $\bar{\eta}$ and (5.8)₆ by \bar{v} , adding the results and taking the real part, we find

$$\frac{d}{dt} \Re(\widehat{v} \bar{\eta}) + l|\widehat{\eta}|^2 - l|\widehat{v}|^2 = \Re(i\xi \widehat{u} \bar{\eta}) - \Re(\widehat{y} \bar{\eta}) + k\Re(i\xi \widehat{\phi} \bar{v}). \quad (5.13)$$

Multiplying the (5.8)₂ by $\bar{\hat{\phi}}$ and (5.8)₅ by $\bar{\hat{u}}$, adding the results and taking the real part, we arrive at

$$\frac{d}{dt} \Re(\hat{u}\bar{\hat{\phi}}) + lk|\hat{u}|^2 - lk|\hat{\phi}|^2 = k\Re(i\xi\hat{\eta}\bar{\hat{u}}) - \gamma\Re(\hat{u}\bar{\hat{\phi}}) + \Re(i\xi\hat{v}\bar{\hat{\phi}}). \quad (5.14)$$

Multiplying (5.8)₁ by $\bar{\hat{y}}$ and (5.8)₄ by $\bar{\hat{v}}$, adding the results and taking the real part, it follows

$$\frac{d}{dt} \Re(\hat{v}\bar{\hat{y}}) + |\hat{y}|^2 - |\hat{v}|^2 = \Re(i\xi\hat{u}\bar{\hat{y}}) - l\Re(\hat{\eta}\bar{\hat{y}}) + a\Re(i\xi\hat{z}\bar{\hat{v}}). \quad (5.15)$$

Multiplying the (5.8)₂ by $a\bar{\hat{z}}$ and (5.8)₃ by $a\bar{\hat{u}}$, adding the results and taking the real part, we observe that

$$\frac{d}{dt} \Re(a\hat{u}\bar{\hat{z}}) = a\Re(i\xi\hat{v}\bar{\hat{z}}) + lka\Re(\hat{\phi}\bar{\hat{z}}) - a\gamma\Re(\hat{u}\bar{\hat{z}}) + a^2\Re(i\xi\hat{y}\bar{\hat{u}}). \quad (5.16)$$

Multiplying (5.8)₂ by $i\xi\bar{\hat{v}}$ and (5.8)₁ by $-i\xi\bar{\hat{u}}$, adding the results and taking the real part, we remark that

$$\frac{d}{dt} \Re(i\xi\hat{u}\bar{\hat{v}}) + \xi^2|\hat{v}|^2 - \xi^2|\hat{u}|^2 = lk\Re(i\xi\hat{\phi}\bar{\hat{v}}) - \gamma\Re(i\xi\hat{u}\bar{\hat{v}}) + \Re(i\xi\hat{y}\bar{\hat{u}}) + l\Re(i\xi\hat{\eta}\bar{\hat{u}}). \quad (5.17)$$

Multiplying (5.8)₆ by $i\xi\bar{\hat{\phi}}$ and (5.8)₅ by $-i\xi\bar{\hat{\eta}}$, adding the results and taking the real part, we obtain

$$\frac{d}{dt} \Re(i\xi\hat{\eta}\bar{\hat{\phi}}) + \xi^2k|\hat{\phi}|^2 - \xi^2k|\hat{\eta}|^2 = l\Re(i\xi\hat{v}\bar{\hat{\phi}}) + lk\Re(i\xi\hat{u}\bar{\hat{\eta}}). \quad (5.18)$$

Multiplying (5.8)₄ by $i\xi\bar{\hat{\phi}}$ and (5.8)₅ by $-i\xi\bar{\hat{y}}$, adding the results and taking the real part, we get

$$\frac{d}{dt} \Re(i\xi\hat{y}\bar{\hat{\phi}}) = -a\xi^2\Re(\hat{z}\bar{\hat{\phi}}) + \Re(i\xi\hat{v}\bar{\hat{\phi}}) + k\xi^2\Re(\hat{\eta}\bar{\hat{y}}) + lk\Re(i\xi\hat{u}\bar{\hat{y}}). \quad (5.19)$$

We finish this section by giving two lemmas and their proof, which will be used in the next sections (see [44]).

Lemma 3.1. *Let σ , p and q be real numbers such that $\sigma > -1$ and $p, q > 0$. Then there exists $C_{\sigma,p,q} > 0$ such that*

$$\int_0^1 \xi^\sigma e^{-qt\xi^p} d\xi \leq C_{\sigma,p,q} (1+t)^{-(\sigma+1)/p}, \quad t \in \mathbb{R}_+. \quad (5.20)$$

Proof. For $0 \leq t \leq 1$, (5.20) is true, for any $C_{\sigma,p,q} \geq \frac{2^{(\sigma+1)/p}}{\sigma+1}$. For $t > 1$, we have

$$\int_0^1 \xi^\sigma e^{-qt\xi^p} d\xi = \int_0^1 \xi^{\sigma+1-p} e^{-qt\xi^p} \xi^{p-1} d\xi = \int_0^1 (\xi^p)^{(\sigma+1-p)/p} e^{-qt\xi^p} \xi^{p-1} d\xi.$$

Taking $\tau = qt\xi^p$. Then

$$\xi^p = \frac{\tau}{qt} \quad \text{and} \quad \xi^{p-1} d\xi = \frac{1}{pqt} d\tau.$$

Replacing in the above integral, we find

$$\int_0^1 (\xi^p)^{(\sigma+1-p)/p} e^{-qt\xi^p} \xi^{p-1} d\xi = \int_0^{qt} \left(\frac{\tau}{qt}\right)^{(\sigma+1-p)/p} e^{-\tau} \frac{1}{pqt} d\tau$$

$$\leq \frac{1}{p(qt)^{(\sigma+1)/p}} \int_0^{+\infty} \tau^{(\sigma+1-p)/p} e^{-\tau} d\tau \leq \frac{2^{(\sigma+1)/p}}{p q^{(\sigma+1)/p}} C_{\sigma,p} (t+1)^{-(\sigma+1)/p},$$

where

$$C_{\sigma,p} = \int_0^{+\infty} \tau^{(\sigma+1-p)/p} e^{-\tau} d\tau,$$

which is a convergent integral, for any $\sigma > -1$ and $p > 0$. This completes the proof of (5.20), where

$$C_{\sigma,p,q} = \max \left\{ \frac{2^{(\sigma+1)/p}}{\sigma+1}, \frac{2^{(\sigma+1)/p}}{p q^{(\sigma+1)/p}} C_{\sigma,p} \right\}.$$

□

Lemma 3.2. For any positive real numbers σ_1 , σ_2 and σ_3 , we have

$$\sup_{|\xi| \geq 1} |\xi|^{-\sigma_1} e^{-\sigma_2 t |\xi|^{-\sigma_3}} \leq (1 + \sigma_1 / (\sigma_2 \sigma_3))^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3}, \quad t \in \mathbb{R}_+. \quad (5.21)$$

Proof. It is clear that (5.21) is satisfied for $t = 0$. Let $t > 0$ and $h(x) = x^{-\sigma_1} e^{-\sigma_2 t x^{-\sigma_3}}$, for $x \geq 1$. Direct and simple computations show that

$$h'(x) = (\sigma_2 \sigma_3 t x^{-\sigma_3} - \sigma_1) x^{-\sigma_1-1} e^{-\sigma_2 t x^{-\sigma_3}}.$$

If $t \geq \sigma_1 / (\sigma_2 \sigma_3)$, then

$$\begin{aligned} h(x) &\leq h((\sigma_2 \sigma_3 t) / \sigma_1)^{1/\sigma_3} = ((\sigma_2 \sigma_3) / \sigma_1)^{-\sigma_1 / \sigma_3} e^{-\sigma_1 / \sigma_3} (1 + 1/t)^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3} \\ &\leq ((\sigma_2 \sigma_3) / \sigma_1)^{-\sigma_1 / \sigma_3} (1 + (\sigma_2 \sigma_3) / \sigma_1)^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3} = (1 + \sigma_1 / (\sigma_2 \sigma_3))^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3}, \end{aligned}$$

which gives (5.21) by taking $x = |\xi|$. If $0 < t < \sigma_1 / (\sigma_2 \sigma_3)$, then

$$h(x) \leq h(1) = e^{-\sigma_2 t} (1+t)^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3} \leq (1 + \sigma_1 / (\sigma_2 \sigma_3))^{\sigma_1 / \sigma_3} (1+t)^{-\sigma_1 / \sigma_3},$$

so, (5.21) holds true, for $x = |\xi|$. □

4 Stability of (5.5)

In this section, we present and show some stability and non-stability results for (5.5) depending on the following conditions:

$$(a-k)(k-1) > 0 \quad \text{and} \quad \max \left\{ \frac{l[l^2(a^2-1)(k^2-1) + k^2]}{l^2(a^2-k^2)(k^2-1) + k^2}, \frac{l(a^2-1)}{a^2-k^2} \right\} < \frac{l^2 a^2 + 1}{l a^2} \quad (5.22)$$

and

$$a = k = \frac{\sqrt{l^2 + 1}}{l}. \quad (5.23)$$

Theorem 4.1. Let \widehat{U} be the solution of (5.5).

1. Assume that (5.22) holds. Then there exist $c, \tilde{c} > 0$ such that

$$\|\widehat{U}(\xi, t)\|^2 \leq \tilde{c} e^{-c f(\xi) t} \|\widehat{U}_0(\xi)\|^2, \quad \xi \in \mathbb{R}, \quad t \in \mathbb{R}_+, \quad (5.24)$$

where

$$f(\xi) = \frac{\xi^2}{\xi^4 + \xi^2 + 1}. \quad (5.25)$$

2. Assume that (5.23) holds. Then $\|\widehat{U}(\xi, t)\|$ does not converge to zero when time t goes to infinity.

Proof. Case 1: (5.22) holds. Let $\alpha_1, \dots, \alpha_9$ be real numbers to be fixed later. Let us define the functionals F_0 and F as follows:

$$F_0(\xi, t) = \Re \left[i \xi \hat{u} \left([(\alpha_3 + k\alpha_4)\xi^2 - l\alpha_7 + lk\alpha_8] \bar{\eta} + [(\alpha_5 - a^2\alpha_6)\xi^2 - \alpha_7 + lk\alpha_9] \bar{y} - \gamma\alpha_7 \bar{v} \right) \right] \\ + \Re \left[\hat{u} \left(\gamma\alpha_4 \xi^2 \bar{\phi} - a\gamma\alpha_6 \xi^2 \bar{z} \right) \right] \quad (5.26)$$

and

$$F(\xi, t) = \Re \left(i \alpha_1 \xi \hat{y} \bar{z} + i \alpha_2 \xi \hat{z} \bar{\eta} + \alpha_3 \xi^2 \hat{v} \bar{\eta} - \alpha_4 \xi^2 \hat{u} \bar{\phi} \right) \\ + \Re \left(\alpha_5 \xi^2 \hat{v} \bar{y} + \alpha_6 a \xi^2 \hat{u} \bar{z} + i \alpha_7 \xi \hat{u} \bar{v} + i \alpha_8 \xi \hat{\eta} \bar{\phi} + i \alpha_9 \xi \hat{y} \bar{\phi} \right). \quad (5.27)$$

By multiplying (5.11)-(5.19) by $\alpha_1, \alpha_2, \alpha_3 \xi^2, -\alpha_4 \xi^2, \alpha_5 \xi^2, \alpha_6 \xi^2, \alpha_7, \alpha_8$ and α_9 , respectively, and adding the obtained identities, we deduce that

$$\frac{d}{dt} F(\xi, t) = F_0(\xi, t) + \Re \left(i A_1 \xi \hat{v} \bar{z} + A_2 \hat{y} \bar{\eta} + i A_3 \xi \hat{\phi} \bar{v} + A_4 \hat{\phi} \bar{z} \right) \\ + A_5 \xi^2 |\hat{v}|^2 + A_6 \xi^2 |\hat{y}|^2 + A_7 \xi^2 |\hat{z}|^2 + A_8 \xi^2 |\hat{\phi}|^2 + A_9 \xi^2 |\hat{\eta}|^2 + A_{10} \xi^2 |\hat{u}|^2, \quad (5.28)$$

where

$$A_1 = a(\alpha_6 - \alpha_5)\xi^2 + \alpha_1 - l\alpha_2, \quad A_2 = -(a\alpha_2 + \alpha_3 + l\alpha_5 - k\alpha_9)\xi^2, \quad A_3 = (k\alpha_3 + \alpha_4)\xi^2 + lk\alpha_7 - l\alpha_8 - \alpha_9, \\ A_4 = (k\alpha_2 + lka\alpha_6 - a\alpha_9)\xi^2, \quad A_5 = l\alpha_3 + \alpha_5 - \alpha_7, \quad A_6 = a\alpha_1 - \alpha_5, \quad A_7 = -a\alpha_1, \\ A_8 = -lk\alpha_4 - k\alpha_8, \quad A_9 = -l\alpha_3 + k\alpha_8 \quad \text{and} \quad A_{10} = lk\alpha_4 + \alpha_7.$$

We choose

$$\alpha_1 = a(\alpha_5 - 1)\xi^2 - l^2 a + \frac{la}{k}\alpha_9, \quad \alpha_2 = -la + \frac{a}{k}\alpha_9, \quad \alpha_3 = la^2 - l\alpha_5 + \frac{k^2 - a^2}{k}\alpha_9, \quad (5.29)$$

$$\alpha_6 = 1 \quad \text{and} \quad \alpha_7 = \frac{1}{k}\alpha_8 + \frac{1}{lk}\alpha_9 - \frac{1}{lk} \left[\alpha_4 - kl\alpha_5 + kla^2 + (k^2 - a^2)\alpha_9 \right] \xi^2. \quad (5.30)$$

Then $A_1 = A_2 = A_3 = A_4 = 0$, and therefore

$$\frac{d}{dt} F(\xi, t) = F_0(\xi, t) + A_5 \xi^2 |\hat{v}|^2 + A_6 \xi^2 |\hat{y}|^2 + A_7 \xi^2 |\hat{z}|^2 + A_8 \xi^2 |\hat{\phi}|^2 + A_9 \xi^2 |\hat{\eta}|^2 + A_{10} \xi^2 |\hat{u}|^2 \quad (5.31)$$

with, according to (5.29) and (5.30),

$$A_5 = (1 - l^2 - \xi^2)\alpha_5 + a^2(l^2 + \xi^2) + \frac{1}{lk}\alpha_4 \xi^2 - \frac{1}{k}\alpha_8 + \left[\frac{k^2 - a^2}{k} \left(l + \frac{1}{l}\xi^2 \right) - \frac{1}{lk} \right] \alpha_9,$$

$$A_6 = a^2(\alpha_5 - 1)\xi^2 - \alpha_5 - l^2 a^2 + \frac{la^2}{k}\alpha_9, \quad A_7 = a^2(1 - \alpha_5)\xi^2 + l^2 a^2 - \frac{la^2}{k}\alpha_9,$$

$$A_8 = -lk\alpha_4 - k\alpha_8, \quad A_9 = l^2 \alpha_5 - l^2 a^2 + k\alpha_8 - \frac{l(k^2 - a^2)}{k}\alpha_9$$

and

$$A_{10} = lk\alpha_4 + \frac{1}{k}\alpha_8 + \frac{1}{lk}\alpha_9 - \frac{1}{lk} \left[\alpha_4 - kl\alpha_5 + kla^2 + (k^2 - a^2)\alpha_9 \right] \xi^2.$$

We remark that the last term in (5.31) and all the terms of F_0 depend on \hat{u} , so they will be estimated using (5.10). Then we need to have negative values of A_5, \dots, A_9 . To do so, first, we make A_5, A_6 and A_7 independent on ξ by choosing

$$\alpha_4 = (a^2 - k^2)\alpha_9 + lk(1 - a^2) \quad \text{and} \quad \alpha_5 = 1. \quad (5.32)$$

Therefore,

$$A_5 = l^2 a^2 + 1 - l^2 - \frac{1}{k} \alpha_8 + \left[\frac{l(k^2 - a^2)}{k} - \frac{1}{lk} \right] \alpha_9, \quad A_6 = - (l^2 a^2 + 1) + \frac{la^2}{k} \alpha_9, \quad (5.33)$$

$$A_7 = l^2 a^2 - \frac{la^2}{k} \alpha_9, \quad A_8 = -k \alpha_8 + lk (k^2 - a^2) \alpha_9 + l^2 k^2 (a^2 - 1) \quad (5.34)$$

and

$$A_9 = l^2(1 - a^2) + k \alpha_8 - \frac{l(k^2 - a^2)}{k} \alpha_9. \quad (5.35)$$

Thanks to the second inequality in (5.22), one can choose

$$\max \left\{ lk, \frac{lk [l^2(a^2 - 1)(k^2 - 1) + k^2]}{l^2(a^2 - k^2)(k^2 - 1) + k^2}, \frac{lk(a^2 - 1)}{a^2 - k^2} \right\} < \alpha_9 < \frac{k(l^2 a^2 + 1)}{la^2},$$

and according to the first inequality in (5.22) and the choice of α_9 , it is possible to choose α_8 such that

$$\max \left\{ \left[l(k^2 - a^2) - \frac{1}{l} \right] \alpha_9 + k [l^2(a^2 - 1) + 1], l [(k^2 - a^2) \alpha_9 + lk(a^2 - 1)] \right\} < \alpha_8 < \frac{l(k^2 - a^2)}{k^2} \alpha_9 + \frac{l^2(a^2 - 1)}{k}.$$

Hence, from (5.33)-(5.35), we observe that the choice of α_8 and α_9 leads to

$$\max \{A_5, \dots, A_9\} < 0,$$

and then, for $c_0 := -\max \{A_5, \dots, A_9\} > 0$, (5.31) implies that

$$\frac{d}{dt} F(\xi, t) \leq F_0(\xi, t) - c_0 \xi^2 (|\hat{v}|^2 + |\hat{y}|^2 + |\hat{z}|^2 + |\hat{\phi}|^2 + |\hat{\eta}|^2) + A_{10} \xi^2 |\hat{u}|^2. \quad (5.36)$$

Because $\alpha_1, \dots, \alpha_9$ are constant, then $A_{10} = C_1 \xi^2 + C_2$, for two constants C_1 and C_2 , and by applying Young's inequality for the terms of F_0 , we get, for any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that

$$F_0(\xi, t) + A_{10} \xi^2 |\hat{u}|^2 \leq \epsilon \xi^2 (|\hat{v}|^2 + |\hat{y}|^2 + |\hat{z}|^2 + |\hat{\phi}|^2 + |\hat{\eta}|^2) + c_\epsilon (\xi^4 + \xi^2 + 1) |\hat{u}|^2. \quad (5.37)$$

By choosing $0 < \epsilon < c_0$, combining (5.36) and (5.37), we obtain, for the positive constant $c_1 = c_0 - \epsilon$,

$$\frac{d}{dt} F(\xi, t) \leq -c_1 \xi^2 \hat{E}(\xi, t) + c_\epsilon (\xi^4 + \xi^2 + 1) |\hat{u}|^2. \quad (5.38)$$

We introduce the *perturbed Energy* L as follows:

$$L(\xi, t) = \alpha_0 \hat{E}(\xi, t) + \frac{1}{\xi^4 + \xi^2 + 1} F(\xi, t), \quad (5.39)$$

where α_0 is a positive constant to be fixed later. Then, from (5.10), (5.38) and (5.39), we have

$$\frac{d}{dt} L(\xi, t) \leq -c_1 f(\xi) \hat{E}(\xi, t) + (c_\epsilon - \gamma \alpha_0) |\hat{u}|^2, \quad (5.40)$$

where f is defined in (5.25). Moreover, using the definitions of \hat{E} , F and L , respectively, and notice that $\alpha_1, \dots, \alpha_9$ are constants, we arrive at, for some $c_2 > 0$ (independent of α_0),

$$|L(\xi, t) - \alpha_0 \hat{E}(\xi, t)| \leq \frac{c_2 (\xi^2 + |\xi|)}{\xi^4 + \xi^2 + 1} \hat{E}(\xi, t) \leq 2c_2 \hat{E}(\xi, t). \quad (5.41)$$

Therefore, for α_0 large enough such that $\alpha_0 > \max\left\{\frac{c_2}{\gamma}, 2c_2\right\}$, it follows from (5.40) and (5.41) that

$$\frac{d}{dt}L(\xi, t) + c_1 f(\xi) \widehat{E}(\xi, t) \leq 0 \quad (5.42)$$

and

$$c_3 \widehat{E}(\xi, t) \leq L(\xi, t) \leq c_4 \widehat{E}(\xi, t), \quad (5.43)$$

where $c_3 = \alpha_0 - 2c_2 > 0$ and $c_4 = \alpha_0 + 2c_2 > 0$. Consequently, a combination of (5.42) and the second inequality in (5.43) leads, for $c = \frac{c_1}{c_4}$, to

$$\frac{d}{dt}L(\xi, t) + c f(\xi) L(\xi, t) \leq 0. \quad (5.44)$$

Finally, by integration of (5.44) with respect to t and using (5.9) and (5.43), estimate (5.24) follows.

Case 2: (5.23) holds. We show that, for any $\xi \in \mathbb{R}$, the matrix $\widehat{\Phi}(\xi)$ defined by (5.7) has at least a pure imaginary eigenvalue; that is

$$\forall \xi \in \mathbb{R}, \exists \lambda \in \mathbb{C} : \Re(\lambda) = 0, \quad \Im(\lambda) \neq 0 \quad \text{and} \quad \det(\lambda I - \widehat{\Phi}(\xi)) = 0,$$

where I denotes the identity matrix. From the definition of the matrix A and L , we have

$$\lambda I - \widehat{\Phi}(\xi) = \begin{pmatrix} \lambda & -i\xi & 0 & 1 & 0 & l \\ -i\xi & \lambda + \gamma & 0 & 0 & -lk & 0 \\ 0 & 0 & \lambda & -ia\xi & 0 & 0 \\ -1 & 0 & -ia\xi & \lambda & 0 & 0 \\ 0 & lk & 0 & 0 & \lambda & -ik\xi \\ -l & 0 & 0 & 0 & -ik\xi & \lambda \end{pmatrix}.$$

A direct computation shows that

$$\begin{aligned} \det(\lambda I - \widehat{\Phi}(\xi)) &= \lambda(\lambda^2 + a^2\xi^2 + 1) [(\lambda + \gamma)(\lambda^2 + k^2\xi^2) + l^2k^2\lambda] \\ &\quad + (\lambda^2 + a^2\xi^2) [\xi^2(\lambda^2 + k^2\xi^2 + l^2k^2) + l^2(\lambda(\lambda + \gamma) + k^2(\xi^2 + l^2))]. \end{aligned}$$

We see that, under condition (5.23), $\lambda = i\sqrt{a^2\xi^2 + l^2 + 1}$ is a pure imaginary eigenvalue of $\widehat{\Phi}(\xi)$. Consequently, according to (5.6), the solution of (5.5) doesn't converge to zero when time t goes to infinity. This implies the second result of Theorem 4.1. \square

5 Stability of (5.3)

This section is devoted to the proof of the following stability result for (5.3):

Theorem 5.1. Assume that (5.22) holds. Let $N, \ell \in \mathbb{N}^*$ such that $\ell \leq N$, $U_0 \in H^N(\mathbb{R}) \cap L^1(\mathbb{R})$ and U be the solution of (5.3). Then, for any $j = 0, \dots, N - \ell$, there exists $c_0 > 0$ such that

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/4-j/2} \|U_0\|_{L^1(\mathbb{R})} + c_0 (1+t)^{-\ell/2} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}_+. \quad (5.45)$$

Proof. From (5.25), we have (low and high frequencies)

$$f(\xi) \geq \begin{cases} \frac{1}{3} \xi^2 & \text{if } |\xi| \leq 1, \\ \frac{1}{3} \xi^{-2} & \text{if } |\xi| > 1. \end{cases} \quad (5.46)$$

Applying Plancherel's theorem and (5.24), it follows that

$$\begin{aligned} \|\partial_x^j U\|_{L^2(\mathbb{R})}^2 &= \left\| \widehat{\partial_x^j U}(x, t) \right\|_{L^2(\mathbb{R})}^2 = \int_{\mathbb{R}} \xi^{2j} \|\widehat{U}(\xi, t)\|^2 d\xi \\ &\leq \tilde{c} \int_{\mathbb{R}} \xi^{2j} e^{-cf(\xi)t} \|\widehat{U}_0(\xi)\|^2 d\xi \\ &\leq \tilde{c} \int_{|\xi| \leq 1} \xi^{2j} e^{-cf(\xi)t} \|\widehat{U}_0(\xi)\|^2 d\xi + \tilde{c} \int_{|\xi| > 1} \xi^{2j} e^{-cf(\xi)t} \|\widehat{U}_0(\xi)\|^2 d\xi := J_1 + J_2. \end{aligned} \quad (5.47)$$

Using (5.20) (with $\sigma = 2j$, $q = \frac{c}{3}$ and $p = 2$) and (5.46), it appears, for the low frequency region, that (where C denotes a positive constant which can be different from inequality to another)

$$J_1 \leq C \|\widehat{U}_0\|_{L^\infty(\mathbb{R})}^2 \int_{|\xi| \leq 1} \xi^{2j} e^{-\frac{c}{3}t\xi^2} d\xi \leq C(1+t)^{-\frac{1}{2}(1+2j)} \|U_0\|_{L^1(\mathbb{R})}^2. \quad (5.48)$$

In the high frequency region, using (5.46), we entail

$$\begin{aligned} J_2 &\leq C \int_{|\xi| > 1} |\xi|^{2j} e^{-\frac{c}{3}t\xi^{-2}} \|\widehat{U}_0(\xi)\|^2 d\xi \\ &\leq C \sup_{|\xi| > 1} \left\{ |\xi|^{-2\ell} e^{-\frac{c}{3}t|\xi|^{-2}} \right\} \int_{\mathbb{R}} |\xi|^{2(j+\ell)} \|\widehat{U}_0(\xi)\|^2 d\xi, \end{aligned}$$

then, using (5.21) (with $\sigma_1 = 2l$, $\sigma_2 = \frac{c}{3}$ and $\sigma_3 = 2$),

$$J_2 \leq C(1+t)^{-\ell} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}^2. \quad (5.49)$$

Consequently, by combining (5.47)-(5.49), we get (5.45). \square

6 Conclusion and comments

We have studied the cauchy problem of Bresse system in the whole space \mathbb{R} and under only one control given by a frictional damping acting on the first equation only. We proved some stability and non-stability results depending on the parameters in the system. More precisely, under condition (5.22), we proved that the system is polynomially stable by showing that the L^2 -norm of the solution decays with the following rate:

$$\|\partial_x^j U\|_{L^2(\mathbb{R})} \leq c_0 (1+t)^{-1/4-j/2} \|U_0\|_{L^1(\mathbb{R})} + c_0 (1+t)^{-\ell/2} \|\partial_x^{j+\ell} U_0\|_{L^2(\mathbb{R})}, \quad t \in \mathbb{R}_+.$$

On the other hand, under condition (5.23), we showed that the solution does not converge to zero when t goes to infinity.

Thereby, it would be interesting to solve the stability question of (5.3) when (5.22) and (5.23) are not satisfied; that is

$$\left\{ \begin{array}{l} a \neq k \quad \text{and} \quad (a-k)(k-1) \leq 0 \\ \text{or} \\ a \neq k \quad \text{and} \quad \max \left\{ \frac{l[l^2(a^2-1)(k^2-1)+k^2]}{l^2(a^2-k^2)(k^2-1)+k^2}, \frac{l(a^2-1)}{a^2-k^2} \right\} \geq \frac{l^2 a^2 + 1}{la^2} \\ \text{or} \\ k \neq \frac{\sqrt{l^2+1}}{l} \quad \text{and} \quad (a-k)(k-1) \leq 0 \\ \text{or} \\ k \neq \frac{\sqrt{l^2+1}}{l} \quad \text{and} \quad \max \left\{ \frac{l[l^2(a^2-1)(k^2-1)+k^2]}{l^2(a^2-k^2)(k^2-1)+k^2}, \frac{l(a^2-1)}{a^2-k^2} \right\} \geq \frac{l^2 a^2 + 1}{la^2}. \end{array} \right. \quad (5.50)$$

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